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Calculations in External Fields in Quantum Chromodynamics. Technical Review.

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Abstract

We review the technique of calculation of operator expansion coefficients. The main emphasis is put on gluon operators which appear in expansion of n-point functions induced by colourless quark currents. Two convenient schemes are discussed in detail: the abstract operator method and the method based on the Fock-Schwinger gauge for the vacuum gluon field. We consider a large number of instructive examples important from the point of view of physical applications.

Introduction

The method of QCD sum rules, originally proposed in works [1], has been intensively developed over the last three-four years (see, for instance, [2] and a short review [3]) and has become now one of the most effective tools for determining the parameters of low-lying hadronic states. Leaving aside theoretical aspects, discussed in great detail in ref. [1] we notice only that the basis of the method is a hypothesis of existence of violent vacuum fields — quark and gluon — which go beyond the framework of perturbation theory. Valence quarks (gluons) injected in the vacuum by electromagnetic or another external current have to develop not in the empty space but in a vacuum medium.

In a certain way we find a domain of intermediate q^2 values — the so called fiducial domain — inside which, on one hand, correlation functions of colourless currents are calculable by expansion in $(G^a_{\mu\nu})_{\text{vac}}/q^2$ ($(G^a_{\mu\nu})_{\text{vac}}$ stands for the characteristic vacuum field) and, on the other hand, the same correlation functions are saturated by the lowest hadronic states with appropriate quantum numbers to a good accuracy. Thus, the structure of the exact QCD vacuum is not explained but, rather, parametrized, and the vacuum gluon field is treated as a given external field (weak and randomly oriented).

In the original works [1] coefficients in front of gluon operators have been determined by virtue of the standard technique appealing to the Feynman diagrams. Soon it became clear, however, that this way of the calculation is extremely inconvenient in gauge theories since, in order to get the final gauge invariant answer one has to deal with a large number of intermediate gauge non-invariant quantities cancelling each other at the end.

The amount of problems solved within the sum rule approach has been growed rapidly, and practical needs have required a new method which would ensure gauge invariance of the gluon operators appearing in the expansion from the very beginning.

To this end two related but not identical computational schemes have been proposed. Historically the first was the method extending Schwinger's operator approach [4] to nonabelian theories. We widely used this formalism in applications; some of its elements are presented in [5]. Another procedure [6] represents actually a branch of the abstract operator method. It concretizes one of its aspects — namely, it exploits the so called Fock-Schwinger gauge for external gluon field [7, 4]. Advantages it gives in constructing the operator expansion in QCD are well-known (e.g. [6, 8]), and there is no need to recall them here. Notice that the operator Schwinger technique works not less (and often even more) effectively than the technique exploiting the Fock-Schwinger gauge. In general, both approaches nicely supplement each other in applications. Sometimes one of them turns to be more convenient while in other problems it is better to use the second method. We feel that exposition of these methods in a form especially suited for QCD will be helpful for theorists studying the sum rule approach and, probably, even for a wider audience.

Presentation of some "favorite" results is not the aim of this review, at least not the main aim. The bulk of the material is of pedagogical character and is arranged in two chapters. The first one is devoted to the abstract operator method. We concentrate here on a single example nicely illustrating all basic elements and devices which we use over several years. The exercise we dwell on is the calculation of determinant Det $\|iD_\mu\gamma_\mu - m\|$ for heavy quarks. Another name for the same quantity is the effective Lagrangian, QCD analogue of the Heisenberg-Euler Lagrangian.

The expansion for the heavy quark loop is constructed in the form of a series in $G^a_{\mu\nu}/m^2$ where $G^a_{\mu\nu}$ stands for the strength tensor of the external gluon field and m is the quark mass. We find coefficients of all gluon operators of dimension 4, 6 and 8.

The second chapter represents a detailed discussion of the variant based on the special choice of the gluon field gauge (the Fock-Schwinger gauge). Choosing a few typical examples we try to explain how one can determine the operator expansion coefficients in various cases. If the reader will master the material he/she will easily reproduce all particular results encountered in original works.

Before proceeding to the review of technical aspects let us recall the general formulation of the problem. For large Euclidean momenta n-point Green functions induced by colourless currents are essentially determined by asymptotic freedom formulae. Nonperturbative vacuum fluctuations manifest themselves as power corrections to these formulae. The concrete structure of vacuum fields is inessential — everything is fixed by average vacuum characteristics such as quark and gluon condensates

$$\langle \mathrm{vac}|\, \bar\psi\psi\, |\mathrm{vac}\rangle, \qquad \langle \mathrm{vac}|\, \frac{\alpha_s}{\pi}\, G^a_{\mu\nu}G^a_{\mu\nu}\, |\mathrm{vac}\rangle .$$

Including into analysis a few first corrections we obtain approximate predictions for parameters of low-lying resonances.

Since we do not explain the vacuum structure but only parametrize corresponding effects, the vacuum fields play the role of external fields. Then the problem reduces to the following. Some colourless currents are given, and one has to calculate the corresponding Green functions in external gluon or quark fields assuming that the field is weak. By saying "weak field" we mean that its average intensity is smaller than the characteristic momentum value so that the expansion in power series is sensible.

Strict formalization of the whole procedure is achieved within the framework of Wilson's operator expansion (OPE) [9]. Consider, for instance, the ordered T product of vector currents

$$\Pi_{\mu\nu}(q) = i \int e^{iqx} d^4x \langle T\{j_\mu(x)\, j_\nu(0)\}\rangle$$

where

$$j_\mu = \bar{q}\gamma_\mu q$$

in the theory with a single massless quark q. For large enough Q^2 ($\equiv -q^2$) the product $T\{j_\mu(x)\, j_\nu(0)\}$ can be expanded in $(x^2)^k$ ($\ln x^2$) and the two-point function $\Pi_{\mu\nu}(q)$ takes the form

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu}) \sum_n C_n(Q^2) \langle O_n\rangle$$

where O_n are local gauge invariant operators, for instance, 1, $\bar{q}q$, $G^a_{\mu\nu}G^a_{\mu\nu}$, ..., while C_n stand for the corresponding coefficient functions. The vacuum matrix elements $\langle O_n\rangle$ measure average vacuum characteristics and are equal, by an order of magnitude, to μ^{d_n}. Here μ is a typical hadronic mass (a few hundred MeV), and d_n denotes the normal dimension of operator O_n (logarithmic factors associated with the anomalous dimensions are neglected). It is evident that the n-th term of the series for $\Pi_{\mu\nu}$ reduces to const $(\mu/Q)^{d_n}$ and for $Q^2 \gg \mu^2$ one can keep only a few first terms.

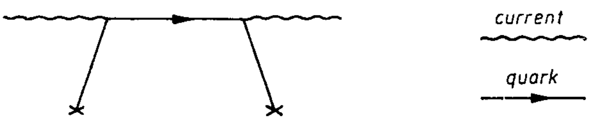


Fig. 1. Two-quark operator in the expansion of the correlation function
$i \int e^{iqx} d^4x \langle \text{vac}|\, T\{j_\mu(x)\, j_\nu(0)\}\, |\text{vac}\rangle$
induced by vector quark currents

Since Wilson's expansion is an operator equality we can sandwich it between any in- and out-states. In particular, taking one-quark state we get the diagram of fig. 1 which fixes the coefficient of $\bar{q}q$ operator. This simple recipe is applicable, in principle, in all cases, but as was already mentioned, it is practically ineffective, especially for gluon operators. More adequate methods are presented below in this review.

One last remark concerning Wilson's expansion. It is strictly proven in perturbation theory [10—12] and can be extended to include non-perturbative effects [5]. Its status in QCD is discussed in detail in ref. [5]. Notice that we use Wilson's expansion *only in momentum space for large* Q. In the literature one can also encounter with another version referring to the coordinate space. Being superficially identical these two formulations are actually not quite equivalent. Indeed, say, the small x expansion of $T\{j_\mu(x)\, j_\nu(0)\}$ contains, among other terms, the operator $j_\mu(0)\, j_\nu(0)$ entering with the *unit* coefficient

$$T\{j_\mu(x)\, j_\nu(0)\}_{x\to 0} = \cdots + j_\mu(0)\, j_\nu(0) + \cdots.$$

In momentum space this would correspond to a delta function, $C(q) \sim \delta^{(4)}(q)$. Thus, we would be dealing with a domain of small q. In such a situation one can expect some subtleties and they indeed are there. A few recent works (see, e.g. [13]) investigating OPE in various simple models with non-perturbative effects question its validity but all violations revealed there refer just to terms whose coefficients are proportional to $\delta^{(4)}(q)$ or derivatives of the delta function (in coordinate space $C(x) \sim (x^2)^k$,

$k = 0, 1, \ldots$). As for normal (surviving at large Q^2) terms of Wilson's expansion, they correctly give the true asymptotic behaviour of Green functions — this fact is indubitable [5]. Within the sum rule method we are never interested in terms with delta-like contributions, although the question deserves, in principle, further investigation.

