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# QCD Sum Rule for the Second Moments of the B-meson Light-Cone-Distribution Amplitude

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Abstract: We have derived the sum rules of the $B$-meson light-cone expansion parameters $\lambda_E^2$ and $\lambda_H^2$ in the framework of heavy-quark effective theory by considering the diagonal quark-antiquark-gluon three-particle current. In this analysis, we include vacuum condensates up to mass dimension seven and find that our new approach lead to more stable sum rule for the parameter $\lambda_H^2$, which reduces the uncertainty compared to previous results. Our result for the parameter $\lambda_E^2$ fully agrees with the latest update that can be found in the literature.

# Contents

# 1 Introduction

QCD sum rules [1–3] have proven to be a powerful and useful method for estimating hadronic observables, which are not accessible by ordinary perturbative approaches. The idea of this method is based on the concept of quark-hadron duality (QHD) [4, 5] and operator-product expansion (OPE) [1, 6–8]. In the spacelike region, processes can be described by a truncated OPE, which separates large distance and small distance contributions. Small distance contributions are encoded in the perturbatively calculable Wilson coefficients, while the large distance contributions are parameterized by a set of local operators, called vacuum condensates. These condensates have the quantum numbers of the QCD vacuum and would vanish in QCD by definition. Physical observables can be described in the timelike region by spectral functions. In order to connect these two regions, where calculations can be performed with standard techniques, one can apply the quark-hadron duality. Over the last decades, many different applications for QCD sum rules have been investigated. The most commonly used nowadays in the literature is to start with a quark condensate of mass dimension three $\langle 0| \bar{q}q |0\rangle$ and include higher dimensional condensates by using combinations of the gluon field-strength tensor $G_{\mu\nu}$ (mass dim. two), spinorial fields (mass dim. $\frac{3}{2}$) and different Lorentz structures containing Dirac matrices. Nevertheless, there is only one unique condensate for each mass dimension up to dimension five. Beyond this mass dimension, vacuum condensates are not uniquely parameterized anymore leading to a larger number of possible contributions. The parameters of interest in this work, which will be denoted as $\lambda^2_{E,H}$, are defined in the heavy-quark effective theory (HQET) [9] and are particularly important for the determination of B-meson distribution amplitudes (CITATIONS). Similar work has already been done for the case of $\pi$- or K-mesons, where analogous parameters have been introduced [10]. The parameters have first been investigated by Grozin and Neubert [11] as a parameterization of the second moment of the B-meson wave function and were further investigated by Nishikawa and Tanaka [12]. Both paper studied these parameters in the framework of HQET, which exploits the large separation in scales between the B-meson and b-quark mass and $\Lambda_{QCD}$, where confinement is supposed to happen. One advantage of HQET in comparison to full QCD is that the gluon from the three-body current does not interact with the heavy quark, which reduces the number of Feynman diagrams in the calculation. Both articles made use of QCD sum rules which have been embedded into the HQET framework [13].

First of all, the parameters are introduced as transition amplitudes between the vacuum and a $\bar{B}$-meson:

$$\langle 0| g_s\bar{q}\,\vec{\alpha}\cdot\vec{E}\,\gamma_5 h_v\,|\bar{B}(v)\rangle = F(\mu)\lambda_E^2 \tag{1.1}$$

$$\langle 0| g_s\bar{q}\,\vec{\sigma}\cdot\vec{H}\,\gamma_5 h_v\,|\bar{B}(v)\rangle = iF(\mu)\lambda_H^2 \tag{1.2}$$

Here, $\alpha^i = \gamma^0\gamma^i\gamma_5$, $\sigma^i$ is the i-th Dirac-$\gamma$ matrix, $E^i = G^{0i}$ is the chromoelectic field and $H^i = -\frac{1}{2}\epsilon^{ijk}G^{jk}$ the chromomagnetic field. The gluon field strength tensor is defined to be $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$ with $g_s$ being the strong coupling constant and $f^{abc}$ denoting the structure functions of the Lie algebra. Furthermore, the fields $q$ indicate light quark fields, whereas the field $h_v$ illustrates the HQET heavy quark field. The HQET coupling constant $F(\mu)$ occurs in the pseudo-scalar or pseudo-vector definition of the B-meson [14] and can be related to the B-meson decay constant in QCD up to one loop order [7]

$$f_B\sqrt{m_B} = F(\mu)K(\mu) = F(\mu)\Big[1 + \frac{C_F\alpha_s}{4\pi}\Big(3\cdot\ln\frac{m_b}{\mu} - 2\Big) + ...\Big] + \mathcal{O}\Big(\frac{1}{m_b}\Big). \tag{1.3}$$

Its explicit $\mu$-dependence has to cancel with the one of the matching prefactor in order to lead to the constant $f_B$. Values for $f_B$ can be found in [15] and estimate this decay constant to be:

$$f_B = (0.1920 \pm 0.0043)\ \text{GeV} \tag{1.4}$$

The coupling constant $F(\mu)$ will be of particular importance in the derivation of the relevant low-energy parameters describing the stability window of the sum rules (Section 3, 4).

As already mentioned, Grozin and Neubert [11] defined the parameters and obtained first estimates by considering all contributions up to dimension five. For this, they considered the off-diagonal sum rule corresponding to the correlation function

$$i\int \mathrm{d}^d x\, \langle 0|\, T\{\bar{q}(0)g_s\Gamma_1 G_{\mu\nu}(0)h_v(0)\bar{h}_v(x)q(x)\}\,|0\rangle\,, \tag{1.5}$$

where the gluon from the three-body current couples to the light quark. In their computation, they investigated all leading order contributions to the sum rules. Up to dimension four, the leading order contribution starts at $\mathcal{O}(\alpha_s)$, while the leading order of mass dimension five starts at $\mathcal{O}(\alpha_s^0)$. They obtained the following values:

$$\lambda_H^2(\mu) = (0.18 \pm 0.07)\ \text{GeV}^2 \tag{1.6}$$
$$\lambda_E^2(\mu) = (0.11 \pm 0.06)\ \text{GeV}^2 \tag{1.7}$$

The extraction of these estimates is connected with a rather large uncertainty, because the sum rule turns out to be unstable with respect to the variation of the Borel parameter, even in the stability window the values have a sizeable dependence on the Borel parameter. Notice that such a dependence is not unexpected, since it is well known [16, 17] that higher dimensional condensates tend to give large contributions to correlation functions including higher dimensional operators.

Consequently, Nishikawa and Tanaka [12] argued in their work that a consistent treatment of all $\mathcal{O}(\alpha_s)$ contributions should resolve the stability problem, which is related to the fact that the operator product expansion (OPE) does not converge for the parameters $\lambda^2_{E,H}$ in [11]. For this analysis, they included the $\mathcal{O}(\alpha_s)$ corrections of the coupling constant $F(\mu)$ as well, which, albeit leading to good convergence of the OPE, obey large higher order perturbative corrections [18, 19]. Moreover, they included as an additional non-perturbative correction the dimension six diagram of $\mathcal{O}(\alpha_s)$ in order to check whether this contribution is still large and calculated the $\mathcal{O}(\alpha_s)$ corrections for the dimension five condensate. After performing a resummation of the large logarithmic contributions, which results into a more stable sum rule and into a more convergent OPE compared to [11], the following values have been obtained:

$$\lambda^2_H(\mu = 1\text{ GeV}) = (0.06 \pm 0.03)\text{ GeV}^2 \tag{1.8}$$

$$\lambda^2_E(\mu = 1\text{ GeV}) = (0.03 \pm 0.02)\text{ GeV}^2 \tag{1.9}$$

Their estimates differ by approximately a factor of three from Eq. (1.7), although the ratio gives nearly the same value.

In order to resolve this tension, we are going to investigate an alternative sum rule, which also allows for predictions of $\lambda^2_H$ and $\lambda^2_E$. Thus, the diagonal sum rule between two quark-antiquark-gluon three-particle currents is examined with up to corrections of mass dimension seven. The advantage of this sum rule is that it is positive definite. Although we are investigating a correlation function which has a higher mass dimension than the function in Equation (1.5), we obtain stable sum rules for these parameters. The analysis also includes all $\mathcal{O}(\alpha_s)$ contributions, which is the leading order in this case.

The paper is organised as follows: In Section 2 we derive the sum rules for this approach by choosing the arbitrary $\Gamma_1$ and $\Gamma_2$ matrices explicitly such that $\lambda^2_H$ and the splitting $\lambda^2_H - \lambda^2_E$ is projected out. Section 3 is devoted to the computation of the various contributions, while Section 4 gives the numerical analysis of the sum rule and shows our results for the parameters. Finally, we conclude in Section 5.

## 2 Derivation of the HQET Sum Rules

In this chapter we are going to derive the HQET sum rules for the diagonal quark-antiquark-gluon three-body current. The starting point for the calculation is the correlation function:

$$\Pi_{\Gamma_1\Gamma_2} = i\int \mathrm{d}^d x\, e^{-i\omega v\cdot x} g_s^2 \langle 0| T\{\bar{q}(0)\Gamma_1 G_{\mu\nu}(0)h_v(0)\bar{h}_v(x)\Gamma_2 G_{\rho\sigma}(x)q(x)\}|0\rangle \tag{2.1}$$

Here, $g_s$ corresponds to the strong coupling constant, $v$ is the velocity of the heavy $B$-meson and the tensor $G_{\mu\nu} = \frac{i}{g_s}[D_\mu, D_\nu]$ is defined to be the field-strength tensor. The spinor field $q(x)$ will generally denote the light quark flavor throughout this work, while the field $h_v(x)$ refers to the heavy quark field in HQET. Notice that at this point we do not require a specific choice of the quantities $\Gamma_1$ and $\Gamma_2$, which indicate an arbitrary combination of Dirac $\gamma$-matrices, but in the following steps it is convenient to choose these matrices such that combinations of the HQET parameters $\lambda^2_{E,H}$ are projected out. Nevertheless, the computation of the perturbative and non-perturbative contributions to the spectral density in Section 3 has been done for general $\Gamma_1$ and $\Gamma_2$, since the projection of these parameters requires two different choices.

Moreover, we are working in the $B$-meson rest frame, where $v = (1,\vec{0})^T$, in order to simplify the calculations.

