Lecture Notes in Physics

Edited by H. Araki, Kyoto, J. Ehlers, München, K. Hepp, Zürich
R. Kippenhahn, München, H. A. Weidenmüller, Heidelberg
and J. Zittartz, Köln

194

P. Pascual
R. Tarrach

QCD: Renormalization for the Practitioner

Springer-Verlag
Berlin Heidelberg New York Tokyo 1984

TABLE OF CONTENTS

PREFACE

These notes correspond to a GIFT (Grupo Interuniversitario de Física Teórica) course which was given by us in autumn 1983 at the University of Barcelona. Their main subject is renormalization in perturbative QCD and only the last chapter goes beyond perturbation theory. They are essentially self contained and their aim is to teach the student the techniques of perturbative QCD and the QCD sum rules. Their scope however is limited. A much larger coverage of QCD is given by a recent book by Yndurain [YN 83]. We both started to learn QCD from Eduardo de Rafael's notes [RA 78]; its influence is conspicuous but the blunders are ours.

We thank Pilar Udina for the typing.

Barcelona, January 1984

P. Pascual

R. Tarrach

I. THE Q.C.D. LAGRANGIAN

Since the establishment of Q.E.D., in the late 40's, as the field theory for describing the electromagnetic interactions of charged leptons with the electromagnetic field much effort has been devoted to find a theory for the strong interactions. In the early 70's appeared quantum chromodynamics (Q.C.D.) as the theory of the strong interactions through a combined effort of many people [GE 72], [FG 72], [FG 73], [GW 73], [PO 73], [WE 73], [WE 73a] .

Q.C.D. is a renormalizable quantum field theory of the strong interactions. Its fundamental fields are Dirac spinor fields describing particles of spin ½ , called quarks, with fractionary electric charge and gauge fields corresponding to chargeless and massless particles of spin 1 , called gluons, which interact with the quarks and among themselves.

Let us denote by $q_\alpha^A(x)$ the quark fields, where the index $A = 1,2,\dots, N_f$ refers to flavor and corresponds to the observed degrees of freedom of existing hadrons such as isotopic spin, strangeness, charm, bottom, etc.. At present 5 flavors are known and usually the corresponding quarks are called: $q^1 = u$ (up) , $q^2 = d$ (down), $q^3 = s$ (strange), $q^4 = c$ (charm) and $q^5 = b$ (bottom). The index $\alpha = 1,2,\dots, N$ refers to the color degrees of freedom. The experimental evidence implies that there are three colors and the usual notation is : $q_1{}^A$ = red, $q_2{}^A$ = blue and $q_3{}^A$ = green. As it is usual for the spin ½ fields the Lagrangian density for massless free quarks can be written as

$$\mathcal{L}_0(x) = \frac{i}{2}\, \bar{q}_\alpha^A(x)\, \gamma^\mu \partial_\mu q_\alpha^A(x) - \frac{i}{2}\left[\partial_\mu \bar{q}_\alpha^A(x)\right] \gamma^\mu q_\alpha^A(x) \qquad (I.1)$$

where a summation on A and α must be understood. We are using the metric $g^{\mu\nu} \equiv (1,-1,-1,-1)$. When no error is possible we will omit the flavor and/or the color indices. We will assume that SU(N) is the color group and the quark fields transform as its fundamental representation. The above given Lagrangian density is clearly invariant under global gauge color transformation. Let us now consider the local gauge color transformation

$$q_\alpha^A(x) \longrightarrow q_\alpha^{A'}(x) = G_{\alpha\beta}(x)\, q_\beta^A(x) \equiv \left[e^{-ig\, T_a\, \Theta_a(x)} \right]_{\alpha\beta} q_\beta^A(x) \quad \text{(I.2)}$$

where $\theta_a(x)$ are real space-time functions, g is a real dimensionless coupling constant and T_a are the generators of SU(N) in its fundamental representation (Appendix A). Under (2) the Lagrangian density $\mathcal{L}_o(x)$ transforms as

$$\mathcal{L}_0(x) \longrightarrow \mathcal{L}_0'(x) = \mathcal{L}_0(x) + \frac{i}{2}\, \bar{q}_\alpha^A(x)\, \gamma_\mu \left[G^+(x)\, \partial^\mu G(x) \right]_{\alpha\beta} q_\beta^A(x)$$

$$- \frac{i}{2}\, \bar{q}_\alpha^A(x)\, \gamma_\mu \left[\left[\partial^\mu G^+(x) \right] G(x) \right]_{\alpha\beta} q_\beta^A(x) \quad \text{(I.3)}$$

In order to obtain a Lagrangian density invariant under local gauge transformations we must substitute in (1) [YM 54] the usual derivative ∂^μ by a covariant one

$$\delta_{\alpha\beta}\, \partial^\mu \longrightarrow D^\mu_{\alpha\beta} \equiv \delta_{\alpha\beta}\, \partial^\mu - ig\, T^a_{\alpha\beta}\, B^\mu_a(x) \quad \text{(I.4)}$$

where $B^\mu_a(x)$ are the (N^2-1) so-called gluon fields. The color indices will be written indistinctly as upper or lower indices. The new Lagrangian density will be invariant under local gauge transformations if $q^A_\alpha(x)$ and $D^\mu_{\alpha\beta}\ q^A_\beta(x)$ transform under (2) in the same way, i.e.

$$D^\mu_{\alpha\beta}\, q^A_\beta(x) \longrightarrow D'^\mu_{\alpha\beta}\, q^{A'}_\beta(x) = G(x)_{\alpha\gamma}\, D^\mu_{\gamma\delta}\, q^A_\delta(x) \tag{I.5}$$

where $D'^\mu_{\alpha\beta}$ denotes (4) with $B^\mu_a(x)$ substituted by the transformed field $B'^\mu_a(x)$. From the last equation we can immediately obtain that the transformed gluon fields can be written in terms of the original ones as

$$\begin{aligned} T^a_{\alpha\delta}\, B'^\mu_a(x) = {} & G_{\alpha\beta}(x)\, T^a_{\beta\gamma}\, B^\mu_a(x)\, G^*_{\delta\gamma}(x) \\ & - \frac{i}{g}\left[\partial^\mu G_{\alpha\gamma}(x)\right] G^*_{\delta\gamma}(x) \end{aligned} \tag{I.6}$$

From (2) and (4) the transformation laws under infinitesimal local gauge transformations turn out to be

$$\begin{aligned} q^A_\alpha(x) &\longrightarrow q'^A_\alpha(x) = q^A_\alpha(x) - i g\, T^a_{\alpha\beta}\, \delta\theta_a(x)\, q^A_\beta(x) \\ B^\mu_a(x) &\longrightarrow B'^\mu_a(x) = B^\mu_a(x) + g f_{abc}\, \delta\theta_b(x)\, B^\mu_c(x) - \partial^\mu \delta\theta_a(x) \end{aligned} \tag{I.7}$$

where $\delta\theta_a(x)$ are the infinitesimal functions characterizing the transformation. By this procedure we have constructed a lagrangian density which is invariant under local gauge transformations:

$$\begin{aligned} \mathcal{L}(x) &= \frac{i}{2}\, \bar q^A_\alpha(x)\, \gamma_\mu\, D^\mu_{\alpha\beta}\, q^A_\beta(x) - \frac{i}{2}\left[D^{\mu *}_{\beta\alpha}\, \bar q^A_\alpha(x)\right]\gamma_\mu\, q^A_\beta(x) = \\ &= \frac{i}{2}\, \bar q^A_\alpha(x)\, \gamma_\mu\, \partial^\mu q^A_\alpha(x) - \frac{i}{2}\left[\partial^\mu \bar q^A_\alpha(x)\right]\gamma_\mu\, q^A_\alpha(x) \\ &\quad + \frac{1}{2}\, g\, \bar q^A_\alpha(x)\, \lambda^a_{\alpha\beta}\, \gamma_\mu\, q^A_\beta(x)\, B^\mu_a(x) \end{aligned} \tag{I.8}$$

which describes the free massless quark fields as well as their interaction with the gluon fields with a universal,real and dimensionless coupling constant g .

Sometimes it is useful to introduce the more compact notation

$$B^{\mu}(x) \equiv i g\, T_a\, B_a^{\mu}(x)$$
$$D^{\mu} \equiv I\,\partial^{\mu} - B^{\mu}(x) \qquad (I.9)$$

where now $B^{\mu}(x)$ and D^{μ} are NxN matrices and I is the corresponding unit matrix. Using this notation the above given transformation laws are

$$q^{A}(x) \longrightarrow q'^{A}(x) = G(x)\, q^{A}(x)$$
$$D^{\mu} q^{A}(x) \longrightarrow D'^{\mu} q'^{A}(x) = G(x)\, D^{\mu} q^{A}(x)$$
$$D^{\mu} \longrightarrow D'^{\mu} = G(x)\, D_{\mu}\, G^{-1}(x) \qquad (I.10)$$
$$B^{\mu}(x) \longrightarrow B'^{\mu}(x) = G(x)\, B^{\mu}(x)\, G^{-1}(x) + [\partial^{\mu} G(x)]\, G^{-1}(x)$$

where $q^{A}(x)$ is a column matrix with elements $q^{A}_{\alpha}(x)$ and $G(x)$ a NxN matrix with elements $G_{\alpha\beta}(x)$.

The Lagrangian density (8) does not fix the equations of motion of the gluonic fields and therefore, without destroying the local gauge invariance, we must add to it some terms in order to complete our theory. Up to this end let us define the antisymmetric field strength tensor $F^{\mu\nu}(x)$

$$F^{\mu\nu}(x) \equiv - [D^\mu , D^\nu] =$$

$$= \partial^\mu B^\nu(x) - \partial^\nu B^\mu(x) - [B^\mu(x), B^\nu(x)] \tag{I.11}$$

which satisfies the Bianchi identity

$$[D^\rho , F^{\mu\nu}] + [D^\mu , F^{\nu\rho}] + [D^\nu , F^{\rho\mu}] = 0 \tag{I.12}$$

The components $F_a^{\mu\nu}(x)$ are defined by

$$F^{\mu\nu}(x) \equiv i g T_a F_a^{\mu\nu}(x)$$

$$F_a^{\mu\nu}(x) = \partial^\mu B_a^\nu(x) - \partial^\nu B_a^\mu(x) + g f_{abc} B_b^\mu(x) B_c^\nu(x) \tag{I.13}$$

where the last term reflects the non abelian character of SU(N). Taking into account the transformation law of D^μ given in (10) and the definition (11) we get immediately

$$F^{\mu\nu}(x) \longrightarrow F'^{\mu\nu}(x) = G(x) F^{\mu\nu}(x) G^{-1}(x) \tag{I.14}$$

and hence

$$Tr [F^{\mu\nu}(x) F_{\mu\nu}(x)] = - \frac{g^2}{2} F_a^{\mu\nu}(x) F^a_{\mu\nu}(x) \tag{I.15}$$

is a scalar under Poincaré transformations and furthermore it is invariant under local gauge transformations. We can add to (8) a term proportional to (15) and in this way we get the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2g^2} \mathrm{Tr}\left[F^{\mu\nu}(x) \, F_{\mu\nu}(x) \right] + \frac{i}{2} \bar{q}^A(x) \gamma_\mu D^\mu q^A(x)$$

$$- \frac{i}{2} \overline{\left[D^\mu q^A(x) \right]} \gamma_\mu q^A(x) \tag{I.16}$$

which describes the kinetic terms for massless quarks and massless gluons as well as the interactions of quarks and gluons and the gluon selfinteractions,all of them characterized by a dimensionless coupling constant g .

As it is well known [YM 54] it is impossible to give mass to the gluons without breaking the local gauge invariance, but mass terms can be added for the quark fields without destroying this invariance. Let us now consider the most general way to give masses to the quark fields without breaking local gauge invariance. Up to this end let us introduce the left and righthanded quark fields defined as

$$q_{\alpha L,R}(x) \equiv \frac{1}{2} \left(I \pm \gamma_5 \right) q_\alpha(x) \tag{I.17}$$

where $q_\alpha(x)$ is a column matrix with N_f rows and $\gamma_5 \equiv -i\gamma^0 \gamma^1 \gamma^2 \gamma$
The most general mass term with the desired properties that can be added to the Lagrangian density is

$$\mathcal{L}_M(x) = q^+_{\alpha L}(x) \gamma^0 M q_{\alpha R}(x) + q^+_{\alpha R}(x) \gamma^0 M^+ q_{\alpha L}(x) \tag{I.18}$$

where M is an arbitrary $N_f \times N_f$ matrix. If $\det M \neq 0$, we can write in a unique way $M = M_H U$ where M_H is the square root of MM^+ and therefore is an hermitian positive defined matrix, while $U = M_H^{-1} M$ is unitary. Then (18) can be written as

$$\mathcal{L}_M(x) = q^+_{\alpha L}(x)\,\gamma^0 M_H\, q'_{\alpha R}(x) + q'^+_{\alpha R}(x)\,\gamma^0 M_H\, q_{\alpha L}(x)$$

where $q'_{\alpha R}(x) \equiv U\, q_{\alpha R}(x)$. Furthermore if $q''_\alpha(x) = q'_{\alpha R}(x) + q_{\alpha L}(x)$

$$\mathcal{L}_M(x) = q^+_{\alpha L}(x)\,\gamma^0 M_H\left[q''_\alpha(x) - q_{\alpha L}(x)\right] + q'^+_{\alpha R}(x)\,\gamma^0 M_H\left[q''_\alpha(x) - q'_{\alpha R}(x)\right] =$$

$$= q^+_{\alpha L}(x)\,\gamma^0 M_H\, q''_\alpha(x) + q'^+_{\alpha R}(x)\,\gamma^0 M_H\, q''_\alpha(x) =$$

$$= \bar{q}''_\alpha(x)\, M_H\, q''_\alpha(x)$$

Since $M_H^+ = M_H$, we can diagonalize this matrix using a unitary matrix V

$$V M_H V^+ = M_D = \begin{vmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_{N_f} \end{vmatrix}$$

and introducing $q'''_\alpha(x) \equiv V\, q''_\alpha(x)$ we can write the mass term as

$$\mathcal{L}_M(x) = \bar{q}'''_\alpha(x)\, M_D\, q'''_\alpha(x) \qquad \text{(I.19)}$$

If det M = 0 we can still write $M = M_H U$ where M_H is the one given above. The diagonalization of M_H still determines the real non-negative values of the diagonal matrix M_D giving the masses. Nevertheless, the matrices U and V are not uniquely determined by this method, unless their unitarity is explicitly imposed.

Let us check that all terms of (16) are invariant under $q_\alpha(x)$ $q'''_\alpha(x)$. The most general form of the terms of (16) involving quark fields is

$$\bar{q}_\alpha(x)\ \gamma^\mu\ A_{\alpha\beta}\ q_\beta(x)$$

Let us prove its invariance

$$\bar{q}_\alpha(x)\ \gamma^\mu\ A_{\alpha\beta}\ q_\beta(x) = [\,q^+_{\alpha L}(x) + q^+_{\alpha R}(x)]\,\gamma^0\gamma^\mu A_{\alpha\beta}\,[\,q_{\beta L}(x) + q_{\beta R}(x)] =$$

$$= q^+_{\alpha L}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\ q_{\beta L}(x) + q^+_{\alpha R}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\ q_{\beta R}(x) =$$

$$= q^+_{\alpha L}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\ q_{\beta L}(x) + q'^+_{\alpha R}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\ q'_{\beta R}(x) =$$

$$= q^+_{\alpha L}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\,[\,q''_\beta(x) - q'_{\beta R}(x)] + q'^+_{\alpha R}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\,[\,q''_\beta(x) - q_{\beta L}(x)] =$$

$$= q^+_{\alpha L}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\ q''_\beta(x) + q'^+_{\alpha R}(x)\ \gamma^0\gamma^\mu\ A_{\alpha\beta}\ q''_\beta(x) =$$

$$\doteq \bar{q}''_\alpha(x)\ \gamma^\mu\ A_{\alpha\beta}\ q''_\beta(x) = \bar{q}'''_\alpha(x)\ \gamma^\mu\ A_{\alpha\beta}\ q'''_\beta(x)$$

Then we can forget about the primes and the desired Lagrangian density can be written as

$$\mathcal{L}(x) = \frac{1}{2g^2}\ \mathrm{Tr}\ [\,F^{\mu\nu}(x)\ F_{\mu\nu}(x)] + \frac{i}{2}\ \bar{q}^A(x)\ \gamma_\mu\ D^\mu q^A(x)$$

$$- \frac{i}{2}\ [\,\overline{D^\mu q^A(x)}\,]\ \gamma_\mu\ q^A(x) - m_A\ \bar{q}^A(x)\ q^A(x) \qquad \text{(I.20)}$$

or explicitly

$$\mathcal{L}(x) = -\frac{1}{4}\ [\,\partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x)\,][\,\partial^\mu B^\nu_a(x) - \partial^\nu B^\mu_a(x)\,]$$

$$
\begin{aligned}
&+ \frac{i}{2} \bar{q}^{A}_{\alpha}(x) \gamma^{\mu} \partial_{\mu} q^{A}_{\alpha}(x) - \frac{i}{2} [\partial_{\mu} \bar{q}^{A}_{\alpha}(x)] \gamma^{\mu} q^{A}_{\alpha}(x) - m_A \bar{q}^{A}_{\alpha}(x) q^{A}_{\alpha}(x) \\
&+ \frac{1}{2} g \bar{q}^{A}_{\alpha}(x) \lambda^{a}_{\alpha\beta} \gamma_{\mu} q^{A}_{\beta}(x) B^{\mu}_{a}(x) \\
&- \frac{1}{2} g f_{abc} [\partial_{\mu} B^{a}_{\nu}(x) - \partial_{\nu} B^{a}_{\mu}(x)] B^{\mu}_{b}(x) B^{\nu}_{c}(x) \\
&- \frac{1}{4} g^{2} f_{abc} f_{ade} B^{b}_{\mu}(x) B^{c}_{\nu}(x) B^{\mu}_{d}(x) B^{\nu}_{e}(x)
\end{aligned}
\tag{I.21}
$$

This is the Lagrangian of classical chromodynamics. The first term represents the kinetic term for the massless gluon fields; the next three terms correspond to the kinetic terms of the quark fields with the possibility of a different mass for each flavor; the fifth term describes the interaction of the quarks with the gluons and the last two terms are the self-interactions of the gluon fields due to the non-abelian character of SU(N) . The equations of motion are, in compact notation,

$$
\begin{aligned}
&[i \gamma_{\mu} D^{\mu} - m_A] q^{A}(x) = 0 \\
&[D^{\mu} , F_{\mu\nu}(x)] = - i g^{2} T_a \sum_{A} \bar{q}^{A}(x) T_a \gamma_{\nu} q^{A}(x)
\end{aligned}
\tag{I.22}
$$

We would like to consider before going on the global symmetries of our Lagrangian density

i) $U_B(1)$

The Lagrangian density (21) is invariant with respect to the set of one parameter transformations

$$
q(x) \longrightarrow q'(x) = e^{-i\theta I} q(x) \tag{I.23}
$$

where θ is a real constant and I is the unit matrix in the color and flavor spaces. To this global gauge transformation there is associated, via Noether's theorem, a baryonic current

$$J^{\mu}(x) = \frac{1}{N}\bar{q}^{A}_{\alpha}(x)\,\gamma^{\mu}\,q^{A}_{\alpha}(x) \tag{I.24}$$

where summation over color and flavor indices must be understood. This is a conserved gauge invariant current

$$\partial_{\mu} J^{\mu}(x) = 0 \tag{I.25}$$

and the associated charge

$$B \equiv \int d^3x \;\; J^{0}(t, \vec{x}) \tag{I.26}$$

is the baryonic charge, generator of the $U_B(1)$ group, which is a constant of motion.

ii) $U_1(1) \otimes U_2(1) \otimes \ldots\ldots \otimes U_{N_f}(1)$

Our Lagrangian density (21) is invariant with respect to each set of uniparametric transformations

$$q^{A}(x) \longrightarrow e^{-i\Theta_A I}\, q^{A}(x) \quad , \quad A = 1, 2, \ldots, N_f \tag{I.27}$$

where Θ_A are real constants and I is the unit matrix in the color space. To each flavor A there is associated a global symmetry $U_A(1)$ and therefore our Lagrangian is invariant under the group $U_1(1) \otimes U_2(1) \otimes \ldots\ldots \otimes U_{N_f}(1)$. The associated gauge invariant currents are

$$J^{A}_{\mu}(x) = \bar{q}^{A}_{\alpha}(x)\,\gamma_{\mu}\,q^{A}_{\alpha}(x) \quad , \quad A = 1, 2, \ldots, N_f \tag{I.28}$$

where a summation over color indices must be understood. These currents are conserved and the corresponding charges are the generators of the group. These symmetries correspond to the separate conservation of each flavor in the strong interactions.

Notice furthermore that if $m_i = m_j$ then $\mathcal{L}(x)$ has a global symmetry larger than $U_i(1) \otimes U_j(1)$; it is invariant under the group of transformations SU(2) acting on the space $(q^i(x)\ ,\ q^j(x))$. If all masses are equal the global symmetry group is $SU(N_f)$.

iii) $SU_L(N_f) \otimes SU_R(N_f)$

Let us now consider the global transformation acting only on the flavor indices

$$
\begin{aligned}
q_\alpha(x) &\longrightarrow q'_\alpha(x) = e^{-i\Theta^A T^A} q_\alpha(x) \\
q_\alpha(x) &\longrightarrow q'_\alpha(x) = e^{-i\Theta^A T^A \gamma_5} q_\alpha(x)
\end{aligned}
\tag{I.29}
$$

where Θ_A is a set of $(N_f^2 - 1)$ real constants and T^A are the generators of $SU(N_f)$ in the fundamental representation. These transformations are global symmetries of our Lagrangian density only if the mass terms are absent: $m_A = 0$. Via Noether's theorem we can associate to the transformations (29) the gauge invariant currents

$$
\begin{aligned}
V_A^\mu(x) &= \bar{q}^{\,y}_\alpha(x)\ \gamma^\mu\ (T^A)_{yz}\ q^z_\alpha(x) \\
A_A^\mu(x) &= \bar{q}^{\,y}_\alpha(x)\ \gamma^\mu \gamma_5\ (T^A)_{yz}\ q^z_\alpha(x)
\end{aligned}
\tag{I.30}
$$

which are the vector and axial vector currents of the current algebra of Gell-Mann [GE 64]. Notice

$$
\begin{aligned}
\partial_\mu V_A^\mu(x) &= i\,(m_y - m_z)\ \bar{q}^{\,y}_\alpha(x)\ (T^A)_{yz}\ q^z_\alpha(x) \\
\partial_\mu A_A^\mu(x) &= i\,(m_y + m_z)\ \bar{q}^{\,y}_\alpha(x)\ \gamma_5\ (T^A)_{yz}\ q^z_\alpha(x)
\end{aligned}
\tag{I.31}
$$

so that the currents are conserved if the quark masses are zero.

Assuming that $q_\alpha^A(x)$ is a quantum field we would like to compute the equal time commutation relations of these currents. All these currents have the general structure $X(x) \equiv q^+(x)\ O\ q(x)$ where O is a matrix acting on color, flavor and spin indices. Let us remember that

$$\delta(x^0-y^0)\ \{ q_i(x)\ ,\ q_j^+(y) \} = \delta_{ij}\ \delta^{(4)}(x-y) \tag{I.32}$$

where the subindex stands for color, flavor and spin components. Then it is immediate to prove that

$$\begin{aligned}\delta(x^0-y^0)\,[\, q^+(x)\ O\ q(x)\ ,\ \ q^+(y)\ O'\ q(y)\,] &= \\ &= q^+(x)\ [\, O,\ O'\,]\ q(x)\ \delta^{(4)}(x-y)\end{aligned} \tag{I.33}$$

In the cases that we are interested in $O = I \otimes \lambda_A/2 \otimes \Gamma$, where I is the unit matrix in the color space, $\lambda_A/2$ are the generators of the $SU(N_f)$ flavor group in its fundamental representation and Γ are matrices acting on the spin indices. It is convenient to introduce

$$\lambda_0 \equiv \sqrt{\frac{2}{N_f}}\ I$$

and then relations (A.6) and (A.12) can be written

$$\lambda_A \lambda_B = [\, d_{ABC} + i f_{ABC}\,]\, \lambda_C \quad , \qquad Tr\,[\, \lambda_A \lambda_B\,] = 2\, \delta_{AB} \tag{I.34}$$

where the indices run from zero to $N_f^{\,2} - 1$. Then

$$\delta(x^0-y^0)\left[q^{+}(x)\ I\otimes\frac{1}{2}\lambda_A\otimes\Gamma\ q(x)\ ,\ q^{+}(y)\ I\otimes\frac{1}{2}\lambda_B\otimes\Gamma'\ q(y)\right]=$$

$$=\frac{i}{2}\ \delta^{(4)}(x-y)\ f_{ABC}\ q^{+}(x)\ I\otimes\frac{1}{2}\lambda_C\otimes\{\Gamma,\Gamma'\}\ q(x)$$

$$+\frac{1}{2}\ \delta^{(4)}(x-y)\ d_{ABC}\ q^{+}(x)\ I\otimes\frac{1}{2}\lambda_C\otimes[\Gamma,\Gamma']\ q(x)$$

since

$$\gamma^{\nu}\gamma^{\lambda}\gamma^{\mu}=\left[S^{\nu\lambda\mu\alpha}+i\,\epsilon^{\nu\lambda\mu\alpha}\gamma_5\right]\gamma_{\alpha}$$

$$S^{\nu\lambda\mu\alpha}\equiv g^{\nu\lambda}g^{\mu\alpha}-g^{\nu\mu}g^{\lambda\alpha}+g^{\nu\alpha}g^{\lambda\mu} \qquad \text{(I.35)}$$

where $\epsilon^{\nu\lambda\mu\alpha}$ is the invariant fully antisymmetric tensor of Levi-Cività defined in such a way that $\epsilon^{0123}=1$. A straightforward calculation gives

$$\delta(x^0-y^0)\left[V^{\mu}_{A}(x),V^{\nu}_{B}(y)\right]=i\delta^{(4)}(x-y)\left[f_{ABC}S^{\mu o\nu\alpha}V^{C}_{\alpha}(x)-d_{ABC}\epsilon^{\mu o\nu\alpha}A^{C}_{\alpha}(x)\right]$$

$$\delta(x^0-y^0)\left[V^{\mu}_{A}(x),A^{\nu}_{B}(y)\right]=i\delta^{(4)}(x-y)\left[f_{ABC}S^{\mu o\nu\alpha}A^{C}_{\alpha}(x)-d_{ABC}\epsilon^{\mu o\nu\alpha}V^{C}_{\alpha}(x)\right] \qquad \text{(I.36)}$$

$$\delta(x^0-y^0)\left[A^{\mu}_{A}(x),A^{\nu}_{B}(y)\right]=i\delta^{(4)}(x-y)\left[f_{ABC}S^{\mu o\nu\alpha}V^{C}_{\alpha}(x)-d_{ABC}\epsilon^{\mu o\nu\alpha}A^{\nu}_{\alpha}(x)\right]$$

In particular if $\nu=0$

$$\delta(x^0-y^0)\left[V^{\mu}_{A}(x),V^{0}_{B}(y)\right]=i\,\delta^{(4)}(x-y)\ f_{ABC}\ V^{\mu}_{C}(x)$$

$$\delta(x^0-y^0)\left[V^{\mu}_{A}(x),A^{0}_{B}(y)\right]=i\,\delta^{(4)}(x-y)\ f_{ABC}\ A^{\mu}_{C}(x) \qquad \text{(I.37)}$$

$$\delta(x^0-y^0)\left[A^{\mu}_{A}(x),A^{0}_{B}(y)\right]=i\,\delta^{(4)}(x-y)\ f_{ABC}\ V^{\mu}_{C}(x)$$

For A, B and C running from 1 to $N_f^2 - 1$ these are the basic commutation relations of Gell-Mann's current algebra [GE 64]. If we introduce the charges

$$Q^A(t) \equiv \int d^3x \; V_A^0(t,\vec{x}) \qquad , \qquad Q_5^A(t) \equiv \int d^3x \; A_A^0(t,\vec{x}) \tag{I.38}$$

which are conserved in the absence of quark masses, we obtain from (37)

$$\begin{aligned}
[\,Q^A(t)\,,\; Q^B(t)\,] &= i\, f_{ABC}\; Q^C(t) \\
[\,Q^A(t)\,,\; Q_5^B(t)\,] &= i\, f_{ABC}\; Q_5^C(t) \\
[\,Q_5^A(t)\,,\; Q_5^B(t)\,] &= i\, f_{ABC}\; Q^C(t)
\end{aligned} \tag{I.39}$$

The combinations

$$Q_L^A(t) \equiv \frac{1}{2}\left[\,Q^A(t) + Q_5^A(t)\,\right] \quad , \quad Q_R^A(t) \equiv \frac{1}{2}\left[\,Q^A(t) - Q_5^A(t)\,\right] \tag{I.40}$$

are the so-called left and right handed charges and they satisfy the commutation relations

$$\begin{aligned}
[\,Q_L^A(t)\,,\; Q_L^B(t)\,] &= i\, f_{ABC}\; Q_L^C(t) \\
[\,Q_R^A(t)\,,\; Q_R^B(t)\,] &= i\, f_{ABC}\; Q_R^C(t) \\
[\,Q_L^A(t)\,,\; Q_R^B(t)\,] &= 0
\end{aligned} \tag{I.41}$$

In the absence of quark masses all theses charges are time independent since the corresponding currents are conserved. In particular Q_L^A

and $Q_R{}^A$ are the generators of $SU_L(N_f) \otimes SU_R(N_f)$ which is a global symmetry of our Lagrangian density.

It should be recalled here that the non-integrated commutation relations of eqs. (36) and (37) are purely formal and specially the time-space and space-space component commutation relations are expected to have additional, highly singular terms, called Schwinger term This should not be anything surprising, as products of currents at the same space-time point are known to be highly singular in quantum field theory.

Before going on we would like to discuss how these last global symmetries are realized in Nature. Let us consider our Lagrangian density without mass terms. There are two options for the chiral symmetry $SU_L(N_f) \otimes SU_R(N_f)$ to be realized on physical states:

i) The Wigner-Weyl realization: the chiral charges annihilate the vacuum

$$Q^A|0\rangle = 0 \quad , \quad Q^A_5\,|0\rangle = 0 \tag{I.42}$$

ii) The Nambu-Goldstone realization, where

$$Q^A|0\rangle \neq 0 \quad , \quad Q^A_5\,|0\rangle \neq 0 \tag{I.43}$$

which means that the vacuum does not share the symmetry of the Lagrangian.

To each of these options there is an associated theorem which applies to charges defined as spatial integrals of local current densities, hence to Q^A and $Q_5{}^A$. These theorems are

i) Coleman's theorem [CO 66]: The realization à la Wigner-Weyl implies that the physical states can be classified according to the irreducible unitary representations of the group generated by the charges which annihilate the vacuum. This means, in our case, that particles should appear in parity doublets degenerated in mass at the chiral limit.

ii) Goldstone's theorem [GO 61], [GS 62]: To each generator of a continuous symmetry which does not annihilate the vacuum there is an associated spin zero massless particle.

Neither of these possibilities seems to be the one realized in Nature. On the one hand parity doublets are not observed. If $N_f = 3$ we see a 0^- and a 1^- octet, but the 0^+ and 1^+ octets more or les degenerate in mass with the 0^- and 1^- have not been found. On the other hand, if we identify the 0^- octet with the Goldstone bosons there doesn't seem to be any 0^+ octet of Goldstone bosons. The picture which has emerged from the study of chiral symmetry problems is a mixed one:

$$Q^A |0\rangle = 0 \qquad , \qquad Q_5^A |0\rangle \neq 0 \tag{I.44}$$

at the limit of chiral symmetry. This implies the existence of a $(N_f^2 - 1)$-plet of zero-mass pseudoscalars, and a set of massive multiplets with degenerate masses. In the case $N_f = 3$ the 0^- octet is the one of Goldstone bosons and the massive multiplets are the 1^- octet, $½^+$ octet, $3/2^+$ decuplet, etc. The symmetry is broken because neither the 0^--octet is formed by exactly massless particles, nor the other multiplets are exactly degenerate in mass. However, the fact that the pion mass is small, as compared to the other hadrons, is associated to an approximate chiral $SU_L(2) \otimes SU_R(2)$ invariance. The successes of P.C.A.C. and current algebra are rooted in this approximate invariance. The diagonal part $SU(2)$ of this chiral $SU_L(2) \otimes SU_R(2)$ group is the isospin group and the fact that isospin invariance is well realized in Nature is associated to the smallness of the u and d quark mass difference as compared to the mass scale Λ associated with perturbative Q.C.D. which will be introduced later on. The qualitative successes of the SU(3) symmetry are also associated to the fact that the strange - non strange quark mass difference is not too large as compared with Λ . To the extend that the other quark flavors have masses much larger than Λ we do not expect to see much of a direct symmetry pattern when we go beyond SU(3) in the spectroscopy of physical states.

iv) $U_A(1)$

In the absence of masses our Lagrangian density has an additional $U_A(1)$ symmetry implemented by the set of uniparametric tranformations

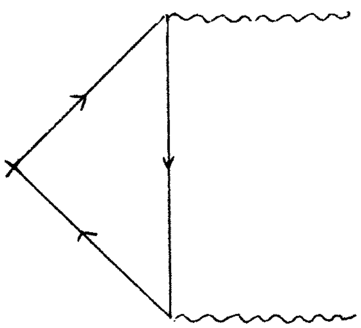
$$q(x) \longrightarrow q'(x) = e^{-i\theta I \gamma_5} q(x) \tag{I.45}$$

where θ is a real constant and I is the unit matrix in the color and flavor space. The associated gauge invariant current is

$$J_5^\mu(x) = \bar{q}_\alpha^A(x) \gamma^\mu \gamma_5 q_\alpha^A(x) \tag{I.46}$$

where a sum over color and flavor indices is implicit. If the quark masses are zero this current is classically conserved and the associated charge is, as usual, the generator of $U_A(1)$. How is this $U_A(1)$ symmetry realized in Nature? A Nambu-Goldstone realization analogous to the other axial charges would imply the existence of a 0^- massless flavor singlet. In the case $N_f = 2$, the natural candidate is the η -particle but $m_\eta/m_\pi \simeq 4$. The symmetry $U_A(1)$ is broken by the same mass terms that break $SU_L(2) \otimes SU_R(2)$ and therefore we must expect that this fourth pseudoscalar meson has a mass similar to the pion More precisely, it is possible to prove [WE 75], using soft pion techniques, that the mass of the new Goldstone boson must be smaller than $\sqrt{2} m_\pi$, and the η particle does not satisfy this relation. Furthermore if we consider $m_s = 0$ we have an exact $SU_L(3) \otimes SU_R(3)$ symmetry and the η is a member of the 0^- octet that contains the pions; we have an additional symmetry $U_A(1)$ and a pseudoscalar flavor singlet meson with mass smaller than $\sqrt{3}\, m_\pi$ is missing. On the other hand a Wigner-Weyl realization would imply that all massive hadrons would appear as parity doublets and this is not so. We face therefore the so-called $U_A(1)$ problem [CR 79] [CO 79].

A first important remark in the solution of this problem is that the Noether current associated to the $U_A(1)$ invariance of our Lagrangian density is in fact not conserved in the quantum theory. This is due to the presence of anomalous terms in the naïve Ward identities, much the same as the Adler-Bell-Jackiw anomaly associated to the $\pi^\circ \rightarrow 2\gamma$ amplitude [BJ 69a][AD 69].These anomalous terms are uniquely determined by the usual triangle diagram. It is possible to prove that



Triangle diagram which gives the anomalous term in the Ward identities associated to the axial baryonic current.

in the limit of quarks with zero mass

$$\partial_\mu J^\mu_5(x) = \frac{g^2}{4\pi}\frac{N_f}{8\pi}\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \tag{I.47}$$

This seems to be the end of the story: if the current is not conserved there is no $U_A(1)$ problem to worry about. Nevertheless life is not so simple; the structure of (47) is such that there is another charge which is conserved and generates the $U_A(1)$ symmetry.

$$\dot{Q}_5(t) = 0 \tag{I.48}$$

and which is gauge invariant in the absence of instantons [BP 75] . To see it, let us notice that the r.h.s. of (47) can be written as a total divergence

$$\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) =$$

$$= \partial_\mu \left\{ 4\epsilon^{\mu\nu\rho\sigma}\left[B^a_\nu(x)\,\partial_\rho B^a_\sigma(x) + \frac{g}{3} f_{abc} B^a_\nu(x) B^b_\rho(x) B^c_\sigma(x)\right]\right\} \tag{I.49}$$

so that we finally have a conserved current

$$\bar{J}^\mu_5(x) \equiv J^\mu_5(x) - \frac{g^2}{8\pi^2} N_f\, \epsilon^{\mu\nu\rho\sigma}\left[B^a_\nu(x)\,\partial_\rho B^a_\sigma(x) + \frac{g}{3} f_{abc} B^a_\nu(x) B^b_\rho(x) B^c_\sigma(x)\right] \tag{I.50}$$

with a new contribution which is, however, gauge-variant. Now recall that the only topological invariant of Q.C.D. is the Pontryagin index or winding number

$$n \equiv \frac{g^2}{64\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \tag{I.51}$$

which being an integral over the whole space-time of a total divergence will depend only on its global topological properties and is zero in the absence of instantons. Now, it is known [BA 75] that in the abse ce of instantons the charge corresponding to the current $\bar{J}_5^{\mu}$, $\bar{Q}_5$, is gauge invariant. Finally, as the new contribution to the current $\bar{J}_5^{\mu}$ commutes with the quark fields, $\bar{Q}_5$ generates the same transformations on the quark fields as Q_5 . Thus in the absence of instantons

$$\bar{Q}_5 \equiv \int d^3x \quad \bar{J}_5^{\,0}(t, \vec{x}) \tag{I.52}$$

is a gauge invariant conserved generator of the $U_A(1)$ symmetry. This leads us back to the unwanted Goldstone boson!

In the presence of instantons, however, n = 1,2,3,... for one, two, three,.. etc. instantons this argument does not go through and we are left with a gauge-variant conserved current $\bar{J}_5^{\mu}$ which will lead to a gauge-variant time independent $\bar{Q}_5$. Its gauge variance, however, might solve the $U_A(1)$ [KS 76] problem. Indeed, 't Hooft [TH 76] has proven that eventually what happens is that the corresponding Goldstone boson does not appear as a zero mass pole of gauge invariant Green functions, but only in gauge variant ones; but these are not relevant for physics, these Goldstone bosons will not be seen in any experiment. In nonabelian gauge theories one thus can have the breaking of a continuous symmetry by the vacuum and nevertheless not have physical massless bosons. For more on this see [CO 79] [CR 79].

We face now the problem of how starting from the classical Lagrangian density given in (21) we can construct the corresponding quantum field theory and here, as in Q.E.D., the method is not straightforward due to the gauge invariance of our theory. As in Q.E.D., there is no problem with the quark fields. We substitute the classical spinor fields by the corresponding quantum ones. If $\psi(x)$ is a generic quark field the corresponding free quantum field can be written in terms of the creation and annihilation operators in the usual way:

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3 \, 2E(\vec{p})} \sum_\lambda \left[u(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} + v(\vec{p},\lambda)\, b^+(\vec{p},\lambda)\, e^{+ip.x} \right] \tag{I.53}$$

where the integration is over the positive sheet of the mass hyperboloid $\Omega_+(m) \doteq \{ p \, ; \, p^2 = m^2 \, , \, p^0 > 0 \}$, being m the mass of the fermion; $d^3 p \,/\, 2E(\vec{p})$ is its invariant measure. λ is a dichotomic variable corresponding to the two possible helicities of the fermion. The fourspinors $u(\vec{p},\lambda)$ and $v(\vec{p},\lambda)$ are the solutions of the equations

$$[\not p - m]\, u(\vec{p},\lambda) = 0 \quad , \quad [\not p + m]\, v(\vec{p},\lambda) = 0 \tag{I.54}$$

with well defined helicity and normalized in such a way that

$$u^+(\vec{p},\lambda)\, u(\vec{p},\lambda) = 2E(\vec{p}) \quad , \quad v^+(\vec{p},\lambda)\, v(\vec{p},\lambda) = 2E(\vec{p}) \tag{I.55}$$

Finally $a(\vec{p},\lambda)$ and $b(\vec{p},\lambda)$ [$a^+(\vec{p},\lambda)$ and $b^+(\vec{p},\lambda)$] are the annihilation [creation] operators for particles and antiparticles which satisfy the anticommutation relations

$$\{ a(\vec{p},\lambda) \, , \, a^+(\vec{p}',\lambda') \} = \{ b(\vec{p},\lambda) \, , \, b^+(\vec{p}',\lambda') \} = (2\pi)^3 \, 2E(\vec{p})\, \delta_{\lambda\lambda'}\, \delta(\vec{p}-\vec{p}') \tag{I.56}$$

being zero the remaining ones. From all that it is easy to show that the free field propagator is

$$i S^{(0)}(x-y) \equiv \underbrace{\psi(x)\,\bar{\psi}(y)} \equiv \langle 0 | T(\psi(x)\,\bar{\psi}(y)) | 0 \rangle =$$

$$= \frac{i}{(2\pi)^4} \int d^4p \, \frac{\not{p} + m}{p^2 - m^2 + i\eta} \, e^{-i p.(x-y)} \equiv$$

$$\equiv \frac{i}{(2\pi)^4} \int d^4p \; S^{(0)}(\not{p}) \; e^{-i p.(x-y)} \tag{I.57}$$

where the expression for the spinor fields is given in (53) and $\eta \downarrow 0$

Let us now turn our attention to the gauge fields. Let us remember what happens in Q.E.D. As it is well known the Lagrangian density for a classical spinor field $\psi(x)$ interacting with the electromagnetic field $A^{\mu}(x)$ is

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \frac{i}{2} \bar{\psi}(x)\, \gamma^{\mu} \partial_{\mu} \psi(x)$$

$$- \frac{i}{2} [\partial_{\mu} \bar{\psi}(x)]\, \gamma^{\mu} \psi(x) - m \bar{\psi}(x)\, \psi(x) - e \bar{\psi}(x)\, \gamma^{\mu} \psi(x)\, A_{\mu}(x)$$

$$F_{\mu\nu}(x) \equiv \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x) \tag{I.58}$$

As before we can quantize the matter field using the canonical method and no problems appear. Let us now see what is the problem with the electromagnetic field. The conjugate momenta of $A_{\rho}(x)$ are given by

$$\pi^{\rho}(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_{\rho}(x))} = F^{\rho 0}(x) \tag{I.59}$$

and due to the fact that $\partial_0 A_0(x)$ does not appear in (58) we have $\pi^0(x) = 0$. But as the canonical equal-time commutation relations are

$$\delta(x^0-y^0)\left[A_\mu(x)\ ,\ A_\nu(y)\right] = \delta(x^0-y^0)\left[\Pi_\mu(x)\ ,\ \Pi_\nu(y)\right] = 0$$

$$\delta(x^0-y^0)\left[A_\mu(x)\ ,\ \Pi_\nu(y)\right] = i\, g_{\mu\nu}\ \delta^{(4)}(x-y) \tag{I.60}$$

it follows that $A_0(x)$ cannot satisfy them but commutes with all other operators, i.e. it behaves as a c-number in contrast to the space components $A_i(x)$ and therefore manifest covariance is then lost. This covariance difficulty is related to the particle content of the field $A_\mu(x)$, which has four components. However, it is expected that it describes photon states which are known to have two independent components since they have zero mass. One way out of this covariance difficulty is to apply the canonical quantization procedure to (58) when the gauge field $A_\mu(x)$ is restricted by a covariant subsidiary condition [RA 76]. The simplest way to implement this is to add to the Lagrangian density (58) the gauge fixing term

$$-\frac{1}{2a}\left[\partial_\mu A^\mu(x)\right]^2 \tag{I.61}$$

where a is the so called gauge parameter.

As it is well known Q.E.D. has infrared divergences which appear in the intermediate steps of the calculations of observable quantities, due to the fact that the photons are massless. A useful way to regulate these divergences is to add to (58) a mass term

$$+\frac{1}{2}\lambda^2 A^\mu(x)\, A_\mu(x) \tag{I.62}$$

Clearly the limit $\lambda \to 0$ must be always well defined at the end of any calculation corresponding to an observable quantity.

Writing all that together the Lagrangian density for Q.E.D. is

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2} \lambda^2 A^\mu(x) A_\mu(x)$$

$$+ \frac{i}{2} \bar{\Psi}(x) \gamma^\mu \partial_\mu \Psi(x) - \frac{i}{2} [\partial_\mu \bar{\Psi}(x)] \gamma^\mu \Psi(x) - m \bar{\Psi}(x) \Psi(x)$$

$$- e \bar{\Psi}(x) \gamma^\mu \Psi(x) A_\mu(x) - \frac{1}{2a} [\partial_\mu A^\mu(x)]^2 \tag{I.63}$$

With this Lagrangian density we can carry out the canonical quantization procedure and for the conjugate momentum of $A_\rho(x)$ we get

$$\Pi^\rho(x) = F^{\rho 0}(x) - \frac{1}{a} g^{\rho 0} [\partial_\mu A^\mu(x)] \tag{I.64}$$

With these new momenta the manifestly covariant canonical equal-time commutation relations (60) do not lead to any problems.

Using (63) we obtain that the equations of motion for the free electromagnetic field are

$$[\Box + \lambda^2] A^\mu(x) - \left(1 - \frac{1}{a}\right) \partial^\mu [\partial_\nu A^\nu(x)] = 0 \tag{I.65}$$

The plane wave solutions $\epsilon^\mu \exp(-i\, k.x)$ of these equations are such that ϵ^μ must satisfy

$$[-k^2 + \lambda^2] \epsilon^\mu + \left(1 - \frac{1}{a}\right) k^\mu [k_\nu \epsilon^\nu] = 0 \tag{I.66}$$

There are three independent solutions satisfying $k^\mu \epsilon_\mu = 0$ and $k^2 = \lambda^2$ and another one with $\epsilon^\mu \,||\, k^\mu$ and $k^2 = a\lambda^2$. A possible choice of these solutions (polarization vectors) is

$$\epsilon^\mu(\vec{k},i) \equiv S^\mu(i) \equiv (0, \vec{S}(i)) \qquad i = 1, 2$$

$$\epsilon^\mu(\vec{k},3) \equiv \left(\frac{1}{\lambda} |\vec{k}| , \frac{k^0}{\lambda|\vec{k}|} \vec{k} \right) \quad , \quad \epsilon^\mu(\vec{k},0) \equiv \frac{1}{\lambda} k^\mu \qquad \text{(I.67)}$$

where S^μ (i) are such that

$$\epsilon^\mu(\vec{k},\sigma) \, \epsilon^*_\mu(\vec{k},\sigma') = g_{\sigma\sigma'} + (a-1) g_{\sigma 0} g_{\sigma' 0} \qquad \text{(I.68)}$$

The photons associated with the first two solutions are the so-called transversal photons which are the only physical ones. The ones associated with $\epsilon^\mu(\vec{k},3)$ are denominated longitudinal photons and the ones associated with the last polarization fourvector are the so-called time-like photons. Some useful relations among the polarization four-vectors are

$$\sum_{\sigma=1,2,3} \epsilon_\mu(\vec{k},\sigma) \, \epsilon^*_\nu(\vec{k},\sigma) = - g_{\mu\nu} + \frac{k_\mu k_\nu}{\lambda^2}$$

$$\epsilon_\mu(\vec{k},0) \, \epsilon^*_\nu(\vec{k},0) = \frac{k_\mu k_\nu}{\lambda^2} \qquad \text{(I.69)}$$

Now the quantized free field can be written as

$$A^\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(\vec{k})} \sum_{\sigma=1}^{3} \left[\epsilon^\mu(\vec{k},\sigma) a(\vec{k},\sigma) e^{-ik x} + \epsilon^{\mu *}(\vec{k},\sigma) a^+(\vec{k},\sigma) e^{+ik.x} \right]$$

$$+ \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\tilde{\omega}(\vec{k})} \left[\epsilon^\mu(\vec{k},0) a(\vec{k},0) e^{-ik.x} + \epsilon^{\mu *}(\vec{k},0) a^+(\vec{k},0) e^{+ik.x} \right] \qquad \text{(I.70)}$$

where $\omega(\vec{k}) = (\vec{k}^2 + \lambda^2)^{1/2}$ and $\tilde{\omega}(k) = (\vec{k}^2 + a\lambda^2)^{1/2}$ and the non-zero commutation relations among the creation and annihilation operators are

$$[a(\vec{k},\sigma) \, , \, a^{+}(\vec{k}',\sigma')] = (2\pi)^3 \, 2\,\omega(\vec{k}) \, \delta_{\sigma\sigma'} \, \delta(\vec{k}-\vec{k}') \quad , \quad \sigma,\sigma' = 1,2,3$$

$$[a(\vec{k},0) \, , \, a^{+}(\vec{k}',0)] = - (2\pi)^3 \, 2\,\tilde{\omega}(\vec{k}) \, \delta(\vec{k}-\vec{k}') \tag{I.71}$$

It is straightforward to check that the equal-time commutation relations given above are satisfied. Furthermore the free field propagator is given by

$$i\,D^{(0)}_{\mu\nu}(x-y) \equiv \underline{A_\mu(x)\,A_\nu}(y) \equiv \langle 0 \,|\, T\,(A_\mu(x)\,A_\nu(y))\,|0\rangle =$$

$$= \frac{i}{(2\pi)^4} \int d^4k \left\{ -g_{\mu\nu} + (1-a)\,\frac{k_\mu k_\nu}{k^2 - a\lambda^2 + i\eta} \right\} \frac{1}{k^2-\lambda^2+i\eta}\, e^{-ik.(x-y)} \equiv$$

$$\equiv \frac{i}{(2\pi)^4} \int d^4k \; D^{(0)}_{\mu\nu}(k)\, e^{-ik.(x-y)} \tag{I.72}$$

which has a well defined limit for $\lambda^2 \to 0$ and this limit is used when the infrared problems are not important or they are regularized by some other prescription. Of course one could have obtained directly from the quadratic terms of the Lagrangian the free field propagators [TV 73] making the introduction of the photon mass unnecessary at this stage.

The gauge fixing term added to the Lagrangian density has allowed us to carry out a covariant quantization. Notice, nevertheless that due to the minus sign appearing in the commutation rules for time-like photons (71) the Fock space of states has an indefinite metric: The one particle states of time-like photons have negative norm. This is not a problem since what it is really important in order to preserve the probabilistic interpretation is that the Fock space of physical states has positive definite norm and this turns out to be the case in Q.E.D., as was first proved by Gupta and Bleuler [GU 50]

[GU 51] [BL 50] [BH 50]. This is due to the fact that the probability of observing longitudinal photons is exactly cancelled by the one of observing time-like photons and the theory is unitary in the subspace of physical (transverse) photons. The practical consequence of all that is that in Q.E.D. when a Feynman diagram is computed we must use as photon propagator the one given in (72) (with $\lambda^2 = 0$ if desired). Nevertheless, if we have external photon lines (obviously transversal) and we are not interested in the photon polarization we can carry out the sum over polarizations using the prescription

$$\sum_{i=1}^{2} \epsilon^{\mu}(\vec{k}, i) \; \epsilon^{\nu *}(\vec{k}, i) \longrightarrow - g^{\mu\nu} \qquad \text{(I.73)}$$

instead of using its full expression.

Let us go back to Q.C.D. It is clear that if we use the canonical quantization prescription for the gluon fields in the Lagrangian density given in (21) we still encounter the problem that the momenta associated with the $B_0^a(x)$ fields are zero and therefore the manifest covariance of the theory is lost. We can try to solve the problem adding to the Lagrangian density (21) a gauge fixing term

$$- \frac{1}{2a} \left[\partial^{\mu} B_{\mu}^{a}(x) \right] \left[\partial^{\nu} B_{\nu}^{a}(x) \right] \qquad \text{(I.74)}$$

which is an obvious generalization of the one used in Q.E.D. Now we can perform, as before, a canonical covariant quantization and we find, another time, that the Fock space of states has indefinite metric. Nevertheless, now it is not true that the restriction to the subspace of transverse gluons is automatically unitary and prescription (73) cannot be used [FE 63][FP 67]. One is forced (see Appendix B) to introduce supplementary non-physical fields, the so-called Faddeev-Popov ghosts, to restore unitarity in the subspace of physical states. The reason behind the necessity of introducing the new fields lies in the self-coupling of the gluon fields that invalidate the Gupta-Bleule method.

The fields needed to restore unitarity on the physical secto of the theory are sets of selfadjoint massless scalar fields $\phi_a(x)$ and $\bar{\phi}_a(x)$, with $a = 1,2,\ldots, N^2-1$, which obey the Fermi-Dirac statistics (and therefore have negative norm in the Fock space) coupled only to the gluon fields. Taking all that into account the Lagrangian density for Q.C.D. can be written as

$$
\begin{aligned}
\mathcal{L}(x) = & -\frac{1}{2}[\partial_\mu B_\nu^a(x)][\partial^\mu B_a^\nu(x) - \partial^\nu B_a^\mu(x)] - \frac{1}{2a}[\partial_\mu B_a^\mu(x)][\partial_\nu B_a^\nu(x)] \\
& + \frac{i}{2}\bar{q}_\alpha^A(x)\gamma^\mu\partial_\mu q_\alpha^A(x) - \frac{i}{2}[\partial_\mu\bar{q}_\alpha^A(x)]\gamma^\mu q_\alpha^A(x) - m_A\bar{q}_\alpha^A(x)\, q_\alpha^A(x) \\
& + \frac{1}{2} g\, \bar{q}_\alpha^A(x)\lambda_{\alpha\beta}^a\gamma^\mu q_\beta^A(x) B_\mu^a(x) \\
& - \frac{1}{2} g\, f_{abc}[\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)] B_b^\mu(x) B_c^\nu(x) \\
& - \frac{1}{4} g^2 f_{abc} f_{ade} B_\mu^b(x) B_\nu^c(x) B_d^\mu(x) B_e^\nu(x) \\
& - [\partial_\mu\bar{\phi}_a(x)]\partial^\mu\phi_a(x) + g f_{abc}[\partial_\mu\bar{\phi}_a(x)]\phi_b(x) B_c^\mu(x)
\end{aligned}
\tag{I.75}
$$

or in a more compact notation

$$
\mathcal{L}(x) = \frac{1}{2g^2}\mathrm{Tr}[F^{\mu\nu}(x) F_{\mu\nu}(x)] + \frac{i}{2}\bar{q}^A(x)\gamma_\mu D^\mu q^A(x)
$$

$$- \frac{i}{2} \left[\overline{D^\mu q^A(x)} \right] \gamma_\mu q^A(x) - m_A \bar{q}^A(x) q^A(x)$$

$$+ \frac{1}{a g^2} \mathrm{Tr} \left\{ [\partial^\mu B_\mu(x)]^2 \right\} + \frac{2}{g^2} \mathrm{Tr} \left\{ \partial_\mu \bar{\Phi}(x) [D^\mu, \phi(x)] \right\} \qquad (I.76)$$

where $\phi(x) = i\, g\, T_a \phi_a(x)$ and similar for $\bar{\Phi}(x)$.

From (75) we can obtain immediatly the equations of motion

$$[i\gamma^\mu \partial_\mu - m_A] q^A_\alpha(x) = -\frac{1}{2} g \lambda^a_{\alpha\beta} \gamma^\mu q^A_\beta(x) B^a_\mu(x)$$

$$i\partial_\mu \bar{q}^A_\alpha(x) \gamma^\mu + m_A \bar{q}^A_\alpha(x) = \frac{1}{2} g \bar{q}^A_\beta(x) \lambda^a_{\beta\alpha} \gamma^\mu B^a_\mu(x)$$

$$\partial_\mu \partial^\mu \phi_a(x) = - g f_{abc} [\partial_\mu B^\mu_b(x)] \phi_c(x) - g f_{abc} B^\mu_b(x) \partial_\mu \phi_c(x)$$

$$\partial_\mu \partial^\mu \bar{\Phi}_a(x) = - g f_{abc} B^\mu_b(x) \partial_\mu \bar{\Phi}_c(x)$$

$$\partial_\nu \partial^\nu B^\mu_a(x) - \left(1 - \frac{1}{a}\right) \partial^\mu [\partial_\nu B^\nu_a(x)] = -\frac{1}{2} g \bar{q}^A_\alpha(x) \lambda^a_{\alpha\beta} \gamma^\mu q^A_\beta(x)$$

$$+ g f_{abc} B^\mu_b(x) \partial_\nu B^\nu_c(x) + 2 g f_{abc} [\partial_\nu B^\mu_b(x)] B^\nu_c(x)$$

$$+ g f_{abc} B^\nu_b(x) \partial^\mu B^c_\nu(x) + g^2 f_{sab} f_{scd} B^b_\nu(x) B^\nu_d(x) B^\mu_c(x)$$

$$- g f_{abc} [\partial^\mu \bar{\Phi}_b(x)] \phi_c(x). \qquad (I.77)$$

The non-zero canonical commutation rules for the physical fields appearing in the Q.C.D. Lagrangian are

$$\delta(x^0 - y^0) \left\{ q^A_\alpha(x), q^{B+}_\beta(y) \right\} = \delta_{\alpha\beta} \delta_{AB} \delta^{(4)}(x-y)$$

$$\delta(x^0 - y^0) \left[B^a_\mu(x), \Pi^b_\nu(y) \right] = i \delta_{ab} g_{\mu\nu} \delta^{(4)}(x-y) \qquad (I.78)$$

where

$$\Pi^b_\nu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial[\partial_0 B^\nu_b(x)]} = \partial_\nu B^0_b(x) - \partial^0 B_{\nu b}(x)$$

$$- \frac{1}{a} g_{\nu 0} \partial_\rho B^\rho_b(x) - g f_{bac} B^0_a(x) B_{\nu c}(x) \tag{I.79}$$

and the free propagators for all fields are

$$i S^{(0)AB}_{\alpha\beta}(x-y) \equiv \underbrace{q^A_\alpha(x) \bar{q}^B_\beta(y)} \equiv \langle 0| T(q^A_\alpha(x) \bar{q}^B_\beta(y))|0\rangle = \delta_{AB} \delta_{\alpha\beta} i S^{(0)}(x-y)$$

$$i D^{(0)\mu\nu}_{ab}(x-y) \equiv \underbrace{B^\mu_a(x) B^\nu_b(y)} \equiv \langle 0|T(B^\mu_a(x) B^\nu_b(y))|0\rangle = \delta_{ab} i D^{(0)\mu\nu}(x-y) \tag{I.80}$$

$$i \tilde{D}^{(0)}_{ab}(x-y) \equiv \underbrace{\phi_a(x) \bar{\phi}_b(y)} \equiv \langle 0| T(\phi_a(x) \bar{\phi}_b(y))|0\rangle =$$

$$= \frac{i}{(2\pi)^4} \delta_{ab} \int d^4q \frac{-1}{q^2 + i\eta} e^{-iq\cdot(x-y)} \equiv \frac{i}{(2\pi)^4} \delta_{ab} \int d^4q\, \tilde{D}^{(0)}(q)\, e^{-iq\cdot(x-y)}$$

where $S^{(0)}(x-y)$ and $D^{(0)\mu\nu}(x-y)$ are given in eqs. (57) and (72) respectively. The ghost field propagator is easiest obtained directly from the Lagrangian [TV 73].

Notice that the Q.C.D. Lagrangian (75) has lost its hermitia character due to the ghost fields. This should not be too surprising, as we are concerned with unitarity in the space of physical states, not in the whole space, and it was the lack of unitarity in the space of physical states which required a nonhermitian term for regaining unitarity in it.

From the Lagrangian density of Q.C.D., and by the standard method, we can obtain the canonical energy-momentum density tensor which turns out to be

$$\tilde{T}^{\mu\nu}(x) = -g^{\mu\nu}\mathcal{L}(x) + \frac{i}{2}\bar{q}^{A}_{\alpha}(x)\gamma^{\mu}\partial^{\nu}q^{A}_{\alpha}(x) - \frac{i}{2}[\partial^{\nu}\bar{q}^{A}_{\alpha}(x)]\gamma^{\mu}q^{A}_{\alpha}(x)$$

$$- F^{\mu\lambda}_{a}(x)\partial^{\nu}B^{a}_{\lambda}(x) - \frac{1}{a}[\partial^{\lambda}B^{a}_{\lambda}(x)]\partial^{\nu}B^{\mu}_{a}(x) \tag{I.81}$$

$$-[\partial^{\mu}\bar{\phi}_{a}(x)]\partial^{\nu}\phi_{a}(x) - [\partial^{\nu}\bar{\phi}_{a}(x)]\partial^{\mu}\phi_{a}(x) + g f_{abc}B^{\mu}_{a}(x)[\partial^{\nu}\bar{\phi}_{b}(x)]\phi_{c}(x)$$

which satisfies

$$\partial_{\mu}\tilde{T}^{\mu\nu}(x) = 0 \tag{I.82}$$

Similarly the canonical angular momentum density tensor is

$$\tilde{M}^{\mu}{}_{\nu\sigma}(x) = \tilde{T}^{\mu}{}_{\nu}(x)\,x_{\sigma} - \tilde{T}^{\mu}{}_{\sigma}(x)\,x_{\nu} + M^{(S)\mu}{}_{\nu\sigma}(x)$$

$$M^{(S)\mu}{}_{\nu\sigma}(x) \equiv -\frac{1}{4}\bar{q}^{A}_{\alpha}(x)[\gamma^{\mu}\sigma_{\nu\sigma} + \sigma_{\nu\sigma}\gamma^{\mu}]q^{A}_{\alpha}(x) + F_{a}{}^{\mu}{}_{\nu}(x)B^{a}_{\sigma}(x)$$

$$- F^{\mu}_{a\sigma}(x)B^{a}_{\nu}(x) + \frac{1}{a}[\partial^{\lambda}B^{a}_{\lambda}(x)][g^{\mu}{}_{\nu}B^{a}_{\sigma}(x) - g^{\mu}{}_{\sigma}B^{a}_{\nu}(x)] \tag{I.83}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^{\mu}, \gamma^{\nu}]$. This tensor satisfies

$$\partial_{\mu}\tilde{M}^{\mu}{}_{\nu\sigma}(x) = 0 \quad , \quad M^{\mu}{}_{\nu\sigma}(x) = -M^{\mu}{}_{\sigma\nu}(x) \tag{I.84}$$

The corresponding time independent charges are the generators of the Poincaré group.

In order to obtain the Belinfante energy-momentum density tensor [BE 39] [BE 40] we define

$$G^{\sigma\mu\nu}(x) \equiv -\frac{1}{2}[M^{(S)\sigma\mu\nu}(x) + M^{(S)\mu\nu\sigma}(x) - M^{(S)\nu\sigma\mu}(x)] \tag{I.85}$$

which satisfies

$$G^{\sigma\mu\nu}(x) = - G^{\mu\sigma\nu}(x) \tag{I.86}$$

and its explicit expression is

$$G^{\sigma\mu\nu}(x) = \frac{i}{8} \bar{q}^A_\alpha(x) [\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu\gamma^\mu] q^A_\alpha(x) - F^{\sigma\mu}_a(x) B^\nu_a(x)$$

$$- \frac{1}{a} [\partial^\lambda B^a_\lambda(x)][g^{\mu\nu} B^\sigma_a(x) - g^{\sigma\nu} B^\mu_a(x)] \tag{I.87}$$

Then the Belinfante energy-momentum density tensor is

$$T^{\mu\nu}(x) \equiv \tilde{T}^{\mu\nu}(x) + \partial_\sigma G^{\sigma\mu\nu}(x) \tag{I.88}$$

and satisfies

$$\partial_\mu T^{\mu\nu}(x) = 0 \quad , \quad T^{\mu\nu}(x) = T^{\nu\mu}(x) \tag{I.89}$$

Using the equations of motion at straightforward calculation gives [FM 74] [FW 74]

$$T^{\mu\nu}(x) = - g^{\mu\nu} \mathcal{L}(x) + \frac{i}{4} \bar{q}^A_\alpha(x) [\gamma^\mu \partial^\nu q^A_\alpha(x) + \gamma^\nu \partial^\mu q^A_\alpha(x)]$$

$$- \frac{i}{4} [\partial^\mu \bar{q}^A_\alpha(x) \gamma^\nu + \partial^\nu \bar{q}^A_\alpha(x) \gamma^\mu] q^A_\alpha(x)$$

$$+ \frac{1}{4} g \bar{q}^A_\alpha(x) \lambda^a_{\alpha\beta} [\gamma^\mu B^\nu_a(x) + \gamma^\nu B^\mu_a(x)] q^A_\beta(x)$$

$$- F^{\mu\lambda}_a(x) F^\nu_{a\lambda}(x) - [\partial^\mu \bar{\Phi}_a(x) \partial^\nu \phi_a(x) + \partial^\nu \bar{\Phi}_a(x) \partial^\mu \phi_a(x)]$$

$$
\begin{aligned}
&+ g\, f_{abc} \left[B^{\mu}_{a}(x)\, \partial^{\nu} \bar{\phi}_{b}(x) + B^{\nu}_{a}(x)\, \partial^{\mu} \bar{\phi}_{b}(x) \right] \phi_{c}(x) \\
&- \frac{1}{a} g^{\mu\nu} \partial_{\sigma} \left\{ \left[\partial_{\lambda} B^{\lambda}_{a}(x) \right] B^{\sigma}_{a}(x) \right\} \\
&+ \frac{1}{a} \left\{ \left[\partial^{\mu} \partial^{\lambda} B^{a}_{\lambda}(x) \right] B^{\nu}_{a}(x) + \left[\partial^{\nu} \partial^{\lambda} B^{a}_{\lambda}(x) \right] B^{\mu}_{a}(x) \right\}
\end{aligned}
\qquad (I.90)
$$

If the equations of motion are used it is easy to check that

$$
\begin{aligned}
T^{\mu}{}_{\mu}(x) = {} & m_{A}\, \bar{q}^{A}_{\alpha}(x)\, q^{A}_{\alpha}(x) + \\
& + 2\, \partial_{\mu} \left\{ \left[\partial^{\mu} \bar{\phi}_{a}(x) \right] \phi_{a}(x) - \frac{1}{a} \left[\partial_{\nu} B^{\nu}_{a}(x) \right] B^{\mu}_{a}(x) \right\}
\end{aligned}
\qquad (I.91)
$$

which shows that $T^{\mu\nu}(x)$ coincides with the improved energy-momentum density tensor [CJ 71], but for the terms needed to carry out a covariant and unitary second quatization.

Finally the Belinfante angular momentum density tensor is

$$
M^{\mu}{}_{\nu\sigma}(x) = T^{\mu}{}_{\nu}(x)\, x_{\sigma} - T^{\mu}{}_{\sigma}(x)\, x_{\nu} \qquad (I.92)
$$

which is related to the canonical by

$$
M^{\mu}{}_{\nu\sigma}(x) = \partial_{\tau} \left[G^{\tau\mu}{}_{\nu}(x)\, x_{\sigma} - G^{\tau\mu}{}_{\sigma}(x)\, x_{\nu} \right] + \tilde{M}^{\mu}{}_{\nu\sigma}(x) \qquad (I.93)
$$

and satisfies eqs. (84) . Obviously the generators of the Poincaré group which we construct from the Belinfante energy-momentum and angular-momentum density tensors are the same as the ones constructed from the canonical ones, as it should.

Consider now the following current

$$
D^{\mu}(x) \equiv x_{\nu}\, T^{\mu\nu}(x) - 2 \left\{ \left[\partial^{\mu} \bar{\phi}_{a}(x) \right] \phi_{a}(x) - \frac{1}{a} \left[\partial_{\nu} B^{\nu}_{a}(x) \right] B^{\mu}_{a}(x) \right\} \qquad (I.94)
$$

which, from eq. (91), has the divergence

$$\partial_\mu D^\mu(x) = m_A \bar{q}^A_\alpha(x) q^A_\alpha(x) \tag{I.95}$$

and is thus conserved in the limit of massless quarks. This is the Belinfante dilatation current. Its canonical counterpart is the Noether current of another symmetry of the Q.C.D. Lagrangian in the limit of massless quarks: dilatation or scale invariance. As we will see later on, in the quantum theory eqs. (91) and (95) will have a further contribution, called the trace anomaly, much the same as what happened in eq. (47) with the axial current.

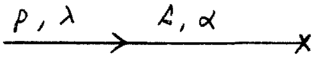
There is still one term allowed by gauge invariance and renormalizability which can be added to the Q.C.D. Lagrangian

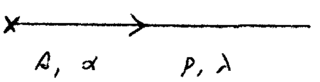
$$\mathcal{L}_\theta(x) \equiv - \frac{\theta g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \tag{I.96}$$

As we saw in (49) this term is a total divergence and leaves therefore the equations of motion unchanged. It is irrelevant for perturbation theory. Its contribution to the Q.C.D. action is proportional to the winding number and it is thus responsible for instanton effects. It influences the structure of the Q.C.D. vacuum through tunneling. We will omit it, because we will work almost exclusively within perturbation theory. Notice that it violates the P and CP symmetries, but these effects can be made very small by taking θ very small. Parity and charge conjugation are otherwise symmetries of the Q.C.D. Lagrangian, as corresponds to a theory of the strong interactions.

Let us finally give the Feynman rules for Q.C.D.

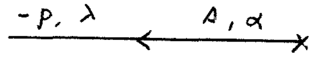
Quark

Incoming  $u^A_\alpha(\vec{p},\lambda)$

Outgoing  $\bar{u}^A_\alpha(\vec{p},\lambda)$

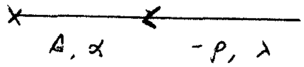
Antiquark

Incoming



$$\bar{v}^{A}_{\alpha}(\vec{p}, \lambda)$$

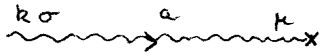
Outgoing



$$v^{A}_{\alpha}(\vec{p}, \lambda)$$

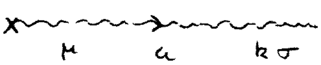
Gluon

Incoming



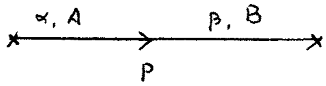
$$\epsilon^{a}_{\mu}(\vec{k}, \sigma)$$

Outgoing



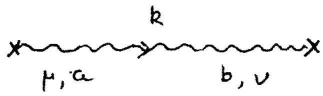
$$\epsilon^{a*}_{\mu}(\vec{k}, \sigma)$$

Quark propagator



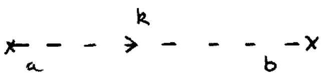
$$+\frac{i}{(2\pi)^4}\,\frac{1}{\not{p} - m_A + i\eta}\,\delta_{\alpha\beta}\,\delta_{AB}$$

Gluon propagator



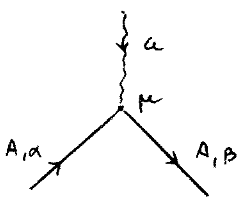
$$-\frac{i}{(2\pi)^4}\left[g_{\mu\nu} - (1-a)\frac{k_\mu k_\nu}{k^2+i\eta}\right]\frac{\delta_{ab}}{k^2+i\eta}$$

Ghost propagator



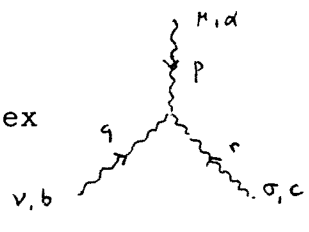
$$-\frac{i}{(2\pi)^4}\,\frac{\delta_{ab}}{k^2+i\eta}$$

Fermionic vertex



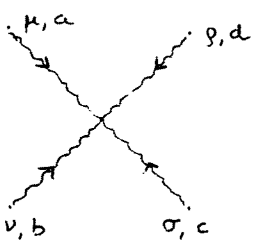
$$+g\left(\frac{\lambda_a}{2}\right)_{\beta\alpha}\gamma^\mu$$

Triple gluon vertex

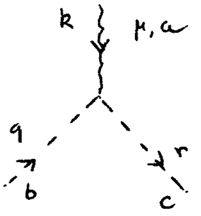


$$-ig\, f_{abc}\left[g_{\mu\nu}(p-q)_\sigma + g_{\nu\sigma}(q-r)_\mu + g_{\sigma\mu}(r-p)_\nu\right]$$

Quartic gluon vertex



$$-g^2\left[f_{abe}f_{cde}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) + f_{ace}f_{bde}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma}) + f_{ade}f_{cbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\sigma\rho})\right]$$

Ghost vertex 

$-ig\, f_{abc}\, r^{\mu}$

The ghost propagator can be taken with opposite sign, as long as one also changes the sign of the ghost vertex. Physical Green's functions are unchanged, because ghosts only appear in loops.