Chapter I. Operator Schwinger Method

1.1. Starting Elements of the Method. Outlining the Problem Chosen as an Example

Quite often in QCD there arises a necessity in calculating the determinant of the Dirac operator $\operatorname{Det} \|iD_\mu\gamma_\mu - m\|$ in an external non-abelian field $A_\mu{}^a$. Recall that $\operatorname{Det} \|iD_\mu\gamma_\mu - m\|$ enters as a pre-exponential factor in the vacuum-to-vacuum transition $\langle 0 \mid 0' \rangle$ determining, in particular, all Green functions of the theory. Indeed, the vacuum-to-vacuum amplitude is representable in the form of the functional integral

$$\langle 0 \mid 0' \rangle = \int DA_\mu D\bar{\psi} D\psi \exp i \left\{ \int d^4x \left[-\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \bar{\psi}(\gamma_\mu P_\mu - m)\,\psi \right] \right\}, \qquad P_\mu = iD_\mu, \tag{1.1}$$

over different configurations of vector and fermion fields. Since the exponent is bilinear in ψ one can perform integration over the fermion degrees of freedom considering the vector field configuration $A_\mu{}^a(x)$ as a given external field. Then

$$\begin{aligned}\langle 0 \mid 0' \rangle &= \int DA_\mu \operatorname{Det} \|\gamma_\mu P_\mu - m\| \exp \left\{ -\frac{i}{4} \int d^4x G^a_{\mu\nu} G^a_{\mu\nu} \right\} \\ &= \int dA_\mu \exp \left\{ i \int d^4x \left[-\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - i \ln \operatorname{Det} \|\mathcal{P} - m\| \right] \right\}.\end{aligned} \tag{1.2}$$

The latter expression is symbolic and requires explanation. The operator $\mathcal{P}$ has several matrix indices: namely, those referring to spinor space, colour space and coordinate space

$$\langle x | \mathcal{P}^{ab}_{\alpha\beta} | y \rangle = (\gamma_\mu)_{\alpha\beta} \left[i\delta^{ab} \frac{\partial}{\partial x_\mu} + g(t^c)_{ab} A_\mu{}^c(x) \right] \delta(x - y) \tag{1.3}$$

where the Dirac matrices are denoted as usual by γ_μ while $(t^c)_{ab}$ stand for generators of the colour group in the fundamental representation; g is the coupling constant. Since x and y are continuous variables the operator at hand has infinite number of matrix elements, and in calculating the determinant of this matrix one has to deal with infinities of different sort. They should be treated with care. It is rather clear that essential part of divergences is connected with the determinant of the free Dirac operator (with no external field). These divergences are trivial, and to get rid of them we shall always consider the ratio

$$\frac{\operatorname{Det} \|(\mathcal{P} - m)_A\|}{\operatorname{Det} \|(\mathcal{P} - m)_{A=0}\|}. \tag{1.4}$$

However, the latter expression still contains a residual logarithmic divergence associated with quark-gluon coupling constant renormalization. (We shall explain the assertion in more detail below.)

In order to regularize the ratio (1.4) in a gauge invariant way we introduce an auxiliary fermion Pauli-Willars field which is quantized as a boson. The completely regularized expression takes the form

$$D_{\text{reg}} \equiv \operatorname{Det} \left\| \frac{(\not{P} - m)_A}{(\not{P} - m)_{A=0}} \frac{(\not{P} - M_R)_{A=0}}{(\not{P} - M_R)_A} \right\| \tag{1.5}$$

where M_R is the regulator mass.

From eq. (1.2) one sees that D_{reg} determines an extra term, to be added to the action of the vector field after integration over the fermion field,

$$\begin{aligned} S_{\text{eff}} &= S_{\text{cl}} + \Delta S\,, \\ \Delta S &= -i \ln D_{\text{reg}} = -i \operatorname{Tr} \ln \left\| \frac{(\not{P} - m)_A}{(\not{P} - m)_{A=0}} \frac{(\not{P} - M_R)_{A=0}}{(\not{P} - M_R)_A} \right\| . \end{aligned} \tag{1.6}$$

Here we use the well-known relation

$$\ln \operatorname{Det} \|A\| = \operatorname{Tr} \ln \|A\| .$$

We shall calculate ΔS expanding in powers of $(G/m^2)^n$ assuming that the field strength tensor $G^a_{\mu\nu}$ is much less than the fermion mass squared. It is important that for large masses the fermion loop at hand is determined by short distances, of order m^{-1}, and so the effective action is representable in the form of a series in local operators

$$\Delta S = \int d^4x \left[C_1 g^2 \left(\ln \frac{M_R{}^2}{m^2} \right) G^a_{\mu\nu} G^a_{\mu\nu} + C_2 \frac{g^3}{m^2} f^{abc} G^a_{\mu\alpha} G^b_{\alpha\beta} G^c_{\beta\mu} + \cdots \right]. \tag{1.7}$$

The dots here stand for the terms of next orders in $1/\mathrm{m}^2$ while the coefficients C_i are numerical constants. The term containing the regulator mass $\ln M_R{}^2/m^2$ has the structure identical to that of original action of the gluon field $A_\mu{}^a$. The sum of these two terms is equal to

$$-\frac{g^2}{4} G^a_{\mu\nu} G^a_{\mu\nu} \left[\frac{1}{g^2} - 4C_1 \ln \frac{M_R{}^2}{m^2} \right].$$

From renormalizability of the theory it is clear that combination in square brackets must reduce to the renormalized charge, $1/g^2_{\text{ren}}$. More strictly, since we are considering here only the fermion loop, coefficient C_1 should correspond to charge renormalization associated with one fermion flavour. Thus, renormalizability of the theory fixes the value of C_1:

$$\frac{1}{g^2_{\text{ren}}} = \frac{1}{g^2} - \frac{1}{16\pi^2} \left(\frac{11}{3} N_c - \frac{2}{3} N_f \right) \ln \frac{M_R{}^2}{m^2}, \qquad C_1 = -\frac{1}{96\pi^2} \tag{1.8}$$

($11/3\, N_c$ is the gluon contribution to the charge normalization for the gauge group $SU(N_c)$; N_f is the number of flavours, in our case $N_f = 1$).

In order to find other coefficients in eq. (1.7) it is convenient to consider the derivative of ΔS over the fermion mass

$$\begin{aligned} m\mathrm{P} &\equiv \left[-\frac{d}{dm} \right] \Delta S = -i \operatorname{Tr} \left[\frac{1}{(\not{P} - m)_A} - \frac{1}{(\not{P} - m)_{A=0}} \right] \\ &= \int d^4x \left[2C_1 g^2 \frac{1}{m} G^2 + 2C_2 g^3 \frac{1}{m^3} f^{abc} G^a_{\mu\alpha} G^b_{\alpha\beta} G^c_{\beta\mu} + \cdots \right]. \end{aligned} \tag{1.9}$$

Quantity $(-d/dm)\,\Delta S$ is simply interpreted in the language of the Feynman diagrams. One can easily recognize in it a fermion loop with a unit vertex in external field (fig. 2). Indeed, in the coordinate representation

$$-i\,\mathrm{Tr}\,\frac{1}{\not{P}-m} = -i\int d^4x\,\mathrm{Tr}_{L+C}\,\langle x|\,\frac{1}{\not{P}-m}\,|x\rangle = -i\int d^4x\,\mathrm{Tr}_{L+C}\,\{G_A(x,x)\}$$

which just corresponds to the graph of fig. 2.

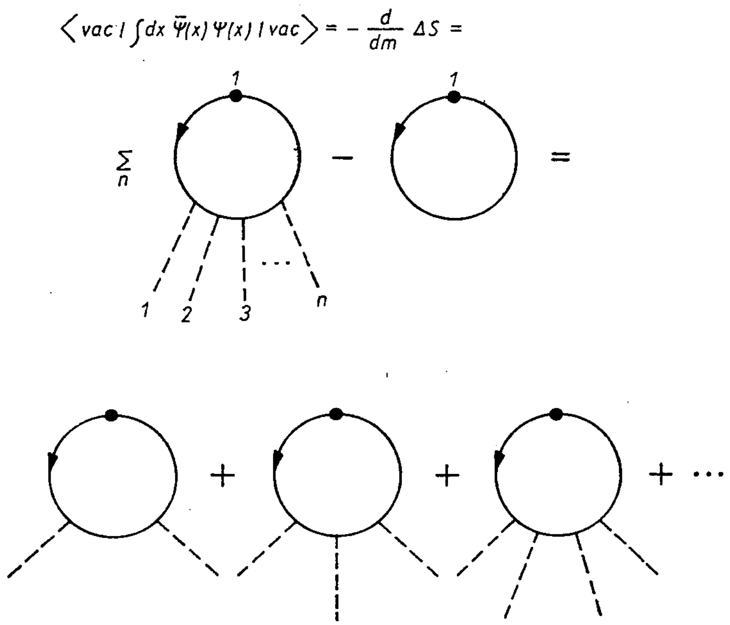


Fig. 2. Graphic representation for the derivative of the effective action (one-loop approximation)

Here $G_A(x, y)$ is the fermion Green function in external field, Tr_{L+C} means the trace operation with respect to spin (Lorenz) and colour indices; $|x\rangle$ is a state vector describing the fermion at the point x.

In further manipulations the following basic commutation relation is used

$$[P_\mu, P_\nu] = igG^a_{\mu\nu}t^a \equiv igG_{\mu\nu}$$

where t^a is the generator of the colour group (for the quark in the fundamental representation of $SU(3)_c$ matrices t^a reduce to the Gell-Mann matrices λ^a, $t^a = \lambda^a/2$). Then

$$\begin{aligned}
\mathrm{P} &= -\frac{d}{m\,dm}\,\Delta S = -\frac{i}{m}\,\mathrm{Tr}\left[\frac{1}{(\not{P}-m)\,(\not{P}+m)}\,(\not{P}+m)\right]_A \\
&\quad + \frac{i}{m}\,\mathrm{Tr}\left[\frac{1}{(\not{P}-m)\,(\not{P}+m)}\,(\not{P}+m)\right]_{A=0} \\
&= -i\,\mathrm{Tr}\left[\frac{1}{P^2-m^2+\dfrac{ig}{2}\,\sigma_{\mu\nu}G_{\mu\nu}} - \frac{1}{(P^2-m^2)_{A=0}}\right]
\end{aligned} \tag{1.10}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$$

and the definition of P is given in eq. (1.9). In deriving eq. (1.10) we exploit the fact that the trace of any odd number of γ matrices vanishes.

The term $ig\sigma G/2$ corresponds to interaction of the fermion magnetic moment with external field $G^a_{\mu\nu}$. Since we would like to find ΔS as a series in $G_{\mu\nu}$ we shall expand eq. (1.10) from the very beginning in σG. Then we arrive at the following series:

$$\mathrm{P} = -i \left\{ \left[\mathrm{Tr} \left(\frac{1}{P^2 - m^2} \right)_A - \mathrm{Tr} \left(\frac{1}{P^2 - m^2} \right)_{A=0} \right. \right.$$
$$\left. + \mathrm{Tr} \frac{1}{P^2 - m^2} \left[-\frac{ig}{2} \sigma G \right] \frac{1}{P^2 - m^2} \left[-\frac{ig}{2} \sigma G \right] \frac{1}{P^2 - m^2} + \cdots \right\}. \tag{1.11}$$

The term of the first order in σG drops out since its trace over the Lorentz indices vanishes.