The next step in the derivation of the sum rules will be to exploit the unitary condition, where the ground state $B$-meson is separated from the continuum excited states:

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\Pi_{\Gamma_1\Gamma_2}(\omega) =& \sum_n (2\pi)^3\delta(\omega - p_n)\langle 0|\bar{q}(0)\Gamma_1 g_s G_{\mu\nu}(0)h_v(0)|n\rangle \\ &\times \langle n|\bar{h}_v(x)\Gamma_2 g_s G_{\rho\sigma}(x)q(x)|0\rangle\, \mathrm{d}\Phi_n \\ =&\,\delta(\omega-\bar{\Lambda})\Theta(\omega^0)\langle 0|\bar{q}(0)\Gamma_1 g_s G_{\mu\nu}(0)h_v(0)|B\rangle \\ &\times \langle B|\bar{h}_v(x)\Gamma_2 g_s G_{\rho\sigma}(x)q(x)|0\rangle\ + \rho^{\mathrm{hadr.}}(\omega)\Theta(\omega - s^{th})\end{aligned} \tag{2.2}$$

In Eq. (2.2), we introduced the binding energy $\bar{\Lambda} = m_B - m_b$, which is one of the important low-energy parameters in this formalism. Furthermore, we separated the full phase space contribution in the first line into a ground state contribution, which will be the dominant contribution, and a continuum contribution described by $\rho^{\mathrm{hadr.}}$ starting at the continuum threshold $s^{th}$. In the case of QCD correlation functions, the exponential in Eq. (2.1) would generally take the form $e^{-iqx}$ with $q$ denoting the external momentum. Due to the fact that there is no spatial component in the

B-meson rest frame, transitions from the ground state to the excited states in Eq. (2.2) are possible by injecting energy $q^0$ into the system. In this work we explicitly chose $q = \omega \cdot v$ such that we end up with the correlation function shown in Eq. (2.1).

The matrix elements occurring in (2.2) can be decomposed in the following way [11, 12]:

$$\langle 0| \bar{q}(0)\Gamma_1 G_{\mu\nu}(0)h_v(0) |B\rangle = \frac{-i}{6}F(\mu)\{\lambda_H^2(\mu) \cdot \mathrm{Tr}[\Gamma_1 P_+ \gamma_5 \sigma_{\mu\nu}] + [\lambda_H^2(\mu) - \lambda_E^2(\mu)] \cdot \mathrm{Tr}[\Gamma_1 P_+ \gamma_5 (iv_\mu\gamma_\nu - iv_\nu\gamma_\mu)]\}. \quad (2.3)$$

Notice that the second decomposition is indeed valid since the B-meson ground state explicitly depends on the velocity $v$ and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ corresponds to the usual antisymmetric Dirac bilinear. In (2.3) we made use of the covariant trace formalism, further investigated in [11, 20].

The next step will be to use the standard dispersion relation, after using residue theorem and the Schwartz reflection principle [1]:

$$\begin{aligned}\Pi_{\Gamma_1\Gamma_2}(\omega) = \frac{1}{\pi}\int_0^\infty \mathrm{d}s \frac{\mathrm{Im}\Pi_{\Gamma_1\Gamma_2}(\omega)}{s-\omega-i0^+} \\ = \frac{1}{\bar{\Lambda}-\omega-i0^+} \langle 0| \bar{q}(0)\Gamma_1 g_s G_{\mu\nu}(0)h_v(0) |B\rangle \langle B| \bar{h}_v(x)\Gamma_2 g_s G_{\rho\sigma}(x) q(x) |0\rangle + \\ \int_{s^{th}}^\infty \mathrm{d}s \frac{\rho^{\mathrm{hadr.}}(s)}{s-\omega-i0^+}\end{aligned} \quad (2.4)$$

We can now move on and evaluate the ground state contribution:

$$\begin{aligned}&\langle 0| \bar{q}(0)\Gamma_1 g_s G_{\mu\nu}(0)h_v(0) |B\rangle \langle B| \bar{h}_v(x)\Gamma_2 g_s G_{\rho\sigma}(x) q(x) |0\rangle \\ &= \frac{-i}{6}F(\mu)\Big[\lambda_H^2(\mu) \cdot \mathrm{Tr}[\Gamma_1 P_+ \gamma_5 \sigma_{\mu\nu}] + [\lambda_H^2(\mu) - \lambda_E^2(\mu)] \cdot \mathrm{Tr}[\Gamma_1 P_+ \gamma_5 (iv_\mu\gamma_\nu - iv_\nu\gamma_\mu)]\Big] \cdot \\ &\frac{-i}{6}F^\dagger(\mu)\Big[\lambda_H^2(\mu) \cdot \mathrm{Tr}[\gamma_5 P_+ \Gamma_2 \sigma_{\rho\sigma}] - (\lambda_H^2(\mu) - \lambda_E^2(\mu)) \cdot \mathrm{Tr}[\gamma_5 P_+ \Gamma_2 (iv_\rho\gamma_\sigma - iv_\sigma\gamma_\rho)]\Big]\end{aligned} \quad (2.5)$$

Notice that the term involving the splitting of both HQET parameter $\lambda_H^2 - \lambda_E^2$ does not change its sign under complex conjugation. Additionally, we introduced the low-energy scale-dependent parameter $F(\mu)$, which is directly related to the decay constant $f_B$ of the B-meson in QCD sum rules [7]. Since the constant $f_B$ does not depend on a scale, we expect a direct cancellation of the $\mu$-dependence in the definition of $F(\mu)$.

In order to derive the sum rules which ultimately determine the parameters $\lambda_{E,H}^2$, we make an explicit choice for the matrices $\Gamma_1$ and $\Gamma_2$ [11]. Following the same

[1] For more details on QCD sum rules or HQET sum rules, see [8, 14]

approach as [12], we choose our gamma matrices $\Gamma_{1,2}$ as

$$\Gamma_{1,\mu\nu} = \mathrm{i}\left(\frac{1}{2}\delta_\alpha^{\ \nu} - 2v_\nu v^\alpha\right)\sigma_{\mu\alpha}\gamma_5 \tag{2.6}$$

$$\Gamma_{2,\rho\sigma} = \mathrm{i}\left(\frac{1}{2}\delta_\beta^{\ \sigma} - 2v_\sigma v^\beta\right)\sigma_{\rho\beta}\gamma_5, \tag{2.7}$$

to obtain the splitting $(\lambda_H^2 - \lambda_E^2)^2$ sum rule. Furthemore, for the projection of $\lambda_H^4$ sum rule we choose

$$\Gamma_{1,\mu\nu} = \mathrm{i}\left(\frac{1}{2}\delta_\alpha^{\ \nu} - v_\nu v^\alpha\right)\sigma_{\mu\alpha}\gamma_5 \tag{2.8}$$

continuing from here

$$\Gamma_{2,\rho\sigma} = \mathrm{i}\left(\frac{1}{2}\delta_\beta^{\ \sigma} - v_\sigma v^\beta\right)\sigma_{\rho\beta}\gamma_5. \tag{2.9}$$

Notice that these choices are Lorentz covariant in comparison to Eq. (1.2), where we have already exploited that we work in the B-meson rest frame. Using the ~~relation~~ definition in Eq. (2.5), and isolating invariant amplitudes we can obtain expressions for $\Pi_H$ and $\Pi_{HE}$:

?? this is more subtle in HQET

$$\Pi_H(\omega) = F(\mu)^2 \cdot \lambda_H^4 \cdot \frac{1}{\bar{\Lambda} - \omega - i0^+} + \int_{s_H^{th}}^{\infty} \mathrm{d}s \frac{\rho_H^{\text{hadr.}}(s)}{s - \omega - i0^+} \tag{2.10}$$

$$\Pi_{HE}(\omega) = F(\mu)^2 \cdot (\lambda_H^2 - \lambda_E^2)^2 \cdot \frac{1}{\bar{\Lambda} - \omega - i0^+} + \int_{s_{HE}^{\text{th}}}^{\infty} \mathrm{d}s \frac{\rho_{HE}^{hadr.}(s)}{s - \omega - i0^+} \tag{2.11}$$

Since we do not know any concrete information about the hadronic spectral density, we make use of the global and semi-local quark-hadron duality (QHD) [4, 5] in order to connect the hadronic spectral density with the spectral density which is described by the operator product expansion (OPE) [1, 6–8]. This is the essential idea of this formalism. In the perturbative regime, where $-\omega \gg 0$, contributions can be calculated according to the usual perturbative series in the coupling $\alpha_s$ due to asymptotic freedom. This region is also called the spacelike Euclidean region, where expansions in small Euclidean distances are possible. Notice that this is essential in the OPE approach in order to rewrite the product of two operators at two spacetime points into a local operator and a matching coefficient, known as Wilson coefficient. This Wilson coefficient parameterizes the UV-physics and can therefore be computed with perturbative methods. However, the perturbative expansion breaks down for $-\omega \approx \Lambda_{\text{QCD}}$ and non-perturbative effects start to dominate. In the approach by [1], these effects were parameterized in terms of a power series of higher-dimensional local condensates as a consequence of the non-trivial QCD vacuum. These condensates vanish in ordinary perturbation theory by definition and carry the quantum

something we know

this is too general for an original paper,

maybe better: short-distance physics

this is not a review

maybe better: nonperturbative effects being power suppressed rise on the top of perturbative for convenience

numbers of the QCD vacuum. ~~Therefore,~~ we show explicitly in Appendix A the expansion and averaging of the correlation function (2.1) in order to obtain the quark condensate $\langle 0|\,\bar{q}q\,|0\rangle$, the gluon condensate $\langle 0|\,G^a_{\mu\nu}G^b_{\rho\sigma}\,|0\rangle$, the quark-gluon condensate $\langle 0|\,\bar{q}g_s\sigma\cdot Gq\,|0\rangle$ and the triple-gluon condensate $\langle 0|\,g_s^3 f^{abc}G^a_{\mu\nu}G^b_{\rho\sigma}G^c_{\alpha\lambda}\,|0\rangle$ with the correct quantum numbers. vacuum matrix elements

?

?? unclear

Although we can handle the Euclidean region, the physical states described by the spectral function in Eq. (2.10) and (2.11) are defined for $\omega > 0$. But since there is no estimate for the hadronic spectral density $\rho_X^{\text{hadr.}}(s)$, we need to make use of two assumptions. First, we assume that the hadronic and the OPE spectral functions coincide at the global level:

this is NOT an assumption global duality follows from asymptotic freedom

$$\Pi_X^{\text{hadr.}} = \Pi_X^{\text{OPE}} \qquad X \in \{H, HE\} \tag{2.12}$$

Moreover, we need to employ the more important semi-local quark-hadron duality, which connects the spectral densities:

$$\int_{s_X^{th}}^{\infty} \mathrm{d}s \frac{\rho_X^{\text{hadr.}}(s)}{s-\omega-\cancel{i0^+}} = \int_{s_X^{\text{th}}}^{\infty} \mathrm{d}s \frac{\rho_X^{\text{OPE}}(s)}{s-\omega-\cancel{i0^+}}, \tag{2.13}$$

where $X$ can be chosen according to (2.12). From a more mathematical point of view, these approximations correspond to an analytic continuation of the variable $\omega$ from strictly negative values to positive values. Equation (2.13) introduces the threshold $s_X^{th}$, another relevant low-energy parameter which needs to be chosen such that the sum rule is stable. In the low-energy region, where non-perturbative effects dominate, the duality relation is largely violated due to strong resonance peaks, while in the high-energy region these peaks become ~~distorted~~ broad and overlapping. Hence, we expect a good agreement in this regime and consequently the cutoff $s_X^{th}$ is introduced. This parameter needs to be chosen such that the sum rule is ~~stable and~~ reliable. Once a stable in reliable sum rule is obtained, the approximations made by QHD are consistent (see Section 4 for more details). So it is necessary to work in the transition region where the condensates are important, but still small enough and local such that perturbative methods can be applied.

statement totally unclear

??

as we discussed stability is not the prior reason

agreement with what ?