Furthermore we must take into account the following rules

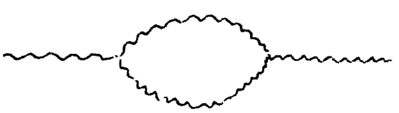
i) For each vertex write a factor $(2\pi)^4\, \delta\left(\sum\limits_{in} p - \sum\limits_{out} p\right)$

ii) Multiply the obtained expression by $(2\pi)^{-4}\, i^{n+1}$, where n is the number of vertices.

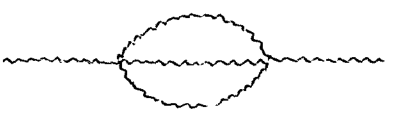
iii) Integrate over all internal momenta and extract a δ - function expressing the total energy-momentum conservati

iv) Multiply by (-1) for each internal loop of quarks or ghosts.

v) Take into account the correct statistical factors



1/2!



1/3!

vi) The arrows along quark and ghost lines should point all into the same directions.

vii) One should put a relative minus sign in contributions coming from diagrams which only differ by the exchange of identical external fermions.

This leads to the T matrix elements, where the S and T matrices are related by

$$S = 1 - i\,(2\pi)^4\,\delta\Big(\sum_{in} p_i - \sum_{out} p_f\Big)\,T \qquad \text{(I.97)}$$

The choices of the gauge parameter $a = 0$ and $a = 1$ are called Landa and Feynman gauge, respectively.

II. BECCHI-ROUET-STORA TRANSFORMATIONS AND SLAVNOV-TAYLOR IDENTITIES

Invariance under global transformations has direct physical consequences, like charge conservation. Invariance under local transformations does not have direct physical consequences because the corresponding conserved current, and its associated charge, are gauge dependent, but it does imply extensive relations among Green's functions. In Q.E.D. the invariance under local gauge transformations implies the Ward-Takahashi identities [WA 50][TA 57], discussed in any textbook of quantum electrodynamics. A consequence of these identities that the equality of bare electric charges implies equality of renormalized electric charges. In non-abelian gauge theories the equalities among Green's functions, consequence of the local gauge invariance of the theory, are the so called Slavnov-Taylor identities [SL 75] [TA 71] and are equally important. Notice however, that as quantizatio has required the breaking of gauge invariance, the Q.C.D. Lagrangian is no longer gauge invariant. There exists however a generalization of gauge invariance to the quantum theory discovered by Becchi, Rouet and Stora. We are going to see how these identities can be easily established using the Becchi-Rouet-Stora (BRS) transformations [BR 74] [BR 75].

The BRS-transformations are the following set of one-parameter transformations

$$q^A_\alpha(x) \longrightarrow q'^A_\alpha(x) = q^A_\alpha(x) + \frac{i}{2}\, g\, \omega\, \lambda^a_{\alpha\beta}\, \phi_a(x)\, q^A_\beta(x)$$

$$\bar{q}^A_\alpha(x) \longrightarrow \bar{q}'^A_\alpha(x) = \bar{q}^A_\alpha(x) - \frac{i}{2}\, g\, \omega\, \bar{q}^A_\beta(x)\, \lambda^a_{\beta\alpha}\, \phi_a(x)$$

(II.1)

$$B^{\mu}_{a}(x) \longrightarrow B'^{\mu}_{a}(x) = B^{\mu}_{a}(x) + g\,\omega\, f_{abc}\, B^{\mu}_{b}(x)\,\phi_c(x) + \omega\,\partial^{\mu}\phi_a(x)$$

$$\phi_a(x) \longrightarrow \phi'_a(x) = \phi_a(x) - \frac{1}{2}\, g\,\omega\, f_{abc}\,\phi_b(x)\,\phi_c(x)$$

$$\bar{\phi}_a(x) \longrightarrow \bar{\phi}'_a(x) = \bar{\phi}_a(x) + \frac{\omega}{a}\,\partial_{\mu} B^{\mu}_{a}(x)$$

for the fields appearing in the Q.C.D. Lagrangian density, where ω is a constant anticommuting quantity, a Grassmann number, so that $\omega^2 = 0$, which may conveniently be regarded as infinitesimal. The fact that ω is an anticommuting quantity guarantees that the original and transformed fields behave, as far as it regards their commutation properties, in the same way since the ghosts fields are anticommu ting quantities. The first three transformation laws correspond to gauge transformations where $\theta_a(x) = -\omega\,\phi_a(x)$. The fact that $\theta_{\alpha}(x)$ must be real implies

$$[\omega\,\phi_a(x)]^{+} = \omega\,\phi_a(x) \tag{II.2}$$

Using this equality it is clear that the second transformation law is a consequence of the first one. The last two transformation laws define how the ghosts fields transform under BRS transformations, leading thus to a generalization of the gauge transformations to the quantum theory.

The classical part of the Lagrangian of Q.C.D. is then obviously BRS-invariant and we only have to consider the gauge fixing and the ghost terms. For these we have the transformation laws

$$[\partial^{\mu} B^{a}_{\mu}(x)][\partial^{\nu} B^{a}_{\nu}(x)] \longrightarrow [\partial^{\mu} B'^{a}_{\mu}(x)][\partial^{\nu} B'^{a}_{\nu}(x)] =$$

$$= [\partial^{\mu} B^{a}_{\mu}(x)][\partial^{\nu} B^{a}_{\nu}(x)] + 2\omega\,[\partial_{\mu} B^{\mu}_{a}(x)]\,\partial_{\nu}\,[D^{\nu}_{ab}\,\phi_b(x)] \tag{II.3}$$

and

$$D^{\mu}_{ab}\ \phi_b(x) \longrightarrow D'^{\mu}_{ab}\ \phi'_b(x) = D^{\mu}_{ab}\ \phi_b(x) \tag{II.4}$$

The last transformation law can be proved in the following way: From (1) it follows immediatly that

$$D'^{\mu}_{ab}\ \phi'_b(x) = D^{\mu}_{ab}\ \phi_b(x) + \frac{1}{2} g^2 \omega\, f_{abc}\, f_{bst}\, B^{\mu}_c(x)\, \phi_s(x)\, \phi_t(x)$$

$$- g^2 \omega\, f_{atb}\, f_{bcs}\, B^{\mu}_c(x)\, \phi_s(x)\, \phi_t(x)$$

where the anticommutative character of the Faddeev-Popov ghost fields has been used. Now we can write the Jacobi identity

$$[f_{atb}\, f_{csb} + f_{tcb}\, f_{asb} + f_{cab}\, f_{tsb}]\ \phi_s(x)\ \phi_t(x) = 0$$

$$\Rightarrow \qquad f_{atb}\, f_{bcs}\, \phi_s(x)\, \phi_t(x) = \frac{1}{2} f_{abc}\, f_{bst}\, \phi_s(x)\, \phi_t(x)$$

and (4) follows. Finally

$$[\partial_\mu \bar{\phi}_a(x)]\, D^{\mu}_{ab}\ \phi_b(x) \longrightarrow [\partial_\mu \bar{\phi}'_a(x)]\, D'^{\mu}_{ab}\ \phi'_b(x) =$$

$$= [\partial_\mu \bar{\phi}_a(x)]\, D^{\mu}_{ab}\ \phi_b(x) + \frac{\omega}{a} [\partial_\mu \partial_\nu B^{\nu}_a(x)]\, D^{\mu}_{ab}\ \phi_b(x) \tag{II.5}$$

From the derived transformation laws it is now clear that under BRS transformations

$$\mathcal{L}(x) \longrightarrow \mathcal{L}'(x) = \mathcal{L}(x) - \partial_\mu \left\{ \frac{\omega}{a} [\partial_\nu B^{\nu}_a(x)]\, D^{\mu}_{ab}\ \phi_b(x) \right\} \tag{II.6}$$

and therefore these transformations are symmetry transformations of the Lagrangian density of Q.C.D.

Before discussing the Slavnov-Taylor identities we would like to introduce the notation for the propagators and vertex functions that we are interested in. As it is well known, in quantum field theory all interesting physical quantities are derivable from the n-point Green's function which are defined as the vacuum expectation value of the time-ordered product of n fields

$$G(x_1, x_2, \ldots, x_m) \equiv \langle \underset{\sim}{0} | T \left(\xi(x_1)\, \xi(x_2) \ldots \xi(x_m) \right) | \underset{\sim}{0} \rangle \tag{II.7}$$

where $\xi(x)$ stands for a generic field in the Heisenberg representation and $|\underset{\sim}{0}\rangle$ is the physical vacuum or ground state of H which is defined by the relations

$$H |\underset{\sim}{0}\rangle = 0 \qquad , \qquad \langle \underset{\sim}{0} | \underset{\sim}{0} \rangle = 1 \tag{II.8}$$

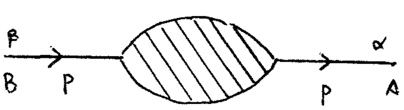
where H is the full Hamiltonian of the theory in the Heisenberg representation. In perturbation theory these Green's functions contain connected as well as disconnected diagrams.

The evaluation of the connected Green's function in perturbation theory is given by the Gell-Mann-Low relation [GL 51]

$$G_c(x_1, x_2, \ldots x_m) = \frac{\langle 0 | T \left(\xi^{(0)}(x_1)\, \xi^{(0)}(x_2) \ldots \xi^{(0)}(x_m) \exp\left[i \int d^4x\, \mathcal{L}^{(0)}_{int}(x) \right] \right) | 0 \rangle}{\langle 0 | T \left(\exp\left[i \int d^4x\, \mathcal{L}^{(0)}_{int}(x) \right] \right) | 0 \rangle} \tag{II.9}$$

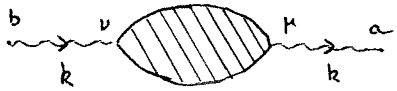
where the subindex c denotes that only the connected graphs are taken into account and it will be omitted if no error is possible. $\mathcal{L}^{(0)}_{int}(x)$ is the interaction Lagrangian density and the superindex (0) means that all fields must be taken as free. Finally $|0\rangle$ is the vacuum annihilated by all the annihilation operators appearing in the expansion of the free fields. The denominator of (9) is precisely the sum of all vacuum to vacuum transition amplitudes and it cancels the disconnected vacuum graphs appearing in the calculation of the numerator.

The two point Green's functions associated with the fields appearing in the theory are the so-called propagators and we will use for their representation in momentum space the following notation



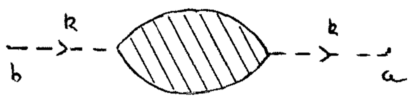
$$i S^{AB}_{\alpha\beta}(\not p) \equiv \int d^4x \, e^{ip.x} \langle \underset{\sim}{0} | T (q^A_\alpha(x) \, \bar q^B_\beta(0)) | \underset{\sim}{0} \rangle \equiv \tag{II.10}$$

$$\equiv i \, \delta_{\alpha\beta} \, \delta_{AB} \, S(\not p)$$



$$i D^{\mu\nu}_{ab}(k) \equiv \int d^4x \, e^{ik.x} \langle \underset{\sim}{0} | T (B^\mu_a(x) \, B^\nu_b(0)) | \underset{\sim}{0} \rangle \equiv \tag{II.11}$$

$$\equiv i \, \delta_{ab} \, D^{\mu\nu}(k) = i \, \delta_{ab} \, D^{\nu\mu}(k) = i \, \delta_{ab} \, D^{\mu\nu}(-k)$$



$$i \tilde D_{ab}(k) \equiv \int d^4x \, e^{ik.x} \langle \underset{\sim}{0} | T (\phi_a(x) \, \bar\phi_b(0)) | \underset{\sim}{0} \rangle \equiv \tag{II.12}$$

$$\equiv i \, \delta_{ab} \, \tilde D(k^2)$$

These are, respectively, the quark, gluon and ghost propagators. In lowest order perturbation theory they turn out to be

$$i S^{(0)AB}_{\alpha\beta}(\not p) = \delta_{AB} \, \delta_{\alpha\beta} \, i \frac{1}{\not p - m_A + i\eta} \tag{II.13}$$

$$i D^{(0)\mu\nu}_{ab}(k) = \delta_{ab} \, i \left[- g_{\mu\nu} + (1-a) \frac{k_\mu k_\nu}{k^2 + i\eta} \right] \frac{1}{k^2 + i\eta} \tag{II.14}$$

$$i \tilde D^{(0)}_{ab}(k) = \delta_{ab} \, (-i) \frac{1}{k^2 + i\eta} \tag{II.15}$$

Let us now proceed to prove the first Slavnov-Taylor identit

Let us start from the trivial identity

$$\langle \underset{\sim}{0} | T \left([\partial_\mu B^\mu_a(x)] \bar{\Phi}_b(y) \right) | \underset{\sim}{0} \rangle = 0 \qquad \text{(II.16)}$$

Since the BRS transformations are symmetry transformations we can writ also

$$\langle \underset{\sim}{0} | T \left([\partial_\mu B'^\mu_a(x)] \bar{\Phi}'_b(y) \right) | \underset{\sim}{0} \rangle = 0$$

Using the transformed fields (1) as well as (16) we obtain

$$\omega \left\{ \frac{1}{a} \langle \underset{\sim}{0} | T \left([\partial_\mu B^\mu_a(x)][\partial_\nu B^\nu_b(y)] \right) | \underset{\sim}{0} \rangle + \langle \underset{\sim}{0} | T \left([\partial^\mu \partial_\mu \phi_a(x)] \bar{\Phi}_b(y) \right) | \underset{\sim}{0} \rangle \right.$$

$$\left. + g f_{acd} \left[\langle \underset{\sim}{0} | T \left([\partial_\mu B^\mu_c(x)] \phi_d(x) \bar{\Phi}_b(y) \right) | \underset{\sim}{0} \rangle + \langle \underset{\sim}{0} | T \left(B^\mu_c(x) [\partial_\mu \phi_d(x)] \bar{\Phi}_b(y) \right) | \underset{\sim}{0} \rangle \right] \right\} =$$

The contribution of the last three terms is zero if use is made of the equations of motion (I.77). Therefore

$$\langle \underset{\sim}{0} | T \left([\partial_\mu B^\mu_a(x)][\partial_\nu B^\nu_b(y)] \right) | \underset{\sim}{0} \rangle = 0 \qquad \text{(II.17)}$$

Hence

$$\int d^4x \, d^4y \; e^{iq\cdot x} \, e^{-iq'\cdot y} \langle \underset{\sim}{0} | T \left([\partial_\mu B^\mu_a(x)][\partial_\nu B^\nu_b(y)] \right) | \underset{\sim}{0} \rangle = 0$$

which can be written as

$$q_\mu q'_\nu \int d^4x\, d^4y\; e^{iq.x} e^{-iq'.y} \langle 0 | T (B^\mu_a(x)\; B^\nu_b(y)) | 0 \rangle$$

$$+ iq_\mu \int d^4x\, d^4y\; e^{iq.x} e^{-iq'.y}\; \delta(x^0 - y^0) \langle 0 | [B^\mu_a(x) , B^0_b(y)] | 0 \rangle$$

$$- \int d^4x\, d^4y\; e^{iq.x} e^{-iq'.y}\; \delta(x^0-y^0) \langle 0 | [B^0_a(x) ,\; \partial_\mu B^\mu_b(y)] | 0 \rangle = 0$$

The second term is zero as consequence of the equal time commutation relations (I.78). Furthermore, and using (I.79), we have

$$\delta(x^0 - y^0) [B^0_a(x) \;,\; \partial_\mu B^\mu_b(y)] = - i a\, \delta_{ab}\, \delta^{(4)}(x-y) \tag{II.18}$$

and hence

$$q_\mu q'_\nu \int d^4y\; e^{-i(q'-q).y} \int d^4x\; e^{iq.(x-y)} \langle 0 | T (B^\mu_a(x-y)\; B^\nu_b(0)) | 0 \rangle$$

$$+ ia\, \delta_{ab} \int d^4y\; e^{-i(q'-q)y} = 0$$

where we have used invariance under space-time translations. Hence

$$\int d^4y\; e^{-i(q'-q).y} \{ i q_\mu q'_\nu D^{\mu\nu}_{ab}(q) + ia\, \delta_{ab} \} = 0$$

and therefore

$$q_\mu q_\nu D^{\mu\nu}_{ab}(q) = - a\, \delta_{ab} \tag{II.19}$$

Since $q_\mu D^{\mu\nu}_{ab}$ is proportional to $q^\nu \delta_{ab}$ we have also

$$q_\mu D^{\mu\nu}_{ab}(q) = - a\, \delta_{ab} \frac{1}{q^2} q^\nu \tag{II.20}$$

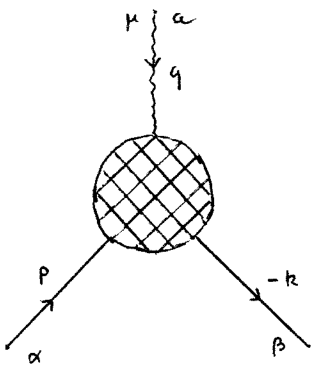
Hence, taking into account (14) , we obtain the Slavnov-Taylor identity

$$q_\mu q_\nu D^{\mu\nu}_{ab}(q) = q_\mu q_\nu D^{(0)\,\mu\nu}_{ab}(q) \tag{II.21}$$

which means that to any order in perturbation theory the non-transverse part of the gluon propagator is equal to the corresponding part of the free propagator. The same result is well known in Q.E.D. for the photon propagator.

Now we will consider the three point Green's functions corresponding to the three triple vertex functions appearing in the Lagrangian density of Q.C.D..We are going to define the corresponding proper vertex functions which are the Green's functions with amputated legs. For their Fourier transform we will introduce the following notation:

i) Fermionic Vertex

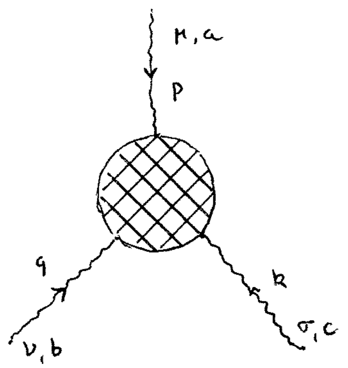


$$i S_{\beta\beta'}(-\not{k})\, ig\, \Gamma^{a'}_{\mu'\beta'\alpha'}(q,k,p)\, i S_{\alpha'\alpha}(\not{p})\, i D^{\mu'\mu}_{a'a}(q) \equiv$$

$$\equiv \int d^4x\, d^4y\, e^{-ik.x}\, e^{-ip.y} \langle \underset{\sim}{0} | T\left(q_\beta(x)\, \bar{q}_\alpha(y)\, B^\mu_a(0)\right) | \underset{\sim}{0} \rangle \tag{II.22}$$

$p + q + k = 0$ The flavor index has been omitted.

ii) Triple Gluon Vertex

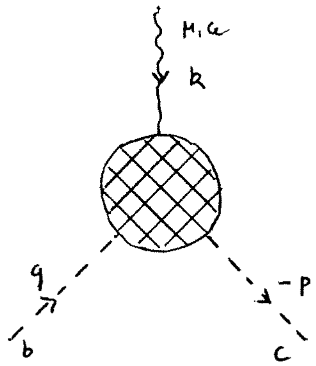


$$ig\, T^{\mu'\nu'\sigma'}_{a'b'c'}(p,q,k)\, i D^{a'a}_{\mu'\mu}(p)\, i D^{b'b}_{\nu'\nu}(q)\, i D^{c'c}_{\sigma'\sigma}(k) \equiv$$

$$\equiv \int d^4x\, d^4y\, e^{-ip.x}\, e^{-iq.y} \langle \underset{\sim}{0} | T\left(B^a_\mu(x)\, B^b_\nu(y)\, B^c_\sigma(0)\right) | \underset{\sim}{0} \rangle \tag{II.23}$$

$p + q + k = 0$

iii) Ghost Vertex



$$ig\, G^{\mu'}_{a'b'c'}(k,q,p)\; i D^{a'a}_{\mu'\mu}(k)\; i\tilde{D}_{b'b}(q)\; i\tilde{D}_{c'c}(p) \equiv$$

$$\equiv \int d^4x\, d^4y\, e^{-ip.x}\, e^{-iq.y} \langle \underset{\sim}{0}|\, T\left(\phi_c(x)\, \bar{\Phi}_b(y)\, B^a_\mu(0)\right)|\underset{\sim}{0}\rangle \qquad \text{(II.24)}$$

$$p + q + k = 0$$

Let us now calculate these vertices in lowest order perturbation theory. We will omit the superindex (o) that denote free fields For the fermionic vertex

$$\int d^4x\, d^4y\, e^{-ik.x}\, e^{-ip\,y} \langle 0|\, T\left(q_\beta(x)\, \bar{q}_\alpha(y)\, B^\mu_a(0)\right)|0\rangle =$$

$$= ig \int d^4x\, d^4y\, d^4z\, e^{-ik.x}\, e^{-ip.y} \langle 0|T\left(q_\beta(x)\, \bar{q}_\alpha(y)\, B^\mu_a(0)\, \bar{q}_{\beta'}(z)\, \gamma_{\mu'} \left(\frac{\lambda^{a'}}{2}\right)_{\beta'\alpha'} q_{\alpha'}(z)\, B^{\mu'}_{a'}(z)\right)|0\rangle$$

$$ig\left(\frac{\lambda^{a'}}{2}\right)_{\beta'\alpha'} \int d^4x\, d^4y\, d^4z\, e^{-ik.x}\, e^{-ip.y}\, \underline{q_\beta(x)\, \bar{q}_{\beta'}(z)}\; \gamma_{\mu'}\; \underline{q_{\alpha'}(z)\, \bar{q}_\alpha(y)}\;\; \underline{B^\mu_a(0)\, B^{\mu'}_{a'}(z)} =$$

$$g\left(\frac{\lambda^{a'}}{2}\right)_{\beta'\alpha'} \frac{1}{(2\pi)^{12}} \int d^4x\, d^4y\, d^4z \int d^4p_1\, d^4p_2\, d^4p_3\, e^{-ik.x}\, e^{-ip.y}\, e^{-ip_1.(x-z)}\, e^{-ip_2.(z-}$$

$$e^{ip_3.z}\, S^{(0)}_{\beta\beta'}(\not{p}_1)\, \gamma_{\mu'}\, S^{(0)}_{\alpha'\alpha}(\not{p}_2)\, D^{(0)\mu'\mu}_{a'a}(p_3) = g\, S^{(0)}_{\beta\beta'}(-\not{k})\, \gamma_{\mu'} \left(\frac{\lambda^{a'}}{2}\right)_{\beta'\alpha'} S^{(0)}_{\alpha'\alpha}(\not{p})\, D^{(0)\mu'\mu}_{a'a}(q)$$

Hence

$$\Gamma^{(0)\,a}_{\mu\,\beta\alpha}(q,k,p) = \gamma_\mu \left(\frac{\lambda^a}{2}\right)_{\beta\alpha} \qquad \text{(II.25)}$$

Now let us consider the triple gluon vertex

$$\int d^4x\, d^4y\, e^{-ip.x}\, e^{-iq.y} \langle 0|T(B^a_\mu(x)\, B^b_\nu(y)\, B^c_\sigma(0))|0\rangle =$$

$$= -g\frac{i}{2} f_{a'b'c'} \int d^4x\, d^4y\, d^4z\, e^{-ip.x}\, e^{-iq.y} \langle 0|T(B^a_\mu(x)\, B^b_\nu(y)\, B^c_\sigma(0)\, [\partial_{\mu'} B^{a'}_{\nu'}(z) - \partial_{\nu'} B^{a'}_{\mu'}(z)]\, B^{\mu'}_{b'}(z)\, B^{\nu'}_{c'}(z))|0\rangle$$

When all possible contractions are carried out we notice that they appear in pairs giving the same contribution. Hence using the equal time commutation rules (I.78) we obtain

$$= -ig f_{a'b'c'} \frac{1}{(2\pi)^{12}} \int d^4x\, d^4y\, d^4z \int d^4p_1\, d^4p_2\, d^4p_3\, e^{-ip.x}\, e^{-iqy}.$$

$$\Big\{ [D^{(0)}_{\mu\nu' aa'}(p_1)\, p_{1\mu'} - D^{(0)}_{\mu\mu' aa'}(p_1)\, p_{1\nu'}]\, D^{(0)\mu'}_{\nu\, bb'}(p_2)\, D^{(0)\nu'}_{\sigma cc'}(p_3)\, e^{-ip_1(x-z)}\, e^{-ip_2\cdot(y-z)}\, e^{+ip_3\cdot z}$$

$$[D^{(0)}_{\nu\nu' ba'}(p_1)\, p_{1\mu'} - D^{(0)}_{\nu\mu' ba'}(p_1)\, p_{1\nu'}]\, D^{(0)\mu'}_{\mu\, ab'}(p_2)\, D^{(0)\nu'}_{\sigma cc'}(p_3)\, e^{-ip_1\cdot(y-z)}\, e^{-ip_2(x-z)}\, e^{+ip_3\cdot z}$$

$$[D^{(0)}_{\sigma\nu' ca'}(p_1)\, p_{1\mu'} - D^{(0)}_{\sigma\mu' ca'}(p_1)\, p_{1\nu'}]\, D^{(0)\mu'}_{\mu\, ab'}(p_2)\, D^{(0)\nu'}_{\nu\, bc'}(p_3)\, e^{+ip_1\cdot z}\, e^{-ip_2\cdot(x-z)}\, e^{-ip_3\cdot(y-z)} \Big\}$$

Now all the integrals can be carried out immediately and we get

$$= -ig f_{a'b'c'}\, D^{(0)aa'}_{\mu\mu'}(p)\, D^{(0)bb'}_{\nu\nu'}(q)\, D^{(0)cc'}_{\sigma\sigma'}(k) [g^{\mu'\nu'}(p-q)^{\sigma'} + g^{\nu'\sigma'}(q-k)^{\mu'} + g^{\sigma'\mu'}(k-p)^{\nu'}]$$

and therefore in lowest order perturbation theory

$$T^{(0)\mu\nu\sigma}_{abc}(p,q,k) = -i f_{abc} [g^{\mu\nu}(p-q)^\sigma + g^{\nu\sigma}(q-k)^\mu + g^{\sigma\mu}(k-p)^\nu] \tag{II.26}$$

Let us finally consider the ghost vertex in lowest order perturbation theory

$$\int d^4x\, d^4y\, e^{-ip.x} e^{-iq.y} \langle 0| T\left(\phi_c(x)\, \bar{\phi}_b(y)\, B^a_\mu(0)\right)|0\rangle =$$

$$= -ig\, f_{a'b'c'} \int d^4x\, d^4y\, d^4z\, e^{-ip.x} e^{-iq.y} \langle 0| T\left(\phi_c(x)\, \bar{\phi}_b(y)\, B^a_\mu(0)\, B^{a'}_{\mu'}(z)\left[\partial^{\mu'}\bar{\phi}_{c'}(z)\right]\phi_{b'}(z)\right)|$$

$$= -\frac{g}{(2\pi)^{12}} f_{a'b'c'} \int d^4x\, d^4y\, d^4z \int d^4p_1\, d^4p_2\, d^4p_3\, D^{(0)\,a'a}_{\mu'\mu}(p_1)\, \tilde{D}^{(0)}_{b'b}(p_3)\, \tilde{D}^{(0)}_{c'c}(p_2)\,.$$

$$e^{-ip\,x} e^{-iq.y} e^{+ip_1 z} e^{-ip_2\cdot(x-z)} e^{-ip_3(z-y)}\; i\,p_2^{\mu'} =$$

$$= ig\, f_{a'b'c'}\, p^{\mu'}\, D^{(0)\,a'a}_{\mu'\mu}(k)\, \tilde{D}^{(0)}_{b'b}(q)\, \tilde{D}^{(0)}_{c'c}(p)$$

and therefore

$$G^{(0)\mu}_{abc}(k, q, p) = +i\, f_{abc}\, p^\mu \qquad \text{(II.27)}$$

Notice that the triple vertex functions are defined so as to reproduce precisely our Feynman rules.

Let us now proceed to derive some properties for these triple vertex functions. Let us start from the trivial identity

$$\langle \underset{\sim}{0}| T\left([\partial^\mu B^a_\mu(x)][\partial^\nu B^b_\nu(y)]\, \bar{\phi}_c(z)\right)|\underset{\sim}{0}\rangle = 0 \qquad \text{(II.28)}$$

The same equation can be written for the transformed fields under a BRS-transformation and using (1) we obtain immediately

$$\langle \underset{\sim}{0}| T\left([\partial^\mu B^a_\mu(x)][\partial^\nu B^b_\nu(y)][\partial^\sigma B^c_\sigma(z)]\right|\underset{\sim}{0}\rangle = 0$$

Using the equal time commutation relations as well as (18) we obtain

$$\partial_x^\mu \partial_y^\nu \partial_z^\sigma \, T\left(B^a_\mu(x)\, B^b_\nu(y)\, B^c_\sigma(z) \right) =$$

$$= T\left([\partial^\mu B^a_\mu(x)][\partial^\nu B^b_\nu(y)][\partial^\sigma B^c_\sigma(z)] \right) \tag{II.29}$$

$$- i\alpha \left[\delta_{ab}\, \delta^{(4)}(x-y)\, \partial^\sigma B^c_\sigma(z) + \delta_{ac}\, \delta^{(4)}(x-z)\, \partial^\nu B^b_\nu(y) + \delta_{bc}\, \delta^{(4)}(y-z)\, \partial^\mu B^a_\mu(x) \right]$$

and hence

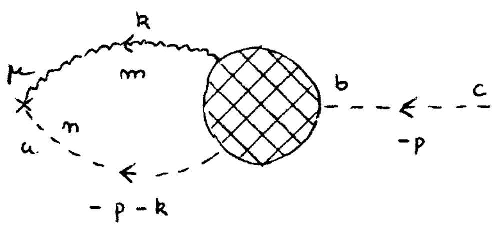
$$\partial_x^\mu \partial_y^\nu \partial_z^\sigma \langle \underset{\sim}{0} | T\left(B^a_\mu(x)\, B^b_\nu(y)\, B^c_\sigma(z) \right) | \underset{\sim}{0} \rangle = 0 \tag{II.30}$$

Fourier transforming one obtains

$$p_\mu\, q_\nu\, k_\sigma\, T^{\mu\nu\sigma}_{abc}(p, q, k) = 0 \tag{II.31}$$

which is clearly satisfied by the lowest order contribution (26). More general formulae of the same type can be found in [TH 71] where the validity of these formal manipulations is discussed.

In order to proceed we need to define some new proper vertex functions



$$\Sigma^{ab}_\mu(p)\, \tilde{D}_{bc}(p) \equiv \tag{II.32}$$

$$\equiv - g \int d^4x\, e^{-ip.x} f_{amn} \langle \underset{\sim}{0} | T\left(B^m_\mu(x)\, \phi_n(x)\, \bar{\Phi}_c(0) \right) | \underset{\sim}{0} \rangle$$

Notice that

$$p^\mu \Sigma^{ab}_\mu(p)\, \tilde{D}_{bc}(p) =$$

$$= ig \int d^4x\, e^{-ip.x} f_{amn}\, \partial_x^\mu \langle \underset{\sim}{0} | T\left(B^m_\mu(x)\, \phi_n(x)\, \bar{\Phi}_c(0) \right) | \underset{\sim}{0} \rangle$$

Using the equal-time commutation relations as well as the equations of motion

$$= -i\int d^4x\, e^{-ip.x}\, \langle 0|\, T\left([\partial_\mu \partial^\mu \phi_a(x)]\, \bar{\Phi}_c(0)\right) |0\rangle =$$

$$= -i\int d^4x\, e^{-ip.x}\, \partial^x_\mu\, \partial_x^\mu\, \langle 0|\, T\left(\phi_a(x)\, \bar{\Phi}_c(0)\right) |0\rangle$$

$$+\, i\int d^4x\, e^{-ip.x}\, \delta(x^0)\, \langle 0|\, \{\partial^0 \phi_a(x)\, ,\, \bar{\Phi}_c(0)\}\, |0\rangle =$$

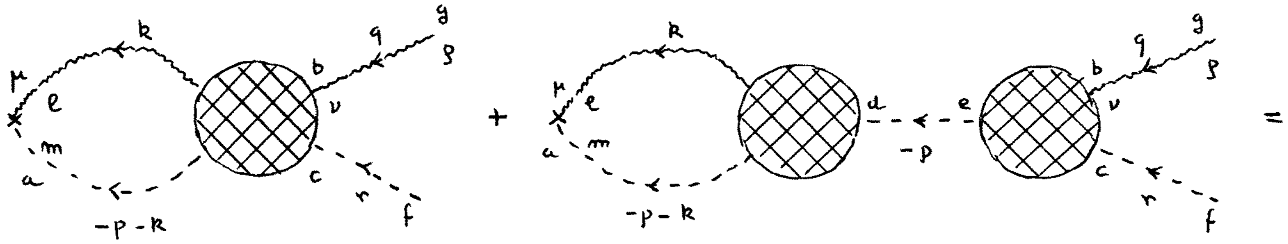
$$= -i\int d^4x\, e^{-ip.x}\, \partial^x_\mu\, \partial_x^\mu\, \langle 0|\, T\left(\phi_a(x)\, \bar{\Phi}_c(0)\right) |0\rangle - \delta_{ac}$$

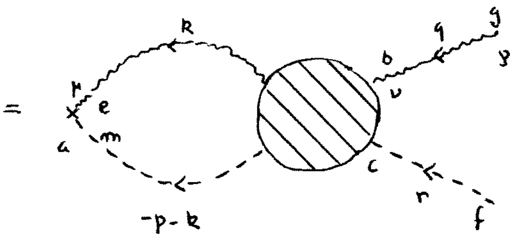
Hence

$$p^\mu\, \Sigma^{ab}_\mu(p)\, \tilde{D}_{bc}(p) = -\delta_{ac} - p^2\, \tilde{D}_{ac}(p) \qquad \text{(II.33)}$$

which is clearly satisfied in lowest order perturbation theory (15).

Furthermore we can define a new proper vertex from the decomposition of a truncated Green's function into its proper parts





$p + q + r = 0$

$$\left[ig\, G^{bca}_{\mu\nu}(q,r,p) + ig\, \Sigma^{ad}_\mu(p)\, \tilde{D}_{de}(p)\, G^{bce}_\nu(q,r,p)\right] i\tilde{D}_{cf}(r)\; iD^{\nu\rho}_{bg}(q) =$$

$$= -g\int d^4x\, d^4y\, e^{-ip.x}\, e^{-ir.y}\, f_{aem}\, \langle 0|\, T\left(B^e_\mu(x)\, \phi_m(x)\, \bar{\Phi}_f(y)\, B^\rho_g(0)\right) |0\rangle \qquad \text{(II.34)}$$

If we multiply by p^μ and use (33)

$$\left\{ ig\, p^{\mu} G^{bca}_{\mu\nu}(q,r,p) + ig\left[-\delta_{ae} - p^2 \tilde{D}_{ae}(p)\right] G^{bce}_{\nu}(q,r,p) \right\} i\tilde{D}_{cf}(r)\; i\, D^{\nu\rho}_{bg}(q) =$$

$$= ig \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} f_{aem} \langle \underset{\sim}{0} | T \left(\left[\partial^{\mu} (B^e_{\mu}(x)\, \phi_m(x))\right] \bar{\Phi}_f(y)\; B^{\rho}_g(0) \right) | \underset{\sim}{0} \rangle =$$

$$= -i \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} \langle \underset{\sim}{0} | T \left(\left[\partial^{\mu} \partial_{\mu} \phi_a(x)\right] \bar{\Phi}_f(y)\; B^{\rho}_g(0) \right) | \underset{\sim}{0} \rangle =$$

$$= i p^2 \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} \langle \underset{\sim}{0} | T \left(\phi_a(x)\; \bar{\Phi}_f(y)\; B^{\rho}_g(0) \right) | \underset{\sim}{0} \rangle$$

where equal-time commutators as well as the equations of motion have been used. Taking into account (24) we obtain immediately

$$p^{\mu} G^{abc}_{\mu\nu}(q,r,p) = G^{abc}_{\nu}(q,r,p) \tag{II.35}$$

Let us now consider the second Slavnov-Taylor identity. We start from the trivial identity

$$\langle \underset{\sim}{0} | T \left(B^a_{\mu}(x)\; \bar{\Phi}_b(y) \left[\partial_{\nu} B^{\nu}{}_c(z) \right] \right) | \underset{\sim}{0} \rangle \tag{II.36}$$

This can be written in terms of the BRS-transformed fields and using (1) we obtain

$$\langle \underset{\sim}{0} | T \left(\left[g f_{aed} B^e_{\mu}(x)\, \phi_d(x) + \partial_{\mu} \phi_a(x) \right] \bar{\Phi}_b(y) \left[\partial_{\nu} B^{\nu}_c(z) \right] \right) | \underset{\sim}{0} \rangle$$

$$+ \frac{1}{a} \langle \underset{\sim}{0} | T \left(B^a_{\mu}(x) \left[\partial_{\nu} B^{\nu}_b(y) \right] \left[\partial_{\rho} B^{\rho}_c(z) \right] \right) | \underset{\sim}{0} \rangle = 0$$

If the equal-time commutation relations are taken into account this can be written as

$$\partial_\nu^z \partial_\mu^x \langle \underset{\sim}{0} | T(\phi_a(x)\ \bar{\Phi}_b(y)\ B_c^\nu(z)) | \underset{\sim}{0} \rangle$$

$$+ \ g\ f_{aed}\ \partial_\nu^z \langle \underset{\sim}{0} | T(B_\mu^e(x)\ \phi_d(x)\ \bar{\Phi}_b(y)\ B_c^\nu(z)) | \underset{\sim}{0} \rangle$$

$$+ \frac{1}{a}\ \partial_\nu^y \partial_\rho^z\ \langle \underset{\sim}{0} | T(B_\mu^a(x)\ B_b^\nu(y)\ B_c^\rho(z)) | \underset{\sim}{0} \rangle = 0$$

Now we can Fourier transform this identity

$$- p^\mu k_\nu \int d^4x\, d^4y\, d^4z\ e^{-ip.x}\ e^{-iq.y}\ e^{-ik.z} \langle \underset{\sim}{0} | T(\phi_a(x)\ \bar{\Phi}_b(y)\ B_c^\nu(z)) | \underset{\sim}{0} \rangle$$

$$+ ik^\nu g\ f_{aed} \int d^4x\, d^4y\, d^4z\ e^{-ip.x}\ e^{-iq.y}\ e^{-ik.z} \langle \underset{\sim}{0} | T(B_e^\mu(x)\ \phi_d(x)\ \bar{\Phi}_b(y)\ B_c^\nu(z)) | \underset{\sim}{0} \rangle$$

$$- \frac{1}{a}\ q_\nu k_\rho \int d^4x\, d^4y\, d^4z\ e^{-ip.x}\ e^{-iq.y}\ e^{-ik.z} \langle \underset{\sim}{0} | T(B_a^\mu(x)\ B_b^\nu(y)\ B_c^\rho(z)) | \underset{\sim}{0} \rangle = 0$$

Using invariance under space-time translations the integral over z can be carried out immediately with a result proportional to $\delta^{(4)}(k+p+q)$ Then for $k + p + q = 0$

$$- p^\mu k^\nu \int d^4x\, d^4y\ e^{-ip.x}\ e^{-iq.y} \langle \underset{\sim}{0} | T(\phi_a(x)\ \bar{\Phi}_b(y)\ B_\nu^c(0)) | \underset{\sim}{0} \rangle$$

$$+ ig\ k^\nu \int d^4x\, d^4y\ e^{-ip.x}\ e^{-iq.y}\ f_{aed} \langle \underset{\sim}{0} | T(B_e^\mu(x)\ \phi_d(x)\ \bar{\Phi}_b(y)\ B_\nu^c(0)) | \underset{\sim}{0} \rangle$$

$$- \frac{1}{a}\ q_\nu k_\rho \int d^4x\, d^4y\ e^{-ip.x}\ e^{-iq.y} \langle \underset{\sim}{0} | T(B_a^\mu(x)\ B_b^\nu(y)\ B_c^\rho(0)) | \underset{\sim}{0} \rangle = 0$$

Taking into account (24), (34) and (23) we get

$$p^\mu k^\nu\ G^{\nu'}_{c'b'a'}(k,q,p)\ D^{c'c}_{\nu'\nu}(k)\ \tilde{D}_{b'b}(q)\ \tilde{D}_{a'a}(p)$$

$$+ k^\nu \left[G^{\mu\nu'}_{c'b'a}(k,q,p) + \Sigma^\mu_{ad}(p)\ \tilde{D}_{de}(p)\ G^{\nu'}_{c'b'e}(k,q,p) \right] \tilde{D}_{b'b}(q)\ D^{c'c}_{\nu'\nu}(k)$$

$$+ \frac{1}{a}\ q_\nu k_\rho\ T^{a'b'c'}_{\mu'\nu'\rho'}(p,q,k)\ D^{\mu'\mu}_{a'a}(p)\ D^{\nu'\nu}_{b'b}(q)\ D^{\rho'\rho}_{c'c}(k) = 0 \qquad \text{(II.37)}$$

Now let us contract this relation with the tensor $(g_{\alpha\mu} - p_\alpha p_\mu / p^2)$. Notice that $\Sigma^\mu(p) \propto p^\mu$ and therefore the term containing Σ^μ as well as the first term give zero when the contraction is carried out. In the last term we can use relation (20) and we get

$$k^\nu \left(g_{\alpha\mu} - \frac{p_\alpha p_\mu}{p^2} \right) G^{\mu\nu'}_{c'b'a}(k,q,p)\, \tilde{D}_{b'b}(q)\, D^{c'c}_{\nu'\nu}(k) =$$

$$= \left(g_{\alpha\mu} - \frac{p_\alpha p_\mu}{p^2} \right) \frac{1}{k^2} q_\nu k^{\rho'}\, T^{a'b'c}_{\mu'\nu'\rho'}(p,q,k)\, D^{\mu'\mu}_{a'a}(p)\, D^{\nu'\nu}_{b'b}(q)$$

and using again (20)

$$\left(g_{\alpha\mu} - \frac{p_\alpha p_\mu}{p^2} \right) G^{\mu\nu}_{cda}(k,q,p)\, \tilde{D}_{db}(q)\, k_\nu =$$

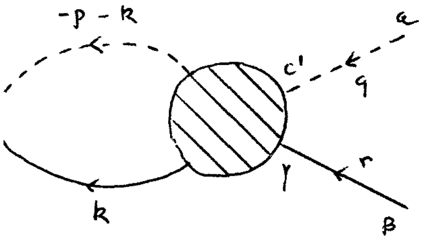
$$= \left(g_{\alpha\mu} - \frac{p_\alpha p_\mu}{p^2} \right) T^{dbc}_{\lambda\nu\rho}(p,q,k)\, D^{\lambda\mu}_{da}(p)\, \frac{k^\rho q^\nu}{q^2} \qquad \text{(II.38)}$$

which is the new Slavnov-Taylor identity. Let us check this relation at lowest order perturbation theory. From (34) we obtain immediately

$$G^{(0)\,abc}_{\mu\nu}(q,r,p) = i f_{abc}\, g_{\mu\nu} \qquad \text{(II.39)}$$

which satisfies (35) at lowest order. Using (14), (15), (26) and (39) the validity of (38) at this order is immediate. In fact an identity stronger than eq. (38) can be obtained starting from the identity (36) without the derivative.

Let us finally present the derivation of the Slavnov-Taylor identity for the quark vertex. We need to introduce



$$H^{c'}_{\alpha\gamma}(p,q,r)\; i\, S_{\gamma\beta}(r)\; i\, \tilde{D}_{c'a}(q) \equiv \qquad \text{(II.40)}$$

$$\equiv -\frac{1}{2}\,[\lambda^c]_{\alpha\gamma}\int d^4x\, d^4y\, e^{-ip\cdot x}\, e^{-ir\cdot y}\, \langle \underset{\sim}{0}|\, T\,(q_\gamma(x)\,\phi_c(x)\,\bar{q}_\beta(y)\,\bar{\phi}_a(0))|$$

where we have omitted the quark flavor index and as usual p + q + r =
At lowest order perturbation theory

$$H^{(0)\,a}_{\alpha\beta}(p,q,r) = \frac{1}{2}\,[\lambda^a]_{\alpha\beta} \qquad \text{(II.41)}$$

Let us now consider the identity

$$\langle \underset{\sim}{0}|\, T\,(q_\alpha(x)\,\bar{q}_\beta(y)\,\bar{\phi}_a(z))\,|\underset{\sim}{0}\rangle = 0 \qquad \text{(II.42)}$$

and proceeding as in the other cases we obtain

$$\frac{i}{2}\, g\,[\lambda^c]_{\alpha\gamma}\, \langle \underset{\sim}{0}|\, T\,(\phi_c(x)\,q_\gamma(x)\,\bar{q}_\beta(y)\,\bar{\phi}_a(z))\,|\underset{\sim}{0}\rangle$$

$$-\frac{i}{2}\, g\,[\lambda^c]_{\gamma\beta}\, \langle \underset{\sim}{0}|\, T\,(q_\alpha(x)\,\bar{q}_\gamma(y)\,\phi_c(y)\,\bar{\phi}_a(z))\,|\underset{\sim}{0}\rangle$$

$$+\frac{1}{a}\, \langle \underset{\sim}{0}|\, T\,(q_\alpha(x)\,\bar{q}_\beta(y)\,\partial_\mu B^\mu_a(z))\,|\underset{\sim}{0}\rangle = 0$$

Now let us Fourier transform this identity and we get

$$\frac{1}{2} g [\lambda^c]_{\alpha\gamma} \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} \langle \underset{\sim}{0} | T(\phi_c(x)\, q_\gamma(x)\, \bar{q}_\beta(y)\, \bar{\Phi}_a(0)) | \underset{\sim}{0} \rangle$$

$$- \frac{1}{2} g [\lambda^c]_{\gamma\beta} \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} \langle \underset{\sim}{0} | T(q_\alpha(x)\, \bar{q}_\gamma(y)\, \phi_c(y)\, \bar{\Phi}_a(0)) | \underset{\sim}{0} \rangle$$

$$+ \frac{1}{a} q_\mu \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} \langle \underset{\sim}{0} | T(q_\alpha(x)\, \bar{q}_\beta(y)\, B^\mu_a(0)) | \underset{\sim}{0} \rangle = 0$$

We can now introduce a vertex conjugate to (40)

$$- \frac{1}{2} [\lambda^c]_{\gamma\beta} \int d^4x\, d^4y\, e^{-ip.x} e^{-ir.y} \langle \underset{\sim}{0} | T(q_\alpha(x)\, \bar{q}_\gamma(y)\, \phi_c(y)\, \bar{\Phi}_a(0)) | \underset{\sim}{0} \rangle \equiv$$

$$\equiv -i\, S_{\alpha\gamma}(-\not{p})\, \bar{H}^c_{\gamma\beta}(-r,-q,-p)\, i\, \tilde{D}_{ca}(-q) \tag{II.43}$$

so that the above identity can be written as

$$H^c_{\alpha\gamma}(p,q,r)\, S_{\gamma\beta}(\not{r})\, \tilde{D}_{ca}(q) - S_{\alpha\gamma}(-\not{p})\, \bar{H}^c_{\gamma\beta}(-r,-q,-p)\, \tilde{D}_{ca}(q) =$$

$$= \frac{1}{a} S_{\alpha\gamma}(-\not{p})\, \Gamma^{a'}_{\mu'\gamma\delta}(q,p,r)\, S_{\delta\beta}(\not{r})\, D^{\mu'\mu}_{a'a}(q)\, q_\mu$$

Using (20) this leads to

$$H^c_{\alpha\gamma}(p,q,r)\, S_{\gamma\beta}(\not{r})\, \tilde{D}_{ca}(q) - S_{\alpha\gamma}(-\not{p})\, \bar{H}^c_{\gamma\beta}(-r,-q,-p)\, \tilde{D}_{ca}(q) =$$

$$= -\frac{1}{q^2} q^\mu\, S_{\alpha\gamma}(-\not{p})\, \Gamma^a_{\mu\gamma\delta}(q,p,r)\, S_{\delta\beta}(\not{r})$$

and hence

$$S^{-1}_{\alpha\gamma}(-\not{p})\, H^a_{\gamma\beta}(p,q,r) - \bar{H}^a_{\alpha\gamma}(-r,-q,-p)\, S^{-1}_{\gamma\beta}(\not{r}) =$$

$$= -\frac{1}{q^2} q^\mu\, \Gamma^b_{\mu\alpha\beta}(q,p,r)\, \tilde{D}^{-1}_{ba}(q) \tag{II.44}$$

At lowest order perturbation theory one finds from (43)

$$\bar{H}^{(0)\,a}_{\alpha\beta}(-r,-q,-p) = \frac{1}{2}\left[\lambda^{a}\right]_{\alpha\beta} \tag{II.45}$$

and one can check (44) at lowest order perturbation theory using (13), (15) and (25). This Slavnov-Taylor identity is a generalization to non-abelian gauge theories of the Ward-Takahashi identity of Q.E.D.

There is also an identity for the quartic gauge vertex which has been derived by 't Hooft [TH 71] but we will not discuss it here. These identities have been also derived in the functional formalism [JL 76].

III. REGULARIZATION AND RENORMALIZATION

The Feynman rules given before allow us to calculate Feynman diagrams at the tree level, i.e. without loops. Nevertheless, as soon as we apply these rules to the calculation of diagrams with loops we find divergent integrals due to the behaviour of the integrands at high virtual momenta and this renders all the perturbation procedure, at least in the present form, meaningless. These divergences associated with high virtual momenta are called ultraviolet divergences. Furthermore, there are other divergences associated with the behaviour of the integrals at low virtual momenta associated with the fact that gluons are massless. These are the so-called infrared divergences.

Q.C.D. is a renormalizable quantum field theory and in practice this means that there exists a well defined set of rules for calculating S-matrix amplitudes which are free of ultraviolet divergences order by order in the interaction coupling constant. Before we can manipulate safely the divergent integrals we must regularize them, i.e. we must give a well defined meaning to the divergent integrals which appear in the direct calculation of diagrams using Feynman rules. The regularization prescription must preserve the gauge invariance of the theory. This is so because otherwise the Slavnov- Taylor identities are no longer valid, but they are needed for proving the renormalizability of the theory.

In Q.E. D. the traditional method of regularization is the Pauli-Villars regularization [PV 49] which is equivalent to substituting the usual Lagrangian density by a regularized one which preserves gauge invariance and contains three new auxiliary spin ½ fields, being two of them quantized according to Bose statistics [IZ 80].

In principle a generalization of this method can be applied to Q.C.D. but it is much more convenient to use dimensional regularization [BG 72] [TV 72] [AS 72] and this will be the method used here. We will introduce now the method of dimensional regularization and more detail can be found in [LE 75].

The crucial point of the method consists in giving a meaning to the divergent integrals by changing the dimension of space-time. Let us consider a typical 4-dimensional integral

$$T \equiv g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{[q^2 - m^2 + i\eta][(q-p)^2 - m^2 + i\eta]}$$

Using the Feynman parametrization method (C.4) we can write

$$T = g^2 \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \frac{1}{\{[q - xp]^2 - m^2 + p^2 x(1-x) + i\eta\}^2}$$

Clearly, this integral is ultraviolet divergent and thus undefined.

If we now change the number of space-time dimensions from 4 to D

$$T = g^2 \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{\{[q - xp]^2 - m^2 + p^2 x(1-x) + i\eta\}^2}$$

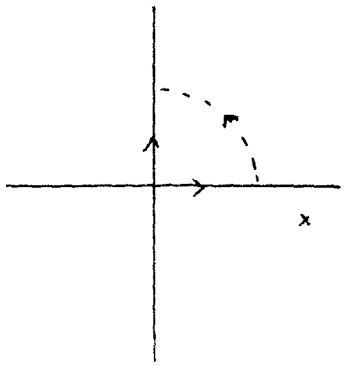
this integral is well defined for $D = 1,2,3$, It is now possible to change $q - xp \longrightarrow q$ and if $a^2 \equiv m^2 - p^2 x(1-x)$ we can write

$$T = g^2 \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2 - a^2 + i\eta]^2} \tag{III.1}$$

Let us now evaluate a generalization of this integral

$$I_m \equiv \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2 - a^2 + i\eta]^m} \tag{III.2}$$

Using the Wick rotation method



$$I_m = \frac{1}{(2\pi)^D} \int dq^0 dq^1 \dots dq^{D-1} \left[q^{0\,2} - \sum_{i=1}^{D-1} q^{i\,2} - a^2 + i\eta \right]^{-m} =$$

$$= i\,(-1)^{-m} (2\pi)^{-D} \int dq^1 \dots dq^{D-1} dq^D \left[q^{D\,2} + \sum_{i=1}^{D-1} q^{i\,2} + a^2 \right]^{-m} =$$

$$= i\,(-1)^{-m} (2\pi)^{-D} \int d^D q \left[q^2 + a^2 \right]^{-m}$$

where the last integral must be performed in a D-dimension Euclidean space. Since the area of the unit sphere in this space is

$$S_D = \frac{2\,\pi^{D/2}}{\Gamma(D/2)} \qquad (\text{III}.3)$$

where $\Gamma(z)$ is the Gamma function , we obtain

$$I_m = \frac{i}{(-1)^m} \frac{1}{2^{D-1}\,\pi^{D/2}} \frac{1}{\Gamma(D/2)} \int_0^\infty dq \frac{q^{D-1}}{(q^2+a^2)^m}$$

where $q = + (\vec{q}^2)^{1/2}$. Hence, with $D = 4 + 2\epsilon$

$$I_m = \frac{i}{(4\pi)^2} (-a^2)^{2-m} \left(\frac{a^2}{4\pi} \right)^{\epsilon} \frac{\Gamma(m-2-\epsilon)}{\Gamma(m)} \qquad (\text{III}.4)$$

A generalization of this formula is given in (C.17). Notice that all these manipulations can be done if $2m > D$. The important point is that the analytic continuation of the Gamma function is unique and this allows us to take (4) as the definition of (2) when the integral is ill-defined. Notice that for n=1 the divergence of the integral if $D \longrightarrow 4$ appears as a pole of the Gamma function.

Using this we can write (1) as

$$T = g^2 \frac{i}{(4\pi)^2} \Gamma(-\epsilon) \frac{1}{(4\pi)^\epsilon} \int_0^1 dx \, [m^2 - p^2 x(1-x)]^\epsilon$$

Since we are interested only in the limit $\epsilon \to 0$ we obtain, using (C.13)

$$T = g^2 \frac{i}{(4\pi)^2} \left\{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma - \int_0^1 dx \ln [m^2 - p^2 x(1-x)] \right\}$$

The last integral can be carried out immediately (C.56) and therefore

$$T = \frac{i}{(4\pi)^2} [g\nu^\epsilon]^2 \left\{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma - \ln \frac{m^2}{\nu^2} + 2 \right.$$
$$\left. - \sqrt{1 - \frac{4m^2}{p^2}} \ln \frac{\sqrt{1 - 4m^2/p^2} + 1}{\sqrt{1 - 4m^2/p^2} - 1} \right\} \tag{III.5}$$

where for convenience we have introduced an arbitrary parameter ν with dimensions of mass.

The great advantage of the dimensional regularization method is that neither new fields nor new couplings have been introduced in the Lagrangian density and hence all symmetry properties, in particular local gauge invariance, are preserved. The only exception is dilatation invariance, because only in D=4 dimensions the coupling constant is dimensionless, and chiral invariance, due to difficulties in defining γ_5 in D dimensions, as we will see immediately. The only change appears in the dimensions of the fields and coupling constant. In a D-dimensional space the Lagrangian density has dimension $[\mathcal{L}(x)] = M^D$, since the action must remain dimensionless. Then from (I.75) we find the following dimensions

$$[q(x)] = [\bar{q}(x)] = M^{\frac{3}{2}+\epsilon} \quad , \quad [B(x)] = M^{1+\epsilon}$$

$$[\phi(x)] = [\bar{\Phi}(x)] = M^{1+\epsilon} \quad , \quad [g] = M^{-\epsilon} \tag{III.6}$$

$$[a] = M^{0} \quad , \quad [m] = M$$

Before going on we would like to make some comments on the Dirac algebra. When the method of dimensional regularization is used we must write the invariant T-matrix element using the Feynman rules given above, but all the algebraic manipulations that must be done before the integrals are evaluated must be carried out in the D-dimensional space. In particular the Dirac algebra is defined by

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \quad , \quad \mu,\nu = 0,1,2,\ldots,D-1 \tag{III.7}$$

Since

$$g^{\mu\nu}\, g_{\mu\nu} = D \tag{III.8}$$

we have

$$\gamma^{\mu}\gamma_{\mu} = D\, I \quad , \quad \gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = -(D-2)\gamma^{\nu}$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma_{\mu} = 4g^{\nu\lambda} I + (D-4)\gamma^{\nu}\gamma^{\lambda} \tag{III.9}$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\rho}\gamma_{\mu} = -2\gamma^{\rho}\gamma^{\lambda}\gamma^{\nu} - (D-4)\gamma^{\nu}\gamma^{\lambda}\gamma^{\rho}$$

We will need also to evaluate traces of products of γ's . It is well known that for a D-dimensional space, with D even, the only irreducible representation of (7) has dimension $f(D) = 2^{D/2}$ and hence $f(D=4) = 4$. Since at the end of our calculation we will put $D = 4$ and $f(D)$ factorizes , its dependence in D is not essential and we can always choose $f(D) = 4$ and this will be the prescription used in all our future calculations [TV 72]. Other choices lead to constant terms which go away after renormalization.

Much more intricate is the problem of the definition of γ_5 in a D-dimensional space. If $D = 4$ the γ_5 is defined as

$$\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\lambda\rho} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \qquad \text{(III.10}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the fully antisymmetric Levi-Cività tensor in the 4-dimensional space with $\epsilon^{0123} = +1$. Clearly γ_5 is an hermitian matrix such that $\gamma_5^2 = I$ and it anticommutes with all γ^μ . These properties allow an equivalent definition of γ_5 (up to a sign) in a $D = 4$ space, i.e. that γ_5 is an hermitian matrix such that $\gamma_5^2 = I$ and which anticommutes with γ^0, γ^1, γ^2 and γ^3 . When $D \neq$ we face a delicate problem: which is the correct generalization of the γ_5 in a D-dimensional space? There are several possible definitions We can generalize (10) to a D-dimensional space writing $\gamma_5 = - i \gamma^0 \gamma^1 \ldots\ldots \gamma^{D-1}$, which commutes with all matrices γ^μ if D is odd and anticommutes if D is even. This is however an unfortunate generalization, precisely because of the dependence on the parity of D of the commuting or anticommuting character, which does not allow interpolations for real values of D . We can define γ_5 as a matrix anticommuting with all γ^μ's . This has the inconvenience that there does not exist a matrix which can be written in terms of the γ^μ's and which anticommutes, for all natural values of D , with all the γ^μ's . Since however one does not need explicit realizations a practical method based on this prescription has been developped [CF 79]. Then in an expression containing several γ_5's on one quark line one moves all of them to one end of the line. If the number of γ_5's is even they will cancel and in fact the method is as good as 't Hooft and Veltman's (see below) and much quicker. If the number of γ_5's is odd one γ_5 will remain at the end and the method leads to ambiguities for closed loops. These can be resolved, at least up to two loops, with the help of the γ_5 - anomaly, but one probably cannot go beyond this [ET 82]. A third definition of γ_5 is that it anticommutes with γ^0, γ^1, γ^2 and γ^3 and commutes with the remaining ones. This is 't Hooft and Veltman's [TV 72] solution, $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, which has been proven not to present problems to any order in perturbation theory [BM 77][BO 80]. This definition leads however to cumbersome computations in which an eye has to be kept on normal Ward identities which are plagued by spurious anomalies [TR 79]. Recently another prescription called dimensional reduction has been introduced [SI 79]. It is based on keeping x^μ and ∂^μ in $D < 4$ dimension but the gauge field $B_a^\mu(x)$ and the γ^μ's in 4 dimensions. There are, however, due to the loss of gauge invariance in 4-D dimensions subtleties which do not help to make the method wieldy [AC 81].

Let us consider now the appearance of divergences in the general case. Let us consider a Lagrangian density

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \sum_i g_i \mathcal{L}_i(x) \qquad \text{(III.11)}$$

where $\mathcal{L}_0(x)$ is the sum of the conventional free Lagrangian densities of the different fields appearing in the theory, while $\mathcal{L}_i(x)$ are the interaction terms where the coupling constants g_i have been written explicitly. Let us assume that $\mathcal{L}_i(x)$ has f_i fermionic fields (clearly f_i must be even), b_i bosonic fields and ∂_i derivatives In a D=4 space-time the dimensions of the fermionic fields, bosonic fields and derivatives are, respectively, $M^{3/2}$, M and M . Since $[\mathcal{L}(x)] = M^4$ the dimensions of the coupling constants are

$$[g_i] = M^{4-d_i} \quad , \quad d_i \equiv \frac{3}{2} f_i + b_i + \partial_i \tag{III.12}$$

In order to proceed we will need two definitions.

i) One particle irreducible diagrams. We will say that a Feyn man diagram is of this type if it is connected and cannot be divided into two disconnected parts cutting only an internal line.

ii) Superficial degree of divergence. To each Feynman diagram corresponds, in general, a multiple integral. The superficial degree of divergence of such diagram is the difference between the number of momenta in the numerator (originated by the derivatives that appear in the vertex functions and by the integration measure variables due to the internal loops) and the number of momenta in the denominator (due to the propagators).

It is rather easy to compute the superficial degree of divergence for a given Feynman diagram. Let us consider a diagram with n_i vertices of the type associated with $\mathcal{L}_i(x)$, $B_E(B_I)$ being the number external (internal) boson lines, $F_E(F_I)$ the number of external (internal) fermion lines and $V = \sum_i n_i$ the total number of vertices. These quantities are related by

$$2B_I + B_E = \sum_i n_i b_i \quad , \quad 2F_I + F_E = \sum_i n_i f_i \tag{III.13}$$

If we consider a theory with particles of spin 0, ½ and 1, the superficial degree of divergence can be written as

$$\vartheta = 4(F_I + B_I) - 4(V-1) - (F_I + 2B_I) + \sum_i n_i \partial_i \tag{III.14}$$

where the first term is due to the integration elements over all internal lines, the second term is related with the Dirac delta functions associated with each vertex and taking into account that one of them gives only the overall energy-momentum conservation. The third term is due to the propagators and the last one to the derivatives appearing in the interaction Lagrangian densities. Using (13) to eliminate B_I and F_I from this expression we can write, taking into account (12)

$$\vartheta = -\sum_i n_i (4 - d_i) - \left(\frac{3}{2} F_E + B_E - 4 \right) \tag{III.15}$$

If $\vartheta = 0,1,2,\ldots$ we will say that the diagram is logarithmically, linearly, quadraticaly divergent. If $\vartheta < 0$ we say that this diagram is superficially convergent. The pejorative term "superficial" is due to the fact that $\vartheta < 0$ does not guarantee the convergence of the integral associated to this diagram. It is possible to prove [WE 60] that a Feynman diagram gives a finite contribution to the invariant T-matrix if this diagram and all its subdiagrams are superficially convergent.

From (15) it is clear that only for theories where the dimensions of all coupling constants are such that $4-d_i \geq 0$ the number of types of superficially divergent diagrams is finite. If one or more of the coupling constants have dimension $(4-d_i) < 0$, then for any values of F_E and B_E we can find a superficially divergent diagram if we take n_i high enough. As we will see later on this implies the need of an infinite number of counterterms to cancel these infinities which renders the theory unpredictive . The theory is not renormalizable.

In Q.C.D. we have just one coupling constant which is dimensionless and therefore the superficial degree of divergence of a Feynman diagram is

$$\mathcal{D} = 4 - \frac{3}{2} F_E - B_E \qquad \text{(III.16)}$$

and hence the only superficially divergent diagrams are (for these counting rules ghosts must be considered bosons)

i) Quark selfenergy : $F_E = 2$, $B_E = 0 \Rightarrow \mathcal{D} = 1$, $\mathcal{D}_{eff} = 0$ $(\not{p}, m)$

ii) Gluon selfenergy : $F_E = 0$, $B_E = 2 \Rightarrow \mathcal{D} = 2$, $\mathcal{D}_{eff} = 0$ $(p^\mu p^\nu - p^2 g^{\mu\nu})$

iii) Ghost selfenergy . $F_E = 0$, $B_E = 2 \Rightarrow \mathcal{D} = 2$, $\mathcal{D}_{eff} = 0$ (p^2)

iv) Ghost vertex : $F_E = 0$, $B_E = 3 \Rightarrow \mathcal{D} = 1$, $\mathcal{D}_{eff} = 0$ (p^μ)

v) Fermionic vertex : $F_E = 2$, $B_E = 1 \Rightarrow \mathcal{D} = 0$, $\mathcal{D}_{eff} = 0$ (III.17)

vi) Triple gluon vertex . $F_E = 0$, $B_E = 3 \Rightarrow \mathcal{D} = 1$, $\mathcal{D}_{eff} = 0$ (p^μ)

vii) Quartic gluon vertex : $F_E = 0$, $B_E = 4 \Rightarrow \mathcal{D} = 0$, $\mathcal{D}_{eff} = 0$

viii) Quartic ghost vertex : $F_E = 0$, $B_E = 4 \Rightarrow \mathcal{D} = 0$

ix) Quartic ghost-gluon vertex : $F_E = 0$, $B_E = 4 \Rightarrow \mathcal{D} = 0$

$\Big\} \; \mathcal{D}_{eff} < 0$

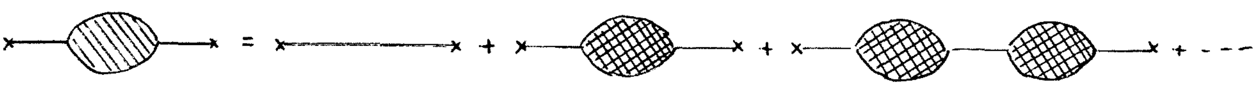
Here $\mathcal{D}_{eff}$ is the degree of divergence which actually remains when the computations are performed. This is due to the fact that certain Green's functions are necessarily proportional to powers of external momenta or masses which decreases correspondingly the degree of divergence, so that $\mathcal{D}_{eff} \leq \mathcal{D}$. We have written in parenthesis in (17) these powers of the momenta or masses. For instance, the quark propagator has to be proportional to $\not p$ or m and $\mathcal{D}_{eff} = \mathcal{D} - 1 = 0$.

Before going on with the renormalization program we would like to calculate the one-loop contribution to these diagrams.

i) <u>Quark selfenergy</u>

The most general form of the quark propagator, defined in (II.10), is

$$i S(\not p) = i S^{(0)}(\not p) + i S^{(0)}(\not p) \left[-i \Sigma(\not p) \right] i S^{(0)}(\not p) + \\ + i S^{(0)}(\not p) \left[-i \Sigma(\not p) \right] i S^{(0)}(\not p) \left[-i \Sigma(\not p) \right] i S^{(0)}(\not p) + \cdots \tag{III.18}$$



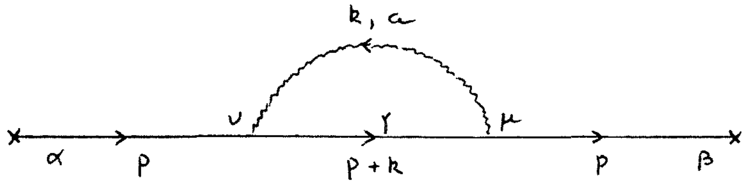
where the bubbles on the r.h.s. denote the contribution of only irreducible diagrams while the bubble on the l.h.s. contains all classes of diagrams. The quantity $[-i \Sigma(\not p)]$ associated with the irreducible diagrams is the so-called quark selfenergy. From (18) we obtain immediately

$$i S(\not p) = i \frac{1}{\not p - m - \Sigma(\not p) + i\eta} \tag{III.19}$$

Frequently we will write $\Sigma(\not p)$ in the following way

$$\Sigma(\not p) = \Sigma_1(p^2) + (\not p - m) \Sigma_2(p^2) \tag{III.20}$$

At the one loop level the only diagram that contributes to th quark selfenergy is



$$-i\,\Sigma^{(2)}_{\beta\alpha}(\not p) = -\,g^2\,\frac{1}{4}\,[\lambda^a\lambda^a]_{\beta\alpha}\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{1}{\not k+\not p-m+i\eta}\,\gamma^\nu$$

$$\cdot\left[g_{\mu\nu}-(1-a)\,\frac{k_\mu k_\nu}{k^2+i\eta}\right]\frac{1}{k^2+i\eta}$$

where we have omitted the flavour index since it never changes in Q.C. interactions. Using (A.19) we can write

$$-i\,\Sigma^{(2)}_{\beta\alpha}(\not p) = -\,\delta_{\beta\alpha}\,g^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[(k+p)^2-m^2+i\eta][k^2+i\eta]}\,\cdot$$

$$\left\{\gamma^\mu(\not k+\not p+m)\gamma_\mu-(1-a)(\not k-\not p+m)-2(1-a)\,\frac{(k.p)\,\not k}{k^2+i\eta}\right\}$$

But $2(k,p) = [(k+p)^2-m^2]-[k^2]-[p^2-m^2]$ and hence

$$-i\,\Sigma^{(2)}_{\beta\alpha}(\not p) = -\,\delta_{\beta\alpha}\,g^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[(k+p)^2-m^2+i\eta][k^2+i\eta]}\,\cdot$$

$$\left\{\gamma^\mu(\not k+\not p+m)\gamma_\mu+(1-a)(\not p-m)+(1-a)(p^2-m^2)\,\frac{\not k}{k^2+i\eta}-(1-a)[(k+p)^2-m^2]\,\frac{\not k}{k^2+i\eta}\right\}$$

The last term is zero when integrated. If we change the integration variable from k^μ to $-k^\mu$ and furthermore introduce $I(\alpha,\beta;p^2,m^2)\equiv I(\alpha,\beta)$ and $I^\mu(\alpha,\beta;p^2,m^2)\equiv p^\mu I'(\alpha,\beta)$ we obtain (see Appendix C)

$$\Sigma^{(2)}_{1\alpha\beta}(p^2) = -i\,\delta_{\alpha\beta}\,(g\nu^{\epsilon})^2\,C_2(R)\,m\,\Big\{\,\big[\,2(1+\epsilon)\,I'(11) + 2\,I(11) -$$

$$- (p^2-m^2)\,I'(12)\,\big] + a\,\big[\,(p^2-m^2)\,I'(12)\,\big]\,\Big\}$$

$$\Sigma^{(2)}_{2\alpha\beta}(p^2) = -i\,\delta_{\alpha\beta}\,(g\nu^{\epsilon})^2\,C_2(R)\,\Big\{\,\big[\,2(1+\epsilon)\,I'(11) - (1+2\epsilon)\,I(11)$$

$$- (p^2-m^2)\,I'(12)\,\big] + a\,\big[-\,I(11) + (p^2-m^2)\,I'(12)\,\big]\,\Big\}$$

and we get immediately for the regularized second order quark self-energy

$$\Sigma^{(2)}_{1\alpha\beta}(p^2) = \delta_{\alpha\beta}\,m\,\frac{(g\nu^{\epsilon})^2}{(4\pi)^2}\,C_2(R)\,\Big\{-3\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\frac{m^2}{\nu^2}\Big] + 4$$

$$+ a\left(1-\frac{m^2}{p^2}\right) - \left(3 - a\,\frac{m^2}{p^2}\right)\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\Big\}$$

(III.21

$$\Sigma^{(2)}_{2\alpha\beta}(p^2) = \delta_{\alpha\beta}\,a\,\frac{(g\nu^{\epsilon})^2}{(4\pi)^2}\,C_2(R)\,\Big\{\,\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\frac{m^2}{\nu^2}$$

$$-1 - \frac{m^2}{p^2} + \left(1-\frac{m^4}{p^4}\right)\ln\left(1-\frac{p^2}{m^2}\right)\Big\}$$

Notice that the selfenergies are analytic functions of p^2 in the Euclidean region $p^2 < 0$ (inthis case it is enough that $p^2 < m^2$) as expected from the analytic properties of Green's functions. The same happens for the gluon and ghost selfenergies.

Another point which is worth stressing is that both Σ_i's have entire dimensions, because $g\,\nu^{\epsilon}$ is dimensionless.