Let us emphasize that the $G_{\mu\nu}$ dependence is present not only in terms with magnetic moment but, implicitly, also in the scalar operator P^2. In particular, the first bracket

$$-im \, \mathrm{Tr} \left[\frac{1}{(P^2 - m^2)_A} - \frac{1}{(P^2 - m^2)_{A=0}} \right] = \frac{1}{2} \frac{d}{dm} \Delta S_{\mathrm{scal}}$$

represents actually a scalar loop in external field and gives an effective action of the scalar particle.

We shall demonstrate how one can construct the corresponding expansion up to terms of the fourth order in $G_{\mu\nu}$. The result was quoted previously in ref. [14]. (For a detailed discussion see ref. [15].) In principle, it can be also extracted from the recent calculations [16] based on other principles[1]). For one heavy fermion the result looks as follows

$$\Delta S_{\mathrm{eff}} = -\frac{1}{32\pi^2} \int d^4x \, \mathrm{Tr}_c \left\{ \frac{2}{3} g^2 G^2_{\mu\nu} \ln \frac{M_R{}^2}{m^2} - \frac{2}{45} ig^3 G_{\mu\nu} G_{\nu\gamma} G_{\gamma\mu} \frac{1}{m^2} \right.$$
$$\left. + \frac{g^4}{18} \left[(G_{\mu\nu} G_{\mu\nu})^2 - \frac{7}{10} \{G_{\mu\alpha}, G_{\alpha\nu}\}_+{}^2 - \frac{29}{70} [G_{\mu\alpha}, G_{\alpha\nu}]_-{}^2 + \frac{8}{35} [G_{\mu\nu}, G_{\alpha\beta}]_-{}^2 \right] \frac{1}{m^4} \right\}.$$

Here $G_{\mu\nu} = G^a_{\mu\nu} t^a$, $\mathrm{Tr}\, t^a t^b = 1/2\, \delta^{ab}$. To avoid confusion commutators (anticommutators) are supplied with subscripts $-$ $(+)$. We keep trace over the colour indices (Tr_c) in the final expression bearing in mind that in such a form the result is valid for any gauge group.

1.2. Determinant of the Klein-Gordon Operator

In this Section we dwell on the calculation of the determinant of the Klein-Gordan operator, or, more exactly, its logarithm

$$\Delta S_{\mathrm{eff}} = i \ln \mathrm{Det} \left\| \frac{(P^2 - m^2)_A}{(P^2 - m^2)_{A=0}} \frac{(P^2 - M_R{}^2)_{A=0}}{(P^2 - M_R{}^2)_A} \right\|$$
$$= i \, \mathrm{Tr} \ln \left\| \frac{(P^2 - m^2)_A}{(P^2 - m^2)_{A=0}} \frac{(P^2 - M_R{}^2)_{A=0}}{(P^2 - M_R{}^2)_A} \right\|. \tag{1.12}$$

[1]) An earlier work [17] contains an erroneous expression for the cubic contribution to ΔS_{eff}. Moreover, the method applied in ref. [17] does not allow one to find the quartic contribution.

This quantity is evidently interesting by itself. Besides that, as we shall see in the next Section, it essentially fixes the determinant of the Dirac operator.

In the language of the Feynman diagrams the effective action (1.12) is representable in the form of an infinite sum — a loop of scalar particle with emission of two, three, etc. gluons. As was mentioned, it is more convenient to calculate the derivative $(d/dm^2)\,\Delta S_{\text{eff}}$ than the effective action itself. We will consider (for the scalar particle)

$$\mathrm{P} = \frac{d}{dm^2}\,\Delta S_{\text{eff}} = -i\,\mathrm{Tr}\left[\frac{1}{(P^2-m^2)_A} - \frac{1}{(P^2-m^2)_{A=0}}\right], \tag{1.13}$$

this quantity corresponds to the same series of graphs with insertion of the unit vertex in all possible ways.

At first sight it is not quite clear how one can work with such objects as traces of infinite-dimensional matrices. Now by virtue of a certain trick we get a few relations for traces in the functional space which allow one to find, with some patience, all terms in expansion of P in $G_{\mu\nu}$. The systematic way of deriving these relations is surprisingly simple. It is sufficient to notice that eq. (1.13) does not change under the shift of P by arbitrary vector q. Indeed,

$$\begin{aligned}
&\mathrm{Tr}\left[\frac{1}{(P^2-m^2)_A} - \frac{1}{(P^2-m^2)_{A=0}}\right] \\
&\equiv \mathrm{Tr}\,\{e^{iq\hat{X}}[(P^2-m^2)_A^{-1} - (P^2-m^2)_{A=0}^{-1}]\,e^{-iq\hat{X}}\} \\
&= \mathrm{Tr}\left\{\left[\frac{1}{(P-q)^2-m^2}\right]_A - \left[\frac{1}{(P-q)^2-m^2}\right]_{A=0}\right\}.
\end{aligned} \tag{1.14}$$

Here $e^{iq\hat{X}}$ generates shifts in the momentum space, $\hat{X}$ is the coordinate operator. When deriving eq. (1.14) it was important that the original expression is regularized and, therefore, is well defined. This relation is equivalent to the fact that in convergent integrals over infinite space one can arbitrarily shift integration variables.

Now the next step reduces to expansion of the right-hand side of eq. (1.14) in q assuming q to be small. One can easily convince oneself that the expansion coefficients are expressed in terms of traces of operators of the Klein-Gordan type. Recall, however, that we start from expression which is q-independent at all. This means that the coefficient in front of any given power of q should vanish identically. As a result, there emerges an infinite set of relations connecting traces of infinite-dimensional operators with each other.

Let us obtain the simplest of these relations. To this end we average eq. (1.14) over direction of the 4-vector q and consider the coefficient of q^2. As was explained, it should vanish. In the formal language the corresponding equality looks as follows:

$$\begin{aligned}
&\mathrm{Tr}\left\{\frac{1}{(P^2-m^2)_A{}^2} - \frac{1}{(P^2-m^2)^2_{A=0}}\right\} \\
&= \mathrm{Tr}\left\{\left[\frac{1}{(P^2-m^2)^2}P_\mu\frac{1}{(P^2-m^2)}P_\mu\right]_A - \left[\frac{1}{(P^2-m^2)^2}P_\mu\frac{1}{P^2-m^2}P_\mu\right]_{A=0}\right\}.
\end{aligned}$$

Now, using the obvious relations

$$\frac{1}{(P^2-m^2)^2}P_\mu\frac{1}{P^2-m^2}P_\mu = \frac{1}{(P^2-m^2)^3}P^2 + \frac{1}{(P^2-m^2)^3}[P^2, P_\mu]\frac{1}{P^2-m^2}P_\mu,$$

$$\frac{1}{(P^2 - m^2)} P_\mu = \frac{1}{P^2 - m^2} P_\mu (P^2 - m^2) \frac{1}{P^2 - m^2}$$
$$= P_\mu \frac{1}{P^2 - m^2} + \frac{1}{P^2 - m^2} [P_\mu, P^2] \frac{1}{P^2 - m^2}$$
$$= P_\mu \frac{1}{P^2 - m^2} - [P^2, P_\mu] \frac{1}{(P^2 - m^2)^2}$$
$$+ \left[P^2 [P^2, P_\mu]\right] \frac{1}{(P^2 - m^2)^3} - \cdots$$

we finally get

$$\mathrm{Tr} \left\{ \frac{1}{(P^2 - m^2)_A{}^3} - \frac{1}{(P^2 - m^2)^3_{A=0}} \right\}$$
$$= -\frac{1}{m^2} \mathrm{Tr} \left\{ \frac{1}{(P^2 - m^2)^4} [P^2, P_\mu] \left\{ P_\mu - [P^2, P_\mu] \frac{1}{(P^2 - m^2)} + \cdots \right\} \right\}. \tag{1.15}$$

The latter relation occupies the central place in our calculations. Indeed, if the external field expansion for eq. (1.15) is constructed it can be immediately rewritten in terms of expansion for ΔS_{eff}. On the other hand, eq. (1.15) has an enormous advantage over the original expression (1.14) — it contains commutators from the very beginning; in other words, the right-hand side is proportional to $G^a_{\mu\nu} G^a_{\mu\nu}$ and higher powers of the external field, so that calculation up to terms $O(G^2)$ is practically finished. Successive use of the trick outlined above allows one to extract terms $O(G^3)$, $O(G^4)$, etc.

Let us proceed now to technical details. We start with the term

$$I_0 = -\frac{1}{m^2} \mathrm{Tr} \left\{ \frac{1}{(P^2 - m^2)^4} [P^2, P_\mu] P_\mu \right\}.$$

The commutator in the numerator can be immediately rewritten in terms of external field. Indeed,

$$[P^2, P_\mu]_- = \{P_\alpha [P_\alpha, P_\mu]_-\}_+ = ig\{P_\alpha G_{\alpha\mu}\}_+ .$$

In our formulation (one heavy quark flavour) external field has no sources, i.e. it is assumed that $[P_\alpha G_{\alpha\mu}]_- = iD_\alpha G_{\alpha\mu} = 0$. Then

$$[P^2, P_\mu]_- P_\mu = 2igG_{\alpha\mu} P_\alpha P_\mu = -g^2 G_{\alpha\mu} G_{\alpha\mu};$$
$$I_0 = \frac{g^2}{m^2} \mathrm{Tr} \left\{ \frac{1}{(P^2 - m^2)^4} G_{\alpha\mu} G_{\alpha\mu} \right\}.$$

If we would like to limit ourselves by G^2 terms in I_0 there would be practically no need in further computations. The needed power of G is explicitly present in I_0 and, therefore, we can now neglect non-commutativity of the operator P. Then the trace over spacial variables is easily computable

$$\mathrm{Tr} \left\{ \frac{1}{(P^2 - m^2)^{n-1}_{A=0}} G^2 \right\} = \mathrm{Tr}_c \int d^4x \, d^4y \langle x| \frac{1}{(P^2 - m^2)^{n-1}_{A=0}} |y\rangle \langle y| G^2 |x\rangle$$
$$= \mathrm{Tr}_c \int d^4x \langle x| (P^2 - m^2)^{-(n-1)}_{A=0} |x\rangle G^2(x);$$

$$\langle x| (P^2 - m^2)^{-(n-1)}_{A=0} |x\rangle = \int \{d^4p/(2\pi)^4\} (p^2 - m^2)^{-(n-1)} = \frac{(-1)^n m^{6-2n}}{16\pi^2 i (n-2)(n-3)}.$$

