$\{\leftarrow, \rightarrow\}$

$B = A_1 \cup A_2 \cup A_3$
$A_1 = \{(\alpha \rightarrow (\beta \rightarrow \alpha))|\alpha, \beta \in \mathrm{WFF}\{\neg, \rightarrow\}\}$
$A_2 = \{((\alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \gamma))|\alpha, \beta, \gamma \in \mathrm{WFF}_{\{\neg, \rightarrow\}}\ \}$
$A_3 = \{((\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha))|\alpha, \beta \in ...\}$
$F = \{MP\}$

$$MP$$
$$\underbrace{\alpha, \alpha \rightarrow \beta}_{\beta}$$
$$\alpha_1, \ldots, a_n \beta$$
$$MP``$$
$$a_n = \beta$$
$$\underbrace{\overbrace{a_1 a_2}^{axi'} a_3}_{\text{Proof series for } a_3} | \ldots a_n$$
$$\vdash \alpha \rightarrow \alpha``$$
$$\beta\alpha \rightarrow \alpha$$

$$(\alpha \rightarrow (\beta \rightarrow \alpha)) \underbrace{\rightarrow}_{\Uparrow} ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \alpha)) A_2$$

$$(\alpha \rightarrow (\beta \rightarrow \alpha)) A_1$$

$$((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \alpha)) MP 1, 2$$

$$\beta$$
$$(\alpha \rightarrow (\underline{\underline{\alpha \rightarrow \alpha}})) \rightarrow (\alpha \rightarrow \alpha))$$

$$(\alpha \rightarrow (\alpha \rightarrow \alpha)), A_1$$

$$(\alpha \rightarrow \alpha) MP\ 4, 3$$
$$\vdash (\alpha \rightarrow \alpha)$$

$$\vdash (\neg\alpha \to (\alpha \to \beta))$$

$$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta))A_3$$

$$(\underline{(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)}) \to (\neg\alpha \to (\underline{(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)}))A_1$$

$$(\neg\alpha \to ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta)))MP\ 1,2$$

$$(\underbrace{\neg\alpha}_{\alpha} \to ((\underbrace{\neg\beta \to \neg\alpha}_{\beta}) \to (\underbrace{\alpha \to \beta}_{\gamma}))) \overbrace{\to}^{1} A_2$$

$$(\underbrace{\neg\alpha}_{\alpha} \to (\underbrace{\neg\beta \to \neg\alpha}_{\beta})) \overbrace{\to}^{2}$$

$$(\underbrace{\neg\alpha}_{\alpha} \to (\underbrace{\alpha \to \beta}_{\gamma}))$$

$$(\neg\alpha \to (\neg\beta \to \neg\alpha)) \to (\neg\alpha \to (\alpha \to \beta)))MP\ 3,4$$

$$(\neg\alpha \to (\neg\beta \to \neg\alpha))A_1$$

$$(\neg\alpha \to (\alpha \to \beta))MP\ 5,6$$
$$\vdash (\neg\alpha \to (\alpha \to \beta))$$

XX
X

$B = A_1 \cup A_2 \cup A_3 \cup X$
$F = \{MP\}$
$a_1, a_2, \ldots, a_n X\beta$
$1 \le i \le n a_n = \beta$
Xa_i
MP“
$X \vdash \beta$

$$\underbrace{\alpha \to \beta, \beta \to \gamma}_{X=\{\alpha\to\beta,\beta\to\gamma|\alpha,\beta,\gamma\in \mathrm{WFF}_{\{\to,\neg\}}\}} \vdash \alpha \to \gamma$$

$$\alpha \to \beta$$

$$\beta \to \gamma$$

$$\{\alpha \to \beta, \beta \to \gamma\} \vdash (\alpha \to \gamma)$$

$$\vdash \beta \vdash \alpha\alpha \in XX \vdash \beta$$

$\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$“

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))A2$$

$$((\beta \to \gamma) \to (\alpha \to (\beta \to \gamma)))A1$$

$$\beta \to \gamma$$

$$(\alpha \to (\beta \to \gamma)), MP\ 2,3$$

$$(\alpha \to \beta) \to (\alpha \to \gamma), MP\ 1,4$$

$$\alpha \to \beta$$

$$\alpha \to \beta, MP\ 5{,}6$$

$\beta \to \gamma, \alpha \to b \vdash \alpha \to \gamma$

$X \vdash \alpha \alpha \in X$

$1.\alpha$
$X \vdash \alpha$
$\alpha \vdash \alpha$
$Y \vdash \alpha X \vdash \alpha \alpha X \subseteq Y$
$X \vdash \alpha$

$$\begin{array}{lcl} a_1[x_1 & & [y_1 \\ \vdots|x_2 & & |y_2 \\ & \Rightarrow & \vdots \\ a_n|\vdots & & \vdots \\ \ [x_n & & [y_n \end{array}$$

$X_{B_1,F} \subseteq X_{B_2,F} B_1 \subseteq B_2$
$X X \vdash \alpha \vdash \alpha$
$Y \vdash \alpha X \alpha$
$Y \vdash \beta X \vdash \beta \beta$
$\underbrace{a_1, \ldots,}_{\text{from } X} \underbrace{a_n}_{\beta} X \vdash \beta$
“X”a_i
Y

$$X = \{\alpha \to \beta, \beta \to \gamma\}$$
$$\delta = \alpha \to \gamma$$
$$X \vdash \delta$$
$$Y = \{\beta, \gamma\}$$

$$y \vdash \alpha \to \beta$$
$$y \vdash \beta \to \gamma$$
$$y \vdash \delta$$

$$v\beta \to (\alpha \to \beta),\ \text{A1}$$

$$\beta y$$

$$\alpha \to \beta$$
$$y \vdash \alpha \to \beta$$

$$(\gamma \to (\beta \to \gamma)),\ \text{A1}$$

$$\gamma y$$

$$\beta \to \gamma$$
$$y \vdash \beta \to \gamma$$

$$X' \vdash \alpha X' \subseteq X X \overbrace{\ X\ }^{\text{(infinite)}} \vdash \alpha$$
$$\boxed{A_1\text{-}A_2}$$
$$\boxed{\text{MP}}$$

$$\alpha, \beta X$$
$$X \vdash \alpha \to \beta X, \alpha \vdash \beta$$
$$\vdash \beta \to \alpha \Leftarrow \beta \vdash \alpha$$
$$X \vdash \alpha \to \beta \Rightarrow$$
$$X, \alpha \vdash \beta$$

$$\alpha$$

$$\alpha \to \beta X$$

$$\beta\ \text{MP}$$
$$X, \alpha \vdash \beta$$