?

Based on the relation in Eq. (2.12), (2.13) and separating the integral over the OPE spectral density by introducing the threshold parameter $s^{\text{th}}$, we end up with the final form of the sum rules:

$$F(\mu)^2 \cdot \lambda_H^4 \frac{1}{\bar{\Lambda}-\omega-i0^+} = \int_0^{s^{th}} \mathrm{d}s \frac{\rho_H^{\text{OPE}}(s)}{s-\omega-i0^+} \tag{2.14}$$

$$F(\mu)^2 \cdot (\lambda_H^2-\lambda_E^2)^2 \frac{1}{\bar{\Lambda}-\omega-i0^+} = \int_0^{s^{th}} \mathrm{d}s \frac{\rho_{HE}^{\text{OPE}}(s)}{s-\omega-i0^+} \tag{2.15}$$

We see that the low-energy parameter $s^{th}$ acts as a cutoff for the higher resonances and the continuum spectrum, such that only the ground state gives sizeable contributions. Finally, we perform a Borel transformation, which removes possible substraction terms and leads further to an exponential suppression of higher resonances and the continuum. Thus, only the lowest-lying ground states gives a dominant contribution to the sum rule and higher resonances can be neglected by assuming a small error. In addition to that the convergence of our sum rule is improved. Generally, the Borel transform can be defined in the following way [8, 14]:

??

wrong, you already took into account them, via duality

???

$$\mathcal{B}_M f(\omega) = \lim_{n\to\infty,-\omega\to\infty} \frac{(-\omega)^{n+1}}{\Gamma(n+1)}\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^n f(\omega), \tag{2.16}$$

where $f(\omega)$ illustrates an arbitrary test function. Furthermore, we keep the ratio $M = \frac{-\omega}{n}$ fixed, $M$ ~~describes~~ denotes the Borel parameter ~~in this context~~.
After applying this transformation, we derive the final form of our sum rule expressions:

$$F(\mu)^2 \cdot \lambda_H^4 e^{-\Lambda/M} = \int_0^{\omega_{th}} \mathrm{d}\omega\ \rho_H^{\mathrm{OPE}}(\omega)\ e^{-\omega/M} = \int_0^{\omega_{\mathrm{th}}} \mathrm{d}\omega\ \frac{1}{\pi}\mathrm{Im}\Pi_H^{\mathrm{OPE}}(\omega)\ e^{-\omega/M} \tag{2.17}$$

$$F(\mu)^2 \cdot (\lambda_H^2 - \lambda_E^2)^2 \cdot e^{-\Lambda/M} = \int_0^{\omega_{th}} \mathrm{d}\omega\ \rho_{HE}^{\mathrm{OPE}}(\omega)\ e^{-\omega/M} = \int_0^{\omega_{\mathrm{th}}} \mathrm{d}\omega\ \frac{1}{\pi}\mathrm{Im}\Pi_{HE}^{\mathrm{OPE}}(\omega)\ e^{-\omega/M} \tag{2.18}$$

These are the HQET sum rules presented in the paper. In order to obtain stable sum rules, the Borel parameter $M$ needs to be chosen accordingly together with the threshold parameter $s_X^{th}$. The next step will be to determine the spectral function $\Pi_X^{\mathrm{OPE}}(s)$, which is given by the operator product expansion:

$$\begin{aligned}\Pi_{\mathrm{X}}^{OPE}(\omega) = {} & C^{\mathrm{X}}_{pert}(\omega) + C^{\mathrm{X}}_{\bar{q}q}\langle\bar{q}q\rangle + C^{\mathrm{X}}_{G^2}\langle\frac{\alpha_s}{\pi}G^2\rangle + C^{\mathrm{X}}_{\bar{q}Gq}\langle\bar{q}g_s\sigma\cdot Gq\rangle \\ & + C^{\mathrm{X}}_{G^3}\langle g_s^3 f^{abc}G^aG^bG^c\rangle + C^{\mathrm{X}}_{\bar{q}qG^2}\langle\bar{q}q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle + \ldots\end{aligned} \tag{2.19}$$

totally unclear termihology

The Wilson coefficients $C$ in Eq. (2.19) describe the UV-physics of the underlying matching and can be calculated perturbatively by the usual QCD approaches (apart from corrections due to the expansion of the local operators, see Section 3). Moreover, we defined a more convenient notation for the condensate contributions:

?

$$\begin{aligned}&\langle\bar{q}q\rangle := \langle 0|\,\bar{q}q\,|0\rangle\,, \langle G^2\rangle := \langle 0|\,G^a_{\mu\nu}G^{a,\mu\nu}\,|0\rangle\,, \langle\bar{q}g_s\sigma\cdot Gq\rangle := \langle 0|\,\bar{q}g_sG^{\mu\nu}\sigma_{\mu\nu}q\,|0\rangle\,, \\ &\langle g_s^3 f^{abc}G^aG^bG^c\rangle := \langle 0|\,g_s^3 f^{abc}G^a_{\mu\nu}G^{b,\nu\rho}G^{c,\mu}_{\rho}\,|0\rangle\,,\end{aligned} \tag{2.20}$$

As previously mentioned, the condensates are uniquely parameterized up to mass dimension five. Starting at dimension six and higher, there occur many different possible contributions, but ~~various~~ some of them are related by QCD equations of motions and

Fierz identities [21] to each other[2]. Note that in the power expansion in Eq. (2.19) we have only stated the dimension six and seven condensates which give a leading order contribution to the parameters $\lambda^2_{E,H}$.

Moreover, there are many estimates for the values of the condensates given in the literature, which have been obtained from various methods like lattice QCD, sum rules [22], but obtaining values for condensates of dimension six and higher is an involved task due to the mixing with lower dimensional condensates. Because of the lack of these values, the vacuum saturation approximation [3] is exploited in many cases, where a full set of intermediate states is introduced into the higher dimensional condensate and the assumption is used that only the ground state gives a dominant contribution. Thus, the higher dimensional condensate will be effectively reduced to a combination of lower dimensional condensates (this has already been done for the dimension seven condensate in Eq. (2.19)). Other approaches assume for instance that there might exist a dimension two condensate of the form $A^a_\mu A^{a,\mu}$ [22] (and citations therein), which is not gauge-invariant. Furthermore, some other ideas argue that the ordinary quark condensate vanishes, while the four-quark condensate gives the first non-zero contribution [23, 24].

**this review mainly discusses**
**“hot and dense QCD”, the only**
**relevant section 3.1 just**
**repeats the well known and old**
**estimates , maybe better to refer**
**to FLAG -2019 foer quark condensate**
**and original papers**
**mentioned there,**
**see also**
**B.~L.~Ioffe,**
**%``Condensates in quantum chromodynamics,''**
**Phys. Atom. Nucl. \textbf{66} (2003), 30-43**
**doi:10.1134/1.1540654**
**[arXiv:hep-ph/0207191 [hep-ph]].**

[2]A list is given for example in the review [22].

26.11.20

of the diagonal correlator of two three-particle currents is

# 3 Computation of the Wilson Coefficients

In this chapter, the leading perturbative and non-perturbative contributions to the ~~spectral~~ function in (2.10) and (2.11) are calculated up to dimension seven. Since the leading order of the diagonal three current correlator if of $\mathcal{O}(\alpha_s)$, we only investigate contributions up to this order in perturbation theory. Condensates of higher order in $\alpha_s$ are expected to become insignificant. Due to this constraint, the number of diagrams reduces enormously. Moreover, we are working in the fixed-point or Fock-Schwinger gauge [25, 26]

correlation

$$x_\mu A^\mu(x) = 0, \tag{3.1}$$

and

where we set the reference point $x_0 = 0$. This reference point would occur in all intermediate steps of the calculation and cancel in the end of the calculation. It ~~turns out~~ that this gauge is particularly useful in ~~HQET~~ computations, ~~because there are many useful relations for the quark and gluon field~~ [1]. ~~One specifically helpful relation is the following~~:

as well known

QCD sum rule

$$A_\mu(x) = \int_0^1 \mathrm{d}u\; u x^\nu G_{\nu\mu}(ux). \qquad \cancel{(3.2)}$$

and its derivatived

Expanding the gluon field strength tensor in $x$, which is necessary for obtaining local condensates, yields a simple relation between the gluon field $A_\mu$ and the field strength tensor $G_{\mu\nu}$. This relation is extremely useful for the calculation of the dimension five condensates including an off-diagonal quark-antiquark-gluon interaction (see Figure 3). Another important property is that this prevents interactions between the heavy quark and the gluon, which can be easily seen by considering the heavy-quark propagator in position space [12]:

unknown terminology

w.r.t. HQET

$$h_v(0)\bar{h}_v(x) = \Theta(-v\cdot x)\delta^{(d-1)}(x_\perp)\cdot \mathrm{P}_+ \cdot \mathcal{P}\exp\left(ig_s\int_{v\cdot x}^0 \mathrm{d}sv\cdot A(sv)\right) \tag{3.3}$$

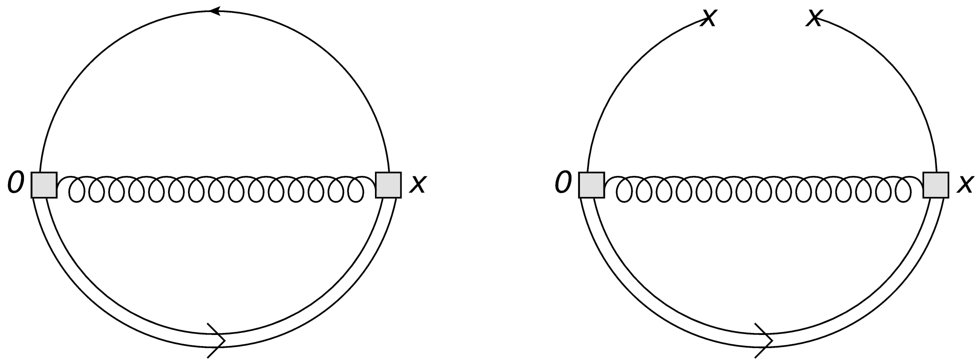
In Eq. (3.3), $x_\perp^\mu = x^\mu - (v\cdot x)v^\mu$, $P_+ = \frac{1+\not{v}}{2}$ denotes the projection operator and $\mathcal{P}$ illustrates the path ordering operator. Besides, there are three additional vanishing subdiagrams depicted in Figure 5 due to this gauge.