As a result we get

$$I_0 = \frac{i}{2^5 \cdot 3 \cdot \pi^2} \frac{g^2}{m^6} \operatorname{Tr}_c G_{\alpha\mu} G_{\alpha\mu} + \cdots. \tag{1.16}$$

However what if we want to make the next step and find which higher-order terms are actually denoted by dots in eq. (1.16)? To make further step we use the already known trick — shift of variable P by a vector q in the quantity

$$\operatorname{Tr} \frac{1}{P^2 - m^2} F(G) \tag{1.17}$$

where $F(G)$ is arbitrary function of the field G. Concentrating ourselves on the coefficient of q^2 we get the following relation

$$\operatorname{Tr}\left\{\frac{1}{(P^2 - m^2)^3} F(G)\right\} = \frac{1}{2^5 \pi^2 i} \frac{1}{m^2} \operatorname{Tr}_c F(G) - g^2 \operatorname{Tr}\left\{\frac{1}{(P^2 - m^2)^5} G^2 F(G)\right\}. \tag{1.18}$$

(The first term in the right-hand side of eq. (1.18) has appeared from the regulator contribution $-\operatorname{Tr}\{(P^2 - m^2)^{-3}_{\Lambda=0} F(G)\}$ which must be added to eq. (1.17) in order to make the whole expression convergent). Differentiating eq. (1.18) with respect to the quark mass one readily gets the trace of any power $\operatorname{Tr}\{(P^2 - m^2)^{-n} F(G)\}$.

The final result for I_0 has the form

$$\begin{aligned} I_0 &= \frac{ig^2}{3 \cdot 2^5 \cdot \pi^2 \cdot m^6} \int d^4x \operatorname{Tr}_c \left\{G_{\alpha\mu} G_{\alpha\mu} - \frac{g^2}{2m^4} (G_{\alpha\mu} G_{\alpha\mu})^2\right\}, \\ [\Delta S_{\text{eff}}]^{(0)} &= -\frac{1}{3 \cdot 2^5 \cdot \pi^2} \int dx \operatorname{Tr}_c \left\{g^2 \ln \frac{M_R{}^2}{m^2} G^2 - \frac{g^4}{24 m^4} (G^2)^2\right\}. \end{aligned} \tag{1.19}$$

Let us consider now the next commutator

$$\begin{aligned} I_1 &= \frac{1}{m^2} \operatorname{Tr}\left\{\frac{1}{(P^2 - m^2)^5} [P^2, P_\mu] [P^2, P_\mu]\right\} = -\frac{4g^2}{m^2} \operatorname{Tr}\left\{\frac{1}{(P^2 - m^2)^5} G_{\alpha\mu} P_\alpha P_\beta G_{\beta\mu}\right\} \\ &= -\frac{4g^2}{m^2} \operatorname{Tr}\left\{\frac{1}{(P^2 - m^2)^5} P_\alpha P_\beta G_{\alpha\mu} G_{\beta\mu}\right\} + \frac{4g^2 i}{m^2} \operatorname{Tr}\left\{\frac{1}{(P^2 - m^2)^5} P_\alpha (D_\beta G_{\alpha\mu} G_{\beta\mu})\right\}. \end{aligned}$$

The second term in the last equality can be omitted since it induces only the terms of an order higher than G^4. Indeed, the matrix element

$$\langle x| \, (P^2 - m^2)^{-5} \, P_\alpha \, |x\rangle,$$

because of the translational invariance, can not depend on x explicitly. The x dependence comes only through the field G and its covariant derivatives. The first non-vanishing term which can emerge in $\langle x| \, (P^2 - m^2)^{-5} \, P_\alpha \, |x\rangle$ is $D_\alpha G^2$. It, clearly, gives rise to $O(G^5)$ contribution in I_1 which is not discussed here. Thus

$$I_1 = -\frac{2g^2}{m^2} \operatorname{Tr} (P^2 - m^2)^{-5} \{P_\alpha, P_\beta\}_+ G_{\alpha\mu} G_{\beta\mu} - \frac{2ig^3}{m^2} \operatorname{Tr} (P^2 - m^2)^{-5} G_{\alpha\beta} G_{\alpha\mu} G_{\beta\mu}.$$

Once more making a shift of P in the regularized expansion for

$$\mathrm{Tr}\,\frac{1}{P^2 - m^2}\,\{P_\alpha, P_\beta\}_+$$

we gain two units of dimension in the field. Omitting straightforward but rather tedious arithmetics we quote here only the answer

$$\begin{aligned} I_1 = &-\frac{ig^2}{3\cdot 2^6\cdot \pi^2\cdot m^6}\,\mathrm{Tr}_c\,G_{\alpha\mu}G_{\alpha\mu} - \frac{g^3}{2^5\cdot 3\cdot \pi^2\cdot m^8}\,\mathrm{Tr}_c\,G_{\alpha\mu}G_{\mu\nu}G_{\nu\alpha} \\ &+ \frac{ig^4}{2^7\cdot 3\cdot \pi^2\cdot m^{10}}\,\mathrm{Tr}_c\,[(G_{\mu\alpha}G_{\mu\alpha})^2 - \{G_{\mu\alpha}G_{\alpha\nu}\}_+{}^2], \end{aligned} \tag{1.20}$$

$$\begin{aligned} [\Delta S_{\mathrm{eff}}]^{(1)} = \int d^4x\,\mathrm{Tr}_c\,\Bigg(&\frac{g^2}{3\cdot 2^6\cdot \pi^2}\ln\frac{M_R{}^2}{m^2}\,G_{\alpha\mu}G_{\alpha\mu} - \frac{ig^3}{2^5\cdot 3^2\cdot \pi^2\cdot m^2}\,G_{\alpha\mu}G_{\mu\nu}G_{\nu\alpha} \\ &- \frac{g^4}{2^9\cdot 3^2\cdot \pi^2\cdot m^4}\,[(G_{\mu\alpha}G_{\mu\alpha})^2 - \{G_{\mu\alpha}G_{\alpha\nu}\}_+{}^2]\Bigg). \end{aligned}$$

Calculation of further commutators proceeds along the lines explained above. Omitting some intermediate elements we formulate basic expressions which allow one to easily reconstruct the whole derivation.

(1) Double commutator[2])

$$\begin{aligned} [\Delta S_{\mathrm{eff}}]^{(2)} = &\frac{ig^3}{2^3\cdot 3^2\cdot 5\cdot \pi^2\cdot m^2}\,\mathrm{Tr}_c\,G^3 \\ &- \frac{g^4}{2^8\cdot 3\cdot 5\cdot \pi^2\cdot m^4}\,\mathrm{Tr}_c\left(\{G_{\mu\alpha}G_{\alpha\nu}\}_+{}^2 + \frac{1}{6}\,[G_{\mu\nu}G_{\alpha\beta}]_-{}^2 - \frac{5}{3}\,[G_{\mu\alpha}G_{\alpha\nu}]_-{}^2\right). \end{aligned} \tag{1.21}$$

(2) Triple commutator

$$\begin{aligned} [\Delta S_{\mathrm{eff}}]^{(3)} = &-\frac{ig^3}{2^5\cdot 3^2\cdot 5\cdot \pi^2\cdot m^2}\,\mathrm{Tr}_c\,G^3 \\ &- \frac{g^4}{2^8\cdot 3^2\cdot 5\cdot \pi^2\cdot m^4}\,\mathrm{Tr}_c\,(-\{G_{\mu\alpha}G_{\alpha\nu}\}_+{}^2 + [G_{\mu\nu}G_{\alpha\beta}]_-{}^2 + 10[G_{\mu\alpha}G_{\alpha\nu}]_-{}^2). \end{aligned} \tag{1.22}$$

(3) Quartic commutator

$$[\Delta S_{\mathrm{eff}}]^{(4)} = \frac{g^4}{2^8\cdot 3^2\cdot 7\cdot \pi^2\cdot m^4}\,\mathrm{Tr}_c(10[G_{\mu\alpha}G_{\alpha\nu}]_-{}^2 + [G_{\mu\nu}G_{\alpha\beta}]_-{}^2). \tag{1.23}$$

(4) 5-Commutator

$$[\Delta S_{\mathrm{eff}}]^{(5)} = -\frac{g^4}{2^9\cdot 3^2\cdot 7\cdot \pi^2\cdot m^4}\,\mathrm{Tr}_c\left(5[G_{\mu\alpha}G_{\alpha\nu}]_-{}^2 + \frac{1}{2}\,[G_{\mu\nu}G_{\alpha\beta}]_-{}^2\right). \tag{1.24}$$

[2]) In eqs. (1.21)—(1.24) integration over d^4x is implied (cf. eq. (1.20)) but not written out explicitly.

Further commutators in eq. (1.15) result in terms $O(G^5)$ and of higher orders. In deriving eqs. (1.21)—(1.24) we have widely used the following relations

$$\langle x| P_\gamma P_\beta P_\alpha (P^2 - m^2)^{-6} |x\rangle = \frac{ig}{2^6 \cdot 3 \cdot 5 \cdot \pi^2 \cdot m^8} [D_\gamma G_{\alpha\beta} + D_\beta G_{\alpha\gamma}],$$

$$\begin{aligned} \langle x| P_\delta P_\gamma P_\beta P_\alpha (P^2 - m^2)^{-7} |x\rangle &= \frac{-i}{2^9 \cdot 3^2 \cdot 5 \cdot \pi^2 \cdot m^6} (g_{\delta\gamma} g_{\beta\alpha} + g_{\delta\beta} g_{\gamma\alpha} + g_{\delta\alpha} g_{\gamma\beta}) \\ &- \frac{g}{2^9 \cdot 3 \cdot 5 \cdot \pi^2 \cdot m^8} (G_{\delta\gamma} g_{\beta\alpha} + G_{\delta\beta} g_{\gamma\alpha} + G_{\gamma\alpha} g_{\delta\beta} \\ &+ G_{\beta\alpha} g_{\delta\gamma} + G_{\delta\alpha} g_{\gamma\beta} + G_{\gamma\beta} g_{\delta\alpha}) + \cdots. \end{aligned}$$

They can be obtained by considering traces of the products "(given operator) $\times$ $F(G)$" where $F(G)$ stands for arbitrary function of the field G. As usual, one should make a shift of P by q.