Some particularly useful values of the gauge parameter are

a = 1 Feynman or Fermi gauge

$a = 0$ Landau gauge

$a = 3$ Yennie gauge (III.22)

$a = \infty$ Unitary gauge

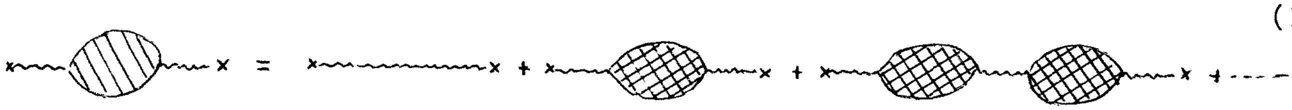
Notice that in the Landau gauge $\Sigma^{(2)}_{\alpha\beta}(p^2) = 0$.

ii) <u>Gluon selfenergy</u>

The most general form of the gluon propagator, defined in (II.11), is

$$i D^{\mu\nu}(k) = i D^{(0)\mu\nu}(k) + i D^{(0)\mu\lambda}(k)\,[i\Pi_{\lambda\rho}(k)]\,i D^{(0)\rho\nu}(k) + \cdots$$

(III.23)



with a notation similar to the one of eq. (18), where $\Pi^{\mu\nu}(k)$ is the gluon self-energy. From this equation

$$i k_\mu D^{\mu\nu}(k) = i k_\mu D^{(0)\mu\nu}(k) + i k_\mu D^{(0)\mu\lambda}(k)\,[i\Pi_{\lambda\rho}(k)]\,i D^{(0)\rho\nu}(k) + \cdots$$

and using (II.20)

$$i k_\mu D^{(0)\mu\lambda}\,[i\Pi_{\lambda\rho}(k)]\,\Big\{ i D^{(0)\rho\nu}(k) + i D^{(0)\rho\tau}(k)\,[i\Pi_{\tau\sigma}(k)]\,i D^{(0)\sigma\nu} + \cdots \Big\} = 0$$

i.e.

$$-a\,\frac{1}{k^2+i\eta}\,k^\lambda\,i\Pi_{\lambda\rho}(k)\,i D^{\rho\nu}(k) = 0 \qquad \text{(III.24)}$$

and hence

$$k_\mu \Pi^{\mu\nu}(k) = k_\nu \Pi^{\mu\nu}(k) = 0 \tag{III.25}$$

Therefore, we can write

$$\Pi^{\mu\nu}(k) = [k^\mu k^\nu - k^2 g^{\mu\nu}] \Pi(k^2) \tag{III.26}$$

Furthermore, eq. (23) can also be written as

$$i D^{\mu\nu}(k) = i D^{(0)\mu\nu}(k) + i D^{(0)\mu\lambda}(k) [i \Pi_{\lambda\rho}(k)] i D^{\rho\nu}(k)$$

and using the explicit form (II.14) as well as (25)

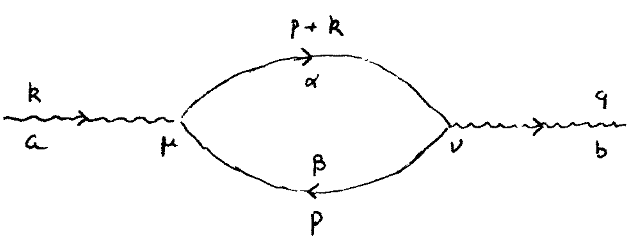
$$[k^2 g^\mu{}_\rho - \Pi^\mu{}_\rho(k)] D^{\rho\nu}(k) = -g^{\mu\nu} + (1-a)\frac{k^\mu k^\nu}{k^2 + i\eta} \tag{III.27}$$

From here it is straightforward to obtain

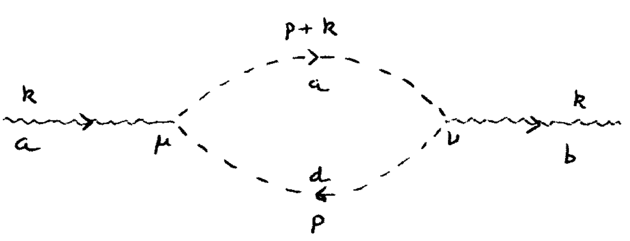
$$D_{\mu\nu}(k) = \left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{1+\Pi(k^2)} - a \frac{k_\mu k_\nu}{k^2} \right] \frac{1}{k^2}$$

$$D^{-1}_{\mu\nu}(k) = -g_{\mu\nu} k^2 + \left(1 - \frac{1}{a}\right) k_\mu k_\nu + (k_\mu k_\nu - k^2 g_{\mu\nu}) \Pi(k^2) \tag{III.28}$$

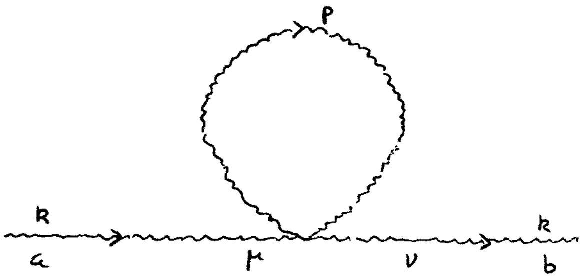
Let us now proceed to calculate the gluon self-energy in second order perturbation theory. The diagrams that contribute are



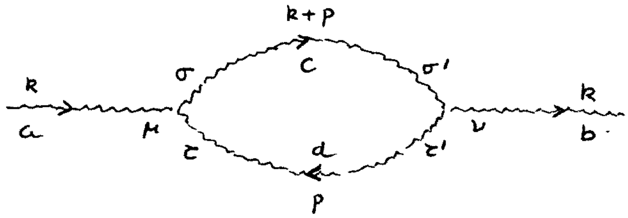
i)



ii)



iii)



iv)

Let us begin with diagram i). Its contribution is

$$i\,\Pi^{(2)\,\mu\nu}_{ab}(k) = -\,g^2\,\frac{1}{4}\,\mathrm{Tr}\,[\lambda_a\lambda_b]\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left\{\gamma^\mu\,\frac{1}{\not p - m + i\eta}\,\gamma^\nu\,\frac{1}{\not p + \not k - m + i\eta}\right\}$$

Since this diagram is the only one involving quark masses it must satisfy the relation (26). If we use (A.12) we can write

$$\Pi^{(2)\,\mu\nu}_{ab}(k) = \delta_{ab}\,[\,k^\mu k^\nu - k^2 g^{\mu\nu}\,]\,\Pi^{(2)}(k^2)$$

$$\Pi^{(2)}(k^2) = g^2\,T(R)\,\frac{(-i)}{(D-1)}\,\frac{1}{k^2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left\{\gamma^\mu\,\frac{1}{\not p - m + i\eta}\,\gamma_\mu\,\frac{1}{\not p + \not k - m + i\eta}\right\}$$

Hence, using, dimensional regularization

$$\Pi^{(2)}(k^2) = -i\,\frac{4g^2\,T(R)}{(D-1)\,k^2}\int\frac{d^Dp}{(2\pi)^D}\,\frac{-(D-2)\,p^2 - (D-2)(k.p) + D\,m^2}{[\,p^2 - m^2 + i\eta\,][\,(p+k)^2 - m^2 + i\eta\,]} =$$

$$= -i\,\frac{2g^2\,T(R)}{(D-1)\,k^2}\int\frac{d^Dp}{(2\pi)^D}\,\frac{(D-2)\,k^2 + 4m^2 - (D-2)\,[\,p^2 - m^2\,] - (D-2)\,[\,(p+k)^2 - m^2\,]}{[\,p^2 - m^2 + i\eta\,][\,(p+k)^2 - m^2 + i\eta\,]} =$$

$$= -i\,\frac{2g^2\,T(R)}{(D-1)\,k^2}\left\{[\,(D-2)\,k^2 + 4m^2\,]\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{[\,p^2 - m^2 + i\eta\,][\,(p+k)^2 - m^2 + i\eta\,]}\right.$$

$$\left. -\,2\,(D-2)\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{[\,p^2 - m^2 + i\eta\,]}\right\}$$

These integrals can be evaluated using (C.12) and (C.17) and we obtain

$$\Pi^{(2)}(k^2) = - \frac{4\,T(R)\,(g\nu^{\epsilon})^2}{3\,(4\pi)^2} \left\{ \frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln \frac{m^2}{\nu^2} - \frac{5}{3} - \frac{4m^2}{k^2} \right.$$

$$\left. + \left(1 + \frac{2m^2}{k^2}\right)\sqrt{1 - \frac{4m^2}{k^2}}\ \ln \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} \right\} \qquad \text{(III.29}$$

for the contribution of one quark flavor.

Let us now consider the contribution of diagram ii). Using Feynman rules

$$i\,\Pi^{(2)\mu\nu}_{ab}(k) = g^2 f_{ade} f_{bcd} \int \frac{d^4p}{(2\pi)^4} \frac{(p+k)^\mu p^\nu}{[p^2 + i\eta][(p+k)^2 + i\eta]}$$

Taking into account (A.16) and changing the integration variable from p^μ to $-p^\mu$ we can write

$$i\,\Pi^{(2)\mu\nu}_{ab}(k) = -\delta_{ab}\, C_2(G)\, g^2 \left\{ \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu p^\nu}{[p^2 + i\eta][(p-k)^2 + i\eta]} \right.$$

$$\left. - k^\mu \int \frac{d^4p}{(2\pi)^4} \frac{p^\nu}{[p^2 + i\eta][(p-k)^2 + i\eta]} \right\}$$

These integrals are given in (C.43) and (C.44) and hence

$$\Pi^{(2)\mu\nu}_{ab}(k) = -\delta_{ab} \frac{C_2(G)\,(g\nu^{\epsilon})^2}{(4\pi)^2} \left\{ \frac{1}{12} k^2 g^{\mu\nu} \left[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(-\frac{q^2}{\nu^2}\right) \right. \right.$$

$$\left. \left. - \frac{8}{3} \right] - \frac{1}{6} k^\mu k^\nu \left[-\frac{1}{\epsilon} + \ln 4\pi - \gamma - \ln\left(-\frac{q^2}{\nu^2}\right) + \frac{5}{3} \right] \right\} \qquad \text{(III.30}$$

which is the desired result.

The tadpole diagram iii) does not contribute as the integral I [0,1;0] is zero (C.82) . Finally the contribution of iv) is

$$i\Pi^{(2)ab}_{\mu\nu}(k) = -\frac{g^2}{2!} f_{adc} f_{bcd} \int \frac{d^4p}{(2\pi)^4} \Big[g_{\mu\sigma}(p+2k)_\tau - g_{\sigma\tau}(2p+k)_\mu$$

$$+ g_{\tau\mu}(p-k)_\sigma \Big] \left[g^{\sigma\sigma'} - (1-a)\frac{(p+k)^\sigma (p+k)^{\sigma'}}{(p+k)^2+i\eta} \right] \frac{1}{(p+k)^2+i\eta} \cdot$$

$$\left[g_{\sigma'\nu}(p+2k)_{\tau'} + g_{\nu\tau'}(p-k)_{\sigma'} - g_{\tau'\sigma'}(2p+k)_\nu \right] \left[g^{\tau\tau'} - (1-a)\frac{p^\tau p^{\tau'}}{p^2+i\eta} \right] \frac{1}{p^2+i\eta}$$

where the statistical factor 1/2! has been included. From (A.16) we obtain $f_{adc}\, f_{bcd} = -C_2(G)\, \delta_{ab}$ and hence

$$i\Pi^{(2)ab}_{\mu\nu}(k) = \delta_{ab}\frac{g^2 C_2(G)}{2} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{[p^2+i\eta][(p+k)^2+i\eta]} \cdot \Big\{ g^{\sigma'}_{\mu}(p+2k)_\tau$$

$$- g^{\sigma'}_{\tau}(2p+k)_\mu + g_{\tau\mu}(p-k)^{\sigma'} - \frac{(1-a)}{(p+k)^2+i\eta}(p+k)^{\sigma'}\Big[(p^2-k^2)g_{\tau\mu} + k_\mu k_\tau$$

$$- p_\mu p_\tau \Big] \Big\} \Big\{ g_{\sigma'\nu}(p+2k)^\tau + g^\tau_\nu (p-k)_{\sigma'} - g^\tau_{\sigma'}(2p+k)_\nu$$

$$- \frac{(1-a)}{p^2+i\eta} p^\tau \Big[(p^2+2k.p)g_{\sigma'\nu} - p_\nu p_{\sigma'} - p_\nu k_{\sigma'} - p_{\sigma'}k_\nu \Big] \Big\} =$$

$$= \delta_{ab}\frac{g^2 C_2(G)}{2} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{[p^2+i\eta][(p+k)^2+i\eta]} \Big\{ \Big[(2p^2+2p.k+5k^2)g_{\mu\nu}$$

$$+ (4D-6)p_\mu p_\nu + (2D-3)p_\mu k_\nu + (2D-3)k_\mu p_\nu + (D-6)k_\mu k_\nu \Big] - \frac{(1-a)}{(p+k)^2+i\eta} \cdot$$

$$\Big[(p^2-k^2)^2 g_{\mu\nu} - (p^2-2k^2)p_\mu p_\nu - (k.p)p_\mu k_\nu - (k.p)k_\mu p_\nu + (2p^2-k^2)k_\mu k_\nu \Big]$$

$$- \frac{(1-a)}{p^2+i\eta}\Big[(p^2+2k.p)^2 g_{\mu\nu} - (p^2+2k.p-k^2)p_\mu p_\nu - (p^2+3k.p)p_\mu k_\nu$$

$$- (p^2+3k.p)k_\mu p_\nu + p^2 k_\mu k_\nu \Big] + \frac{(1-a)^2}{[(p+k)^2+i\eta][k^2+i\eta]} \Big[k^4 p_\mu p_\nu$$

$$- k^2 (k.p)\, p_\mu k_\nu - k^2 (k.p)\, k_\mu p_\nu + (k.p)^2 k_\mu k_\nu] \Big\}$$

Let us now take into account that $2(p.k) = (p+k)^2 - p^2 - k^2$ and if we eliminate all terms giving zero contribution when integrated we obtain

$$i\,\Pi^{(2)ab}_{\mu\nu}(k) = \delta_{ab} \frac{g^2 C_2(G)}{2} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2+i\eta][(p+k)^2+i\eta]} \Big\{ [4k^2 g_{\mu\nu}$$

$$+ (4D-6) p_\mu p_\nu + (2D-3)(p_\mu k_\nu + k_\mu p_\nu) + (D-6) k_\mu k_\nu] - \frac{(1-a)}{(p+k)^2+i\eta} \cdot [k^4 g_{\mu\nu}$$

$$+ 2k^2 p_\mu p_\nu - \frac{1}{2}[(p+k)^2 - k^2](p_\mu k_\nu + k_\mu p_\nu) - k^2 k_\mu k_\nu] - \frac{(1-a)}{p^2+i\eta} [k^4 g_{\mu\nu}$$

$$+ 2k^2 p_\mu p_\nu + \frac{1}{2}[p^2 + 3k^2](p_\mu k_\nu + k_\mu p_\nu) + p^2 k_\mu k_\nu] + \frac{(1-a)^2}{[p^2+i\eta][(p+k)^2+i\eta]} \cdot$$

$$\cdot [k^4 p_\mu p_\nu - \frac{1}{2} k^2 [(p+k)^2 - p^2 - k^2](p_\mu k_\nu + k_\mu p_\nu) + \frac{1}{4}(k^4 - 2p^2(p+k)^2$$

$$- 2k^2 (p+k)^2 + 2p^2 k^2) k_\mu k_\nu] \Big\}$$

Using elementary symmetry properties of these integrals and performing when needed, the change of variables $p^\mu \longrightarrow -(p+k)^\mu$ we get

$$i\,\Pi^{(2)ab}_{\mu\nu}(k) = \delta_{ab} \frac{g^2 C_2(R)}{2} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2+i\eta][(p+k)^2+i\eta]} \cdot$$

$$\cdot \Big\{ [4k^2 g_{\mu\nu} + (4D-6) p_\mu p_\nu + 2(2D-3) k_\mu p_\nu + (D-6) k_\mu k_\nu]$$

$$- \frac{(1-a)}{p^2+i\eta} [2k^4 g_{\mu\nu} + 4k^2 p_\mu p_\nu + 2(p^2+3k^2) k_\mu p_\nu + 2p^2 k_\mu k_\nu]$$

$$+ \frac{(1-a)^2}{[p^2+i\eta][(p+k)^2+i\eta]} [k^4 p_\mu p_\nu - k^2 [(k+p)^2 - p^2 - k^2] k_\mu p_\nu$$

$$+ \frac{1}{4}(k^4 - 2p^2(p+k)^2) k_\mu k_\nu] \Big\}$$

All the needed integrals are given in Appendix C and after a straigh forward calculation the result is

$$\Pi^{(2)\mu\nu}_{ab}(k) = \delta_{ab}\,\frac{(g\nu^{\epsilon})^2 C_2(G)}{2(4\pi)^2}\Big\{ k^2 g^{\mu\nu}\Big(-\frac{19}{6}\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{k^2}{\nu^2}\Big)\Big]$$

$$+\frac{58}{9}\Big) + k^{\mu}k^{\nu}\Big(\frac{11}{3}\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{k^2}{\nu^2}\Big)\Big] - \frac{67}{9}\Big) \tag{III.31}$$

$$+ (k^{\mu}k^{\nu} - k^2 g^{\mu\nu})\Big((1-a)\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{k^2}{\nu^2}\Big) + 2\Big] - \frac{1}{2}(1-a)^2\Big)\Big\}$$

Since this is the only diagram dependent on the gauge parameter the terms containing it must have the tensorial structure $(k^{\mu}k^{\nu} - k^2 g^{\mu\nu})$ and this is so. Adding (29), (30) and (31) we obtain for the gluon self-energy in second order perturbation theory

$$\Pi^{(2)\mu\nu}_{ab}(k) = \delta_{ab}\,(k^{\mu}k^{\nu} - k^2 g^{\mu\nu})\,\Pi^{(2)}(k^2)$$

$$\Pi^{(2)}(k^2) = \frac{(g\nu^{\epsilon})^2}{\pi(4\pi)}\Big\{\frac{C_2(G)}{4}\Big[\Big(\frac{13}{6} - \frac{a}{2}\Big)\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{k^2}{\nu^2}\Big)\Big]$$

$$-\frac{97}{36} - \frac{1}{2}a - \frac{1}{4}a^2\Big] - \frac{T(R)}{3}\sum_{A}\Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\frac{m_A^2}{\nu^2}$$

$$-\frac{5}{3} - \frac{4m_A^2}{k^2} + \Big(1 + \frac{2m_A^2}{k^2}\Big)\sqrt{1 - \frac{4m_A^2}{k^2}}\,\ln\frac{\sqrt{1 - 4m_A^2/k^2} + 1}{\sqrt{1 - 4m_A^2/k^2} - 1}\Big]\Big\} \tag{III.32}$$

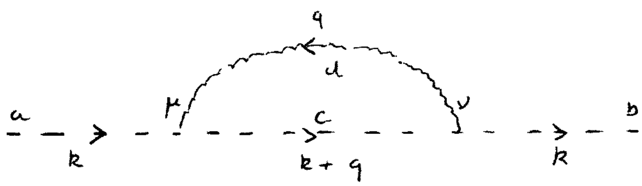
where the existence of several flavors has been taken into account.

iii) <u>Ghost selfenergy</u>

As in the last two cases we can separate in the ghost propagator the proper contributions and we will write

$$i\tilde{D}(k) = i\tilde{D}^{(0)}(k) + i\tilde{D}^{(0)}(k)\,[\,i\tilde{\Pi}(k^2)\,]\,i\tilde{D}^{(0)}(k) + \cdots \tag{III.33}$$

where $\tilde{D}(k)$ is defined in (II.12) and $\tilde{D}^{(0)}(k)$ is given in (II.15). The only diagram contributing to the ghost selfenergy at the one loop level is



and using Feynman rules

$$i\tilde{\Pi}^{(2)}_{ab}(k^2) = -g^2 f_{dac} f_{dcb} \int \frac{d^4q}{(2\pi)^4} \frac{(k+q)^\mu k^\nu}{[q^2+i\eta][(q+k)^2+i\eta]} \left[g_{\mu\nu} - (1-a)\frac{q_\mu q_\nu}{q^2+i\eta} \right] =$$

$$= \delta_{ab}\, g^2 C_2(G) \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2+i\eta][(q+k)^2+i\eta]} \left\{ k^2 + (k.q) - (1-a)\frac{(k.q)^2 + q^2(k.q)}{q^2+i\eta} \right\}$$

Using that $2(k.q) = (q+k)^2 - k^2 - q^2$ and eliminating the terms that give zero contribution when integrated, we obtain

$$i\tilde{\Pi}^{(2)}_{ab}(k^2) = \delta_{ab}\, g^2 C_2(G) \int \frac{d^Dq}{(2\pi)^D} \frac{1}{[q^2+i\eta][(q+k)^2+i\eta]} \left\{ \frac{1}{2}k^2 - \frac{(1-a)}{4}\frac{k^4}{q^2+i\eta} \right\}$$

and using the integrals given in the Appendix C we find

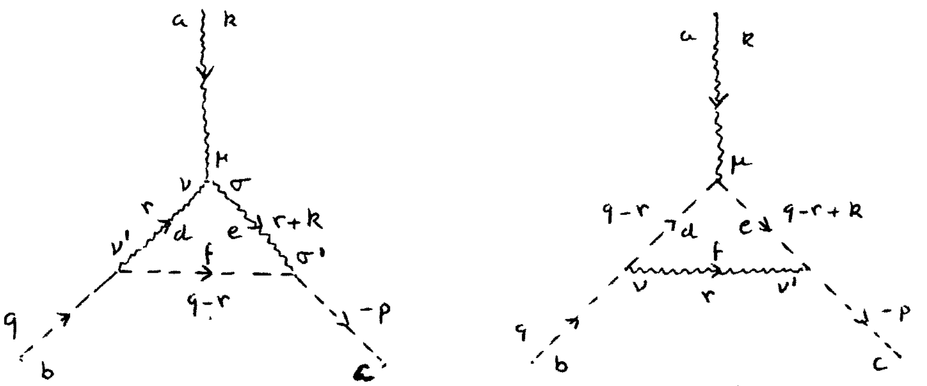
$$\tilde{\Pi}^{(2)}_{ab}(k^2) = -\delta_{ab}\frac{(g\nu^\epsilon)^2}{(4\pi)^2} C_2(G)\, k^2 \left\{ \left(\frac{3-a}{4}\right)\left[\frac{1}{\epsilon} - \ln 4\pi + \right.\right.$$

$$\left.\left. + \gamma + \ln\left(-\frac{k^2}{\nu^2}\right)\right] - 1 \right\} \tag{III.34}$$

which is finite in the Yennie gauge.

iv) Ghost Vertex

The proper ghost vertex has been defined in (II.24) and its lowest order contribution has been given in (II.27). The diagrams that contribute to this vertex at the one loop level are



i) ii)

The contribution of these diagrams is

$$G^{\mu}_{abc}(i) = -g^2 f_{ade} f_{dbf} f_{efc} \int \frac{d^4r}{(2\pi)^4} (q-r)^{\nu'} p^{\sigma'} \frac{1}{(q-r)^2+i\eta} \cdot$$

$$\cdot \left[(k-r)^{\sigma} g^{\mu\nu} + (2r+k)^{\mu} g^{\nu\sigma} - (r+2k)^{\nu} g^{\sigma\mu} \right] \cdot$$

$$\cdot \left[g_{\nu\nu'} - (1-a)\frac{r_\nu r_{\nu'}}{r^2+i\eta} \right] \frac{1}{r^2+i\eta} \left[g_{\sigma\sigma'} - (1-a)\frac{(r+k)_\sigma (r+k)_{\sigma'}}{(r+k)^2+i\eta} \right] \frac{1}{(r+k)^2+i\eta} \qquad \text{(III.35}$$

$$G^{\mu}_{abc}(ii) = -g^2 f_{ade} f_{fbd} f_{fec} \int \frac{d^4r}{(2\pi)^4} (q-r)^{\nu} p^{\nu'} (q-r+k)^{\mu} \cdot$$

$$\left\{ g_{\nu\nu'} - (1-a)\frac{r_\nu r_{\nu'}}{r^2+i\eta} \right\} \frac{1}{r^2+i\eta} \; \frac{1}{(q-r)^2+i\eta} \; \frac{1}{(q-r+k)^2+i\eta} \qquad \text{(III.36}$$

Using the results of Appendix A

$$- f_{ade} f_{dbf} f_{efc} = \frac{1}{2} C_2(G) f_{abc}$$

and

$$G^{\mu}_{abc}(k,q,p) \equiv f_{abc}\, G^{\mu}(k,q,p) \tag{III.37}$$

with

$$G^{\mu}(k,q,p) = G_1(k^2,q^2,p^2)\,p^{\mu} + G_2(k^2,q^2,p^2)\,q^{\mu} \tag{III.38}$$

where only G_1 is divergent. In order to simplify as much as possible we are going to assume $q^{\mu} = 0$ and then

$$G^{\mu}(i) = \frac{1}{2} g^2 C_2(G) \int \frac{d^4k}{(2\pi)^4} (p+k)^{\nu'} p^{\sigma'} \frac{1}{(k+p)^2+i\eta} \left[k^{\sigma} g^{\mu\nu} - 2k^{\mu} g^{\nu\sigma} + k^{\nu} g^{\sigma\mu} \right] .$$

$$\cdot \left[g_{\nu\nu'} - (1-a)\frac{k_{\nu}k_{\nu'}}{k^2+i\eta} \right] \frac{1}{k^2+i\eta} \left[g_{\sigma\sigma'} - (1-a)\frac{k_{\sigma}k_{\sigma'}}{k^2+i\eta} \right] \frac{1}{k^2+i\eta}$$

$$G^{\mu}(ii) = \frac{1}{2} g^2 C_2(G) \int \frac{d^4k}{(2\pi)^4} (p+k)^{\nu} p^{\nu'} (p+k)^{\mu} \left[g_{\nu\nu'} - (1-a)\frac{k_{\nu}k_{\nu'}}{k^2+i\eta} \right] .$$

$$\cdot \frac{1}{k^2+i\eta} \; \frac{1}{(p+k)^2+i\eta} \; \frac{1}{(p+k)^2+i\eta}$$

Hence using dimensional regularization and the fact that $2(p.k) = (k+p)^2 - k^2 - p^2$ we obtain

$$G^{\mu}(i) = \frac{1}{2} g^2 C_2(G) \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2+i\eta]^2[(k+p)^2+i\eta]} \Big\{ \Big[-p^2p^{\mu} + \frac{1}{2}k^2k^{\mu}$$

$$- \frac{3}{2}p^2k^{\mu} \Big] - (1-a)\frac{1}{k^2+i\eta} \Big[-p^2k^2p^{\mu} - \frac{1}{2}p^4k^{\mu} - \frac{1}{2}p^2k^2k^{\mu} \Big] \Big\}$$

$$G^{\mu}(ii) = \frac{1}{2} g^2 C_2(G) \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2+i\eta][(k+p)^2+i\eta]^2} \Big\{ \Big[\frac{1}{2}(p+k)^2p^{\mu} + \frac{1}{2}(p+k)^2k^{\mu}$$

$$+ \frac{1}{2}p^2p^{\mu} + \frac{1}{2}p^2k^{\mu} \Big] - (1-a)\frac{1}{k^2+i\eta} \Big[\frac{1}{4}p^4p^{\mu} + \frac{1}{4}p^4k^{\mu} - \frac{1}{2}(p+k)^2p^{\mu} -$$

$$- \frac{1}{2}(p+k)^2k^{\mu} \Big] \Big\}$$

The needed integrals can be found in the Appendix C and we get

$$G^{\mu}(i) = \frac{i(g\nu^{\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{4} p^{\mu} \left\{ -\frac{3a}{2}\left[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(-\frac{p^2}{\nu^2}\right)\right] + \frac{a}{2} + \frac{3}{2} \right\}$$

$$G^{\mu}(ii) = \frac{i(g\nu^{\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{4} p^{\mu} \left\{ -\frac{a}{2}\left[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(-\frac{p^2}{\nu^2}\right)\right] + \frac{a}{2} + \frac{3}{2} \right\}$$

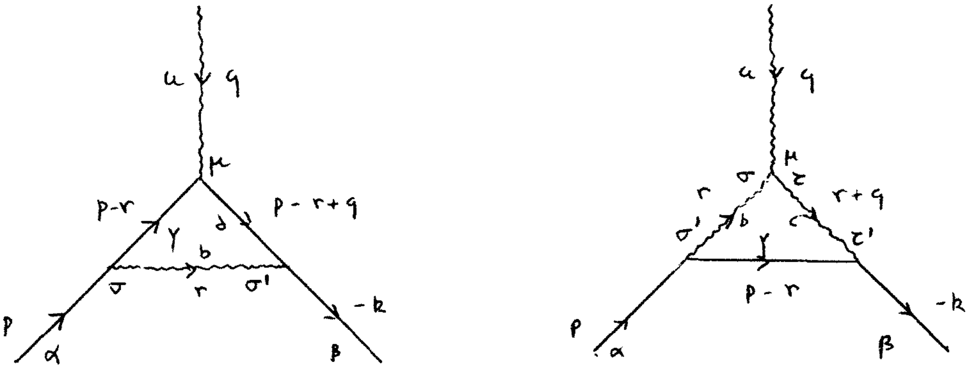
and therefore the one loop contribution to the ghost vertex is

$$G^{\mu}_{abc}(0,-p,p) = i f_{abc}\, p^{\mu} \frac{(g\nu^{\epsilon})^2}{(4\pi)^2} \frac{C_2(G)}{4} \left\{ -2a\left[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(-\frac{p^2}{\nu^2}\right)\right] + a + 3 \right\} \qquad \text{(III.39)}$$

which is the desired result. One may wonder here about whether the limit $q^{\mu} \to 0$ might have introduced infrared divergences [SY 71], which in dimensional regularization again show up as poles in $1/\epsilon$ mixing with the poles of ultraviolet origin [GM 73], [MS 75]. This is not so. Somehow, and we say so because we do not know of any theorem which states this result, the IR divergences show up in the limit $q^{\mu} \to 0$ only in those amplitudes which go with tensorial structures which precisely go to zero in this limit. In our case this means in $G_2(k^2,0,p^2)$, but not in $G_1(k^2,0,p^2)$. But only G_1 is U.V. divergent. One can anyhow check this by comparing with the result of the computation of the vertex in the symmetric euclidean point [PT 80].

v) Fermionic Vertex

The proper fermionic vertex has been defined in (II.22) and its lowest order contribution has been given in (II.25). The diagrams that contribute to this vertex at the one loop level are



The contributions of these diagrams are

$$\Gamma^{a}_{\mu\beta\alpha}(i) = -ig^2\left(\frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right)_{\beta\alpha}\int\frac{d^4r}{(2\pi)^4}\,\gamma^{\sigma'}\frac{1}{\not p+\not q-\not r-m+i\eta}\,\gamma_\mu\,\frac{1}{\not p-\not r-m+i\eta}\,\gamma^{\sigma}$$

$$\left[g_{\sigma\sigma'}-(1-a)\frac{r_\sigma r_{\sigma'}}{r^2+i\eta}\right]\frac{1}{r^2+i\eta} \quad \text{(III.40)}$$

$$\Gamma^{a}_{\mu\beta\alpha}(ii) = g^2 f_{abc}\left(\frac{\lambda_c}{2}\frac{\lambda_b}{2}\right)_{\beta\alpha}\int\frac{d^4r}{(2\pi)^4}\,\gamma^{\tau'}\frac{1}{\not p-\not r-m+i\eta}\,\gamma^{\sigma'}\left[g_{\sigma\sigma'}-(1-a)\frac{r_\sigma r_{\sigma'}}{r^2+i\eta}\right]$$

$$\frac{1}{r^2+i\eta}\left[g_{\tau\tau'}-(1-a)\frac{(r+q)_\tau(r+q)_{\tau'}}{(r+q)^2+i\eta}\right]\frac{1}{(r+q)^2+i\eta}\left[g^{\sigma}{}_{\mu}(q-r)^{\tau}+g^{\sigma\tau}(2r+q)_\mu-g^{\tau}{}_{\mu}(r+2q)^{\sigma}\right] \quad \text{(III.41)}$$

Using (A.21) and (A.22) we obtain

$$\left(\frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right)_{\beta\alpha} = -\frac{1}{2N}\left(\frac{\lambda_a}{2}\right)_{\beta\alpha}, \qquad f_{abc}\left(\frac{\lambda_c}{2}\frac{\lambda_b}{2}\right)_{\beta\alpha} = -i\frac{N}{2}\left(\frac{\lambda_a}{2}\right)_{\beta\alpha}$$

and therefore both terms have the same colour structure as the lowest order term. The results in Appendix A allow us to see that this result is completely general. Then we can write

$$\Gamma^{a}_{\mu\beta\alpha}(p,q,k) = \left(\frac{\lambda_a}{2}\right)_{\beta\alpha}\Gamma_\mu(p,q,k) \quad \text{(III.42)}$$

The Lorentz structure of $\Gamma_\mu(p,q,k)$ is rather complex and allows as many as twelve form factors:

$$\Gamma_\mu(p,q,k) = \gamma_\mu \Gamma_1 + p_\mu \Gamma_2 + k_\mu \Gamma_3 + \gamma_\mu \not{p}\, \Gamma_4 + \gamma_\mu \not{k}\, \Gamma_5 + p_\mu \not{p}\, \Gamma_6$$

$$+ p_\mu \not{k}\, \Gamma_7 + k_\mu \not{p}\, \Gamma_8 + k_\mu \not{k}\, \Gamma_9 + \gamma_\mu \not{p}\not{k}\, \Gamma_{10} + p_\mu \not{p}\not{k}\, \Gamma_{11} + k_\mu \not{k}\not{p}\, \Gamma_{12} \tag{III.43}$$

where $\Gamma_i \equiv \Gamma_i(p^2,q^2,k^2)$. Of course in a renormalizable field theory at the one loop level the only ultraviolet divergent form factor is Γ_1 . We will be interested here only in this ultraviolet divergent part and this allows us to put $m = 0$ and $q^\mu = 0$; $m = 0$ because UV divergences are mass independent and for $q^\mu = 0$ the same comment a previously applies. Then (40) and (41) can be written as

$$\Gamma_\mu(i) = i g^2 \frac{1}{2N} \int \frac{d^4k}{(2\pi)^4}\, \gamma^{\sigma'} \frac{1}{\not{p}-\not{k}+i\eta}\, \gamma_\mu\, \frac{1}{\not{p}-\not{k}+i\eta}\, \gamma^\sigma \left[g_{\sigma\sigma'} - (1-a)\frac{k_\sigma k_{\sigma'}}{k^2+i\eta} \right] \frac{1}{k^2+i\eta}$$

$$\Gamma_\mu(ii) = - i g^2 \frac{N}{2} \int \frac{d^4k}{(2\pi)^4}\, \gamma^{\tau'} \frac{1}{\not{p}-\not{k}+i\eta}\, \gamma^{\sigma'} \left[g_{\sigma\sigma'} - (1-a)\frac{k_\sigma k_{\sigma'}}{k^2+i\eta} \right] \frac{1}{k^2+i\eta}$$

$$\cdot \left[g_{\tau\tau'} - (1-a)\frac{k_\tau k_{\tau'}}{k^2+i\eta} \right] \frac{1}{k^2+i\eta} \left[- g^\sigma_\mu k^\tau + 2 g^{\sigma\tau} k_\mu - g^\tau_\mu k^\sigma \right]$$

i.e.

$$\Gamma_\mu(i) = i g^2 \frac{1}{2N} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2+i\eta][(k-p)^2+i\eta]^2} \Big\{ \left[-2(D-2)(\not{p}-\not{k})(p-k)_\mu + \right.$$

$$\left. + (D-2)(k-p)^2 \gamma_\mu \right] - (1-a)\frac{1}{k^2+i\eta} \left[k^2(k-p)^2 \gamma_\mu - 2p^2 k_\mu \not{k} \right.$$

$$\left. + 2k^2 k_\mu \not{p} + 4(p.k) p_\mu \not{k} - 2k^2 p_\mu \not{k} - 2k^2 p_\mu \not{p} \right] \Big\}$$

$$\Gamma_\mu(ii) = - i g^2 \frac{N}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2+i\eta]^2[(k-p)^2+i\eta]} \Big\{ \left[-2p_\mu \not{k} + 2k_\mu \not{p} - 2(p.k)\gamma_\mu \right.$$

$$\left. + 2k^2 \gamma_\mu - 2(D-2)(\not{p}-\not{k})k_\mu \right] - (1-a)\frac{1}{k^2+i\eta} \left[2k^4 \gamma_\mu - 2k^2 k_\mu \not{k} \right.$$

$$\left. - 2k^2 p_\mu \not{k} + 4(p.k) k_\mu \not{k} - 2(p.k) k^2 \gamma_\mu \right] \Big\}$$

Using $2(p.k) = p^2 + k^2 - (k-p)^2$ and eliminating all terms that clearly do not contribute to $\Gamma_1(p^2)$ we get

$$\Gamma_\mu(i) = i g^2 \frac{1}{2N} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2+i\eta][(k-p)^2+i\eta]^2} \Big\{ [-2(D-2) \not k k_\mu + (D-2)(k-p)^2 \gamma_\mu]$$

$$- (1-a) \frac{1}{k^2+i\eta} [k^2 (k-p)^2 \gamma_\mu - 2p^2 k_\mu \not k] \Big\}$$

$$\Gamma_\mu(ii) = - i g^2 \frac{N}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2+i\eta]^2[(k-p)^2+i\eta]} \Big\{ [-p^2 \gamma_\mu + k^2 \gamma_\mu + 2(D-2) \not k k_\mu]$$

$$- (1-a) \frac{1}{k^2+i\eta} [k^4 \gamma_\mu - p^2 k^2 \gamma_\mu + 2p^2 k_\mu \not k] \Big\}$$

All needed integrals can be found in the Appendix C and therefore

$$\Gamma_\mu(i) = - \gamma_\mu \frac{(g\nu^\epsilon)^2}{(4\pi)^2} \frac{1}{2N} \Big\{ - a \Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{p^2}{\nu^2}\Big)\Big] + a \Big\}$$

$$\Gamma_\mu(ii) = \gamma_\mu \frac{(g\nu^\epsilon)^2}{(4\pi)^2} \frac{N}{2} \frac{3}{2} (1+a) \Big\{ - \Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{p^2}{\nu^2}\Big)\Big] + 1 \Big\}$$

and hence at the one loop level the regularized fermionic vertex is

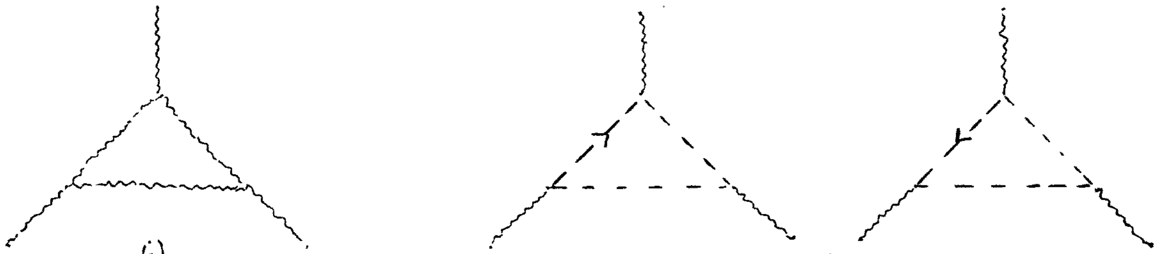
$$\Gamma^{(2)a}_{\mu\beta\alpha}(p,0,-p) = \Big(\frac{\lambda^a}{2}\Big)_{\beta\alpha} \gamma_\mu \frac{(g\nu^\epsilon)^2}{(4\pi)^2} \cdot \Big[- a\, C_2(R) - \frac{C_2(G)}{4}(3+a)\Big] \cdot$$

$$\cdot \Big\{ \Big[\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\Big(-\frac{p^2}{\nu^2}\Big)\Big] - 1 \Big\} + \cdots \qquad \text{(III.44)}$$

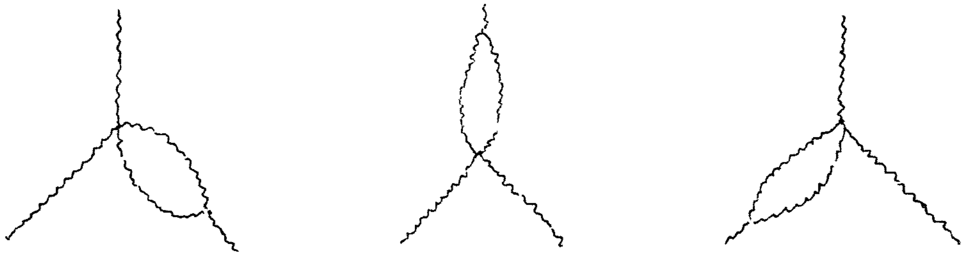
where the dots denote ultraviolet convergent parts that have not been calculated.

vi) <u>Triple gluon vertex</u>

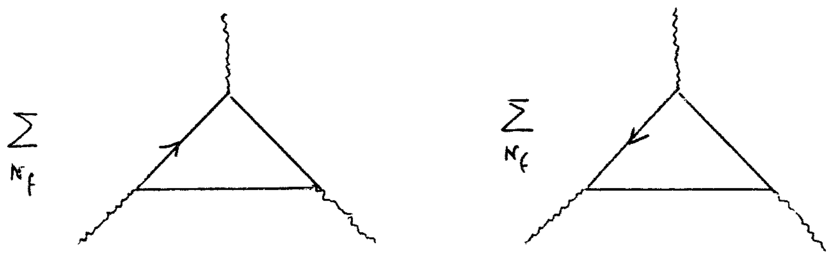
This vertex has been defined in (II.23) and its lowest order contribution has been given in (II.26). The diagrams that contribute to it at the one loop level are



i) ii)



iii)



iv)

The calculations of the ultraviolet divergent part of these diagrams i quite lengthy and we will not give the details of the calculation. If we write

$$T^{(2)\,\mu\nu\sigma}_{abc}(p,q,k) = -i f_{abc}\left[g^{\mu\nu}(p-k)^{\sigma} + g^{\nu\sigma}(p-k)^{\mu} + g^{\sigma\mu}(k-p)^{\nu}\right] T_1^{(2)}(p^2,q^2,k^2) + \text{------} \qquad \text{(III.45}$$

where the dots correspond to further tensorial structures, all the ultraviolet divergent parts appear in $T_1^{(2)}$ and for each set of diagrams of the last figure the contributions proportional to $1/\epsilon$ are

$$T_1^{(2)}(i) = \frac{(g\nu^\epsilon)^2}{(4\pi)^2}\frac{C_2(G)}{8}\left\{-\frac{4}{\epsilon} - \frac{9a}{\epsilon}\right\}$$

$$T_1^{(2)}(ii) = \frac{(g\nu^\epsilon)^2}{(4\pi)^2}\frac{C_2(G)}{8}\left\{\frac{1}{3\epsilon}\right\}$$

$$T_1^{(2)}(iii) = \frac{(g\nu^\epsilon)^2}{(4\pi)^2}\frac{C_2(G)}{8}\left\{\frac{15}{\epsilon} + \frac{3a}{\epsilon}\right\}$$

$$T_1^{(2)}(iv) = \frac{(g\nu^\epsilon)^2}{(4\pi)^2} N_f \left\{-\frac{2}{3\epsilon}\right\}$$

and therefore the total contribution at the one loop level is

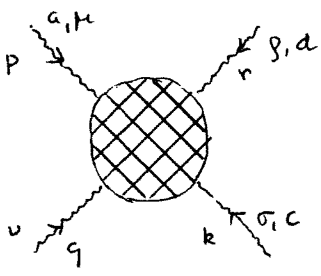
$$T_1^{(2)} = \frac{(g\nu^\epsilon)^2}{(4\pi)^2}\left\{ C_2(G)\left[\frac{17}{12}\frac{1}{\epsilon} - \frac{3a}{4}\frac{1}{\epsilon}\right] - \right. \tag{III.46}$$

$$\left. - \frac{4N_f}{3} T(R)\frac{1}{\epsilon}\right\} + \text{finite parts}$$

which is the desired result.

vii) Quartic gluon vertex

The proper quartic gluon vertex is defined by



$$i g^2\, T^{\mu'\nu'\sigma'\rho'}_{a'b'c'd'}(p,q,k,r)\; iD^{a'a}_{\mu'\mu}(p)\; iD^{b'b}_{\nu'\nu}(q)$$

$$iD^{c'c}_{\sigma'\sigma}(k)\; iD^{d'd}_{\rho'\rho}(r) = \int d^4x\, d^4y\, d^4z\; e^{-ip.x}\, e^{-iq.y}$$

$p+q+k+r=0$

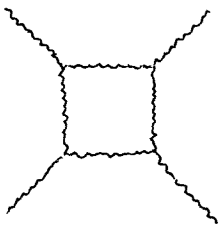
$$. \; e^{-ik.z} \langle \underset{\sim}{0}| T\left(B^a_\mu(x)\, B^b_\nu(y)\, B^c_\sigma(z)\, B^d_\rho(0)\right)|\underset{\sim}{0}\rangle \tag{III.47}$$

It is easy to check that the lowest order contribution is just

$$T^{(0)\mu\nu\sigma\rho}_{abcd} = - \Big[f_{abe}\, f_{cde} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma} \right) +$$

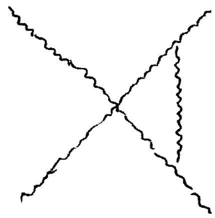
$$+ f_{ace}\, f_{bde} \left(g^{\mu\nu} g^{\sigma\rho} - g^{\mu\rho} g^{\nu\sigma} \right) + f_{ade}\, f_{cbe} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\sigma\rho} \right) \Big] \qquad \text{(III.48)}$$

The diagrams at the one loop level are



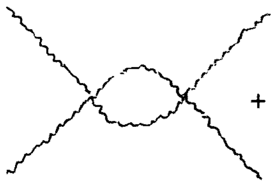
+ 2 permutations

i)



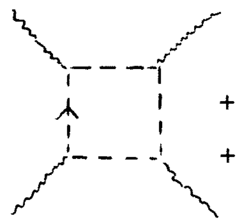
+ 5 permutations

ii)



+ 2 permutations

iii)

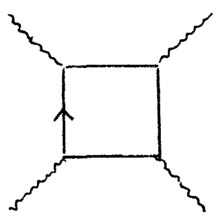


+ 2 permutations

+ three other diagrams with opposite sense of the ghost line

iv)

$\sum_{N_f}$



+ 2 permutations

+ three other diagrams with opposite sense of the fermion line.

v)

If we write

$$T^{(2)\mu\nu\sigma\rho}_{abcd} = T^{(0)\mu\nu\sigma\rho}_{abcd}\; T_1^{(2)}\left(p^2, q^2, k^2, r^2, p.q, p.k\right) + \text{----} \qquad \text{(III.49)}$$

where the dots denote other tensorial (color and space-time) structure All the ultraviolet divergent terms appear in $T_1^{(2)}$ and they turn out to be [PT 80]

$$T_1^{(2)} = - \frac{(g\nu^{\epsilon})^2}{(4\pi)^2} \left\{ C_2(G)\left(-\frac{2}{3} + a\right)\frac{1}{\epsilon} + \frac{4N_f}{3} T(R) \frac{1}{\epsilon} \right\} \tag{III.50}$$

Up to now we have seen how to regularize dimensionally the ultraviolet divergences appearing in the theory. In particular we have obtained the expression for the regularized propagators at the one loop level as well as the divergent parts for all divergent vertices appearing in Q.C.D. at the one loop level. Now we must proceed to renormalize the theory, i.e. to give a well defined prescription to eliminate all divergent parts in such a way that we obtain finite results for the Green's functions, in the limit $\epsilon \to 0$, to any order in perturbation theory.

Formally the elimination of the divergent parts can be achieve adding counterterms to the initial Q.C.D. Lagrangian density corresponding to each superficially divergent diagram appearing in the theory. This means to substitute $\mathcal{L}(x)$ given in (I.75) by $\mathcal{L}(x) + \mathcal{L}_c(x)$ where

$$\begin{aligned}
\mathcal{L}_c(x) = {} & C_{3YM} \frac{1}{2} [\partial_\mu B_\nu^a(x)][\partial^\mu B_a^\nu(x) - \partial^\nu B_a^\mu(x)] + C_6 \frac{1}{2a} [\partial_\mu B_a^\mu(x)][\partial_\nu B_a^\nu(x)] \\
& - \frac{i}{2} C_{2F}\, \bar{q}_\alpha^A(x) \gamma^\mu \partial_\mu q_\alpha^A(x) + \frac{i}{2} C_{2F} [\partial_\mu \bar{q}_\alpha^A(x)] \gamma^\mu q_\alpha^A(x) + C_4 m_A\, \bar{q}_\alpha^A(x) q_\alpha^A(x) \\
& - C_{1F} \frac{1}{2} g\, \bar{q}_\alpha^A(x) \lambda_{\alpha\beta}^a \gamma^\mu q_\beta^A(x) B_\mu^a(x) \\
& + C_{1YM} \frac{1}{2} g f_{abc} [\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)] B_b^\mu(x) B_c^\nu(x) \\
& + C_5 \frac{1}{4} g^2 f_{abc} f_{ade} B_\mu^b(x) B_\nu^c(x) B_d^\mu(x) B_e^\nu(x) \\
& + \tilde{C}_3 [\partial_\mu \bar{\Phi}_a(x)] \partial^\mu \phi_a(x) - \tilde{C}_1 g f_{abc} [\partial_\mu \bar{\Phi}_a(x)] \phi_b(x) B_c^\mu(x)
\end{aligned} \tag{III.51}$$

The proof of renormalizability consists in showing that the C_i, understood as a power series in g^2, are all the extra terms needed to remove the ultraviolet divergences of the theory at an arbitrary order of the perturbation expansion. The renormalization of gauge theories has been discussed by 't Hooft and Veltman [TV 72] [TV 72a] and Lee and Zinn-Justin [LE 72] [LZ 72] [LZ 73] [ZI 75] .

Notice that since all divergent terms calculated above are proportional to g^2 it is clear that all C_i start with a g^2 term and therefore all terms in (51) can be thought as new perturbative terms in the Lagrangian density with Feynman rules immediately derivable. For instance if we try to compute the quark propagator up to orde g^2 we have two new diagrams corresponding to the terms C_{2F} and C_4 and therefore we must add to $\Sigma^{(2)}_{\alpha\beta}(\not p)$ given in (21) the new terms

$$\Delta \Sigma^{(2)}_{\alpha\beta}(\not p) = \delta_{\alpha\beta} \left[C_{2F} \not p - C_4 m \right] \tag{III.52}$$

and now we must choose C_{2F} and C_4 in such a way that the new self-energy is finite, which can be done in many different ways. We will return to this point later on.

We will introduce now the renormalization constants Z_i defined as

$$Z_i \equiv 1 - C_i \tag{III.53}$$

and therefore

$$\mathcal{L}(x) + \mathcal{L}_C(x) = -\frac{Z_{3YM}}{2}\left[\partial_\mu B^a_\nu(x)\right]\left[\partial^\mu B^\nu_a(x) - \partial^\nu B^\mu_a(x)\right] - \frac{Z_6}{2a}\left[\partial_\mu B^\mu_a(x)\right]\left[\partial_\nu B^\nu_a(x)\right]$$

$$+ \frac{i Z_{2F}}{2} \bar{q}^A_\alpha(x) \gamma^\mu \partial_\mu q^A_\alpha(x) - \frac{i Z_{2F}}{2} [\partial_\mu \bar{q}^A_\alpha(x)] \gamma^\mu q^A_\alpha(x) - Z_4 m_A \bar{q}^A_\alpha(x) q^A_\alpha(x)$$

$$+ Z_{1F} \frac{1}{2} g \bar{q}^A_\alpha(x) \lambda^a_{\alpha\beta} \gamma^\mu q^A_\beta(x) B^a_\mu(x)$$

$$- Z_{1YM} \frac{1}{2} g f_{abc} [\partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x)] B^\mu_b(x) B^\nu_c(x)$$

$$- Z_5 \frac{1}{4} g^2 f_{abc} f_{ade} B^b_\mu(x) B^c_\nu(x) B^\mu_d(x) B^\nu_e(x)$$

$$- \tilde{Z}_3 [\partial_\mu \bar{\phi}_a(x)] \partial^\mu \phi_a(x) + \tilde{Z}_1 g f_{abc} [\partial_\mu \bar{\phi}_a(x)] \phi_b(x) B^\mu_c(x) \qquad \text{(III.54}$$

Notice that we are only considering flavor independent renormalization schemes where Z_{1F}, Z_{2F} and Z_4 are not dependent on the flavor index.

Let us now introduce the bare fields, bare coupling constants as well as the bare mass and gauge parameter according to the definitions

$$B^\mu_{a0}(x) \equiv Z^{1/2}_{3YM} B^\mu_a(x) \quad , \quad q^A_{\alpha 0}(x) \equiv Z^{1/2}_{2F} q^A_\alpha(x)$$

$$\phi_{a0}(x) \equiv \tilde{Z}_3^{1/2} \phi_a(x) \quad , \quad \bar{\phi}_{a0}(x) \equiv \tilde{Z}_3^{1/2} \bar{\phi}_a(x) \qquad \text{(III.55}$$

$$g_{0YM} \equiv Z_{1YM} Z^{-3/2}_{3YM} g \quad , \quad \tilde{g}_0 \equiv \tilde{Z}_1 \tilde{Z}_3^{-1} Z^{-1/2}_{3YM} g$$

$$g_{0F} \equiv Z_{1F} Z^{-1/2}_{3YM} Z^{-1}_{2F} g \quad , \quad g_{0S} \equiv Z_5^{1/2} Z^{-1}_{3YM} g$$

$$m_{A0} \equiv Z_4 Z^{-1}_{2F} m_A \quad , \quad a_0 \equiv Z_6^{-1} Z_{3YM} a$$

Recall that quantities in both sides are defined in a D-dimensional space with dimensions given in (6). The fields and parameters at the right hand side will be called renormalized from now on. These parameters are finite and the fields give rise to finite Green's functions even in D=4 dimensions.

Then the renormalized Lagrangian density

$$\mathcal{L}_R(x) \equiv \mathcal{L}(x) + \mathcal{L}_c(x) \tag{III.56}$$

can be written in terms of the bare quantities as

$$\mathcal{L}_R(x) = -\frac{1}{2}[\partial_\mu B^a_{\nu o}(x)][\partial^\mu B^\nu_{ao}(x) - \partial^\nu B^\mu_{ao}(x)] - \frac{1}{2a_o}[\partial_\mu B^\mu_{ao}(x)][\partial_\nu B^\nu_{ao}(x)]$$

$$+\frac{i}{2}\bar{q}^A_{\alpha o}(x)\gamma^\mu\partial_\mu q^A_{\alpha o}(x) - \frac{i}{2}[\partial_\mu \bar{q}^A_{\alpha o}(x)]\gamma^\mu q^A_{\alpha o}(x) - m_{Ao}\bar{q}^A_{\alpha o}(x) q^A_{\alpha o}(x)$$

$$+\frac{1}{2} g_{oF}\, \bar{q}^A_{\alpha o}(x)\lambda^a_{\alpha\beta}\gamma^\mu q^A_{\beta o}(x) B^\mu_{ao}(x)$$

$$-\frac{1}{2} g_{oYM} f_{abc}[\partial_\mu B^a_{\nu o}(x) - \partial_\nu B^a_{\mu o}(x)] B^\mu_{bo}(x) B^\nu_{co}(x)$$

$$-\frac{1}{4} g^2_{os} f_{abc} f_{ade} B^b_{\mu o}(x) B^c_{\nu o}(x) B^\mu_{do}(x) B^\nu_{eo}(x)$$

$$-[\partial_\mu \bar{\Phi}_{ao}(x)]\partial^\mu \Phi_{ao}(x) + \tilde{g}_o f_{abc}[\partial_\mu \bar{\Phi}_{ao}(x)]\Phi_{bo}(x) B^\mu_{co}(x) \tag{III.57}$$

This Lagrangian density is not exactly of the same type as the initial one (I.75) since it is not invariant under the BRS transformation unles

$$g_{oYM} = \tilde{g}_o = g_{oF} = g_{os} \equiv g_o \tag{III.58}$$

This means that if the renormalization procedure must preserve gauge invariance the renormalization constants cannot be completely arbitrary but must satisfy the following relations

$$g_{0YM} = \tilde{g}_0 \qquad \Longrightarrow \qquad \frac{Z_{3YM}}{Z_{1YM}} = \frac{\tilde{Z}_3}{\tilde{Z}_1}$$

$$g_{0YM} = g_{0F} \qquad \Longrightarrow \qquad \frac{Z_{3YM}}{Z_{1YM}} = \frac{Z_{2F}}{Z_{1F}} \qquad \text{(III.59)}$$

$$g_{0YM} = g_{0S} \qquad \Longrightarrow \qquad Z_S = Z_{1YM} \frac{Z_{1YM}}{Z_{3YM}}$$

which are also called Slavnov-Taylor identities, since we will show that they are direct consequence of the Slavnov-Taylor identities considered in Chapter II.

Before going on we would like to discuss different renormalization schemes, i.e. different methods of choosing the constants C_i in such a way that at the end of our calculations we have finite Green' functions. Let us consider, for instance, the quark selfenergy at the one loop level with the counterterms added. If we forget about the color and flavor indices, since it is diagonal in both quantities, we can write

$$\Sigma_R^{(2)}(\not p) = \Sigma_1^{(2)}(p^2) + (\not p - m)\, \Sigma_2^{(2)}(p^2) + C_{2F}\, \not p - C_4\, m =$$

$$= \left[\Sigma_1^{(2)}(p^2) - (C_4 - C_{2F})\, m \right] + (\not p - m) \left[\Sigma_2^{(2)}(p^2) + C_{2F} \right] \qquad \text{(III.60)}$$

where $\Sigma_i^{(2)}(p^2)$ are given in (21).

The minimal substraction renormalization scheme or MS-scheme [TH 73] [CM 74] [CO 75] [BM 77] is defined in the following way:

Defining the dimensionless coupling

$$\alpha \equiv \frac{(g\, \nu^{\epsilon})^2}{4\pi}$$

all the C_i's are power series expansions in α of the type

$$C_i = \sum_{j=1}^{\infty} \sum_{k=1}^{j} C_{i,k}^{(2j)} \frac{1}{\epsilon^k} \left(\frac{\alpha}{\pi}\right)^j \tag{III.61}$$

$C_{i,k}^{(2j)}$ being at most a function of the gauge parameter a . In words, the coefficients are just single, double, etc,... poles in $1/\epsilon$. In the case at hand

$$\Sigma_R^{(2)}(\not p) = m \left\{ \frac{\alpha}{\pi} \frac{C_2(R)}{4} \left[-3\left(\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\frac{m^2}{\nu^2}\right) + 4 + a\left(1 - \frac{m^2}{p^2}\right) \right.\right.$$

$$\left.\left. - \left(3 - a\frac{m^2}{p^2}\right)\left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\right] - [C_4 - C_{2F}] \right\}$$

$$+ (\not p - m)\left\{ \frac{\alpha}{\pi}\, a\, \frac{C_2(R)}{4} \left[\left(\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\frac{m^2}{\nu^2}\right) - 1 - \frac{m^2}{p^2} \right.\right.$$

$$\left.\left. + \left(1 - \frac{m^4}{p^4}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\right] + C_{2F} \right\} \tag{III.62}$$

Now we choose C_{2F} and C_4 in such a way that they cancel the $1/\epsilon$ terms and therefore at the one loop level

$$Z_{2F}^{(2)} = 1 + \frac{\alpha}{\pi}\, a\, \frac{C_2(R)}{4} \frac{1}{\epsilon} \tag{III.63}$$

$$Z_4^{(2)} = 1 + \frac{\alpha}{\pi} (3 + a) \frac{C_2(R)}{4} \frac{1}{\epsilon} \tag{III.64}$$

and therefore the renormalized selfenergy is

$$\Sigma_{1R}^{(2)}(p^2) = \frac{\alpha}{\pi}\ \frac{C_2(R)}{4}\, m \left\{ -3\left(-\ln 4\pi + \gamma + \ln\frac{m^2}{\nu^2}\right) + 4 + a\left(1 - \frac{m^2}{p^2}\right) \right.$$

$$\left. - \left(3 - a\frac{m^2}{p^2}\right)\left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\right\}$$

$$\Sigma_{2R}^{(2)}(p^2) = \frac{\alpha}{\pi}\ \frac{C_2(R)}{4}\, a \left\{ \left(-\ln 4\pi + \gamma + \ln\frac{m^2}{\nu^2}\right) - 1 - \frac{m^2}{p^2} \right. \tag{III.65}$$

$$\left. + \left(1 - \frac{m^4}{p^4}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\right\}$$

which is the desired result.

In general it is not usual to work explicitly with the counter terms and it is much more convenient to proceed in the following way: taking into account (55) as well as the definition of the quark propagator (II.10) we can write, with $\alpha_0 \equiv (g_0\ \nu^{\epsilon})^2/4$

$$S_R(\not p\,; m, a, \alpha) = \lim_{\epsilon\to 0}\left\{ Z_{2F}^{-1}\ S_0(\not p\,; m_0, a_0, \alpha_0; \epsilon)\right\} \tag{III.66}$$

where in the r.h.s. m_0 , a_0 and g_0 have to be substituted by their expressions in terms of the renormalized quantities (55) and the limit taken order by order in the coefficients of the expansion in α . Explicitly for the inverse quark propagator one has from (66)

$$\left\{1 - \Sigma_{2R}(p^2; m, a, \alpha)\right\}\left\{\not p - m\left[1 + \frac{\Sigma_{1R}(p^2; m, a, \alpha)/m}{1 - \Sigma_{2R}(p^2; m, a, \alpha)}\right]\right\} =$$

$$= \lim_{\epsilon\to 0}\left\{ Z_{2F}\left\{1 - \Sigma_2(p^2; m_0, a_0, \alpha_0; \epsilon)\right\}\left\{\not p - m_0\left[1 + \frac{\Sigma_1(p^2; m_0, a_0, \alpha_0; \epsilon)/m_0}{1 - \Sigma_2(p^2; m_0, a_0, \alpha_0; \epsilon)}\right]\right\}\right\} \tag{III.67}$$

and from where, taking into account

$$m = Z_4^{-1}\ Z_{2F}\ m_0 \tag{III.68}$$

one finds

$$1 - \Sigma_{2R}(p^2; m, a, \alpha) = \lim_{\epsilon\to 0}\left\{ Z_{2F}\left[1 - \Sigma_2(p^2; m_0, a_0, \alpha_0; \epsilon)\right]\right\} \tag{III.69}$$

and

$$[1 - \Sigma_{2R}(p^2; m, a, \alpha) + \Sigma_{1R}(p^2; m, a, \alpha)/m] =$$

$$= \lim_{\epsilon \to 0} \{ Z_4 [1 - \Sigma_2(p^2; m_0, a_0, \alpha_0; \epsilon) + \Sigma_1(p^2; m_0, a_0, \alpha_0; \epsilon)/m_0] \} \qquad \text{(III.70)}$$

In the MS renormalization scheme Z_{2F} and Z_4 are chosen in such a way as to cancel exactly the poles in $1/\epsilon$ appearing in the square brackets of the right-hand side of the last equations. Using (21) we reobtain immediately (63), (64) and (65).

Let us now consider the gluon propagator. From (55) and (II.11) we can write

$$D_R^{\mu\nu}(k) = \lim_{\epsilon \to 0} \{ Z_{3YM}^{-1} D_0^{\mu\nu}(k) \} \quad , \quad a = Z_6 Z_{3YM}^{-1} a_0 \qquad \text{(III.71)}$$

where $D_0^{\mu\nu} \equiv D^{\mu\nu}(k; m_0, a_0, \alpha_0; \epsilon)$ and (28) allows us to write

$$-\frac{1}{a} k^\mu k^\nu + (k^\mu k^\nu - k^2 g^{\mu\nu}) [1 + \Pi_R(k^2; m, a, \alpha)] =$$

$$= \lim_{\epsilon \to 0} \{ - Z_{3YM} \frac{1}{a_0} k^\mu k^\nu + (k^\mu k^\nu - k^2 g^{\mu\nu}) Z_{3YM} [1 + \Pi(k^2; m_0, a_0, \alpha_0; \epsilon)] \}$$

Hence

$$Z_6 = 1 \qquad \text{(III.72)}$$

which is clearly true to all orders in perturbation theory and is an immediate consequence of the Slavnov-Taylor identity (II.20). Indeed if (II.20) is to be true not only for bare quantities but also for renormalized ones (72) follows immediately. Furthermore

$$\left[1 + \Pi_R (k^2; m, a, \alpha)\right] = \lim_{\epsilon \to 0} \left\{ Z_{3YM} \left[1 + \Pi (k^2; m_0, a_0\, \alpha_0; \epsilon)\right]\right\} \tag{III.73}$$

In second order perturbation theory

$$\left[1 + \Pi_R^{(2)} (k^2; m, a, \alpha)\right] = \lim_{\epsilon \to 0} \left\{ Z_{3YM} \left[1 + \Pi^{(2)} (k^2; m_0, a_0, \alpha_0; \epsilon)\right]\right\}$$

In the MS-scheme we choose Z_{3YM} in such a way that it cancels exactly the pole in $1/\epsilon$ of the quantity in the square brackets of the r.h.s. of the last equation , i.e.

$$Z_{3YM}^{(2)} = 1 - \frac{\alpha}{\pi} \left\{ \frac{C_2(G)}{4} \left(\frac{13}{6} - \frac{a}{2} \right) - \frac{T(R)}{3} N_f \right\} \frac{1}{\epsilon} \tag{III.74}$$

and the one loop renormalized gluon selfenergy is given by

$$\Pi_R^{(2)}(k^2) = \frac{\alpha}{\pi} \left\{ \frac{C_2(G)}{4} \left[\left(\frac{13}{6} - \frac{a}{2} \right) \left(-\ln 4\pi + \gamma + \ln\left(-\frac{k^2}{\nu^2}\right)\right) - \frac{97}{36} - \frac{1}{2} a - \frac{1}{4} a^2 \right] - \frac{T(R)}{3} \sum_A \left[\left(-\ln 4\pi + \gamma + \ln\left(\frac{m_A^2}{\nu^2}\right)\right) - \frac{5}{3} - \frac{4m_A^2}{k^2} + \left(1 + \frac{2m_A^2}{k^2}\right) \sqrt{1 - \frac{4m_A^2}{k^2}} \ln \frac{\sqrt{1 - 4m_A^2/k^2} + 1}{\sqrt{1 - 4m_A^2/k^2} - 1} \right] \right\} \tag{III.75}$$

Let us consider finally the ghost propagator defined in (II.12 Using (55) we can write

$$\tilde{D}_R (k) = \lim_{\epsilon \to 0} \left\{ \tilde{Z}_3^{-1} \tilde{D}_0 (k) \right\} \tag{III.76}$$

and hence

$$\left[1 - \frac{1}{k^2} \tilde{\Pi}_R (k^2; m, a, \alpha)\right] = \lim_{\epsilon \to 0} \left\{ \tilde{Z}_3 \left[1 - \frac{1}{k^2} \tilde{\Pi} (k^2; m_0, a_0, \alpha_0; \epsilon)\right]\right\} \tag{III.77}$$

and proceeding as before we obtain in the MS-scheme

$$Z_3^{(2)} = 1 - \frac{\alpha}{\pi} \frac{C_2(G)}{4} \frac{3-a}{4} \frac{1}{\epsilon} \tag{III.78}$$

and

$$\tilde{\Pi}_R^{(2)}(k^2) = -\frac{\alpha}{\pi} k^2 \frac{C_2(G)}{4} \left\{ \left(\frac{3-a}{4}\right)\left(-\ln 4\pi + \gamma + \ln\left(-\frac{k^2}{\nu^2}\right)\right) - 1 \right\} \tag{III.79}$$

Let us now consider the vertex functions defined in (II.22), (II.23), (II.24) and (47). Using (55) we obtain

$$\begin{aligned} \Gamma^a_{\mu\beta\alpha,R}(q,k,p) &= \lim_{\epsilon\to 0} \left\{ Z_{1F}\, \Gamma^a_{0\mu\beta\alpha}(q,k,p) \right\} \\ T^{\mu\nu\sigma}_{abc,R}(p,q,k) &= \lim_{\epsilon\to 0} \left\{ Z_{1YM}\, T^{\mu\nu\sigma}_{0abc}(p,q,k) \right\} \\ G^{\mu}_{abc,R}(k,q,p) &= \lim_{\epsilon\to 0} \left\{ \tilde{Z}_1\, G^{\mu}_{0abc}(k,q,p) \right\} \\ T^{\mu\nu\sigma\rho}_{abcd,R}(p,q,k,r) &= \lim_{\epsilon\to 0} \left\{ Z_5\, T^{\mu\nu\sigma\rho}_{0abcd}(p,q,k,r) \right\} \end{aligned} \tag{III.80}$$

In the MS-scheme we choose the renormalization constants in (80) in such a way that they cancel exactly the poles in the expressions of the vertices of the r.h.s..Using (39), (44),(46) and (50) we get immediately

$$Z_{1F}^{(2)} = 1 + \frac{\alpha}{\pi} \frac{1}{4} \left\{ a\, C_2(R) + (3+a) \frac{C_2(G)}{4} \right\} \frac{1}{\epsilon}$$

$$Z_{1YM}^{(2)} = 1 - \frac{\alpha}{\pi} \frac{1}{4} \left\{ \left(\frac{17}{12} - \frac{3a}{4}\right) C_2(G) - \frac{4T(R)}{3} N_f \right\} \frac{1}{\epsilon}$$

$$\tilde{Z}_1^{(2)} = 1 + \frac{\alpha}{\pi} \frac{1}{4} a \frac{C_2(G)}{2} \frac{1}{\epsilon}$$

$$Z_5^{(2)} = 1 + \frac{\alpha}{\pi} \frac{1}{4} \left\{ \left(-\frac{2}{3} + a \right) C_2(G) + \frac{4T(R)}{3} N_f \right\} \frac{1}{\epsilon} \tag{III.81}$$

and with this we have computed, at the one loop level, all the renormalization constants in the MS subtraction scheme. Now it is easy to check that the Slavnov-Taylor identities (59) are satisfied. The formulae (66), (71), (76) and (80) are valid to all orders in perturbation theory . Once the one loop renormalization constants are known they allow to compute their two loop expressions. Higher orders are computed similarly . Let us here just give the two loop results [ET 79],[TA 81].

$$\begin{aligned} Z_{2F} = 1 &+ \frac{\alpha}{\pi} \frac{C_2(R)}{4} a \frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2 \left\{ \left[\frac{C_2(R)}{64} \left(C_2(G) (3a + a^2) \right.\right.\right. \\ &\left.\left. + C_2(R)\, 2a^2 \right) \right] \frac{1}{\epsilon^2} + \left[\frac{C_2(R)}{64} \left(C_2(G) \left(\frac{25}{2} + 4a + \frac{a^2}{2} \right) \right.\right. \\ &\left.\left.\left. - C_2(R)\, 3 - T(R)\, 4 N_f \right) \right] \frac{1}{\epsilon} \right\} \end{aligned} \tag{III.82}$$

$$\begin{aligned} Z_{3YM} = 1 &+ \frac{\alpha}{\pi} \left[\frac{C_2(G)}{8} \left(-\frac{13}{3} + a \right) + \frac{T(R)}{3} N_f \right] \frac{1}{\epsilon} \\ &+ \left(\frac{\alpha}{\pi}\right)^2 \left\{ \left[\frac{C_2(G)}{32} \left(C_2(G) \left(-\frac{13}{4} - \frac{17}{12} a + \frac{1}{2} a^2 \right) \right.\right.\right. \\ &\left.\left. + 2T(R) \left(1 + \frac{2}{3} a \right) N_f \right) \right] \frac{1}{\epsilon^2} + \left[\frac{C_2^2(G)}{128} \left(-\frac{59}{2} + \frac{11}{2} a + a^2 \right) \right. \\ &\left.\left. + \frac{1}{8} C_2(R) T(R) N_f + \frac{5}{32} C_2(G) T(R) N_f \right] \frac{1}{\epsilon} \right\} \end{aligned} \tag{III.83}$$

$$\tilde{Z}_3 = 1 + \frac{\alpha}{\pi} \frac{C_2(G)}{16} (-3+a) \frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2 \Bigg\{ \Bigg[\frac{C_2^2(G)}{512} (-35 + 3a^2)$$

$$+ \frac{1}{32} C_2(G) T(R) N_f \Bigg] \frac{1}{\epsilon^2} + \Bigg[\frac{C_2(G)}{192} \Bigg(\frac{C_2(G)}{8} (-95 - 3a)$$

$$+ 5 T(R) N_f \Bigg) \Bigg] \frac{1}{\epsilon} \Bigg\} \tag{III.84}$$

$$\tilde{Z}_1 = 1 + \frac{\alpha}{\pi} \frac{C_2(G)}{8} a \frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2 \Bigg\{ \Bigg[\frac{C_2^2(G)}{64} \left(\frac{3}{2} a + a^2 \right) \Bigg] \frac{1}{\epsilon^2}$$

$$+ \Bigg[\frac{C_2^2(G)}{256} (5a + a^2) \Bigg] \frac{1}{\epsilon} \Bigg\} \tag{III.85}$$

$$Z_4 = 1 + \frac{\alpha}{\pi} \frac{C_2(R)}{4} (3+a) \frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2 \Bigg\{ \Bigg[\frac{C_2(R)}{8} \Bigg(\frac{C_2(G)}{8} (22 + 3a + a^2)$$

$$+ C_2(R) \left(\frac{9}{4} + \frac{3}{2} a + \frac{1}{4} a^2 \right) - T(R) N_f \Bigg) \Bigg] \frac{1}{\epsilon^2}$$

$$+ \Bigg[\frac{C_2(R)}{16} \Bigg(\frac{C_2(G)}{8} \left(\frac{269}{3} + 8a + a^2 \right) - \frac{8}{3} T(R) N_f \Bigg) \Bigg] \frac{1}{\epsilon} \Bigg\} \tag{III.86}$$

All the other renormalization constants can be obtained from them using the Slavnov-Taylor identities .