Generally, all diagrams can be evaluated in position space like in [11, 12], but in this work we chose to work in momentum space ~~and use~~ with the usual Feynman rules. We make use of dimensional regularization ~~as the regulator~~ for ~~our~~ the loop integrals with $d = 4 - 2\epsilon$. Moreover, for the computation of the Wilson coefficient in Eq. (2.19) we use an in-house program, which is based on FeynCalc tensor decomposition [27–29] and the technique of integration-by-parts of LiteRed [30, 31] to reduce it to master integrals. We cross-check the results by comparing the results, which have been also computed by hand.

?

comment: in the sum rules terminology "Wilson coeffs" is usually not used, but this is a matter of taste

~~Consequently, we are now able~~ to evaluate the diagrams in Figure 1-5 following we the standard procedures and using standard techniques [3]. Gathering all results for the Wilson coefficients introduced in Eq. (2.19), we get




(a) (b)

**Figure 1**: Feynman diagrams for the perturbative and $\langle \bar{q}q \rangle$ condensate contribution. The double line denotes the heavy quark propagator. **, the wavy line ..., the thin line..**

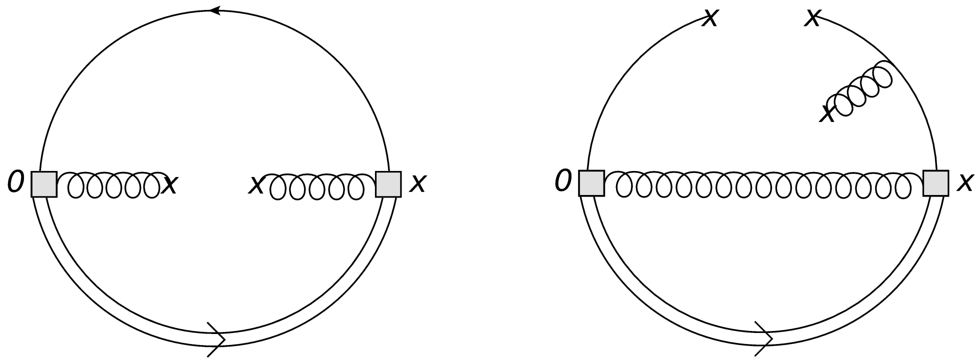
$$C^{\mathrm{X}}_{\mathrm{pert}}(\omega) = \frac{1}{2} \cdot \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \not{v}] \cdot \frac{\alpha_s}{\pi^3} C_F N_c \cdot \omega^{6-4\epsilon} \Bigg[ (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \cdot \left( \frac{1}{720\epsilon} + \frac{13}{1200} + \frac{i\pi}{180} \right) + (-g_{\nu\sigma} v_\mu v_\rho + g_{\mu\sigma} v_\nu v_\rho + g_{\nu\rho} v_\mu v_\sigma - g_{\mu\rho} v_\nu v_\sigma) \cdot \left( \frac{1}{360\epsilon} + \frac{73}{3600} + \frac{i\pi}{90} \right) \Bigg] + \mathcal{O}(\epsilon) \tag{3.4}$$

$$C^{\mathrm{X}}_{\bar{q}q}(\omega) = - \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma}] \cdot \frac{\alpha_s}{\pi} C_F \cdot \omega^{3-2\epsilon} \Bigg[ (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \cdot \left( - \frac{1}{12\epsilon} - \frac{2}{9} - \frac{i\pi}{6} \right) + (-g_{\nu\sigma} v_\mu v_\rho + g_{\mu\sigma} v_\nu v_\rho + g_{\nu\rho} v_\mu v_\sigma - g_{\mu\rho} v_\nu v_\sigma) \cdot \left( - \frac{1}{6\epsilon} - \frac{13}{36} - \frac{i\pi}{3} \right) \Bigg] + \mathcal{O}(\epsilon) \tag{3.5}$$

$$C^{\mathrm{X}}_{G^2}(\omega) = \frac{1}{2} \cdot \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \not{v}] \cdot \omega^{2-2\epsilon} \cdot (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \cdot \left( \frac{1}{24\epsilon} + \frac{19}{144} + \frac{i\pi}{12} \right) + \mathcal{O}(\epsilon) \tag{3.6}$$

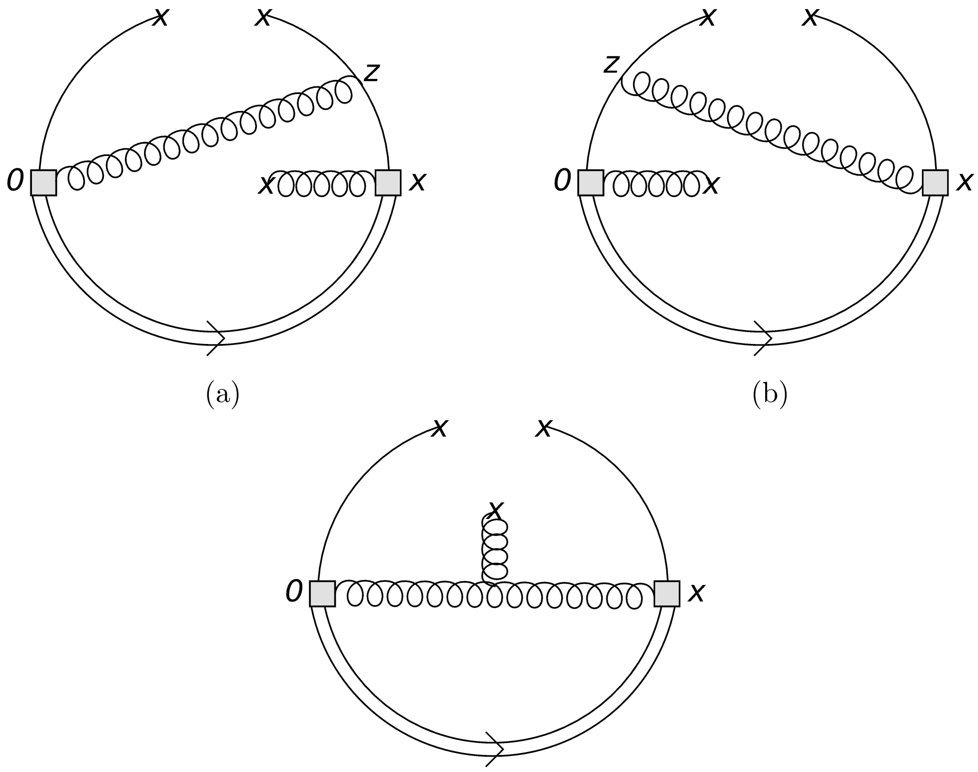
$$C^{\mathrm{X}}_{\bar{q}Gq,1}(\omega) = - \frac{1}{4} \cdot \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma}] \cdot \frac{\alpha_s}{\pi} C_F \cdot \omega^{1-2\epsilon} \cdot (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \cdot \left( - \frac{1}{6\epsilon} - \frac{13}{36} - \frac{i\pi}{3} \right) + \mathcal{O}(\epsilon) \tag{3.7}$$

[3] All diagrams in this work have been created with JaxoDraw [32].




(a) (b)

**Figure 2**: ~~Figure~~ (a) ~~shows~~ the Feynman diagram for the dimension four contribution, ~~while Figure~~ (b) ~~is~~ a schematicall~~y~~ illustration of the dimension five condensate originating from the higher order expansion of the dimension three contribution in Figure 1. The double line denotes the heavy quark propagator.




(a) (b)

(c)

**Figure 3**: Feynman diagrams for ~~all~~ dimension five condensate contributions, ~~which can be evaluated with standard perturbative methods~~. ~~The double line denotes the heavy quark propagator.~~

comment: figure captions do not need detailed explanations

$$C^{\mathrm{X}}_{\bar{q}Gq,2}(\omega) = \frac{\alpha_s}{4\pi} C_F \cdot \omega^{1-2\epsilon} \Bigg[ \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \sigma_{\mu\nu}\sigma_{\rho\sigma}] \cdot \Big( -\frac{1}{24\epsilon} - \frac{19}{144} - \frac{i\pi}{12} \Big) - \\ \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \not{v} \mathrm{i}(v_\mu \gamma_\nu - v_\nu \gamma_\mu)\sigma_{\rho\sigma}] \cdot \Big( -\frac{1}{24\epsilon} - \frac{13}{144} - \frac{i\pi}{12} \Big) \Bigg] + \mathcal{O}(\epsilon) \quad (3.8)$$

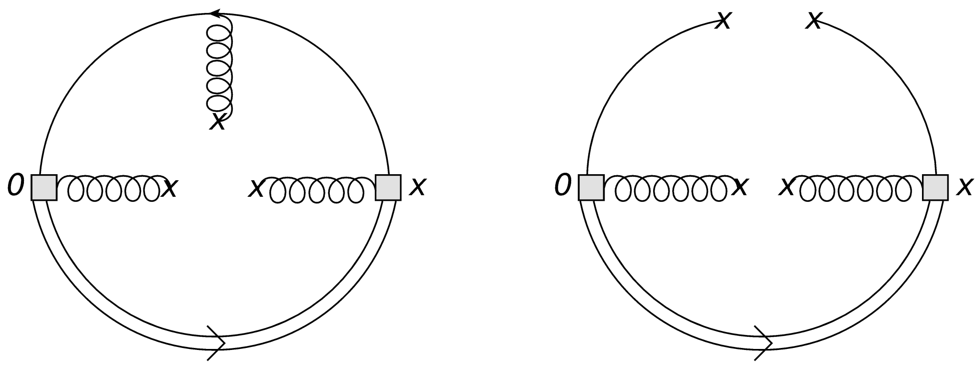
$$C^{\mathrm{X}}_{\bar{q}Gq,3}(\omega) = \frac{\alpha_s}{4\pi} C_F \cdot \omega^{1-2\epsilon} \Bigg[ \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \sigma_{\mu\nu}\sigma_{\rho\sigma}] \cdot \Big( -\frac{1}{24\epsilon} - \frac{19}{144} - \frac{i\pi}{12} \Big) + \\ \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \sigma_{\mu\nu} \mathrm{i}(v_\rho \gamma_\sigma - v_\sigma \gamma_\rho)\not{v}] \cdot \Big( -\frac{1}{24\epsilon} - \frac{13}{144} - \frac{i\pi}{12} \Big) \Bigg] + \mathcal{O}(\epsilon) \quad (3.9)$$

$$C^{\mathrm{X}}_{\bar{q}Gq,4}(\omega) = \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} (A_{\nu\rho\mu\sigma} + B_{\mu\nu\rho\sigma})] \cdot \frac{\alpha_s}{96\pi} C_A C_F \cdot \omega^{1-2\epsilon} \cdot i\pi \quad (3.10)$$

$$A_{\nu\rho\mu\sigma} := i(v_\nu v_\rho \sigma_{\mu\sigma} - g_{\nu\rho}\sigma_{\mu\sigma} + \frac{1}{2}\sigma_{\mu\beta} v^\beta g_{\nu\rho} v_\sigma) - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + \\ \{(\mu \leftrightarrow \nu), (\rho \leftrightarrow \sigma)\} \quad (3.11)$$

$$B_{\mu\nu\rho\sigma} := -\frac{1}{2} v_\mu g_{\nu\rho} \sigma_{\sigma\beta} v^\beta - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + \{(\mu \leftrightarrow \nu), (\rho \leftrightarrow \sigma)\} \quad (3.12)$$

The total Wilson coefficient for the mass dimension five condensate is given by the sum of the four previous contributions, namely $C^{\mathrm{X}}_{\bar{q}Gq} = \sum_{k=1}^{4} C^{\mathrm{X}}_{\bar{q}Gq,k}$.