It is worth noting that computation of multiple commutators $[P^2, [P^2, \ldots [P^2, P_\mu] \ldots]]$ represents essentially a very simple rather lengthy and cumbersome operation of purely arithmetical character. The operation seems to be easily formalizable for analytical computer calculations. In Appendix A we give for completeness such commutators up to $\left[P^2[P^2[P^2[P^2[P^2, P_\mu]]]]\right]$.

Finally we are able to write out the final answer for $(\Delta S_{\text{eff}})_{\text{scal}}$:

$$(\Delta S_{\text{eff}})_{\text{scal}} = i \operatorname{Tr} \ln \left\| \frac{(P^2 - m^2)_A}{(P^2 - m^2)_{A=0}} \cdot \frac{(P^2 - M_R{}^2)_{A=0}}{(P^2 - M_R{}^2)_A} \right\| = \int d^4x (L_{\text{eff}})_{\text{scal}}, \tag{1.25}$$

$$\begin{aligned} (L_{\text{eff}})_{\text{scal}} &= -\frac{g^2}{3 \cdot 2^6 \cdot \pi^2} \ln \left(\frac{M_R{}^2}{m^2} \right) \operatorname{Tr}_c G_{\alpha\mu} G_{\alpha\mu} - \frac{ig^3}{2^5 \cdot 3^2 \cdot 5 \cdot \pi^2 \cdot m^2} \operatorname{Tr}_c G_{\mu\alpha} G_{\alpha\beta} G_{\beta\mu} \\ &+ \frac{g^4}{2^9 \cdot 3^2 \cdot \pi^2 \cdot m^4} \operatorname{Tr}_c \Big((G_{\mu\alpha} G_{\mu\alpha})^2 + \frac{1}{5} \{G_{\mu\alpha} G_{\alpha\nu}\}_+{}^2 \\ &+ \frac{1}{7} [G_{\mu\alpha} G_{\alpha\nu}]_-{}^2 + \frac{1}{70} [G_{\mu\nu} G_{\alpha\beta}]_-{}^2 \Big) + \cdots. \end{aligned}$$

The infinite logarithm $\ln (M_R{}^2/m^2)$ in front of $\operatorname{Tr}_c G^2$ is absorbed, as was explained above, in the charge renormalization.

1.3. Determinant of the Dirac Operator

Let us return now to the determinant D of the Dirac operator and the corresponding effective action:

$$D_{\text{reg}} = \operatorname{Det} \left\| \frac{(\not{P} - m)_A}{(\not{P} - m)_{A=0}} \frac{(\not{P} - M_R)_{A=0}}{(\not{P} - M_R)_A} \right\|.$$

$$\Delta S_{\text{eff}} = -i \ln D_{\text{reg}} = -i \operatorname{Tr} \ln \left\| \frac{(\not{P} - m)_A}{(\not{P} - m)_{A=0}} \frac{(\not{P} - M_R)_{A=0}}{(\not{P} - M_R)_A} \right\|.$$

As was explained in Sec. 1.1. the derivative $(d/dm)\, \Delta S_{\text{eff}}$ is expressed in terms of $(d/dm^2)\, (\Delta S_{\text{eff}})_{\text{scal}}$ — and the latter was already found — and an extra contribution coming from interaction with the fermion magnetic moment σG. We write out here once more the relevant expansion

$$\begin{aligned}
\mathrm{P} = -\frac{d}{m\,dm}\,\Delta S_{\mathrm{eff}} &= -i\,\mathrm{Tr}\left\{\frac{1}{(P^2-m^2)_A} - \frac{1}{(P^2-m^2)_{A=0}}\right\} \\
&-i\,\mathrm{Tr}\,\frac{1}{(P^2-m^2)^2}\left(-\frac{ig}{2}\,\sigma G\right)\frac{1}{(P^2-m^2)}\left(-\frac{ig}{2}\,\sigma G\right) \\
&-i\,\mathrm{Tr}\,\frac{1}{(P^2-m^2)^2}\left(-\frac{ig}{2}\,\sigma G\right)\frac{1}{(P^2-m^2)}\left(-\frac{ig}{2}\,\sigma G\right)\frac{1}{(P^2-m^2)}\left(-\frac{ig}{2}\,\sigma G\right) \\
&-i\,\mathrm{Tr}\,\frac{1}{(P^2-m^2)^2}\left(-\frac{ig}{2}\,\sigma G\right)\frac{1}{(P^2-m^2)}\left(-\frac{ig}{2}\,\sigma G\right)\frac{1}{(P^2-m^2)}\left(-\frac{ig}{2}\,\sigma G\right) \\
&\times\frac{1}{(P^2-m^2)}\left(-\frac{ig}{2}\,\sigma G\right).
\end{aligned} \tag{1.26}$$

Next terms of expansion contain explicitly the fifth and higher powers of G and will not be considered here.

It is worth noting that the most essential part of the work has been already done in the previous Section where we have learned to handle expressions of the type

$$\mathrm{Tr}\,\{(P^2-m^2)^{-n}\,F(G)\}, \tag{1.27}$$

$F(G)$ being any function of the field G. Therefore, if one manages to place all (σG) encountered in eq. (1.26) in the rightmost position, the problem is reduced to (1.27). How can one interchange (σG) and $(P^2-m^2)^{-1}$? Let us show this considering successively term by term in eq. (1.26).

The simplest one is the last term. It explicitly contains G to the fourth power and, therefore, one can neglect non-commutativity of operators P and G. Then

$$\mathrm{P}_4 = -i\,\mathrm{Tr}\,(P^2-m^2)^{-5}\left(-\frac{ig}{2}\,\sigma G\right)^4 = -\frac{g^4}{3\cdot 2^{10}\cdot\pi^2\cdot m^6}\int d^4x\,\mathrm{Tr}_{C+L}\,(\sigma G)^4. \tag{1.28}$$

The only thing to be done now is taking the trace over the Lorentz indices of eight γ matrices. The operation is trivial and we quote the corresponding result without any comments:

$$\mathrm{Tr}_{C+L}\,(\sigma G)^4 = \mathrm{Tr}_c\,\{48(G_{\mu\alpha}G_{\mu\alpha})^2 + 8[G_{\mu\nu}G_{\alpha\beta}]_-^{\,2} - 16\{G_{\mu\nu}G_{\nu\alpha}\}_+^{\,2} - 48[G_{\mu\alpha}G_{\alpha\nu}]_-^{\,2}\}. \tag{1.29}$$

In the next term, P_3, which contains (σG) to the third power it is necessary to account for non-commutativity of P. Let us rewrite P_3 in the following identical form:

$$\begin{aligned}
\mathrm{P}_3 = \frac{g^3}{8}\,\mathrm{Tr}\,\{&(P^2-m^2)^{-4}\,(\sigma G)^3 - (P^2-m^2)^{-6}\big([P^2,\sigma G]_-\,(\sigma G)\,[P^2,\sigma G]_- \\
&+ [P^2,\sigma G]_-\,[P^2,(\sigma G)^2]_- + [P^2,(\sigma G)^2]_-\,[P^2,\sigma G]_-\big) \\
&+ (P^2-m^2)^{-5}\big([P^2,\sigma G]_-\,(\sigma G)^2 - (\sigma G)^2\,[P^2,\sigma G]_-\big) + \cdots\}.
\end{aligned} \tag{1.30}$$

Here dots refer to higher commutators resulting in terms $O(G^5)$ and of higher orders. It is worth noting that such identical rearrangement of terms ensuring the rightmost position for $(\sigma G)^3$ can be done different ways so that the explicit expression for P_3 may be superficially different from eq. (1.30).

Commutators are calculable with no difficulty. For instance,

$$[P^2, \sigma G]_- = \{P_\alpha[P_\alpha, \sigma G]_-\}_+ = -D^2\sigma G + 2i(D_\alpha \sigma G)\, P_\alpha ,$$

etc.

As a result

$$\mathrm{P}_3 = \frac{ig^3}{2^8 \cdot 3 \cdot \pi^2 \cdot m^4} \int d^4x\, \mathrm{Tr}_{C+L}\, (\sigma G)^3 - \frac{g^4}{2^4 \cdot 3 \cdot \pi^2 \cdot m^6} \int d^4x\, \mathrm{Tr}_C\, [G_{\mu\alpha} G_{\alpha\nu}]_-{}^2 ;$$

$$\mathrm{Tr}_{C+L}\, (\sigma G)^3 = 2^5\, \mathrm{Tr}_C\, G^3 . \tag{1.31}$$

The last thing to be done is to find P_2 — the piece containing double interaction with magnetic moment,

$$\mathrm{P}_2 = \frac{ig^2}{4} \mathrm{Tr}\, (P^2 - m^2)^{-2}\, \sigma G (P^2 - m^2)^{-1}\, \sigma G = \frac{ig^2}{4} \mathrm{Tr}\, \{(P^2 - m^2)^{-3}\, (\sigma G)^2\}$$
$$- \frac{ig^2}{4} \mathrm{Tr} \left\{ (P^2 - m^2)^{-4} \left(\sigma G + \frac{1}{(P^2 - m^2)} [P^2, \sigma G]_- \right.\right.$$
$$\left.\left. + \frac{1}{(P^2 - m^2)^2} [P^2[P^2, \sigma G]_-]_- + \cdots \right) [P^2, \sigma G]_- \right\} . \tag{1.32}$$

Since we already know how to ensure two extra powers of G in any expression of the type $\mathrm{Tr}\, (P^2 - m^2)^{-n}\, F(G)$ we immediately write out the first term

$$\frac{ig^2}{4} \mathrm{Tr}\, (P^2 - m^2)^{-3}\, (\sigma G)^2$$
$$= -\frac{g^2}{2^4 \cdot \pi^2 \cdot m^2} \int d^4x\, \mathrm{Tr}_C\, G_{\alpha\mu} G_{\alpha\mu} + \frac{g^4}{2^5 \cdot 3 \cdot \pi^2 \cdot m^6} \int d^4x\, \mathrm{Tr}_C\, (G_{\alpha\mu} G_{\alpha\mu})^2 .$$

In order to extract all operators of order G^4 in the second term, eq. (1.32), one must account for all commutators up to $[P^2[P^2[P^2, \sigma G]]]$. Using the procedure described above the reader will easily find them (see Appendix A).