Another useful renormalization scheme in QCD is the modified minimal subtraction scheme ($\overline{\text{MS}}$ - scheme) in which we take into account that $1/\epsilon$ appear always in the combination $(1/\epsilon - \ln 4\pi + \gamma)$ and we define the Z_i in such a way that this combination is eliminated from the renormalized Green's functions. At the one loop level the Z_i in the $\overline{\text{MS}}$ - scheme are obtained from the corresponding ones in the MS - scheme through the substitution $1/\epsilon \longrightarrow (1/\epsilon - \ln 4\pi + \gamma)$ while the renormalized Green's functions are obtained from the ones in the MS - scheme eliminating the $(-\ln 4\pi + \gamma)$ terms. At higher orders in perturbation theory there are different ways of defining the scheme, and the passage from one to the other is less trivial.

Let us now discuss another renormalization scheme: the μ - scheme. In the μ - scheme the renormalized Green's functions are obtained from the regularized ones subtracting from them its value with all momenta taken in some arbitrarily chosen Euclidean point. Let us begin considering the quark selfenergy. The μ -scheme renormalized expressions are

$$\Sigma_{iR} (p^2; \mu^2) = \Sigma_i (p^2; \epsilon) - \Sigma_i (p^2 = -\mu^2; \epsilon) \tag{III.87}$$

and hence

$$\Sigma^{(2)}_{1,R} (p^2, \mu^2) = \frac{\alpha(\mu)}{\pi} m(\mu) \frac{C_2(R)}{4} \Big\{ - a(\mu) m^2(\mu) \left(\frac{1}{p^2} + \frac{1}{\mu^2} \right)$$

$$- \left(3 - a(\mu) \frac{m^2(\mu)}{p^2}\right) \left(1 - \frac{m^2(\mu)}{p^2}\right) \ln \left(1 - \frac{p^2}{m^2(\mu)}\right)$$

$$+ \left(3 + a(\mu) \frac{m^2(\mu)}{\mu^2}\right) \left(1 + \frac{m^2(\mu)}{\mu^2}\right) \ln \left(1 + \frac{\mu^2}{m^2(\mu)}\right) \Big\} \tag{III.88}$$

$$\Sigma^{(2)}_{2,R} (p^2, \mu^2) = \frac{\alpha(\mu)}{\pi} a(\mu) \frac{C_2(R)}{4} \Big\{ - m^2(\mu) \left(\frac{1}{p^2} + \frac{1}{\mu^2} \right)$$

$$+ \left(1 - \frac{m^4(\mu)}{p^4}\right) \ln \left(1 - \frac{p^2}{m^2(\mu)}\right) - \left(1 - \frac{m^4(\mu)}{\mu^4}\right) \ln \left(1 + \frac{\mu^2}{m^2(\mu)}\right) \Big\}$$

where the renormalized quantities depend on the point where the subtraction has been carried out.

In the language of renormalization constants we find from (66 and (68) that

$$Z_{2F}^{(2)} = 1 + \Sigma_2^{(2)} (p^2 = -\mu^2 ; \epsilon)$$

$$Z_4^{(2)} = 1 + \Sigma_2^{(2)} (p^2 = -\mu^2 ; \epsilon) - \Sigma_1^{(2)} (p^2 = -\mu^2 ; \epsilon) / m(\mu) \quad \text{(III.89)}$$

The same is done for all the other superficially divergent functions.

The μ-scheme seems to be physically more meaningful than the MS -or $\overline{\text{MS}}$- schemes in as much as it consists in absorbing the higher-order corrections to a lowest-order structure in the parameters and fields of the theory, so that at least at a Euclidean point there are no explicit higher-order corrections. These are absorbed by the redefinitions of the coupling constant, the masses and the gauge parameter. It is then hoped that the coefficients of the higher-order corrections computed with the parameters defined through a momentum-spac subtraction will not be too large, so that the series converges for reasonable momenta. The advantages from the point of view of physical interpretation of the μ-scheme or Euclidean momentum-space subtraction scheme are also seen in the decoupling theorem [SY 73] [AC 75], that states that flavors with mass larger than the energy one is working at are irrelevant. This can only be immediately seen in this schem as it is the only one which is quark-mass and therefore flavor dependent.

Nevertheless this flavor dependence is often more a nuisance that an advantage and, furthermore, complicates the calculations very much and we are not going to proceed to its study. Weinberg [WE 73b] has proposed a scheme (W-scheme) which somehow lies inbetween the μ and the MS schemes: one performs a subtraction at a Euclidean point, but putting all quark masses equal to zero. It is thus flavor independent but still a momentum-space subtraction. Its computational difficulty lies half way between the complexity of the μ-scheme and the simplicity of the MS-cheme.

Let us now proceed to study the renormalization constants in the W-scheme. From (21), (32) and (34) and the above given discussion we find immediately

$$Z_{2F}^{(2)} = 1 + \frac{\alpha(\mu)}{\pi} \frac{C_2(R)}{4} a(\mu) \left[\frac{1}{\tilde{\varepsilon}} - 1 \right]$$

$$Z_4^{(2)} = 1 + \frac{\alpha(\mu)}{\pi} \frac{C_2(R)}{4} \left[(3 + a(\mu)) \frac{1}{\tilde{\varepsilon}} - 4 - 2a(\mu) \right]$$

$$Z_{3YM}^{(2)} = 1 + \frac{\alpha(\mu)}{\pi} \left\{ \frac{C_2(G)}{8} \left[\left(-\frac{13}{3} + a(\mu) \right) \frac{1}{\tilde{\varepsilon}} + \frac{97}{18} + a(\mu) \right. \right. \qquad \text{(III.90}$$

$$\left. \left. + \frac{1}{2} a^2(\mu) \right] + T(R) N_f \frac{1}{3} \left(\frac{1}{\tilde{\varepsilon}} - \frac{5}{3} \right) \right\}$$

$$\tilde{Z}_3^{(2)} = 1 + \frac{\alpha(\mu)}{\pi} \frac{C_2(G)}{4} \left[-\frac{1}{4} (3 - a(\mu)) \frac{1}{\tilde{\varepsilon}} + 1 \right]$$

where

$$\frac{1}{\tilde{\varepsilon}} \equiv \frac{1}{\varepsilon} - \ln 4\pi + \gamma + \ln \frac{\mu^2}{\nu^2}$$

All the other renormalization constants, at the one loop level, can be found in [PT 80] where also the role played by the Slavnov-Taylor identities is discussed.

Before finishing this chapter we would like to return to the Slavnov-Taylor identities. We have already seen that the identity (II. implies $Z_6 = 1$ to all orders in perturbation theory. Consider now th second Slavnov-Taylor identity (II.38). It is clear from the first one (II.35) that $G^{abc}_{\mu\nu}$ renormalizes exactly as $G^{ab\nu}_{\nu}$. Then (II.38) implies from (71), (76) and (80)

$$\tilde{Z}_1 \tilde{Z}_3^{-1} = Z_{1YM} Z_{3YM}^{-1} \tag{III.91}$$

which is the first identity of (59). The other identities of (59) can be obtained along the same lines, although one needs to develop the identities deduced in Chapter II further.

IV. THE RENORMALIZATION GROUP

Renormalization invariance is the statement that physical observables must be independent of the renormalization scheme which one chooses for their theoretical calculation. For historical reasons [SP 53] [GL 54] the renormalization invariance is usually denominated invariance under the renormalization group. In fact, as we shall see, the group structure is only defined within classes of renormalization.

Let us assume that we have chosen a particular renormalization scheme, say R , and let us denote by $\Gamma_R(....)$ the R-renormalized Green's function. Its relation with the bare Green's function $\Gamma(...)$ is

$$\Gamma_R(-----) = Z(R)\ \Gamma(-----) \tag{IV.1}$$

where Z(R) denotes the appropiate product of renormalization constants defined in the R-scheme. Let us now choose another renormalization scheme R' , then

$$\Gamma_{R'}(----) = Z(R')\ \Gamma(----) \tag{IV.2}$$

From (1) and (2) we obtain

$$\begin{aligned} \Gamma_{R'}(----) &= Z(R',R)\ \Gamma_R(----) \\ Z(R',R) &= Z(R')/Z(R) \end{aligned} \tag{IV.3}$$

Let us consider the set of all possible Z(R',R) with R and R' arbitrary, Among the elements of this set there exists a composition law

$$Z(R'', R) = Z(R'', R')\, Z(R', R) \tag{IV.4}$$

To each element $Z(R',R)$ we can associate an inverse

$$Z^{-1}(R', R) = Z(R, R') \tag{IV.5}$$

and we can define a unit element

$$Z(R, R) = 1 \tag{IV. 6}$$

Notice that the composition law is not defined for two arbitrary elements of the group: the product $Z(R_i, R_j)\, Z(R_k, R_e)$ is not, in general, an element of the group unless $R_j = R_k$. A set of transformations obeying these relations define a grupoid structure [AJ 67] [AJ 68] [AJ 69] [SU 77].

The study of the differential form of the renormalization invariance was initiated by Stueckelberg and Peterman [SP 53] and Gell-Mann and Low [GL 54]. Later on, the study of scaling in quantum field theory, motivated by the experimental evidence of Bjorken scaling [BJ 69] in deep inelastic scattering in electron-proton collisions, originated the Callan-Symanzik equations [CA 70] [SY 70] which are a powerful method for studying the restrictions imposed by renormalization invariance on the behaviour at small distances of the Green's functions. More general methods that can be applied to arbitrary Green's functions have been formulated by 't Hooft [TH 73] and Weinberg [WE 73]. The central idea in these methods is to treat in a similar way the coupling constant, mass and gauge parameter renormalization constants; i.e. g, m_A and a as coupling constants of different terms in the Lagrangian density.

Let us consider a Green's function $G(x_1,x_2,\ldots,x_N)$ with n_{YM}, n_F and $\tilde{n}$ gluon, quark and ghost external lines, respectively. Notice

$$[G(x_1,x_2,\ldots x_N)] = M^{n_{YM}+\tilde{n}+\frac{3}{2}n_F}$$

$$n_{YM}+\tilde{n}+n_F = N \qquad \text{(IV.7)}$$

The dimension of its Fourier transform $\hat{G}(p_1,p_2,\ldots,p_N)$, where the total energy-momentum conservation delta function has been eliminated, is

$$[\hat{G}(p_1,p_2,\ldots p_N)] = M^{4-3n_{YM}-3\tilde{n}-\frac{5}{2}n_F} \qquad \text{(IV.8)}$$

We will use, in general, the corresponding amputated Green's function and for it

$$[\Gamma(p_1,p_2,\ldots p_N)] = M^{4-n_{YM}-\tilde{n}-\frac{3}{2}n_F} \equiv M^{d_\Gamma} \qquad \text{(IV.9)}$$

Furthermore the relation between the renormalized and bare Green's functions is

$$\Gamma_R(p_1,p_2,\ldots p_N;\alpha,a,m_A;\mu) = \lim_{\epsilon\to 0}\left\{Z_\Gamma(\mu,\epsilon)\;\Gamma_0(p_1,p_2,\ldots,p_N;\alpha_0,a_0,m_{A0};\epsilon)\right\}$$

$$Z_\Gamma(\mu,\epsilon) \equiv Z_{3YM}^{-n_{YM}/2}(\mu,\epsilon)\;\tilde{Z}_3^{-\tilde{n}/2}(\mu,\epsilon)\;Z_{2F}^{-n_F/2}(\mu,\epsilon) \qquad \text{(IV.10)}$$

Here μ is either the subtraction point in the momentum subtraction and Weinberg schemes, or the mass scale ν in the MS and $\overline{\text{MS}}$ schemes.

Obviously the bare Green's function is μ -independent and we can write

$$\mu \frac{d}{d\mu} \Gamma_0 (p_1, p_2, \dots p_N ; \alpha_0, a_0, m_{A_0}; \epsilon) = 0 \tag{IV.11}$$

and using (10)

$$\left[\mu \frac{\partial}{\partial \mu} + \mu \frac{d\alpha}{d\mu} \frac{\partial}{\partial \alpha} + \sum_A \frac{\mu}{m_A} \frac{d m_A}{d\mu} m_A \frac{\partial}{\partial m_A} + \mu \frac{da}{d\mu} \frac{\partial}{\partial a} \right] .$$

$$\Gamma_R (p_1 p_2 \dots p_N ; \alpha, a, m_A, \mu) = \frac{1}{Z_\Gamma} \mu \frac{d Z_\Gamma}{d\mu} \Gamma_R (p_1, p_2 \dots , p_N ; \alpha, a, m_A ; \mu) \tag{IV.12}$$

By this method we have introduced a set of universal functions, which depend on α , $x_A \equiv m_A/\mu$ and a which can be written as

$$\mu \frac{d\alpha}{d\mu} \equiv \alpha \beta (\alpha, a, x_A) \qquad , \qquad \frac{\mu}{m_A} \frac{d m_A}{d\mu} \equiv - \gamma_A (\alpha, a, x_A)$$

$$\mu \frac{da}{d\mu} \equiv a \, \delta (\alpha, a, x_A)$$

$$\frac{\mu}{Z_{3YM}} \frac{d Z_{3YM}}{d\mu} \equiv \gamma_{YM} (\alpha, a, x_A) \quad , \quad \frac{\mu}{Z_{2F}} \frac{d Z_{2F}}{d\mu} \equiv \gamma_F (\alpha, a, x_A) \tag{IV.13}$$

$$\frac{\mu}{\tilde{Z}_3} \frac{d \tilde{Z}_3}{d\mu} \equiv \tilde{\gamma} (\alpha, a, x_A)$$

These functions are universal in the sense that (12) is valid for any Green's function with the same coefficients in front of each derivative. From the fact that Q.C.D. is a renormalizable theory it follows that these universal functions are non-singular in the limit $\epsilon \to 0$. This is obvious for the first three functions of (13) because they correspond to derivatives of renormalized parameters, but it is less so for the last three renormalization group functions. It can be proved writing (12) for the renormalized Green's functions associated with the diagrams of the primitive divergences and solving the algebraic system so obtained. Using (13) we can write

$$\Big[\mu \frac{\partial}{\partial \mu} + \beta(\alpha)\,\alpha \frac{\partial}{\partial \alpha} + \delta(\alpha)\, a \frac{\partial}{\partial a} - \sum_A \gamma_A(\alpha)\, x_A \frac{\partial}{\partial x_A}$$

$$- \gamma_\Gamma(\alpha)\Big]\, \Gamma_R(p_1, p_2, \dots, p_N;\ \alpha, a, m_A;\ \mu) = 0 \qquad \text{(IV. 14}$$

$$n_{YM}\,\gamma_{YM}(\alpha) + n_F\,\gamma_F(\alpha) + \tilde{m}\,\tilde{\gamma}(\alpha) \equiv -2\gamma_\Gamma(\alpha)$$

where we have written explicitly only the α -dependence of the univer functions just introduced.

It is evident that the explicit form of these universal functions depends on the renormalization scheme used. In the μ -scheme they have nontrivial dependences on α , a and x_A , while in the MS , $\overline{\text{MS}}$ or Weinberg schemes they are independent of x_A . Furthermore we will see that the β function in the MS and $\overline{\text{MS}}$ schemes is gauge independent while this is not so in other schemes.

Dimensional analysis puts another constrain on the Green's function. Taking into account (9) we can write

$$\Gamma_R(p_1, p_2, \dots, p_N;\ \alpha, a, m_A;\ \mu) \equiv \mu^{d_\Gamma}\, \psi(p_1/\mu, \dots\ p_N/\mu;\ \alpha, a, x_A) \qquad \text{(IV.15)}$$

where ψ is a dimensionless function of its dimensionless arguments. This allows us to write

$$\Gamma_R(\xi p_1, \ldots, \xi p_N; \alpha, a, m_A; \mu) = \mu^{d_\Gamma} \psi(\xi p_1/\mu, \ldots, \xi p_N/\mu; \alpha, a, x_A) =$$

$$= \xi^{d_\Gamma} (\mu/\xi)^{d_\Gamma} \psi(p_1/(\mu/\xi), \ldots, p_N/(\mu/\xi); \alpha, a, x_A)$$

i.e.

$$\Gamma_R(\xi p_1, \ldots, \xi p_N; \alpha, a, m_A; \mu) = \xi^{d_\Gamma} \Gamma_R(p_1, \ldots p_N; \alpha, a, m_A/\xi; \mu/\xi) \tag{IV.16}$$

Hence

$$\left\{ \xi \frac{\partial}{\partial \xi} \Gamma_R(\xi p_1, \ldots, \xi p_N; \alpha, a, m_A; \mu) \right\} =$$

$$= \xi \frac{\partial}{\partial \xi} \left\{ \xi^{d_\Gamma} \Gamma_R(p_1, \ldots, p_N; \alpha, a, m_A/\xi; \mu/\xi) \right\} =$$

$$= \left\{ d_\Gamma - \mu \frac{\partial}{\partial \mu} - \sum_A x_A \frac{\partial}{\partial x_A} \right\} \Gamma_R(\xi p_1, \ldots, \xi p_N; \alpha, a, m_A; \mu)$$

and therefore we can write

$$\left[\xi \frac{\partial}{\partial \xi} + \sum_A x_A \frac{\partial}{\partial x_A} + \mu \frac{\partial}{\partial \mu} - d_\Gamma \right] \Gamma_R(\xi p_1, \ldots, \xi p_N; \alpha, a, m_A; \mu) = 0 \tag{IV.17}$$

Using this as well as (14)

$$\Big[\xi \frac{\partial}{\partial \xi} - \beta(\alpha)\, \alpha \frac{\partial}{\partial \alpha} - \delta(\alpha)\, a \frac{\partial}{\partial a} + \sum_A [1 + \gamma_A(\alpha)] x_A \frac{\partial}{\partial x_A}$$

$$- d_\Gamma + \gamma_\Gamma(\alpha) \Big] \Gamma_R(\xi p_1, \ldots, \xi p_N; \alpha, a, m_A; \mu) = 0$$

Introducing

$$t \equiv \ln \xi \qquad \Longleftrightarrow \qquad \xi = e^{t} \tag{IV.18}$$

the last equation can be written as

$$\Big[- \frac{\partial}{\partial t} + \beta(\alpha)\, \alpha \frac{\partial}{\partial \alpha} + \delta(\alpha)\, a \frac{\partial}{\partial a} - \sum_A \left[1 + \gamma_A(\alpha) \right] x_A \frac{\partial}{\partial x_A}$$

$$+ d_r - \gamma_r(\alpha) \Big] \Gamma_R (e^t p_1, \ldots, e^t p_N ; \alpha, a, m_A ; \mu) = 0 \tag{IV.19}$$

which is the fundamental equation of the renormalization group. Let us recall that the universal functions appearing here are not only functions of α but also of a and x_A .

The general solution of (19) can be obtained via the method of characteristics to solve linear partial differential equations. First one must solve the system of coupled ordinary differential equations

$$\frac{d\bar{\alpha}(t)}{dt} = \bar{\alpha}(t)\, \beta \left[\bar{\alpha}(t), \bar{a}(t), \bar{x}_A(t) \right] \quad , \quad \bar{\alpha}(0) = \alpha$$

$$\frac{d\bar{a}(t)}{dt} = \bar{a}(t)\, \delta \left[\bar{\alpha}(t), \bar{a}(t), \bar{x}_A(t) \right] \quad , \quad \bar{a}(0) = a \tag{IV.20}$$

$$\frac{d\bar{x}_A(t)}{dt} = - \left\{ 1 + \gamma_A \left[\bar{\alpha}(t), \bar{a}(t), \bar{x}_A(t) \right] \right\} \bar{x}_A(t) \quad , \quad \bar{x}_A(0) = x_A$$

The quantities $\bar{\alpha}(t)$, $\bar{a}(t)$ and $\bar{x}_A(t)$ are called, respectively, running (or effective) coupling constant, running gauge parameter and running masses (in units of μ). All of them are functions of t and of the initial conditions α , a and x_A .

Let us now see how once these equations are solved we can obtain the general solution of (19) . In order to simplify as much as possible the argument let us skip the dependence in a and x_A ; then

(19) can be written as

$$\left[-\frac{\partial}{\partial t}+\beta(\alpha)\,\alpha\,\frac{\partial}{\partial\alpha}-\hat{\gamma}_{\Gamma}(\alpha)\right]\Gamma_R(e^t p_i\,;\,\alpha\,;\,\mu)=0 \qquad \text{(IV.21)}$$

where $\hat{\gamma}_{\Gamma}(\alpha)\equiv\gamma_{\Gamma}(\alpha)-d_{\Gamma}$. Let us now write the same equation with $\alpha\to\bar{\alpha}\ (-t)$

$$\left[-\frac{\partial}{\partial t}+\beta[\bar{\alpha}(-t)]\,\bar{\alpha}(-t)\frac{\partial}{\partial\bar{\alpha}(-t)}-\hat{\gamma}_{\Gamma}[\bar{\alpha}(-t)]\right]\Gamma_R(e^t p_i\,;\,\bar{\alpha}(-t)\,;\,\mu)=0$$

and using

$$\frac{d\bar{\alpha}(t)}{dt}=\bar{\alpha}(t)\,\beta[\bar{\alpha}(t)]\quad,\quad \bar{\alpha}(0)=\alpha \qquad \text{(IV.22)}$$

we obtain

$$\left[-\frac{\partial}{\partial t}-\frac{d\bar{\alpha}(-t)}{dt}\frac{\partial}{\partial\bar{\alpha}(-t)}-\hat{\gamma}_{\Gamma}[\bar{\alpha}(-t)]\right]\Gamma_R(e^t p_i\,;\,\bar{\alpha}(-t)\,;\,\mu)=0$$

This can be written as

$$\left[\frac{d}{dt}+\hat{\gamma}_{\Gamma}[\bar{\alpha}(-t)]\right]\Gamma_R(e^t p_i\,;\,\bar{\alpha}(-t)\,;\,\mu)=0$$

and its solution is

$$\Gamma_R(e^t p_i;\,\bar{\alpha}(-t,\alpha)\,;\,\mu)=\Gamma_R(p_i\,;\,\alpha\,;\,\mu)\exp\left\{-\int_{-t}^{0}dt'\,\hat{\gamma}_{\Gamma}[\bar{\alpha}(t',\alpha)]\right\} \qquad \text{(IV.23)}$$

where the initial conditions have been written explicitly. An importan property of eq. (22) is that its r.h.s. does not depend explicitly on t and therefore

$$\bar{\alpha}(t) = \bar{\alpha}\,[\,t - t_1\,,\ \bar{\alpha}(t_1)\,] \tag{IV.24}$$

being t_1 the new initial value of the parameter. Let us consider now $t = 0\ ,\ t_1\ ,\ t_2$; we can write

$$\bar{\alpha}(t_2) = \bar{\alpha}\,[\,t_2\,,\ \bar{\alpha}(0) = \alpha\,] \quad , \quad \bar{\alpha}(t_2) = \bar{\alpha}\,[\,t_2 - t_1\ ;\ \bar{\alpha}(t_1)\,] \tag{IV.25}$$

and taking $t_2 = 0$ and $t_1 = t$ we obtain

$$\bar{\alpha}\,[\,0,\ \alpha\,] = \alpha = \bar{\alpha}\,[\,-t\,,\ \bar{\alpha}(t)\,] \tag{IV.26}$$

Let us return to eq. (23); if we replace α by $\bar{\alpha}\,(t)$ we have

$$\Gamma_R\,(e^t p_i\ ;\ \bar{\alpha}[-t,\bar{\alpha}(t)]\ ;\ \mu) = \Gamma_R\,(p_i\ ;\ \bar{\alpha}(t)\ ;\ \mu)\ \exp\left\{ - \int_{-t}^{0} dt'\ \hat{\gamma}_r\,[\,\bar{\alpha}(t',\bar{\alpha}(t))\,]\right\}$$

and using (25) and (26) we obtain finally

$$\Gamma_R\,(e^t p_i\ ;\ \alpha\ ;\ \mu) = \Gamma_R\,(p_i\ ;\ \bar{\alpha}(t)\ ;\ \mu)\ \exp\left\{ - \int_0^t dt'\ \hat{\gamma}_r\ [\,\bar{\alpha}(t')\,]\right\} \tag{IV.27}$$

which is a particular case of the general solution

$$\Gamma_R\,(\xi p_1, \ldots, \xi p_N\ ;\ \alpha\,,\ a\,, m_A\ ;\ \mu\,) = \xi^{d_r}\,\Gamma_R\,(p_1, \ldots, p_N\ ;\ \bar{\alpha}(t)\,,\ \bar{a}(t)\,, \bar{m}_A(t)\,;\ \mu)\,\cdot$$

$$\cdot\ \exp\left\{ - \int_0^t dt'\ \gamma_r\ [\,\bar{\alpha}(t')\,]\right\} \tag{IV.28}$$

where $\xi \equiv e^t$.

Equation (28) is the fundamental result of the renormalizatio group equations and it tells us that the behaviour of the Green's func tions when all external momenta are multiplied by a scale factor, for fixed value of μ , is given by the flux of the running parameters of the theory as functions of the scale variable; furthermore a global scale factor appears that is not the naïvely expected ξ^{d_r} but

$$\exp \left\{ t d_r - \int_0^t dt' \, \gamma_r [\bar{\alpha}(t', \alpha)] \right\} \tag{IV.29}$$

and for this reason γ_r is called the anomalous dimension.

In order to understand the relevance of eq. (28) we must analyze the role of the zeros of the function $\beta(\alpha)$. For simplicity we will forget its possible dependence on the gauge parameter and on th masses. In this case the equation for the running coupling constant is given by

$$\frac{d \bar{g}(t)}{dt} = \beta_g [\bar{g}(t)] \quad , \quad \bar{g}(0) = g \tag{IV.30}$$

(this is equation (22) written in terms of the coupling constant). Let us assume that g_1 is a simple zero of the equation $\beta_g [g] = 0$ and therefore for values of g near g_1 we can write

$$\beta_g (g) = \beta_1 (g - g_1) \tag{IV.31}$$

and solving (30) in this approximation we get

$$\bar{g}(t) = g_1 + (g - g_1) e^{\beta_1 t} \tag{IV.32}$$

Hence:

i) If $\beta_1 > 0$ and g lies near g_1 (more technically: if g lies in the domain of attraction of the zero or fixed point g_1, A study of eq. (30) proves that this is so if β_g has no zero in the interval $(g_1, g]$) then in the infrared limit ($\xi \to 0$, $t \to -\infty$) we get $\bar{g}(t) \longrightarrow g_1$ and we say that g_1 is an infrared stable fixed point (IRSFP) and this point regulates the IR behaviour of the theory.

ii) If $\beta_1 < 0$ and g lies in the domain of attraction of the fixed point g_1 then in the ultraviolet limit ($\xi \to \infty$, $t \to \infty$) we get $\bar{g}(t) \longrightarrow g_1$ and we say that g_1 is an ultraviolet stable fixed point (UVSFP) and this point regulates the UV behaviour of the theory.

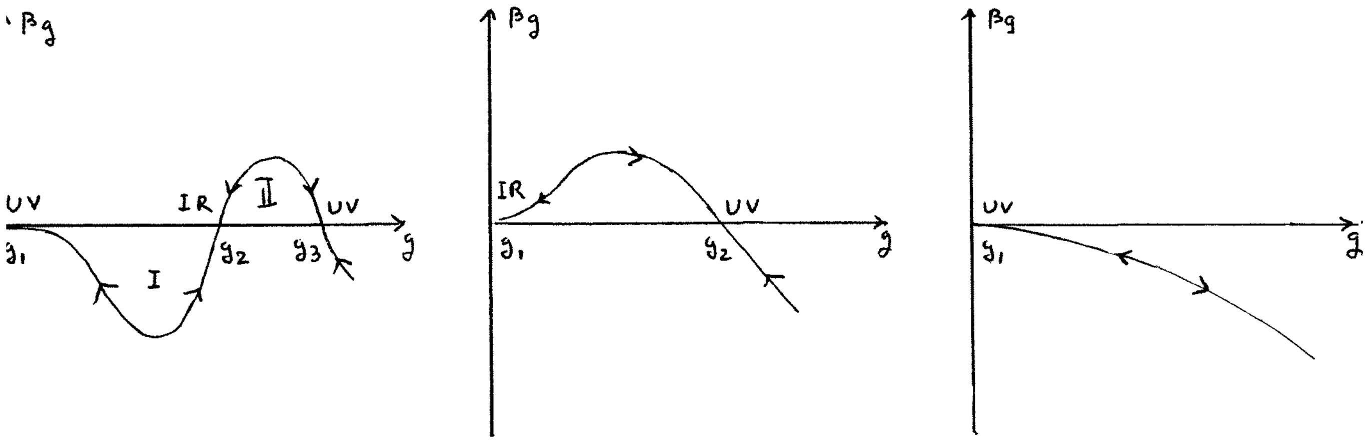
Analogous results are obtained for zeroes of odd order, i.e. $\beta_g(g) = \beta_1 (g-g_1)^n$ where n is odd. For zeroes of even order further analysis is required. For instance if $\beta_g(g) = \beta_1 (g-g_1)^2$, solving equation (30) we find

$$\bar{g}(t) = g_1 + \frac{g-g_1}{1-(g-g_1)\beta_1 t}$$

and in this case it is necessary to know the sign of $\beta_1 (g-g_1)$ in order to determine the IR or UV behavior of the field theory.

Furthermore a theory is said to have an infrared stable origin if the origin is an IRSFP . Since, as it will be seen, in perturbation theory $\beta_g(g = 0) = 0$ all single coupling theories where g lies in the domain of attraction of the origin are either asymptoticall free or infrared stable. For theories with multiple coupling constants, the situation is more complicated, because the origin may be UV stable for some couplings and IR stable for others.

These ideas and definitions are illustrated in this figure,



a) b) c)

which describes some examples of how $\beta_g(g)$ might look like. Of course, for a given theory, the only place where we know how to calculate $\beta_g(g)$ is near $g = 0$, using perturbation theory and it will be seen that $\beta_g \sim g^3$; its subsequent behaviour is pure speculation. Fig. a) illustrates the possible behavior of an asymptotically free theory and distinguishes two distinct regions. If $0 = g_1 < g < g_2$ (Region I), then in the limit $t \to \infty$ $\bar{g}(t) \longrightarrow g_1 = 0$ and in the limit $t \to -\infty$, $\bar{g}(t) \longrightarrow g_2$. So asymptotically the effective coupling constant vanishes (the theory presents asymptotic freedom), while at low momenta $\bar{g}(t)$ approaches the fixed point g_2. On the other hand if $g_2 < g < g_3$ (Region II) then the theory will not display asymptotic freedom, instead in the limit $t \to \infty$, $\bar{g}(t) \to g_3 \neq 0$. So it is not enough that the origin is an UVSFP, the coupling constant must also lie in its domain of attraction if a theory is to exhibit asymptotic freedom. The fact that if g lies in region I the theory is asymptotically free is a direct consequence of eq. (27). Fig. b) describes what might happen in Q.E.D., an infrared stable theory, if it posseses an UVSFP.

In Fig. c) we illustrate what we hope takes place in Q.C.D. In this case g is always in the domain of attraction of the UVSFP at the origin and in addition there is no IRSFP .

Let us now consider the calculation, in Q.C.D., of the universal functions introduced above in the MS-scheme. Recall that μ corresponds to ν in this scheme. From (III.55) we can write

$$\alpha \equiv Z_\alpha^{-1} \alpha_0 \quad , \quad Z_\alpha \equiv \tilde{Z}_1^{2} \tilde{Z}_3^{-2} Z_{3YM}^{-1} \tag{IV.33}$$

Using (III.83) to (III.85) we find

$$Z_\alpha = 1 + \frac{\alpha}{\pi}\left[\frac{11}{12} C_2(G) - \frac{T(R)N_f}{3}\right]\frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2\left\{\left[\frac{121}{144} C_2^2(G) - \frac{11}{18} C_2(G)T(R)N_f \right.\right.$$

$$\left.\left. + \frac{T^2(R)N_f^2}{9}\right]\frac{1}{\epsilon^2} + \left[\frac{17}{48} C_2^2(G) - \frac{5}{24} C_2(G)T(R)N_f - \frac{1}{8} C_2(R)T(R)N_f\right]\frac{1}{\epsilon}\right\} \tag{IV.34}$$

where the fact that Z_α is independent of the gauge parameter is true to all orders in perturbation theory in the MS and $\overline{\text{MS}}$ schemes [CW 74] [TV 72] . Then we can write (13)

$$\alpha\beta(\alpha;\epsilon) = \nu \frac{d\alpha}{d\nu} \tag{IV.35}$$

and hence using (33) and recalling that α_0 is proportional to $\nu^{2\epsilon}$

$$\alpha\beta(\alpha,\epsilon) = 2\epsilon\alpha - \alpha\frac{\nu}{Z_\alpha}\frac{\partial Z_\alpha}{\partial\nu}$$

We keep here the $\alpha\epsilon$ term which only appears in schemes where the subtraction point is the regularization mass scale, because there is no finite order α term. Since all the ν dependence in Z_α is due to its dependence on the renormalized coupling constant, we can write using (35)

$$\alpha \beta(\alpha;\epsilon) = 2\epsilon\alpha - \alpha^2 \beta(\alpha;\epsilon) \frac{1}{Z_\alpha} \frac{\partial Z_\alpha}{\partial \alpha} \tag{IV.36}$$

where Z_α can be expanded as

$$Z_\alpha = 1 + \sum_{m=1}^{\infty} \frac{Z_{\alpha,m}(\alpha)}{\epsilon^m} \tag{IV.37}$$

Then, since Q.C.D. is a renormalizable theory, we have

$$\alpha \beta(\alpha;\epsilon) = 2\alpha\epsilon + \alpha \beta(\alpha) \tag{IV.38}$$

Substituting (37) and (38) in eq. (36) and equalling identical powers in ϵ we obtain

$$\beta(\alpha) = -2\alpha \frac{\partial Z_{\alpha,1}(\alpha)}{\partial \alpha}$$

(IV.39)

$$2\alpha \frac{\partial Z_{\alpha,m+1}(\alpha)}{\partial \alpha} = -\beta(\alpha) \left\{ Z_{\alpha,m}(\alpha) + \alpha \frac{\partial Z_{\alpha,m}(\alpha)}{\partial \alpha} \right\} \qquad n = 1,2\cdots$$

where the first equation can be used to calculate $\beta(\alpha)$ and the second ones must be satisfied identically and they guarantee that $\beta(\alpha;\epsilon)$ has no poles in ϵ. Using (34) we obtain

$$\beta(\alpha) = \frac{\alpha}{\pi}\beta_1 + \left(\frac{\alpha}{\pi}\right)^2 \beta_2 + \cdots$$

$$\beta_1 = -\frac{11}{6} C_2(G) + \frac{2}{3} T(R) N_f$$

$$\beta_2 = \frac{1}{2}\left[-\frac{17}{6} C_2^2(G) + \frac{5}{3} C_2(G) T(R) N_f + C_2(R) T(R) N_f \right] \tag{IV.40}$$

$$\beta_3 = \frac{1}{32}\Big[-\frac{2857}{54} C_2^3(G) + \frac{1415}{27} C_2^2(G)\, T(R)\, N_f - \frac{158}{27} C_2(G)\, T^2(R)\, N_f^2$$

$$+ \frac{205}{9} C_2(G)\, C_2(R)\, T(R)\, N_f - \frac{44}{9} C_2(R)\, T^2(R)\, N_f^2 - 2\, C_2^2(R)\, T(R)\, N_f \Big]$$

The β_2 has been calculated in [CA 74] [JO 74] and β_3 is given in [TV 80]. The second equation of (39) , for n = 1 , allows us to check the $(\alpha/\pi)^2 (1/\epsilon^2)$ term in the expression for Z_α given in (34).

Let us now proceed to the calculation of $\delta(\alpha, a)$. Let us recall that

$$a = Z_{3YM}^{-1}\, a_0 \tag{IV.41}$$

where the result (III.72) has been used. Furthermore

$$a\, \delta(\alpha, a) = \nu \frac{da}{d\nu} \tag{IV.42}$$

Hence

$$a\, \delta(\alpha, a) = -\frac{a}{Z_{3YM}}\, \nu \frac{d Z_{3YM}}{d\nu} \tag{IV.43}$$

and all the dependence of Z_{3YM} in ν comes from its dependence in α and a . Then

$$\delta(\alpha, a) = -\frac{1}{Z_{3YM}} \left(\beta(\alpha;\epsilon)\, \alpha \frac{\partial Z_{3,YM}}{\partial \alpha} + \delta(\alpha, a)\, a \frac{\partial Z_{3,YM}}{\partial a} \right) \tag{IV.44}$$

The fact that Q.C.D. is renormalizable tells us that $\delta(\alpha,a)$ is independe of ϵ . Using an expansion analogous to (37) we find immediately

$$\delta(\alpha, a) = -2\alpha \frac{\partial Z_{3YM,1}(\alpha, a)}{\partial \alpha} \tag{IV.45}$$

$$2\alpha \frac{\partial Z_{3YM,n+1}(\alpha,a)}{\partial\alpha} = -\left[\delta(\alpha,a) + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha} + \delta(\alpha,a)\, a\frac{\partial}{\partial a} \right] Z_{3YM,n}(\alpha,a)$$

$$n = 1,2,3\ldots$$

As before the first equation allows us to compute $\delta(\alpha,a)$ and the second ones must be identities which confirm that δ is independent of ϵ . Using (III.83) we have

$$\delta(\alpha,a) = \frac{\alpha}{\pi}\,\delta_1(a) + \left(\frac{\alpha}{\pi}\right)^2 \delta_2(a) + \ldots$$

$$\delta_1(a) = \frac{C_2(G)}{4}\left(\frac{13}{3} - a\right) - \frac{2}{3}T(R)N_f \tag{IV.46}$$

$$\delta_2(a) = \frac{C_2^2(G)}{32}\left(\frac{59}{2} - \frac{11}{2}a - a^2\right) - \frac{1}{2}C_2(R)T(R)N_f - \frac{5}{8}C_2(G)T(R)N_f$$

In order to compute $\gamma(\alpha,a)$ we must take into account

$$m = Z_m^{-1} m_0 \quad , \qquad Z_m = Z_4 Z_{2F}^{-1} \tag{IV.47}$$

where Z_m is calculated in [TA 81] at the two loop level.

$$Z_m = 1 + \frac{\alpha}{\pi}\frac{3}{4}C_2(R)\frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2 \Big\{ \left[\frac{9}{32}C_2^2(R) + \frac{11}{32}C_2(R)C_2(G) - \frac{1}{8}C_2(R)T(R)N_f \right]\frac{1}{\epsilon^2} + \left[\frac{3}{64}C_2^2(R) + \frac{97}{192}C_2(R)C_2(G) - \frac{5}{48}C_2(R)T(R)N_f \right]\frac{1}{\epsilon} \Big\} \tag{IV.48}$$

The fact that Z_m is independent of the gauge parameter is true to all orders in perturbation theory [CW 74][TV 72] . Furthermore

$$\gamma(\alpha) = -\frac{\nu}{m}\frac{dm}{d\nu} \tag{IV.49}$$

Proceeding as before

$$\gamma(\alpha) = 2\alpha \frac{\partial Z_{m,1}(\alpha)}{\partial\alpha}$$

(IV.50)

$$2\alpha \frac{\partial Z_{m,m+1}(\alpha)}{\partial\alpha} = \left[\gamma(\alpha) - \alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right] Z_{m,m}(\alpha) \quad , n = 1,2,3,\ldots$$

Hence

$$\gamma(\alpha) = \frac{\alpha}{\pi}\gamma_1 + \left(\frac{\alpha}{\pi}\right)^2 \gamma_2 + \ldots$$

$$\gamma_1 = \frac{3}{2} C_2(R) \qquad \text{(IV.51)}$$

$$\gamma_2 = \frac{3}{16} C_2^2(R) + \frac{97}{48} C_2(R) C_2(G) - \frac{5}{12} C_2(R) T(R) N_f$$

We will also be interested in the anomalous dimensions of renormalization constants defined in general as

$$\gamma_Z \equiv \frac{\nu}{Z}\frac{dZ}{d\nu} \qquad \text{(IV.52)}$$

i.e.

$$Z\gamma_Z(\alpha,a) = \beta(\alpha;\epsilon)\,\alpha\frac{\partial Z}{\partial\alpha} + \delta(\alpha,a)\,a\frac{\partial Z}{\partial a} \qquad \text{(IV.53)}$$

and using for Z expansions as (37)

$$\gamma_Z(\alpha,a) = 2\alpha \frac{\partial Z_1(\alpha,a)}{\partial\alpha}$$

(IV.54)

$$2\alpha \frac{\partial Z_{m+1}(\alpha,a)}{\partial\alpha} = \left[\gamma_Z(\alpha,a) - \beta(\alpha)\,\alpha\frac{\partial}{\partial\alpha} - \delta(\alpha,a)\,a\frac{\partial}{\partial a}\right] Z_m(\alpha,a) \quad , n = 1,2,3\ldots$$

where the first equation allows us to calculate $\gamma_Z(\alpha, a)$ and the other equations allows us to check that the anomalous dimension is independent of ϵ . If

$$\gamma_Z(\alpha, a) = \frac{\alpha}{\pi}\gamma_{Z,1}(a) + \left(\frac{\alpha}{\pi}\right)^2 \gamma_{Z,2}(a) + \cdots \tag{IV.55}$$

we have for the anomalous dimensions defined in (13)

$$\begin{aligned} \gamma_{F,1}(a) &= \frac{1}{2} C_2(R)\, a \\ \gamma_{F,2}(a) &= \frac{C_2(R)}{16}\left[C_2(G)\left(\frac{25}{2} + 4a + \frac{1}{2}a^2\right) - 3C_2(R) - 4T(R)N_f\right] \end{aligned} \tag{IV.56}$$

$$\begin{aligned} \tilde{\gamma}_1(a) &= C_2(G)\frac{1}{8}(-3 + a) \\ \tilde{\gamma}_2(a) &= \frac{C_2(G)}{48}\left[C_2(G)\frac{1}{8}(-95 - 3a) + 5\,T(R)N_f\right] \end{aligned} \tag{IV.57}$$

$$\begin{aligned} \gamma_{YM,1}(a) &= C_2(G)\frac{1}{4}\left(-\frac{13}{3} + a\right) + \frac{2}{3}T(R)N_f \\ \gamma_{YM,2}(a) &= \frac{C_2^2(G)}{32}\left(-\frac{59}{2} + \frac{11}{2}a + a^2\right) + \frac{1}{2}C_2(R)T(R)N_f + \frac{5}{8}C_2(R)T(R)N_f \end{aligned} \tag{IV.58}$$

Notice that $\gamma_{YM}(\alpha, a) \equiv -\delta(\alpha, a)$, since $Z_6 = 1$.

Before going on we would like to prove some properties of the β-function. We will begin proving that β is gauge independent in the MS scheme [CW 74] [GR 76] . Let us start from a dimensionless Green's function Γ associated to any physical process like an S-matri element. It will therefore be renormalization point and gauge independent. Then we can write the following equations in the MS-scheme

$$\left[\nu\frac{\partial}{\partial\nu}+\beta\alpha\frac{\partial}{\partial\alpha}+\delta a\frac{\partial}{\partial a}-\gamma x\frac{\partial}{\partial x}\right]\Gamma_R(\alpha,a,x)=0$$

$$\left[\frac{\partial}{\partial a}+\rho\alpha\frac{\partial}{\partial\alpha}-\sigma x\frac{\partial}{\partial x}\right]\Gamma_R(\alpha,a,x)=0 \tag{IV.59}$$

where

$$\alpha\rho\equiv\frac{d\alpha}{da}\quad,\qquad x\sigma\equiv-\frac{dx}{da} \tag{IV.60}$$

where these derivatives are to be taken with α_o and x_o held fixed, and we have assumed a unique mass term $(x\equiv m/\nu)$ in order to simplify the notation. Let us now consider the commutator of the operators appearing in (59) applied to the renormalized Green function

$$\left\{\left[\frac{\partial(\alpha\beta)}{\partial a}+\alpha\rho\frac{\partial(\alpha\beta)}{\partial\alpha}-\alpha\beta\frac{\partial(\alpha\rho)}{\partial\alpha}-\alpha\delta\frac{\partial(\alpha\rho)}{\partial a}\right]\frac{\partial}{\partial\alpha}\right.$$

$$+\left[\frac{\partial(a\delta)}{\partial a}+\alpha\rho\frac{\partial(a\delta)}{\partial\alpha}\right]\frac{\partial}{\partial a}$$

$$\left.+\left[-\frac{\partial\gamma}{\partial a}-\alpha\rho\frac{\partial\gamma}{\partial\alpha}+\alpha\beta\frac{\partial\sigma}{\partial\alpha}+a\delta\frac{\partial\sigma}{\partial a}\right]x\frac{\partial}{\partial x}\right\}\Gamma_R(\alpha,a,x)=0 \tag{IV.61}$$

where we have used $\partial\beta/\partial x=\partial\gamma/\partial x=\partial\delta/\partial x=\partial\sigma/\partial x=\partial\rho/\partial x=0$ because the UV divergences, which determine β, γ and δ, as well as σ and ρ, are mass-independent as they are all logarithmic. Using this equation we can eliminate $\partial\Gamma/\partial a$ in the second equation of (59) and we get

$$\left\{\left[\left(\frac{\partial}{\partial a}+\alpha\rho\frac{\partial}{\partial\alpha}\right)(\beta-\rho a\delta)-(\beta-\rho a\delta)\alpha\frac{\partial\rho}{\partial\alpha}\right]\alpha\frac{\partial}{\partial\alpha}\right.$$

$$\left.-\left[\left(\frac{\partial}{\partial a}+\alpha\rho\frac{\partial}{\partial\alpha}\right)(\gamma-\sigma a\delta)-(\beta-\rho a\delta)\alpha\frac{\partial\sigma}{\partial\alpha}\right]x\frac{\partial}{\partial x}\right\}\Gamma_R(\alpha,a,x)=0 \tag{IV.62}$$

But Γ_R obeys only equations (59) since we have not imposed further conditions on the Green's function and therefore (62) must be trivially satisfied and hence

$$\tilde{D}\tilde{\beta} - \tilde{\beta}\alpha \frac{\partial \rho}{\partial \alpha} = 0 \quad , \quad \tilde{D}\tilde{\gamma} - \tilde{\beta}\alpha \frac{\partial \sigma}{\partial \alpha} = 0$$

$$\tilde{D} \equiv \frac{\partial}{\partial a} + \alpha\rho\frac{\partial}{\partial \alpha} \quad , \quad \tilde{\beta} \equiv \beta - \rho a \delta \quad , \quad \tilde{\gamma} \equiv \gamma - \sigma a \delta \qquad \text{(IV.63)}$$

and therefore if $\partial \Gamma_R / \partial a$ is obtained from the second equation (59) and substituted in the first we get

$$\left[\nu \frac{\partial}{\partial \nu} + \tilde{\beta}\alpha \frac{\partial}{\partial \alpha} - \tilde{\gamma}\, x \frac{\partial}{\partial x} \right] \Gamma_R (\alpha, a, m) = 0 \qquad \text{(IV.64)}$$

which shows that the physical consequences of the renormalization grou equations are gauge independent.

In the MS-scheme

$$\alpha_0 = Z_\alpha \alpha = \alpha \left\{ 1 + \sum_{m=1}^{\infty} \frac{Z_{\alpha,m}}{\epsilon^m} \right\}$$

and therefore

$$\rho = -\frac{1}{Z_\alpha}\frac{d Z_\alpha}{d a} = -\frac{1}{Z_\alpha}\left\{ \frac{\partial Z_{\alpha,1}}{\partial a}\frac{1}{\epsilon} + \frac{\partial Z_{\alpha,2}}{\partial a}\frac{1}{\epsilon^2} + \cdots \right\}$$

Then

$$\rho \left(1 + \frac{Z_{\alpha,1}}{\epsilon} \right) = -\frac{\partial Z_{\alpha,1}}{\partial a}\frac{1}{\epsilon} + O(1/\epsilon^2)$$

This equation is true if and only if $\rho = 0$, since it is independent of ϵ. Using now eq. (63)

$$\frac{\partial \beta}{\partial a} = 0$$

and therefore β must be gauge independent. A similar reasoning proves that $\sigma = 0$ and also γ turns out to be gauge independent.

Notice that in general the coefficients of these universal functions, when expanded in powers of (α/π), are renormalization scheme dependent but later on we will prove that this is not so for β_1 which is scheme independent for a large class of schemes, and even for β_2 for a smaller class of schemes.

Let us now calculate in the MS-scheme and at the two loop level the running constants defined by eq. (20). Let us start with the running coupling constant at the one loop level; we must solve the equation

$$\frac{d\bar{\alpha}(t)}{dt} = \frac{\beta_1}{\pi}\bar{\alpha}^2(t) \quad , \quad \bar{\alpha}(0) = \alpha \quad , \quad \beta_1 = -\frac{11}{6}C_2(G) + \frac{2}{3}T(R)N_f \tag{IV.65}$$

Notice that for three colors β_1 has a zero in $N_f = 33/2$ and it is negative if the number of flavors is $N_f \leq 16$. From (65) we obtain immediately

$$\bar{\alpha}(t) = \frac{\alpha}{1 - \frac{\alpha}{\pi}\beta_1 t} \tag{IV.66}$$

Notice that the behaviour of $\bar{\alpha}(t)$ depends critically of the sign of β_1 : If $N_f \leq 16$ then $\beta_1 < 0$ and $\bar{\alpha}(t)$ decreases for increasing values of t, while if $N_f > 16$, $\beta_1 > 0$ and $\bar{\alpha}(t)$ increases for increasing values of t. If $\beta_1 < 0$ the theory will be asymptotically free: In that case the Green's functions when all external momenta are scaled by a common factor increasing to infinity are determined by a theory where $\bar{\alpha}(t) \longrightarrow 0$, as can be seen immediately from eq. (28).

In Q.E.D. the value of β_1 turns out to be $\beta_1 = 2/3$ $(C_2(G) \equiv 0\ ,\ C_2(R) \equiv T(R)N_f \equiv 1)$ and therefore it is not asymptotically free. At shorter and shorter distances the effective coupling constants of QED becomes larger and larger and eventually it may enter a non-perturbative regime. The high precision measurements of (g-2) for the muon test the photon propagator at short distances and these results can be thought as a measurement of $\bar{\alpha}_{QED}(t)$ and they agree with the theoretical predictions [LR 74][CN 77]. On the other hand the theory is stable at long distances in the sense that at low energies there is a smooth transition of the quantized perturbation theory to the classical theory. This is guaranteed by Thirring's low energy theorem [TH 50] which states that order by order in renormalized perturbation theory, with on-shell renormalization, the Compton scattering cross-section for photons with incident energies smaller than the electron mass coincides with the classical Thomson cross section. In fact it is this limit which in Q.E.D. provides an interpretation of th on-shell renormalized coupling constant. The on-shell renormalized coupling constant and the on-shell renormalized mass are respectively identified with the classical charge and mass of the electron.

In Q.C.D. we do not know of classical observables (equivalent to α_{QED} and m in Q.E.D.) in terms of which we can express the renormalized parameters of the theory. How do we fix the renormalized parameters? In order to show that this is not an academic question let us consider the following situation: suppose that in order to compare the results of an experiment, say experiment 1, with the theory we have fixed the renormalized parameters subtracting at an Euclidean point ν_1 :

Experiment 1 : parametrized by $\alpha(\nu_1)$, $m(\nu_1)$

Somebody else who wants to compare the results of another experiment, say experiment 2, with theory may find it more convenient to use another set of renormalized parameters

Experiment 2 : parametrized by $\alpha(\nu_2)$, $m(\nu_2)$

It is now clear that in order to be able to predict something for the experiment 2 from the results of experiment 1 we must know how to express the parameters $(\alpha(\nu_2), m(\nu_2))$ in terms of $(\alpha(\nu_1), m(\nu_1))$. This is precisely what the renormalization group does for us. The crucial point in relation with α is equation (20)

$$\frac{d\bar{\alpha}(t)}{dt} = \bar{\alpha}(t)\ \beta\,[\bar{\alpha}(t)] \tag{IV.67}$$

which can be written as

$$t = \int_{\alpha}^{\bar{\alpha}(t)} dz\ \frac{1}{z\,\beta(z)} \equiv \psi\,[\bar{\alpha}(t)] + \text{const.} \tag{IV.68}$$

which is an implicit expression for $\bar{\alpha}(t)$ in terms of t and $\alpha(\nu)$. Furthermore from the equation which defines the β-function (13) we obtain

$$\frac{1}{2}\ln\nu^2 = \psi\,[\alpha] + \text{cte} \tag{IV.69}$$

This implies that

$$\frac{1}{2}\ln\nu^2 - \psi\,[\alpha] = \text{const} \equiv \frac{1}{2}\ln\Lambda^2 \tag{IV.70}$$

must be renormalization group invariant since it is precisely the arbitrary constant of the general solution of the differential equation. At the one loop level $\beta(z) = z\beta_1/\pi$, and therefore $\psi(z) = -\pi/z\beta_1$, and hence

$$\frac{1}{2}\ln\nu^2 + \frac{\pi}{\beta_1\,\alpha(\nu)} = \frac{1}{2}\ln\Lambda^2 \qquad \text{(IV.71)}$$

where Λ is renormalization group invariant. Instead of parametrizing the physical observables in terms of $\alpha(\nu)$, which depends on the arbitrary choice of the renormalization point ν , we can use Λ which is independent of ν . Writing

$$t \equiv \frac{1}{2}\ln\left(-\frac{q^2}{\nu^2}\right) \qquad \text{(IV.72)}$$

we obtain, using (66) ,

$$\bar{\alpha}\left[\frac{1}{2}\ln\left(-\frac{q^2}{\nu^2}\right), \alpha(\nu)\right] = \frac{\pi}{-\frac{1}{2}\beta_1\ln\left(-\frac{q^2}{\Lambda^2}\right)} \qquad \text{(IV.73)}$$

This effective coupling constant is renormalization group invariant. Clearly this expression is a good approximation only if $(-q^2)\gg\Lambda^2$ and $\beta_1 < 0$. Notice that for $q^2 = -\nu^2$ this effective coupling constant coincides with $\alpha(\nu)$.

Let us study now the higher order corrections to eq. (73). Before proceeding we would like to prove that the coefficients β_1 and β_2 in the expansion of $\beta(\alpha)$ in power series of α/π are renormalization group invariants for classes of schemes where they do not depend on m nor on a [LR 81] [ET 82a]. Let $\beta_I(\alpha_I)$ and $\beta_{II}(\alpha_{II})$ be the β functions calculated in two different renormaliza tion schemes. Since by hypothesis β does not depend on m nor a the first of eqs. (20) decouples, and can be integrated separately. Thus only one integration constant, Λ, enters. Λ of course depends on the scheme. If we introduce the dimensionless scales $\hat{\nu}_I \equiv \nu/\Lambda_I$ and $\hat{\nu}_{II} \equiv \nu/\Lambda_{II}$, one can then relate the coupling constants of both schemes expanding one in terms of the other

$$\alpha_{II}(\hat{\nu}_{II}) = \alpha_I(\hat{\nu}_I)\left\{1 + A\frac{\alpha_I(\hat{\nu}_I)}{n} + B\left(\frac{\alpha_I(\hat{\nu}_I)}{n}\right)^2 + C\left(\frac{\alpha_I(\hat{\nu}_I)}{n}\right)^3 + \cdots\right\} \quad \text{(IV.74)}$$

Notice that the coefficients A,B,C, etc do not depend on $\hat{\nu}_I$, because all the dependence in ν of Z_α comes in only through $\alpha(\nu)$. Now acting with $\nu\, d/d\nu$ on (74) one gets

$$\alpha_{II}\beta_{II}(\alpha_{II}) = \alpha_I\beta_I(\alpha_I)\left\{1 + 2A\frac{\alpha_I}{n} + 3B\left(\frac{\alpha_I}{n}\right)^2 + 4C\left(\frac{\alpha_I}{n}\right)^3 + \cdots\right\} \quad \text{(IV.75)}$$

and recalling the expansion of β in powers of α/n one finds with the help of (74)

$$\begin{aligned} \beta_{II1} &= \beta_{I1} \equiv \beta_1 \\ \beta_{II2} &= \beta_{I2} \equiv \beta_2 \\ \beta_{II3} &= \beta_{I3} - A\beta_2 - A^2\beta_1 + B\beta_1 \end{aligned} \quad \text{(IV.76)}$$

which is the desired result. Let us stress again that it is only true for renormalization schemes for which the renormalized coupling constan is gauge parameter independent. This does not happen, e.g., for Weinberg's scheme. The study of the renormalization group invariant coefficients for other renormalization group functions can be found in [ET 82a].

It has been observed by 't Hooft [TH 77] that if $\beta(\alpha)$ has no fixed point in $(0,\infty)$ then it is always possible to find a renormalization prescription such that $\beta_n = 0$ for $n > 2$.

The precision of the deep inelastic scattering experiments nowadays implies that many theoretical results are needed at the two loop level. In particular we need to know at the two loop level the effective coupling constant. We have

$$\psi(z) = \int \frac{dz}{z\,\beta(z)} = \frac{n}{\beta_1}\int dz \frac{1}{z^2\left(1+\frac{\beta_2}{\beta_1}\frac{z}{n}\right)} =$$

$$= \frac{n}{\beta_1}\int dz \left\{ \frac{1}{z^2} - \frac{\beta_2}{\beta_1}\frac{1}{n}\frac{1}{z} + \frac{\beta_2^2}{\beta_1^2}\frac{1}{n^2}\frac{1}{1+\frac{\beta_2}{\beta_1}\frac{1}{n}z} \right\}$$

i.e.

$$\psi(z) = \frac{n}{\beta_1}\left\{ -\frac{1}{z} + \frac{\beta_2}{\beta_1}\frac{1}{n}\ln\frac{1+\frac{\beta_2}{\beta_1}\frac{1}{n}z}{z} \right\} \tag{IV.77}$$

and we can write as before

$$\frac{1}{2}\ln \nu^2 - \psi[\alpha(\nu)] = \frac{1}{2}\ln\Lambda^2 \tag{IV.78}$$

where Λ is renormalization group invariant. Later on we will discuss the role of the constant that can be added to the r.h.s. of (77). Now we will proceed to determine $\bar{\alpha}(t)$ at the two loop level. From (68)

$$t = \psi[\bar{\alpha}(t,\alpha(\nu))] - \psi[\alpha(\nu)] \tag{IV.79}$$

Let us expand $\psi(z)$ given in eq. (77) in power series in z neglecting terms $O(z)$. It would be meaningless to keep the terms of $O(z)$ since they depend on β_3 and we are calculating at the two loop level

$$\psi(z) = \frac{n}{\beta_1}\left\{ -\frac{1}{z} - \frac{\beta_2}{\beta_1}\frac{1}{n}\ln z + \ldots \right\} \tag{IV.80}$$

Hence

$$t = \frac{n}{\beta_1}\left\{ \frac{1}{\alpha(\nu)} - \frac{1}{\bar{\alpha}(t)} \right\} - \frac{\beta_2}{\beta_1^2}\ln\frac{\bar{\alpha}(t)}{\alpha(\nu)} \tag{IV.81}$$

From (66) we have that at the one loop level

$$\bar{\alpha}^{(2)}(t) = \frac{\alpha(\nu)}{1 - \frac{\alpha(\nu)}{n}\beta_1 t} \tag{IV.82}$$

which can be used in the logarithmic term of (81) and we obtain

$$\bar{\alpha}(t) = \bar{\alpha}^{(2)}(t)\left\{ 1 + \bar{\alpha}^{(2)}(t)\frac{\beta_2}{\beta_1}\frac{1}{n}\ln\left(1 - \frac{\alpha(\nu)}{n}\beta_1 t\right)\right\}^{-1} \tag{IV.83}$$

and therefore at the order that we are interested in we have

$$\bar{\alpha}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}, \alpha(\nu)\right] = \bar{\alpha}^{(2)}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}, \alpha(\nu)\right]\left\{ 1 - \right.$$

$$\left. - \bar{\alpha}^{(2)}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}, \alpha(\nu)\right]\frac{\beta_2}{\beta_1}\frac{1}{n}\ln\left[1 - \frac{\alpha(\nu)}{n}\beta_1\frac{1}{2}\ln\frac{-q^2}{\nu^2}\right]\right\} \tag{IV.84}$$

$$\bar{\alpha}^{(2)}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}, \alpha(\nu)\right] = \frac{\alpha(\nu)}{1 - \frac{\alpha(\nu)}{n}\beta_1\frac{1}{2}\ln\frac{-q^2}{\nu^2}}$$

which is the exact sum of the leading and next to leading terms in the perturbative expansion.

The combination of $\alpha(\nu)$ and ν which is renormalization group invariant can be choosen as [BF 77]

$$\frac{1}{2}\ln\nu^2 + \frac{\Pi}{\beta_1}\frac{1}{\alpha(\nu)} - \frac{\beta_2}{\beta_1^2}\ln\frac{1+\frac{\beta_2}{\beta_1}\frac{1}{\Pi}\alpha(\nu)}{\alpha(\nu)} =$$

$$= \frac{1}{2}\ln\Lambda^2 - \frac{\beta_2}{\beta_1^2}\ln\left(-\frac{\beta_1}{\Pi}\right) \tag{IV.85}$$

where a convenient constant has been added to (77). Introducing (73)

$$\bar{\alpha}^{(2)}(q^2/\Lambda^2) \equiv \frac{\Pi}{-\beta_1\frac{1}{2}\ln\left(-\frac{q^2}{\Lambda^2}\right)} \tag{IV.86}$$

equation (84) can be written as

$$\bar{\alpha}^{(2)}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}\,,\,\alpha(\nu)\right] = \alpha(\nu)\left\{1 - \frac{\alpha(\nu)}{\Pi}\beta_1\frac{1}{2}\ln(-q^2) + \frac{\alpha(\nu)}{\Pi}\beta_1\frac{1}{2}\ln\nu^2\right\}^{-1} =$$

$$= \alpha(\nu)\left\{1 - \frac{\alpha(\nu)}{\Pi}\beta_1\frac{1}{2}\ln(-q^2) + \frac{\alpha(\nu)}{\Pi}\beta_1\left[\frac{1}{2}\ln\Lambda^2 - \frac{\beta_2}{\beta_1^2}\ln\left(-\frac{\beta_1}{\Pi}\right)\right.\right.$$

$$\left.\left. - \frac{\Pi}{\beta_1}\frac{1}{\alpha(\nu)} + \frac{\beta_2}{\beta_1^2}\ln\frac{1+\frac{\beta_2}{\beta_1}\frac{1}{\Pi}\alpha(\nu)}{\alpha(\nu)}\right]\right\}^{-1} =$$

$$= \alpha(\nu)\left\{-\frac{\alpha(\nu)}{\Pi}\beta_1\frac{1}{2}\ln\frac{-q^2}{\Lambda^2} + \frac{\alpha(\nu)}{\Pi}\frac{\beta_2}{\beta_1}\ln\frac{1+\frac{\beta_2}{\beta_1}\frac{\alpha(\nu)}{\Pi}}{-\beta_1\frac{\alpha(\nu)}{\Pi}}\right\}^{-1}$$

At the level that we are interested in we can expand the last logarithm and we get

$$\bar{\alpha}^{(2)}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}\,,\,\alpha(\nu)\right] = \frac{\Pi}{-\beta_1\frac{1}{2}\ln\frac{-q^2}{\Lambda^2} + \frac{\beta_2}{\beta_1}\ln\left(-\frac{\Pi}{\beta_1\alpha(\nu)}\right)} \tag{IV.87}$$

and expanding in powers of $1/\ln(-q^2)$

$$\bar{\alpha}^{(2)}\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}\,,\,\alpha(\nu)\right] = \bar{\alpha}^{(2)}(q^2/\Lambda^2)\left\{1 - \bar{\alpha}^{(2)}(q^2/\Lambda^2)\frac{1}{\Pi}\frac{\beta_2}{\beta_1}\ln\left(-\frac{\Pi}{\beta_1\alpha(\nu)}\right)\right\} \tag{IV.88}$$

Furthermore for the logarithm that appears in (84) we can write

$$\ln\left[1 - \frac{\alpha(\nu)}{\pi}\beta_1 \frac{1}{2}\ln\frac{-q^2}{\nu^2}\right] = \ln\left\{1 - \frac{\alpha(\nu)}{\pi}\beta_1\frac{1}{2}\ln(-q^2) + \right.$$

$$\left. + \frac{\alpha(\nu)}{\pi}\beta_1\left[\frac{1}{2}\ln\Lambda^2 - \frac{\beta_2}{\beta_1^2}\ln\left(-\frac{\beta_1}{\pi}\right) - \frac{\pi}{\beta_1}\frac{1}{\alpha(\nu)} + \frac{\beta_2}{\beta_1^2}\ln\frac{1+\frac{\beta_2}{\beta_1}\frac{1}{\pi}\alpha(\nu)}{\alpha(\nu)}\right]\right\} =$$

$$= \ln\left\{-\frac{\alpha(\nu)}{\pi}\beta_1\frac{1}{2}\ln\frac{-q^2}{\Lambda^2} + \frac{\alpha(\nu)}{\pi}\frac{\beta_2}{\beta_1}\ln\left(-\frac{\pi}{\beta_1\alpha(\nu)}\right)\right\} =$$

$$= \ln\left(-\frac{\alpha(\nu)}{\pi}\beta_1\right) + \ln\left(\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right) + \cdots \qquad \text{(IV.89)}$$

Using (88) and (89) the first equation of (84) can be written as

$$\bar{\alpha}_s\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}, \alpha(\nu)\right] = \bar{\alpha}^{(2)}(q^2/\Lambda^2)\left\{1 - \right.$$

$$\left. - \frac{\beta_2}{\beta_1}\frac{1}{\pi}\bar{\alpha}^{(2)}(q^2/\Lambda^2)\ln\left(\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right)\right\} \qquad \text{(IV.90)}$$

$$\bar{\alpha}^{(2)}(q^2/\Lambda^2) \equiv \frac{\pi}{-\beta_1\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}}$$

which is the desired result, i.e. the effective coupling constant at the two loop level. Notice that $\bar{\alpha}(t)$ at the one or the two loop levels are such that they go to infinity when $(-q^2) \to \Lambda^2$. This is the limit in which perturbation theory makes no sense. Therefore perturbation theory can not give any information for $q^2 \to 0$ and therefore it is unable to answer the question of the possibility of confinement. The great advantage of $\bar{\alpha}(t)$ over $\alpha(\nu)$ for large value of $(-q^2)$ is that while the expansion in powers of $\alpha(\nu)$ is really an expansion in powers of $\alpha(\nu)\ln(-q^2/\nu^2)$ and therefore is meaninless, the expansion in powers of $\bar{\alpha}(t)$ has no logarithmic terms and therefore is meaningful at high energies where $\bar{\alpha}(t) \to 0$. The $\bar{\alpha}(t)$ is really the summation of the terms $\alpha(\nu)\ln(-q^2/\nu^2)$ at leading order.

Let us now turn our attention to the effective mass parameter In the MS-scheme we have (20)

$$\frac{d\bar{x}_A(t)}{dt} = -\left[1 + \gamma\left[\bar{\alpha}(t)\right]\right]\bar{x}_A(t) \qquad , \quad \bar{x}_A(0) = x_A \tag{IV.91}$$

and therefore

$$\bar{x}_A(t) = x_A \exp\left\{ -\left[t + \int_0^t dt' \, \gamma\left[\bar{\alpha}(t', \alpha)\right]\right]\right\} \tag{IV.92}$$

At the one loop level we have, using (51) and (73)

$$\bar{x}_A(t) = x_A\left(-\frac{\nu^2}{q^2}\right)^{1/2} \exp\left\{ \frac{\gamma_1}{\beta_1}\int_0^t dt' \frac{1}{t' + \frac{1}{2}\ln\left(\frac{\nu^2}{\Lambda^2}\right)}\right\}$$

Performing this integral and taking into account (73) and the fact that $\bar{\alpha}\left[0, \alpha(\nu)\right] = \alpha(\nu)$ we obtain

$$\bar{x}_A(t) = x_A\left[\frac{\pi}{-\beta_1\,\alpha(\nu)}\right]^{\gamma_1/(-\beta_1)}\left(-\frac{\nu^2}{q^2}\right)^{1/2} \frac{1}{\left[\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right]^{\gamma_1/(-\beta_1)}} \tag{IV.93}$$

In the same way that we have introduced Λ we would like to introduc a renormalization group invariant mass parameter $\hat{m}_A$ defined, at the one loop level, as

$$\hat{m}_A = m_A(\nu)\left[\frac{\pi}{-\beta_1\,\alpha(\nu)}\right]^{\gamma_1/(-\beta_1)} \tag{IV.94}$$

and then we can write

$$\bar{x}_A\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}, x_A(\nu)\right] = \frac{\hat{m}_A}{\sqrt{-q^2}} \frac{1}{\left[\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right]^{\gamma_1/(-\beta_1)}} \tag{IV.95}$$

and if

$$\overline{m}_A(-q^2) \mp \sqrt{-q^2}\;\bar{x}_A\left[\frac{1}{2}\ln\frac{-q^2}{\nu^2}\;,\;x_A(\nu)\right] \tag{IV}$$

we obtain

$$\overline{m}_A(-q^2) = \hat{m}_A\;\frac{1}{\left[\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right]^{\gamma_1/(-\beta_1)}} \tag{IV}$$

which is the desired result. Notice that from (95) and with q^2 =
we get

$$\frac{m_A(\nu)}{m_B(\nu)} = \frac{\hat{m}_A}{\hat{m}_B} \tag{IV}$$

i.e. the ratio of the renormalized masses appearing in the Lagran
density is equal to the ratio of invariant masses.

Let us now prove that $\hat{m}_A$ is renormalization group inva
We can write (13)

$$\frac{\nu}{m_A}\;\frac{d\,m_A}{d\nu} = -\,\gamma\left[\alpha(\nu)\right]$$

and at the one loop level

$$\int\frac{d\,m_A}{m_A} = -\,\frac{\gamma_1}{\pi}\int d\nu\;\frac{\alpha(\nu)}{\nu} + \text{const}$$

Using (73) with $q^2 = -\nu^2$

$$\ln m_A(\nu) = \frac{\gamma_1}{\beta_1}\int d(\nu/\Lambda)\;\frac{1}{(\nu/\Lambda)\;\ln(\nu/\Lambda)} + \text{const}$$

and writing the constant, which is renormalization group invariant, as $\ln \hat{m}_A$ we get

$$\ln \frac{m_A(\nu)}{\hat{m}_A} = \frac{\gamma_1}{\beta_1} \int d(\nu/\Lambda) \frac{1}{(\nu/\Lambda)\ln(\nu/\Lambda)} = \frac{\gamma_1}{\beta_1} \ln\left(\ln \frac{\nu}{\Lambda}\right) \qquad \text{(IV.99)}$$

which, if use is made of (73), can be written as (94).

At the two loop level and defining the invariant mass through the equation

$$\ln m_A(\nu) + \frac{\gamma_1}{\beta_1} \ln\left(-\frac{\beta_1 \alpha(\nu)}{\pi}\right) - \left(\frac{\beta_2 \gamma_1}{\beta_1^2} + \frac{\gamma_2}{-\beta_1}\right)\frac{\alpha(\nu)}{\pi} = \ln \hat{m}_A \qquad \text{(IV.100)}$$

we obtain [TA 81]

$$\overline{m}_A(-q^2) = \frac{\hat{m}_A}{\left[\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right]^{\gamma_1/(-\beta_1)}} \left\{ 1 + \frac{\gamma_2 - \frac{\gamma_1\beta_2}{\beta_1}}{\beta_1^2} \frac{1}{\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}} - \right.$$
$$\left. - \frac{\gamma_1 \beta_2 \ln\left[\frac{1}{2}\ln\frac{-q^2}{\Lambda^2}\right]}{\beta_1^3 \, \frac{1}{2}\ln\frac{-q^2}{\Lambda^2}} \right\} \qquad \text{(IV.101)}$$

It is interesting to compare this situation with the corresponding result for the effective coupling constant. In the later case since β_1 and β_2 are independent of the renormalization scheme(within a certain class of schemes) the value of Λ is independent of the substraction point and of the renormalization scheme. In the first case γ_1 is independent of the renormalization scheme but this is not so for γ_2 and therefore it is not clear if $\hat{m}_A$ at the two loop level is only renormalization point invariant or also renormalization scheme invariant if the dependence on the renormalization scheme of $m_A(\nu)$ and $\alpha(\nu)$ cancels that of γ_2. But even if it is not so, it is clear from (100) that the possible renormalization scheme dependence of $\hat{m}_A$ would be multiplicative. Since the same happens to the arbitrariness in its definition one can conclude that the ratio of invariant masses of different flavour $\hat{m}_A/\hat{m}_B$ is renormalization scheme independent and uniquely defined in Q.C.D.

