서울대학교
SEOUL NATIONAL UNIVERSITY

Homework 2. (2022-24052).

1. (a) Show that the reachability Grammian satisfies the matrix differential equation: $\frac{d}{dt} W_r(t_0,t) = A(t) W_r(t_0,t) + W_r(t_0,t) A^T(t) + B(t) B^T(t)$, $W_r(t_0,t_0) = 0$.

A: For Leibniz Law, we have:

$$\frac{d}{dt}\int_{g(t)}^{h(t)} F(x,t)\cdot dx = F(h,t)\cdot \dot{h} - F(g,t)\cdot \dot{g} + \int_{g(t)}^{h(t)} \frac{\partial F}{\partial t}\cdot dx.$$

For the reachability Grammian $W_r(t_0,t)$, where $g(t) = c$.

Thus $\frac{d}{dt}\int_c^{h(t)} F(x,t)\,dx = F(h,t)\cdot \dot{h} + \int_c^{h(t)} \frac{\partial F}{\partial t}\,dx.$

$$\Rightarrow \frac{d}{dt} W_r(t_0,t) = W_r(t,t)\cdot 1 + \int_{t_0}^{t} \frac{\partial W_r(t_0,t)}{\partial t}\cdot dx.$$

$$= \Phi(t,t)\cdot B(t)\cdot B^T(t)\cdot \Phi^T(t,t) + \int_{t_0}^{t} (A(t)\Phi(t,\tau)B(t)B^T(t)\Phi^T(t,\tau) + \Phi(t,\tau)\cdot B(t)\cdot B^T(t)\cdot \Phi^T(t,\tau)\cdot A^T(t))\cdot d\tau$$

Because of the transition matrix property: $\Phi(t,t) = I$.

$$\Rightarrow \frac{d}{dt} W_r(t_0,t) = B(t)\cdot B^T(t) + \int_{t_0}^{t} (A(t)\Phi(t,\tau)B(t)B^T(t)\Phi^T(t,\tau) + \Phi(t,\tau)B(t)B^T(t)\Phi^T(t,\tau)A(t))d\tau$$

$$= B(t)\cdot B^T(t) + A(t) W_r(t,t_0) + W_r(t,t_0) A^T(t).$$

(b) Show that the inverse of the reachability Grammian satisfies:

$$\frac{d}{dt} W_r^{-1}(t_0,t) = -A^T(t) W_r^{-1}(t_0,t) - W_r^{-1}(t_0,t) A(t) - W_r^{-1}(t_0,t) B(t) B^T(t) W_r^{-1}(t_0,t)$$

for values of t such that the inverse exists.

A: From derivative definition, for inverse matrix:

$$\frac{d W_r^{-1}(t_0,t)}{dt} = \lim_{\Delta t \to 0} \frac{(W_r + \Delta W_r)^{-1} - W_r^{-1}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(W_r+\Delta W_r)^{-1}\cdot W_r\cdot W_r^{-1} - (W_r+\Delta W_r)^{-1}\cdot (W_r+\Delta W_r) W_r^{-1}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(W_r+\Delta W_r)^{-1}\cdot(-\Delta W_r) W_r^{-1}}{\Delta t}$$

$$= -W_r^{-1}\cdot \partial_t W_r\, W_r^{-1} = -W_r^{-1}\cdot (\tfrac{d}{dt} W_r) W_r^{-1}$$

From answer (a).

$$\Rightarrow \frac{d}{dt} W_r^{-1}(t_0,t) = -W_r^{-1}\cdot(A(t)\cdot W_r(t,t_0) + W_r(t,t_0) A^T(t) + B(t) B^T(t))\cdot W_r^{-1}$$

$$= -W_r^{-1}(t,t_0)\cdot A(t) - A^T(t)\cdot W_r^{-1}(t,t_0) - W_r^{-1}(t,t_0)\cdot B(t)\cdot B^T(t)\cdot W_r^{-1}(t,t_0).$$

for t.