The final result includes, among others, operators containing high covariant derivatives, e.g. $(D \ldots DG)^2$. One can pose a question: "is it always possible to reduce such expressions (with covariant derivatives) to expressions constructed from the field strength tensor itself?" The answer is:

All Lorentz-singlet gauge-invariant gluon operators of dimension 8 and 6 are reducible to operators containing only G's by virtue of equation of motion and the Bianchi identity. For instance,

$$G_{\mu\nu} D^4 G_{\mu\nu} = -4g^2 (G_{\alpha\mu} G_{\alpha\nu})^2 .$$

Indeed,

$$G_{\mu\nu} D^4 G_{\mu\nu} = D^2 G_{\mu\nu} D^2 G_{\mu\nu} + \text{inessential total derivatives.}$$

Now, the Bianchi identity $D_\alpha G_{\mu\nu} + D_\nu G_{\alpha\mu} + D_\mu G_{\nu\alpha} = 0$ implies

$$(D^2 G_{\mu\nu})\, (D_\alpha D_\alpha G_{\mu\nu}) = 2(D^2 G_{\mu\nu})\, (D_\alpha D_\mu G_{\alpha\nu}) = 2(D^2 G_{\mu\nu})\, [D_\alpha D_\mu]_-\, G_{\alpha\nu}$$
$$= -2ig (D^2 G_{\mu\nu})\, (G_{\alpha\mu} G_{\alpha\nu}) = -4g^2 (G_{\alpha\mu} G_{\alpha\nu})^2 .$$

When deriving these formulae we have used the fact that $D_\mu G_{\mu\nu} = 0$ and omitted the terms which are full derivatives since they give no contribution in the integral for ΔS_{eff}. The number of terms in P_2 to be rewritten in such a way in terms of G's is rather large. We would not like to dwell on conceptually simple intermediate steps and reproduce directly the answer

$$\mathrm{P}_2 = \mathrm{Tr}_c \int d^4x \left\{ -\frac{g^2}{2^4 \cdot \pi^2 \cdot m^2} G^2 + \frac{g^4}{2^5 \cdot 3 \cdot \pi^2 \cdot m^6} (G^2)^2 \right.$$
$$\left. + \frac{g^4}{2^3 \cdot 3 \cdot 5 \cdot \pi^2 \cdot m^6} [G_{\mu\alpha} G_{\alpha\nu}]_-^2 + \frac{g^4}{2^6 \cdot 3 \cdot 5 \cdot \pi^2 \cdot m^6} [G_{\mu\nu} G_{\alpha\beta}]_-^2 \right\}.$$

Let us assemble now together all pieces, P_2, P_3, P_4 and add $-i\,\mathrm{Tr}\,(P^2 - m^2)^{-1}$ for scalar particles. Recall that the term $-i\,\mathrm{Tr}\left((P^2 - m^2)_A^{-1} - (P^2 - m^2)_{A=0}^{-1}\right)$ in eq. (1.26) differs from the result given in sec. 1.2 by a factor of 4. This factor is due to the fact that the symbol Tr in eq. (1.26) includes also the trace over the Lorentz indices, and $\mathrm{Tr}_L\, 1 = 4$.

Now, the Heisenberg-Euler Lagrangian ΔS_{eff} is reconstructed by integration $\Delta S_{\text{eff}} = -1/2 \int_{m^2}^{M_R^2} dm^2\, \mathrm{P}(m^2)$. The result was quoted in sec. 1.1.

1.4. Concluding Remarks

The operator method of calculation in external fields originally proposed by Schwinger enormously simplifies computations as compared to the standard technique of Feynman integrals. Its advantages in constructing the operator expansion are obvious to everybody who deals with the sum rules and it is enjoying now wider and wider popularity in theoretical audience.

From other applications of the method let us mention the wonderful work by 'T HOOFT [18] who has found the fermion determinant in the instanton field. In this work it has been shown for the first time how one can introduce (basing on the notion of external field) the Pauli-Villars regulators for the gauge field with no violation of gauge invariance of the original non-abelian theory. Later on it turned out possible to generalize the method in supersymmetric theories and construct explicitly supersymmetric Pauli-Villars regularization [19]. Needless to say that although we discuss here only one example, ΔS_{eff} for one heavy fermion, just the same technique is applicable in other problems. Say, we are interested in the two-point function

$$\Pi^{(5)}(q) = i \int e^{iqx}\, d^4x \langle T\{\bar\psi(x)\, i\gamma_5 \psi(x),\, \bar\psi(0)\, i\gamma_5 \psi(0)\}\rangle$$

where ψ is the heavy quark field.

First of all we rewrite it identically as follows

$$\Pi^{(5)}(q^2) = i \int d^4x\, \mathrm{Tr}_{L+C} \left(i\gamma_5 \langle x| \frac{1}{\left(P^2 - m^2 + \frac{ig}{2}\sigma G\right)} (\not P + m) |y\rangle \right.$$
$$\left. \times\, i\gamma_5 \langle y| \frac{1}{(P+q)^2 - m^2 + \frac{ig}{2}\sigma G} (\not P + \not q + m) |x\rangle \right)_{y=0}.$$

Then we expand in (σG) up to required order, take the trace over the Lorentz indices and integrate over d^4y. As a result we find that $\Pi^{(5)}(q^2 = 0) \int d^4y$, $(d\Pi^{(5)}/dq^2)_{q^2=0} \int d^4y$ etc. are expressed in terms of traces of scalar operators of the type

$$\mathrm{Tr}\,(P^2 - m^2)^{-1}\left(G(P^2 - m^2)^{-1}\right)^k P_\alpha (P^2 - m^2)^{-1}\left(G(P^2 - m^2)^{-1}\right)^{k'} P_\alpha .$$

Further manipulations are already known.

Chapter II. Method Based on the Fock-Schwinger Gauge

2.1. The Fock-Schwinger Gauge

Correlation functions of colourless sources are gauge invariant. Any gauge condition imposed on the vacuum gluon field (considered as given) will eventually lead to one and the same final answer. Hence the choice of the gauge is, first of all, the question of convenience. The gauge proposed in electrodynamics by Fock and then, independently, by Schwinger [7, 4] is as follows

$$(x - x_0)\, A_\mu{}^a(x) = 0 \tag{2.1}$$

where $A_\mu{}^a$ is the four-potential and x_0 is an arbitrary point in the space playing the role of a gauge parameter. This gauge was rediscovered several times in chromodynamics [20], and its virtues were revealed in subsequent works.

Let us notice from the very beginning that condition (2.1) is invariant under the scale transformation and inversion of coordinates, but breaks the translational symmetry. The latter restores itself in gauge invariant quantities. In other words, parameter x_0 should cancel out in all correlation functions induced by colourless sources. This fact — cancellation of x_0 dependence — may serve as an additional check of correctness of calculations (see, e.g. [21]). We shall not exploit this useful property, however, and shall put $x_0 = 0$ in all following considerations,

$$x_\mu A_\mu{}^a(x) = 0 \tag{2.2}$$

getting, as a reward, compact formulae for quark and gluon propagators.

The most important consequence from eq. (2.2) is the following: potential $A_\mu{}^a(x)$ is expressable directly in terms of the gluon field strength tensor [20]. Namely,

$$A_\mu{}^a(x) = \int_0^1 \alpha\, d\alpha\, G^a_{\varrho\mu}(\alpha x)\, x_\varrho . \tag{2.3}$$

The derivation of this relation is quite straightforward. Start from identity

$$A_\mu(y) = \frac{\partial}{\partial y_\mu}\left(A_\varrho(y)\, y_\varrho\right) - y_\varrho \frac{\partial A_\varrho(y)}{\partial y_\mu}, \tag{2.4}$$

where, due to eq. (2.2) the first term in the right-hand side vanishes. The second term can be, evidently, rewritten as follows

$$-y_\varrho G_{\mu\varrho} - y_\varrho \frac{\partial A_\mu(y)}{\partial y_\varrho}. \tag{2.5}$$

Combining eqs. (2.4), (2.5) we get

$$A_\mu(y) + y_\varrho \frac{\partial A_\mu(y)}{\partial y_\varrho} = y_\varrho G_{\varrho\mu}(y). \tag{2.6}$$

Substitute now $y = \alpha x$. One immediately sees that the left-hand side of eq. (2.6) reduces to a full derivative $(d/d\alpha)\big(\alpha A_\mu(\alpha x)\big)$. Integrating over α from 0 to 1 we arrive at eq. (2.3).

For our purposes — and we are interested here in the expansion in local operators $O_n(0)$ — it is much more convenient to deal with another representation for $A_\mu^a(x)$ which stems [22] from eq. (2.3):

$$A_\mu(x) = \frac{1}{2\cdot 0!} x_\varrho G_{\varrho\mu}(0) + \frac{1}{3\cdot 1!} x_\alpha x_\varrho\big(D_\alpha G_{\varrho\mu}(0)\big) + \frac{1}{4\cdot 2!} x_\alpha x_\beta x_\varrho\big(D_\alpha D_\beta G_{\varrho\mu}(0)\big) + \cdots, \tag{2.7}$$

where D as usual denotes the covariant derivative. Eq. (2.7) shows that $A_\mu^a(x)$ is expressed directly in terms of the field strength tensor at the origin and its covariant derivatives.