Let us finally consider the effective gauge parameter. We must consider

$$\nu \frac{d a(\nu)}{d\nu} = a(\nu)\ \delta\left[\alpha(\nu), a(\nu)\right] \tag{IV.102}$$

where at the one loop level

$$\delta(\alpha, a) = \frac{\alpha}{\pi}\delta_1 \quad , \quad \delta_1 = \left(\frac{13}{12} C_2(G) - \frac{2}{3} T(R) N_f\right) - \frac{C_2(G)}{4} a \tag{IV.103}$$

and hence if $\tilde{\delta}_1 \equiv (13\ C_2(G)/12 - 2T(R)\ N_f/3\ )$

$$\frac{1}{\tilde{\delta}_1}\left(\frac{da}{a} - \frac{da}{a - \frac{4}{C_2(G)}\tilde{\delta}_1}\right) - \frac{1}{\pi}\frac{d\nu}{\nu}\alpha(\nu) = 0$$

Using (71)

$$\frac{1}{\tilde{\delta}_1} \ln \frac{a(\nu)}{a(\nu) - \frac{4}{C_2(G)}\tilde{\delta}_1} + \frac{1}{\beta_1}\ln\left(-\frac{\pi}{\beta_1 \alpha(\nu)}\right) = \ln \hat{a} \tag{IV.104}$$

where $\hat{a}$ is a renormalization group independent quantity. And hence

$$\hat{a} = \frac{a(\nu)}{1 - \frac{C_2(G)}{4\tilde{\delta}_1} a(\nu)}\left(-\frac{\pi}{\beta_1 \alpha(\nu)}\right)^{\tilde{\delta}_1/\beta_1}$$

Since $\bar{a}(-q^2 = \nu^2) = a(\nu)$

$$\bar{a}(q^2/\Lambda^2) = \frac{\hat{a}}{\left[\frac{1}{2}\ln\left(-\frac{q^2}{\Lambda^2}\right)\right]^{\delta}}\left\{1 + \frac{C_2(G)}{4\tilde{\delta}_1}\frac{\hat{a}}{\left[\frac{1}{2}\ln\left(-\frac{q^2}{\Lambda^2}\right)\right]^{\delta}}\right\}^{-1} \tag{IV.105}$$

$$\tilde{\delta}_1 \equiv \frac{13}{12} C_2(G) - \frac{2}{3} T(R) N_f \ , \qquad \delta \equiv -\frac{1}{2}\frac{13 C_2(G) - 4N_f}{11 C_2(G) - 2N_f}$$

Notice that

$$\bar{a}\,(q^2/\Lambda^2) \xrightarrow[-q^2/\Lambda^2 \to \infty]{} 0 \qquad \text{if} \qquad \delta > 0$$

$$\bar{a}\,(q^2/\Lambda^2) \xrightarrow[-q^2/\Lambda^2 \to \infty]{} \frac{4\,\tilde{\delta}_1}{C_2(G)} \qquad \text{if} \qquad \delta < 0 \tag{IV.106}$$

Notice that for $N=3$ and $N_f = 0$ $\delta = -39/66$. It increases with the number of flavors and takes the value $\delta \mp 0$ for $N_f = 39/4$. It still increases further until $\delta = +\infty$ for $N_f = 33/2$; for even larger values of N_f , $\delta < 0$.

There are two excepcional choices of the initial gauge for which

$$\bar{a}\,(q^2/\Lambda^2) = a \tag{IV.107}$$

the Landau gauge $a = 0$ and the peculiar gauge $a = 4\,\tilde{\delta}_1/C_2(G)$. There is a difference however between these two gauges. In the case of the Landau gauge $\bar{a} = 0$ to all orders in perturbation theory whereas this is non true for the other gauge.

V. RENORMALIZATION OF COMPOSITE OPERATORS. ANOMALIES.

Since only colourless states have been observed in Nature, it is clear that the successes of quantum gauge field theory in explaining the physical reality will be correlated to the understanding one develops of gauge invariant operators in field theory. Unfortunately gauge invariant operators are either composite local ones, or nonlocal path dependent ones. Since we will not give up in these notes the local approach, we need to extend the renormalization theory to composite operators. Of course, because of the short distance singularities which characterize the operator (or field) product expansion, the very definition of the local composite operator is a whole subject by itself [ZI 71],[ZI 73]. However, only insertions of composite operators into Green's functions are relevant, and composite local operators within Green's functions are no new subject: perturbation theory is an expansion in the coupling constant times a local composite operator, the interaction lagrangian. If perturbation theory is understood, there is absolutely no new ingredient in the understanding of a Green's function with an insertion of a local composite operator within perturbation theory. Now, exactly as the UV divergences associated with a given vertex in perturbation theory are not absorbed by the renormalization of the component fields of the interaction lagrangian, but a renormalization of the coupling constant is necessary, the same happens with composite operators: they may require an additional renormalization beyond the one of their component fields. This is the subject of this chapter.

In QCD there is a one-to-one correspondence of a bare field with its renormalized counterpart. This is because all the fields which enter the quantum action have different quantum numbers: they cannot be mixed under renormalization. This is no longer so with composite operators, there exist usually several of them with the same dimensions and quantum numbers. They therefore can and usually do mix under renormalization.

We will not present here the general theory of the renormalization of composite operators but instead shortly review the rules of operation when one works in a background field gauge (appendix D) stressing the advantages of this procedure. We recommend the work of Kluberg-Stern and Zuber for those readers who wish to go into more detail [KZ 75] . In the background field gauge the renormalization of a classical (not involving ghost fields) gauge invariant operator which does not vanish by virtue of the classical equations of motion (class I operators) involves other class I operators and operators which are gauge invariant but vanish if one uses the classical equations of motion (class II operators) but not non gauge invariant operators of the same dimensions, quantum numbers and Lorentz structure (class III operators). Furthermore, class II and III operators are renormaliz among themselves (see nevertheless [ES 83]) but their renormalization is only of marginal importance. For operators of class I one has in a general gauge (in compact matrix notation)

$$O = Z_{I} O_{oI} + Z_{II} O_{oII} + Z_{III} O_{oIII} \qquad (V.1)$$

Both the non-diagonal renormalization constant matrices Z_{II} and Z_{III} are gauge dependent and the great advantage of the background field gauge is that for graphs with only external quark and background gauge fields one only needs gauge invariant counterterms, i.e.

$$Z_{III} = 0 \qquad \text{(V.2)}$$

One can thus determine the gauge independent renormalization matrix Z_I and the corresponding anomalous dimensions by just considering gauge invariant operators (classes I and II). As a byproduct Z_{II} is also obtained, but its gauge dependence makes it as irrelevant for Physics as Z_{III}.

The deep reason for the result (2) of the background field gauge can be traced back to the explicit background field gauge invariance in the background field gauge of the effective action, which has not been broken in the process or quantization and renormalization (appendix D).

Let us start now with the easiest application of the above mentioned results. We will always work in the MS scheme.

1. Consider the gauge invariant vector and axial vector currents of the current algebra given in eq. (I.30),

$$\begin{aligned} V^{\mu}_{A0}(x) &= \bar{q}^{B}_{\alpha 0}(x)\,\gamma^{\mu}\,(T^{A})_{BC}\,q^{C}_{\alpha 0}(x) \\ A^{\mu}_{A0}(x) &= \bar{q}^{B}_{\alpha 0}(x)\,\gamma^{\mu}\,\gamma_5\,(T^{A})_{BC}\,q^{C}_{\alpha 0}(x) \end{aligned} \qquad \text{(V.3)}$$

Recall that they are pseudoconserved, that is that the symmetry which generates these currents is softly broken, i.e. by terms in the lagrangian involving operators of dimension less than four. In QCD this is necessarily the quark mass term and chiral symmetry is the softly broken symmetry. We will now prove that both these pseudo-conserved currents as also their divergences do not get renormalized. We will follow Preparata and Weisberger [PW 68] in the proof, which we will do for the vector current; the extension to the axial current is straightforward.

Consider the 1PI Green's function corresponding to

$$\langle 0 | \, T \, (q^{B}_{\alpha o}(x) \, V^{\mu}_{A o}(y) \, \bar{q}^{C}_{\beta o}(0)) \, | 0 \rangle \tag{V.4}$$

in momentum space. We will denote it by $\Gamma^{\mu \, BC}_{o \; A}(p, p') \, \delta_{\alpha\beta}$ where p and p' are the incoming and outgoing quark momenta. The Ward identity

$$\begin{aligned} -i (p' - p)_{\mu} \, \Gamma^{\mu \, BC}_{o A}(p, p') &= \\ &= (T^{A})_{BC} \left[S^{-1}_{oC}(\not{p}) - S^{-1}_{oB}(\not{p}') \right] + \Gamma^{BC}_{oA}(p, p') \end{aligned} \tag{V.5}$$

follows immediately, where Γ^{BC}_{oA} (p,p') is the 1PI Green's function of the divergence of V^{μ}_{A} ; $\partial_{\mu} V^{\mu}_{A} \equiv m \, D_{A}$, m being any quark mass. Now V^{μ}_{A} renormalizes multiplicatively, because there is no other operator of dimension three with the same quantum numbers, and the same happens with D_A :

$$\begin{aligned} V^{\mu}_{A}(x) &= Z_V \, V^{\mu}_{Ao}(x) \\ D_A(x) &= Z_D \, D_{Ao}(x) \end{aligned} \tag{V.6}$$

Thus, writing the Ward identity in terms of renormalized fields one finds

$$\begin{aligned} -i (p' - p)_{\mu} \, Z_V^{-1} \, \Gamma^{\mu \, BC}_{\; A}(p, p') &= \\ &= (T^{A})_{BC} \left[S^{-1}_{C}(\not{p}) - S^{-1}_{B}(\not{p}') \right] + Z_m Z_D^{-1} \, \Gamma^{BC}_{A}(p, p') \end{aligned} \tag{V.7}$$

Putting the quark of flavour B or C or both on the mass shell one obtains immediately

$$Z_V = Z_D Z_m^{-1} = 1 \tag{V.8}$$

which finishes our proof. The currents of eq. (3) and their divergences are thus renormalization group invariant quantities. There is only one loophole , the possible presence of anomalies in the Ward identity. But there are no anomalies for these currents and we will anyhow come back to them later on in this chapter.

2. Let us now study in another way the renormalization of $(\bar{q}^A_{\alpha_0} \; q^A_{\alpha_0})(x)$, where A is not summed over. Consider a zero momentum insertion of $(\bar{q}^A_{\alpha_0} \; q^A_{\alpha_0})(x)$ into a Green's function. In momentum space, a zero momentum insertion

$$(\bar{q}^A_{\alpha_0} \; q^A_{\alpha_0})^\sim \equiv \int d^Dx \; (\bar{q}^A_{\alpha_0} \; q^A_{\alpha_0})(x) \tag{V.9}$$

is equivalent to acting with $i\partial/\partial m_{A_0}$ on the bare Green's function, and since renormalization does not depend on the momentum of the insertion, if follows that

$$m_{A_0} \; (\bar{q}^A_{\alpha_0} \; q^A_{\alpha_0})(x) = m_A \; (\bar{q}^A_{\alpha} \; q^A_{\alpha})(x) \tag{V.10}$$

i.e. that the above expression is renormalization group invariant to all orders in perturbation theory.

Notice that this result together with the nonrenormalization result for the chiral currents implies that $(\bar{q}^A_{\alpha_0} \; q^B_{\alpha_0})(x)$ renormalizes the same way for A=B than for $A \neq B$. This might surprise some reader, because the short distance singularities of $\bar{q}^A_{\alpha_0}(y) \; q^B_{\alpha_0}(x)$ when $y \to x$ are much stronger for A=B than for $A \neq B$, because in the first case the operator product expansion starts with the identity operator whereas in the second case not. But this is not what matters, in the OPE

$$\lim_{y \to x} T\left(\bar{q}^A_{\alpha o}(y)\, q^B_{\alpha o}(x)\right) = C_o(y-x)\,\delta_{AB} + C_1(y-x)\left(\bar{q}^A_{\alpha o}\, q^B_{\alpha o}\right)(x) \qquad \text{(V.11)}$$
$$+ \text{ higher dimension operators}$$

it is precisely the composite local operator of coefficient function $C_1(y-x)$ we are considering, and its renormalization has nothing to do with the existence or not of the first term in the OPE.

3. Let us now consider a slightly more complicated case: the renormalization of $(F^{\mu\nu}_{a_o}\, F_{a_o\mu\nu})(x)$. We will do it first in pure QCD (no quark fields). We will use the background field gauge Feynman rules as given in (D.35). We will work in the Landau background field gauge $a=0$, because then one does not need the renormalization of the gauge parameter and because of other advantages which will become evident immediately. Recall that this implies that terms proportional to $1/a$ have to be kept until they are cancelled or vanish.

It is not difficult to convince oneself that this operator is the only one of dimension four which is gauge invariant in pure QCD, thus

$$(F\,F)(x) = Z_{FF}\,(F_o\,F_o)(x) \qquad \text{(V.12)}$$

In order to compute Z_{FF} it is enough to consider the Green's function $\langle A^{\mu}_a(x)\ (FF)_o(0)\ A^{\nu}_b(y)\rangle$ in momentum space, which we write in terms of renormalized background fields so that one does not have to consider external field renormalization. The notation we use for the composite unrenormalized operator written in terms of renormalized fields is

$$(FF)_0 = Z_\alpha (F_0 F_0) \tag{V.13}$$

where (D.36,37) have been used.

The Feynman rules for the insertion of $-\frac{1}{4} i\, (FF)_0$ of zero momentum are given by

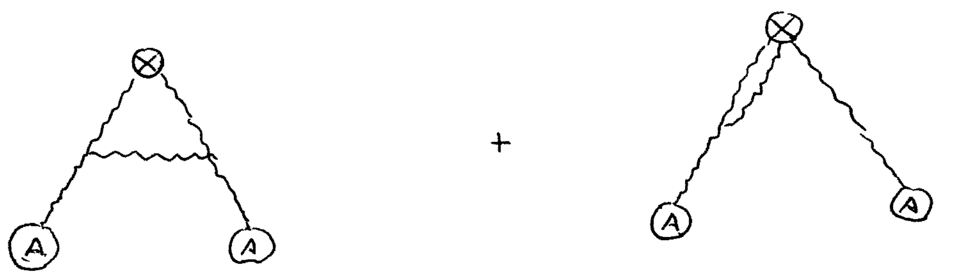
$$- i\, \delta_{ab} (p^2 g_{\mu\nu} - p_\mu p_\nu) \tag{V.14}$$

for an insertion on a gluon propagator of momentum p , and by the ordinary three -and four- gluon vertices for quantum fields for insertions on three -and four- gluon vertices independently of whether background fields flow into these or not.

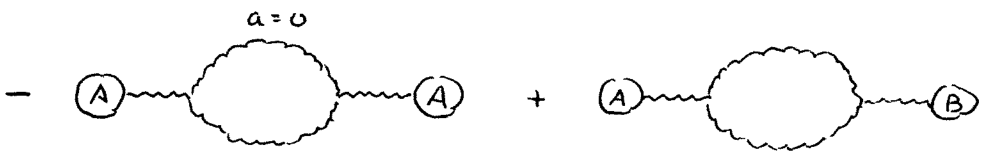
The diagrams which contribute, at the one loop level to the Green's function in momentum space

$$\langle A^\mu_a \, (FF)_0 \, A^\nu_b \rangle = Z_{FF}^{-1} Z_\alpha \langle A^\mu_a \, (FF) \, A^\nu_b \rangle \tag{V.15}$$

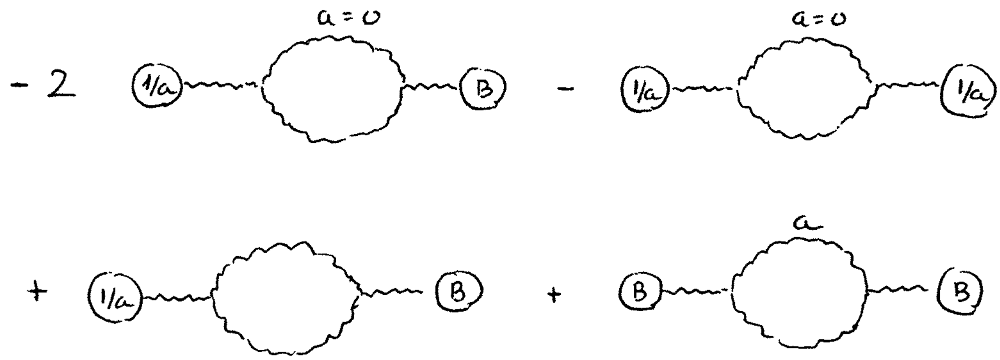
are the following



where the cross is the zero momentum insertion of $(FF)_0$. From eq. (14) one follows that the insertion of $(FF)_0$ into a gluon propagator transforms it into the same propagator in the Landau gauge but with opposite sign. Recalling the above given comment on the gluon vertex insertion, the previous diagrams are equal to



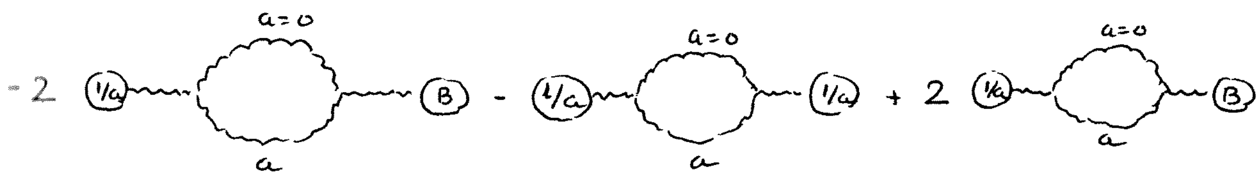
Notice that now the second diagram has an external quantum field B and that proper account has been taken of the weight factors. Now recalling that the difference in the Feynman rules for A and B fields are the terms $\frac{1}{a}$ (compare (D.35) with the Feynman rules given in chapter I), the above diagrams go over to



The last diagram does not contribute in the Landau gauge. Finally, as the $\frac{1}{a}$ terms always go with momenta corresponding to quantum fields and which carry the Lorentz index of precisely the same field, they just pick out the a-term when contracted with the corresponding propagator according to

$$\frac{1}{a} \, p^{\nu} \, \frac{g_{\mu\nu} - (1-a)\frac{p_\nu p_\mu}{p^2}}{p^2} = \frac{p_\mu}{p^2} \tag{V.16}$$

This then leads to the identity of the last diagrams with



where in the last diagram the fact that we work in the Landau gauge has been used. Only the second diagram is left over, but this is proportional to $\frac{1}{a}$ and thus vanishes.

This concludes our one-loop proof. It is purely diagrammatic and shows that there is no renormalization of $(FF)_0$ at this level,

$$Z_{FF}^{(2)} = Z_{\alpha}^{(2)} \tag{V.17}$$

so that

$$\alpha^{(2)} (FF)^{(2)} = \alpha_0 (F_0 F_0) \tag{V.18}$$

is renormalization group invariant.

4. Let us now consider the renormalization of the same operator but in the presence of a massless quark field. There is then a new independent scalar gauge invariant operator of dimension 4 and which therefore mixes with (FF) : $i\bar{q}\not{D}q$. At non-zero momentum there is still another one: $i\partial^{\mu}(\bar{q}\ \gamma_{\mu}\ q)$; however, we will only consider zero momentum insertions in order to keep things manageable.

Our notation (recall we are always referring to zero momentum insertions) is $O_1 = -\frac{1}{4} i\ FF$ and $O_2 = -\bar{q}\not{D}q$ and we have to compute the renormalization constants of

$$O_1 = Z_{11}\ O_{10}^{0} + Z_{12}\ O_{20}^{0} \tag{V.19}$$

where we use the notation O^o_{jo} for bare operators written in terms of bare fields and

$$O_{1o} = Z_\alpha O^o_{1o} \quad , \quad O_{2o} = Z^{-1}_{2F} O^o_{2o} \tag{V.20}$$

for bare operators written in terms of renormalized fields. The renormalization constants Z_{11} and Z_{12} will be obtained from the divergent parts of the insertion of the operators O^o_{10} and O^o_{20} at zero momentum into the two background field and the two quark field Green's functions. Then, for renormalized fields

$$\begin{aligned} \langle A^\mu_a O_1 A^\nu_b \rangle &= Z^{-1}_\alpha Z_{11} \langle A^\mu_a O_{1o} A^\nu_b \rangle + Z_{12} \langle A^\mu_a O^o_{2o} A^\nu_b \rangle \\ \langle q\, O_1\, \bar q \rangle &= Z_{11} \langle q\, O^o_{1o}\, \bar q \rangle + Z_{2F} Z_{12} \langle q\, O_{2o}\, \bar q \rangle \end{aligned} \tag{V.21}$$

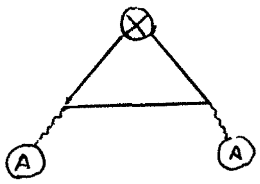
The Feynman rules for an insertion of O_{20} of zero momentum on a quark propagator of momentum p are given by

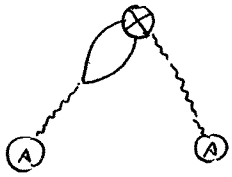
$$i \not{p} \tag{V.22}$$

and for an insertion on a quark-quark-gluon vertex, independently on whether the gluon is a background or quantum gauge field by

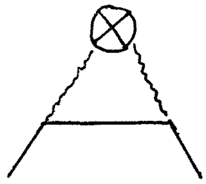
$$i g \frac{1}{2} \lambda^a \gamma_\mu \tag{V.23}$$

To lowest order the diagrams which contribute to $\langle A^\mu_a O_{10} A^\nu_b \rangle$ are the same ones as considered in the previous section. The diagrams which contribute to $\langle A^\mu_a O^o_{2o} A^\nu_b \rangle$ are

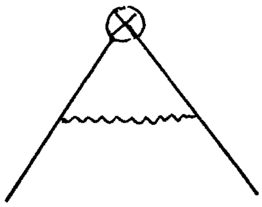


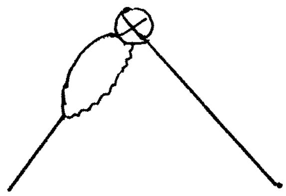


There are two of each type and from eqs. (22,23) one sees that they cancel. The diagram which contributes to $\langle q \quad O^{o}_{1o} \, \bar{q} \rangle$ is



which is equivalent to the quark self-energy diagram with changed sign in the Landau gauge, which is again zero (III.21). Finally the diagrams which contribute to $\langle q \quad O_{20} \quad \bar{q} \rangle$ are





and they again do not contribute in the Landau gauge. Since also in this gauge $Z_{2F}^{(2)} = 1$ the solution of eq. (21) is

$$Z_{11}^{(2)} = Z_{\alpha}^{(2)} \quad , \qquad Z_{12}^{(2)} = 0 \tag{V.24}$$

where we have used the fact that $Z_{12}^{(0)} = 0$ and not 1 , or equivalently

$$\langle q \; O_1 \; \bar{q} \rangle^{(0)} = 0 \tag{V.25}$$

Thus (FF) does not mix with $(\bar{q} \not{D} q)$ and eq. (18) still holds at zero momentum.

5. Consider now a massive quark. There is then a new operator which enters the game $m\bar{q}q$. One now has to consider the three operators $O_1 \equiv -\frac{1}{4} i \, FF$, $O_2 = -\bar{q} \, (\not{D} + im)q$ and $O_3 = im\bar{q}q$.

We want to compute the renormalization constants of

$$O_1 = Z_{11}\, O_{10}^{o} + Z_{12}\, O_{20}^{o} + Z_{13}\, O_{30}^{o} \tag{V.26}$$

Since the Z's are mass independent we already know $Z_{11}^{(2)}$ and $Z_{12}^{(2)}$ from the previous section. It remains to compute $Z_{13}^{(2)}$. To do so consider

$$\langle q\, O_1\, \bar{q} \rangle = Z_{11} \langle q\, O_{10}^{o}\, \bar{q} \rangle + Z_{2F} Z_{12} \langle q\, O_{20}\, \bar{q} \rangle + Z_{2F} Z_{13} \langle q\, O_{30}\, \bar{q} \rangle \tag{V.27}$$

with

$$\begin{aligned} O_{20} &\equiv -\left(\bar{q}\,(\not{D}_o + i m_o)\, q\right)_o \equiv Z_{2F}^{-1}\, O_{20}^{o} \\ O_{30} &\equiv i m_o\, (\bar{q}\, q)_o \equiv Z_{2F}^{-1}\, O_{30}^{o} \end{aligned} \tag{V.28}$$

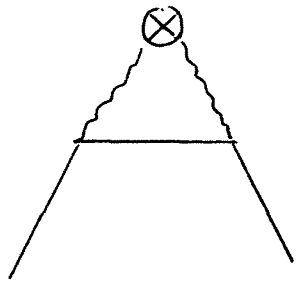
The Feynman rule for an insertion of O_{20} of zero momentum into a quark propagator of momentum p is

$$i\,(\not{p} - m_o) \tag{V.29}$$

and of O_{30} into a quark propagator is

$$i m_o \tag{V.30}$$

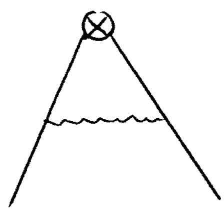
At the one loop level the calcualtion of $\langle q\, O_{10}^{o}\, \bar{q} \rangle$ corresponds to the diagram



which now is not zero because of the non-vanishing quark mass. Indeed it is given by $i \quad \Sigma_0^{(2)}(\not p, \; a=0)$ which from eq. (III.21) has the divergent part

$$- i m_0 \; \frac{\gamma_1}{2} \; \frac{\alpha_0}{\pi} \; \frac{1}{\epsilon} \tag{V.31}$$

with γ_1 given by eq. (IV.51) . One does not need to calculate $\langle q \; O_{20} \; \bar{q} \rangle$ but $\langle q \; O_{30} \; \bar{q} \rangle$ is needed. It is given by the diagram



and recalling that the insertion of $im_0(\bar{q}q)_0$ is equivalent to $- m_0 \; \partial / \partial m_0$ the result is $i m_0 \, (\partial/\partial m_0) \, \Sigma_0^{(2)}(\not p \, , \, a=0)$, the divergent part of which is given by the divergent part of $i \; \Sigma_{01}^{(2)}(p^2)$, which, as before, is again given by eq. (31) . Putting things togethe

$$Z_{13}^{(2)} = \frac{\gamma_1}{2} \; \frac{\alpha}{\pi} \; \frac{1}{\epsilon} \tag{V.32}$$

We thus find, for zero momentum insertions,

$$(FF)^{(2)} = \left(1 - \frac{\alpha}{\pi} \; \frac{\beta_1}{2\epsilon}\right) (F_0 F_0) \; - \; 2\gamma_1 \; \frac{\alpha}{\pi} \; \frac{1}{\epsilon} \; m_0 \, (\bar{q}_0 q_0) \tag{V.33}$$

and FF is no longer multiplicatively renormalizable. One can however easily build a renormalization group invariant expression from eq. (33). It is, as one can easily check

$$\frac{1}{4}\beta_1 \frac{\alpha^{(2)}}{\pi}(FF)^{(2)} + \gamma_1 \frac{\alpha^{(2)}}{\pi} m^{(2)} (\bar{q}q)^{(2)} = \\ = \frac{1}{4}\beta_1 \frac{\alpha_0}{\pi}(F_0 F_0) + \gamma_1 \frac{\alpha_0}{\pi} m_0 (\bar{q}_0 q_0) \tag{V.34}$$

with β_1 given by (IV.40) .

This expression is precisely the lowest order trace anomaly, as we will see immediately.

The generalization of these results to the two loop level is performed in reference [TA 82].

6. Recall from eq. (I.95) that dilatation invariance is only broken by the quark mass term of the QCD lagrangian. Also, and using the Belinfante dilatation current and the Belinfante energy momentum tensor ,the classical lagrangian of chromodynamics leads to the divergence equation

$$\partial_\mu D^\mu = T^\mu{}_\mu \tag{V.35}$$

Of course, by studying divergences of Green's functions with insertion of the dilatation current one will obtain the Ward identities relating them to the Green's functions with insertions of the trace of the energy momentum tensor plus the typical terms coming from differentiating the time ordering. These Ward identities, however, are wrong!

It is not our purpose to review here the breaking of scale invariance in quantum field theory and we refer the reader to Coleman's lecture [CO 73a] ,neither will we study the trace anomaly in QCD and we refer to Collins, Duncan and Joglekar or Nielsen [CD 77] [NI 77] for this purpose. However we will not skip the subject completely .

It is not difficult to understand why Ward identities based on dilatation invariance might break down in the quantum theory. The reason lies in the fact that Ward identities are proved formally, without taking into account the UV singularities of the theory. But if one first regularizes the theory, inevitably a mass scale is introduced, and scale invariance is broken by this mass scale. The Ward identities are then anomalous, new terms appear originated by the regularization of the theory.

It has been proven [CD 77], [NI 77] that the trace of the bare Belinfante energy momentum tensor between physical states (on the mass shell) and at non-zero momentum is

$$T_0{}^{\mu}{}_{\mu} = (1+\gamma)\sum_A m_A (\bar{q}^A_\alpha q^A_\alpha) + \frac{1}{4}\beta (F^{\mu\nu}_a F_{\mu\nu a}) \tag{V.36}$$

where the first term of the rhs is the canonical trace and the two terms proportional to the renormalization group functions are the trace anomaly. The trace anomaly is renormalization group invariant, as is not surprising since $T_0{}^{\mu}{}_{\mu}$ is given for the bare theory and the canonical trace is, as we saw previously, also renormalization point independent. There are however subtleties related to the fact that the trace anomaly is not homogeneous in α which make that renormalization group invariance does not imply that the functional dependence in the bare theory is the same as the one in the renormalized theory; we refer again to the literature at this point [TA 82].

It is now immediate to check that indeed eq. (34) is nothing but the lowest order trace anomaly, as at this order renormalization group invariance implies functional identity of the bare and the renormalized expressions.

One might wonder why the trace anomaly does not spoil the renormalizability of QCD. The reason is simple: the dilatation current is not coupled to the gauge fields in the QCD Lagrangian and therefore the trace anomaly will not appear in any of the Slavnov-Taylor identities of the Green's functions which one needs to prove renormalizability.

7. We would not like to finish this chapter without adding some comments on the other anomaly, the γ_5, chiral, triangle or Adler-Bell-Jackiw anomaly eq. (I.47) . It reads, in terms of bare quantities

$$\partial_\mu J^\mu_{5o}(x) = 2i \sum_A m_{Ao}\, \bar{q}^A_{\alpha o}(x)\, \gamma_5\, q^A_{\alpha o}(x) + \frac{\alpha_o}{8\pi} N_f\, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu o}(x)\, F^a_{\rho\sigma o}(x) \quad (V.37)$$

There are several differences between the two anomalies which make the renormalization of $\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$ different from the one of $F^a_{\mu\nu}(x)\ F^{\mu\nu a}(x)$:

i) the appearance of the renormalization group functions in the trace anomaly, and thus of powers of any order in α, which makes it impossible to write in a compact form the r.h.s. of eq. (36) in terms of bare quantities, as is written the r.h.s. of eq. (37).

ii) the appearance of γ_5 in the triangle anomaly, and thus of the problem of how to treat it in D dimensions.

iii) the fact that the axial anomaly, the last term of the r.h.s. of eq. (37) , can be written as a total divergence, eq. (I.49), whereas this is not so for the trace anomaly, so that its renormalization cannot be studied in the zero momentum limit.

We will not go into more detail here, and refer again to literature [ET 82] , [JL 82] , but just quote the main results which refer to the renormalization of the composite operators which appear in the equation of the triangle anomaly. The renormalization group invariant expressions are

$$m_{A0}\, \bar{q}^{A}_{\alpha 0}\, \gamma_5\, q^{A}_{\alpha 0} = m_A\, (\bar{q}^{A}_{\alpha}\, \gamma_5\, q^{A}_{\alpha}) \tag{V.38}$$

which we knew already from our previous results, and

$$\partial^\mu \sum_A \bar{q}^{A}_{\alpha 0}\, \gamma_\mu \gamma_5\, q^{A}_{\alpha 0} - \frac{\alpha_0}{8\pi} N_f\, \epsilon^{\mu\nu\rho\tau} F^{a}_{\mu\nu 0}\, F^{a}_{\rho\tau 0} = \tag{V.39}$$

$$= \partial^\mu \sum_A (\bar{q}^{A}_{\alpha}\, \gamma_\mu \gamma_5\, q^{A}_{\alpha}) - \frac{\alpha}{8\pi} N_f\, (\epsilon^{\mu\nu\rho\tau} F^{a}_{\mu\nu}\, F^{a}_{\rho\tau})$$

and under renormalization the anomaly mixes with the divergence of the axial current. The above eqs. imply immediately that the anomaly equation takes the same form in terms of bare or renormalized quantities. This is in the context of composite operators the meaning of the nonrenormalizability of the anomaly, or Adler-Bardeen's theorem [AB 69].

The origin of this anomaly lies in the fact that chiral symmetry, the symmetry behind the pseudo-conservation of the axial current, is broken by quark mass terms. It is, as the dilatation-symmetry, a symmetry related to the absence of dimensional parameters in the theory. Thus regularization is expected to break it. The only diagrams however which require regularization, contain the axial current and involve only internal quark lines are diagrams having only external gluons and the axial current ; those are the only ones in which the anomaly shows up. Other diagrams if they require regularization they always involve internal gluons, and one can then always interpret the mass scales introduced by regularization as belonging to the gluon sector, and not to the quark sector, so that regularization does not spoil chiral symmetry. This is the reason why the anomaly in eq. (37) is only of order α .

The axial anomaly does not spoil the renormalizability of QCD either. The reason is again the fact that the axial current is not coupled to the gluon field in the QCD Lagrangian.

VI. Q.C.D. SUM RULES

As a consequence of asymptotic freedom the theoretical results obtained from Q.C.D. can be compared easily with the experimental situation for the so called hard processes: at short distances the effective coupling constant $\bar{\alpha}$ becomes small and the interaction can be treated perturbatively. On the other hand, any comprehensive theory of the strong interaction must include large-distance dynamics as well. In particular quark interaction within hadrons is strong, since it binds quarks into unseparable groups. At present there is no accurate quantitative framework within QCD for dealing with the strong interaction regime and such fundamental problems as the evaluation of the hadron spectrum is beyond the reach of the actual techniques.

In the last few years a great deal of effort has been made towards the construction of new tools for doing reliable computations in the non-perturbative region of Q.C.D..Most of the efforts to obtain quantitative results can be divided into two categories [PA 80]: "brute force" computations and analytic calculations. Brute force computations are usually very long and the results can be obtained only after spending a lot of computer time. They are mainly based on lattice gauge theories [WI 74][KO 79] and they are producing nowadays very promising results [HM 81].

There are several approaches using analytical calculations. For several years much effort has been devoted to the search for classical solutions of non-abelian field theories with the hope that a semiclassical approach may shed some light on the underlying quantum world and that classical configurations of fields that make the action stationary play an important role in the problem of confinement

[CH 78], but now there is some feeling that this is a dead end. Nowadays physicists feel that the most interesting analytic approach is based on the idea of writing equations for $\overline{W}(C)$, the vacuum expectation value of the Wilson loop

$$\overline{W}(C) \equiv \langle W(C) \rangle \quad , \qquad W(C) \equiv \mathrm{Tr} \left\{ P \left[\exp i \oint_C dx^\mu B_\mu(x) \right] \right\} \tag{VI.1}$$

One obtains equations [DD 77] [NA 78] [PO 78] [GN 79] [CH 79] which are rather difficult to be solved. However it is known [MM 79], [MM 81] that rather impressive simplifications are present in the limit in which the number of colours N becomes infinite. This fact enables us to write simple closed equations for SU(N) in the limit $N \to \infty$. Although these equations seem formidable there is some hope that they might be solved, at least approximately.

An interesting new approach was opened in 1979 [SV 79] which is less fundamental than the last ones in the sense that it does not try to solve the problem of confinement but assumes that confinement exists. In practice the effects of confinement can be described through the use of a few parameters, the so called condensates, and this allows to obtain many hadronic properties through an appropiate use of sum rules.

One of the main ingredients of this approach is the operator product expansion (O.P.E.) [WI 69]. Wilson proposed a short distance operator product expansion of the following form

$$A(x)\, B(0) \underset{x^\mu \to 0}{\sim} \sum_n C_n(x)\, O_n(0) \tag{VI.2}$$

where A and B are local operators. The $C_n(x)$ are c-number functions which can have singularities on the light cone of the form

$[x^2 - i\epsilon x^0]^{-p}$, p being any real number. They can also involve logarithms of x^2 . In general, the complete expansion involves an infinite number of non singular operators O_m but to any finite order in x only a finite number of these operators contribute. The expansion is valid in the weak sense: One must sandwich the product A(x) B(o) between fixed initial and final states. Similar expansions exist for time ordered products, commutators or whatever you have. These OPE exist for the free scalar and spinor field theories and for renormalized interacting fields to all orders in perturbation theory. In every case they are valid for any elementary or composite local fields: A and B can be elementary scalar or spinor fields or local currents or the stress-energy tensor or any local Wick product in a free field theory.

The nature of the singularities of the functions $C_n(x)$ is determined, in general, by the exact and broken symmetries of the theory. The most crucial of these symmetries is broken scale invariance. The free scalar and spinor field theories with zero mass are exactly scale invariant. Mass terms and renormalizable interactions treated in perturbation theory break the symmetry but the ghost of scale invariance still governs the behavior of the singular functions [WI 69]. Exact scale invariance means that the field theory is invariant under a one-parameter group of transformations $U(\lambda)$. The local operator $O_m(x)$ transforms as

$$U^{+}(\lambda)\ O_m(x)\ U(\lambda) = \lambda^{d(O_m)}\ O_m(\lambda x) \qquad \text{(VI.3)}$$

In free field theories the constant $d(O_n)$ is the canoncial dimension of the field, i.e. $[O_n(x)] = M^{d(O_n)}$, which is determined from the commutation relations. In particular for a free scalar field

$d(\phi) = 1$, while for a free spin ½ field $d(\psi) = 3/2$.

In an exactly scale-invariant theory the behaviour of the functions $C_n(x-y)$ is determined, except for some constants, by scale invariance. Performing a scale transformation in

$$A(x)\, B(y) \underset{x^\mu \to y^\mu}{\sim} \sum_m C_m(x-y)\, O_m(y) \tag{VI.4}$$

we obtain

$$\lambda^{d(A)+d(B)} A(\lambda x)\, B(\lambda y) \underset{x^\mu \to y^\mu}{\sim} \sum_m C_m(x-y)\, \lambda^{d(O_m)} O_m(\lambda y) \tag{VI.5}$$

But

$$\lambda^{d(A)+d(B)} A(\lambda x)\, B(\lambda y) \underset{x^\mu \to y^\mu}{\sim} \lambda^{d(A)+d(B)} \sum_m C_m(\lambda x - \lambda y)\, O_m(\lambda y) \tag{VI.6}$$

and comparing the last two expressions we get when $x^\mu \to y^\mu$

$$\lambda^{d(A)+d(B)} \sum_m C_m(\lambda x - \lambda y)\, O_m(\lambda y) = \sum_m C_m(x-y)\, \lambda^{d(O_m)} O_m(\lambda y) \tag{VI.7}$$

If the fields $O_n(x)$ are linearly independent, which can always be arranged, one must have

$$C_m(\lambda x - \lambda y) = \lambda^{d(O_m) - d(A) - d(B)} C_m(x-y) \tag{VI.8}$$

This equation tells us that $C_n(x-y)$ must be a homogeneous functi of order $d(O_n) - d(A) - d(B)$ in $(x-y)$. This property as well the Lorentz transformation properties of $C_n(x-y)$ determine the b haviour of this function completely, except for one or more consta [SC 71]. In particular the strength of the singularity when $x^\mu \to y$

is determined by the dimension $d(A) + d(B) - d(O_n)$. The function C_n can be singular, in this limit, only if $d(A) + d(B) > d(O_n)$.

Let us now comment briefly the effects of introducing the interactions [WI 69] [ZI 71] . It is still true that the C-number funcions $C_n(x)$ have scaling behaviours, as $x^\mu \longrightarrow 0$, which can be summarized with the rules

$$C_m(x) \underset{x^\mu \to 0}{\sim} x^{-\lambda_m} , \qquad \lambda_m \equiv d(A) + d(B) - d(O_m) \tag{VI.9}$$

except that the operator dimensions are no longer given by naîve counting of the mass dimensions of the operators: they become "anomalous" in general [CO 73] (recall Chap. IV). There are, nevertheless, some operators which retain their naîve dimensions: these include the identity I, and the operators generating symmetries of the theory, such as currents $J^\mu(x)$ or the energy-momentum tensor $T^{\mu\nu}$ related to Lorentz invariance (recall chapter V). Let us, for instance, consider the electromagnetic current $J^\mu(x)$; as it is well known the integral over all space of $J^0(x)$ gives the charge and hence $d(J^0) = 3$ and by Lorentz covariance it is clear the $d(J^\mu) = 3$. Similarly, the fact that the fourmomentum has dimension 1 allows us to conclude that $d(T^{\mu\nu}) = 4$. Something more on the validity of the O.P.E. for Yang-Mills field theories will be said later on.

It seems to us more pedagogical to present the method for a particular case rather than in general. Let us consider the calculation of the ρ^0 meson mass using the Q.C.D. sum rules. The first step is to choose a local composite operator in terms of the quark fields with the same quantum numbers as the ρ^0 , i.e. an interpolating field for this meson. In this case the obvious choice is the isovector part of the electromagnetic current

$$J^{\mu}_{(\rho)}(x) \equiv \frac{1}{2} : \bar{u}(x) \gamma^{\mu} u(x) - \bar{d}(x) \gamma^{\mu} d(x) : \qquad \text{(VI.10)}$$

where $u(x)$ $[d(x)]$ denote the u $[d]$ -quark field and a summation in the color index must be understood in each term. We will keep in this chapter the normal ordering notation $:\ :$ that reminds us that these composite operators are regular. They are the ones which appear in the OPE eq. (2) and also the ones of chapter V. Let us now introduce the two point spectral function

$$\Pi^{\mu\nu}_{(\rho)}(q) \equiv i \int d^4x \, e^{iqx} \langle \underset{\sim}{0} | T (J^{\mu}_{(\rho)}(x) \, J^{\nu}_{(\rho)}(0)) | \underset{\sim}{0} \rangle \qquad \text{(VI.11)}$$

where $|\underset{\sim}{0}\rangle$ is the physical vacuum of the theory. Since (10) is a conserved current we can write

$$\Pi^{\mu\nu}_{(\rho)}(q) \equiv (q^{\mu} q^{\nu} - q^2 g^{\mu\nu}) \Pi_{(\rho)}(q^2)$$

$$\Pi_{(\rho)}(q^2) = - \frac{i}{(D-1) q^2} \int d^Dx \, e^{iq.x} \langle \underset{\sim}{0} | T (J^{\mu}_{(\rho)}(x) \, J_{(\rho)\mu}(0)) | \underset{\sim}{0} \rangle \qquad \text{(VI.12)}$$

where D is the number of space-time dimensions. Notice that the dimensions of the operators appearing here are $d(J^{\mu}_{\rho}) = M^{D-1}$, $d(\Pi_{(\rho)}) = M^{D-4}$. Now let us use the operator product expansion; then we can write

$$\lim_{q \to \infty} \Pi_{(\rho)}(q^2) = - \frac{i}{(D-1) q^2} \sum_{m} \langle \underset{\sim}{0} | O_m(0) | \underset{\sim}{0} \rangle \int d^Dx \, e^{iqx} \, C_m(x) \qquad \text{(VI.13)}$$

It should be noticed here that the passage from the limit $x^{\mu} \to 0$ of eq. (2) to the limit $q^{\mu} \to \infty$ of this equation is strictly correct only for euclidean values of q, $q^2 < 0$, for which $q^\mu \to \infty$ implies $q^2 \to -\infty$. For large values of q^μ the behaviour of the integral appearing in the r.h.s. of this equation is

$$[q^2]^{-[d(O_m)+2-D]/2} \tag{VI.14}$$

up to logarithmic terms. The operators $O_n(0)$ are conveniently classified according to their Lorentz spin and dimension $d(O_n)$. We need to consider only spin-zero operators since only these can contribute to the vacuum expectation value appearing in eq. (13). Furthermore if we are going to neglect in the calculation of $\Pi_{(\rho)}(q^2)$ all terms that in the limit $q^\mu \to \infty$ decrease faster than $[q^2]^{-N}$, $N > 0$, it will be necesary to consider in (13) only the operators $O_n(x)$ such that

$$d(O_m) \leq (D-4) + 2N \tag{VI.15}$$

We will be interested in the lowest dimension scalar operators and these are

Dimension 0 : I (the unit operator)

Dimension 4 : $:m_A \bar{q}^A_\alpha(x)\, q^A_\alpha(x):$ (no summation in flavours)

$:F^{\mu\nu}_a(x)\, F^a_{\mu\nu}(x):$ (VI.16)

Dimension 6 : $:\bar{q}_\alpha(x)\,\Gamma\, q_\alpha(x)\,\bar{q}_\beta(x)\,\Gamma\, q_\beta(x):$

$:\bar{q}_\alpha(x)\,\Gamma\,(\lambda^a)_{\alpha\beta}\, q_\beta(x)\,\bar{q}_\gamma(x)\,\Gamma\,(\lambda^a)_{\gamma\delta}\, q_\delta(x):$

$:m_A \bar{q}^A(x)\,\lambda^a \sigma_{\mu\nu}\, q^A(x)\, F^{\mu\nu}_a(x):$

$:f_{abc} F^{\mu\nu}_a(x)\, F^b_{\nu\rho}(x)\, F^{c\ \rho}_{\mu}(x):$

where the first two operators of dimension 6 have been given only for one flavour and the Γ are any combination of Dirac matrices and therefore for a given flavour there are five such quantities ($I, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma_5, \gamma_5$) that render a scalar operator. Any other gauge invariant scalar operator of the same dimension can be reduced to these with the help of the equations of motion. Notice that the use of the equations of motion is legitimate because in eq. (13) the physical vacuum expectation values are taken and the equations of motion are satisfied by physical states.

Within standard perturbation theory only the unit operator would survive in eq. (13) , but the non perturbative effects induce non-vanishing vacuum expectation values for other operators as well and they are the so called condensates. Therefore the non-perturbative effects of QCD introduce power corrections of the type $1/[q^2]^N$, $N \geqslant 1$, to the perturbative calculation. This vacuum condensates, once renormalized, depend on the renormalization point μ and are, therefore, not a covenient way of characterizing the vacuum. Instead, one can build with them renormalization group invariant quantities, once a choice of a renormalization scheme prescription has been made, which are μ -independent and lead thus to a more meaningful description of the physical vacuum. For the two lowest dimensional vacuum condensates, these expressions are well known [CD 77] [TA 82] (recall Chapter V):

$$\phi_1 \equiv m \langle \underset{\sim}{0} | : \bar{q}_\alpha(0)\, q_\alpha(0) : | \underset{\sim}{0} \rangle$$

$$\phi_2 \equiv m\gamma \langle \underset{\sim}{0} | : \bar{q}_\alpha(0)\, q_\alpha(0) : | \underset{\sim}{0} \rangle + \frac{1}{4}\beta \langle \underset{\sim}{0} | : F^{\mu\nu}_a(0)\, F^a_{\mu\nu}(0) : | \underset{\sim}{0} \rangle \qquad \text{(VI.17)}$$

where we consider only one flavour; γ and β are respectivelly the quark mass anomalous dimension and the Callan-Symanzik β function. The quantities ϕ_1 and ϕ_2 , given in eq. (17), are known to be μ -independent to all orders in perturbation theory. For dimension 5 and 6 operators the corresponding study of renormalization group invariance is done in Ref. [NT 83].

Our calculation is based on the OPE and its validity is by no means obvious.The problem is that non-perturbative effects are included, while the standard derivation of the operator expansion relie heavily on the Feynman graph analysis [SY 71a][CA 72][BR 70]. The effec of the non-perturbative terms in Q.C.D. without quarks is twofold [SV 79]: i) they induce non-vanishing vacuum expectation values, such as $\langle \underset{\sim}{0} | : F_a^{\mu\nu}(o) \; F^a_{\mu\nu}(o) : | \underset{\sim}{0} \rangle$, which in the standard perturbation theory vanish by definition, ii) they break down the operator expansion itself, starting from some power $[q^2]^{-d_{cr}}$, where $d_{cr} = 6$ One can argue [SV 79] that the presence of quarks affects the critical value d_{cr} , but not the very fact of the operator product expansion breakdown.

Expansion (13) along with the vacuum-to-vacuum matrix elements of the operators involved provide the basis for the Q.C.D. predictions for the two-point spectral function considered. An alternative form for this quantity is given by the general dispersion relations which give the polarization operators in terms of the observable cross sections. Equating the two representations we get QCD sum rules.

Let us now begin considering the calculation of $\Pi_{(\rho)}(Q^2)$ in the framework of Q.C.D..We will begin with the calculation of the perturbative contribution. The lowest order perturbative contribution is

$$\Pi_{(\rho)}(q^2) = -\frac{i}{4q^2(D-1)} \int d^D x \, e^{iq.x} \langle 0| T\left(:\bar{u}(x)\gamma^\mu u(x) - \bar{d}(x)\gamma^\mu d(x): \right.$$

$$\left. :\bar{u}(0)\gamma_\mu u(0) - \bar{d}(0)\gamma_\mu d(0):\right)|0\rangle \qquad \text{(VI.18)}$$

Using Wick's theorem we obtain

$$\Pi_{(\rho)}(q^2) = \frac{i}{4q^2(D-1)} (\gamma^\mu)_{ij}(\gamma_\mu)_{ke} \int d^D x \, e^{iq.x} \left\{ \underline{u_{j\alpha}(x)\bar{u}_{k\beta}(0)} \cdot \right. \qquad \text{(VI.19)}$$

$$\left. \cdot \, \underline{u_{e\beta}(0)\,\bar{u}_{i\alpha}(x)} + (u \to d) \right\}$$

Let us remember (I.57)

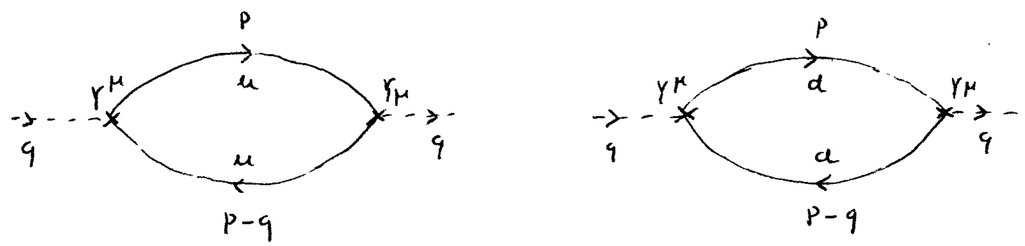
$$\underline{\psi^A_{i\alpha}(x)\,\bar{\psi}^B_{j\beta}(y)} = i\,\delta_{\alpha\beta}\,\delta_{AB}\, S^A_{ij}(x-y) =$$

$$= \delta_{\alpha\beta}\,\delta_{AB}\, i \int \frac{d^4p}{(2\pi)^4} S^A_{ij}(p)\, e^{-ip.(x-y)} \,, \qquad S^A(p) \equiv \frac{1}{\not{p} - m_A + i\eta} \qquad \text{(VI.20)}$$

Introducing this expression in (19) we obtain immediately

$$\Pi_{(\rho)}(q^2) = -\frac{N i}{4q^2(D-1)} \int \frac{d^D p}{(2\pi)^D} \left\{ \mathrm{Tr}\left[\gamma^\mu S^u(p)\gamma_\mu S^u(p-q)\right] + (u \to d) \right\} \qquad \text{(VI.21)}$$

and the corresponding Feynman diagrams are



Since we are dealing with light quarks we can neglect terms of order $m^2_{u,d}/q^2$ and therefore

$$\Pi_{(\rho)}(q^2) = \frac{i\,2N(D-2)}{q^2(D-1)} \int \frac{d^Dp}{(2\pi)^D} \frac{p^2 - p\cdot q}{[p^2 + i\eta][(p-q)^2 + i\eta]} \qquad \text{(VI.22)}$$

Using dimensional regularization and the integrals of Appendix C

$$\Pi_{(\rho)}(q^2) = \frac{N}{8\pi^2}\left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(2+\epsilon)\Gamma(-\epsilon)}{(3+2\epsilon)\Gamma(2+2\epsilon)} =$$

$$= \frac{1}{8\pi^2}\left\{-\frac{1}{\epsilon} + \ln 4\pi - \gamma - \ln\left(-\frac{q^2}{\nu^2}\right) + \frac{5}{3}\right\} \qquad \text{(VI.23)}$$

which is the desired result. Since once the sum rule has been obtained we will try to improve it using the Borel transformation (see below) and that is equivalent to carry out an infinite number of derivatives in both members of the sum rules, all polynomials in q^2 will give zero contribution and therefore they do not need to be calculated. In eq. (23), for instance, only the logarithmic term $\ln(-q^2)$ is needed.

Let us now consider the next order term in perturbation theory. Its contribution is

$$\Pi_{(\rho)}(q^2) = -\frac{i}{4q^2(D-1)}\frac{i^2}{2!}\int d^4x\, d^4y\, d^4z\, e^{iq.x} \langle 0|\, T\left(J^{\mu}_{(\rho)}(x)\, J_{(\rho)\mu}(0)\, L_I(y)\, L_I(z)\right)|0\rangle$$

$$= \frac{i g^2}{32 q^2(D-1)} (\gamma^{\mu})_{ij} (\gamma_{\mu})_{k\ell} (\gamma^{\lambda})_{mr} (\gamma^{\rho})_{st} (\lambda^a)_{\gamma\delta} (\lambda^b)_{\epsilon\tau} \int d^4x\, d^4y\, d^4z\; e^{iq.x}$$

$$\text{(VI.24)}$$

$$\Big\{\langle 0|\; T\Big(:\bar{u}_{\alpha i}(x)\, u_{\alpha j}(x):\; :\bar{u}_{\beta k}(0)\, u_{\beta \ell}(0):\; :\bar{u}_{\gamma m}(y)\, u_{\delta r}(y)\, B^a_{\lambda}(y):$$

$$:\bar{u}_{\epsilon s}(z)\, u_{\tau t}(z)\, B^b_{\rho}(z):\Big)|0\rangle \;+\; (u \to d)\Big\}$$

and applying Wick's theorem

$$\Pi_{(\rho)}(q^2) = -\frac{i g^2}{16(D-1)q^2}\int d^4x\, d^4y\, d^4z\; e^{iq\cdot x}\,(\gamma^\mu)_{ij}(\gamma_\mu)_{ke}(\gamma^\lambda)_{mr}(\gamma^\rho)_{st}(\lambda^a)_{\gamma\delta}(\lambda^a)_{\epsilon}$$

$$\begin{aligned}
&\underline{B^a_\lambda(y)\, B^b_\rho(z)}\Big\{\underline{u_{\beta e}(0)\bar u_{\alpha i}(x)}\;\underline{u_{\alpha j}(x)\bar u_{\epsilon s}(z)}\;\underline{u_{\tau t}(z)\bar u_{\gamma m}(y)}\;\underline{u_{\delta r}(y)\bar u_{\beta k}(0)}\\
&+\underline{u_{\beta e}(0)\bar u_{\epsilon s}(z)}\;\underline{u_{\tau t}(z)\bar u_{\gamma m}(y)}\;\underline{u_{\delta r}(y)\bar u_{\alpha i}(x)}\;\underline{u_{\alpha j}(x)\bar u_{\beta k}(0)}\\
&+\underline{u_{\beta e}(0)\bar u_{\gamma m}(y)}\;\underline{u_{\delta r}(y)\bar u_{\alpha i}(x)}\;\underline{u_{\alpha j}(x)\bar u_{\epsilon s}(z)}\;\underline{u_{\tau t}(z)\bar u_{\beta k}(0)} + u\to d\Big\}
\end{aligned}\tag{VI.25}$$

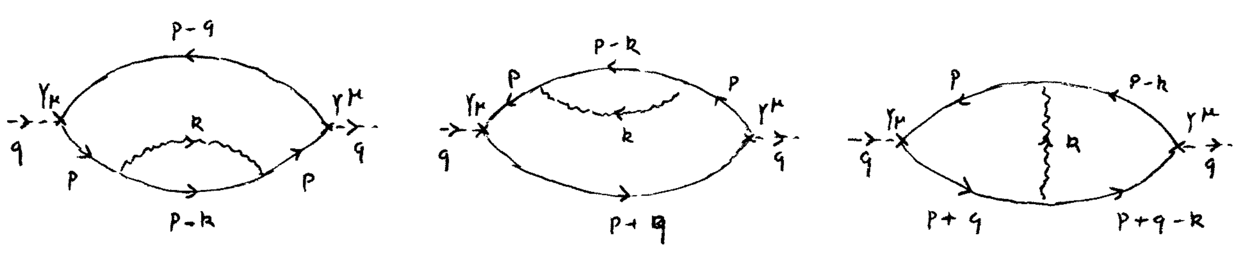
Taking into account that

$$\begin{aligned}
&\underline{B^a_\mu(x)\; B^b_\nu(y)} = i\,\delta_{ab}\, D_{\mu\nu}(x-y) = i\,\delta_{ab}\int\frac{d^4k}{(2\pi)^4}\, D_{\mu\nu}(k)\, e^{-ik.(x-y)}\\
&D_{\mu\nu}(k) \equiv \frac{1}{k^2+i\eta}\left[-g_{\mu\nu} + (1-a)\frac{k_\mu k_\nu}{k^2+i\eta}\right]
\end{aligned}\tag{VI.26}$$

as well as eq. (20) we obtain

$$\begin{aligned}
\Pi_{(\rho)}(q^2) = \frac{g^2}{(D-1)\,q^2}\int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\, D_{\lambda\rho}(k)\Big\{ &\mathrm{Tr}\left[S^u(p-q)\gamma_\mu S^u(p)\gamma^\rho S^u(p-k)\gamma^\lambda S^u(p)\gamma^\mu\right]\\
&+\mathrm{Tr}\left[S^u(p)\gamma^\rho S^u(p-k)\gamma^\lambda S^u(p)\gamma_\mu S^u(p+q)\gamma^\mu\right]+\\
&+\mathrm{Tr}\left[S^u(p)\gamma^\lambda S^u(p-k)\gamma_\mu S^u(p-k+q)\gamma^\rho S^u(p+q)\gamma^\mu\right] + (u\to d)\Big\}
\end{aligned}\tag{VI.27}$$

which correspond to the Feynman diagrams



where the circulating quark can be a u or a d quark. Neglecting quark masses and taking into account that the first two diagrams give an identical contribution we obtain in the Feynman gauge (recall our Green's function is gauge invariant)

$$\Pi_{(\rho)}(q^2) = -\frac{2g^2}{(D-1)q^2}\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\left\{ 2\,\frac{\mathrm{Tr}\left[(\not p-\not q)\gamma_\mu \not p\gamma^\rho(\not p-\not k)\gamma_\rho \not p\gamma^\mu\right]}{k^2[p^2]^2(p-k)^2(p-q)^2}\right. \tag{VI.28}$$

$$\left. + \frac{\mathrm{Tr}\left[\not p\gamma_\rho \not k\gamma_\mu(\not k-\not q)\gamma^\rho(\not p-\not q)\gamma^\mu\right]}{k^2p^2(p-k)^2(k-q)^2(p-q)^2}\right\}$$

Using dimensional regularization and the formulae of Appendix C a straightforward calculation gives

$$\Pi_{(\rho)}(q^2) = -\frac{2g^2}{q^2(3+2\epsilon)}\frac{q^2}{32\pi^4}\left\{6\zeta(3)+\frac{1}{2}+\left(-\frac{q^2}{4\pi\nu^2}\right)^{2\epsilon}\frac{\Gamma^3(1+\epsilon)\Gamma(-\epsilon)(1+\epsilon)}{(1+2\epsilon)}\cdot\right.$$

$$\left.\cdot\left[\frac{2(1+\epsilon)^2\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)\Gamma(3+3\epsilon)} + \frac{(3-2\epsilon)\Gamma(1+\epsilon)\Gamma(-\epsilon)}{2\Gamma(1+2\epsilon)\Gamma(2+2\epsilon)} - \frac{4\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)\Gamma(2+3\epsilon)}\right]\right\}$$

and therefore (in this chapter α will be written α_s , and α will be the fine structure constant)

$$\Pi_{(\rho)}(q^2) = -\frac{\alpha_s}{8\pi^3}\ln(-q^2) + \mathrm{const} \tag{VI.29}$$

where only the needed terms have been taken into account. From (23) and (29) the contribution to $\Pi_{(\rho)}(q^2)$ from the perturbative expansion is

$$\Pi_{(\rho)}(q^2) = -\frac{1}{8\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln(-q^2) \tag{VI.30}$$

where we have neglected higher order terms, terms of the type $m^2_{u,d}/q$ and constants that will disappear when the Borel transformation is carried out.There is a point which we would like to stress here, although it is not relevant to what follows. $\Pi_{(\rho)}(q^2)$ is UV divergen already to lowest order perturbation theory, in other words , it does not have a tree diagram contribution. This is charateristic of Green's functions of composite operators and no fields (recall in the previous chapter we considered Green's functions of one composite operator and several fields). This U.V. divergence has nothing to do with the ones which are eliminated renormalizing the composite operator, and they cannot be taken away by multiplicative renormalization. They require a new renormalization by fixing the value of the Green's functions at a certain euclidean point. In the dispersion relation language this corresponds to the need of a substraction constant. Notice, along the same line, that the two loop perturbative contribution eq. (29) has only simple poles and no double poles, $1/\epsilon^2$. This has to be so because double poles imply the existence of $(1/\epsilon)\ln(-q^2)$ terms. Which neither can be taken away by renormalization of parameters of the one loop contribution eq. (23) , because there are no parameters, nor by an overall subtraction, because of the $\ln(-q^2)$. This is however impossible in a renormalizable field theory.

Let us now turn our attention to the calculation of the non-perturbative contributions, beginning with the quark condensates. Let us consider the expression (18); we must apply Wick's theorem but we will leave a $q\bar{q}$ pair without contraction, i.e.

$$\Pi_{(\rho)}(q^2) = -\frac{i}{4(D-1)q^2}(\gamma^\mu)_{ij}(\gamma_\mu)_{ke}\int d^4x\, e^{iq.x}\Big\{\underbrace{u_{\alpha j}(x)\bar{u}_{\beta k}}(0)\cdot$$

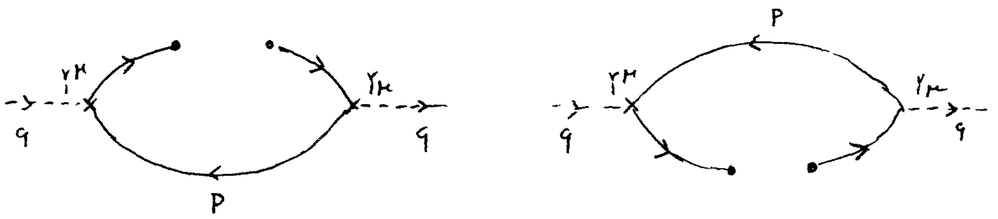
$$\cdot\langle\underset{\sim}{0}|:\bar{u}_{\alpha i}(x)u_{\beta e}(0):|\underset{\sim}{0}\rangle + \underline{u_{\beta e}(0)\bar{u}_{\alpha i}}(x)\langle\underset{\sim}{0}|:\bar{u}_{\beta k}(0)u_{\alpha j}(x):|\underset{\sim}{0}\rangle + (u\to d)\Big\} \quad \text{(VI.31)}$$

and taking into account (20) we get

$$\Pi_{(\rho)}(q^2) = \frac{1}{4(D-1)q^2} \int d^4x \int \frac{d^4p}{(2\pi)^4}\, e^{-i(p-q).x} \Big\{ \langle \underset{\sim}{0}|:\bar{u}_{\alpha i}(x)\, u_{\alpha j}(0):|\underset{\sim}{0}\rangle \,[\gamma^\mu S^u(p)\gamma_\mu$$

$$+ \langle \underset{\sim}{0}|:\bar{u}_{\alpha i}(x)\, u_{\alpha j}(0):|\underset{\sim}{0}\rangle \,[\gamma^\mu S^u(p)\gamma_\mu]_{ij} \;+\; u \to d \Big\} \qquad \text{(VI.32)}$$

which can be represented diagrammatically by



where the circulating quark can be a u or a d quark and —• •— represents the condensate. Let us consider the quark condensate; expanding in powers of x^μ we can write

$$\langle \underset{\sim}{0}|:\bar{q}^{A}_{\alpha i}(x)\, q^{B}_{\beta j}(0):|\underset{\sim}{0}\rangle = \langle \underset{\sim}{0}|:\bar{q}^{A}_{\alpha i}(0)\, q^{B}_{\beta j}(0):|\underset{\sim}{0}\rangle +$$

$$+ \langle \underset{\sim}{0}|:[\partial_\mu \bar{q}^{A}_{\alpha i}(0)]\, q^{B}_{\beta j}(0):|\underset{\sim}{0}\rangle\, x^\mu + \cdots \qquad \text{(VI.33)}$$

Since only scalars can have a non-zero vacuum expectation value, we have

$$\langle \underset{\sim}{0}|:\bar{q}^{A}_{\alpha i}(0)\, q^{B}_{\beta j}(0):|\underset{\sim}{0}\rangle = \delta_{AB}\,\delta_{\alpha\beta}\,\delta_{ij}\, A \qquad \text{(VI.34)}$$

and summing over color and spinorial indices we get

$$4N\,\delta_{AB}\, A = \langle \underset{\sim}{0}|:\bar{q}^{A}(0)\, q^{A}(0):|\underset{\sim}{0}\rangle\, \delta_{AB} \equiv \langle \bar{q}^{A} q^{A}\rangle\, \delta_{AB} \qquad \text{(VI.35)}$$

and hence

$$A = \frac{1}{12} \langle \bar{q}^A q^A \rangle \tag{VI.36}$$

We have to work out now the next term in the expansion (33). However as the derivative is an ordinary one the vacuum expectation value is not gauge invariant. As we want to relate them to the condensates, which are gauge invariant, it is convenient to work out all the nonperturbative contributions in the Schwinger [SC 70] , fixed point [CR 80] or coordinate gauge [FS 76],[SH 80],[DS 81]

$$x^\mu B_\mu (x) = 0 \tag{VI.37}$$

In this gauge $B^a_\mu (x)$ can be expressed directly in terms of the field strength tensor,namely

$$B^a_\mu (x) = \int_0^1 d\alpha \; \alpha \; F^a_{\rho\mu} (\alpha x) x^\rho \tag{VI.38}$$

In order to prove (38) let us start from the identity

$$B^a_\mu (y) = \frac{\partial}{\partial y^\mu} \left[B^a_\rho (y) y^\rho \right] - y^\rho \frac{\partial B^a_\rho (y)}{\partial y^\mu}$$

The first term on the r.h.s. drops off because of the gauge condition and the second can be written in the form

$$- y^\rho F^a_{\mu\rho} (y) - y^\rho \frac{\partial B^a_\mu (y)}{\partial y^\rho}$$

so that

$$B_\mu^a(y) + y^\rho \frac{\partial B_\mu^a(y)}{\partial y^\rho} = y^\rho F_{\rho\mu}^a(y)$$

Now substitute $y = \alpha x$. The l.h.s. of the last equation is a total derivative

$$\frac{d}{d\alpha}\left[\alpha\, B_\mu^a(\alpha x)\right] = \alpha x^\rho F_{\rho\mu}^a(\alpha x)$$

Integrating from 0 to 1 one obtains eq. (38). If we now expand $F_{\rho\mu}^a(\alpha x)$ in eq. (38) around $x^\mu = 0$ one gets

$$B_\mu^a(x) = \sum_{m=0}^{\infty} \frac{1}{m!\,(m+2)}\, x^\omega x^{\omega_1} \ldots x^{\omega_m} \partial_{\omega_1} \ldots \partial_{\omega_m} F_{\omega\mu}^a(0) \qquad \text{(VI.39)}$$

Notice that because of the gauge condition

$$x^{\omega_1} \partial_{\omega_1} F_{\omega\mu}(0) = x^{\omega_1}\left[D_{\omega_1}(0),\ F_{\omega\mu}(0)\right]$$

since $B_{\omega_1}(0) = 0$. Also

$$x^{\omega_1} x^{\omega_2} \partial_{\omega_1} \partial_{\omega_2} F_{\omega\mu}(0) = x^{\omega_1} x^{\omega_2}\left[D_{\omega_1}(0),\left[D_{\omega_2}(0),\ F_{\omega\mu}(0)\right]\right]$$

because $x^{\omega_1} x^{\omega_2} \partial_{\omega_1} B_{\omega_2}(0) = 0$, etc. Thus (39) can be written substituting ordinary by covariant derivatives as

$$B_\mu(x) = \sum_{m=0}^{\infty} \frac{1}{m!\,(m+2)}\, x^\omega x^{\omega_1} \ldots x^{\omega_m} \cdot \left[D_{\omega_1}(0),\left[D_{\omega_2}(0),\left[\ldots\left[D_{\omega_m}(0),\ F_{\omega\mu}(0)\right]\right]\ldots\right]\right. \qquad \text{(VI.40)}$$

By the very same argument the quark fields can be Taylor expanded in terms of covariant derivatives

$$q(x) = \sum_{m=0}^{\infty} \frac{1}{m!} x^{\omega_1} \ldots x^{\omega_m} D_{\omega_1}(0) \ldots D_{\omega_m}(0)\, q(0)$$

$$\bar{q}(x) = \sum_{m=0}^{\infty} \frac{1}{m!} x^{\omega_1} \ldots x^{\omega_m} \bar{q}(0)\, D^{+}_{\omega_1}(0) \ldots D^{+}_{\omega_m}(0) \qquad \text{(VI.41)}$$

where $\bar{q}(0)\, \partial^{+}_{\omega_1} \equiv \partial_{\omega_1} \bar{q}(0)$. With the help of eqs. (40,41) any non local nongauge-invariant normal product of fields can be written in terms of local gauge invariant normal products.

Let us now go on with the last term in the rhs of (33) where now the ordinary derivative is a covariant one. Lorentz invariance implies

$$\langle \underset{\sim}{0} | : (\bar{q}^{A}_{i}\, D^{+}_{\mu})_{\alpha}(0)\, q^{B}_{\beta j}(0) : | \underset{\sim}{0} \rangle = B\, \delta_{AB}\, \delta_{\alpha\beta}\, (\gamma_\mu)_{ji}$$

$$\Rightarrow \langle \underset{\sim}{0} | : (\bar{q}^{A}\, D^{+}_{\mu}\, \gamma^{\mu})_{k\alpha}(0)\, q^{B}_{\beta j}(0) : | \underset{\sim}{0} \rangle = B\, D\, \delta_{AB}\, \delta_{\alpha\beta}\, \delta_{jk} \qquad \text{(VI.42)}$$

and using the equations of motion (I.22)

$$i m_A \langle \underset{\sim}{0} | : \bar{q}^{A}_{\alpha k}(0)\, q^{B}_{\beta j}(0) : | \underset{\sim}{0} \rangle = D\, B\, \delta_{\alpha\beta}\, \delta_{AB}\, \delta_{kj}$$

using (34) and (36) leads to

$$B = \frac{i}{12 D} m_A \langle \bar{q}^{A} q^{A} \rangle \qquad \text{(VI.43)}$$

Using (33), (36) and (43) one can write

$$\langle \underset{\sim}{0} | : \bar{q}^{A}_{\alpha i}(x)\, q^{B}_{\beta j}(0) : | \underset{\sim}{0} \rangle =$$

$$= \frac{1}{12} \langle \bar{q}^{A} q^{A} \rangle\, \delta_{\alpha\beta}\, \delta_{AB} \left[\delta_{ij} + \frac{i}{D} m_A\, x_\mu\, (\gamma^\mu)_{ji} \right] + \cdots \qquad \text{(VI.44)}$$

where the neglected terms give either higher dimensional condensates or contributions of order $\frac{m_A^2}{q^2} \langle \bar{q}^{A}\, q^{A} \rangle$.