(a) (b)

**Figure 4**: Feynman diagrams for the dimension six and dimension seven condensate, which contribute to the leading order estimate of $\lambda^2_{E,H}$. ~~The double line denotes the heavy quark propagator.~~

$$
\begin{aligned}
C^{\mathrm{X}}_{G^3}(\omega) = B_{\mu\lambda\rho\nu\sigma\alpha} \cdot \omega^{-2\epsilon} \Bigg[ & \mathrm{Tr}[-2i \cdot \Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \not{v} \sigma^{\lambda\alpha}] \cdot \Big( \frac{1}{6144\pi^2\epsilon} + \frac{13}{36864\pi^2} + \frac{i\pi}{3072\pi^2} \Big) \\
& + \mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} ((d-2) v^\alpha \gamma^\lambda - 2 v^\lambda \gamma^\alpha)] \cdot \Big( \frac{1}{6144\pi^2\epsilon} + \frac{13}{36864\pi^2} + \frac{i\pi}{3072\pi^2} \Big) \\
& + \mathrm{Tr}[2 \cdot \Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma} \not{v} v^\lambda v^\alpha] \cdot \Big( \frac{1}{3072\pi^2\epsilon} + \frac{7}{18432\pi^2} + \frac{i\pi}{1536\pi^2} \Big) \Bigg] + \mathcal{O}(\epsilon),
\end{aligned}
\tag{3.13}
$$

where the expression $B_{\mu\lambda\rho\nu\sigma\alpha}$ is defined in Appendix A.

$$
C^{\mathrm{X}}_{\bar{q}qG^2}(\omega) = -\,\mathrm{Tr}[\Gamma_1^{\mu\nu} P_+ \Gamma_2^{\rho\sigma}] \cdot \frac{1}{\omega + i0^+} \cdot \frac{\pi^2}{2N_c d(d-1)} \cdot (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \tag{3.14}
$$

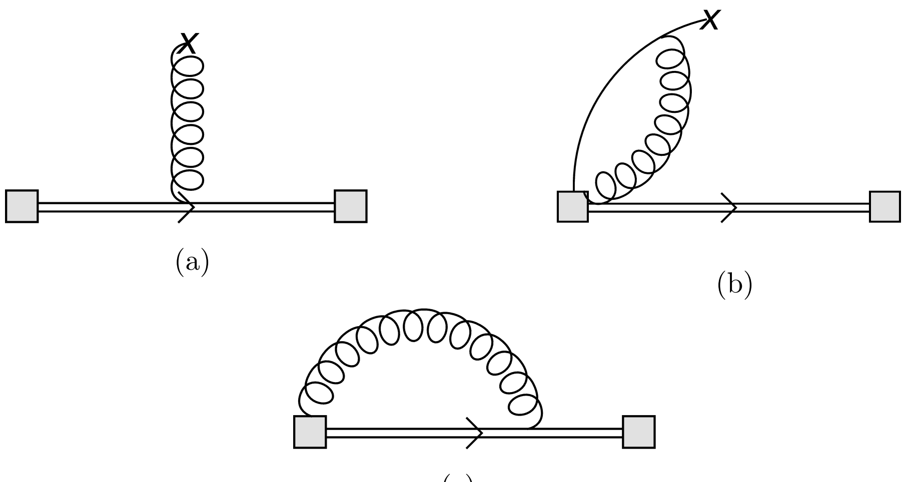


**Figure 5**: Vanishing subdiagrams in the Fock-Schwinger gauge.

According to Eq. (2.10) and (2.11), we still need to take the imaginary part of these diagrams. We chose to compute directly the loop diagrams and take the imaginary part of the resulting expression. Following Cutkosky rules, another approach would be to perform the calculation by considering all possible cuts for the diagrams. Apart from the diagrams in Figure 3, the diagrams are finite. The off-diagonal diagrams in Figure 3 include both a three-particle and a two-particle cut, where the latter requires a non-trivial renormalization procedure [33].

Notice that the diagrams in these figures can generally be calculated by using ordinary perturbative methods, while the right diagram in Figure 2 denotes a contribution arising from higher order expansions of the quark condensate in Eq. (A.1). Moreover, the diagrams contributing to the quark-gluon condensate in Figure 3 obey the same off-diagonal structure as the contributions in [11, 12] and hence a cross-check is possible after replacing the quark condensate by the quark-gluon condensate and keeping in mind that the Lorentz structures differ.

In our notation $\alpha_s$ is defined to be $g_s^2/(4\pi)$ and $C_F$, $C_A$ denote the two quadratic Casimir operators in a general SU(N) gauge group, satisfying:

$$\begin{aligned} C_A &= N_c \\ C_F &= \frac{N_c^2-1}{2N_c} \end{aligned} \tag{3.15}$$

with $N_c$ illustrating the number of colors. Depending on the conventions used during the computation, all results need to be multiplied by a factor of $4^{-\epsilon}$ (or $4^{-2\epsilon}$ in the perturbative case), which contributes to the finite part and does not change the imaginary part. ?

By taking the imaginary part of all Wilson coefficients discussed above and plugging the results into Eq. (2.17), (2.18) and performing the integration over $\omega$ up to the threshold parameter $\omega_{th}$, we obtain **the final expressions for sum rules**

$$\begin{aligned} F(\mu)^2\cdot(\lambda_H^2-\lambda_E^2)^2 e^{-\frac{\bar\Lambda}{M}} = {} & \frac{\alpha_s N_c C_F}{30\pi^3}\cdot 720M^7\cdot G_6\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F}{\pi}\cdot\langle\bar q g_s\sigma\cdot Gq\rangle\cdot M^2\cdot G_1\left(\frac{\omega_{th}}{M}\right) - \\ & \frac{\alpha_s C_F C_A^2}{8\pi N_c}\cdot\langle\bar q g_s\sigma\cdot Gq\rangle\cdot M^2\cdot G_1\left(\frac{\omega_{th}}{M}\right) - \frac{\pi^2}{2N_c}\langle\bar q q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle \end{aligned} \tag{3.16}$$

$$\begin{aligned} F(\mu)^2\cdot\lambda_H^4 e^{-\frac{\bar\Lambda}{M}} = {} & \frac{\alpha_s N_c C_F}{60\pi^3}\cdot 720M^7\cdot G_6\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F}{\pi}\langle\bar q q\rangle\cdot 6\cdot M^4\cdot G_3\left(\frac{\omega_{th}}{M}\right) + \\ & \frac{1}{4}\langle\frac{\alpha_s}{\pi}G^2\rangle\cdot 2\cdot M^3\cdot G_2\left(\frac{\omega_{th}}{M}\right) - \frac{3\alpha_s C_F}{4\pi}\cdot\langle\bar q g_s\sigma\cdot Gq\rangle\cdot M^2\cdot G_1\left(\frac{\omega_{th}}{M}\right) - \\ & \frac{\alpha_s C_F C_A^2}{8\pi N_c}\cdot\langle\bar q g_s\sigma\cdot Gq\rangle\cdot M^2\cdot G_1\left(\frac{\omega_{th}}{M}\right) + \frac{\langle g_s^3 f^{abc}G^aG^bG^c\rangle}{64\pi^2}\cdot M\cdot \\ & G_0\left(\frac{\omega_{th}}{M}\right) - \frac{\pi^2}{4N_c}\langle\bar q q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle \end{aligned} \tag{3.17}$$

$$\begin{aligned} F(\mu)^2\cdot\lambda_E^4 e^{-\frac{\bar\Lambda}{M}} = {} & \frac{\alpha_s N_c C_F}{60\pi^3}\cdot 720M^7\cdot G_6\left(\frac{\omega_{th}}{M}\right) + \frac{\alpha_s C_F}{\pi}\langle\bar q q\rangle\cdot 6\cdot M^4\cdot G_3\left(\frac{\omega_{th}}{M}\right) - \\ & \frac{1}{4}\langle\frac{\alpha_s}{\pi}G^2\rangle\cdot 2\cdot M^3\cdot G_2\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F}{2\pi}\cdot\langle\bar q g_s\sigma\cdot Gq\rangle\cdot M^2\cdot G_1\left(\frac{\omega_{th}}{M}\right) - \\ & \frac{\langle g_s^3 f^{abc}G^aG^bG^c\rangle}{64\pi^2}\cdot M\cdot G_0\left(\frac{\omega_{th}}{M}\right) - \frac{\pi^2}{4N_c}\langle\bar q q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle \end{aligned} \tag{3.18}$$

For convenience, we introduced the function

$$G_n(x) := 1-\sum_{i=0}^{n}\frac{x^i}{i!}e^{-x}. \tag{3.19}$$

**continuum ratio defined for each contribution as:**

**R_n(\omega_th,M)=1-G_n(omega_th/M)/G_n(\infty/M)**

**to be discussed in numerical section**

comment: the sum rule for F(\mu) is known for ages , long before Nishikawa

Further information on the calculation of the correlation functions for the decay constant $F(\mu)$ can be found in Ref. [12]. We will just state the result up to $\mathcal{O}(\alpha_s)$ corrections without any renormalization group improvement. At this point it is hence also necessary to include the dimension six contribution coming from the expansion of the quark condensate and using the vacuum saturation approximation.

$$F^2(\mu) \cdot e^{-\frac{\bar{\Lambda}}{M}} = \frac{N_c M^3}{\pi^2} \int_0^{\omega_{th}/M} \mathrm{d}x \; x^2 e^{-x} \left(1 + \frac{3 C_F \alpha_s}{2\pi} \left(\ln\left(\frac{\mu}{2Mx}\right) + \frac{17}{6} + \frac{2\pi^2}{9}\right)\right) - \langle \bar{q} q \rangle \left(1 + \frac{3 C_F \alpha_s}{2\pi}\right) + \frac{1}{16 M^2} \langle \bar{q} g_s G \cdot \sigma q \rangle + \frac{\pi C_F \alpha_s}{72 N_c M^3} \langle \bar{q} q \rangle^2 \tag{3.20}$$

The parameters $\lambda_{E,H}^2$ can be derived by dividing Eq. (3.16), (3.17) by (3.20) and taking the square root in the case of $\lambda_H^4$. In order to derive an estimate for $\lambda_E^2$, it is necessary to substract the sum rule for $\lambda_H^2$ by the sum rule for the splitting $\lambda_H^2 - \lambda_E^2$. Note that the perturbative expression contributes in both sum rules with a positive sign, which is expected in the case of a diagonal sum rule. For large values of the Borel parameter $M$, the condensates become less important and the perturbative contribution describes the sum rule completely.