Let us outline the proof of eq. (2.7). The key point is the following observation:

$$x_{\alpha_1} \dots x_{\alpha_n}[\partial_{\alpha_1} \dots \partial_{\alpha_n} G_{\varrho\mu}]_{x=0} = x_{\alpha_1} \dots x_{\alpha_n}[D_{\alpha_1} \dots D_{\alpha_n} G_{\varrho\mu}]_{x=0}, \tag{2.8}$$

or, in other words, the ordinary derivatives at the origin may be substituted by covariant derivatives.

Indeed, expanding four-potential in eq. (2.2) we notice that for any x

$$x_\mu\left[A_\mu^a(0) + x_{\alpha_1}\,\partial_{\alpha_1} A_\mu^a(0) + \frac{1}{2}\,x_{\alpha_1}x_{\alpha_2}\,\partial_{\alpha_1}\,\partial_{\alpha_2} A_\mu^a(0) + \cdots\right] = 0, \tag{2.9}$$

and hence

$$\begin{aligned} &x_\mu A_\mu^a(0) = 0,\\ &x_\mu x_{\alpha_1}\partial_{\alpha_1} A_\mu^a(0) = 0,\\ &x_\mu x_{\alpha_1} x_{\alpha_2}\partial_{\alpha_1}\partial_{\alpha_2} A_\mu^a(0) = 0, \dots \end{aligned} \tag{2.10}$$

Now one sees that

$$x_{\alpha_1}\,\partial_{\alpha_1} G^a_{\varrho\mu}(0) = x_{\alpha_1} D_{\alpha_1} G^a_{\varrho\mu}(0),$$

$$x_{\alpha_1}x_{\alpha_2}\,\partial_{\alpha_1}\,\partial_{\alpha_2} G^a_{\varrho\mu}(0) = x_{\alpha_1}x_{\alpha_2}\,\partial_{\alpha_1} D_{\alpha_2} G^a_{\varrho\mu}(0) = x_{\alpha_1}x_{\alpha_2} D_{\alpha_1} D_{\alpha_2} G^a_{\varrho\mu}(0),$$

etc. Further details are given in ref. [22] where the proof is based on the method of mathematical induction.

In order to obtain eq. (2.7) we expand $G(\alpha x)$ in eq. (2.3) in αx, apply eq. (2.8) and integrate over α.

Quite analogous representation is valid for the fermion field

$$\psi(x) = \psi(0) + x_{\alpha_1} D_{\alpha_1}\psi(0) + \frac{1}{2}\,x_{\alpha_1}x_{\alpha_2} D_{\alpha_1} D_{\alpha_2}\psi(0) + \cdots. \tag{2.11}$$

It is worth emphasizing one more that expansions (2.7), (2.11) contain only gauge covariant quantities. Just this fact ensures advantages of the given computational scheme over others.

2.2. The Quark Propagator. General Expression

Correlation functions of currents are determined by quark and gluon propagators, however, unlike the ordinary Feynman diagrams, these propagators describe propagation in external gluon (quark) fields. Concentrate at first on the gluon fields leaving aside for a while effects of quark condensate.

Let the quark be emitted at the point y and annihilated at the point x. Propagator

$$S(x, y) = \langle x| \frac{1}{\not{P} - m} |y\rangle = -i\langle T\{\psi(x), \bar{\psi}(y)\}\rangle \tag{2.12}$$

satisfies the standard equation

$$\left(i \frac{\partial}{\partial x_\mu} \gamma_\mu + gA_\mu(x)\, \gamma_\mu - m\right) S(x, y) = \delta^{(4)}(x - y), \tag{2.13}$$

where $A_\mu(x)$ is the vacuum field (2.7), $A_\mu = 1/2\, \lambda^a A_\mu{}^a$. If the field $A_\mu(x)$ is small in comparison with characteristic distances $(x - y)$ — and we always work only in these conditions — then $S(x, y)$ is representable in the form of the standard series:

$$\begin{aligned} iS(x, y) = {} & iS^{(0)}(x - y) + g \int d^4z iS^{(0)}(x - z)\, i\not{A}(z)\, iS^{(0)}(z - y) \\ & + g^2 \int d^4z'\, d^4z iS^{(0)}(x - z')\, i\not{A}(z')\, iS^{(0)}(z' - z)\, i\not{A}(z)\, iS^{(0)}(z - y) \\ & + \cdots, \end{aligned} \tag{2.14}$$

where $S^{(0)}(x - y)$ is the free quark propagator. The meaning of this expansion is quite transparent. The quark emitted at the point y and annihilated at the point x scatters 0, 1, 2, ... times off the vacuum field (2.7).

It would be in order here to draw the reader's attention to a subtlety. The propagator (2.14) is translationally non-invariant due to two reasons. First of all, the field A_μ depends on coordinate. In a sense this dependence is fictitious since the vacuum field, after averaging, is clearly invariant under translations. The true source of the symmetry breaking is the gauge condition (2.2) according to which the origin plays a special role. When one works with gauge-dependent quantities one should be careful since, for instance, $S(x, y)$ and $S(x - y, 0)$ are generally speaking absolutely different functions. After fixing the reference frame one should consistently perform all computations just in this frame. Only in final results for n-point functions of colourless currents we return to original situation — the answer is one and the same for any choice of the reference frame.

We will not dwell any more on general relations, although one could go rather far along these lines (Chapter I). Instead, we turn to particular examples and consider all cases one may encounter with in applications. Recall that for light quarks (u, d, s) the mass is a small parameter, $m_q \ll \mu$ (μ stands for characteristic hadronic scale). Starting in the zeroth approximation from massless quarks, we, then, account for corrections proportional to m_q perturbatively. On the contrary, for c, b and t quarks $m_Q \gg \mu$, which also results in simplifications. Two different limiting cases, $m_q \ll \mu$ and $m_Q \gg \mu$ are treated separately.

2.3. Heavy Quarks

If the quark mass m_Q is sufficiently large the corresponding quark condensate is not an independent parameter, and the problem completely reduces to a motion in external gluon field. Assume that we are interested in the two-point function of vector currents $\langle J_\mu(x), J_\nu(y)\rangle$ where

$$J_\mu(x) = \bar{Q}(x)\,\gamma_\mu Q(x)\,. \tag{2.15}$$

The current is defined in such a way that it contains no derivatives — an important fact allowing one to choose from the very beginning the origin, say, at the point y. This reference frame will be called below standard. Then

$$\begin{aligned}\Pi_{\mu\nu}(q) &= i\int e^{iqx}\,d^4x\langle T\{J_\mu(x)\,J_\nu(0)\}\rangle\\ &= i\int e^{iqx}\,d^4x\,\mathrm{Tr}\,\{\gamma_\mu S(x,0)\,\gamma_\nu S(0,x)\}\,.\end{aligned} \tag{2.16}$$

In this Chapter the symbol Tr means the trace operation with respect *to Lorentz and colour indices only.*

As is well known, if $m_Q \neq 0$ it is much more convenient to work in the momentum space. Since $S(x, 0)$ depends on a single variable, x, one can perform the following Fourier transformation

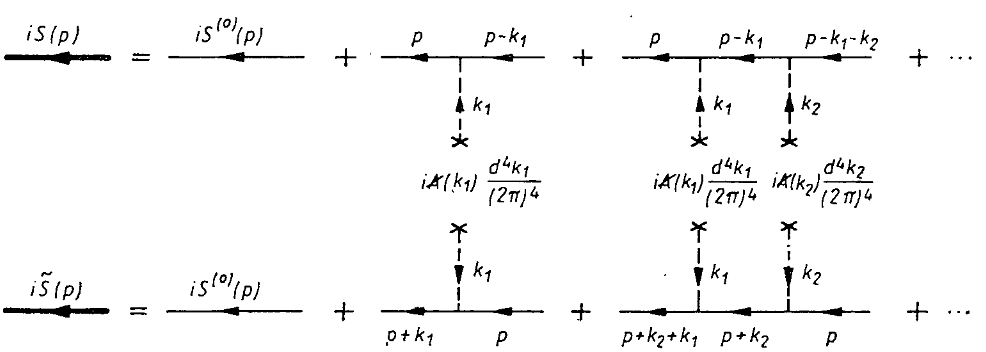
$$\begin{aligned}S(p) &= \int S(x,0)\,e^{ipx}\,d^4x,\\ \tilde{S}(p) &= \int S(0,x)\,e^{-ipx}\,d^4x,\end{aligned} \tag{2.17}$$

and in terms of these functions

$$\Pi_{\mu\nu}(q) = i\int \mathrm{Tr}\,\{\gamma_\mu S(p)\,\gamma_\nu\tilde{S}(p-q)\}\,\frac{d^4p}{(2\pi)^4} \tag{2.18}$$

(see fig. 3). Notice that $S(p)$ and $\tilde{S}(p)$, generally speaking, do not coincide. If it were not for this subtlety, the expression would be superficially just the same as for the loop of free quarks.

The general expansion (2.14) after Fourier transformation turns into the following graphic series



(2.19)

where $A_\mu(k)$ is the Fourier transform of the vacuum field,

$$A_\mu(k) = \int A_\mu(z)\, e^{ikz}\, d^4z = \frac{-i(2\pi)^4}{2}\, G_{\varrho\mu}(0)\, \frac{\partial}{\partial k_\varrho}\, \delta^{(4)}(k)$$
$$+ \frac{(-i)^2\,(2\pi)^4}{3}\, \big(D_\alpha G_{\varrho\mu}(0)\big)\, \frac{\partial^2}{\partial k_\varrho\, \partial k_\alpha}\, \delta^{(4)}(k) + \cdots. \tag{2.20}$$

Each gluon line is accompanied by integration. Since $A_\mu(k)$ reduces to derivatives of the delta function the integration is performed trivially. Actually, it is equivalent to differentiation of all propagators preceding the given vertex (in the case of $\tilde{S}(p)$) or following it (in the case of $S(p)$).