Let us now see how a typical integral appearing in (32) can be calculated using (44)

$$I = \int d^4x \int \frac{d^4p}{(2\pi)^4} \langle \underset{\sim}{0} | : \bar{q}^A_{\alpha i}(x)\, q^B_{\beta j}(0) : | \underset{\sim}{0} \rangle \; F_{ij}(p)\, e^{-i(p-q)\cdot x} =$$

$$= \frac{1}{12} \delta_{AB}\, \delta_{\alpha\beta}\, \delta_{ij} \langle \bar{q}^A q^A \rangle \int d^4x \int \frac{d^4p}{(2\pi)^4} F_{ij}(p)\, e^{-i(p-q)\cdot x} +$$

$$+ \frac{i}{12D} \delta_{AB}\, \delta_{\alpha\beta} (\gamma^\mu)_{ji}\, m_A \langle \bar{q}^A q^A \rangle \int d^4x \int \frac{d^4p}{(2\pi)^4} F_{ij}(p)\, x_\mu\, e^{-i(p-q)\cdot x} + \dots = \tag{VI.45}$$

$$\frac{1}{12} \delta_{AB}\, \delta_{\alpha\beta} \langle \bar{q}^A q^A \rangle \left\{ \mathrm{Tr}\,[F(q)] - \frac{1}{D} m_A \left\{ -\frac{\partial}{\partial p^\mu} \mathrm{Tr}\,[F(p)\gamma^\mu] \right\}_{p=q} + \dots \right\}$$

which is the needed result. Then

$$\Pi_{(\rho)}(q^2) = \frac{1}{6q^2} \left\{ \frac{1}{4} \langle \bar{u}u \rangle \,\mathrm{Tr}\,[\gamma^\mu S^u(q)\gamma_\mu] - \right.$$

$$\left. - \frac{1}{16} m_u \langle \bar{u}u \rangle \left[-\frac{\partial}{\partial p^\lambda} \mathrm{Tr}\,[\gamma^\mu S^u(p)\gamma_\mu \gamma^\lambda] \right]_{p=q} + \dots + (u \to d) \right\}$$

Taking into account that

$$-\frac{\partial}{\partial p^\lambda} S(p) = S(p)\,\gamma_\lambda\, S(p) \tag{VI.46}$$

and keeping only the dominant terms in masses we get that the quark condensate contribution is

$$\Pi_{(\rho)}(q^2) = \frac{1}{2} \frac{1}{(-q^2)^2} \left[m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle \right] \tag{VI.47}$$

Let us now consider the contribution of the gluon condensate which is also an operator of dimension four. Our starting point is eq. (24). We must contract all the quark fields and we get

$$\Pi_{(\rho)}(q^2) = -\frac{i g^2}{16(D-1)\,q^2}\,(\gamma^\rho)_{ij}\,(\gamma_\mu)_{ke}\,(\gamma^\lambda)_{mr}\,(\gamma^\rho)_{st}\,(\lambda^a)_{\gamma\delta}\,(\lambda^b)_{\varepsilon\tau}\,\cdot$$

$$\int d^4x\, d^4y\, d^4z\; e^{iq\cdot x}\;\langle \underset{\sim}{0} |\, : B^a_\lambda(y)\, B^b_\rho(z) :\, | \underset{\sim}{0} \rangle \,\cdot$$

$$\Big\{ \underline{u_{\beta e}(0)\,\bar u_{\alpha i}(x)}\;\; \underline{u_{\alpha j}(x)\,\bar u_{\gamma m}(y)}\;\; \underline{u_{\tau t}(z)\,\bar u_{\beta k}(0)}\;\; \underline{u_{\delta r}(y)\,\bar u_{\varepsilon s}(z)} \;+$$

$$+\, \underline{u_{\alpha j}(x)\,\bar u_{\beta k}(0)}\;\; \underline{u_{\delta r}(y)\,\bar u_{\alpha i}(x)}\;\; \underline{u_{\beta e}(0)\,\bar u_{\varepsilon s}(z)}\;\; \underline{u_{\tau t}(z)\,\bar u_{\gamma m}(y)} \;+$$

$$+\, \underline{u_{\delta r}(y)\,\bar u_{\alpha i}(x)}\;\; \underline{u_{\alpha j}(x)\,\bar u_{\varepsilon s}(z)}\;\; \underline{u_{\tau t}(z)\,\bar u_{\beta k}(0)}\;\; \underline{u_{\beta e}(0)\,\bar u_{\gamma m}(y)} \;+\; (u \leftrightarrow d) \Big\}$$

Taking into account eq. (20)

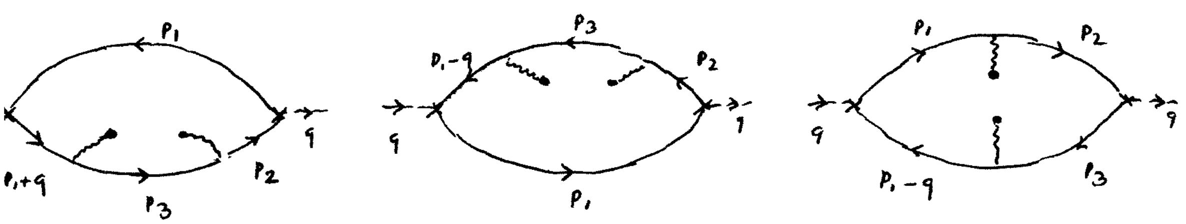
$$\Pi_{(\rho)}(q^2) = -\frac{i g^2}{8(D-1)\,q^2}\int d^4y\, d^4z \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{d^4p_3}{(2\pi)^4}\;\langle \underset{\sim}{0} |\, : B^a_\lambda(y)\, B^a_\rho(z) :\, | \underset{\sim}{0} \rangle$$

$$\Big\{ \mathrm{Tr}\big[\gamma^\mu S^u(p_1+q)\,\gamma^\lambda S^u(p_3)\,\gamma^\rho S^u(p_2)\,\gamma_\mu S^u(p_1)\big]\, e^{i(q+p_1-p_3)\cdot y}\, e^{i(p_3-p_2)\cdot z}$$

$$+\, \mathrm{Tr}\big[\gamma^\mu S^u(p_1)\,\gamma_\mu S^u(p_2)\,\gamma^\rho S^u(p_3)\,\gamma^\lambda S^u(p_1-q)\big]\, e^{i(-p_1+p_3+q)\cdot y}\, e^{i(p_2-p_3)\cdot z}$$

$$+\, \mathrm{Tr}\big[\gamma^\mu S^u(p_1)\,\gamma^\rho S^u(p_2)\,\gamma_\mu S^u(p_3)\,\gamma^\lambda S^u(p_1-q)\big]\, e^{i(q-p_1+p_3)\cdot y}\, e^{i(p_1-p_2)\cdot z} + (u \to d)\Big\} \qquad \text{(VI.48)}$$

which corresponds to the diagrams



and therefore the first two diagrams must give the same contribution

The vacuum expectation value appearing in eq. (48) can now be written using (40) as

$$\langle \underset{\sim}{0} | \, B^a_\lambda(y) \, B^a_\rho(z) : |\underset{\sim}{0}\rangle = \frac{1}{4} y^\nu z^\tau \langle \underset{\sim}{0} | : F^a_{\nu\lambda}(0) \, F^a_{\tau\rho}(0) : |\underset{\sim}{0}\rangle + \cdots$$

$$= \frac{1}{4D(D-1)} y^\nu z^\tau \left[g_{\nu\tau} g_{\lambda\rho} - g_{\nu\rho} g_{\lambda\tau} \right] \langle \underset{\sim}{0} | : F^a_{\mu\sigma}(0) \, F^{\mu\sigma}_a(0) : |\underset{\sim}{0}\rangle + \cdots \equiv \qquad \text{(VI.49)}$$

$$= \frac{1}{4D(D-1)} y^\nu z^\tau \left[g_{\nu\tau} g_{\lambda\rho} - g_{\nu\rho} g_{\lambda\tau} \right] \langle FF \rangle + \cdots$$

Now we can substitute this expression in eq. (48) and we obtain

$$\Pi_{(\rho)}(q^2) = -\frac{i g^2}{32 D (D-1)^2 q^2} \int d^4y \, d^4z \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \langle FF \rangle \left[g_{\nu\tau} g_{\lambda\rho} - g_{\nu\rho} g_{\lambda\tau} \right] y^\nu z^\tau$$

$$\Big\{ 2 \, \mathrm{Tr} \left[\gamma^\mu S^u(p_1+q) \, \gamma^\lambda S^u(p_3) \, \gamma^\rho S^u(p_2) \, \gamma_\mu S^u(p_1) \right] e^{i(q+p_1-p_3)\cdot y} \, e^{i(p_3-p_2)\cdot z}$$

$$+ \mathrm{Tr} \left[\gamma^\mu S^u(p_1) \, \gamma^\rho S^u(p_2) \, \gamma_\mu S^u(p_3) \, \gamma^\lambda S^u(p_1-q) \right] e^{i(q-p_1+p_3) y} \, e^{i(p_1-p_2)\cdot z} + (u \to d) \Big\}$$

and hence

$$\Pi_{(\rho)}(q^2) = -\frac{i g^2}{32 D (D-1)^2 q^2} \langle FF \rangle \left[g_{\nu\tau} g_{\lambda\rho} - g_{\nu\rho} g_{\lambda\tau} \right] \int \frac{d^4p_1}{(2\pi)^4} \cdot$$

$$\cdot \Big\{ 2 \left[\frac{\partial}{\partial p_{1\nu}} \frac{\partial}{\partial p_{2\tau}} \mathrm{Tr} \left[\gamma^\mu S^u(p_1+q) \, \gamma^\lambda S^u(p_3) \, \gamma^\rho S^u(p_2) \, \gamma_\mu S^u(p_1) \right] \right]_{p_2 = p_3 = p_1+q} +$$

$$+ \left[\frac{\partial}{\partial p_{3\nu}} \frac{\partial}{\partial p_{2\tau}} \mathrm{Tr} \left[\gamma^\mu S^u(p_1) \, \gamma^\rho S^u(p_2) \, \gamma_\mu S^u(p_3) \, \gamma^\lambda S^u(p_1-q) \right] \right]_{\substack{p_2 = p_1 \\ p_3 = p_1 - q}} + (u \to d) \Big\}$$

and using eq. (46) we can carry out the derivatives and the traces can be evaluated easily since the quark masses can be neglected. A straightforward calculation gives

$$\Pi_{(g)}(q^2) = -\frac{i g^2 (D-2)}{16 D (D-1)^2 q^2} \langle FF \rangle \int \frac{d^4 p}{(2\pi)^4} \left\{ 8(3D-2) \frac{1}{p^4 (p-q)^2} + \right.$$

$$\left. + 8(D-2)(D-3) \frac{q^2}{p^6 (p-q)^2} + 4(D^2 - 11D + 14) \frac{q^2}{p^4 (p-q)^4} + 8(D-2) \frac{q^4}{p^6 (p-q)^4} \right\}$$

All these integrals can be carried out immediately with the help of the formulae of Appendix C and we obtain

$$\Pi_{(g)}(q^2) = \frac{1}{24 (-q^2)^2} \left\langle \frac{\alpha_s}{\pi} FF \right\rangle \tag{VI.50}$$

Up to now we have calculated the perturbative contribution to $\Pi_{(g)}(q^2)$ up to terms of order α_s included as well as the contribution of all condensates of dimension four. Let us now consider the contribution of the condensate of four quark fields. Let us forget for the moment any possible contribution coming from the expression of eq. (18) . Let us now turn our attention to the second order perturbative result, which can be written as

$$\Pi_{(g)}(q^2) = \frac{i g^2}{32 (D-1) q^2} (\gamma^\mu)_{ij} (\gamma_\mu)_{\alpha e} (\gamma^\lambda)_{mr} (\gamma^\rho)_{st} (\lambda^a)_{r\delta} (\lambda^b)_{\epsilon z} \int d^4x\, d^4y\, d^4z\, e^{iq.x}$$

$$\sum_{ABCD} \langle \underset{\sim}{0} | T \left(:\epsilon_A \bar{q}^A_{\alpha i}(x) q^A_{\alpha j}(x):, :\epsilon_B \bar{q}^B_{\beta k}(0) q^B_{\beta e}(0):, \right.$$

$$\left. :\bar{q}^C_{\gamma m}(y) q^C_{\delta r}(y) B^a_\lambda(y):, :\bar{q}^D_{\epsilon s}(z) q^D_{z t}(z) B^b_\rho(z): \right) | \underset{\sim}{0} \rangle$$

where $\epsilon_u = +1$, $\epsilon_d = -1$ and zero for any other flavor. Now we must contract the gluon fields as well as two pairs of quark fields. As only connected diagrams must be taken into account one finds

$$\Pi_{(q)}(q^2) = \frac{i g^2}{16(D-1)q^2} (\gamma^\mu)_{ij} (\gamma_\mu)_{ke} (\gamma^\lambda)_{mr} (\gamma^\rho)_{st} (\lambda^a)_{\gamma\delta} (\lambda^b)_{\epsilon\tau} \int d^Dx\, d^Dy\, d^Dz\, e^{iq\cdot x}$$

$$\sum_{ABCD} \epsilon_A \epsilon_B \underline{B^a_\lambda(y)\, B^b_\rho(z)} \Big\{ - \underline{q^C_{\delta r}(y)\, \bar q^A_{\alpha i}(x)}\; \underline{q^D_{\tau t}(z)\, \bar q^B_{\beta k}(0)} \langle \underset{\sim}{0} | : \bar q^C_{\gamma m}(y)\, \bar q^D_{\epsilon s}(z)\, q^A_{\alpha j}(x)\, q^B_{\beta e}(0) : | \underset{\sim}{0} \rangle$$

$$+ \underline{q^C_{\delta r}(y)\, \bar q^A_{\alpha i}(x)}\; \underline{q^B_{\beta e}(0)\, \bar q^D_{\epsilon s}(z)} \langle \underset{\sim}{0} | : \bar q^B_{\beta k}(0)\, \bar q^C_{\gamma m}(y)\, q^A_{\alpha j}(x)\, q^D_{\tau t}(z) : | \underset{\sim}{0} \rangle$$

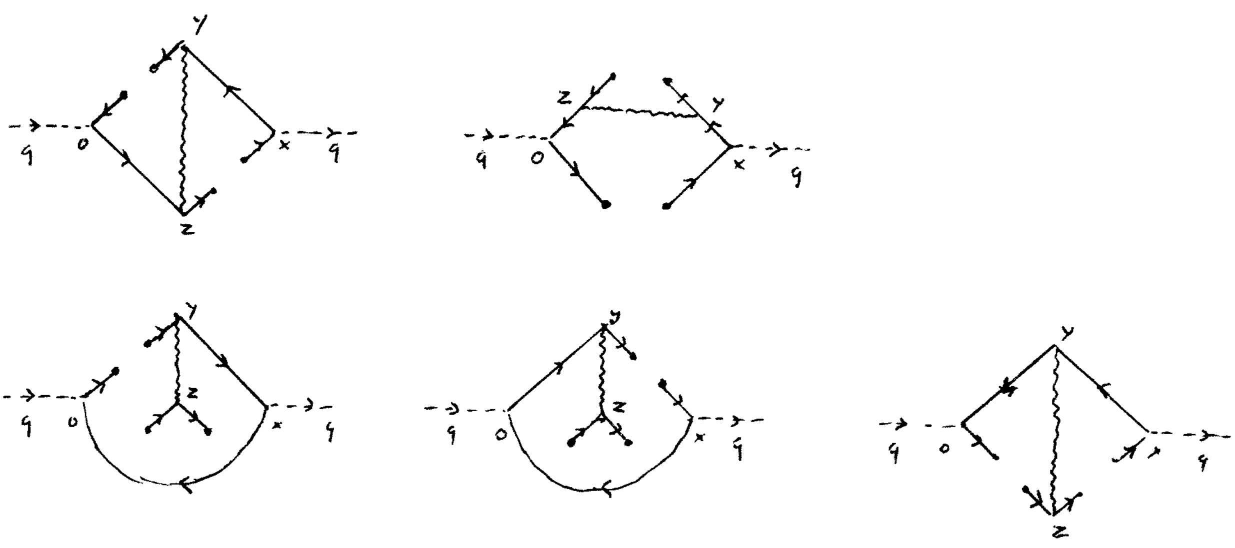
$$+ \underline{q^B_{\beta e}(0)\, \bar q^A_{\alpha i}(x)}\; \underline{q^A_{\alpha j}(x)\, \bar q^C_{\gamma m}(y)} \langle \underset{\sim}{0} | : \bar q^B_{\beta k}(0)\, \bar q^D_{\epsilon s}(z)\, q^C_{\delta r}(y)\, q^D_{\tau t}(z) : | \underset{\sim}{0} \rangle$$

$$+ \underline{q^B_{\beta e}(0)\, \bar q^A_{\alpha i}(x)}\; \underline{q^C_{\delta r}(y)\, \bar q^B_{\beta k}(0)} \langle \underset{\sim}{0} | : \bar q^C_{\gamma m}(y)\, \bar q^D_{\epsilon s}(z)\, q^A_{\alpha j}(x)\, q^D_{\tau t}(z) : | \underset{\sim}{0} \rangle$$

$$+ \underline{q^C_{\delta r}(y)\, \bar q^A_{\alpha i}(x)}\; \underline{q^B_{\beta e}(0)\, \bar q^C_{\gamma m}(y)} \langle \underset{\sim}{0} | : \bar q^B_{\beta k}(0)\, \bar q^D_{\epsilon s}(z)\, q^A_{\alpha j}(x)\, q^D_{\tau t}(z) : | \underset{\sim}{0} \rangle$$

$$+ \text{the same terms with } \bar q \leftrightarrow q \Big\} . \qquad \text{(VI.51)}$$

Their diagrammatic representation is given in the figure.



The diagrams where quarks and antiquarks are exchanged are the same but the sense of the fermionic lines reversed. We will denote by $\Pi^{(1)}_{(p)}(q^2)$ the contribution of the first two diagrams and their reversed and by $\Pi^{(2)}_{(p)}(q^2)$ the contribution of the remaining ones. Let us calculate $\Pi^{(1)}_{(p)}(q^2)$. Taking into account eqs. (20) and (26) as well as (A.15)

$$\Pi^{(1)}_{(p)}(q^2) = \frac{g^2}{8(D-1)q^2}\int d^4x\, d^4y\, d^4z \int \frac{d^4k}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\sum_{A,B}\epsilon_A\epsilon_B\frac{1}{k^2}$$

$$\Big\{ [\gamma^\lambda S^A(p_1)\gamma^\mu]_{mj}\,[\gamma_\lambda S^B(p_2)\gamma_\mu]_{se}\; e^{i(q+p_1)\cdot x}\, e^{-i(k+p_1)\cdot y}\, e^{+i(k-p_2)\cdot z}$$

$$\Big[\langle \underset{\sim}{0}|:\bar q^A_{\beta m}(y)\,\bar q^B_{\alpha s}(z)\,q^A_{\alpha j}(x)\,q^B_{\beta e}(0):|\underset{\sim}{0}\rangle - \frac{1}{3}\langle \underset{\sim}{0}|:\bar q^A_{\alpha m}(y)\,\bar q^B_{\beta s}(z)\,q^A_{\alpha j}(x)\,q^B_{\beta e}(0):|\underset{\sim}{0}\rangle\Big]$$

$$- [\gamma_\mu S^B(p_2)\gamma_\lambda]_{kt}\,[\gamma^\lambda S^A(p_1)\gamma^\mu]_{mj}\; e^{i(q+p_1)\cdot x}\, e^{-i(k+p_1)x}\, e^{i(k+p_2)z}$$

$$\Big[\langle \underset{\sim}{0}|:\bar q^B_{\alpha k}(0)\,\bar q^A_{\gamma m}(y)\,q^A_{\alpha j}(x)\,q^B_{\gamma t}(z):|\underset{\sim}{0}\rangle - \frac{1}{3}\langle \underset{\sim}{0}|:\bar q^B_{\beta k}(0)\,\bar q^A_{\alpha m}(y)\,q^A_{\alpha j}(x)\,q^B_{\beta t}(0):|\underset{\sim}{0}\rangle\Big]$$

$$+ \text{ inverted diagrams } \Big\} \quad . \qquad \text{(VI.52)}$$

In order to deal with the four quark condensate we are going to assume the vacuum saturation hypothesis [SV 79] , i.e.

$$\langle \underset{\sim}{0}|:\bar q^A_{\alpha a}(x)\,\bar q^B_{\beta b}(y)\,q^C_{\gamma c}(z)\,q^D_{\delta d}(t):|\underset{\sim}{0}\rangle \simeq$$

$$\simeq \langle \underset{\sim}{0}|:\bar q^A_{\alpha a}(x)\,q^D_{\delta d}(t):|\underset{\sim}{0}\rangle\langle \underset{\sim}{0}|:\bar q^B_{\beta b}(y)\,q^C_{\gamma c}(z):|\underset{\sim}{0}\rangle \qquad \text{(VI.53)}$$

$$- \langle \underset{\sim}{0}|:\bar q^A_{\alpha c}(x)\,q^C_{\gamma c}(z):|\underset{\sim}{0}\rangle\langle \underset{\sim}{0}|:\bar q^B_{\beta b}(y)\,q^D_{\delta d}(t):|\underset{\sim}{0}\rangle$$

and furthermore neglecting quark masses

$$\langle 0 | \; \bar{q}^A_{\alpha a}(x) \; \bar{q}^B_{\beta b}(y) \; q^C_{\gamma c}(z) \; q^D_{\delta d}(t) : | \underset{\sim}{0} \rangle =$$

$$= \frac{1}{144} \left[\delta_{AD} \delta_{BC} \delta_{\alpha\delta} \delta_{\beta\gamma} \delta_{ad} \delta_{bc} - \delta_{AC} \delta_{BD} \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{ac} \delta_{bd} \right] \langle \bar{q}^A q^A \rangle \langle \bar{q}^B q^B \rangle \qquad \text{(VI.54)}$$

Using this relation we obtain easily, assuming $\langle \bar{d}d \rangle = \langle \bar{u}u \rangle$ (isospin symmetry) and including the inverted diagrams,

$$\Pi^{(1)}_{(\rho)}(q^2) = - \frac{g^2}{144 \times 12 \; q^8} \langle \bar{u}u \rangle^2 \Big\{ 16 \, \mathrm{Tr} \left[\gamma^\lambda \not{q} \gamma^\mu \gamma_\lambda \not{q} \gamma_\mu \right]$$

$$- 16 \, \mathrm{Tr} \left[\gamma_\mu \not{q} \gamma_\lambda \gamma^\lambda \not{q} \gamma^\mu \right] \Big\}$$

and therefore

$$\Pi^{(1)}_{(\rho)}(q^2) = \frac{16}{9} \pi \alpha_s \langle \bar{u}u \rangle^2 \frac{1}{q^6} \qquad \text{(VI.55)}$$

The computation of $\Pi^{(2)}_{(\rho)}(q^2)$ cannot proceed along the same lines from eq. (51) because it involves a zero momentum gluon propagator. Perturbation theory is meaningless at this order. Instead one stays at a lower order in perturbation theory, so that no zero momentum propagators appear, and generates these diagrams by using the equations of motion in the lower dimensional condensates in order to get the higher dimensional quartic quark condensates. Let us show explicitely what we mean. Consider first the third and fourth quartic quark condensate diagrams. They can be generated by the triple derivative term of the quark condensate. Indeed expanding up to fourth order (33) requires the computation of

$$\langle \underset{\sim}{0} | : \left(\bar{q}^A_i(0) \, D^+_\mu(0) \, D^+_\nu(0) \right)_\alpha \, q^B_{\beta j}(0) : | \underset{\sim}{0} \rangle = C_1 \, \delta_{AB} \, \delta_{\alpha\beta} \, \delta_{ij} \, g_{\mu\nu}$$

$$+ C_2 \, \delta_{AB} \, \delta_{\alpha\beta} \, (\sigma_{\mu\nu})_{ji} \qquad \text{(VI.56)}$$

where only the symmetric part is actually of interest to us. From here contracting with γ^{μ} and using the eqs. of motion one gets as in (42)

$$i m_A \langle \underset{\sim}{0} | : (\bar{q}^A_k (0)\, D^+_\nu (0))_\alpha\, q^B_{\beta j} (0) : | \underset{\sim}{0} \rangle =$$

$$= \delta_{AB}\, \delta_{\alpha\beta}\, (\gamma_\nu)_{jk}\, [\, C_1 + i\, C_2\, (1 - D)\,]$$

which using (43) gives

$$C_1 + i\, C_2\, (1 - D) = i m_A\, B \tag{VI.57}$$

On the other hand (56) can be written using translation invariance as

$$\langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha} (0)\, (D_\mu (0)\, D_\nu (0)\, q^B_j (0))_\beta : | \underset{\sim}{0} \rangle =$$

$$= \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha} (0)\, (F_{\nu\mu} (0)\, q^B_j (0))_\beta : | \underset{\sim}{0} \rangle + \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha} (0)\, (D_\nu (0)\, D_\mu (0)\, q^B_j (0))_\beta : | \underset{\sim}{0} \rangle \tag{VI.58}$$

where (I.11) has been used. But Lorentz covariance allows to write the first term of the rhs of (58) as

$$\langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha} (0)\, (F_{\nu\mu} (0)\, q^B_j (0))_\beta : | \underset{\sim}{0} \rangle = E\, \delta_{AB}\, \delta_{\alpha\beta}\, (\sigma_{\nu\mu})_{ji} \tag{VI.59}$$

which immediately leads to

$$E = \frac{1}{12\, D\, (D-1)} \langle \bar{q}^A\, \sigma_{\nu\mu}\, F^{\nu\mu}\, q^A \rangle \tag{VI.60}$$

Putting together (56) to (58) and (60) one finds

$$C_1 = i\left(\frac{1-D}{2}E + m_A B\right)$$
$$C_2 = -\frac{1}{2}E \tag{VI.61}$$

Notice that as expected on dimensional grounds the only new condensate is the mixed one and the quartic quark condensate does not appear. It appears however in

$$\langle \underset{\sim}{0} | : (\bar{q}^A_i(0)\, D^+_\mu(0)\, D^+_\nu(0)\, D^+_\rho(0))_\alpha\, q^B_{\beta j}(0) : | \underset{\sim}{0}\rangle =$$
$$\cdot\, \delta_{AB}\,\delta_{\alpha\beta}\left(F_1 g_{\mu\nu}\gamma_\rho + F_2 g_{\mu\rho}\gamma_\nu + F_3 g_{\nu\rho}\gamma_\mu + F_4 \gamma_\mu\gamma_\nu\gamma_\rho\right)_{ji} \tag{VI.62}$$

Indeed a contraction with γ^μ and separately with γ^ρ and use of the eqs. of motion gives immediately

$$F_1 + F_2 + D(F_3 + F_4) = i m_A C_1$$
$$-F_1 + F_2 + (D-4)F_4 = -m_A C_2 \tag{VI.63}$$
$$F_1 = F_3$$

Furthermore translation invariance allows one to write (62) as

$$-\langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha}(0)\,(D_\mu(0)\, D_\nu(0)\, D_\rho(0)\, q^B_j(0))_\beta : | \underset{\sim}{0}\rangle =$$
$$= \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha}(0)\,(D_\mu(0)\, F_{\nu\rho}(0)\, q^B_j(0))_\beta : | \underset{\sim}{0}\rangle - \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha}(0)\,(D_\mu(0)\, D_\rho(0)\, D_\nu(0)\, q^B_j(0))_\beta : | \underset{\sim}{0}\rangle =$$
$$= \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha}(0)\,([D_\mu(0), F_{\nu\rho}(0)]\, q^B_j(0))_\beta : | \underset{\sim}{0}\rangle + \tag{VI.64}$$
$$+ \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha}(0)\,(F_{\nu\rho}(0)\, D_\mu(0)\, q^B_j(0))_\beta : | \underset{\sim}{0}\rangle - \langle \underset{\sim}{0} | : \bar{q}^A_{i\alpha}(0)\,(D_\mu(0)\, D_\rho(0)\, D_\nu(0)\, q^B_j(0))_\beta : | \underset{\sim}{0}\rangle$$

Lorentz covariance and the Bianchi identities (I.12) allow one to write

$$\langle \underset{\sim}{0} | : \bar{q}^{A}_{i\alpha}(0) \left([D_\mu(0), F_{\nu\rho}(0)]\, q^{B}_{j}(0) \right)_\beta : | \underset{\sim}{0} \rangle = G\, \delta_{AB}\, \delta_{\alpha\beta} \left(g_{\mu\nu} \gamma_\rho - g_{\mu\rho} \gamma_\nu \right)_{ji}$$

$$\langle \underset{\sim}{0} | : \bar{q}^{A}_{i\alpha}(0) \left(F_{\nu\rho}(0)\, D_\mu(0)\, q^{B}_{j}(0) \right)_\beta : | \underset{\sim}{0} \rangle = \tag{VI.65}$$

$$= \delta_{AB}\, \delta_{\alpha\beta} \left[H_1 \left(g_{\mu\nu} \gamma_\rho - g_{\mu\rho} \gamma_\nu \right) + H_2\, \sigma_{\nu\rho}\, \gamma_\mu \right]_{ji}$$

Contraction of the first expression with $g^{\mu\nu}$ and the use of the eqs. of motion leads to

$$G = - i \frac{g^2}{12\, D(D-1)} \langle \bar{q}^A \gamma^\rho \frac{\lambda^a}{2} q^A \sum_c \bar{q}^c \frac{\lambda^a}{2} \gamma_\rho q^c \rangle \tag{VI.66}$$

Contraction of the second expression of (65) with γ_μ and the use of the eqs. of motion gives, using (59)

$$2 H_1 + i (D-4) H_2 = m_A E \tag{VI.67}$$

Now the l.h.s. of the second expression of eq. (65) can be written also as

$$\langle \underset{\sim}{0} | : \bar{q}^{A}_{i\alpha}(0) \left([F_{\nu\rho}(0), D_\mu(0)]\, q^{B}_{j}(0) \right)_\beta : | \underset{\sim}{0} \rangle -$$

$$- \langle \underset{\sim}{0} | : \left(\bar{q}^{A}_{i}(0)\, D^{+}_{\mu}(0) \right)_\alpha \left(F_{\nu\rho}(0)\, q^{B}_{j}(0) \right)_\beta : | \underset{\sim}{0} \rangle$$

which using (65), contracting with γ_μ and using eqs. of motion as well as (59) gives

$$2 H_1 - i D H_2 = - 2 G - m_A E \tag{VI.68}$$

Eqs. (67) and (68) immediately give

$$H_1 = \frac{1}{D-2}\left(\frac{4-D}{2}\,G + m_A\,E\right)$$
$$H_2 = -\frac{i}{D-2}\,(G + m_A\,E) \tag{VI.69}$$

Putting all this back into (64) gives the relationships

$$F_1 - F_2 = G + H_1 - 2i\,H_2$$
$$F_4 = \frac{i}{2}\,H_2 \tag{VI.70}$$

which together with (63) allows one to obtain F_i , $i = 1$ to 4 . For our purposes we only need the symmetrized part of (62) which is proportional to $\sum_{i=1}^{4} F_i$ and which, for $D = 4$ is

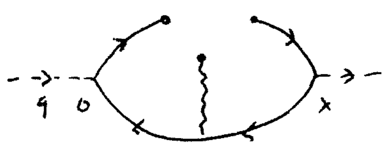
$$\sum_{i=1}^{4} F_i\,(D=4) = \frac{1}{4}\left(-G + 3m_A\,E - 2m_A^2\,B\right) \tag{VI.71}$$

With all this we can now extend (44) to fourth order

$$\begin{aligned}\langle \underset{\sim}{0} | : \bar{q}^A_{\alpha i}(x)\, q^B_{\beta j}(0) : | \underset{\sim}{0} \rangle = \frac{1}{12}\,\delta_{AB}\,\delta_{\alpha\beta}\Big[\Big(&\delta_{ij}\left(1 - \frac{x^2}{8}m_A^2\right)\\ &+ \frac{i}{4}\,m_A\,x^\mu(\gamma_\mu)_{ji}\left(1 - \frac{x^2}{12}m_A^2\right)\Big)\langle \bar{q}^A q^A\rangle -\\ &- \frac{i}{16}\,x^2\left(\delta_{ij} + \frac{i}{6}\,m_A\,x^\mu(\gamma_\mu)_{ji}\right)\langle \bar{q}^A\,\sigma_{\mu\nu}F^{\mu\nu}q^A\rangle\\ &+ \frac{i}{288}\,x^2\,x^\mu(\gamma_\mu)_{ji}\,g^2\langle \bar{q}^A\gamma^\rho\frac{\lambda^a}{2}q^A\sum_c \bar{q}^c\gamma_\rho\frac{\lambda^a}{2}q^c\rangle\Big]\end{aligned} \tag{VI.72}$$

The last term of this expression shows clearly that the quark condensate generates a quartic quark condensate. Putting (72) into eq. (32) allows to compute the contribution which comes from the third and forth quartic quark condensate diagrams. A straightforward computation which does not contain any new ingredients gives a zero result. However, if one computes $q_\mu\, q_\nu\, \Pi^{\mu\nu}_{(\rho)}(q)$ one obtains a non-vanishing result! This subclass of diagrams is not transverse.

Let us now compute the last diagram. In it the quartic quark condensate is generated by the mixed condensate. It therefore requires the computation of the diagram



which corresponds to

$$\Pi_{(\rho)}(q^2) = -\frac{g}{4\,q^2(D-1)}\, \lambda^a_{\alpha\delta}\, \gamma^\mu_{mb}\, \gamma_{\mu cd}\, \gamma^\rho_{ef} \int d^4x\, d^4z\, e^{iqx} \int \frac{d^4p_1}{(2\pi)^4}\, \frac{d^4p_2}{(2\pi)^4} \tag{VI.73}$$

$$\delta_{\alpha\beta}\, \delta_{\gamma\delta}\, e^{-ip_1(x-z)}\, e^{-ip_2 z}\, S^u_{be}(p_1)\, S^u_{fc}(p_2)\, \langle 0| : \bar u_{\beta m}(x)\, u_{\gamma d}(0)\, B^a_\rho(z) : |0\rangle$$

$$+ \quad (u \to d)$$

Now using (40) and (41) one gets for the vacuum expectation value

$$\langle 0| : \bar q^A_m(x)\, B_\rho(z)\, \bar q^B_d(0) : |0\rangle = \frac{1}{2} z^\mu \langle 0| : \bar q^A_m(0)\, F_{\mu\rho}(0)\, \bar q^B_d(0) : |0\rangle$$

$$+ \frac{1}{2} x^\nu z^\mu \langle 0| : \bar q^A_m(0)\, D^+_\nu(0)\, F_{\mu\rho}(0)\, \bar q^B_d(0) : |0\rangle + \tag{VI.74}$$

$$+ \frac{1}{3} z^\nu z^\mu \langle 0| : \bar q^A_m(0)\, [D_\nu(0), F_{\mu\rho}(0)]\, \bar q^B_d(0) : |0\rangle + \cdots$$

The only new vacuum expectation value is the second one which can be written as

$$\langle \underset{\sim}{0} | : \bar{q}^A_m (0)\, D_\nu (0)\, F_{\mu\rho} (0)\, \bar{q}^B_d (0) : | \underset{\sim}{0} \rangle =$$

$$- \langle \underset{\sim}{0} | : \bar{q}^A_m (0) \left[D_\nu (0) \,,\, F_{\mu\rho} (0) \right] \bar{q}^B_d (0) : | \underset{\sim}{0} \rangle -$$

$$- \langle \underset{\sim}{0} | : \bar{q}^A_m (0)\, F_{\mu\rho} (0)\, D_\nu (0)\, \bar{q}^B_d (0) : | \underset{\sim}{0} \rangle$$

which are both known from (65). Putting all this together one gets (D = 4)

$$\langle \underset{\sim}{0} | : \bar{q}^A_m (x)\, B_\rho (z)\, \bar{q}^B_d (0) : | \underset{\sim}{0} \rangle = \delta_{AB} \frac{z^\mu}{96} \Big\{ \Big(\sigma_{\mu\rho} - \frac{m_A}{2} (x_\mu \gamma_\rho - x_\rho \gamma_\mu)$$

$$+ i \frac{m_A}{2} x^\nu \sigma_{\mu\rho} \gamma_\nu \Big)_{dm} \langle \bar{q}^A \sigma_{\omega\tau} F^{\omega\tau} q^A \rangle +$$

$$+ \Big(i \left(- \frac{2}{3} z_\mu \gamma_\rho + \frac{2}{3} z_\rho \gamma_\mu \right) + \frac{1}{2} x^\nu \gamma_\nu \sigma_{\mu\rho} \Big)_{dm} \cdot \tag{VI.75}$$

$$\cdot g^2 \langle \bar{q}^A \gamma^\tau \frac{\lambda^a}{2} q^A \sum_c \bar{q}^c \gamma_\tau \frac{\lambda^a}{2} q^c \rangle \Big\}$$

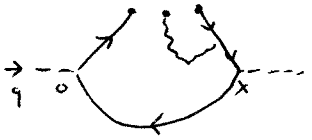
The last term gives the quartic quark condensate contribution. Putting it back into (73) and performing all the traces and integrals gives

$$\Pi^{(2)}_{(\rho)} (q^2) = - \frac{32}{81} \frac{\alpha_s \pi}{(q^2)^3} \langle \bar{u} u \rangle^2 \tag{VI.76}$$

where factorization and SU(2) symmetry have been used.

It is not difficult to compute the nontransverse part $q^\mu q^\nu \Pi_{\mu\nu(\rho)} (q)$ corresponding to this diagram. It is not zero and precisely cancels the previously encountered contribution.

The mixed condensate contribution is easily computed with the help of the previous formulae. One part comes from the last diagram and can be immediately obtained with the help of (75): the result is zero, but a nontransverse part appears. The other contribution comes from the diagrams



and one can compute it with the help of (72). It is also zero but cancels the nontransverse part of the first contribution. The mixed condensate only enters at order m^2/q^2.

Taking into account all the obtained results we get from Q.C.D. the following information $(-q^2 \equiv Q^2)$

$$\Pi_{(\rho)}(Q^2) = -\frac{1}{8\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln Q^2 + \frac{1}{2}\left[m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle\right]\frac{1}{Q^4}$$
$$+\frac{1}{24}\left\langle\frac{\alpha_s}{\pi}FF\right\rangle\frac{1}{Q^4} - \frac{112}{81}\pi\alpha_s\langle\bar{u}u\rangle^2\frac{1}{Q^6} + \cdots \qquad \text{(VI.77)}$$

where we have neglected constants, higher order terms in the O.P.E., radiative corrections to the condensates and m^2/Q^2 terms.

Let us now turn our attention to the needed dispersion relation. If $J^\mu(x)$ denotes the electromagnetic current and we define

$$\Pi^{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle \underset{\sim}{0}|T(J^\mu(x), J^\nu(0))|\underset{\sim}{0}\rangle \equiv [q^\mu q^\nu - q^2 g^{\mu\nu}]\,\Pi(q^2) \qquad \text{(VI.78)}$$

its absortive part is

$$\tilde{\Pi}^{\mu\nu}(q) \equiv \frac{1}{2\pi} \, \mathrm{Im} \, \Pi^{\mu\nu}(q) = \frac{1}{4\pi i} \left[\Pi^{\mu\nu}(q) - \Pi^{\mu\nu *}(q) \right] \tag{VI.79}$$

and

$$\tilde{\Pi}^{\mu}{}_{\mu}(q) = -\frac{3q^2}{2\pi} \, \mathrm{Im} \, \Pi(q^2) \tag{VI.80}$$

Furthermore

$$R(s) \equiv \frac{\sigma(e^+e^- \to \gamma \to \text{hadrons})}{\sigma(e^+e^- \to \gamma \to \mu^+\mu^-)} = 12\pi \, \mathrm{Im} \, \Pi(s) \tag{VI.81}$$

since

$$\begin{aligned} \sigma(e^+e^- \to \gamma \to \text{hadrons}) &= -\frac{32\pi^3\alpha^2}{3q^4} \tilde{\Pi}^{\mu}{}_{\mu}(q^2) \\ \sigma(e^+e^- \to \gamma \to \mu^+\mu^-) &= \frac{4\pi\alpha^2}{3q^2} \end{aligned} \tag{VI.82}$$

Now we can write the subtracted dispersion relation

$$-\frac{d\Pi(Q^2)}{dQ^2} = \frac{1}{\pi} \int_0^\infty ds \, \frac{\mathrm{Im}\,\Pi(s)}{(s+Q^2)^2} \tag{VI.83}$$

and using (80) and restricting it to the isospin T=1 channel we have

$$\frac{1}{12\pi^2} \int_0^\infty ds \, \frac{R^{(T=1)}(s)}{(s+Q^2)^2} = -\frac{d\Pi_{(\rho)}(Q^2)}{dQ^2} \tag{VI.84}$$

From here and the result (77) we get the desired sum rule

$$\int_0^{\infty} ds \, \frac{R^{(I=1)}(s)}{(s+Q^2)^2} = \frac{3}{2Q^2}\Big\{ 1 + \frac{\alpha_s}{\pi} + 8\pi^2 \left[m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle \right] \frac{1}{Q^4} +$$

$$+ \frac{2\pi^2}{3} \left\langle \frac{\alpha_s}{\pi} FF \right\rangle \frac{1}{Q^4} - \frac{896}{27} \pi^3 \alpha_s \langle \bar{u}u \rangle^2 \frac{1}{Q^6} + \cdots \Big\} \qquad \text{(VI.85)}$$

The right hand side is exact to lowest order in the strong coupling constants α_s and all quantities appearing there must be taken as the renormalized ones at some point ν , introduced via renormalization. In principle this result can be improved through the use of the renormalization group. The net effect is that all quantities must be substituted by the corresponding running ones at the point Q^2 (see however [NT 83]).

It is well known that the integral appearing in the left hand side of eq. (85) has a large contribution due to the ρ-meson- and now we would like to use the sum rule derived above to predict the properties of this meson. In order to enhance as much as possible the contribution of the ρ -meson to the lhs of eq. (85) we must take $Q^2 \to 0$, while the rhs is a good approximation only in the limit $Q^2 \to \infty$. We can take a compromise value of $Q^2 \simeq 1\text{GeV}^2$ for which the r.h.s. of (85) is not a very bad approximation, but it is clear that the saturation of the r.h.s. by the ρ -meson is a rather poor approximation. If we wish good values for the ρ -meson parameters we must do something better.

A possible solution [SV 79] is to apply to both terms of the sum rule the Borel transformation

$$\hat{B} \equiv \lim_{\substack{Q^2 \to \infty,\ N \to \infty \\ Q^2/N = M^2 = \text{fixed}}} \frac{1}{\Gamma(N)} (-Q^2)^N \left(\frac{d}{dQ^2} \right)^N \qquad \text{(VI.86)}$$

If $f(Q^2) \equiv (Q^2 + s)^{-\beta}$ it is easy to check (see Appendix E) that

$$\hat{f}(M^2) \equiv \hat{B} f = \frac{1}{\Gamma(\beta)} \frac{1}{(M^2)^\beta} e^{-s/M^2} \qquad \text{(VI.87)}$$

This is a special case of the general theorem: If $F(1/M^2)$ is a function which Laplace transformed is $f(Q^2)$ i.e.

$$f(Q^2) = \int_0^\infty d\left(\frac{1}{M^2}\right) F\left(\frac{1}{M^2}\right) e^{-Q^2/M^2} \qquad \text{(VI.88)}$$

then

$$\hat{f}(M^2) \equiv \hat{B} f = \frac{1}{M^2} F\left(\frac{1}{M^2}\right) \qquad \text{(VI.89)}$$

Using the Borel transformation in our sum rule we obtain

$$\int_0^\infty ds\, R^{(T=1)}(s)\, e^{-s/M^2} = \frac{3}{2} M^2 \Big\{ 1 + \frac{\alpha_s}{\pi} + 4\pi^2 \left[m_u \langle \bar{u} u \rangle + m_d \langle \bar{d} d \rangle \right] \frac{1}{M^4}$$

$$+ \frac{\pi^2}{3} \left\langle \frac{\alpha_s}{\pi} FF \right\rangle \frac{1}{M^4} - \frac{448}{81} \pi^3 \alpha_s \langle \bar{u} u \rangle^2 \frac{1}{M^6} + \cdots \Big\} \qquad \text{(VI.90)}$$

Notice that we have neglected the dependence on Q^2 of the masses and coupling constant when Borel transforming, but if taken into account they originate new terms of the same order of magnitude as the neglected ones. Now all the running quantities must be taken at M^2. Through the Borel transformation we have improved both sides of the sum rule: For $M^2 \simeq 1\ \text{GeV}^2$ is is clear that the l.h.s. is dominated by the ρ-meson, while the r.h.s. is improved due to the factor $1/\Gamma(\beta)$ appearing in the transform of $1/(Q^2)^\beta$.

In the narrow width approximation [BR 79]

$$R^{(I=1)}(s) = \frac{12\pi^2 m_\rho^2}{f_\rho^2}\,\delta(s - m_\rho^2) \tag{VI.91}$$

where the experimental values are

$$m_\rho^2 = 0.602(5)\ GeV^2 \quad , \quad \frac{f_\rho^2}{4\pi} = 2.36(18) \tag{VI.92}$$

Then our sum rule reads

$$\frac{12\pi^2 m_\rho^2}{f_\rho^2}\, e^{-m_\rho^2/M^2} = \frac{3}{2} M^2 \Big\{ 1 + \frac{\bar{\alpha}_s(M^2)}{\pi} + 4\pi^2 [m_u \langle \bar{u}u\rangle + m_d \langle \bar{d}d\rangle] \frac{1}{M^4}$$

$$+ \frac{\pi^2}{3} \langle \frac{\alpha_s}{\pi} FF \rangle \frac{1}{M^4} - \frac{448}{81} \pi^3 \alpha_s \langle \bar{u}u\rangle^2 \frac{1}{M^6} + \cdots \Big\} \tag{VI.93}$$

For the running coupling constant we take

$$\bar{\alpha}_s(M^2) = \frac{4\pi}{9} \frac{1}{\ln(M^2/\Lambda^2)} \quad , \quad \Lambda = 0.1\ GeV \tag{VI.94}$$

which is the one corresponding to 3 flavours. For the condensates we will neglect the Q^2 dependence and we will take the values [NP 83] [SV 79]

$$\begin{aligned} m_u \langle \bar{u}u\rangle + m_d \langle \bar{d}d\rangle &= -1.7 \times 10^{-4}\ GeV^4 \\ \langle \frac{\alpha_s}{\pi} FF \rangle &= 1.2 \times 10^{-2}\ GeV^4 \\ \pi^3 \alpha_s \langle \bar{u}u\rangle^2 &= 5.2 \times 10^{-3}\ GeV^6 \end{aligned} \tag{VI.95}$$

Hence

$$\frac{12\pi^2 m_\rho^2}{f_\rho^2} e^{-m_\rho^2/M^2} = \frac{3}{2} M^2 \left\{ 1 + \frac{\bar{\alpha}_s(M^2)}{\pi} + \frac{0.033\ \mathrm{GeV}^4}{M^4} - \frac{0.029\ \mathrm{GeV}^6}{M^6} \right\} \quad \text{(VI.96)}$$

Notice that the M^{-4} and M^{-6} terms are comparable for $M^2 \simeq (1\mathrm{GeV})^2$ and one can think that this means the breaking of the whole expansion at these values of M^2. In fact, this is not so. The point is that the coefficient of M^{-6} is anomalously large as compared to the one of M^{-4}. The reason is readily traced back to the fact that the term in $1/M^6$ comes from Born graphs while the $1/M^4$ comes from loop graphs.This "anomaly" does not repeat itself at higher orders [SV 79] .

From (96) and its derivative with respect to M^2 we obtain

$$m_\rho^2 = M^2 \frac{1 + \frac{\bar{\alpha}_s(M^2)}{\pi} - \frac{0.033\ \mathrm{GeV}^4}{M^4} + \frac{0.058\ \mathrm{GeV}^6}{M^6} + \cdots}{1 + \frac{\bar{\alpha}_s(M^2)}{\pi} + \frac{0.033\ \mathrm{GeV}^4}{M^4} - \frac{0.029\ \mathrm{GeV}^6}{M^6}} \quad \text{(VI.97)}$$

This result is clearly wrong for low values of M^2 where the Q.C.D. calculation is meaningless and also for large values of M^2 where the ρ-meson dominance hypothesis fails . Nevertheless we expect that for $M^2 \simeq (1\mathrm{GeV})^2$ the value of m_ρ^2 turns out to be practically independent of M^2 and this is so. The quantity m_ρ^2 when ploted versus M^2 present a wide minimum which appear at $M^2 \simeq (0.73\mathrm{GeV})^2$ and the corresponding value of m_ρ^2 is $m_\rho^2 = 0.71\ \mathrm{GeV}^2$ in good agreement with the experimental mass (92). Furthermore from these values and eq. (96) we obtain $f_\rho^2/4\pi = 2.15$ to be compared with the experimental value given in (92).

Further applications of these sum rules for mesons can be found in [SV 79]. They have been applied to baryons [EP 83] [IO 81] [CD 81] [CD 82] with good success.

APPENDIX A.- SU(N)

Let T_a $(a= 1,2,\ldots, N^2-1)$ be the SU(N) generators, which close a Lie algebra

$$[T_a, T_b] = i f_{abc} T_c$$

$$\mathrm{Tr}[T_a] = 0 \tag{A.1}$$

f_{abc} being real and totally antisymmetric, normalized in such a way that

$$f_{abc} f_{dbc} = N \delta_{ad} \tag{A.2}$$

The gluon fields transform as its adjoint (regular) representation and then

$$(T_a)_{bc} = -i f_{abc} \tag{A.3}$$

The quark fields transform under SU(N) as the fundamental (lowest-dimensional cogradient) representation and in this case

$$T_a = \frac{1}{2} \lambda_a \tag{A.4}$$

where λ_a are hermitian, traceless, NxN matrices generalizing the Gell-Mann matrices of SU(3) and which satisfy

$$[\lambda_a, \lambda_b] = i 2 f_{abc} \lambda_c$$

$$\{\lambda_a, \lambda_b\} = \frac{4}{N} \delta_{ab} 1 + 2 d_{abc} \lambda_c \tag{A.5}$$

where d_{abc} are real and totally symmetric. Hence

$$\lambda_a \lambda_b = \frac{2}{N} \delta_{ab} I + d_{abc} \lambda_c + i f_{abc} \lambda_c \tag{A.6}$$

By simple algebraic manipulations it is easy to prove the following useful relations

$$d_{abb} = 0 \tag{A.7}$$

$$d_{abc} d_{dbc} = \left(N - \frac{4}{N} \right) \delta_{ad} \tag{A.8}$$

$$f_{abr} f_{cdr} = \frac{2}{N} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) + d_{acr} d_{dbr} - d_{adr} d_{bcr} \tag{A.9}$$

$$f_{abr} d_{car} + f_{acr} d_{abr} + f_{adr} d_{bcr} = 0 \tag{A.10}$$

$$f_{abr} f_{cdr} + f_{acr} f_{dbr} + f_{adr} f_{bcr} = 0 \tag{A.11}$$

Then from (A.6) and (A.9)

$$\mathrm{Tr}\,[\lambda_a \lambda_b] = 2 \delta_{ab} \equiv 4\, T(R)\, \delta_{ab} \tag{A.12}$$

$$\mathrm{Tr}\,[\lambda_a \lambda_b \lambda_c] = 2 (d_{abc} + i f_{abc}) \tag{A.13}$$

$$\begin{aligned} \mathrm{Tr}\,[\lambda_a \lambda_b \lambda_c \lambda_d] = {} & \frac{4}{N} (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \\ & + 2 (d_{abr} d_{cdr} - d_{acr} d_{dbr} + d_{adr} d_{bcr}) \\ & + 2i (d_{abr} f_{cdr} - d_{acr} f_{dbr} + d_{adr} f_{bcr}) \end{aligned} \tag{A.14}$$

In order to compute the needed traces in the adjoint representation we must remember [CV 76]

$$(\lambda_a)_{\alpha\beta}(\lambda_a)_{\gamma\delta} = 2\left[\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}\right] \tag{A.15}$$

and using (A.3) a straightforward calculation leads to

$$\mathrm{Tr}_{ad}\,[T_a T_b] = N\,\delta_{ab} \tag{A.16}$$

$$\mathrm{Tr}_{ad}\,[T_a T_b T_c] = i\,\frac{N}{2}\,f_{abc} \tag{A.17}$$

$$\mathrm{Tr}_{ad}\,[T_a T_b T_c T_d] = \delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + $$

$$+\frac{N}{4}\left(d_{abr}d_{cdr} - d_{acr}d_{dbr} + d_{adr}d_{bcr}\right) \tag{A.18}$$

Some other useful relations that can be easily derived are

$$(\lambda_a)_{\alpha\beta}(\lambda_a)_{\beta\gamma} = 2\,\frac{N^2-1}{N}\,\delta_{\alpha\gamma} \equiv 4\,C_2(R)\,\delta_{\alpha\gamma} \tag{A.19}$$

$$(T_a)_{bc}(T_a)_{cd} = N\,\delta_{bd} \equiv C_2(G)\,\delta_{bd} \tag{A.20}$$

$$(\lambda_b\lambda_a\lambda_b)_{\alpha\beta} = -\frac{2}{N}(\lambda_a)_{\alpha\beta} \tag{A.21}$$

$$(\lambda_a\lambda_b)_{\alpha\beta}(T_b)_{ca} = N\,(\lambda_c)_{\alpha\beta} \tag{A.22}$$

Let us now consider a vertex with n external gluon lines, then its contribution to the invariant T-matrix element can be writte as $V_{a_1 a_2 \dots a_n}$, where only the colour indices have been explicited. This can always be written as [PT 80]

$$V_{a_1 a_2 \dots a_m} = \sum_i V^{(i)} S^{(i)}_{a_1 a_2 \dots a_m} \tag{A.23}$$

where the tensors $S^{(i)}_{a_1 a_2 \dots a_n}$ $(i = 1,2,\dots)$ form a basis constructed uniquely from δ_{ab}, f_{abc} and d_{abc}. Let us try to characterize this basis for the lowest values of n. For $n = 1$ the base does not exist. For $n = 2$ the basis contains only one element : δ_{ab}. For $n = 3$ the elements of the basis are f_{abc} and d_{abc}. Finally let us consider the situation for $n = 4$. Let us characterize an irreducible representation of SU(N) by its Young diagram $(\lambda_1, \lambda_2, \dots, \lambda_{N-1})$ where $\lambda_j + \lambda_{j+1} + \dots + \lambda_{N-1}$ denotes the number of boxes in the row j. Then one can prove that the Clebsch-Gordan series for the product of two adjoint representations can be written as

$$N = 3 \qquad (1,1) \otimes (1,1) = (0,0) \oplus (1,1) \oplus (1,1) \oplus (3,0) \oplus (0,3) \oplus (2,2)$$

$$N = 4 \qquad (1,0,1) \otimes (1,0,1) = (0,0,0) \oplus (1,0,1) \oplus (1,0,1) \oplus (0,2,0)$$
$$\oplus (2,1,0) \oplus (0,1,2) \oplus (2,0,2)$$

$$N \geq 5 \qquad (1,0,\dots,0,1) \otimes (1,0,\dots,0,1) = (0,0,\dots,0,0) \oplus (1,0,\dots,0,1)$$
$$\oplus (1,0,\dots,0,1) \oplus (0,1,\dots,1,0) \oplus (2,0,\dots,1,0) \oplus$$
$$+ (0,1,\dots,0,2) \oplus (2,0,\dots,0,2)$$

where the dots denote (N-5) zeroes. If we characterize the irreducible representations by their dimension we can write $(N \geq 3)$

$$(N^2-1)\otimes(N^2-1) = 1 \oplus (N^2-1) \oplus (N^2-1) \oplus \frac{1}{4}N^2(N+1)(N-3)$$

$$\oplus \frac{1}{4}(N^2-4)(N^2-1) \oplus \overline{\frac{1}{4}(N^2-4)(N^2-1)} \oplus \frac{1}{4}N^2(N-1)(N+3) \tag{A.24}$$

Clearly for $N = 3$ the fourth term in the r.h.s. disappears. For one external gluon line (n=1) there are no invariant tensors. For n=2 since $(N^2-1)\otimes(N^2-1)$ contains only once the trivial representation, there is only one irreducible tensor: δ_{ab} . For n=3 and since $(N^2-1)\otimes(N^2-1)\otimes(N^2-1)$ contains only twice the trivial representation there are two invariant tensors: d_{abc} and f_{abc} . For n=4 and $N \geq 4$ the Clebsh-Gordon series for $(N^2-1)\otimes(N^2-1)\otimes(N^2-1)\otimes(N^2-1)$ contains nine times the trivial representation, and therefore the independent invariant tensors can be chosen as

$$\begin{array}{lll} \delta_{ab}\,\delta_{cd} & \delta_{ac}\,\delta_{db} & \delta_{ad}\,\delta_{bc} \\ & & \\ d_{abr}\,d_{cdr} & d_{acr}\,d_{dbr} & d_{adr}\,d_{bcr} \\ & & \\ d_{abr}\,f_{cdr} & d_{acr}\,f_{dbr} & d_{adr}\,f_{bcr} \end{array} \tag{A.25}$$

If N=3 the same reasoning proves that there are only eight invariant tensors and there must therefore exist a relation among these nine ten sors and it is

$$\delta_{ab}\,\delta_{cd} + \delta_{ac}\,\delta_{db} + \delta_{ad}\,\delta_{bc} =$$

$$= 3\left[d_{abr}\,d_{cdr} + d_{acr}\,d_{dbr} + d_{adr}\,d_{bcr} \right] \tag{A.26}$$

For $N = 3$ an explicit representation of the λ_a is

$$\lambda_1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \lambda_2 = \begin{vmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \lambda_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\lambda_4 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \qquad \lambda_5 = \begin{vmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{vmatrix} \qquad \lambda_6 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \tag{A.27}$$

$$\lambda_7 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{vmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

Furthermore

$$\begin{aligned} f_{123} &= +1 \\ f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = +\frac{1}{2} \\ f_{458} &= f_{678} = +\frac{\sqrt{3}}{2} \end{aligned} \tag{A.28}$$

$$\begin{aligned} d_{118} &= d_{228} = d_{338} = -d_{888} = +\frac{1}{\sqrt{3}} \\ d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = +\frac{1}{2} \\ d_{448} &= d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \end{aligned} \tag{A.29}$$

The components of these tensors which cannot be obtained by permutations of indices of the above given ones are zero.

APPENDIX B.- PATH INTEGRALS

Local gauge invariant theories are difficult to quantize because the gauge fields are gauge dependent quantities and therefore exhibit an extra non-dynamical degree of freedom which must be dealt with. The most convenient method for overcoming this difficulty is Feynman's path integral formalism [FE 48][FE 50][FH 65]. We present in this Appendix a brief outline of these methods. We have made extensive use of [AL 73][ZI 75][MP 78][FS 80] where more details can be found.

As it is well known in the path integral formalism the transition amplitude, in Quantum Mechanics, from an initial state characterized by coordinates $(q_1, q_2, \ldots, q_N)$ at the time t to a final state with coordinates $(q_1', q_2', \ldots, q_N')$ at the instant t' is given by $(\hbar = 1)$

$$\langle q_1', q_2', \ldots q_N' t' | q_1 q_2 \ldots q_N t \rangle = \lim_{n \to \infty} \prod_{\alpha=1}^{N} \int \prod_{i=1}^{n} dq_\alpha(t_i) \int \prod_{i=1}^{n+1} \frac{dp_\alpha(t_i)}{2\pi} \cdot$$

$$\exp\left[i \sum_{j=1}^{n+1} \left\{ \sum_{\alpha=1}^{N} p_\alpha(t_j)[q_\alpha(t_j) - q_\alpha(t_{j-1})] - \epsilon H\left(p(t_j), \frac{q(t_j) + q(t_{j-1})}{2}\right)\right\}\right] \quad (B.1)$$

where H is the Hamiltonian of the problem, $t_j \equiv j\epsilon + t$, $\epsilon \equiv$ $\equiv (t' - t)/(n+1)$, $t_o \equiv t$ and $t_{n+1} \equiv t'$. This can be written in a compact form as

$$\langle q't' | q t \rangle^J = \int [dq] \int [dp] \exp\left\{ i \int_t^{t'} d\tau \left[p(\tau)\dot{q}(\tau) - H(\tau) + J(\tau) q(\tau) \right\} \right. \quad (B.2)$$

where we have furthermore introduced a classical external source $J(t)$ It is now simple to prove that

$$\langle q't' | T(Q(t_1) \ldots Q(t_m)) | qt \rangle =$$

$$= \int [dq] \int [dp]\, q(t_1) \ldots q(t_m) \exp \left\{ i \int_t^{t'} d\tau \left[p(\tau)\dot{q}(\tau) - H(\tau) \right] \right\} \tag{B.3}$$

where $T(Q(t_1)\ldots.Q(t_n))$ is the time ordered product of Heisenberg position operators. Supposing that the source acts between t_i and t_f we find ($t < t_i < t_f < t'$)

$$\langle q't' | qt \rangle^J = \int dq_i\, dq_f \langle q't' | q_f t_f \rangle \langle q_f t_f | q_i t_i \rangle^J \langle q_i t_i | qt \rangle \tag{B.4}$$

On the other hand the transition amplitude can be written in terms of energy eigenstates as

$$\langle q't' | qt \rangle = \sum_m \phi_m(q') \phi_m^*(q) \exp \left\{ -i E_m (t'-t) \right\} \tag{B.5}$$

which in the limit $t' \to -i\infty$ or $t \to +i\infty$ is dominated by the groun state. One then obtains immediately that

$$\lim_{\substack{t' \to -i\infty \\ t \to +i\infty}} \frac{\langle q't' | qt \rangle^J}{\exp[-iE_0(t'-t)]\, \phi_0^*(q)\, \phi_0(q')} =$$

$$= \int dq_i \int dq_f\, \phi_0^*(q_f t_f) \langle q_f t_f | q_i t_i \rangle^J \phi_0(q_i t_i) \equiv W[J] \tag{B.6}$$

which is nothing but the ground state to ground state transition ampli tude in presence of a source J. Now from this expression one obtains by functional differentiation ($t_f > t_1, \ldots, t_m > t_i$)

$$\left. \frac{\delta^m W[J]}{\delta J(t_1) \ldots \delta J(t_m)} \right|_{J=0} =$$

$$= i^m \int dq_i \int dq_f\, \phi_0^*(q_f t_f)\, \phi_0(q_i t_i) \langle q_f t_f | T(Q(t_1) \ldots Q(t_m)) | q_i t_i \rangle \tag{B.7}$$

which is the ground state expectation value of a time ordered product of Heisenberg position operators; in field theory these will be the Green's functions. Notice that in a less well-defined manner also the physical limit $t' \to \infty$, $t \to -\infty$ picks out the ground state if

E_n is substituted by $E_n - i\eta$ with $\eta \downarrow 0$; the same happens in field theory in the Minkowskian region. All these equations can be immediately generalized to field theory. Let us begin considering a neutral scalar field $\phi(x)$. Let us subdivide the space into cubes of dimension ϵ^3 and label them by an integer α . We define the α-th coordinate $q_\alpha(t) \equiv \phi_\alpha(t)$ by

$$\phi_\alpha(t) \equiv \frac{1}{\epsilon^3} \int_{V_\alpha} d^3x \; \phi(t, \vec{x}) \tag{B.8}$$

where the integration is over the α-th cell of volume ϵ^3 . We can also write the Lagrangian as

$$L(t) \equiv \int d^3x \mathcal{L}(x) \longrightarrow \sum_{\alpha=1}^{N} \epsilon^3 \mathcal{L}_\alpha \left(\dot{\phi}_\alpha(t), \phi_\alpha(t), \phi_{\alpha \pm s}(t) \right) \tag{B.9}$$

where N is the total number of cells, $\dot{\phi}_\alpha(t)$ is the average of $\partial\phi(t,x)/\partial t$ over the α-th cell and $\phi_{\alpha \pm s}$ is the average value of the field in the neighboring cells ($\alpha \pm s$), which are needed to express the space derivatives of the field appearing in $\mathcal{L}(x)$. We define the canonical conjugate momentum $p_\alpha(t)$ as

$$p_\alpha(t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha(t)} = \epsilon^3 \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\phi}_\alpha(t)} \equiv \epsilon^\alpha \Pi_\alpha(t) \tag{B.10}$$

The Hamiltonian can be written as

$$H = \sum_\alpha p_\alpha(t)\,\dot{\phi}_\alpha(t) - L(t) \equiv \sum_\alpha \epsilon^3 \mathcal{H}_\alpha$$

$$\tag{B.11}$$

$$\mathcal{H}_\alpha\left(\Pi_\alpha(t), \phi_\alpha(t), \phi_{\alpha \pm s}(t)\right) = \Pi_\alpha(t)\,\dot{\phi}_\alpha(t) - \mathcal{L}_\alpha$$

In field theory we can write an expression equivalent to (B.1) as

$$\lim_{\substack{n\to\infty \\ \epsilon\downarrow 0}} \prod_{\alpha=1}^{N} \int \prod_{i=1}^{n} d\phi_\alpha(t_i) \int \prod_{i=1}^{n+1} \frac{\epsilon^3 d\Pi_\alpha(t_i)}{2\pi} \exp\Big[i \sum_{j=1}^{n+1} \epsilon \sum_{\alpha} \epsilon^3 .$$

$$\Big\{ \Pi_\alpha(t_j) \frac{\phi_\alpha(t_j) - \phi_\alpha(t_{j-1})}{\epsilon} - \mathcal{H}_\alpha\Big(\Pi_\alpha(t_j), \frac{\phi_\alpha(t_j)+\phi_\alpha(t_{j-1})}{2}, \frac{\phi_{\alpha\pm s}(t_j)+\phi_{\alpha\pm s}(t_{j-1})}{2}\Big)\Big\}\Big]$$

$$\equiv \int [\mathcal{D}\phi] \int [\mathcal{D}\Pi] \exp\Big\{ i \int_t^{t'} d\tau \int d^3x \Big[\Pi(\tau,\vec{x}) \frac{\partial \phi(\tau,\vec{x})}{\partial \tau} - \mathcal{H}(\tau,\vec{x})\Big]\Big\} \qquad \text{(B.12)}$$

where $\Pi(t,\vec{x}) \equiv \partial\mathcal{L}/\partial\dot{\phi}(t,\vec{x})$ and its cell average is just $\Pi_\alpha(t)$. This is the transition amplitude from an initial field configuration characterized by $\{\phi_\alpha(t_0) \equiv \phi_\alpha(t)\}$ to a final one given by $\{\phi_\alpha(t_{n+1}) = \phi_\alpha(t')\}$.

In field theory, all physical quantities are derivable from Green's functions, which are in turn derivable from vacuum-to-vacuum transition amplitudes in the presence of classical external sources. The physical vacuum plays the same role as the ground state in Quantum Mechanics. As in that case the fundamental quantity is

$$W[J] \equiv \int [\mathcal{D}\phi] \int [\mathcal{D}\Pi] \exp\Big\{ i \int d^4x \big[\Pi(x)\dot{\phi}(x) - \mathcal{H}(x) + \tfrac{1}{2} i\eta\, \phi^2(x) + J(x)\phi(x)\big]\Big\} \text{(B.13)}$$

where a J-independent factor has been omitted and the imaginary time limit has been substituted by a real one plus a new term in the Hamiltonian. The extra term proportional to $\eta \downarrow 0$ indicates how to rotate the time-integration contour to pick out the correct limit. We will return to this point.

From (13) we obtain

$$\frac{1}{W[0]} \left. \frac{\delta^m W[J]}{\delta J(x_1)\,\delta J(x_2) \ldots\ldots \delta J(x_m)} \right|_{J=0} = i^m \frac{\langle \underset{\sim}{0} | T(\phi(x_1)\phi(x_2) \ldots \phi(x_m)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} \equiv$$

$$\equiv i^m \, G(x_1, x_2, \ldots, x_m) \tag{B.14}$$

where $|\underset{\sim}{0}\rangle$ is the physical normalized vacuum. $G(x_1,x_2,\ldots,x_n)$ is the n-point complete Green's function, i.e. the vacuum expectation value of the time-ordered product of n fields. The word complete means that they include the contributions from disconnected diagrams, which are simply products of lower order Green's functions. The connec ed Green's functions are given by

$$G_c(x_1, x_2, \ldots x_m) = (-i)^m \left. \frac{\delta^m \ln W[J]}{\delta J(x_1)\ \delta J(x_2) \ldots \delta J(x_m)} \right|_{J=0} \tag{B.15}$$

or in another usual notation

$$W[J] \equiv \exp\left\{ i Z_1 [J] \right\}$$

$$G_c(x_1, x_2, \ldots, x_m) = (-i)^{m-1} \left. \frac{\delta^m Z[J]}{\delta J(x_1)\ \delta J(x_2) \ldots \delta J(x_m)} \right|_{J=0} \tag{B.16}$$

The fact that the Green's functions may be defined by (14) was first discovered by Schwinger [SC 51] and does not depend on the path-integral formula (13) for $W[J]$, but the path integral provides not only a simple proof of (14), but an explicit formula for computing $W[J]$.

Let us check that (16) gives the connected Green's function in a simple case

$$G_c(x_1,x_2,x_3) = -\left.\frac{\delta^3 Z[J]}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)}\right|_{J=0} = i\left.\frac{\delta^3 \ln W[J]}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)}\right|_{J=0} =$$

$$= \frac{i}{W[J]}\left.\frac{\delta^3 W[J]}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)}\right|_{J=0} - \frac{i}{W[J]^2}\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\left.\frac{\delta W[J]}{\delta J(x_3)}\right|_{J=0} -$$

$$- \frac{i}{W[J]^2}\frac{\delta^2 W[J]}{\delta J(x_2)\,\delta J(x_3)}\left.\frac{\delta W[J]}{\delta J(x_1)}\right|_{J=0} - \frac{i}{W[J]^2}\frac{\delta^2 W[J]}{\delta J(x_3)\delta J(x_1)}\left.\frac{\delta W[J]}{\delta J(x_2)}\right|_{J=0} +$$

$$+ 2\frac{i}{W[J]^3}\frac{\delta W[J]}{\delta J(x_1)}\frac{\delta W[J]}{\delta J(x_2)}\left.\frac{\delta W[J]}{\delta J(x_3)}\right|_{J=0} =$$

$$= \frac{\langle \underset{\sim}{\varrho} | T(\phi(x_1)\phi(x_2)\phi(x_3)) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle} - \frac{\langle \underset{\sim}{\varrho} | T(\phi(x_1)\phi(x_2)) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle}\frac{\langle \underset{\sim}{\varrho} | \phi(x_3) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle} -$$

$$- \frac{\langle \underset{\sim}{\varrho} | T(\phi(x_2)\phi(x_3)) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle}\frac{\langle \underset{\sim}{\varrho} | \phi(x_1) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle} - \frac{\langle \underset{\sim}{\varrho} | T(\phi(x_3)\phi(x_1)) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle}\frac{\langle \underset{\sim}{\varrho} | \phi(x_2) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle} +$$

$$+ 2\frac{\langle \underset{\sim}{\varrho} | \phi(x_1) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle}\frac{\langle \underset{\sim}{\varrho} | \phi(x_2) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle}\frac{\langle \underset{\sim}{\varrho} | \phi(x_3) | \underset{\sim}{\varrho}\rangle}{\langle \underset{\sim}{\varrho} | \underset{\sim}{\varrho}\rangle}$$

which is the desired result as can be seen immediately.

Let us now consider the case in which the Hamiltonian densit is of the form

$$\mathcal{H}(x) = \frac{1}{2}\pi^2(x) + f[\phi(x), \vec{\nabla}\phi(x)] \tag{B.17}$$

then the $\pi(x)$ integrations can be carried out explicitly, and we obtain from (13)

$$W[J] = N\int \mathcal{D}[\phi]\exp\left\{ i\int d^4x\,[\mathcal{L}(x) + J(x)\phi(x)]\right\}$$

$$\mathcal{L}(x) \equiv \frac{1}{2}[\partial_0\phi(x)]^2 - f[\phi(x), \vec{\nabla}\phi(x)] + \frac{1}{2}i\eta\,\phi^2(x) \tag{B.18}$$

where N is a J-independent factor which is unimportant when we are interested in the Green's functions. For gauge theories $\mathcal{H}(x)$ is not of the form given in (17) and we will be forced to work with a modified Lagrangian. Nevertheless in order to get some familiarity with this method let us go on with our neutral scalar field assuming the Lagrangian density is given by

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x)$$

$$\mathcal{L}_0(x) = \frac{1}{2}\left[\partial_\mu \phi(x)\, \partial^\mu \phi(x) - m^2 \phi^2(x)\right] \quad , \quad \mathcal{L}_I(x) = \mathcal{L}_I\left[\phi(x)\right] \tag{B.19}$$

which leads to a Hamiltonian density of the form (17) and therefore (18) can be used.