(c). Show that the equation:

$$W_r(t_0,t_1) = \Phi(t_1,t)\cdot W_r(t_0,t)\cdot \Phi^T(t_1,t) + W_r(t,t_1) \quad \text{for } \forall t.$$

A: For $W_r(t,t_1) = \int_t^{t_1} \Phi(\tau,t_1) B(\tau) B^T(\tau) \Phi(\tau,t_1)\cdot d\tau.$

$$\Phi(t_1,t)\cdot W_r(t_0,t)\Phi^T(t_1,t) = \Phi(t_1,t)\cdot\int_{t_0}^{t} \Phi(t_0,\tau) B(\tau) B^T(\tau) \Phi^T(t_0,\tau)\cdot d\tau\cdot \Phi^T(t_1,t)$$

$$= \int_{t_0}^{t} \Phi(\tau,t_1)\cdot B(\tau)\cdot B^T(\tau)\cdot \Phi(\tau,t_1)\cdot d\tau.$$

Thus $W_r(t_0,t_1) = \Phi(t_1,t)\, W_r(t_0,t)\, \Phi^T(t_1,t) + W_r(t,t_1).$

$$= \int_{t_0}^{t} \Phi(\tau,t_1)\cdot B(\tau) B^T(\tau)\cdot \Phi(\tau,t_1)\cdot d\tau \quad \text{for } \forall t.$$

河明山

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(d) Establish properties of the observability grammian $W_o(t_0, t_1)$ corresponding to the properties of $W_r(t_0, t_1)$ in above 3 subproblems.

A: As we know, the observability grammian is defined with:

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(t_0, \tau) \cdot C^T(\tau) C(\tau) \cdot \Phi(t_0, \tau)\, d\tau.$$

From (a): $\frac{d}{dt} W_o(t_0, t_1) = (A^T(t) W_o(t_0, t_1) + W_o(t_0, t_1) A(t) + C^T(t) \cdot C(t)) \cdot (-1).$

From (b): $\frac{d}{dt} W_o^{-1}(t_0, t_1) = (-A(t) W_o^{-1}(t_0, t_1) - W_o^{-1}(t_0, t_1) A^T(t) - W_o^{-1}(t_0, t_1) \cdot C^T(t) \cdot C(t) \cdot W_o^{-1}(t_0, t_1)) \cdot (-1).$

From (c): $W_o(t_0, t_1) = \Phi(t_1, t) \cdot W_o(t_0, t_0) \Phi^T(t_1, t) + W_o(t, t_1)$ for $\forall t$.

where (-1) is caused by ~~invers~~ transform of matrix.

2. Consider the following linear system.

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} u.$$

Check the controllability of the system at $\forall t_0$. According to the cond.

(a). $b_1(t)$ and $b_2(t)$ are nonzero constants.

A: $b_1(t) = C_1$; $b_2(t) = C_2$ where $C_1 > 0$ and $C_2 > 0$.

We can check the rank of controllability matrix. which

$$P = [B \quad AB] = \begin{bmatrix} C_1 & C_1 \\ C_2 & 2C_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \Rightarrow \text{rank}(P) = 2.$$

Thus the system is controllable.

(b). $b_1(t) = e^{t}$ and $b_2(t) = e^{2t}$.

A: For the LTV system. we can check the rank of the $[M_0(t_1). M_1(t_1) \cdots]$ where $M_{m+1}(t) = -A(t) M_m(t) + \frac{d}{dt} M_m(t)$. $M_0(t) = B(t)$.

Thus. $\text{rank}\left(\begin{bmatrix} e^{t} & 0 \\ e^{2t} & 0 \end{bmatrix}\right) = 1$

Thus the system is not controllable.

(c). $b_1(t) = \begin{cases} \sin^2 t & \text{if } t \in [2k\pi, (2k+1)\pi], \; k = 0, \pm 1, \pm 2, \cdots \\ 0 & \text{otherwise} \end{cases}$ and $b_2(t) = b_1(t + \pi)$

A: The input signal is ~~seg~~ picewise function.

For the $t \in [2k\pi, (2k+1)\pi]$, $k = 0, \pm 1, \pm 2 \cdots$

~~The controllability matrix is~~

~~$P = [B \quad AB] =$~~

Cause of the system is LTV, we need to check the rank of the matrix $[M_0(t_1). M_1(t_1) \cdots]$. $M_0(t_1) = B(t_1)$.