up to here

## 4 Numerical Analysis

In this section we compute the HQET parameters according to Eq. (3.16) and (3.17) following the procedure described in Section 3. The numerical inputs for the necessary parameters and constants are given in Table 1. In Figure 6 we plot the splitting $(\lambda_H^2 - \lambda_E^2)^2$ individually for each order in the power expansion. We are explicitly considering the difference of both parameters instead of $\lambda_E^4$, because the
? splitting provides better convergence and a more reliable estimate of $\lambda_E^2$. Moreover, the dimension three, four and six condensates do not contribute to this specific choice. The terms corresponding to the dimension five condensate provide the largest
contribution? enhancement and beyond this dimension the power expansion is expected to converge, which is indicated from the small contribution of mass dimension seven.
Similarly, we plot higher dimensional contributions for $\lambda_H^4$ in Figure 7, but with the difference that each power correction enhances the total value of $\lambda_H^4$. Again, the dimension five contribution leads to the biggest upshift in Figure 7. The fact that correlation functions with higher Fock states experience large contributions from high ?? dimensional condensates for small values of the Borel parameter $M$ is a well known fact and has been studied in many occasions [12, 16, 17]. Nonetheless, as already observed in the splitting, contributions from dimensions greater than five become smaller indicating convergence of the OPE. Figure 10 and 11 show the values of $\lambda_H^2$ and $\lambda_H^2 - \lambda_E^2$ as a function of $M$, following the variation of $\omega_{th}$ according to [12]. Here, it can be explicitly seen that in the highly non-pertubative regime with small $M$ the condensate contributions become dominant and therefore the sum rule unreliable. To find the stability window of the threshold $\omega_{th}$, we vary the function $F(\mu)$ in Eq.

optimal window

specify N of loops and input

| Parameters | Value | Ref. |
|---|---|---|
| $\alpha_s(1\ \text{GeV})$ | 0.47 | [34] |
| $\cancel{C_F}$ | 4/3 | |
| $\cancel{N_c}$ | 3 | |
| $\langle \bar{q}q \rangle$ | $(-0.240 \pm 0.015)^3\ \text{GeV}^3$ | [35] |
| $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ | $(0.012\ \pm 0.004)\ \text{GeV}^4$ | [22] |
| $\langle \bar{q} g G \cdot \sigma q \rangle \,/\, \langle \bar{q}q \rangle$ | $(0.8\ \pm 0.2)\ \text{GeV}^2$ | [36] |
| $\langle g_s^3 f^{abc} G^a G^b G^c \rangle$ | $(0.045 \pm 0.01)\ \text{GeV}^6$ | [3] |
| $\bar{\Lambda}$ | $(0.55 \pm 0.06)\ \text{GeV}$ | [37] |

trivial

GMOR + (m_u+m_d)

normalization scale ?

0.012+0.006-0.012

**Table 1**: List of the numerical inputs, which will be used in our analysis.

(3.20) for different values of $\omega_{th}$, see Figure 8. As we can see, the decay constant $F(\mu)$ stabilizes in the interval $0.8\,\text{GeV} \leq \omega_{th} \leq 1.0\,\text{GeV}$. In order to see whether our threshold choice gives reasonable results, we compute the physical decay constant $f_B$ by using Eq. (1.3), see Figure 9.

better plot separate d=0 (pert), 5,7

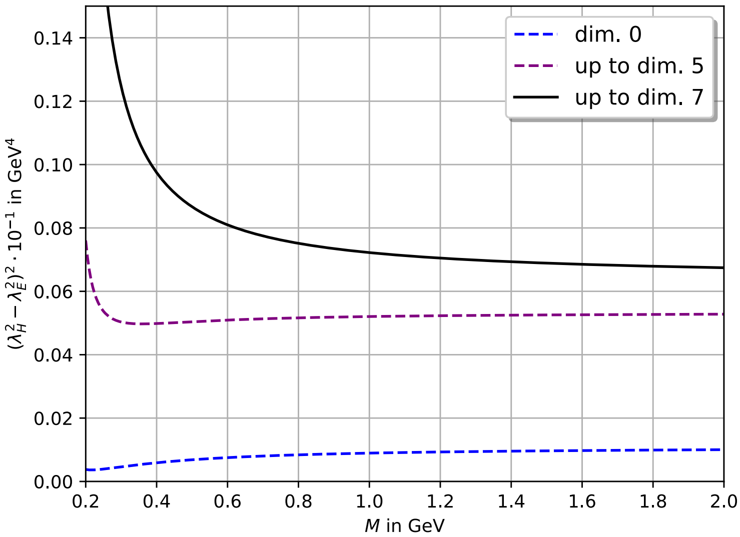



**Figure 6**: Comparison of the central values of $(\lambda_H^2 - \lambda_E^2)^2$ for higher dimensional contributions with threshold $\omega_{th} = 1$ GeV. The Borel parameter is investigated in the interval $M \in [0.2,\ 2.0]$ GeV.

plot d=0,3,4,5,6,7,
separately
and the sum

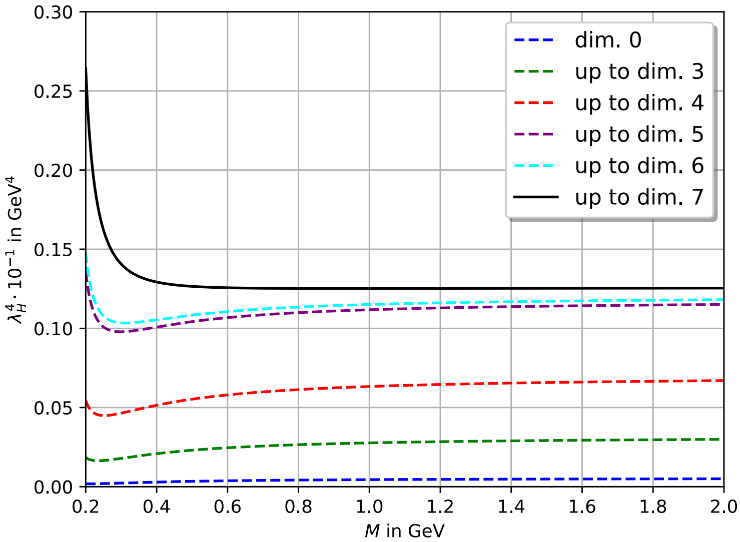



**Figure 7**: Comparison of the central values of $\lambda_H^4$ for higher dimensional contributions with threshold $\omega_{th} = 1$ GeV. The Borel parameter is investigated in the interval $M \in [0.2,\ 2.0]$ GeV.

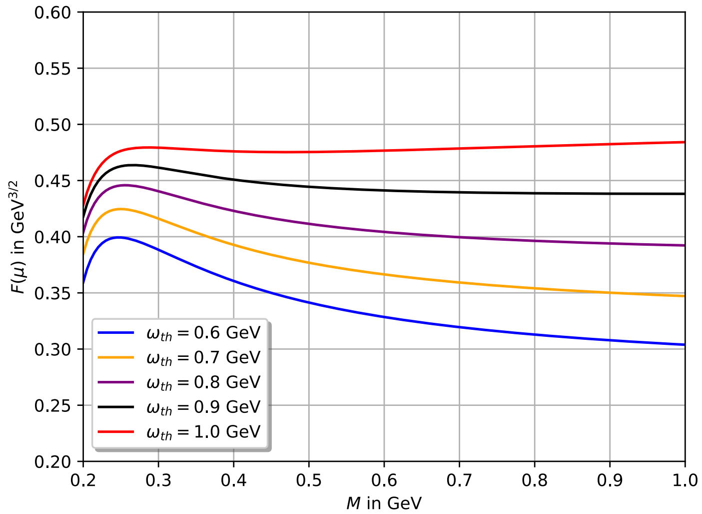



**Figure 8**: Comparison of the central values of the decay constant $F(\mu)$ for different values of $\omega_{th}$. The Borel parameter is investigated in the interval $M \in [0.2,\, 1.0]$ GeV and the binding energy takes the value $\bar{\Lambda} = 0.55$ GeV.

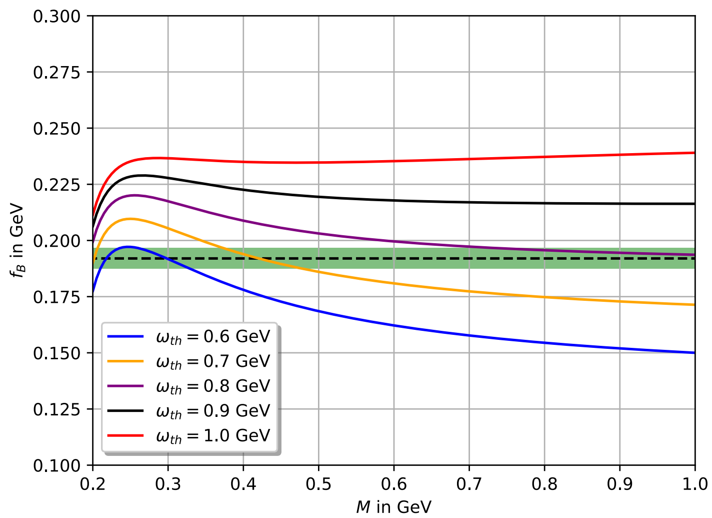



**Figure 9**: Comparison of the central values of the physical decay constant $f_B$ with different values of $\omega_{th}$. The Borel parameter is investigated in the interval $M \in [0.2,\, 1.0]$ GeV and the binding energy takes the value $\bar{\Lambda} = 0.55$ GeV. The dashed line indicates the lattice result and the shaded green area illustrates its corresponding uncertainty.

We observe in Figure 9 that for $0.6\,\text{GeV} \lesssim M$ the curve stabilizes and converges to the value of lattice QCD for $\omega_{th} = 0.8$ GeV. However, since we consider leading order contribution with the addition of only corrections up to dimension seven, we assume a conservative uncertainty of $15\% - 20\%$ from the lattice result of $f_B$, which is fulfilled within the interval $0.8\,\text{GeV} \lesssim \omega_{th} \lesssim 1.0\,\text{GeV}$ [4].

In Figure 10, 11 we plot the central values of $\lambda_H^2$ and $(\lambda_H^2 - \lambda_E^2)$ for different values of the threshold $\omega_{th}$ according to its ~~stability window~~. estimated interval We find that for $1.0\,\text{GeV} \lesssim M$ the nonperturbative contributions are dominant compared to the perturbative contribution. Here, the perturbative term contributes approximately $\sim 20\%$ to the OPE of the Borel sum rule. Moreover, for $M \lesssim 2.0\,\text{GeV}$ the sum rule converges very well. Hence, our choice for the stability window $1.0\,\text{GeV} \lesssim M \lesssim 2.0\,\text{GeV}$.