The functional $W[J]$ of eq. (18) is in general an ill-defined integral even in the lattice approximation. The Osterwalder-Schrader theorem of axiomatic field theory [OS 73] [OS 75] indicates that if one can construct a well-behaved field theory in the Euclidean space $(\tau, \vec{x})$ [SY 69] obeying certain appropiate axioms, then one gets the corresponding field theory in the Minkowski space $(x^0, \vec{x})$ as the analytic continuation of the former as $\tau = ix^0$, which obeys Wightman's axioms. Thus any ambiguities should be resolved by appealing to the Euclidicity postulate, namely that the Green's functions (16) are the analytic continuation of those defined by the well-defined functional integral in the Euclidean field theory

$$W_E[J] = N \int \mathcal{D}[\phi] \exp\left\{ - \int d^3x\, d\tau \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial \tau}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + m^2\phi^2 - \mathcal{L}_I[\phi] - J\phi \right] \right\} \tag{B.20}$$

Notice that if $-\mathcal{L}_I[\phi]$ is bounded from below, as it will be assumed, the quantity in the square brackets in the exponent of (20) is also bounded from below and the integral is well defined. In other words the Euclidicity postulate implies that when computations are performed in the Euclidean space-time, the generating functionals $W_E[J]$ are well defined and can be analytically continued to Minkowski space and that the amplitudes found in this way correspond to the physical ones. It has been proved that this procedure actually works in perturbation theory for all renormalizable theories.

The Euclidicity postulate determines the boundary conditions to be imposed on propagators. For our problem it implies that in Minkowski space we must add a damping term as the one introduced in (13). Let us begin considering the free field case where the generating functional $W_0[J]$ is given by

$$W_0[J] = \int \mathcal{D}[\phi] \exp\left\{ i \int d^4x \left[\frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} i\eta \phi^2 + J\phi \right] \right\} \tag{B.21}$$

where here, and in the future, we have neglected inessential multiplicative factors (factors that are J-independent). The explicit form of (21) is

$$W_0[J] = \lim_{\epsilon \downarrow 0} \int \prod_\alpha d\phi_\alpha \exp\left\{ i \left[\sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \frac{1}{2} \phi_\alpha K_{\alpha\beta} \phi_\beta + \sum_\alpha \epsilon^4 J_\alpha \phi_\alpha \right] \right\} \tag{B.22}$$

where α labels space-time cells of dimension ϵ^4 and the matrix $K_{\alpha\beta}$ is such that

$$\lim_{\epsilon \downarrow 0} K_{\alpha\beta} = \left[-\partial_\mu \partial^\mu - m^2 + i\eta \right] \delta^{(4)}(x-y) \tag{B.23}$$

where $\alpha \to x$ and $\beta \to y$ when $\epsilon \downarrow 0$ and $\sum_\alpha \epsilon^4 \to \int d^4x$. Since

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} dx_i \, \exp\left\{ -\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i \right\} =$$

$$= \frac{(2\pi)^{n/2}}{[\det A]^{1/2}} \exp\left\{ \frac{1}{2} \sum_{i,j=1}^{n} b_i (A^{-1})_{ij} b_j \right\} \tag{B.24}$$

where the matrix A has positive definite real part, we can perform cavalierly the integrations over $\{\phi_\alpha\}$ in (22) by going forth and back to the Euclidean space and, up to inessential factors, obtain

$$W_0[J] = \lim_{\epsilon \downarrow 0} \frac{1}{[\det K]^{1/2}} \exp\left\{ -\frac{i}{2} \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 J_\alpha \frac{1}{\epsilon^8} (K^{-1})_{\alpha\beta} J_\beta \right\} \tag{B.25}$$

where

$$\sum_\gamma K_{\alpha\gamma} (K^{-1})_{\gamma\beta} = \delta_{\alpha\beta} \tag{B.26}$$

and as $\epsilon \downarrow 0$ we have

$$\frac{1}{\epsilon^4} \delta_{\alpha\beta} \longrightarrow \delta^{(4)}(x-y) \quad , \quad \sum_\alpha \epsilon^4 \longrightarrow \int d^4x \tag{B.27}$$

With the definition

$$\frac{1}{\epsilon^8} (K^{-1})_{\alpha\beta} \longrightarrow \Delta_F (x-y) \tag{B.28}$$

we can write eq. (26) as

$$\sum_\gamma \epsilon^4 K_{\alpha\gamma} \left(\frac{1}{\epsilon^8} (K^{-1})_{\gamma\beta} \right) = \frac{1}{\epsilon^4} \delta_{\alpha\beta}$$

$$\longrightarrow \quad [-\partial_\mu \partial^\mu - m^2 + i\eta] \Delta_F (x-y) = \delta^{(4)}(x-y) \tag{B.29}$$

and hence

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int d^4q \; \frac{e^{-iq\cdot x}}{q^2 - m^2 + i\eta} \tag{B.30}$$

which is the usual free Feynman propagator. Neglecting inessential factors our final result is

$$W_0[J] = \exp\left\{-\frac{i}{2}\int d^4x \int d^4y \; J(x)\,\Delta_F(x-y)\,J(y)\right\} \tag{B.31}$$

Let us now consider the interacting case (19) including the damping factor $m^2 \longrightarrow m^2 - i\eta$; then

$$W[J] = \int \mathcal{D}[\phi] \exp\left\{ i \int d^4x \, [\mathcal{L}_0(x) + \mathcal{L}_I(x) + J(x)\,\phi(x)] \right\} =$$

$$= \exp\left[i \int d^4x \, \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] \int \mathcal{D}[\phi] \exp\left\{ i\int d^4x \, [\mathcal{L}_0(x) + J(x)\,\phi(x)]\right\}$$

$$= \exp\left[i \int d^4x \, \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] \exp\left\{-\frac{i}{2}\int d^4y \int d^4z \; J(y)\,\Delta_F(y-z)\,J(z)\right\} \tag{B.32}$$

This equation is the basis of the Feynman-Dyson expansion of the Green functions of our theory. To carry it out we will use the Volterra expansion

$$\exp\left\{ i \int d^4x \, \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\} = \sum_{m=0}^{\infty} \frac{i^m}{m!} \left[\int d^4x \, \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]^m \tag{B.33}$$

and Wick's theorem is simply the rule for functional differentiation

$$\frac{\delta J(x)}{\delta J(y)} = \delta^{(4)}(x-y) \tag{B.34}$$

As an application of all that let us study, in the theory

$$\mathcal{L}_I(x) = -\frac{\lambda}{4!}\phi^4(x) \qquad , \tag{B.35}$$

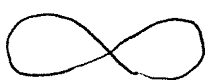
the complete two-point Green's function $G(x_1,x_2)$ at first order in λ. In the order that we are interested in

$$W[J] = \left\{ 1 - i\frac{\lambda}{4!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4 \right\} \cdot$$

$$\cdot \exp\left\{ -\frac{i}{2}\int d^4y \int d^4z \; J(y)\,\Delta_F(y-z)\,J(z) \right\}$$

Hence

$$W[0] = 1 - i\frac{\lambda}{4!}\int d^4x \; 3\,[\,i\Delta_F(0)\,]^2 =$$

$= 1 \; +$ 

and

$$\left.\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = -\frac{i}{2}\left[\Delta_F(x_1-x_2) + \Delta_F(x_2-x_1)\right] +$$

$$+ i\frac{\lambda}{4!}\int d^4x \left\{ 3\,[\,i\Delta_F(0)]^2 \left(\frac{i}{2}\right)\left[\Delta_F(x_1-x_2) + \Delta_F(x_2-x_1)\right] \right\} +$$

$$+ i\frac{\lambda}{4!}\int d^4x \left\{ 3\cdot 2\,[\,i\Delta_F(0)]\;2\,\frac{i}{2}\left[\Delta_F(x_1-x) + \Delta_F(x-x_1)\right]\frac{i}{2}\left[\Delta_F(x_2-x) + \Delta_F(x-x_2)\right]\right\}$$

$$= -i\Delta_F(x_1-x_2) + i\frac{\lambda}{8}\int d^4x \;[\,i\Delta_F(0)]^2\,[\,i\Delta_F(x_1-x_2)] +$$

$$+ i\frac{\lambda}{2}\int d^4x \,[\,i\Delta_F(0)]\,[\,i\Delta_F(x_1-x)]\,[\,i\Delta_F(x-x_2)] =$$

$=$  $+$  $+$ 

We obtain, using (14), that in the one loop approximation

$$G(x_1, x_2) = i\,\Delta_F\,(x_1 - x_2) \;-$$

$$- \; i\,\frac{\lambda}{2} \int d^4x \,[\, i\,\Delta_F\,(0)\,]\,[\, i\,\Delta_F\,(x_1 - x)\,]\,[\, i\,\Delta_F\,(x - x_2)\,]$$

Notice that the fact that there are no disconnected parts in $G(x_1, x_2)$ is due to the fact that $W_0\,[J]$ is quadratic in J. Before going on we would like to comment on an important point. In performing the integral (32) we have separated in the exponent of the integrand a special part composed of a quadratic $\phi\,K\,\phi$ and a linear $J\,\phi$ term. It is possible to prove [ZI 75] that the result (32) will still be obtained using different linear and quadratic forms and this establishes the uniqueness of the functional path integral.

An introduction to functional integration can be found in [GY 60]. A rigorous treatment of the method is given in [WM 79].

If we need to deal with fermion fields many of the results given above can be generalized if use is made of Grassmann variables. We will present only a few result that we will need later on; more on Grassmann variables can be found in [BE 66]. Let us consider a set of quantities $\theta_1, \theta_2, \ldots, \theta_N$ which satisfy the anticommutation relations

$$\{\,\theta_i\,,\;\theta_j\,\} = 0 \qquad , \qquad i, j = 1, 2, \ldots N \tag{B.36}$$

The algebra generated by these quantities is the so-called Grassmann algebra G_N. notice that $\theta_i^{\,2} = 0$ for $i = 1,2,\ldots,N$. It is obviou that G_N, considered as a linear space, has dimension 2^N. A convenient base in G_N is formed by the 2^N monomials

$$1, \theta_1, \theta_2, \ldots, \theta_N, \theta_1\theta_2, \theta_1\theta_3, \ldots, \theta_{N-1}\theta_N, \ldots\ldots, \theta_1\theta_2 \ldots \theta_N \tag{B.37}$$

We can always choose the p-degree monomial $\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p}$ appearing in (37) in such a way that $i_1 < i_2 < \ldots < i_p$. Notice that any element $f(\theta) \in G_N$ can be written as

$$f(\theta) = f_0 + \sum_{i_1} f_1(i_1)\,\theta_{i_1} + \sum_{i_1 i_2} f_2(i_1, i_2)\,\theta_{i_1}\theta_{i_2} + \\ + \ldots + \sum_{i_1 \ldots i_N} f_N(i_1, i_2, \ldots i_N)\,\theta_{i_1}\theta_{i_2}\ldots\theta_{i_N} \tag{B.38}$$

which has unique coefficients $f_p(i_1, i_2 \ldots i_p)$ if they are chosen fully antisymmetric.

We can define the right and left derivatives on any element of G_N . Both are defined as linear operators in G_N and therefore it is enough to give their action on the elements of the basis

$$\left[\frac{\partial}{\partial\theta_s}\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p}\right] = \delta_{si_1}\theta_{i_2}\ldots\theta_{i_p} - \delta_{si_2}\theta_{i_1}\ldots\theta_{i_p} + \ldots + (-1)^{p-1}\delta_{si_p}\theta_{i_1}\ldots\theta_{i_{p-1}} \tag{B.39}$$

$$\left[\theta_{i_1}\theta_{i_2}\ldots\theta_{i_p}\frac{\partial}{\partial\theta}\right] = \delta_{si_p}\theta_1\ldots\theta_{i_{p-1}} - \delta_{si_{p-1}}\theta_{i_1}\ldots\theta_{i_p} + \ldots + (-1)^{p-1}\delta_{si_1}\theta_{i_2}\ldots\theta_{i_p}$$

It is immediate to obtain the needed derivation rules by using (38) an these definitions, as for instance

$$\left[\frac{\partial}{\partial\theta_1}\left[\frac{\partial}{\partial\theta_2}f\right]\right] = -\left[\frac{\partial}{\partial\theta_2}\left[\frac{\partial}{\partial\theta_1}f\right]\right] \tag{B.40}$$

In order to define the integration we will introduce the symbols $d\theta_1$, $d\theta_2$,..., $d\theta_N$ which must satisfy the anticommutation rules

$$\{d\theta_i, d\theta_j\} = \}\theta_i, d\theta_j\} = 0 \qquad , \quad i, j = 1, 2, \ldots, N \tag{B.41}$$

and we define the following simple integrals as

$$\int d\theta_i = 0 \qquad , \qquad \int d\theta_i \, \theta_i = 1 \qquad , \quad i = 1, 2, \ldots, N \tag{B.}$$

Using (38) we can calculate now any multiple integral. In partic it is easy to prove that

$$\int d\theta_N \ldots d\theta_1 \, f(\theta) = N! \; f_N(1, 2, \ldots N) \tag{B.}$$

Let us now see how a change of the integration variables must be formed. Let us consider a non-singular matrix (K_{ij}) $i,j = 1,2,.$ and define new Grassmann variables $\xi_1 \ldots . \xi_N$ by

$$\theta_i = \sum_{j=1}^{N} K_{ij} \, \xi_j \tag{B.}$$

If we define

$$d\theta_i = \sum_{j=1}^{N} (K^{-1})_{ji} \, d\xi_j \tag{B.}$$

we guarantee

$$\int d\theta_i \, \theta_j = \int d\xi_i \, \xi_j = \delta_{ij} \tag{B.}$$

Then it follows immediately that

$$\begin{aligned} \theta_1 \theta_2 \ldots \theta_N &= [\det K] \, \xi_1 \xi_2 \ldots \xi_N \\ d\theta_N \, d\theta_{N-1} \ldots d\theta_1 &= [\det K^{-1}] \, d\xi_N \, d\xi_{N-1} \ldots d\xi_1 \end{aligned} \tag{B.}$$

and therefore

$$\int d\theta_N \dots d\theta_1 \, f(\theta) = [\det K^{-1}] \int d\xi_N \dots d\xi_1 \, f[\theta(\xi)] \tag{B.48}$$

Using all that it is easy to calculate

$$I \equiv \int d\theta_N \dots d\theta_1 \exp\left\{ \sum_{i,j=1}^{N} \theta_i A_{ij} \theta_j \right\}$$

where $\{A_{ij}\}$ is an antisymmetric matrix. Let us assume, for the moment, that $A_{ij} = A_{ij}^*$. It is wellknown that we can find an orthogonal matrix S such that

$$S^T A S = \begin{vmatrix} 0 & +\lambda_1 & 0 & 0 & \cdots \\ -\lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & +\lambda_2 & \cdots \\ 0 & 0 & -\lambda_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \equiv \tilde{A}$$

Let us now carry out the lineal transformation $\theta_i = \sum_{j=1}^{N} S_{ij} \xi_j$; using (47) and the fact that det $S = \mp 1$ we obtain

$$I = \int d\xi_N \dots d\xi_1 \exp\left\{ \sum_{i,j=1}^{N} \xi_i (\tilde{A})_{ij} \xi_j \right\}$$

Hence

$$I = \begin{cases} \int d\xi_N \dots d\xi_1 \exp\left\{ 2\left[\lambda_1 \xi_1 \xi_2 + \lambda_2 \xi_3 \xi_4 + \dots + \lambda_{\frac{N}{2}} \xi_{N-1} \xi_N \right] \right\} = 2^{N/2} \lambda_1 \lambda_2 \dots \lambda_{N/2} & N = \text{even} \\ \int d\xi_N \dots d\xi_1 \exp\left\{ 2\left[\lambda_1 \xi_1 \xi_2 + \lambda_2 \xi_3 \xi_4 + \dots + \lambda_{\frac{N-1}{2}} \xi_{N-2} \xi_{N-1} \right] \right\} = 0 & N = \text{odd} \end{cases}$$

and hence

$$\int d\theta_N \cdots d\theta_1 \exp\left\{\sum_{i,j=1}^{N} \theta_i A_{ij} \theta_j\right\} = [\det 2A]^{1/2}, \qquad A_{ij} = -A_{ji} \tag{B.49}$$

This equation is also valid for matrices with complex elements: Eq. (49) has been proved for real matrices. The l.h.s. of this equation is a sum of monomials in A_{ij} and therefore this is also true for the r.h.s. .Both are equal for real values of the arguments and therefore also fór complex values.

For a given Grassmann algebra G_N we can define $\bar\theta_1, \bar\theta_2, \ldots$ $\ldots, \bar\theta_N$ in such a way that $\theta_i \longleftrightarrow \bar\theta_i$ is an involution of the algebra on itself with the following properties

$$\begin{aligned}
&i) \quad \overline{(\bar\theta_i)} = \theta_i \\
&ii) \quad \overline{(\theta_i \theta_j)} = \bar\theta_j \bar\theta_i \\
&iii) \quad \overline{(\lambda \theta_i)} = \lambda^* \bar\theta_i , \qquad \lambda \text{ any complex number}
\end{aligned} \tag{B.50}$$

The elements of the Grassmann algebra with involution are $\theta_1, \ldots, \theta_N$, $\bar\theta_1, \ldots, \bar\theta_N$, anticommuting each one with the others. Defining integration with respect to the new elements as in (42) we can prove, following techniques similar to the used in the proof of (49), that

$$\int d\bar\theta_N d\theta_N \cdots d\bar\theta_1 d\theta_1 \exp\left\{-\sum_{i,j=1}^{N} \bar\theta_i B_{ij} \theta_j\right\} = \det B \tag{B.51}$$

for any matrix antisymmetric or not. In fact the properties shown in (50) are not required for the proof of (51). Notice that the minus sign in the exponential can be eliminated by exchanging the $d\bar\theta$'s with $d\theta$'s .

For our purpose we need to generalize all that to a Grassmann algebra with an infinite number of generators. This can be done rigorously [BE 66] but we will proceed in a completely formal way. Let us consider a Hilbert space and let $u_n(x)$ be a complete orthonormal set. Let us introduce the Grassmann function

$$\theta(x) \equiv \sum_{n=0}^{\infty} u_n(x)\, \theta_n \tag{B.52}$$

where θ_n are Grassmann variables. Clearly

$$\{ \theta(x) \,,\, \theta(y) \} = 0 \tag{B.53}$$

The elements $\theta(x)$ considered as generators, generate an algebra called the Grassmann algebra G . In complete analogy with (38) any element $f \in G$ can be written in a unique way as

$$f = \sum_{n=0}^{\infty} \int dx_1\, dx_2 \ldots dx_N\, \theta(x_1)\, \theta(x_2) \ldots \theta(x_n)\, f_n(x_1, x_2, \ldots x_n) \tag{B.54}$$

where $f_n\,(x_1, x_2, \ldots . x_n)$ are fully antisymmetric functions.

In analogy with (39) we can define the right and left derivatives, being their action on the elements of the basis

$$\left[\frac{\delta}{\delta\theta(x)} \theta(x_1) \ldots \theta(x_n) \right] = \delta(x-x_1)\, \theta(x_2) \ldots \theta(x_n) - \ldots\ldots + (-1)^{n-1} \delta(x-x_n)\, \theta(x_1) \ldots \theta(x_{n-1})$$

(B. 55)

$$\left[\theta(x_1) \ldots \theta(x_n) \frac{\delta}{\delta\theta(x)} \right] = \delta(x-x_n)\, \theta(x_1) \ldots \theta(x_{n-1}) - \ldots\ldots + (-1)^{n-1} \delta(x-x_1)\, \theta(x_2) \ldots \theta(x_n)$$

and from these definitions the operation rules for derivatives are immediately derived.

We can extend the integration. We will denote

$$\mathcal{D}[\theta(x)] \equiv \lim_{N\to\infty} d\theta_N \cdots d\theta_2 \, d\theta_1 \tag{B.56}$$

Let us now consider the integral

$$I \equiv \int \mathcal{D}[\theta(x)] \exp\left\{ \int dx\,dy \;\; \theta(x)\, A(x,y)\, \theta(y) \right\}$$

where $A(x,y) = - A(y,x)$. We can write formally $A_{ij} = \int dxdy\; u_i(x)$ $A(x,y)\; u_j(y) = - A_{ji}$, then

$$I = \lim_{N\to\infty} \int d\theta_N \cdots d\theta_1 \;\exp\left\{ \sum_{i,j=1}^{N} \theta_i A_{ij} \theta_j \right\}$$

and using (49)

$$I = \lim_{N\to\infty} \left[\det 2A_N \right]^{1/2}$$

where A_N is the NxN matrix. We will write

$$\int \mathcal{D}[\theta(x)] \exp\left\{ \int dx\,dy \;\; \theta(x)\, A(x,y)\, \theta(y) \right\} = \left[\det 2A\right]^{1/2} , \quad A(x,y) = -A(y,x) \tag{B.57}$$

Notice that this result is independent of the $\{u_n(x)\}$ used in order to define the basis since a change of the complete orthonormal set is implemented by a unitary transformation.

Furthermore in a Grassmann algebra with an involution we will write

$$\mathcal{D}[\bar{\theta}(x)]\mathcal{D}[\theta(x)] \equiv \lim_{N\to\infty} d\bar{\theta}_N \, d\theta_N \cdots d\bar{\theta}_1 \, d\theta_1 \tag{B.58}$$

and the generalization of (51) is

$$\int \mathcal{D}[\bar{\theta}(x)]\mathcal{D}[\theta(x)] \exp\left\{-\int dx\,dy\, \bar{\theta}(x)\, B(x,y)\, \theta(y)\right\} = \det B \qquad \text{(B.59)}$$

where B(x,y) is arbitrary.

Let us now consider the quantization of non abelian gauge theories. Let us start with a pure gauge field theory without matter couplings with classical Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{4} F^{a}_{\mu\nu}(x)\, F^{\mu\nu}_{a}(x)$$

$$F^{a}_{\mu\nu}(x) \equiv \partial_\mu B^{a}_{\nu}(x) - \partial_\nu B^{a}_{\mu}(x) + g f_{abc} B^{b}_{\mu}(x)\, B^{c}_{\nu}(x) \qquad \text{(B.60)}$$

We might expect that Green's functions are generated by the generating functional

$$W[J^{a}_{\mu}] \equiv \int [\mathcal{D}B^{a}_{\mu}] \exp\left\{ i\int d^4x \left[\mathcal{L}(x) + J_a^{\mu}(x)\, B_\mu^a(x)\right]\right\} \qquad \text{(B.61)}$$

where $J^{\mu}_{a}(x)$ is the classical source of the gauge field. Notice, before going on, that we start directly from the integrated expression of the generating functional, which however is in principle only true for hamiltonians of the type (17). We will use this starting point, because it allows a manifestly covariant quantization, and study the modifications it needs. One can also use the Coulomb gauge, build the Hamiltonian and construct $W[J^{a}_{\mu}]$ in a canonical way [FS 80] [AL 7 but manifest Lorentz covariance is lost. Both procedures are equivalent and lead to the same S-matrix elements, but the covariant one is handier for perturbative calculations.

Let us also mention here that the problems which show up in the canonical formalism for theories with derivative couplings due to the fact that the interaction hamiltonian is not just the opposite of the interaction Lagrangian[LU 68] are avoided in the functional formalism. Expression (61) is however not correct for gauge theories because the "kinetic" part of the Lagrangian density is singular. Indeed

$$\int d^4x\, \mathcal{L}_0(x) = -\frac{1}{4}\int d^4x\,[\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)][\partial^\mu B_a^\nu - \partial^\nu B_a^\mu(x)] \equiv$$

$$\equiv -\frac{1}{2}\int d^4x\, d^4y\; B_\mu^a(x)\; K^{\mu\nu}(x,y)\; B_\nu^a(y) \tag{B.62}$$

with

$$K^{\mu\nu}(x,y) \equiv [-g^{\mu\nu}\partial_\lambda\partial^\lambda + \partial^\mu\partial^\nu]\,\delta^{(4)}(x-y) \tag{B.63}$$

but $K^{\mu\nu}(x,y)$ is singular and cannot be inverted. In fact

$$\partial_\mu\, K^{\mu\nu}(x,y) = 0 \tag{B.64}$$

and it is essentially a projection operator for the transverse components of $B_\mu^a(x)$. This means in particular that the Euclidean version of the functional integral has no Gaussian damping factor with respect to the variation of the non-transverse components of $B_\mu^a(x)$ and it is thus meaningless.

More generally, the action is invariant under the gauge tran formation (I.10).

$$B^\mu(x) \longrightarrow B'^\mu(x) = G(x)\,B^\mu(x)\,G^{-1}(x) + [\partial^\mu G(x)]\,G^{-1}(x) \tag{B.65}$$

which means that it is constant on the orbits of the gauge group which are formed by all the gauge transformed of a fixed field $B_\mu^a(x)$. The path integral for the vacuum to vacuum amplitude $W[J_\mu^a]$ diverges since along those orbits the action does not provide damping. Faddeev and Popov [FP 67] pointed out that $W[0]$ is thus proportional to the "volume" of the orbits and this factor should be extracted before defining $W[J_\mu^a]$. For gauge fields the path integral should be performed not over all variations of the gauge fields, but rather over distinct orbits of $B_\mu^a(x)$.

To implement this idea we shall choose a "hypersurface" in the manifold of all fields which intersects each orbit only once. This means that if

$$f_a\,[\,B_\mu^b(x)\,;\,x\,] = 0 \qquad , \quad a=1,2,\ldots,(N^2-1) \tag{B.66}$$

is the equation of the hypersurface, in each set of equivalent fields there actually exists a unique set $B_\mu^b(x)$ which satisfies the condition (66) . This set, considered as representative of the class, characterizes uniquely the true physical configuration. Conditions (66) are really a set of non-linear equations for $G(x)$ or for the parameters $\Theta_a(x)$ which characterize the gauge transformation in the sense that for a given field configuration it is always possible to find one and only one equivalent field which satisfies (66) . In this sense conditions (66) are the so-called gauge fixing conditions. Another requirement that is less fundamental, although important practically, is that the equations (66) must not be too complicated and should give a sufficiently explicit solution $G(x)$, at least in the framework of perturbation theory.

A necessary condition for the solvability of (66) is the non-degeneracy of the Jacobian corresponding to an infinitesimal gauge transformation (I.7): If $B_\mu^b(x)$ is a given field configuration which does not satisfy (66), in order to find a solution of these equations we must be able to solve for the $\delta\Theta_a(x)$ the equations

$$f_a\left[B_\mu^b(x) - g f_{bdc} B_\mu^d(x)\,\delta\Theta_c(x) - \partial_\mu \delta\Theta_b(x);\ x\right] = 0 \tag{B.67}$$

which can be written as

$$f_a[B_\mu^b(x);x] - \int d^4y\, \frac{\delta f_a[B_\mu^b(x);x]}{\delta B_\nu^c(y)}\left[\delta_{cd}\partial_\nu + g f_{ced} B_\nu^e(y)\right]\delta\Theta_d(y) = 0 \tag{B.68}$$

this can be solved if and only if

$$\det M_f \equiv \det\left\{ - \frac{\delta f_a[B_\mu^b(x), x]}{\delta B_\nu^c(y)}\left[\delta_{cd}\partial_\nu - g f_{cde} B_\nu^e(y)\right]\right\} =$$

$$= \det\left\{ - \frac{\delta f_a[B_\mu^b(x);x]}{\delta B_\nu^c(y)}\left[D_\nu(y)\right]_{cd}\right\} \neq 0 \tag{B.69}$$

which is the so-called admissibility condition for the gauge fixing conditions.

An example of all that is the so called Hamilton or time-like gauge characterized by the following gauge fixing conditions

$$n_\mu B_a^\mu(x) = 0 \quad , \quad n_\mu n^\mu > 1 \quad , \quad a = 1, 2, \ldots (N^2-1) \tag{B.70}$$

In particular we can choose $n^\mu \equiv (1,0,0,0)$. For an arbitarily give field configuration the equation for $G(x)$ that allows us to obtain the equivalent field configuration that satisfies (67) is immediately obtained using (65) and it is

$$\frac{\partial G(t,\vec{x})}{\partial t} = - G(t,\vec{x})\, B^0(t,\vec{x})$$

which has the unique solution

$$G(t,\vec{x}) = T \exp\left\{ - \int_{-\infty}^{t} ds\, B^0(s,\vec{x}) \right\} \tag{B.71}$$

where T is the time ordering operator. This gauge is interesting because it is also admissible beyond the scope of perturbation theory, which is the only thing guaranteed by (67) . Nevertheless it is non-covariant and this is not convenient for carrying out explicit calculations.

Common choices of the gauge fixing conditions are

$$\begin{array}{lll} \partial_\mu B^\mu_a(x) = 0 & & \text{Lorentz gauge} \\ \partial_\mu B^\mu_a(x) = A'_a(x) & & \text{Generalized Lorentz gauge} \\ n_\mu B^\mu_a(x) = 0 , & n^2 < 0 & \text{Axial gauge} \\ n_\mu B^\mu_a(x) = 0 , & n^2 = 0 & \text{Light-like gauge} \\ n_\mu B^\mu_a(x) = 0 , & n^2 > 0 & \text{Hamilton or time-like gauge} \\ \vec{\nabla}\cdot \vec{B}_a(x) = 0 & & \text{Coulomb gauge} \end{array} \tag{B.72}$$

and the corresponding axial, light-like, Hamilton and Coulomb generalized gauges, where $A_a(x)$ are arbitrary functions. All of them satisfy the admissibility condition (69) for small field configurations, but in some cases the uniqueness condition may fail for large fields beyond the domain of perturbation theory.

In the Lorentz gauge the admissibility condition is

$$\det \{ \delta_{ab} \Box - g f_{abc} \partial^\mu B_\mu^c(y) \} \neq 0 \tag{B.73}$$

which for small g reduces to $\det(\delta_{ab} \Box) \neq 0$, whereas in the axial gauge the condition is

$$\det \{ \delta_{ab} n^\mu \partial_\mu - g f_{abc} n^\mu B_\mu^c(y) \} = \det (\delta_{ab} n^\mu \partial_\mu) \neq 0 \tag{B.74}$$

We refer to [FS 80] for a discussion of these results. Notice that if we have a field configuration which satisfies (66) the infinitesimal gauge transformation which leaves (66) invariant is, for small g in the Lorentz gauge, $\Box \delta \Theta_a(x) = 0$. In Euclidean space the solution is $\delta \Theta_a(x) = C_a + C_a^\mu x_\mu$, which for small g generates a gauge transformation $B_\mu^b(x) \rightarrow B_\mu^b(x) - C_\mu^b$. This however changes the boundary conditions, but recall that in the path integral formalism one integrates over all the paths with the same boundary conditions. This is the reason why one condition (66) is enough for quantizing gauge field

Before going on we will review briefly a few simple facts about group representations. Let us denote by u an element of the gauge group that in our case is SU(N). Its elements can be characterized by (N^2-1) real parameters Θ_a, $a = 1,2,\ldots(N^2-1)$. Let $G(u)$ be the representation of u acting on the gauge fields that correspond to the adjoint representation of SU(N). For infinitesimal transformations

$$G(u) = I - i g T_a \Theta_a + O(\Theta^2) \tag{B.75}$$

where T_a is the adjoint representation of the Lie algebra of SU(N). Clearly if $u, u' \in SU(N)$, then $u.u' \in SU(N)$ and $G(u)\, G(u') =$

$G(uu')$. The invariant Hurwitz measure over the group is an integration measure on the group space which is invariant in the sense $du' = d(u'u)$. In the neighborhood of the identity we may always choose

$$du = \prod_a d\theta_a \tag{B.76}$$

Since we are dealing with a local gauge group θ_a is a space-time function and therefore this is also the case for u and $G(u)$.

After all these preliminaries let us proceed to quantize our gauge field. Let us define $\Delta_f [B^a_\mu]$ by

$$\Delta_f [B^b_\mu] \int \prod_x du(x) \prod_{x,a} \delta[f_a\{B^{b,u}_\mu(x) ; x\}] = 1 \tag{B.77}$$

where $B^{b,u}_\mu(x)$ is the u-transform of $B_\mu^b(x)$. Now we can verify easily that $\Delta_f [B^a_\mu]$ is gauge invariant:

$$\Delta_f^{-1} [B^{b,u}_\mu] = \int \prod_x du'(x) \prod_{x,a} \delta[f_a\{B^{b,u'u}_\mu(x) ; x\}] =$$

$$= \int \prod_x d(u'(x)u(x)) \prod_{x,a} \delta[f_a\{B^{b,u'u}_\mu(x) : x\}] =$$

$$= \int \prod_x du''(x) \prod_{x,a} \delta[f_a\{B^{b,u''}_\mu(x) ; x\}] = \Delta_f^{-1} [B^b_\mu]$$

In (61) we gave a naîve expression for $W[J^a_\mu]$ and using (77) we can write

$$W[0] = \int \prod_x du(x) \int [\mathcal{D} B^a_\mu] \, \Delta_f [B^a_\mu] \prod_{x,b} \delta[f_b\{B^{a,u}_\mu(x) ; x\}] .$$

$$\cdot \exp\{i \int d^4x \, \mathcal{L}(x)\}$$

We can perform the gauge transformation $B^a_\mu(x) \longrightarrow B^{a,u^{-1}}_\mu(x)$ and since the action and the functional integration measure are invariant under these (because of their form, eq. (65))

$$W[0] = \int \prod_x du(x) \int [\mathcal{D} B_\mu^a]\, \Delta_f\, [B_\mu^a] \prod_{x,b} \delta [f_b \{ B_\mu^a(x) ; x \}]\, \exp \{ i \int d^4x\, \mathcal{L}(x) \}$$

and the integrand of the group integration is independent of $u(x)$. This is the crucial observation of Faddeev-Popov, who saw that $\int \prod_x du(x)$ is an infinite factor independent of the fields and is an inessential factor. This allows us to define

$$W[J_\mu^a] \equiv \int [\mathcal{D} B_\mu^a]\, \Delta_f\, [B_\mu^a] \prod_{x,b} \delta [f_b \{ B_\mu^a(x) ; x \}] \cdot$$

$$\cdot \exp \{ i \int d^4x\, [\mathcal{L}(x) + J_\mu^a(x)\, B_a^\mu(x)] \} \tag{B.78}$$

This is the desired solution to our problem since now the Dirac delta function guarantees that only one field configuration contributes to each orbit.

Now we must rewrite this expression in a somewhat more convenient way. Let us begin calculating $\Delta_f [B_\mu^a]$. Since it appears in (78) multiplied by a Dirac delta function it suffices to compute it only for $B_\mu^a(x)$ which satisfy (66). As we have seen before for infinitesimal gauge transformations

$$f_a [B_\mu^{b,u}(x), x] = f_a [B_\mu^b(x) ; x] + \int d^4y\, M_{f\,ac}(x,y)\, \theta_c(y) + O(\theta^2) =$$

$$= \int d^4y\, M_{f\,ac}(x,y)\, \theta_c(y) + O(\theta^2)$$

where M_f is defined in (69) and the fields appearing in it are the ones satisfying the gauge fixing condition. From (77)

$$\Delta_f^{-1} [B_\mu^b] = \int \prod_{x,a} \{ d\,\theta_a(x)\ \delta [\int d^4y\ M_{fac}(x,y)\,\theta_c(y)] \}$$

and hence

$$\Delta_f [B_\mu^b] = \det M_f \tag{B.79}$$

Introducing the Grassmann fields $\bar{\phi}_a(x)$, $\phi_b(x)$, a,b = 1,2,... (N^2-1) , which are the so-called Faddeev-Popov ghost fields, and using eq. (59) we can write the generating functional, up to unessential factors, as

$$W[J_\mu^a] = \int [\mathcal{D} B_\mu^a][\mathcal{D}\bar{\phi}_b][\mathcal{D}\phi_c] \prod_{x,a} \delta [f_a \{ B_\mu^b(x) ; x \}]$$

$$\exp \{ i \int d^4x [\mathcal{L}(x) + J_\mu^a(x)\, B^\mu_a(x)] + i \int d^4x\, d^4y\ \bar{\phi}_b(x)\, M_{f,bc}(x,y)\,\phi_c(y) \} \tag{B.80}$$

If we use the Lorentz gauge $(f \equiv L)$ then we obtain immediately (73)

$$M_{Lab}(x,y) = \delta^{(4)}(x-y) [\delta_{ab}\,\partial_\mu \partial^\mu - g\, f_{abc}\,\partial_\mu\, B_c^\mu(x)] \tag{B.81}$$

and therefore

$$W[J_\mu^a] = \int [\mathcal{D} B_\mu^c][\mathcal{D}\bar{\phi}_b][\mathcal{D}\phi_c] \prod_{x,a} \delta [\partial_\mu B^\mu_a(x)] \,.$$

$$\exp \{ i \int d^4x [-\frac{1}{4} F_a^{\mu\nu}(x)\, F_{\mu\nu}^a(x) - \partial_\mu \bar{\phi}_a(x)\, \partial^\mu \phi_a(x)$$

$$+ g\, f_{abc} [\partial_\mu \bar{\phi}_a(x)]\, \phi_b(x)\, B_c^\mu(x) + J_\mu^a(x)\, B^\mu_a(x)] \} \tag{B.82}$$

For the axial gauge $(f \equiv A)$

$$M_{Aab}(x,y) = \delta^{(4)}(x-y)\,[\,\delta_{ab}\, n_\mu \partial^\mu\,] \tag{B.83}$$

and no ghost-gauge field coupling appears. The ghosts are then disposable.

Let us now consider the Dirac delta function appearing in this expression. First, let us pass from the Lorentz gauge to the generalized Lorentz gauge (GL)

$$\partial_\mu B^\mu_a(x) = A_a(x) \tag{B.84}$$

where $A_a(x)$ are arbitrary functions. The corresponding function $\Delta_{GL}[B_\mu{}^a]$ coincides on the hypersurface (84) with det M , where M is given by equation (81). Thus (82) can be identically rewritten substituting in the argument of the Dirac delta function $\partial_\mu B^\mu_a(x)$ by $\partial_\mu B^\mu_a(x) - A_a(x)$. Because of (77) the functional W[0] does not depe on $A_a(x)$ we can integrate over $A_a(x)$ with a weight function (which makes the integral convergent in Euclidean space)

$$\exp\left\{-\frac{i}{2a}\int d^4x\, A_a(x)\, A^a(x)\right\} \tag{B.85}$$

where a is an arbitrary constant. This leads to our final result: in the covariant Lorentz gauge the generating functional for a gauge field without matter coupling is

$$W[J^a_\mu, \bar{\xi}_a, \xi_a] = \int [\mathcal{D} B^a_\mu][\mathcal{D}\bar{\phi}_a][\mathcal{D}\phi_a] \exp \left\{ i \int d^4x \left[-\frac{1}{4} F^{\mu\nu}_a(x) F^a_{\mu\nu}(x) \right.\right.$$

$$- \partial_\mu \bar{\phi}_a(x) \partial^\mu \phi_a(x) + g f_{abc} \partial_\mu \bar{\phi}_a(x) \phi_b(x) B^\mu_c(x) - \frac{1}{2a} [\partial_\mu B^\mu_a(x)][\partial_\nu B^\nu_a(x)]$$

$$\left.\left. + J^a_\mu(x) B^\mu_a(x) + \bar{\xi}_a(x) \phi_a(x) + \bar{\phi}_a(x) \xi_a(x) \right] \right\} \tag{B.86}$$

where we have added the external ghost sources $\xi_a(x)$ and $\bar{\xi}_a(x)$ which are anticommuting quantities. No problem arises when the matter fields are included if they are coupled in a gauge invariant way.

We have given a formal derivation of (86) which has full meaning only in its Euclidean version; if, nevertheless, we use directly (86) we must add damping terms for all fields analogous to the ones appearing in (13) . It is also clear from our derivation that in doing perturbation theory the global sign of the ghost lagrangian is arbitrary.

More on all that and the study of other gauge fixing conditions can be found in [FS 80].

APPENDIX C.- FEYNMAN INTEGRALS

We are going to give formulae, in this Appendix, that ar needed in order to calculate the integrals appearing in the eval tion of the invariant T-matrix elements.

i) Feynman's parametrization

Different forms of Feynman's parametrization are

$$\frac{1}{a_1 a_2 \cdots a_m} = (m-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{m-2}} dx_{m-1} \frac{}{[a_1 x_{m-1} + a_2 (x_{m-2} - x_{m-1}) + \cdots + a_m (1 - x_1)]^m} \quad \text{(C}$$

$$= (m-1)! \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_{m-1} \frac{u_1^{m-2}\, u_2^{m-3} \cdots u_{m-2}}{[a_1 u_1 \cdots u_{m-1} + a_2 u_1 \cdots u_{m-2}(1-u_{m-1}) + \cdots + a_{m-1} u_1 (1-u_2) + a_m} \quad \text{(C}$$

$$= (m-1)! \int_0^1 dz_1 \int_0^1 dz_2 \cdots \int_0^1 dz_m \frac{\delta(1 - z_1 - z_2 - \cdots - z_m)}{[a_1 z_1 + a_2 z_2 + \cdots + a_m z_m]^m} \quad \text{(C}$$

where the second one is particularly useful. Particularizing (1) $n = 2$ and $n = 3$ and repeated derivation with respect to the gives the results

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1} (1-x)^{\beta-1}}{[ax + b(1-x)]^{\alpha+\beta}} \quad \text{(C}$$

$$\frac{1}{a^\alpha b^\beta c^\gamma} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(c)} \int_0^1 dx \; x \int_0^1 dy \frac{(xy)^{\alpha-1} [x(1-y)]^{\beta-1} (1-x)^{\gamma-1}}{[axy + bx(1-y) + c(1-x)]^{\alpha+\beta+\gamma}} \quad \text{(C}$$

where $\Gamma(z)$ is the usual Euler gamma function [AS 65].

ii) Gamma function

The gamma function is defined by Euler's integral

$$\Gamma(z) \equiv \int_0^\infty dt\; t^{z-1} e^{-t} \quad , \quad Re\, z > 0 \quad \text{(C}$$

It is more convenient for us to use the Weierstrass analytic continuation

$$\Gamma(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+z)} + \int_1^{\infty} dt\; t^{z-1} e^{-t} \tag{C.7}$$

which tells us that $\Gamma(z)$ is analytic in the entire z-plane but for simple poles at $z = 0,-1,-2,\ldots$. Since the analytic continuation is unique we can consider that the r.h.s. of (7) is the definition of the gamma function when the integral appearing in (6) does not exist .

Some useful properties of the gamma function are

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{C.8}$$

and some special values are

$$\Gamma(m+1) = m! \quad , \; m = \text{integer} \qquad ; \qquad \Gamma(1/2) = \sqrt{\pi} \tag{C.9}$$

Furthermore if $|z| < 1$ a useful expansion is

$$\Gamma(1+z) = \exp\left\{ -\gamma z + \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} z^m \right\} =$$

$$= 1 - \gamma z + \frac{1}{2}\left[\gamma^2 + \zeta(2)\right] z^2 - \frac{1}{6}\left[\gamma^3 + 3\gamma\,\zeta(2) + 2\zeta(3)\right] z^3 + \cdots \tag{C.10}$$

where $\gamma = 0.577\,215\,664\,9\ldots$ is Euler's constant and $\zeta(n)$ is Riemann's Zeta function

$$\zeta(s) \equiv \sum_{k=1}^{\infty} \frac{1}{k^s} \quad , \qquad Re\, s > 1 \tag{C.}$$

Some special values are

$$\zeta(2) = \frac{\pi^2}{6} \quad , \qquad \zeta(3) = 1.202\ 056\ 903\ 1 \ldots$$
$$\zeta(4) = \frac{\pi^4}{90} \tag{C.}$$

From (8) and (10) we obtain also

$$\Gamma(z) = \frac{1}{z} - \gamma + \frac{1}{2}\left[\gamma^2 + \zeta(2)\right] z - \frac{1}{6}\left[\gamma^3 + 3\gamma\zeta(2) + 2\zeta(3)\right] z^2 + \cdots$$

(C.

$$\frac{1}{\Gamma(1+z)} = 1 + \gamma z + \frac{1}{2}\left[\gamma^2 - \zeta(2)\right] z^2 + \frac{1}{6}\left[\gamma^3 - 3\gamma\zeta(2) + 2\zeta(3)\right] z^3 + \cdots$$

Sometimes it is useful to introduce the Beta function

$$B(z,w) \equiv \int_0^1 dt\ t^{z-1}(1-t)^{w-1} \quad , \ Re\, z > 0, \quad Re\, w > 0 \tag{C.}$$

which turns out to be

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{C.}$$

which can be used as definition when the integral (14) does not e

From (10)

$$B(1+az, 1+bz) = \frac{1}{1+(a+b)z} \exp\left\{ \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}\left[a^m + b^m - (a+b)^m\right] z^m \right\}$$

(C.

$$= 1 - (a+b)z + \left[(a+b)^2 - ab\,\zeta(2)\right] z^2 +$$

$$+ (a+b)\left[-(a+b)^2 + ab\,\zeta(2) + ab\,\zeta(3)\right] z^3 + \cdots$$

iii) Basic formula

In dimensional regularization a basic formula is $(\eta \downarrow 0)$

$$\Gamma(D, \alpha, \beta) \equiv \int \frac{d^D k}{(2\pi)^D} \frac{[k^2]^\alpha}{[k^2 - a^2 + i\eta]^\beta} =$$

$$= \frac{i}{(4\pi)^2} (-a^2)^{\alpha-\beta+2} \left(\frac{a^2}{4\pi}\right)^\epsilon \frac{\Gamma(2+\alpha+\epsilon)\,\Gamma(\beta-\alpha-2-\epsilon)}{\Gamma(\beta)\,\Gamma(2+\epsilon)} \tag{C.17}$$

where $D = 4 + 2\epsilon$ is the space-time dimension and α and β are arbitrary real numbers.

iv) Gegenbauer Polynomials

These polynomials are now widely used in the calculation of Feynmann integrals [CK 80] [RO 67]. Let us begin giving some propertie and later on we will see how they can be used to calculate some needed integrals.

The polynomial $C_n^\lambda(x)$ of degree n which is defined by the generating function [BE 53] [GR 65]

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{m=0}^{\infty} C_m^\lambda(x)\, t^m \tag{C.18}$$

is called the Gegenbauer polynomial of degree n and order λ. We will consider only $\lambda > 0$. Its general expression is

$$C_m^\lambda(x) = (-2)^{-m} (1-x^2)^{-\lambda+1/2} \frac{\Gamma(2\lambda+m)\,\Gamma(\lambda+1/2)}{m!\,\Gamma(2\lambda)\,\Gamma(\lambda+1/2+m)} \frac{d^m}{dx^m} (1-x^2)^{m+\lambda-1/2} =$$

$$= \sum_{k=0}^{[m/2]} \frac{(-1)^k\,\Gamma(\lambda+m-k)}{k!\,(m-2k)!\,\Gamma(\lambda)} (2x)^{m-2k} \tag{C.19}$$

$$C_m^\lambda(\cos\theta) = \sum_{k=0}^{m} \frac{\Gamma(\lambda+k)\,\Gamma(\lambda+m-k)}{k!\,(m-k)!\,\Gamma^2(\lambda)} \cos(m-2k)\theta$$

where $[m/2]$ means integer part of n/2 . Some special cases are

$$C_0^\lambda(x) = 1 \quad , \quad C_2^\lambda(x) = -\lambda + 2\lambda(\lambda+1)x^2$$

$$C_1^\lambda(x) = 2\lambda x \quad , \quad C_3^\lambda(x) = -2\lambda(\lambda+1)x + \frac{4}{3}\lambda(\lambda+1)(\lambda+2)x^3$$

$$C_m^\lambda(x=1) = \frac{\Gamma(n+2\lambda)}{n!\,\Gamma(2\lambda)} \quad , \quad C_n^\lambda(x=0) = \begin{cases} 0 \quad , & n = \text{odd} \\ \dfrac{(-1)^m\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)} \quad , & n = 2m = \text{even} \end{cases} \tag{C.2}$$

$$C_m^{1/2}(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1) = P_m(x)$$ Legendre Polynomi

$$C_n^1(\cos\theta) = \sum_{k=0}^{n}\cos(n-2k)\theta = \frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta)$$ Chebyshev Polynom

Furthermore

$$C_m^\lambda(-x) = (-1)^m C_m^\lambda(x) \tag{C.2}$$

The Gegenbauer polynomial $C_n^\lambda(x)$ is the polynomic solut of the differential equation

$$(1-x^2)\frac{d^2y(x)}{dx^2} - (2\lambda+1)x\frac{dy(x)}{dx} + n(n+2\lambda)y(x) = 0 \tag{C.2}$$

with adequate normalization. These polynomials satisfy the recurre formula

$$2(m+\lambda)x\,C_m^\lambda(x) = (m+1)C_{m+1}^\lambda(x) + (m+2\lambda-1)C_{m-1}^\lambda(x) \tag{C.2}$$

and their Clebsch-Gordan series is

$$C_\ell^\lambda(x)\, C_m^\lambda(x) = \sum_{k=|\ell-m|}^{\ell+m} \frac{k!\,(k+\lambda)\,\Gamma(g+2\lambda)}{\Gamma^2(\lambda)\,\Gamma(g+\lambda+1)\,\Gamma(k+2\lambda)} \cdot$$

$$\cdot\, \frac{\Gamma(g-\ell+\lambda)\,\Gamma(g-m+\lambda)\,\Gamma(g-k+\lambda)}{\Gamma(g-\ell+1)\,\Gamma(g-m+1)\,\Gamma(g-k+1)}\; C_k^\lambda(x) \tag{C.24}$$

where $\ell + n + k = 2g$, and g = integer. Other useful relations are

$$(1-ax)^{-\rho} = \frac{\Gamma(\lambda)}{\Gamma(\rho)} \sum_{m=0}^{\infty} \frac{(m+\lambda)\,\Gamma(m+\rho)}{\Gamma(m+\lambda+1)} \left(\frac{a}{2}\right)^n \cdot \tag{C.25}$$

$$\cdot\, {}_2F_1\left(\frac{\rho+m}{2}, \frac{\rho+m+1}{2}, \lambda+m+1; a^2\right) C_m^\lambda(x)$$

where ${}_2F_1(a,b,c;z)$ is the usual Gauss hypergeometric series [AS 65] and

$$x^m = \frac{\Gamma(\lambda)\, m!}{2^m} \sum_{j=0}^{[m/2]} \frac{(m-2j+\lambda)}{j!\,\Gamma(1+\lambda+m-j)}\; C_{m-2j}^\lambda(x) \tag{C.26}$$

Some special cases of the last formula are

$$x = \frac{1}{2\lambda}\, C_1^\lambda(x)$$

$$x^2 = \frac{1}{2\lambda(\lambda+1)} \left[C_2^\lambda(x) + \lambda\, C_0^\lambda(x) \right] \tag{C.27}$$

$$x^3 = \frac{3}{4\lambda(\lambda+1)(\lambda+2)} \left[C_3^\lambda(x) + (\lambda+1)\, C_1^\lambda(x) \right]$$

Let us now consider a D-dimensional Euclidean space and let us define λ as

$$D \equiv 4 + 2\epsilon \equiv 2(\lambda+1) \quad \Rightarrow \quad \lambda = 1+\epsilon \tag{C.28}$$

If p^μ and q^μ are arbitrary vectors in this space, then using some properties of the hypergeometric series, eq. (25) leads to

$$(p^2 - 2p.q + q^2)^{-\rho} = \frac{1}{[|p||q|]^{\rho}} \frac{\Gamma(\lambda)}{\Gamma(\rho)} \sum_{m=0}^{\infty} \frac{\Gamma(\rho+m)}{\Gamma(\lambda+m)} [T(p,q)]^{\rho+m} .$$

$$_2F_1(\rho-\lambda, \rho+m, \lambda+m+1; [T(p,q)]^2) \; C_m^{\lambda}(\hat{p}.\hat{q}) \tag{C.29}$$

$$x \equiv |x| \equiv [x^2]^{1/2}, \qquad \hat{x}^{\mu} \equiv \frac{x^{\mu}}{x}, \qquad T(p,q) = \min\left(\frac{p}{q}, \frac{q}{p}\right)$$

A special case of this formula is

$$(p^2 - 2p.q + q^2)^{-\lambda} = \frac{1}{[\max(q,p)]^{2\lambda}} \sum_{m=0}^{\infty} [T(p,q)]^m \; C_m^{\lambda}(\hat{p}.\hat{q}) \tag{C.30}$$

Let us denote by $d\Omega_k$ the solid angle in this space

$$d^D k \equiv k^{D-1} \, dk \, d\Omega_k \equiv S_D \, (k)^{2\lambda+1} \, dk \, d\hat{k}$$

$$S_D \equiv \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \tag{C.31}$$

being S_D the area of the sphere of radius 1 in this space; then

$$\int d\hat{k} \; C_{m_1}^{\lambda}(\hat{a}.\hat{k}) \, C_{m_2}^{\lambda}(\hat{b}.\hat{k}) = \delta_{m_1 m_2} \frac{\lambda}{m_1 + \lambda} \; C_{m_1}^{\lambda}(\hat{a}.\hat{b}) \tag{C.32}$$

In the process of calculating integrals other useful formula are

$$\frac{1}{[k^2]^{\alpha}} = \frac{\Gamma(\lambda+1-\alpha)}{\pi^{\lambda+1}\,\Gamma(\alpha)} \int d^D x \; \frac{e^{2ik.x}}{[x^2]^{\lambda+1-\alpha}} \tag{C.33}$$

$$e^{2ip.x} = \Gamma(\lambda) \sum_{m=0}^{\infty} i^m (m+\lambda) \; C_m^{\lambda}(\hat{x}.\hat{p}) \; (p^2x^2)^{m/2} \; j_{\lambda+m}(p^2x^2) \tag{C.34}$$

where

$$j_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\ \Gamma(n+\alpha+1)} z^n \tag{C.35}$$

An important property is

$$\int_0^\infty dz\ z^{2b+1} j_a(z^2) = \frac{\Gamma(b+1)}{2\Gamma(a-b)} \quad , \quad Re\,b > -1 \ , \quad Re\,a > 2Re\,b + \frac{1}{2} \tag{C.36}$$

All these formulae are useful to calculate Feynman integrals by the Gegenbauer method.

v) Integrals $I^{(\mu_1,\mu_2\ldots,\mu_n)}$

Let us define in the Minkowskian D-dimension space

$$I^{(\mu_1\mu_2\cdots\mu_n)}(p;r,s) \equiv \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq}{(2\pi)^D} \frac{q^{(\mu_1\mu_2\cdots\mu_n)}}{[q^2+i\eta]^r\,[(q-p)^2+i\eta]^s} \tag{C.37}$$

being ν a mass dimension scale parameter and $q^{(\mu_1\mu_2\cdots\mu_n)}$ is constructed starting from the n-rank symmetric tensor $q^{\mu_1}\ q^{\mu_2}\ \ldots.q^{\mu_n}$ and adding the needed terms to make it traceless if $n \geq 2$, i.e.

$$q^{(\mu_1)} \equiv q^{\mu_1}$$

$$q^{(\mu_1\mu_2)} \equiv q^{\mu_1} q^{\mu_2} - \frac{1}{D} q^2 g^{\mu_1\mu_2} \tag{C.38}$$

$$q^{(\mu_1\mu_2\mu_3)} \equiv q^{\mu_1}q^{\mu_2}q^{\mu_3} - \frac{1}{D+2} q^2 \left[g^{\mu_1\mu_2} q^{\mu_3} + g^{\mu_2\mu_3} q^{\mu_1} + g^{\mu_3\mu_1} q^{\mu_2} \right]$$

and so on. Inverting these equations we obtain

$$q^{\mu_1} = q^{(\mu_1)}$$

$$q^{\mu_1} q^{\mu_2} = q^{(\mu_1 \mu_2)} + \frac{1}{D} q^2 g^{\mu_1 \mu_2} \tag{C.39}$$

$$q^{\mu_1} q^{\mu_2} q^{\mu_3} = q^{(\mu_1 \mu_2 \mu_3)} + \frac{1}{D+2} q^2 \left[g^{\mu_1 \mu_2} q^{(\mu_3)} + g^{\mu_2 \mu_3} q^{(\mu_1)} + g^{\mu_3 \mu_1} q^{(\mu_2)} \right]$$

Let us now evaluate (37) using the technique of Gegenbauer polynomials. If we carry out a Wick notation ($p^0 \longrightarrow ip^D$) we can write this integral in a D-dimension Euclidean space as

$$I^{(\mu_1 \mu_2 \cdots \mu_m)} = \frac{i}{(-1)^{r+s} \nu^{2\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{q^{(\mu_1 \mu_2 \cdots \mu_m)}}{q^{2r} (p-q)^{2s}}$$

where the $q^{(\mu_1 \mu_2 \cdots \mu_m)}$ still include the i's introduced by the Wick rotation. Using (33) we can write this expression as

$$= \frac{i}{(-1)^{r+s} \nu^{2\epsilon}} \frac{\Gamma(\lambda+1-r)\,\Gamma(\lambda+1-s)}{\pi^D\, \Gamma(r)\, \Gamma(s)} \int \frac{d^D q}{(2\pi)^D} q^{(\mu_1 \mu_2 \cdots \mu_m)} \int d^D x\, d^D y \frac{e^{2iq.x}\; e^{2i(p-q).y}}{x^{2(\lambda+1-r)}\; y^{2(\lambda+1-s)}}$$

$$= \frac{i}{(-1)^{r+s} \nu^{2\epsilon}} \frac{\Gamma(\lambda+1-r)\,\Gamma(\lambda+1-s)}{2^D \pi^{2D}\, \Gamma(r)\Gamma(s)} \int d^D x\, d^D y \frac{e^{2ip y}}{x^{2(\lambda+1-r)}\; y^{2(\lambda+1-s)}} \frac{\partial_x^{(\mu_1 \mu_2 \cdots \mu_m)}}{(2i)^m} \int d^D q\, e^{2iq.(x}$$

The last integral can be done immediately with the result $\pi^D \delta^{(D)}(x$ and integrating by parts we obtain

$$= \frac{i}{(-1)^{r+s} (2\pi)^D \nu^{2\epsilon}} \frac{\Gamma(\lambda+1-r)\,\Gamma(\lambda+1-s)}{\Gamma(r)\;\; \Gamma(s)} \int d^D x \frac{e^{2ip.x}}{x^{2(\lambda+1-s)}} \frac{(-1)^m}{(2i)^m} \left\{ \partial^{(\mu_1 \mu_2 \cdots \mu_m)} \frac{1}{x^{2(\lambda+1-r)}} \right\}$$

Now it is easy to check that

$$\frac{(-1)^m}{2^m}\left\{\partial^{(\mu_1\mu_2\cdots\mu_m)}\frac{1}{x^{2a}}\right\} = \frac{\Gamma(m+a)}{\Gamma(a)}\frac{x^{(\mu_1\mu_2\cdots\mu_m)}}{x^{2(m+a)}} \tag{C.40}$$

Hence

$$I^{(\mu_1\mu_2\cdots\mu_m)} = \frac{i}{(-1)^{r+s}(2\pi)^D\nu^{2\epsilon}}\frac{\Gamma(m+\lambda+1-r)\,\Gamma(\lambda+1-s)}{\Gamma(r)\,\Gamma(s)}\frac{(-1)^m}{2^m}\partial_p^{(\mu_1\cdots\mu_m)}\int d^Dx\,\frac{e^{2ip\cdot x}}{x^{2(m+2\lambda+2-r-s)}}$$

Using another time (33) and (40), and returning to the Minkowskian notation, we get finally

$$I^{(\mu_1\mu_2\cdots\mu_m)}(p;r,s) \equiv \frac{1}{\nu^{2\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{q^{(\mu_1\mu_2\cdots\mu_m)}}{[q^2+i\eta]^r\,[(q-p)^2+i\eta]^s} = \tag{C.41}$$

$$= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{[p^2]^{r+s-2}}\,p^{(\mu_1\mu_2\cdots\mu_m)}\,\frac{\Gamma(r+s-2-\epsilon)\,\Gamma(m+2-r+\epsilon)\,\Gamma(2-s+\epsilon)}{\Gamma(r)\,\Gamma(s)\,\Gamma(m+4-r-s+2\epsilon)}$$

which is the desired result. Using (39) we obtain the useful expressions

$$I(p;r,s) \equiv \frac{1}{\nu^{2\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{[q^2+i\eta]^r\,[(q-p)^2+i\eta]^s} =$$

$$= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{p^{2(r+s-2)}}\,\frac{\Gamma(2-r+\epsilon)\,\Gamma(2-s+\epsilon)\,\Gamma(r+s-2-\epsilon)}{\Gamma(r)\,\Gamma(s)\,\Gamma(4-r-s+2\epsilon)} \tag{C.42}$$

$$I^{\mu}(p;r,s) \equiv \frac{1}{\nu^{2\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{q^\mu}{[q^2+i\eta]^r\,[(q-p)^2+i\eta]^s} =$$

$$= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{p^{2(r+s-2)}}\,p^\mu\,\frac{\Gamma(3-r+\epsilon)\,\Gamma(2-s+\epsilon)\,\Gamma(r+s-2-\epsilon)}{\Gamma(r)\,\Gamma(s)\,\Gamma(5-r-s+2\epsilon)} \tag{C.43}$$

$$I^{\mu\nu}(p;r,s) \equiv \frac{1}{\nu^{2\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{q^\mu q^\nu}{[q^2+i\eta]^r\,[(q-p)^2+i\eta]^s} =$$

$$= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{p^{2(r+s-2)}}\left\{g^{\mu\nu}p^2\,\frac{\Gamma(3-r+\epsilon)\,\Gamma(3-s+\epsilon)\,\Gamma(r+s-3-\epsilon)}{2\,\Gamma(r)\,\Gamma(s)\,\Gamma(6-r-s+2\epsilon)}\right. +$$

$$\left. + p^\mu p^\nu\,\frac{\Gamma(4-r+\epsilon)\,\Gamma(2-s+\epsilon)\,\Gamma(r+s-2-\epsilon)}{\Gamma(r)\,\Gamma(s)\,\Gamma(6-r-s+2\epsilon)}\right\} \tag{C.44}$$

In the following tables we give the values of these integrals or low values of r and s , in the limit $\epsilon \to 0$ in terms of

$$\frac{1}{\hat{\epsilon}} \equiv \frac{1}{\epsilon} - \ln(4\pi) + \gamma + \ln\left(-\frac{p^2}{\nu^2}\right) \tag{C.45}$$

r	s	$\frac{(4\pi)^2}{i} p^{2(r+s-2)} I(p;r,s)$
1	1	$-\frac{1}{\hat{\epsilon}} + 2$
2	1	$+\frac{1}{\hat{\epsilon}}$
3	1	-1
2	2	$+\frac{2}{\hat{\epsilon}} - 2$
4	1	$-\frac{1}{3}$
3	2	$+\frac{2}{\hat{\epsilon}} - 5$

Table C-1.- Values of $I(p;r,s) = I(p;s,r)$ in the limit $\epsilon \to 0$.

r	s	$I^\mu(p;r,s)$
1	1	$-\frac{1}{2\hat{\epsilon}} + 1$
2	1	$+1$
1	2	$+\frac{1}{\hat{\epsilon}} - 1$
3	1	$+\frac{1}{2\hat{\epsilon}} - \frac{1}{2}$
2	2	$+\frac{1}{\hat{\epsilon}} - 1$
1	3	$-\frac{1}{2\hat{\epsilon}} - \frac{1}{2}$

Table C-2.- The last column gives the coeffient that multiplies $i(4\pi)^{-2} p^{-2(r+s-2)} p^\mu$ in $I^\mu(p;r,s)$ in the limit $\epsilon \to 0$

r	s	A	B
1	1	$+\frac{1}{12\hat{\epsilon}} - \frac{2}{9}$	$-\frac{1}{3\hat{\epsilon}} + \frac{13}{18}$
2	1	$-\frac{1}{4\hat{\epsilon}} + \frac{1}{2}$	$+\frac{1}{2}$
1	2	$-\frac{1}{4\hat{\epsilon}} + \frac{1}{2}$	$+\frac{1}{\hat{\epsilon}} - \frac{3}{2}$
3	1	$+\frac{1}{4\hat{\epsilon}} - \frac{1}{4}$	$+\frac{1}{2}$
2	2	$+\frac{1}{2}$	$+\frac{1}{\hat{\epsilon}} - 2$
1	3	$+\frac{1}{4\hat{\epsilon}} - \frac{1}{4}$	$-\frac{1}{\hat{\epsilon}} + \frac{1}{2}$

TAble C-3.- We give for $\epsilon \to 0$ the values of

$$I^{\mu\nu}(p;r,s) = i\,(4\pi)^{-2}\, p^{-2(r+s-2)}\left[A\,p^2 g^{\mu\nu} + B\,p^\mu p^\nu\right]$$

Furthermore, repeated use of equation (42) allows us to calculate integrals such as

$$I(\alpha,\beta,\gamma,\delta) \equiv \frac{1}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{1}{[k_1^2+i\eta]^\alpha\,[(k_1-q)^2+i\eta]^\beta\,[(k_1-k_2)^2+i\eta]^\gamma\,[(k_2-q)^2+i\eta]^\delta}$$

$$= \frac{i^2}{(4\pi)^4}\left(-\frac{q^2}{4\pi\nu^2}\right)^{2\epsilon}\frac{1}{q^{2(\alpha+\beta+\gamma+\delta-4)}}\,\frac{\Gamma(\gamma+\delta-2-\epsilon)}{\Gamma(\gamma)\,\Gamma(\delta)}\,\frac{\Gamma(\alpha+\beta+\gamma+\delta-4-2\epsilon)}{\Gamma(\alpha)\,\Gamma(\beta+\gamma+\delta-2-\epsilon)}$$

$$\cdot\frac{\Gamma(2-\gamma+\epsilon)\,\Gamma(2-\delta+\epsilon)}{\Gamma(4-\gamma-\delta+2\epsilon)}\,\frac{\Gamma(2-\alpha+\epsilon)\,\Gamma(4-\beta-\gamma-\delta+2\epsilon)}{\Gamma(6-\alpha-\beta-\gamma-\delta+3\epsilon)} \qquad (C.46)$$

vi) Two and three loop integrals

As another example of how to use the method of Gegenbauer's polynomials let us compute

$$I \equiv \frac{1}{\nu^{4\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{[q^2+i\eta][(q-p)^2+i\eta][(q-k)^2+i\eta][k^2+i\eta][(k-p)^2+i\eta]}$$

As before we perform a Wick rotation and use, repeatedly, the formula (33) in order to obtain

$$= \frac{1}{\nu^{4\epsilon}} \left[\frac{\Gamma(\lambda)}{\pi^{\lambda+1}} \right]^5 \frac{1}{(2\pi)^{2D}} \int d^D x_1\, d^D x_2\, d^D x_3\, d^D x_4\, d^D x_5 \; \frac{1}{x_1^{2\lambda}\, x_2^{2\lambda}\, x_4^{2\lambda}\, x_5^{2\lambda}\, (x_2 - x_3)^{2\lambda}} \cdot$$

$$\cdot \int d^D q\, d^D k\, e^{2iq\cdot x_1}\, e^{2i(p-q)\cdot x_2}\, e^{2i(k-p)\cdot(x_2-x_3)}\, e^{2i(k-q)\cdot x_4}\, e^{2ik\cdot x_5}$$

The last two integrals can be done immediately and using the Dir delta functions obtained we can perform the integration over x_4 and x_5

$$= \frac{1}{\nu^{4\epsilon}} \left[\frac{\Gamma(\lambda)}{\pi^{\lambda+1}} \right]^5 \frac{1}{4^D} \int d^D x_1\, d^D x_2\, d^D x_3 \; \frac{e^{2ip\cdot x_3}}{x_1^{2\lambda}\, x_2^{2\lambda}\, (x_1 - x_2)^{2\lambda}\, (x_3 - x_1)^{2\lambda}\, (x_2 - x_3)^{2\lambda}}$$

Now we can use eq. (30) to expand the three terms in the denomin of the type $(x_i - x_j)^{2\lambda}$ and use eq. (34) to expand the exponent then

$$= \frac{1}{\nu^{4\epsilon}} \frac{\Gamma^3(\lambda)/\lambda^3}{\pi^{2(\lambda+1)}\, 2^{4\lambda+1}} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{m=0}^{\infty} i^r (\lambda + r)\, p^r \int dx_1\, dx_2\, dx_3\, x_1^{2\lambda+1}\, x_2^{2\lambda+1}\, x_3^2$$

$$\cdot \frac{x_3^r\, [T(x_1,x_2)]^s\, [T(x_1,x_3)]^m\, [T(x_2,x_3)]^t}{x_1^{2\lambda}\, x_2^{2\lambda}\, [\max(x_1,x_2)]^{2\lambda}\, [\max(x_1,x_3)]^{2\lambda}\, [\max(x_2,x_3)]^{2\lambda}}\; j_{\lambda+r}(p^2 x$$

$$\cdot \int d\hat{x}_1\, d\hat{x}_2\, d\hat{x}_3\; C_s^\lambda(\hat{x}_1 \cdot \hat{x}_2)\, C_m^\lambda(\hat{x}_1 \cdot \hat{x}_3)\, C_t^\lambda(\hat{x}_2 \cdot \hat{x}_3)\, C_r^\lambda(\hat{x}_3 \cdot \hat{p})$$

The angular integrals can be carried out immediately by repeated of eq. (32) with the result

$$\left[\frac{\lambda}{m+\lambda} \right]^2 \frac{\Gamma(m+2\lambda)}{m!\; \Gamma(2\lambda)}\; \delta_{sm}\, \delta_{tm}\, \delta_{r0}$$

and hence

$$I = \frac{1}{\nu^{4\epsilon}} \quad \frac{\Gamma^3(\lambda)}{\pi^{2(\lambda+1)}\, 2^{4\lambda+1}\, \Gamma(2\lambda)} \sum_{m=0}^{\infty} \frac{\Gamma(m+2\lambda)}{(m+\lambda)^2\, m!} \cdot$$

$$\cdot \int dx_1\, dx_2\, dx_3\; x_1\, x_2\, x_3^{2\lambda+1} \frac{[T(x_1,x_2)]^m\, [T(x_1,x_3)]^m\, [T(x_2,x_3)]^m}{[\max(x_1,x_2)]^{2\lambda}\, [\max(x_1,x_3)]^{2\lambda}\, [\max(x_2,x_3)]^{2\lambda}}\; j_\lambda(p^2 x_3^2)$$

The p^2 dependence can be obtained dimensionally and it turns out to be p^{2D-10} ; taking this into account it is enough to calculate the integral with $p^2 = 1$. Using eq. (36) a straightforward calculation gives for this integral

$$\frac{1}{2^2} \frac{\Gamma(3-2\lambda)}{\Gamma(3\lambda-2)} \left\{ \frac{1}{(m+1)(m+2-\lambda)} + \frac{1}{(m+1)(m-1+2\lambda)} + \frac{1}{(m+2\lambda-1)(m+3\lambda-2)} \right\}$$

hence returning to the Minkowskian notation we obtain

$$I = \frac{i^2}{(4\pi)^4} \frac{1}{p^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^{2\epsilon} \frac{2\,\Gamma^3(1+\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1+3\epsilon)\,\Gamma(2+2\epsilon)} \sum_{m=0}^{\infty} \frac{\Gamma(m+2+2\epsilon)}{m!\,(m+1+\epsilon)^2}$$

$$\left\{ \frac{1}{(m+1)(m+1-\epsilon)} + \frac{1}{(m+1)(m+1+2\epsilon)} + \frac{1}{(m+1+2\epsilon)(m+1+3\epsilon)} \right\}$$

In order to carry out the summation of this series let us rewrite it as

$$I = \frac{i^2}{(4\pi)^4} \frac{1}{p^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^{2\epsilon} \frac{\Gamma^3(\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1+3\epsilon)\,\Gamma(2+2\epsilon)} \sum_{m=0}^{\infty} \frac{\Gamma(m+2+2\epsilon)}{m!}$$

$$\cdot \left\{ -\frac{1}{m+1} + \frac{1}{m+1+2\epsilon} + \frac{1}{2}\frac{1}{m+1-\epsilon} - \frac{1}{2}\frac{1}{m+1+3\epsilon} \right\}$$

Furthermore it is well known [AS 65]

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+a)}{m!\,(m+b)} = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(1-a)}{\Gamma(b+1-a)} \qquad \text{(C.47)}$$

and we obtain immediately

$$\frac{1}{\nu^{4\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{[q^2+i\eta][(q-p)^2+i\eta][(q-k)^2+i\eta][k^2+i\eta][(k-p)^2+i\eta]}=$$

$$=\frac{i^2}{(4\pi)^4}\frac{1}{p^2}\left(-\frac{p^2}{4\pi\nu^2}\right)^{2\epsilon}\frac{\Gamma(-1-2\epsilon)\Gamma^2(\epsilon)\Gamma(1+\epsilon)\Gamma(-2\epsilon)}{\Gamma(1+3\epsilon)}\left\{\frac{\Gamma(1+3\epsilon)}{\Gamma(\epsilon)}-\frac{\Gamma(1-\epsilon)}{\Gamma(-3\epsilon)}+\frac{2}{\Gamma(-2\epsilon)}\right\} \tag{C.48}$$

which is the desired result. By identical methods we can show that [CK 80]

$$I(\alpha,\beta)\equiv\frac{1}{\nu^{4\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{1}{[q^2+i\eta]^\alpha[(q-p)^2+i\eta]^\beta[(q-k)^2+i\eta][k^2+i\eta][(k-p)^2+i\eta]}=$$

$$=\frac{i^2}{(4\pi)^4}\frac{1}{p^{2(\alpha+\beta-1)}}\left(-\frac{p^2}{4\pi\nu^2}\right)^{2\epsilon}\frac{\Gamma(-1-2\epsilon)\Gamma(1-\alpha+\epsilon)\Gamma(1-\beta+\epsilon)\Gamma(1+\epsilon)\Gamma(\alpha+\beta-2-2\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(3-\alpha-\beta+3\epsilon)}\cdot$$

$$\left\{\frac{\Gamma(3-\alpha-\beta+3\epsilon)}{\Gamma(2-\alpha-\beta+\epsilon)}-\frac{\Gamma(-1+\alpha+\beta-\epsilon)}{\Gamma(-2+\alpha+\beta-3\epsilon)}+\left[\frac{\Gamma(\alpha)}{\Gamma(-1+\alpha-2\epsilon)}-\frac{\Gamma(2-\alpha+2\epsilon)}{\Gamma(1-\alpha)}+(\alpha\to\beta)\right]\right\} \tag{C.49}$$

which reduces to (48) for $\alpha=\beta=1$. In this case and in the limit of small ϵ we have

$$I(1,1)=\frac{i^2}{(4\pi)^4}\frac{1}{p^2}\left(-\frac{p^2}{4\pi\nu^2}\right)^{2\epsilon}\frac{\Gamma^3(1+\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+3\epsilon)}\,6\,.$$
$$\left\{\zeta(3)+[\zeta(4)-\zeta^2(2)-2\zeta(3)]\epsilon+\cdots\right\} \tag{C.50}$$

Suppose we further modify the integral on the l.h.s. of (49), e.g. as

$$\frac{1}{[(q-k)^2+i\eta]}\longrightarrow\frac{1}{[(q-k)^2+i\eta]^2}$$

In x space

$$\frac{1}{[x_1-x_2]^{2\lambda}}\longrightarrow\frac{1}{[x_1-x_2]^{2(\lambda-1)}}$$

Since [BE 53]

$$(m+\lambda)\, C^{\lambda-1}_{m+1}(x) = (\lambda-1)\,[\,C^{\lambda}_{m+1}(x) - C^{\lambda}_{m-1}(x)\,] \tag{C.51}$$

the calculations will proceed as before, and the result can again be represented in a closed form similar to (49).

