

$$\{\leftarrow, \rightarrow\}$$

$$\begin{aligned} B &= A_1 \cup A_2 \cup A_3 \\ A_1 &= \{(\alpha \rightarrow (\beta \rightarrow \alpha)) | \alpha, \beta \in \mathbf{WFF}\{\neg, \rightarrow\}\} \\ A_2 &= \{((\alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \gamma)) | \alpha, \beta, \gamma \in \mathbf{WFF}\{\neg, \rightarrow\} \} \\ A_3 &= \{((\neg \alpha \rightarrow \neg \beta) \rightarrow (\beta \rightarrow \alpha)) | \alpha, \beta \in \dots\} \\ F &= \{MP\} \end{aligned}$$

$$\begin{aligned} &MP \\ &\underbrace{\alpha, \alpha \rightarrow \beta}_{\beta} \\ &\alpha_1, \dots, a_n \beta \\ &MP“ \\ &a_n = \beta \\ &\overbrace{a_1 a_2 a_3}^{axi'} \mid \dots a_n \\ &\text{Proof series for } a_3 \\ &\quad \vdash \alpha \rightarrow \alpha“ \\ &\beta \alpha \rightarrow \alpha \\ &(\alpha \rightarrow (\beta \rightarrow \alpha)) \underbrace{\rightarrow}_{\uparrow} ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \alpha)) A_2 \\ &(\alpha \rightarrow (\beta \rightarrow \alpha)) A_1 \\ &((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \alpha)) MP1, 2 \\ &(\alpha \rightarrow (\underline{\underline{\alpha \rightarrow \alpha}})) \rightarrow (\alpha \rightarrow \alpha) \beta \\ &(\alpha \rightarrow (\alpha \rightarrow \alpha)), A_1 \\ &(\alpha \rightarrow \alpha) MP\ 4, 3 \\ &\quad \vdash (\alpha \rightarrow \alpha) \end{aligned}$$

$$\begin{aligned}
& \vdash (\neg\alpha \rightarrow (\alpha \rightarrow \beta)) \\
& (\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta))A_3 \\
& \frac{((\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\neg\alpha \rightarrow ((\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)))A_1}{(\neg\alpha \rightarrow ((\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)))MP\ 1,2} \\
& (\underbrace{\neg\alpha}_{\alpha} \rightarrow ((\underbrace{\neg\beta \rightarrow \neg\alpha}_{\beta} \rightarrow (\underbrace{\alpha \rightarrow \beta}_{\gamma}))) \xrightarrow{1} A_2 \\
& (\underbrace{\neg\alpha}_{\alpha} \rightarrow (\underbrace{\neg\beta \rightarrow \neg\alpha}_{\beta})) \xrightarrow{2} \\
& (\underbrace{\neg\alpha}_{\alpha} \rightarrow (\underbrace{\alpha \rightarrow \beta}_{\gamma})) \\
& ((\neg\alpha \rightarrow (\neg\beta \rightarrow \neg\alpha)) \rightarrow (\neg\alpha \rightarrow (\alpha \rightarrow \beta)))MP\ 3,4 \\
& (\neg\alpha \rightarrow (\neg\beta \rightarrow \neg\alpha))A_1 \\
& (\neg\alpha \rightarrow (\alpha \rightarrow \beta))MP\ 5,6 \\
& \vdash (\neg\alpha \rightarrow (\alpha \rightarrow \beta))
\end{aligned}$$

XX  
X

$$\begin{aligned}
B &= A_1 \cup A_2 \cup A_3 \cup X \\
F &= \{MP\} \\
a_1, a_2, \dots, a_n X \beta \\
1 \leq i \leq na_n &= \beta \\
Xa_i \\
MP^{\omega} \\
X \vdash \beta \\
\alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash \alpha \rightarrow \gamma \\
X &= \{\alpha \rightarrow \beta, \beta \rightarrow \gamma \mid \alpha, \beta, \gamma \in WFF_{\{\rightarrow, \neg\}}\} \\
&\alpha \rightarrow \beta \\
&\beta \rightarrow \gamma \\
&\{\alpha \rightarrow \beta, \beta \rightarrow \gamma\} \vdash (\alpha \rightarrow \gamma) \\
&\vdash \beta \vdash \alpha \alpha \in XX \vdash \beta
\end{aligned}$$

$$\begin{array}{c}
\alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash \alpha \rightarrow \gamma \text{“}\\
(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))A2\\
((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \rightarrow \gamma)))A1\\
\beta \rightarrow \gamma\\
(\alpha \rightarrow (\beta \rightarrow \gamma)),MP\ 2,3\\
(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma),MP\ 1,4\\
\alpha \rightarrow \beta\\
\alpha \rightarrow \beta,MP\ 5,6\\
\beta \rightarrow \gamma, \alpha \rightarrow \beta \vdash \alpha \rightarrow \gamma\\
X \vdash \alpha \alpha \in X
\end{array}$$

$$\begin{array}{c}
1.\alpha\\
X \vdash \alpha\\
\alpha \vdash \alpha\\
Y \vdash \alpha X \vdash \alpha \alpha X \subseteq Y\\
X \vdash \alpha
\end{array}$$

$$\begin{array}{ccc}
a_1[x_1 & & [y_1\\
\vdots | x_2 & & | y_2\\
& \Rightarrow & \vdots\\
a_n | \vdots & & \vdots\\
[x_n & & [y_n
\end{array}$$

$$\begin{array}{c}
X_{B_1,F} \subseteq X_{B_2,F} B_1 \subseteq B_2\\
XX \vdash \alpha \vdash \alpha\\
Y \vdash \alpha X \alpha\\
Y \vdash \beta X \vdash \beta \beta\\
\overbrace{a_1, \dots, a_n}^{\text{from } X} \underbrace{X}_{\beta} \vdash \beta\\
\text{“}X\text{” } a_i\\
Y
\end{array}$$

$$\begin{array}{l} X = \{\alpha \rightarrow \beta, \beta \rightarrow \gamma\} \\ \delta = \alpha \rightarrow \gamma \\ X \vdash \delta \\ Y = \{\beta, \gamma\} \end{array}$$

$$\begin{array}{l} y \vdash \alpha \rightarrow \beta \\ y \vdash \beta \rightarrow \gamma \\ y \vdash \delta \end{array}$$

$$v\beta \rightarrow (\alpha \rightarrow \beta), \text{ A1}$$

$$\beta y$$

$$\begin{array}{l} \alpha \rightarrow \beta \\ y \vdash \alpha \rightarrow \beta \end{array}$$

$$(\gamma \rightarrow (\beta \rightarrow \gamma)), \text{ A1}$$

$$\begin{array}{l} \gamma y \\ \beta \rightarrow \gamma \\ y \vdash \beta \rightarrow \gamma \end{array}$$

$$\begin{array}{l} \text{(infinite)} \\ X' \vdash \alpha X' \subseteq XX \quad \overbrace{X}^{\text{infinite}} \vdash \alpha \\ \boxed{A_1-A_2} \\ \boxed{\text{MP}} \end{array}$$

$$\begin{array}{l} \alpha, \beta X \\ X \vdash \alpha \rightarrow \beta X, \alpha \vdash \beta \\ \vdash \beta \rightarrow \alpha \Leftarrow \beta \vdash \alpha \\ X \vdash \alpha \rightarrow \beta \Rightarrow \\ X, \alpha \vdash \beta \end{array}$$

$$\alpha$$

$$\alpha \rightarrow \beta X$$

$$\begin{array}{l} \beta \text{ MP} \\ X, \alpha \vdash \beta \end{array}$$