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$$\operatorname{rank}\begin{bmatrix} \sin^2 t & 2\sin t\cdot\cos t - \sin^2 t \\ \sin^2 t\cdot(t+\pi) & 2\sin t\cos t(t+\pi) + \sin^2 t - 2\sin^2 t(t+\pi) \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \sin^2 t & 2\sin t\cos t - \sin^2 t \\ 0 & \sin^2 t - \sin^2 t(t+\pi) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \sin^2 t & 0 \\ 0 & \sin^2 t - \sin^2 t(t+\pi) \end{bmatrix} = 2.$$

Thus in the first segment, the system is controllable.

For $t \in$ otherwise.

$\operatorname{rank}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 < 2$, which means the system is not controllable in this segment.

Its system is composed of the controllable and uncontrollable segment. But as the definition of the controllability, the input can make the system reach the any state. Thus the system can be controlled in $t \in [2k\pi, (2k+1)\pi]$, which means the system is controllable.

3. Find the Kalman decomposition of the following system and its minimal realization using the decomposed structure. All calculation procedures should be specified.

$$\dot{x} = \begin{bmatrix} 5 & 3 & 5 & 4 \\ -8 & -6 & 0 & 8 \\ 3 & 3 & -2 & 3 \\ -1 & -1 & -5 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} u.$$

$$y = [2 \ 1 \ 0 \ 2]x.$$

A: The controllability matrix G_c can be computed by $[B \ AB \ A^2B \ A^3B]$.

$G_c = \begin{bmatrix} 1 & 3 & 7 & 5 \\ -2 & -4 & -8 & -16 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, The rank of G_c is $2 < 4$, which means that the system is not controllable.

The observability matrix G_o can be computed by $[C \ AC \ A^2C \ A^3C]^T$

$G_o = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & -2 & 0 & 0 \\ 16 & 12 & 0 & 16 \\ -32 & -40 & 0 & -32 \end{bmatrix}$, The rank of G_o is $2 < 4$. which means that the system is not observable.

Then choose two independent ~~eigen~~ basis from ~~G_c~~ $R(G_c)$.

$\boxed{[1 \ -2 \ 0 \ 1]^T}$ and $\boxed{[3 \ -4 \ 0 \ 1]^T}$

Then choose two independent basis from $N(G_o)$.

In this, we need to consider the one of the basises need to be the dependent of basis with $R(G_c)$.

$\Rightarrow -2\cdot[1 \ -2 \ 0 \ 1]^T + 1\cdot[3 \ -4 \ 0 \ 1] = \boxed{[1 \ 0 \ 0 \ -1]^T}$

$\boxed{[0 \ 0 \ 1 \ 0]^T}$ Check: $G_o\cdot[1 \ 0 \ 0 \ -1]^T = 0^T$ and $G_o[0 \ 0 \ 1 \ 0]^T = 0^T$

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Then we need to construct.

$$P^{-1} = [\, Q_{co} \ Q_{c\bar{o}} \ Q_{\bar{c}o} \ Q_{\bar{c}\bar{o}} \,].$$

$Q_{co} = R(G_c) \cap R(G_o) = $ controllable and observable $= [1 \ -2 \ 0 \ 1]^T$ which is in the $R(G_c)$ and $R(G_o)$.

$Q_{c\bar{o}} = R(G_c) \cap N(G_o) = $ controllable and unobservable $= [1 \ 0 \ 0 \ -1]^T$ which is the dependent value of $R(G_c)$ and in $N(G_o)$.

$Q_{\bar{c}o} = N(G_c) \cap R(G_o) = [0 \ 0 \ 0 \ 1]^T$ which is in $R(G_o)$ and independent of $R(G_c)$.