The uncertainties of $\lambda_H^2$ and the splitting $(\lambda_H^2 - \lambda_E^2)$ are determined by varying each input parameter individually according to their uncertainty, see Table 1. For the strong coupling constant we use the two-loop expression with $\Lambda_{\text{QCD}}^{(4)} = 0.31$ GeV to obtain $\alpha_s(1\,\text{GeV}) = 0.47$. We vary $\Lambda_{\text{QCD}}^{(4)}$ in the interval $0.29\,\text{GeV} \lesssim \Lambda_{\text{QCD}}^{(4)} \lesssim 0.33\,\text{GeV}$, which corresponds to the running coupling $\alpha_s(1\,\text{GeV}) = 0.44 - 0.5$. In the last step, we square each uncertainty in quadrature:

$$
\begin{aligned}
(\lambda_H^2 - \lambda_E^2) = &\Bigg[0.084 + \begin{pmatrix} +0.003 \\ -0.001 \end{pmatrix}_{\omega_{th}} + \begin{pmatrix} +0.002 \\ -0.001 \end{pmatrix}_{M} + \begin{pmatrix} +0.006 \\ -0.006 \end{pmatrix}_{\langle \bar{q}q \rangle} \\
&+ \begin{pmatrix} +0.004 \\ -0.004 \end{pmatrix}_{\langle \frac{\alpha_s}{\pi} G^2 \rangle} + \begin{pmatrix} +0.001 \\ -0.001 \end{pmatrix}_{\langle \bar{q} g G \cdot \sigma q \rangle}\Bigg] \,\text{GeV}^2 \\
&= (0.084 \pm 0.008)\ \text{GeV}^2, \qquad (4.1)\\
\lambda_H^2 = &\Bigg[0.112 + \begin{pmatrix} +0.001 \\ -0.000 \end{pmatrix}_{\omega_{th}} + \begin{pmatrix} +0.005 \\ -0.005 \end{pmatrix}_{\langle \bar{q}q \rangle} + \begin{pmatrix} +0.007 \\ -0.007 \end{pmatrix}_{\langle \frac{\alpha_s}{\pi} G^2 \rangle} \\
&+ \begin{pmatrix} +0.006 \\ -0.006 \end{pmatrix}_{\langle \bar{q} g G \cdot \sigma q \rangle}\Bigg] \,\text{GeV}^2 \\
&= (0.112 \pm 0.011)\ \text{GeV}^2, \qquad (4.2)
\end{aligned}
$$

Generally, there are many uncertainties entering the computation of the parameters $\lambda_{E,H}^2$. The errors based on the input parameters given in Table 1 provide one source for the total uncertainty. In addition to that there is another uncertainty on the low-energy threshold parameter $\omega_{th}$, which works as a cutoff for the nonperturbative regime. ??? By considering the variation of the sum rules for different values of this parameter it is possible to obtain an estimate for its contribution to the total uncertainty. Besides these contributions, there are other uncertainties ~~involved during the computation of the sum rules~~ due to several approximations and systematic

[4] In addtion to that, we determined the interval for the threshold parameter $\omega_{th}$ by taking the derivative with respect to the Borel paramter $M$ $-\partial/\partial M$ in Eq. (3.16) and (3.17) and dividing these expressions by the sum rules in Eq. (3.16) and (3.17) respectively. This gives an estimate for the parameter $\bar{\Lambda}$ and needs to be compatible with the value stated in table 1. Both methods give the same interval.

an option: take last

errors. Since we truncated the perturbative series at $\mathcal{O}(\alpha_s)$ and the power corrections at dimension seven, we introduce another error which is more complicated to determine. Moreover, there is also an intrinsic uncertainty caused by the sum rule approach, for instance generated by the use of the quark-hadron duality. The total uncertainties stated in Eq. (4.1) and (4.2) only list those quantities, which give deviations from the central values up to three digits after decimal. The variation of the strong coupling constant $\alpha_s$ and the dimension six condensate $\langle g_s^3 f^{abc} G^a G^b G^c \rangle$ does not change the central value significantly within three decimals. Therefore, the uncertainty can be neglected for both cases $\lambda_H^2$ and $(\lambda_H^2 - \lambda_E^2)$. A conservative estimate of the uncertainties, leads to the following final results:

??

here R <<[12] ! quote R

$$\lambda_E^2(1\ \mathrm{GeV}) = (0.03 \pm 0.02)\ \mathrm{GeV}^2 \tag{4.3}$$

$$\lambda_H^2(1\ \mathrm{GeV}) = (0.11 \pm 0.02)\ \mathrm{GeV}^2. \tag{4.4}$$

Note that, our result for $\lambda_E^2$ is in full agreement with the result in [12]. Additionally, our result for $\lambda_H^2$ tends towards the result in [12]. A short comment is necessary at this point in order to discuss the deviations from the original estimates in [11]: One important difference in our analysis is that we included the $\mathcal{O}(\alpha_s)$ corrections for the HQET coupling constant $F(\mu)$, see Eq. (3.20). These contributions are known to be huge and question the convergence of the perturbative expansion in general [18]. First with the help of some simplified models [38] and later by explicit calculation of the order $\alpha_s^2$ corrections it has been shown [19] that the perturbative series becomes convergent. Nevertheless, these sizeable contributions combined with $\mathcal{O}(\alpha_s)$ corrections of the dimension five condensate, the dimension six condensate and RGE improvement lead to the values of [12], which deviate from the first estimate of [11] by a factor of three and are in good agreement with our values. Consequently, the inclusion of all order $\alpha_s$ contributions into the sum rule is necessary for good convergence of the OPE.

?

unrealistic !

Another point we would like to emphasize is related to the direct projection of $\lambda_E^4$. It is possible to choose $\Gamma_1^{\mu\nu}, \Gamma_2^{\rho\sigma}$ such that $\lambda_E^4$ is projected out, the choice can be read off Eq. (2.7) and (2.9). However, the sum rule is much more unstable compared to the splitting sum rule, see Figure 14. The reason for the instability stems from the large contribution of dimension three, four and six, since they contribute with a negative sign to the value of the sum rule. Moreover, most contributions of the dominant quark-gluon dimension five condensate do not give any contribution to this sum rule, apart from the dimension three correction (Figure 2) . One advantage of the splitting sum rule is that these contributions cancel completely, see Figure 6. The leading-order contribution and power corrections of mass dimension five and seven lead to an enhancement of the values. Still, in order to obtain a reasonable sum rule for $\lambda_E^4$, we have to choose a lower stability window for the threshold $\omega_{th}$ and Borel parameter $M$, namely $0.55\,\mathrm{GeV} \lesssim \omega_{th} \lesssim 0.65\,\mathrm{GeV}$ and $0.8\,\mathrm{GeV} \lesssim M \lesssim 1.0\,\mathrm{GeV}$. By repeating the uncertainty analysis similar to Eq. (4.1), we reproduce the value:

$$\lambda_E^2 = (0.03 \pm 0.02)\ \mathrm{GeV}^2. \tag{4.5}$$

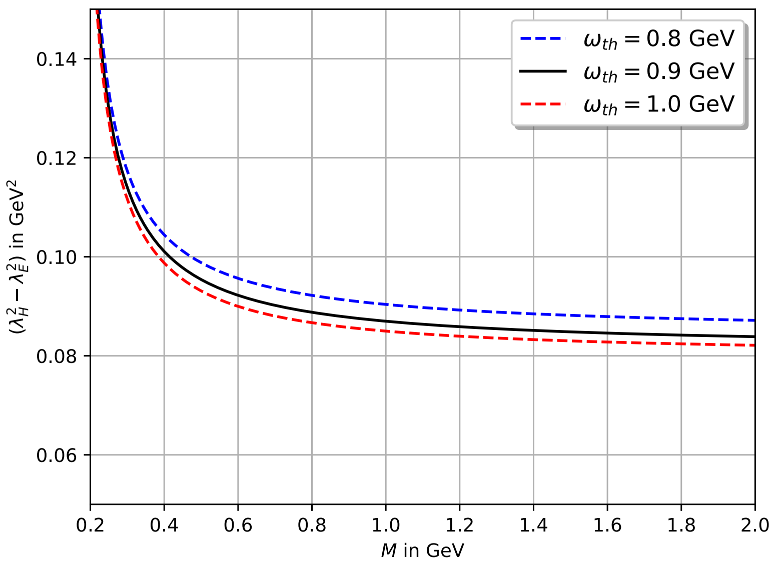



**Figure 10**: Borel sum rule for $(\lambda_H^2 - \lambda_E^2)$ based on Eq. (3.16) for the stability window $0.8\,\text{GeV} \lesssim \omega_{th} \lesssim 1.0\,\text{GeV}$.

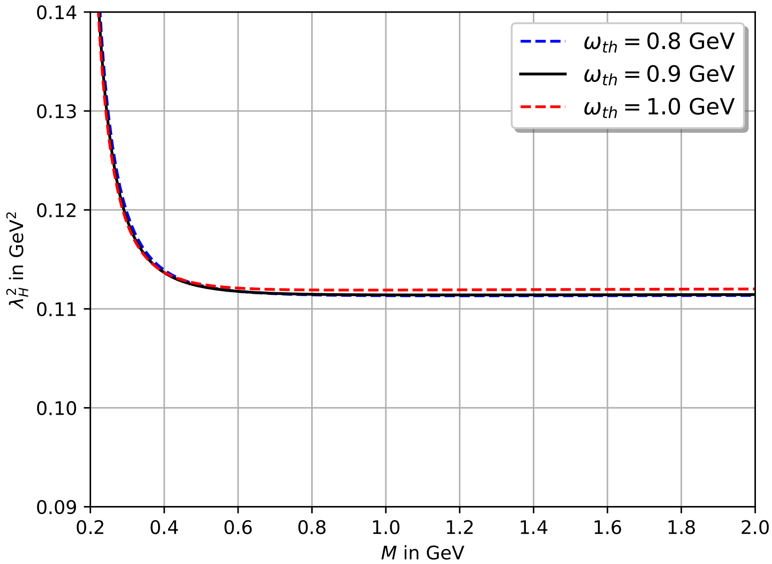



**Figure 11**: Borel sum rule for $\lambda_H^2$ based on Eq. (3.17) for the stability window $0.8\,\text{GeV} \lesssim \omega_{th} \lesssim 1.0\,\text{GeV}$.