Another useful type of integrals that can be calculated in the same way is [CK 80]

$$I(\alpha,\beta,\gamma) \equiv \frac{1}{\nu^{4\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{d^D k}{(2\pi)^D} \frac{1}{[q^2+i\eta]^{\alpha}[(q-p)^2+i\eta][(q-k)^2+i\eta]^{\beta}[k^2+i\eta]^{\gamma}[(k-p)^2+i\eta]}$$

$$= \frac{i^2}{(4\pi)^4} \frac{1}{p^{2(\alpha+\beta+\gamma-2)}} \left(-\frac{p^2}{4\pi\nu^2}\right)^{2\epsilon} \frac{\Gamma(2-\alpha+\epsilon)\Gamma(2-\beta+\epsilon)\Gamma(2-\gamma+\epsilon)\,\Gamma(1+\epsilon)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(2+2\epsilon)} \cdot \tag{C.52}$$

$$\sum_{m,n=0}^{\infty} \frac{(-1)^m\,\Gamma(2+n+2\epsilon)\,\Gamma(-2+m+n+\alpha+\beta+\gamma-2\epsilon)}{m!\,n!\,(n+1+\epsilon)\,\Gamma(4-m-\alpha-\beta-\gamma+3\epsilon)\,\Gamma(2+m+n+\epsilon)} \cdot$$

$$\left\{ \frac{1}{(n+\beta)(m+n+\alpha+\beta-1-\epsilon)} + \frac{1}{(m+n+\alpha)(m+n+\alpha+\beta-1-\epsilon)} + \frac{1}{(m+n+\alpha)(2+n-\beta+2\epsilon)} + (\alpha\to\gamma) \right\}$$

This series is convergent in the region ($A \equiv \alpha+\beta+\gamma$) $A < 4+3\epsilon$, $A < 4+2\epsilon$, $A < 5+\epsilon$. The summation of it seems to be non-trivial. This integral appears, for instance, in the evaluation of the β -function at the three loop level.

vii) <u>Logarithmic integrals</u>

When the Feynman parametrization method is used we need some integrals that will be presented here. We have

$$\int dx\, x^n \ln(ax-b) = \tag{C.53}$$

$$= \frac{1}{(n+1)} \left\{ \left[x^{n+1} - \left(\frac{b}{a}\right)^{n+1} \right] \ln(ax-b) - \sum_{\nu=1}^{n+1} \left(\frac{b}{a}\right)^{n+1-\nu} \frac{x^{\nu}}{\nu} \right\}$$

and hence

$$\int_0^1 dx \; x^m \ln x = - \frac{1}{(m+1)^2} \qquad (C.54)$$

Another useful class of integrals are

$$I_m \equiv \int_0^1 dx \; x^m \ln \left[u - x(1-x) \right] \qquad (C.55)$$

Using (53) we obtain

$$I_0 = -2 + \ln u + \sqrt{1-4u} \; \ln \frac{\sqrt{1-4u}+1}{\sqrt{1-4u}-1}$$

$$I_1 = \frac{1}{2} \; I_0$$

$$I_2 = \frac{1}{3} \left\{ -\frac{13}{6} + 2u + \ln u + (1-u)\sqrt{1-4u} \; \ln \frac{\sqrt{1-4u}+1}{\sqrt{1-4u}-1} \right\} \qquad (C.56)$$

$$I_3 = \frac{1}{2} \left[I_0 - 3 I_1 + 3 I_2 \right]$$

and so on. Notice that since the argument of the logarithm appearing in (55) is invariant under $x \leftrightarrow (1-x)$ only the integrals correspond ing to even values of n must be calculated explicitly.

It is interesting to introduce the Spence function or dilogarithm defined as [LE 58]

$$L_2(z) \equiv - \int_0^z dt \; \frac{\ln(1-t)}{t} \qquad (C.57)$$

Equation (57) holds for general complex values of z . When z is rea and greater than unity the logarithm is complex, and there is a branch cut from $z=1$ to ∞ . In order to give the function a definite value we shall take that value of arg $(1-t)$ such that $-\pi < \arg(1-t) \leq \pi$. When z is real , $= x > 1$, say, we have $\ln(1-t) = i\pi + \ln(t-1)$, so that $L_2(x + i0)$ has $-i\pi \ln x$ as its imaginary part when x is greater than unity. We have

$$L_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad , \qquad |z| \le 1 \tag{C.58}$$

Some special values are

$$L_2(1) = \frac{\pi^2}{6} \qquad\qquad L_2(-1) = -\frac{\pi^2}{12}$$

$$L_2(1/2) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2 \qquad\qquad L_2(2+i0) = \frac{\pi^2}{4} - i\pi \ln 2 \tag{C.59}$$

Useful relations are

$$L_2(z) + L_2(1-z) + \ln z \ln(1-z) = \frac{\pi^2}{6} \tag{C.60}$$

$$L_2\left(-\frac{1}{x}\right) + L_2(-x) + \frac{1}{2}\ln^2(x) = -\frac{\pi^2}{6} \quad , \qquad x > 0$$

$$L_2\left(\frac{1}{x}\right) + L_2(x) + \frac{1}{2}\ln^2 x = \frac{\pi^2}{3} - i\pi \ln x \quad , \qquad x > 1 \tag{C.61}$$

$$L_2(x) + L_2\left(\frac{x}{x-1}\right) = \frac{1}{2}\pi^2 - 2i\pi \ln x + i\pi \ln(x-1) - \frac{1}{2}\ln^2(x-1) \quad , \qquad x > 1$$

$$L_2(x) + L_2\left(-\frac{x}{1-x}\right) = -\frac{1}{2}\ln^2(1-x) \quad , \qquad x < 1 \tag{C.62}$$

$$L_2\left(\frac{1}{1+x}\right) - L_2(-x) = \frac{\pi^2}{6} - \frac{1}{2}\ln(1+x)\ln\left(\frac{1+x}{x^2}\right) \tag{C.63}$$

$$L_2(1-x) - L_2(1/x) = \frac{1}{2}\ln x \ln\frac{x}{(x-1)^2} - \frac{\pi^2}{6} \tag{C.64}$$

$$L_2(x) + L_2(-x) = \frac{1}{2} L_2(x^2) \qquad \text{(C.}$$

$$\frac{1}{m} L_2(x^m) = \sum_{r=1}^{m} L_2\left(x\, e^{2\pi r i/m}\right) \quad , \; m = 1, 2, 3, \ldots \qquad \text{(C.}$$

Formulae (61) can be used to reduce any dilogarithm to one with a gument $|z| \leq 1$ and therefore can be evaluated numerically usin (58). Nevertheless it is more useful to use

$$L_2(z) = \frac{1}{1+z} \left\{ 2(1-z)\ln(1-z) + z\left[3 + \sum_{n=1}^{\infty} \frac{z^n}{n^2(n+1)^2} \right] \right\} \qquad \text{(C.}$$

Some useful integrals related to dilogarithms are

$$\int_0^x dt \, \frac{\ln(1+t^2)}{t} = -\frac{1}{2} L_2(-x^2)$$

$$\int_0^r dt \, \frac{\ln(1-t+t^2)}{t} = -\frac{1}{3} L_2(-r^3) + L_2(-r) \quad , \; r > 0$$

$$\int_0^r dt \, \frac{\ln(1+t+t^2)}{t} = -\frac{1}{3} L_2(r^3) + L_2(r) \quad , \; r > 0$$

$$\int_0^1 dt \, \frac{\ln t}{t - 1/x} = L_2(x) \qquad \text{(C.}$$

Other useful integrals are $\left[a_4 \equiv \sum_{n=1}^{\infty} (1/2^n n^4) = 0.517\,479\,061\,6\ldots \right]$

$$\int_0^1 dx \, \frac{\ln^2 x \, \ln(1+x)}{1+x} = -\frac{3}{2}\zeta^2(2) + 4a_4 + \frac{7}{2}\zeta(3)\ln 2 - \zeta(2)\ln^2 2 + \frac{1}{6}\ln^4$$

$$\int_0^1 dx \, \frac{\ln x \, \ln^2(1+x)}{1+x} = -\frac{4}{5}\zeta^2(2) + 2a_4 + \frac{7}{4}\zeta(3)\ln 2 - \frac{1}{2}\zeta(2)\ln^2 2 + \frac{1}{12}$$

$$\int_0^1 dx \, \frac{\ln^2 x \, \ln(1+x)}{x} = -\frac{1}{3}\int_0^1 dx \, \frac{\ln^3 x}{1+x} = \frac{7}{10}\zeta^2(2)$$

$$\int_0^1 dx\ \frac{\ln(1+x)\ L_2(-x)}{x} = \frac{1}{8}\ \zeta^2(2)$$

$$\int_0^1 dx\ \frac{\ln(1+x)\ L_2(-x)}{1+x} = \frac{6}{5}\ \zeta^2(2) - 3a_4 - \frac{21}{8}\ \zeta(3)\ln 2 + \frac{1}{2}\ \zeta(2)\ln^2 2 - \frac{1}{8}\ln^4 2$$

$$\int_0^1 dx\ \frac{\ln x\ L_2(-x)}{1+x} = \frac{13}{8}\ \zeta^2(2) - 4a_4 - \frac{7}{2}\ \zeta(3)\ln 2 + \zeta(2)\ln^2 2 - \frac{1}{6}\ln^4 2$$

$$\int_0^1 dx\ \frac{\ln^2(1+x)\ln(1-x)}{1+x} = -\frac{4}{5}\ \zeta^2(2) + 2a_4 + 2\zeta(3)\ln 2 - \zeta(2)\ln^2 2 + \frac{1}{3}\ln^4 2 \quad . \tag{C.69}$$

Another function that appears in the evaluation of parametric integrals is

$$S_1(z) \equiv \sum_{k=1}^{\infty} \frac{z}{k(z+k)} \quad ; \quad S_\ell(z) = \zeta(\ell) - \sum_{k=1}^{\infty} \frac{1}{(z+k)^\ell} \ , \ \ell = 2,3,4\ldots \tag{C.70}$$

If $z = n =$ integer they can be reduced to

$$S_\ell(n) = \sum_{k=1}^{n} \frac{1}{k^\ell} \qquad , \ \ell = 1,2,3\ldots \tag{C.71}$$

Clearly

$$S_1(z+1) = S_1(z) + \frac{1}{z+1} \tag{C.72}$$

$$S_\ell(\infty) = \zeta(\ell) \qquad , \quad \ell = 2,3,4,\ldots \tag{C.73}$$

$$\frac{d S_\ell(z)}{dz} = \ell\,[\zeta(\ell+1) - S_{\ell+1}(z)] \qquad \ell = 1,2,3,\cdots \tag{C.74}$$

These functions are related with the Psi function

$$\psi(z) \equiv \frac{d \ln \Gamma(z)}{dz} \tag{C.75}$$

Since

$$\psi(z) = -\gamma - \frac{1}{z} + S_1(z) \qquad \text{(C.76}$$

$$\frac{d^\ell \psi(z)}{dz^\ell} = \ell!\,(-1)^{\ell+1}\left[\zeta(\ell+1) - S_{\ell+1}(z-1)\right] \qquad \text{(C.77}$$

Then some other useful integrals are

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1}\ln x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left[S_1(\alpha-1) - S_1(\alpha+\beta-1)\right]$$

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1}\ln^2 x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left\{\left[S_1(\alpha-1) - S_1(\alpha+\beta-1)\right]^2 + S_2(\alpha+\beta-1) - S_2(\alpha-1)\right\} \qquad \text{(C.78}$$

$$\int_0^1 dx\,\frac{1-x^\alpha}{1-x} = S_1(\alpha)$$

The first of these integrals can be obtained derivating (14) with respect to z . If we derive it with respect to β

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1}\ln x\,\ln(1-x) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left\{S_2(\alpha+\beta-1) - \zeta(2) + \left[S_1(\alpha-1) - S_1(\alpha+\beta-1)\right]\left[S_1(\beta-1) - S_1(\alpha+\beta-1)\right]\right\} \qquad \text{(C.79}$$

From it we obtain

$$\int_0^1 dx\, x^{\alpha-1}\ln x\,\ln(1-x) = \frac{S_1(\alpha)}{\alpha^2} + \frac{S_2(\alpha)}{\alpha} - \frac{\zeta(2)}{\alpha}$$

$$\int_0^1 dx\,\frac{x^\alpha}{1-x}\ln x\,\ln(1-x) = \zeta(2)S_1(\alpha) - S_1(\alpha)S_2(\alpha) - S_3(\alpha) + \zeta(3) \qquad \text{(C.80}$$

Finally

$$\int_0^1 dx \; \frac{x^{\alpha-1}}{1-x} \, \ln^m x = (-1)^m \, m! \left[\zeta(m+1) - S_{m+1}(\alpha-1) \right] \tag{C.81}$$

viii) <u>Integrals with mass terms</u>

From (C.17) it is immediate to find

$$I(\alpha,\beta;m^2) \equiv \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \; \frac{[k^2]^\alpha}{[k^2 - m^2 + i\eta]^\beta} =$$

$$= \frac{i}{(4\pi)^2} \, (-m^2)^{\alpha-\beta+2} \left(\frac{m^2}{4\pi\nu^2} \right)^\epsilon \frac{\Gamma(2+\alpha+\epsilon)\,\Gamma(\beta-\alpha-2-\epsilon)}{\Gamma(\beta)\;\Gamma(2+\epsilon)} \; . \tag{C.82}$$

Furthermore using the tensor $k^{(\mu_1\mu_2\dots\mu_m)}$ introduced in (37) we have $(m \geq 1)$

$$I^{(\mu_1\mu_2\dots\mu_m)}(\alpha,\beta;m^2) \equiv \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \; \frac{[k^2]^\alpha}{[k^2 - m^2 + i\eta]^\beta} \, k^{(\mu_1\mu_2\dots\mu_m)} = 0 \tag{C.83}$$

This tells us that all integrals of this type where the integrand is proportional to $k^{\mu_1} k^{\mu_2} \dots k^{\mu_{2m+1}}$ are zero and that the ones with the integrands proportional to $k^{\mu_1} k^{\mu_2} \dots k^{\mu_{2m}}$ can be easily obtained in terms of (82) carrying out the substitutions

$$k^{\mu_1} k^{\mu_2} \longrightarrow \frac{1}{D} \, k^2 g^{\mu_1\mu_2}$$

$$k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4} \longrightarrow \frac{1}{D(D+2)} [k^2]^2 \left\{ g^{\mu_1\mu_2} g^{\mu_3\mu_4} + g^{\mu_1\mu_3} g^{\mu_4\mu_2} + \right.$$
$$\left. + g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right] \tag{C.84}$$

and so on.

Using (4) it is immediate to obtain

$$I(\alpha,\beta;q^2;m^2) \equiv \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-q)^2 - m^2 + i\eta]^\alpha \, [k^2 + i\eta]^\beta} =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-qx)^2 + q^2 x(1-x) - m^2 x + i\eta]^{\alpha+\beta}} =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + q^2 x(1-x) - m^2 x + i\eta]^{\alpha+\beta}}$$

$$= \frac{i}{(4\pi)^2}\left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon} \frac{\Gamma(\alpha+\beta-2-\epsilon)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{[q^2]^{\alpha+\beta-2}} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1}\left[x(1-x) - \frac{m^2}{q^2}x\right]^{2+\epsilon-\alpha-\beta} \quad \text{(C.8}$$

Using

$$\int_0^1 dx\, x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b,c;z) \quad \text{(C.8}$$

we get immediately

$$I(\alpha,\beta,q^2;m^2) = \frac{i}{(4\pi)^2}\frac{1}{[q^2]^{\alpha+\beta-2}} \frac{\Gamma(\alpha+\beta-2-\epsilon)\,\Gamma(2+\epsilon-\beta)}{\Gamma(\alpha)\,\Gamma(2+\epsilon)}\left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon} \cdot$$

$$\cdot\left(1-\frac{m^2}{q^2}\right)^{2+\epsilon-\alpha-\beta} {}_2F_1\left(\alpha+\beta-2-\epsilon,\ 2+\epsilon-\beta,\ 2+\epsilon;\ \frac{1}{1-\frac{m^2}{q^2}}\right) \quad \text{(C.8}$$

Then it is easy to check either from (87) or directly from (85) th

in the limit $\epsilon \to 0$

$$I(1,1;q^2;m^2) = \frac{i}{(4\pi)^2}\left\{-\frac{1}{\epsilon} + \ln 4\pi - \gamma - \ln\left(-\frac{q^2}{\nu^2}\right) - \frac{m^2}{q^2}\ln\left(-\frac{m^2}{q^2}\right) - \right.$$

$$\left. - \left(1-\frac{m^2}{q^2}\right)\ln\left(1-\frac{m^2}{q^2}\right) + 2\right\} \quad \text{(C.8}$$

$$I(1,2;q^2;m^2) = \frac{i}{(4\pi)^2}\frac{1}{q^2-m^2}\left\{\frac{1}{\epsilon} - \ln 4\pi + \gamma + \ln\left(-\frac{q^2}{\nu^2}\right) - \frac{m^2}{q^2}\ln\left(-\frac{m^2}{q^2}\right.\right.$$

$$\left. + \left(1+\frac{m^2}{q^2}\right)\ln\left(1-\frac{m^2}{q^2}\right)\right\}$$

The integrals with $\alpha > 1$ and $\beta = 1$ or 2 can be obtained derivin (88) with respect to m^2.

Similarly

$$I^{\mu}(\alpha, \beta; q^2; m^2) \equiv \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{k^{\mu}}{[(k-q)^2 - m^2 + i\eta]^{\alpha} [k^2 + i\eta]^{\beta}} =$$

$$= q^{\mu} \frac{i}{(4\pi)^2} \frac{1}{[q^2]^{\alpha+\beta-2}} \frac{\Gamma(\alpha+\beta-2-\epsilon)\,\Gamma(3+\epsilon-\beta)}{\Gamma(\alpha)\;\Gamma(3+\epsilon)} \left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon} \cdot$$

$$\cdot \left(1 - \frac{m^2}{q^2}\right)^{2+\epsilon-\alpha-\beta} {}_2F_1\left(\alpha+\beta-2-\epsilon,\; 3+\epsilon-\beta,\; 3+\epsilon\; ; \frac{1}{1-\frac{m^2}{q^2}}\right) \tag{C.89}$$

In particular we obtain

$$I^{\mu}(1,1;q^2;m^2) = q^{\mu} \frac{i}{(4\pi)^2} \frac{1}{2} \left\{ -\frac{1}{\epsilon} + \ln 4\pi - \gamma - \ln\left(-\frac{q^2}{\nu^2}\right) - \frac{m^2}{q^2}\left(2 - \frac{m^2}{q^2}\right) \ln\left(-\right.\right.$$

$$\left. - \left(1 - 2\frac{m^2}{q^2} + \left(\frac{m^2}{q^2}\right)^2\right) \ln\left(1 - \frac{m^2}{q^2}\right) - \frac{m^2}{q^2} + 2 \right\} \tag{C.90}$$

$$I^{\mu}(1,2;q^2;m^2) = q^{\mu} \frac{i}{(4\pi)^2} \frac{1}{q^2} \left\{ -\frac{m^2}{q^2} \ln\left(-\frac{m^2}{q^2}\right) + \frac{m^2}{q^2} \ln\left(1 - \frac{m^2}{q^2}\right) + 1 \right\}$$

Let us now consider the integrals

$$\tilde{I}(\alpha, \beta; q^2; m^2) \equiv \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-q)^2 - m^2 + i\eta]^{\alpha} [k^2 - m^2 + i\eta]^{\beta}} =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(k-xq)^2 + q^2 x(1-x) - m^2 + i\eta]^{\alpha+\beta}} =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \frac{1}{\nu^{2\epsilon}} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + q^2 x(1-x) - m^2 + i\eta]^{\alpha+\beta}} =$$

$$= \frac{i}{(4\pi)^2} \left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon} \frac{\Gamma(\alpha+\beta-2-\epsilon)}{\Gamma(\alpha)\,\Gamma(\beta)} \frac{1}{[q^2]^{\alpha+\beta-2}} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \left[x(1-x) - \frac{m^2}{q^2}\right]^{2-\alpha-\beta+\epsilon} \tag{C.91}$$

The only integral of this family that is divergent, if we consider that α and β are integers, is $\alpha = \beta = 1$. Then

$$\tilde{I}(1,1;q^2;m^2) = \frac{i}{(4\pi)^2}\left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon}\Gamma(-\epsilon)\int_0^1 dx\,\left[x(1-x)-\frac{m^2}{q^2}\right]^{\epsilon} =$$

$$= \frac{i}{(4\pi)^2}\left(-\frac{q^2}{4\pi\nu^2}\right)^{\epsilon}\Gamma(-\epsilon)\left\{1+\epsilon\ln(-1)+\epsilon\int_0^1 dx\,\ln\left[\frac{m^2}{q^2}-x(1-x)\right]+O(\epsilon^2)\right\}$$

and using (51) we obtain, in the limit $\epsilon \to 0$,

$$\tilde{I}(1,1;q^2;m^2) = \frac{i}{(4\pi)^2}\left\{-\frac{1}{\epsilon}+\ln 4\pi-\gamma-\ln\left(\frac{m^2}{\nu^2}\right)-\right.$$

$$\left.-\sqrt{1-\frac{4m^2}{q^2}}\,\ln\frac{\sqrt{1-4m^2/q^2}+1}{\sqrt{1-4m^2/q^2}-1}+2\right\} \qquad \text{(C.92)}$$

If $\alpha+\beta > 2$ then (91) can be written as

$$\tilde{I}(\alpha,\beta;q^2;m^2) = \frac{i}{(4\pi)^2}\,\frac{\Gamma(\alpha+\beta-2)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{1}{[q^2]^{\alpha+\beta-2}}\int_0^1 dx\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\left[x(1-x)-\frac{m^2}{q^2}\right]^{\alpha+\beta-2}} \qquad \text{(C.93)}$$

In particular

$$\tilde{I}(2,1;q^2;m^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{q^2\sqrt{1-4m^2/q^2}}\,\ln\frac{\sqrt{1-4m^2/q^2}+1}{\sqrt{1-4m^2/q^2}-1}\right\} \qquad \text{(C.94)}$$

Notice furthermore that $\tilde{I}(\alpha,\beta;q^2;m^2) = \tilde{I}(\beta,\alpha;q^2;m^2)$.

APPENDIX D.- BACKGROUND FIELD GAUGE

Recall that in the conventional functional approach to field theory the generating functional is for pure QCD (appendix B)

$$W[J] = \int [\mathcal{D} B_\mu^a] \det \left| \frac{\delta f_a}{\delta \theta_c} \right| \exp \left\{ i \int d^4x \left[\mathcal{L}(B) - \frac{1}{2a} (f_a)^2 + J_a^\mu B_\mu^a \right] \right\} \quad (D.1)$$

where the gauge field is denoted by B_μ^a and θ_c is the parameter of an infinitesimal gauge transformation. The functional derivatives of $W[J]$ with respect to J_μ^a at $J_\mu^a = 0$ are the disconnected Green's functions of the theory. The connected ones are generated by

$$Z[J] \equiv -i \ln W[J] \quad (D.2)$$

One defines the effective action by making the Legendre transformation

$$\Gamma[\bar{B}] = Z[J] - \int d^4x \, J_\mu^a \bar{B}_a^\mu \quad (D.3)$$

where

$$\bar{B}_\mu^a \equiv \frac{\delta Z[J]}{\delta J_a^\mu} \quad (D.4)$$

which is the vacuum expectation value of B_μ^a in presence of a classical source J_μ^a . Notice that it is in fact the classical field, in the WBK approximation, as can be seen writing $B_\mu^a = \bar{B}_\mu^a + \delta B_\mu^a$ and noticing that the vacuum expectation value of δB_μ^a is zero. This happens if the exponent in the generating functional is quadratic in

δB^a_μ , as happens in the WBK approximation if $\bar{B}^a_\mu$ is a stationary path. Of course

$$\bar{B}^a_\mu \Big|_{J=0} = 0 \tag{D.5}$$

Notice that

$$W[J] = \exp \left\{ i \left[\Gamma[\bar{B}] + \int d^4x \, J^a_\mu \bar{B}^{\mu a} \right] \right\} \tag{D.6}$$

which justifies the denomination effective action.

The functional derivatives of $\Gamma[\bar{B}]$ with respect to $\bar{B}^a_\mu$ at $\bar{B}^a_\mu = 0$ are the proper (one-particle irreducible) Green's functions. Indeed, recall

$$\frac{\delta \Gamma[\bar{B}]}{\delta \bar{B}^a_\mu} = - J^\mu_a \tag{D.7}$$

which is the dual of eq. (D.4), so that

$$\frac{\delta \Gamma[\bar{B}]}{\delta \bar{B}^a_\mu} \Big|_{\bar{B}^a_\mu = 0} = 0 \tag{D.8}$$

For the second variation one obtaines from eq. (D.4)

$$\frac{\delta^2 Z[J]}{\delta J^\mu_a \, \delta J^\nu_b} \frac{\delta J^\nu_b}{\delta \bar{B}^\rho_c} = \delta_{ac} \, g_{\mu\rho} \tag{D.9}$$

where here and in the following the space-time arguments and integration over them are not explicitly written. Then from eq. (D.7), with

$$\Sigma_{\mu\nu, ab}[J] \equiv \frac{\delta^2 Z[J]}{\delta J^\mu_a \, \delta J^\nu_b} \tag{D.10}$$

one finds

$$\frac{\delta^2 \Gamma[\bar{B}]}{\delta \bar{B}^a_\mu \, \delta \bar{B}^b_\nu} = - \left[\Sigma^{-1} [J] \right]^{\mu\nu}_{ab} \tag{D.11}$$

which is indeed for $J^\mu_a = \bar{B}^\mu_a = 0$ the proper two point function. For the third derivative, from eq. (D.9) one obtains

$$\frac{\delta^3 Z[J]}{\delta J^\mu_a \, \delta J^\nu_b \, \delta J^\rho_c} \frac{\delta J^\rho_c}{\delta \bar{B}^\lambda_d} \frac{\delta J^\nu_b}{\delta \bar{B}^\tau_e} + \frac{\delta^2 Z[J]}{\delta J^\mu_a \, \delta J^\nu_b} \frac{\delta^2 J^\nu_b}{\delta \bar{B}^\lambda_d \, \delta \bar{B}^\tau_e} = 0 \tag{D.12}$$

so that

$$\frac{\delta^3 \Gamma[\bar{B}]}{\delta \bar{B}^a_\mu \, \delta \bar{B}^b_\nu \, \delta \bar{B}^c_\rho} = \left[\Sigma^{-1}[J] \right]^{aa'}_{\mu\mu'} \left[\Sigma^{-1}[J] \right]^{bb'}_{\nu\nu'} \left[\Sigma^{-1}[J] \right]^{cc'}_{\rho\rho'} \frac{\delta^3 Z[J]}{\delta J^{a'}_{\mu'} \, \delta J^{b'}_{\nu'} \, \delta J^{c'}_{\rho'}} \tag{D.13}$$

which for $J^\mu_a = \bar{B}^\mu_a = 0$ is the proper three point function, etc.

Define now

$$\tilde{W}[J,A] = \int [\mathcal{D} B^a_\mu] \det \left[\frac{\delta f_a}{\delta \theta_b} \right] \exp \left\{ i \int d^4x \left[\mathcal{L} (A+B) - \frac{1}{2\alpha} (f_a)^2 + J^a_\mu B^{\mu a} \right] \right\} \tag{D.14}$$

where A^μ_a, not being integrated over, is the classical background field, and θ_b refers to the infinitesimal gauge transformation

$$\delta B^\mu_a = g f_{abc} \, \delta\theta_b \, (A^\mu_c + B^\mu_c) - \partial^\mu \delta \theta_a \tag{D.15}$$

As in the conventional approach one defines

$$\tilde{Z}[J,A] = -i \ln \tilde{W}[J,A] \tag{D.16}$$

and

$$\tilde{\Gamma}[\tilde{B}, A] = \tilde{Z}[J, A] - \int d^4x \, J_\mu^a \, \tilde{B}^{\mu a} \tag{D.17}$$

with

$$\tilde{B}_a^\mu = \frac{\delta \tilde{Z}[J, A]}{\delta J_\mu^a} \tag{D.18}$$

We now choose the background field gauge condition

$$f^a = \partial^\mu B_\mu^a + g f_{abc} A_\mu^b B^{\mu c} \equiv (D_A^\mu B_\mu)^a \tag{D.19}$$

where

$$(D_A^\mu)_{ab} = \partial^\mu \delta_{ab} - g f_{abc} A_c^\mu \tag{D.20}$$

is the covariant derivative with respect to gauge transformations in the background field. By making the change of variables

$$B_a^\mu \longrightarrow B^\mu_{\ a} + g f_{abc} \theta_b B^\mu_c \tag{D.21}$$

one can easily show that $\tilde{W}[J,A]$ and hence $\tilde{Z}[J,A]$ are invariant under the infinitesimal transformations

$$\begin{aligned} \delta A_\mu^a &= g f_{abc} \delta\theta_b A_\mu^c - \partial_\mu \delta\theta_a \\ \delta J_\mu^a &= g f_{abc} \delta\theta_b J_\mu^c \end{aligned} \tag{D.22}$$

with this gauge fixing term. Indeed the measure is invariant, as well as $\mathcal{L}(A+B)$ and $J_\mu^a B_a^\mu$. On the other hand, from eq. (D.19)

$$\delta f_a = g f_{abc}\, \delta\theta_b\, (\partial_\mu B^\mu_c) - g^2 f_{adc} f_{ceb} A^b_\mu\, \delta\theta_d\, B^\mu_e = g f_{abc}\, \delta\theta_b\, f_c \tag{D.23}$$

but

$$\delta(f_a^2) = 0 \tag{D.24}$$

Finally, since eqs. (D.23) and (D.24) imply an orthogonal transformation in the space of the adjoint index a, the last factor, $\det[\delta f_a/\delta\theta_b]$, is also invariant because the determinant of an orthogonal transformation is 1.

It then follows that $\tilde{\Gamma}[\tilde{B},A]$ is invariant under the infinitesimal transformation

$$\delta A^a_\mu = g f_{abc}\, \delta\theta_B\, A^c_\mu - \partial_\mu \delta\theta_a$$

$$\delta \tilde{B}^a_\mu = g f_{abc}\, \delta\theta_b\, \tilde{B}^c_\mu \tag{D.25}$$

in the background field gauge. In particular $\tilde{\Gamma}[0,A]$ is an explicitly gauge invariant functional of A, since the first of eqs. (D.25) is just the ordinary gauge transformation of the background field. The quantity $\tilde{\Gamma}[0,A]$ is the gauge-invariant effective action which one computes in the background field method. We will now show that $\tilde{\Gamma}[0,A]$, which up to this point looks as a rather artificial construction, is the usual effective action $\Gamma[\bar{B}]$ with $\bar{B}=A$, calculated in a unconventional gauge which depends on A. Thus, $\tilde{\Gamma}[0,A]$ can be used to generate the proper Green's functions and thus the S-matrix elements of a gauge theory exactly as the usual effective action is employed. It is furthermore explicitly gauge invariant, which has several advantages as we will see.

We will now derive relationships between W,Z and Γ and $\tilde{W}$, $\tilde{Z}$ and $\tilde{\Gamma}$. Consider the change of variable $B_a^\mu \rightarrow B_a^\mu - A_a^\mu$ in eq. (D.14) . One then finds in the background field gauge

$$\tilde{W}[J,A] = W[J] \exp\left\{ -i \int d^4x \, J_\mu^a A_a^\mu \right\} \tag{D.26}$$

with $W[J]$ evaluated in the gauge

$$f^a = \partial^\mu B_\mu^a - \partial^\mu A_\mu^a + g f_{abc} A_\mu^b B^{\mu c} \tag{D.27}$$

and where $\delta f^a / \delta \theta^b$ refers to the standard gauge transformations in the quantum field

$$\delta B_a^\mu = g f_{abc} \delta\theta_b B_c^\mu - \partial^\mu \delta\theta_a \tag{D.28}$$

Notice that because of the presence of A_μ^a in the gauge of eq. (D. $W[J]$ in eq. (D.26) is actually a functional of A as well. From eq. (D.26)

$$\tilde{Z}[J,A] = Z[J] - \int d^4x \, J_\mu^a A_a^\mu \tag{D.29}$$

and again $Z[J]$ depends on A through the gauge fixing condition. Taking derivatives with respect to J we find

$$\tilde{B}_a^\mu = \bar{B}_a^\mu - A_a^\mu \tag{D.30}$$

Finally, performing a Legendre transformation on eq. (D.29) we find

$$\tilde{\Gamma}(\tilde{B}, A) = \Gamma[\bar{B}]\Big|_{\bar{B}=\tilde{B}+A} \tag{D.31}$$

and putting $\tilde{B} = 0$

$$\tilde{\Gamma}[0, A] = \Gamma[\bar{B}]\Big|_{\bar{B}=A} \tag{D.32}$$

which is what we wanted to prove. Recall that $\Gamma[\bar{B}]$ depends on A through the gauge-fixing term.

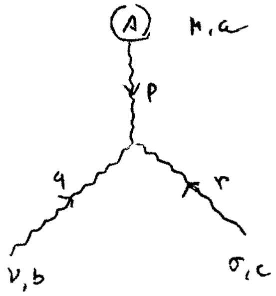
We are now ready for determining the Feynman rules. What is left is to write the determinant factor appearing in the functional integral in terms of ghost fields. From eqs. (D.19) and (D.15) one obtains immediately

$$\frac{\delta f_a}{\delta \theta_b} = -\delta_{ab}\,\Box + g\, f_{abc}\, A^c_\mu \partial^\mu - g\, f_{abc}\, \overleftarrow{\partial}^\mu (A^c_\mu + B^c_\mu) + g^2 f_{amc} f_{cbm} A^m_\mu (A^\mu_m + B^\mu_m) \tag{D.33}$$

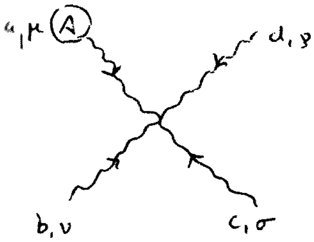
and finally (cf. eq. B.80)

$$\mathcal{L}_{ghost} = \bar{\phi}_a M_{B,ab} \phi_b \tag{D.34}$$

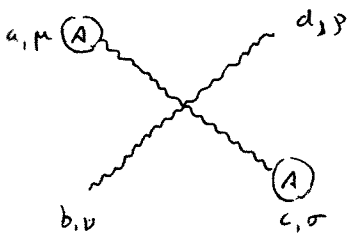
with $M_{B,ab}$ given by eq. (D.33) . One can now read off the Feynman rules. Since the effective action involves only one-particle-irreducible diagrams, vertices with only one outgoing quantum line will never contribute. We therefore do not include them in the Feynman rules. The new ones, involving the background field, are



$$-ig\, f_{abc} \left[g_{\mu\nu} \left(p - q - \tfrac{1}{\alpha} r\right)_\sigma + g_{\nu\sigma} (q - r)_\mu + g_{\sigma\mu} \left(r - p + \tfrac{1}{\alpha} q\right)_\nu \right]$$



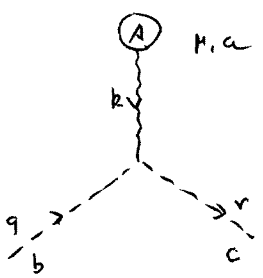
the same as the quartic gluon vertex



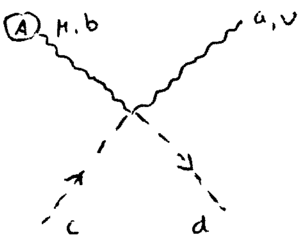
$$-g^2 \left[f_{abe} f_{cde} \left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma} + \tfrac{1}{\alpha} g_{\mu\nu} \right.\right.$$

$$+ f_{ace} f_{bde} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}\right)$$

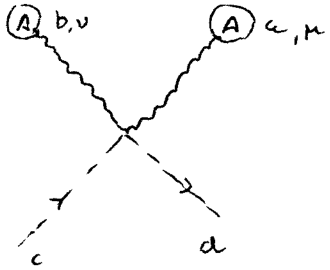
$$+ f_{ade} f_{cbe} \left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\sigma\rho} + \tfrac{1}{\alpha} g_{\mu\rho}\right.$$



$$-ig\, f_{abc} \, (r + q)^\mu$$



$$g^2 f_{ace} f_{abe}\, g_{\mu\nu}$$



$$g^2 \left(f_{ace} f_{dbe} + f_{bce} f_{dae} \right) g_{\mu\nu}$$

Consider now the UV divergences which appear in $\tilde{\Gamma}[0,A]$. Since it is the sum of all one-particle-irreducible diagrams with A-fields on external legs and B-fields inside loops one only needs to perform the A-field renormalization, the coupling constant renormalization and the gauge parameter renormalization. This last one can be avoided by working in the Landau gauge. However the limit $a=0$ can only be performed when the calculations are at a stage where all the $1/a$ factors from the vertices have cancelled.

Because explicit gauge invariance is retained in the background field method, the renormalization constants Z_A and Z_g are related. The infinities appearing in the gauge invariant effective action $\tilde{\Gamma}[0,A]$ must take the gauge invariant form of a divergent constant times $F_{\mu\nu}^{a} F_{a}^{\mu\nu}$, $F_{\mu\nu}^{a}$ being the field strength of the background gauge field. This is a requirement analog to the one which demands that the Slavnov-Taylor identities be satisfied also for the renormalized theory. But the bare field strength is

$$F_{\mu\nu 0}^{a} = Z_A^{1/2} \left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} Z_g Z_A^{1/2} A_\mu^b A_\nu^c \right] \qquad (D.36)$$

Here we have renormalized the composite field product $A_\mu^b A_\nu^c$ just by renormalizing its fields, because the fields are classical. Now, eq. (D.36) will only take the gauge covariant form of a constant times $F_{\mu\nu}^{a}$ if

$$Z_g = Z_A^{-1/2} \tag{D.37}$$

This leads then to the relation between the coupling constant and the background field renormalization in the background field gauge. Notice that eq. (D.37) is an identity well known for QED : in the background field gauge the renormalization of a nonabelian gauge theory resembles the one of an abelian gauge theory.

Eq. (D.37) simplifies the computation of the β -function in QCD, since it is now enough to know the background field renormalization and this only requires the study of the background field two poi function. The details of this computation can be followed in a reference which we have followed almost verbatim in this appendix [AB 81

APPENDIX E.- SUM RULES IN QUANTUM MECHANICS

We would like to discuss here sum rules, analogous to the ones used in Q.C.D., in the framework of Quantum Mechanics [BB 81] [NS 81].

Let us consider a particle of mass M under the influence of a potential $V(\vec{x})$. The Schrödinger equation for stationary states is

$$H\psi(\vec{x}) \equiv \left[-\frac{\hbar^2}{2M} \Delta + V(\vec{x}) \right] \psi(\vec{x}) = E\psi(\vec{x}) \tag{E.1}$$

Let $\psi_\alpha(x)$ be the eigenfunction corresponding to the eigenvalue E_α and, in order to simplify the notation, we will assume that H has only discrete eigenvalues. The resolvent of H is defined as

$$G(z) \equiv \frac{1}{zI - H} \tag{E.2}$$

where **z** is an arbitrary complex number. Its matrix elements, in the position representation, are

$$G(\vec{x},\vec{y};z) \equiv \langle \vec{x} | G(z) | \vec{y} \rangle = \sum_\alpha \frac{\psi_\alpha(\vec{x})\,\psi_\alpha^*(y)}{z - E_\alpha} \tag{E.3}$$

Let us now introduce

$$M(E) \equiv \left[\frac{d}{dE} G(\vec{x},\vec{y};-E) \right]_{\vec{x}=\vec{y}=0} , \quad E>0 \tag{E.4}$$

Using (3) we can write

$$\sum_\alpha \frac{|\psi_\alpha(\vec{0})|^2}{(E + E_\alpha)^2} = M(E) , \quad E>0 \tag{E.5}$$

which is the fundamental equation of our analysis.

In general $M(E)$ can be calculated only in the framework of perturbation theory and eq. (5) can be used to relate some pro ties of the bound states to a perturbative calculation. In order calculate $M(E)$ let us write $H \equiv H_0 + V$, then

$$G(z) = \frac{1}{zI - H_0 - V} = \sum_{n=0}^{\infty} \frac{1}{zI - H_0}\left[V\frac{1}{zI-H_0}\right]^n \equiv \sum_{n=0}^{\infty} G^{(n)}(z) \qquad \text{(E.6}$$

where $G^{(0)}(z) \equiv G_0(Z)$ is the resolvent corresponding to the fr particles and, as it is well known,

$$G_0(\vec{x},\vec{y};z) = -\frac{M}{2\pi^2\hbar^2 i|\vec{x}-\vec{y}|}\int_{-\infty}^{+\infty} dp\,\frac{1}{p^2-2Mz}\, e^{ip|\vec{x}-\vec{y}|/\hbar} \qquad \text{(E.7}$$

In particular

$$G_0(\vec{x},\vec{y};-E) = -\frac{M}{2\pi\hbar^2|\vec{x}-\vec{y}|}\, e^{-\sqrt{2ME}\,|\vec{x}-\vec{y}|/\hbar} \quad , \quad E>0 \qquad \text{(E.8}$$

Furthermore the matrix element of the general term in (6) is

$$\langle\vec{x}|G^{(n)}|\vec{y}\rangle = \int d^3x_1 \ldots d^3x_n \; G_0(\vec{x},\vec{x}_1;z)\,V(\vec{x}_1)\,G_0(\vec{x}_1,\vec{x}_2;z)\,V(\vec{x}_2)\ldots$$

$$\ldots V(\vec{x}_n)\,G_0(\vec{x}_n,\vec{y};z) \equiv G^{(n)}(\vec{x},\vec{y};z) \qquad \text{(E.9}$$

If we introduce

$$M^{(n)}(E) \equiv \left[\frac{d}{dE}\,G^{(n)}(\vec{x},\vec{y};-E)\right]_{\vec{x}=\vec{y}=0} \qquad \text{(E.1}$$

our fundamental equation can be written as

$$\sum_{\alpha} \frac{|\psi_\alpha(\bar{0})|^2}{(E+E_\alpha)^2} = \sum_{n=0}^{\infty} M^{(n)}(E) \quad , \quad E>0 \tag{E.11}$$

where the relation between the bound state properties and the perturbative calculation is clearly shown.

The first two terms of this expansion are obtained easily and they turn out to be

$$M^{(0)}(E) = \frac{M^{3/2}}{2^{3/2}\,\pi\,\hbar^3\,E^{1/2}}$$

$$M^{(1)}(E) = M^{(0)}(E)\left\{-\frac{4M}{\hbar^2}\int_0^\infty dr\, r\, V(r)\, e^{-2\sqrt{2ME}\, r/\hbar}\right\} \tag{E.12}$$

where the first order terms has been given for central potentials.

The main question now is if the calculation of the first perturbative terms of the expansion appearing in eq. (11) allows us to extract some useful information on the properties of bound states. Up to this end let us consider the isotropic harmonic oscillator

$$V(r) \equiv \frac{1}{2} M\omega^2 r^2 \tag{E.13}$$

then $\alpha \equiv (n,l,m)$ and

$$E_{n\ell m} = \hbar\omega\left[2(n-1)+\ell+\frac{3}{2}\right] \quad , \quad |\psi_{n\ell m}(\bar{0})|^2 = \delta_{\ell 0}\,\delta_{m0}\,\frac{(2n-1)!!}{2^{n-1}(n-1)!}\left(\frac{M\omega}{\hbar\pi}\right)^{3/2} \tag{E.14}$$

Equation (11) can be written as

$$\frac{2^{3/2}}{\pi^{1/2}}\left(\frac{\hbar\omega}{E}\right)^{3/2}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{2^{n-1}(n-1)!}\,\frac{1}{\left[1+\frac{\hbar\omega}{E}(2n-1/2)\right]^2} =$$

$$= 1 - \frac{3}{16}\left(\frac{\omega\hbar}{E}\right)^2 + \cdots \tag{E.15}$$

Let us now study if eq. (11) is useful to obtain information on the ground state energy. It is clear from eq. (11) that only in the limit of small values of E the contribution of the ground state term might dominate the sum appearing in the l.h.s, but even in this limit we don't expect strong dominance since the other terms contribution are killed only by terms of order $(E_{g.s.}/E_\alpha)^2$. From eq. (15) it is also clear that the calculation of only a few terms of the perturbative expansion is only a good approximation to M(E) in the limit $E \to \infty$. The situation is quite hopeless and the only thing we can do is to study numerically if a few terms saturate the l.h.s. of eq. (15) for a compromise energy of $E = h\omega$. The first seven energy levels give a contribution to the left hand side of eq. (15) of 0.58 to be compared to the value 0.81 of the right hand side. The obtained sum rule is useless.

The only way to save this sum rule is transforming it in such a way that the ground state dominates the l.h.s. and that the rate of convergence of the perturbative expansion is improved. Up to this end we apply to both terms of the equality (11) the so-called Borel transformation operator

$$\hat{B} \equiv \lim_{\substack{E\to\infty,\ N\to\infty \\ E/N \equiv 1/\tau = \text{fixed}}} \frac{1}{(N-1)!} (-E)^N \left(\frac{d}{dE}\right)^N \tag{E.16}$$

We need to consider the application of $\hat{B}$ to $f(E) = (E+\alpha)^{-\beta}$, $\beta > 0$. Then

$$\hat{f}(\tau) \equiv \hat{B} f = \lim_{\substack{E\to\infty,\ N\to\infty \\ E/N = 1/\tau = \text{fixed}}} \frac{1}{(N-1)!} (-E)^N \left(\frac{d}{dE}\right)^N (E+\alpha)^{-\beta} =$$

$$= \lim_{\substack{E\to\infty,\ N\to\infty \\ E/N = 1/\tau = \text{fixed}}} \frac{1}{\Gamma(N)} (-E)^N (-1)^N \frac{\Gamma(\beta+N)}{\Gamma(\beta)} (E+\alpha)^{-(\beta+N)} =$$

$$= \lim_{N\to\infty} \tau^\beta \frac{1}{\left(1+\frac{\alpha\tau}{N}\right)^\beta} \frac{\Gamma(N+\beta)}{N^\beta\,\Gamma(\beta)\,\Gamma(N)} \frac{1}{\left(1+\frac{\alpha\tau}{N}\right)^N} =$$

$$= \tau^\beta \frac{1}{\Gamma(\beta)} e^{-\alpha\tau}$$

i.e.

$$\hat{f}(\tau) = \hat{B}\,(E+\alpha)^{-\beta} = \tau^\beta \frac{1}{\Gamma(\beta)} e^{-\alpha\tau} \tag{E.17}$$

Using this result we obtain

$$\hat{B} \sum_\alpha \frac{|\psi_\alpha(\vec{o})|^2}{(E+E_\alpha)^2} = \tau^2 \sum_\alpha |\psi_\alpha(\vec{o})|^2 e^{-E_\alpha\tau} \tag{E.18}$$

and the contribution of the ground state is clearly dominating in the limit $\tau \to \infty$. The right hand side of eq. (11) is an expansion with general term of the form a_n/E^n and by the Borel transformation this gets transformed into $a_n\,\tau^n/\Gamma(n)$ and the convergence of the series expansion is clearly improved. Our sum rule is now

$$\sum_\alpha |\psi_\alpha(\vec{o})|^2 e^{-E_\alpha\tau} = \mathcal{M}(\tau) \tag{E.19}$$

where $\mathcal{M}(\tau)$ must be computed by perturbative methods.

Let us recall that the retarded propagator is

$$G_+(\vec{x}_f,t_f;\vec{x}_i,t_i) \equiv \langle \vec{x}_f | e^{-iH(t_f-t_i)/\hbar} | \vec{x}_i \rangle \quad , \quad t_f > t_i \tag{E.20}$$

and hence

$$G_+ (\vec{x}_f, t_f ; \vec{x}_i, t_i) = \sum_\alpha \psi_\alpha (\vec{x}_f) \psi^*_\alpha (\vec{x}_i) \, e^{-i E_\alpha (t_f - t_i)/\hbar} \tag{E.21}$$

and by analytical continuation we get

$$\mathcal{M}(\tau) = G_+ (\vec{0}, -i\hbar\tau ; \vec{0}, 0) \tag{E.22}$$

and our sum rule is

$$\sum_\alpha |\psi_\alpha (\vec{0})|^2 e^{-E_\alpha \tau} = G_+ (\vec{0}, -i\hbar\tau ; \vec{0}, 0) \tag{E.23}$$

where the retarded propagator must be computed through a power series expansion.

Let us now return to the harmonic oscillator. In this case we know the exact expression of G_+ and therefore

$$\mathcal{M}(\tau) = \left[\frac{M\omega}{2\pi\hbar \sinh (\hbar\omega\tau)} \right]^{3/2} \tag{E.24}$$

and our sum rule reads

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{n-1} (n-1)!} e^{-\hbar\omega\tau (2n - 1/2)} = \frac{1}{[2 \sinh (\hbar\omega\tau)]^{3/2}} =$$

$$= \frac{1}{(2\hbar\omega\tau)^{3/2}} \left\{ 1 - \frac{1}{4} (\hbar\omega\tau)^2 + \frac{19}{480} (\hbar\omega\tau)^4 - \frac{631}{120\,960} (\hbar\omega\tau)^6 + \cdots \right\} \tag{E.25}$$

The l.h.s. is dominated by the ground state term in the limit $\tau \to \infty$, while a finite number of terms in the perturbative expansion is a good approximation only in the limit $\tau \to 0$. If we take as a compromise value $\tau = 1/\hbar\omega$ then the ground state level gives a contribution of 0.223 which practically saturates the sum rule since the r.h.s. is 0.278 . The situation has clearly improved.

Since the sum rule (23) seems adequate to obtain information on bound states from a perturbative calculation let us see a possible way to proceed in order to calculate $|\psi_\alpha(\vec{0})|^2$ and E_α for the ground state of the harmonic oscillator. For $y = \hbar\omega\tau$ large enough the l.h.s. of our sum rule is dominated by the ground state contribution and we can write

$$|\psi_{g.s}(\vec{0})|^2 e^{-E_{g.s.}y} = \frac{1}{[2\sinh y]^{3/2}} = \frac{1}{(2y)^{3/2}}\left\{1 - \frac{1}{4}y^2 + \frac{19}{480}y^4 \cdots\right\} \qquad \text{(E. 26)}$$

Taking derivatives with respect to y and dividing the obtained expression by the initial one we get

$$E_{g.s.} = \frac{3}{2}\,\frac{1}{\tanh y} = \frac{3}{2y}\sum_{m=0}^{\infty}\frac{2^{2m}B_{2m}}{(2m)!}\,y^{2m} =$$

$$= \frac{3}{2y}\left\{1 + \frac{1}{3}y^2 - \frac{1}{45}y^4 + \frac{2}{945}y^6 - \frac{1}{4725}y^8 + \frac{2}{93555}y^{10}\cdots\right\} \qquad \text{(E.27)}$$

where B_{2m} are Bernouilli's numbers.

It is clear that, when we calculate a finite number of terms in the perturbative expansion, this expression is valid only if y is large enough for the ground state to dominate the l.h.s. of eq. (25) and furthermore not so large that the neglected terms in the perturbative expansion give an important contribution. Then we expect that if we plot $E_{g.s.}$ versus y we will find a region of values of y , neither too large nor too small, for which $E_{g.s.}$ is practically independent of y , and this is what actually happens.

A possible way to precise the value of $E_{g.s.}$, when the first N terms of the perturbative expansion are used, is to study $E_{g.s.}(y)$ and determine the point where $E_{g.s.}(y)$ presents a minimum or an inflection point. For N even we find only one minimum and for N odd, larger than 1, we find only one inflection point. Using this criteria and eq. (27) and (26) we find the results given in the following table, which are not in bad agreement with the real values $E_{g.s.} = 3/2$ and $|\psi_{g.s.}(\bar{o})|^2 = 1$.

N	Value of y for the minimum or inflec. point	$E_{g.s.}$	$\lvert\psi_{g.s.}(\bar{o})\rvert^2$
2	1.732	1.732	1.101
3	1.968	1.492	1.254
4	2.043	1.584	1.070
5	2.052	1.536	1.165
6	2.237	1.550	1.058

REFERENCES

AB 69 S.L. ADLER and W.A. BARDEEN, Phys. Rev. 182,1517(1969).

AB 81 L.F. ABBOTT, Nuclear Physics B185,189(1981).

AC 75 T. APPELQUIST and J. CARAZZONE, Phys. Rev. D11,2856(1975).

AC 81 G. ALTARELLI, G. CURCI, G. MARTINELLI and S. PETRARCA, Nuclear Physics B187,461(1981).

AD 69 S.L. ADLER, Phys. Rev. 177,2426(1969).

AL 73 E.S. ABERS and B.W. LEE, Physics Reports 9C,1(1973).

AJ 67 M. ASTAUD and B. JOUVET, C.R. Acad. Sci. 264,1633(1967).

AJ 68 M. ASTAUD and B. JOUVET, Nuovo Cim. 53A,841(1968).

AJ 69 M. ASTAUD and B. JOUVET, Nuovo Cim. 63A,5(1969).

AS 65 M. ABRAMOWITZ and I. STEGUN, "Handbook of Mathematical Function Dover Pub. Inc. N.Y., 1965.

AS 72 J. ASHMORE, Nuovo Cim. Lett. 4,289(1972).

BA 75 W.A. BARDEEN, Nuclear Physics B75,246(1975).

BB 81 J.S. BELL and R. BERTLMANN, Nuclear Physics B177,218(1981).

BE 39 F.J. BELINFANTE, Physica 6,887(1939).

BE 40 F.J. BELINFANTE, Physica 7, 305(1940).

BE 53 H. BATEMAN and A. ERDELYI, "Higher Transcendental Functions", Mc Graw-Hill, N.Y. 1953.

BE 66 F.A. BEREZIN, "The Method of Second Quantization", Academic Press, N.Y. 1966.

BF 77 A.J. BURAS, E.G. FLORATOS, D.A. ROSS and C.J. SACHRAJDA, Nuclear Physics B131,308(1977).

BG 72 C.G. BOLLINI and J.J. GIAMBIAGI, Nuovo Cim. 12B,20(1972).

BH 50 K. BLEULER and W. HEITLER, Progr. Theor. Phys. 5,600(1950).

BJ 69 J.D. BJORKEN, Phys. Rev. 179,1547(1969).

BJ 69a J.S. BELL and R. JACKIW, Nuovo Cim. 60A, 47(1969).

BL 50 K. BLEULER, Helv. Phys. Acta 23,567(1950).

BM 77 P. BREITENLOHNER and D. MAISON, Comm. Math. Phys. 52,11,39 (1977).

BO 80 G. BONNEAU, Nuclear Physics B171,477(1980).

BP 75 A.A. BELAVIN, A.M. POLYAKOV, A.S. SCHWARTZ and YU.S. TYUPKIN, Physics Letters 59B,85(1975).

BR 70 R. BRANDT, Fortschr. Phys. 18,249(1970).

BR 74 C. BECCHI, A. ROUET and R. STORA, Phys. Letters 52B,344(1974).

BR 75 C. BECCHI, A. ROUET and R. STORA, Comm. Math. Phys. 42,127(1975

BR 79 A. BRAMON, "Lectures on Electromagnetic Interactions of Hadrons GIFT Lectures, Univ. Autónoma de Barcelona, 1979.

CA 70 C.G. CALLAN, Phys. Rev. D2,1541(1970).

CA 72 C.G. CALLAN, Phys. Rev. D5,3302(1972).

CA 74 W. CASWELL, Phys. Rev. Letters 33,244(1974).

CD 77 J.C. COLLINS, A. DUNCAN and S.D. JOGLEKAR, Phys. Rev. D16, 438(1977).

CD 81 Y. CHUNG, H.G. DOSCH, M. KREMER and D. SCHALL, Physics Letters, 102B,175(1981).

CD 82 Y. CHUNG, H.G. DOSCH, M. KREMER and D. SCHALL, Nuclear Physics, B197,55(1982).

CF 79 M. CHANOWITZ, M. FURMAN and I. HINCHLIFFE, Nuclear Physics, B159,225(1979).

CH 78 A. CHAKRBARTI, "Introduction to Classical Solutions of Yang-Mills Field Equations", Talks given at the "Rencontre de Rabat", May 1978.

CH 79 E. CORRIGAN and H. HASSLACHER, Physics Letters, 81B,181(1979).

CJ 71 S. COLEMAN and R. JACKIW, Ann. Phys. (N.Y.)67,552(1971).

CK 80 K.G. CHETYRKIN, A.L. KATAEV and F.Y. TKACHOV, Nuclear Physics B174,345(1980).

CM 74 J.C. COLLINS and A.J. MACFARLANE, Phys. Rev. D10,120(1974).

CN 77 J. CALMET, S. NARISON, M. PERROTET and E. de RAFAEL, Rev. Mod. Phys. 69,21(1977).

CO 66 S. COLEMAN, J. Math. Phys. 7,787(1966).

CO 73 S. COLEMAN, "Renormalization and Symmetry: A Review for Non-Specialists", Intern. Summer School of Physics, "Ettore Majorana", 1971(1973).

CO 73a S. COLEMAN, "Dilatations", Intern. School of Subnuclear Physics "Ettore Majorana", 1971(1973).

CO 75 J.C.COLLINS, Nuclear Physics B92,477(1975).

CO 79 S. COLEMAN, "The Uses of Instantons", Intern. School of Subnuclear Physics, "Ettore Majorana", 1977(1979).

CR 79 R.J. CREWTHER, Riv. Nuovo Cim. 2,63(1979).

CR 80 C. CRONSTROEM, Physics Letters, 90B,267(1980).

CV 76 P. CVITANOVIC, Phys. Rev. D14,1536(1976).

CW 74 W.E. CASWELL and F. WILCZEK , Physics Letters, 49B,291(1974).

DD 77 G. DE ANGELIS, D. DE FALCO and F. GUERRA, Lett. Nuovo Cim. 19,55(1977).

DS 81 M. DUBOVICOV and A. SMILGA, Nuclear Physics, B185,109(1981).

EP 83 D. ESPRIU, P. PASCUAL and R. TARRACH, Nuclear Physics,B214,285(

ES 83 D. ESPRIU, Phys. Rev. D28,349(1983).

ET 79 E. Sh. EGORYAN and O.V. TARASOV, Theor. Math. Phys. 41,26(1979)

ET 82 D. ESPRIU and R. TARRACH, Zeitschr. f. Physik C16,77(1982).

ET 82a D. ESPRIU and R. TARRACH, Phys. Rev. D25,1073(1982).

FE 48 R.P. FEYNMAN, Rev. Mod. Phys. 20,267(1948).

FE 50 R.P. FEYNMAN, Phys. Rev. 80,440(1950).

FE 63 R.P. FEYNMAN, Acta Phys. Polon. 26,697(1963).

FG 72 H. FRITZSCH and M. GELL-MANN, Proc. XVI Intern. Conf. on High Energy Physics, Batavia 1972(1972).

FG 73 H. FRITZSCH, M. GELL-MANN and H. LEUTWYLER, Physics Letters, 47B,367(1973).

FH 65 R.P. FEYNMANN and A.R. HIBBS, "Quantum Mechanics and Path Integrals", Mc Graw-Hill, N.Y. 1965.

FM 74 D.Z. FREEDMAN, I.J. MUZINICH and E.J. WEINBERG, Ann. Phys. (N.Y. 87,95(1974).

FP 67 L.D. FADDEEV and Y.N. POPOV, Physics Letters 25B,29(1967).

FS 76 V. FATEEV, A. SCHWARTZ and Y. TYUPKIN, preprint FIAN-155, (Moscow 1976).

FS 80 L.D. FADDEEV and A.A. SLAVNOV, "Gauge Fields. Introduction to Quantum Theory", The Benjamin/Cummings Publ.Co. Inc. Reading, Massachusetts, 1980.

FW 74 C.Z. FREEDMAN and E.J. WEINBERG, Annals Physics (N.Y.) 87, 354(1974).

GE 64 M. GELL-MANN, Physics 1,63(1964).

GE 72 M. GELL-MANN, Acta Phys. Austriaca Suppl.IX,733(1972).

GL 51 M. GELL-MANN and F. LOW, Phys. Rev. 84,350(1951).

GL 54 M. GELL-MANN and F. LOW, Phys. Rev. 95,1300(1954).

GM 73 R. GASTMANS and R. MEULDERMANS, Nuclear Physics B63,277(1973).

GN 79 J.L. GERVAIS and A. NEVEAU, Physics Letters, 80B,255(1979).

GO 61 J. GOLDSTONE, Nuovo Cim. 19,154(1961).

GR 65 I.S. GRADSHTEIN and I.M. RYZHIK, "Tables of Integrals, Series and Products", Academic Press, New York and London (1965).

GR 76 D.J. GROSS in "Methods in Field Theory", Les Houches, 1975, Editors : R. Balian and J. Zinn-Justin, North-Holland, (1976)

GS 62 J. GOLDSTONE, A. SALAM and S. WEINBERG, Phys. Rev. 127,965(1962)

GU 50 S.N.GUPTA, Proc. Phys. Soc. 63,681(1950).

GU 51 S.N. GUPTA, Proc. Phys. Soc. 64,850(1951).

GW 73 D.J. GROSS and F. WILCZEK, Phys. Rev. D8,3633(1973).

GY 60 I.M. GEL'FAND and A.M. YAGLOM, Journal Math. Phys. 1,48(1960).

HM 81 H. HAMBER, E. MARINARI, G. PARISI and C. REBBI, Physics Letters 108B,314(1981).

IO 81 B.L. IOFFE, Nuclear Physics B188,317(1981), Err. Nuclear Physic B191,599(1981).

IZ 80 C. ITZYKSON and J.B. ZUBER, "Quantum Field Theory", Mc Graw-Hill, New York 1980.

JL 76 S.D. JOGLEKAR and B.W. LEE, Annals Physics (N.Y.) 97,160(1976).

JL 82 D.R.T. JONES and J.P. LEVEILLE, Nuclear Physics B206,473(1982).

JO 74 D.R.T. JONES, Nuclear Physics B75,531(1974).

KO 79 J.B. KOGUT, Rev. Mod. Phys.51,659(1979).

KS 76 J.B. KOGUT and L. SUSSKIND, Phys. Rev. D11,3594(1976).

KZ 75 H. KLUBERG-STERN and J.B. ZUBER, Phys. Rev. D12,3159(1975).

LE 58 L. LEWIN, "Dilogarithms and Associated Functions", MacDonald, London 1958.

LE 72 B.W. LEE, Phys. Rev. D5,823(1972).

LE 75 G. LEIBRANDT, Rev. Mod. Phys. 47,849(1975).

LR 74 B. LAUTRUP and E. DE RAFAEL, Nuclear Physics B70,317(1974).

LR 81 C.H. LLEWELLYN-SMITH, G.G. ROSS and F.J. WHEATER, Nuclear Physics B177,263(1981).

LU 68 D. LURIE, "Particles and Fields", Interscience N.Y. 1968.

LZ 72 B.W. LEE and J. ZINN-JUSTIN, Phys. Rev. D5,3137(1972).

LZ 73 B.W. LEE and J. ZINN-JUSTIN, Phys. Rev. D7,1049(1972).

MM 79 YU.M. MAKEENKO and A.A. MIGDAL, Physics Letters, 88B,135(1979).

MM 81 YU. M. MAKEENKO and A.A. MIGDAL, Nuclear Physics, B188, 269(1981).

MP 78 W. MARCIANO and N. PAGELS, Physics Reports 36C,138(1978).

MS 75 W. MARCIANO and A. SIRLIN, Nuclear Physics B88,86(1975).

NA 78 Y. NAMBU, Physics Letters 80B,372(1978).

NI 77 N.K. NIELSEN, Nuclear Physics, B120,212(1977).

NP 83 S. NARISON, N. PAVER, E. DE RAFAEL and D. TRELEANI, Nuclear Physics B212,365(1983).

NS 81 V.A. NOVIKOV, M.A. SHIFMAN, A.I. VAINSHTEIN and V.I. ZAKHAROV, Nuclear Physics B191,301(1981).

NT 83 S. NARISON and R. TARRACH, Physic Letters, 125B,217(1983).

OS 73 K. OSTERWALDER and R. SCHRADER, Comm. Math. Phys. 31,83(1973).

OS 75 K. OSTERWALDER and R. SCHRADER, Comm. Math. Phys. 42,281(1975).

PA 80 G. PARISI, "Recent Progress in Gauge Theories" in XX Conf. on High Energy Physics, Madison 1980.

PO 73 N.D. POLITZER, Phys. Rev. Letters 30,1346(1973).

PO 78 A.M. POLIAKOV, Physics Letters 82B,247(1978).

PT 80 P. PASCUAL and R. TARRACH, Nuclear Physics B174,123(1980), Err. B181,546(1981).

PV 49 W. PAULI and F. VILLARS, Rev. Mod. Phys. 21,434(1949).

PW 68 G. PREPARATA and W.I. WEISBERGER, Phys. Rev. 175,1965(1968)

RA 76 E. DE RAFAEL, "Lectures on Quantum Electrodynamics", Universidad Autónoma Barcelona 1976.

RA 78 E. DE RAFAEL, "Quantum Chromodynamics as a Theoretical Framework of the Hadronic Interactions", Ecole d'Eté de Physique des Particules. Gif-sur-Yvette, 1978;Springer L.N.P.118(1980

RO 67 J.L. ROSNER, Annals Physics 44,11(1967).

SC 51 J. SCHWINGER, Proc. Nat. Acad. Sci, 37,452(1951).

SC 70 J. SCHWINGER, "Particles, Sources and Fields", Addison Wesley, 1970.

SC 71 E.T. SCHREIER, Phys. Rev. D3,980(1971).

SH 80 M.A. SHIFMAN, Nuclear Physics B173,13(1980).

SI 79 W. SIEGEL, Physics Letters 84B,193(1979).

SL 75 A.A. SLAVNOV, Sov. Jour. Part. Nucl. 5,303(1975).

SP 53 E.C.G. STUECKELBERG and A. PETERMAN, Helv. Phys. Acta, 26,499(1953).

SU 77 K. SUZUKI, Comm. Math. Phys. 57,193(1977).

SV 79 M.A. SHIFMAN, A.I. VAINSHTEIN and V.I. ZAKHAROV, Nuclear Physic B147,385,448,519(1979).

SY 69 K.SYMANZIK, Proc. Intern. School of Physics "Enrico Fermi", Course XLV, ed. R. Jost, Academic Press, N.Y. 1969.

SY 70 K. SYMANZIK, Comm. Math. Phys. 18,227(1970).

SY 71 K. SYMANZIK, "Small Distance Behaviour in Field Theory", Springer Tracts in Modern Physics 57,Berlin, Heidelberg, New York (1971).

SY 71a K. SYMANZIK, Comm. Math. Phys. 23,49(1971).

SY 73 K. SYMANZIK, Comm. Math. Phys. 34,7(1973).

TA 57 Y. TAKAHASHI, Nuovo Cim. 6,370(1957).

TA 71 J.C. TAYLOR; Nuclear Physics B33,436(1971).

TA 81 R. TARRACH, Nuclear Physics B183,384(1981).

TA 82 R. TARRACH, Nuclear Physics B196,45(1982).

TH 50 W. THIRRING, Phil. Mag. 41,113(1950).

TH 71 G. 'T HOOFT, Nuclear Physics B33,173(1971).

TH 73 G. 'T HOOFT, Nuclear Physics B61,455(1973).

TH 76 G. 'T HOOFT, Phys. Rev. Letters 37,8(1976).

TH 77 G. 'T HOOFT, "Some Observations on Quantum Chromodynamics", Orbir Scientiae 1977, Miami (1977).

TR 79 T.L. TRUEMAN, Physics Letters 88B,331(1979).

TV 72 G. 'T HOOFT and M. VELTMAN, Nuclear Physics B44,189(1972).

TV 72a G. 'T HOOFT and M. VELTMAN, Nuclear Physics B50,318(1972).

TV 73 G. 'T. HOOFT and M. VELTMAN, "Diagrammar" CERN Yellow Report 73-9(1973).

TV 80 O.V. TARASOV, A.A. VLADIMIROV and A. YU. ZHARKOV, Physics Letters 93B,429(1980).

WA 50 J.C. WARD, Phys. Rev. 78,182(1950).

WE 60 S. WEINBERG, Phys. Rev. 118,838(1960).

WE 73 S. WEINBERG, Phys. Rev. Letters, 31,494(1973).

WE 73a S. WEINBERG, Phys. Rev. D8,4482(1973).

WE 73b S. WEINBERG, Phys. Rev. D8,3497(1973).

WE 75 S. WEINBERG, Phys.Rev. D11,3583(1975).

WI 69 K.G. WILSON, Phys. Rev. 179,1499(1969).

WI 74 K.G. WILSON, Phys. Rev. D10,2445(1974).

WM 79 C. DE WITT-MORETTE, A. MAHESHWARI and B. NELSON, Physics Reports C50,255(1979).

YM 54 C.N. YANG and R.L. MILLS, Phys. Rev. 86,191(1954).

YN 83 F.J. YNDURAIN, "Quantum Chromodynamics", Springer Verlag (1983)

ZI 71 W. ZIMMERMANN, in "Lectures on Elementary Particles and Quantum Field Theory", 1970, Brandeis, eds. S. Deser, M. Grisaru and H. Pendleton, MIT Press, Cambridge, Mass. 1971.

ZI 73 W. ZIMMERMANN, Annals Physics 77,536,570(1973).

ZI 75 J.ZINN-JUSTIN, "Renormalization of Gauge Theories", 1974 Bonn International Summer Institute, Lecture Notes in Physics, 37,Springer, Berlin, Heidelberg, New York (1975).