$Q_{\bar{c}\bar{o}} = N(G_c) \cap N(G_o) = [0 \ 0 \ 1 \ 0]^T.$

$$\Rightarrow P^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

Then $$\bar{A} = PAP^{-1} = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 1 & 1 & 0 & 5 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & \bar{A}_{\bar{c}o} & 0 \\ & & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix}$$

$$\bar{B} = PB = [1 \ 0 \ 0 \ 0]^T.$$

$$\bar{C} = CP^{-1} = [2 \ 0 \ 2 \ 0].$$

$\Rightarrow$ "Kalman Decomposition"

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 1 & 1 & 0 & 5 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u.$$

$$y = [2 \ 0 \ 2 \ 0] \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix}$$

The minimal realization of the system is the controllable and observable part. which means. $\bar{A}_{co} = 2.$

$\Rightarrow \dot{\bar{x}}_{co} = 2\bar{x}_{co} + u.$ and $y = 2 \cdot \bar{x}_{co}.$

Then
$$\begin{aligned} H(s) &= C(sI - A)^{-1}B + D. \\ &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + D \\ &= 2 \cdot (s-2)^{-1} \cdot 1 + 0. \\ &= \frac{2}{(s-2)} \end{aligned}$$

Thus the minimal realization of this system is.

$$H(s) = \frac{2}{(s-2)}.$$

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4. Consider the following transfer function.

$$H(s) = \frac{4s^2 - 2s - 6}{2s^5 + 6s^4 + 8s^3 + 7s^2 + 4s + 1}$$

(a). Find a minimal realization of $H(s)$.

$$H(s) = \frac{4\,\cancel{(x+1)}\,(x - \frac{3}{2})}{(x+1)^{\cancel{2}\,1}\,(2x^3 + 2x^2 + 2x + 1)}$$

$$= \frac{4\,(x - \frac{3}{2})}{(x+1)\,(2x^3 + 2x^2 + 2x + 1)}$$

$$= \frac{4x - \frac{3}{2}}{2x^4 + 4x^3 + 4x^2 + 3x + 1}$$

with Canonical Form, the state space equation can be described as.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -3 & -4 & -4 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u.$$