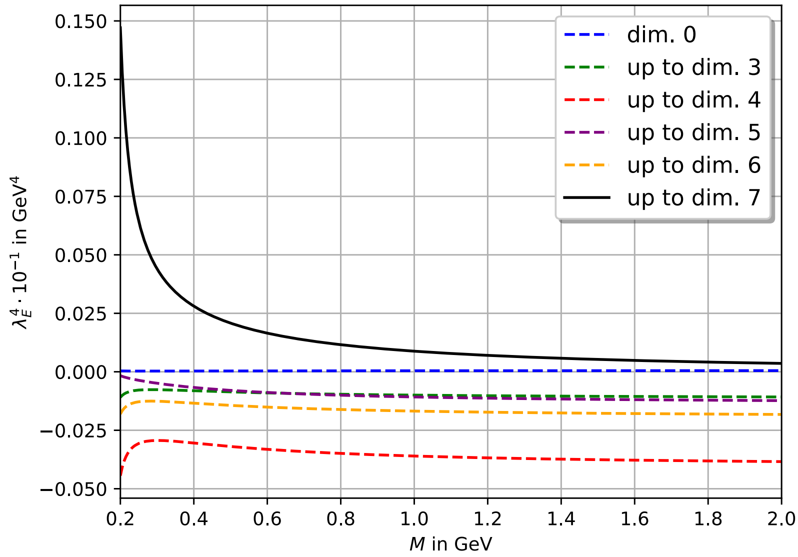



**Figure 12**: Comparison of the central values of $\lambda_E^4$ for higher dimensional contributions with threshold $\omega_{th} = 0.60$ GeV. The Borel parameter is investigated in the interval $M \in [0.2,\, 2.0]\,$GeV.

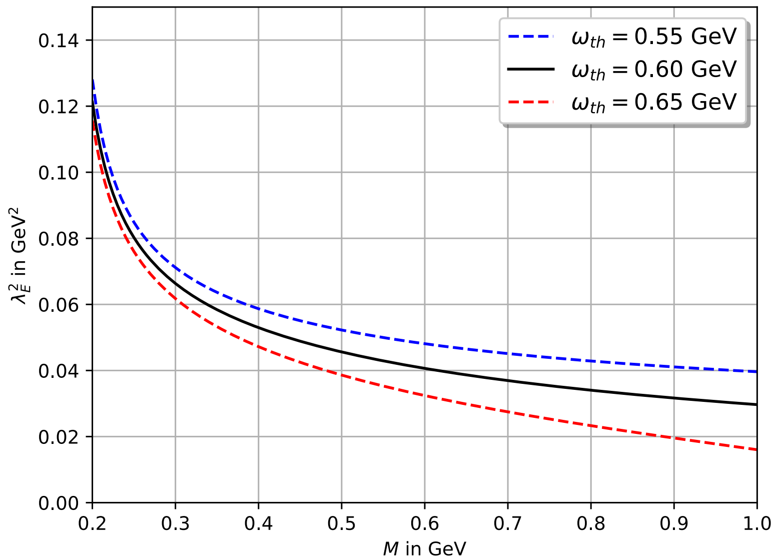



**Figure 13**: Borel sum rule for $\lambda_E^2$ based on Eq. ?) for the stability window $0.55\,\text{GeV} \lesssim \omega_{th} \lesssim 0.65\,\text{GeV}$.

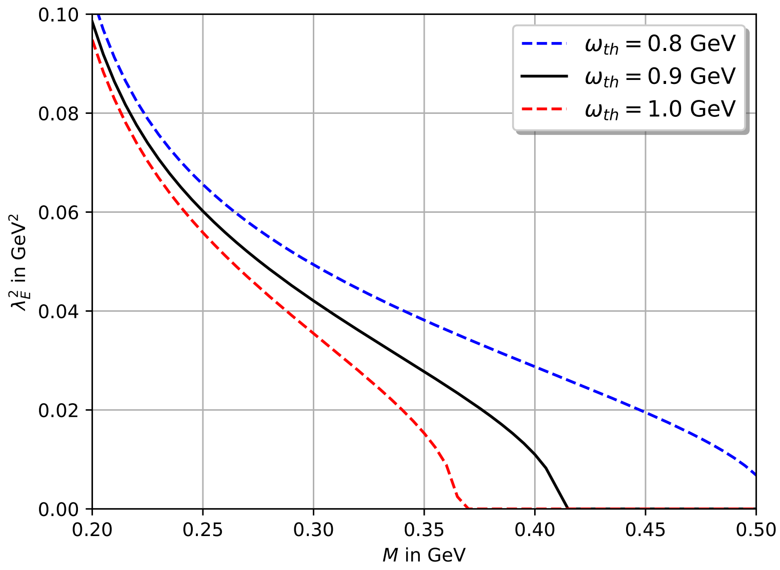



**Figure 14**: Borel sum rule for $\lambda_E^2$ based on Eq. ?) for the stability window $0.8\,\mathrm{GeV} \lesssim \omega_{th} \lesssim 1.0\,\mathrm{GeV}$.

## 5 Conclusion

In this work we derived estimates for the HQET parameters $\lambda^2_{E,H}$ ~~with leading order contribution $\mathcal{O}(\alpha_s)$~~ in ~~the strong coupling constant by~~ including condensates up to mass dimension seven as parameterizations of the non-perturbative nature of the QCD vacuum. Contrary to previous works [11, 12], the diagonal three-body quark-antiquark-gluon current has been investigated. Although it is known that correlation functions of high mass dimension lead to instabilities and poor convergence, we observe that the OPE seems to converge when we include condensates with mass dimension six and seven. In order to allow for a complete $\mathcal{O}(\alpha_s)$ treatment of the sum rule, we have considered the HQET coupling constant $F(\mu)$ with $\mathcal{O}(\alpha_s)$ accuracy as well. These contributions are known to be huge and indicate one reason for the deviation of the original estimates for the parameters $\lambda^2_{E,H}$. Our new estimates for the HQET parameters $\lambda^2_{E,H}$ are: ?

$$\lambda^2_E(1\text{ GeV}) = (0.03 \pm 0.02)\text{ GeV}^2 \tag{5.1}$$
$$\lambda^2_H(1\text{ GeV}) = (0.11 \pm 0.02)\text{ GeV}^2. \tag{5.2}$$

Hence, we observe that once we consider all order $\alpha_s$ contributions and include condensates up to mass dimension seven (with leading order contribution in $\alpha_s$) the OPE is expected to converge and our values in equation (5.2) get into the region of [12]. While $\lambda^2_E$ fully agrees with the estimate in [12], $\lambda^2_H$ is larger than their estimate. With these new estimates, calculations of B-meson distribution amplitudes and ... can be carried out without the need to choose either Grozin/Neuberts estimate or the estimate by Nishikawa/Tanaka and obtain sizeable differences in the result. ? ??

**We suggest an alternative diagonal sum rule. Advantage: positive definite, better duality, Disadvantage**

**large contributions of higher dimension operators, due to larger dimension of the current produxt**

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## A Parametrization of the QCD Condensates

Here we present the condensates that we have used in the work. All results are based on [39] if not stated otherwise. First, we Taylor expand the following matrix element:

$$\begin{aligned}\langle 0|\, \bar{q}(0)\Gamma_1 P_+ \Gamma_2\, q(x)\,|0\rangle &= \langle 0|\, \bar{q}(0)\Gamma_1 P_+ \Gamma_2 q(0)\,|0\rangle + x^\mu \langle 0|\, \bar{q}(0)\Gamma_1 P_+ \Gamma_2 D_\mu q(0)\,|0\rangle \\ &+ \frac{1}{2} x^\mu x^\nu \langle 0|\, \bar{q}(0)\Gamma_1 P_+ \Gamma_2 D_\mu D_\nu q(0)\,|0\rangle + \cdots \end{aligned} \tag{A.1}$$

The first term in Eq.(A.1) corresponds to the quark condensate.

$$\langle 0|\, \bar{q}^i_\alpha(0)\Gamma_{1,\alpha\beta} P_{+,\beta\gamma} \Gamma_{2,\gamma\delta}\, q^j_\delta(0)\,|0\rangle = \frac{1}{4N_c} \cdot \mathrm{Tr}[\Gamma_1 P_+ \Gamma_2]\, \langle \bar{q}q\rangle\, \delta^{ij}, \tag{A.2}$$

where $(i, j)$ are color indices and $(\alpha, \beta, \gamma, \delta)$ are spinor indices. The second term in Eq.(A.1) does not contribute. Making use of the Dirac equation, we can rewrite the covariant derivative as:

$$\not{D} q = -i m_q q. \tag{A.3}$$

In HQET we assume $m_q = 0$ for light quarks.

Before we consider the third term in more detail, we parametrize the dimension five matrix element:

$$\langle 0|\, \bar{q}^i_\alpha(0) g_s G_{\mu\nu}(0) q^j_\delta(0)\,|0\rangle = \langle 0|\, \bar{q} g_s \sigma \cdot G q\,|0\rangle \cdot \frac{1}{4N_c d(d-1)} \cdot \delta^{ij} \cdot (\sigma_{\mu\nu})_{\delta\alpha} \tag{A.4}$$

The third term in Eq.(A.1) corresponds to the quark-gluon condensate.

$$\frac{1}{2} x^\mu x^\nu \cdot \langle 0|\, \bar{q}^i_\alpha(0) D_\mu D_\nu q^j_\delta(0)\,|0\rangle = \frac{x^2}{16 N_c d}\, \delta^{ij} \delta_{\alpha\delta} \cdot \langle 0|\, \bar{q} g_s \sigma \cdot G q\,|0\rangle \tag{A.5}$$

The gluon-gluon condensate can be parametrized as:

$$\langle 0|\, G^a_{\mu\nu} G^b_{\rho\sigma}\,|0\rangle = \frac{1}{d(d-1)(N_c^2-1)} \langle G^2\rangle \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) \tag{A.6}$$

Next is the parametrization of the triple-gluon condensate, which was denoted as $B_{\mu\lambda\rho\nu\sigma\alpha}$ in Eq. (3.13). The decomposition of the triple-gluon condensate has been investigated in [40]:

$$\begin{aligned}\langle g_s^3 f^{abc} G^a_{\mu\nu} G^b_{\rho\sigma} G^c_{\alpha\lambda}\rangle = \frac{\langle g_s^3 f^{abc} G^a G^b G^c\rangle}{d(d-1)(d-2)} \cdot \Big( & g_{\mu\lambda} g_{\rho\nu} g_{\sigma\alpha} + g_{\mu\sigma} g_{\rho\alpha} g_{\lambda\nu} + g_{\rho\lambda} g_{\mu\alpha} g_{\nu\sigma} + g_{\alpha\nu} g_{\mu\rho} g_{\sigma\lambda} - \\ & g_{\mu\sigma} g_{\rho\lambda} g_{\alpha\nu} - g_{\mu\lambda} g_{\rho\alpha} g_{\nu\sigma} - g_{\rho\nu} g_{\mu\alpha} g_{\sigma\lambda} - g_{\sigma\alpha} g_{\mu\rho} g_{\nu\lambda} \Big)\end{aligned} \tag{A.7}$$

Since the complete prefactor in front of the brackets in eq. (A.7) is already included in Eq. (3.13), $B_{\mu\lambda\rho\nu\sigma\alpha}$ is given by the expression in the square brackets.

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