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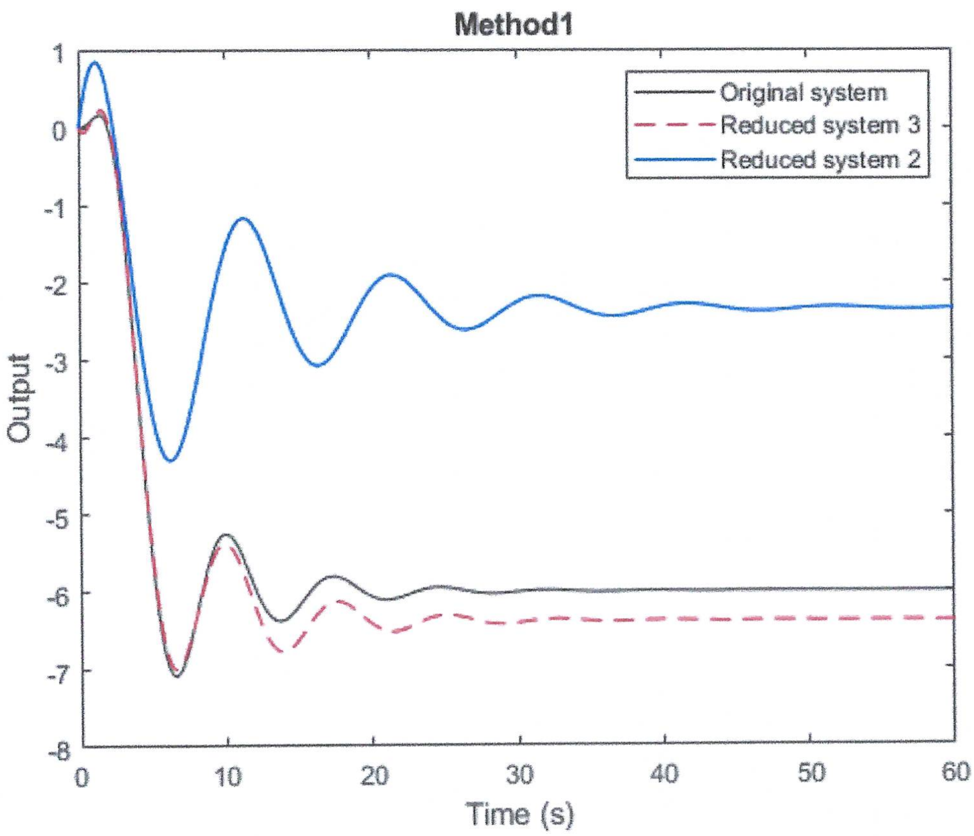
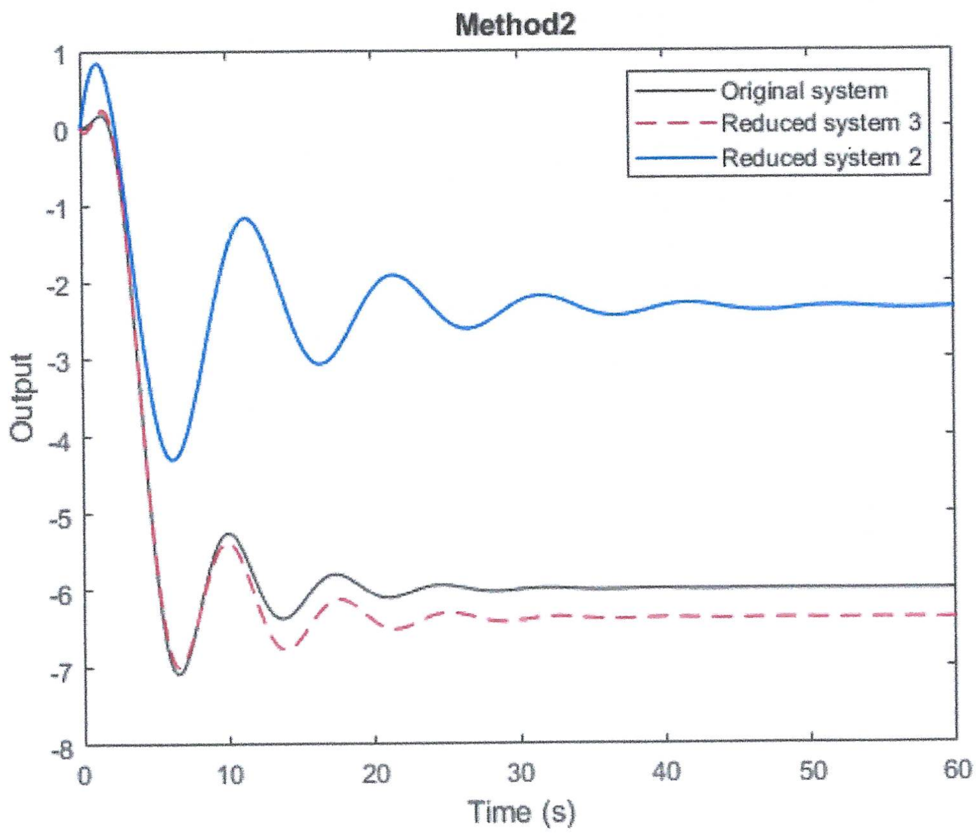
```
%Method1:
num = [4 -2 -6];     % numerator coefficients of transfer function
den = [2 6 8 7 4 1]; % denominator coefficients of transfer function
% Convert transfer function to state-space model
[A, B, C, D] = tf2ss(num, den);
sys = ss(A, B, C, D);
% Compute the controllability and observability gramians
Wc = gram(sys, 'c');
Wo = gram(sys, 'o');
% Compute the SVD of the gramians
[Uc,Sc,Vc] = svd(Wc);
R = sqrtm(Sc)*Uc';
[Uo,So,Vo] = svd(R*Wo*R');
mSigma = sqrtm(So);
% Determine the balancing transformation matrix
P = mSigma*Uo'*inv(R');
% Compute the balanced realization
Ab = P * A * inv(P);
Bb = P * B;
Cb = C * inv(P);
% Define the desired reduced order
% Truncate the balanced realization to obtain the reduced-order system\
A_red_3 = Ab(1:3,1:3);
B_red_3 = Bb(1:3,:);
C_red_3 = Cb(:,1:3);

A_red_2 = Ab(1:2,1:2);
B_red_2 = Bb(1:2,:);
C_red_2 = Cb(:,1:2);
% Convert the reduced-order system back to standard state-space form
sys_red_3 = ss(A_red_3,B_red_3,C_red_3,D);
sys_red_2 = ss(A_red_2,B_red_2,C_red_2,D);
% Compare the step response of the original and reduced systems
t = 0:0.1:60;
u = ones(size(t));
[y,t] = step(sys,t);
[y_red_3,~] = step(sys_red_3,t);
[y_red_2,~] = step(sys_red_2,t);
figure;
subplot(1,2,1);
plot(t,y,'k',t,y_red_3,'r--',t,y_red_2,'b-');
legend('Original system','Reduced system 3','Reduced system 2');
xlabel('Time (s)');
ylabel('Output');
title('Method1');
%Method2:
% Define the system transfer function
s = tf('s');
G = (4*s^2-2*s-6)/(2*s^5+6*s^4+8*s^3+7*s^2+4*s+1);
% Convert to state-space representation
sys = ss(G);
% Balance the system
sys_bal_3 = balancmr(sys,3);
% Perform balanced truncation with desired order
order = 3; % Choose the desired order
sys_red_3 = balred(sys_bal_3,order);
order = 2; % Choose the desired order
sys_bal_2 = balancmr(sys,2);
sys_red_2 = balred(sys_bal_2,order);
% Compute step response of the original and reduced systems
t = linspace(0,60,1000); % Time vector for simulation
[y,t] = step(sys,t); % Step response of original system
[y_red_3,t] = step(sys_red_3, t);
[y_red_2,t] = step(sys_red_2,t);
```

```
% Plot step response comparison
subplot(1,2,2);
plot(t,y,'k',t,y_red_3,'r--',t,y_red_2,'b-');
legend('Original system','Reduced system 3','Reduced system 2');
xlabel('Time (s)');
ylabel('Output');
title('Method2');
```





I use two different method to answer this two question in homework2-4(b)&(c). Actually, the first method is used the equations which are mentioned in the slides and I code it step by step. Then, I use the `balancmr` and `balred` two functions to complete balance realization.

In homework2-4 (b) and (c), I used two different methods to answer the questions. Firstly, I followed the equations mentioned in the slides and coded it step by step. Later I used the `balancmr` and `balred` functions to complete the balance realization.

In question (a), it was stated that the transfer function has pole-zero cancellation. The result showed that the minal realization of H(s) is of rank 4, which is why question (b) started with 3 states instead of 4.

Then, I plotted the step response of the balanced realization models with 3 and 2 orders separately. The results showed that the more reduction, the more error is introduced to the original system. However, the realization with 3 states was closer to the original than the realization with 2 states. Additionally, the balanced realization did not determine the direction, meaning that the 2 states realization was higher than the original, while the 3 states realization was lower than the original.

Other Question Part Code:

```
A = [5 3 5 4; -8 -6 0 -8; 3 3 -2 3; -1 -1 -5 0];
B = [1; -2; 0; 1];
C = [2 1 0 2];
disp(A*B)
disp(A*A*B)
disp(A*A*A*B)

C_ = [1 3 7 15; -2 -4 -8 -16; 0 0 0 0; 1 1 1 1];
disp(rank(C_))
C_2 = [1  0 0 0;
      -2  2 0 0;
       0  0 0 1;
       1  -2 1 0];
disp(rank(C_2))
disp(inv(C_2))

disp(inv(C_2)*A*C_2)
disp(inv(C_2)*B)

disp(C*A)
disp(C*A*A)
disp(C*A*A*A)
O = [2 1 0 2; 0 -2 0 0; 16 12 0 16; -32 -40 0 -32];
rank(O)
O_2 = [2    1 0 1;
       0   -2 0 0;
       16  12 1 0;
      -32 -40 0 1];
rank(O_2)

% Define transfer function
num = [4 -2 -6];
den = [2  6  8 7 4 1];
sys_tf = tf(num, den);

% Find minimal realization
sys_minreal = minreal(sys_tf);

% Display transfer functions
disp('Original transfer function:')
tf(sys_tf)

disp('Minimal realization:')
tf(sys_minreal)
```