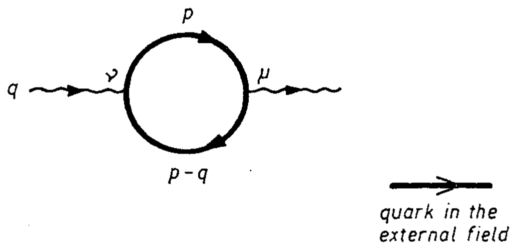


Fig. 3. Correlation function of quark currents in the external field

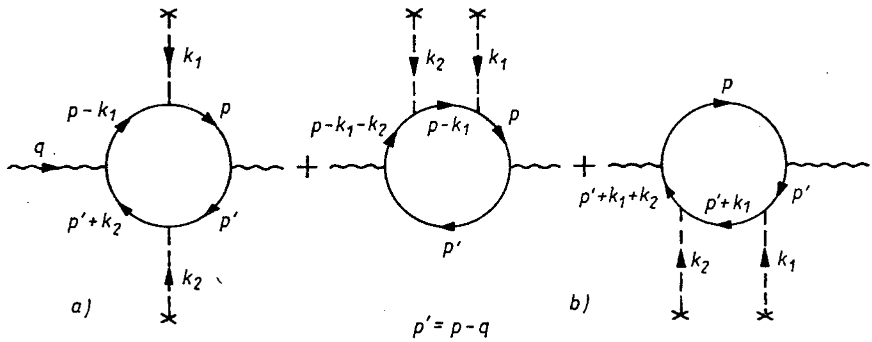


Fig. 4. G^2 correction in the correlation function of quark currents (heavy quarks)

To demonstrate in detail how the procedure works we will calculate, following ref. [6c], the G^2 correction on the two-point function (2.18). Operator G^2, evidently, can appear only in the diagrams where the external field $A(k)$ is inserted twice (fig. 4). Moreover, it is sufficient to keep only the first term of expansion for $A(k)$,

$$A_\alpha(k_1) = -\frac{i(2\pi)^4}{2}\, G_{\varrho\alpha}(0)\, \frac{\partial}{\partial k_{1\varrho}}\, \delta^{(4)}(k_1),$$

$$A_\beta(k_2) = -\frac{i(2\pi)^4}{2}\, G_{\sigma\beta}(0)\, \frac{\partial}{\partial k_{2\sigma}}\, \delta^{(4)}(k_2).$$

Using the fact that

$$\langle G^a_{\mu\nu}(0)\, G^b_{\alpha\beta}(0)\rangle = \frac{1}{96}\, \delta^{ab}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})\, \langle G^c_{\varrho\sigma}G^c_{\varrho\sigma}\rangle \tag{2.21}$$

and taking the colour trace we get

$$[\Pi_{\mu\nu}(q)]_a = -\frac{i}{96} \langle g^2 G^a_{\mu\nu} G^a_{\mu\nu} \rangle (g_{\varrho\sigma} g_{\alpha\beta} - g_{\varrho\beta} g_{\sigma\alpha}) \int \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial k_{1\varrho}} \frac{\partial}{\partial k_{2\sigma}} \mathrm{Tr}_L$$
$$\times \left\{ \gamma_\mu \frac{1}{\not p - m} \gamma_\alpha \frac{1}{\not p - \not k_1 - m} \gamma_\nu \frac{1}{\hat{\not p}' + \not k_2 - m} \gamma_\beta \frac{1}{\hat{\not p}' - m} \right\}_{k_1 = k_2 = 0},$$
$$[\Pi_{\mu\nu}(q)]_b = -\frac{i}{48} \langle g^2 G^a_{\mu\nu} G^a_{\mu\nu} \rangle (g_{\varrho\sigma} g_{\alpha\beta} - g_{\varrho\beta} g_{\sigma\alpha}) \int \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial k_{1\varrho}} \frac{\partial}{\partial k_{2\sigma}} \mathrm{Tr}_L$$
$$\times \left\{ \gamma_\mu \frac{1}{\not p - m} \gamma_\alpha \frac{1}{\not p - \not k_1 - m} \gamma_\beta \right.$$
$$\left. \times \frac{1}{\not p - \not k_1 - \not k_2 - m} \gamma_\nu \frac{1}{\not p' - m} \right\}_{k_1 = k_2 = 0},$$

where we have accounted for the fact that two diagrams of fig. 4b give equal contributions.

Due to current conservation the two-point function (2.18) is transversal,

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2),$$

and, therefore, with no loss of information one can contract indices μ and ν which results in further simplifications:

$$[\Pi_{\mu\mu}]_a = -i\langle g^2 G^2 \rangle \int \frac{d^4p}{(2\pi)^4} \frac{pp'}{(p^2 - m^2)^2 (p'^2 - m^2)^2}, \tag{2.22}$$

$$[\Pi_{\mu\mu}]_b = -i4m^2\langle g^2 G^2 \rangle \int \frac{d^4p}{(2\pi)^4} \frac{pp' - 2p^2}{(p^2 - m^2)^4 (p'^2 - m^2)}. \tag{2.23}$$

As a result, the problem is reduced to the simplest Feynman integrals. Their computation is performed by standard methods and yields.

$$\Pi^{(G)}_{\mu\nu}|_{\text{heavy quarks}, 1^-} = (q_\mu q_\nu - q^2 g_{\mu\nu}) \frac{1}{48} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{1}{Q^4}$$
$$\times \left\{ \frac{3(a+1)(a-1)^2}{a^2} \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a}-1} - \frac{3a^2 - 2a + 3}{a^2} \right\}, \tag{2.24}$$
$$a = 1 + (4m_Q^2/Q^2)$$

— the expression used in [1] in analysis of charmonium.

It is evident that the algorithm described above is easily generalized on gluon operators of higher dimension. Thus, in recent works [16, 21] coefficients in front of G^3 and G^4 in the two-point function (2.18) have been found by this technique. Certainly, the procedure is not less effective for other two-point functions. For instance, the G^2 correction in the correlation function of pseudoscalar currents $\bar{Q}i\gamma_5 Q$ is as follows

$$\Pi^{(G)}|_{\text{heavy quarks}, 0^-} = \frac{1}{48} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{1}{4m_Q^2}$$
$$\times \left\{ \frac{3(3a+1)(a-1)^2}{a^2} \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a}+1}{\sqrt{a}-1} - \frac{9a^2 + 4a + 3}{a^2} \right\}. \tag{2.25}$$

The careful reader, probably, has noticed that one of the simplifying devices is non-universal. We bear in mind the choice of the origin coinciding with coordinate of one of the currents. If the current contains derivatives, say, $\bar{Q}\vec{D}_\mu\gamma_5 Q$, it is impossible to put x or y equal to zero from the very beginning since the quark loop (in external field) is determined now by derivatives of the type $(\partial/\partial x_\alpha)\, S(x, y)$ and $(\partial/\partial y_\alpha)\, S(x, y)$. First of all, we must perform differentiation. This causes no difficulties, though. Instead of eq. (2.14), we have now

$$\begin{aligned}\frac{\partial}{\partial y_\alpha} iS(x, y) &= \frac{\partial}{\partial y_\alpha} iS^{(0)}(x-y) + g\int d^4z iS^{(0)}(x-z)\, i\not{A}(z)\, \frac{\partial}{\partial y_\alpha} iS^{(0)}(z-y) + \cdots \\ &= -\frac{\partial}{\partial x_\alpha} iS^{(0)}(x-y) + g\int d^4z iS^{(0)}(x-z)\, i\not{A}(z)\left(-\frac{\partial}{\partial z_\alpha}\right) \\ &\quad\times iS^{(0)}(z-y) + \cdots,\end{aligned} \tag{2.26}$$

and this last expression already permits $y = 0$. Moreover, Fourier transformation of eq. (2.26) is well-defined,

$$\begin{aligned}&\int d^4x e^{ipx}\left[\frac{\partial}{\partial y_\alpha} iS(x, y)\right]_{y=0} \\ &= ip_\alpha iS^{(0)}(p) + g\int iS^{(0)}(p)\, i\not{A}(k)\, \frac{d^4k}{(2\pi)^4}\, i(p-k)_\alpha\, iS^{(0)}(p-k) + \cdots;\end{aligned} \tag{2.27}$$

all further manipulations go along the lines sketched above.

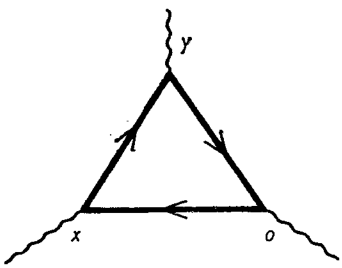


Fig. 5. Three-point function in the external field. The standard choice of the reference frame

Analysis of three-point functions would require additional, although rather trivial, modification of the algorithm. Even under the standard choice of the reference frame (fig. 5) in one of the propagators both coordinates are different from zero. We leave the problem of transition to the momentum space in this case to the reader as a simple exercise. The answer is illustrated in fig. 6. Let us especially call the reader's attention to the fact that momentum k_2 penetrated on the "foreign" line — the line corresponding to the first propagator, which never happens in two-point functions under the standard choice of the reference frame (i.e. $y = 0$).

Concluding this section let us make one more remark concerning the dependence on the quark mass. Varying this parameter — passing from heavy quarks to light ones — we eventually find ourselves in the domain $m \ll \mu$ where some diagrams become infrared — unstable even if all external momenta are euclidean and large. Consider, for instance, the graphs depicted in fig. 4b; the corresponding contribution to $\Pi_{\mu\mu}$ is given in eq. (2.23). At first sight it is proportional to m^2. One can easily convince oneself, however, that at $Q^2 \gg m^2$ characteristic virtual momenta in integral (2.23) $p^2 \sim m^2$ (and not $p^2 \sim Q^2$!), the integral is of order $1/m^2$, and the answer turns out to be of order m^0 — the situation not dangerous for heavy quarks since $p^2 \sim m^2 \gg \mu^2$ is the domain where calculations are reliable. On the contrary, for light quarks, $m^2 \ll \mu^2$, the result obtained in this way, evidently, has nothing to do with correct one. Indeed, when deriving eq. (2.23) we have assumed that the interval between consequent scatterings off the vacuum field is small, and in between the scatterings the quarks move as if they are free. If $p^2 \sim m^2 \ll \mu^2$ both assumptions become wrong. Propagation of the soft quark in these conditions is determined entirely by confinement dynamics. The graphs of the type of fig. 4b are absolutely senseless, and, instead of them, we draw the diagram of fig. 1 where the crosses mark soft lines. The corresponding effect is parametrized as $\langle \bar{u}u \rangle$ or $\langle \bar{d}d \rangle$.

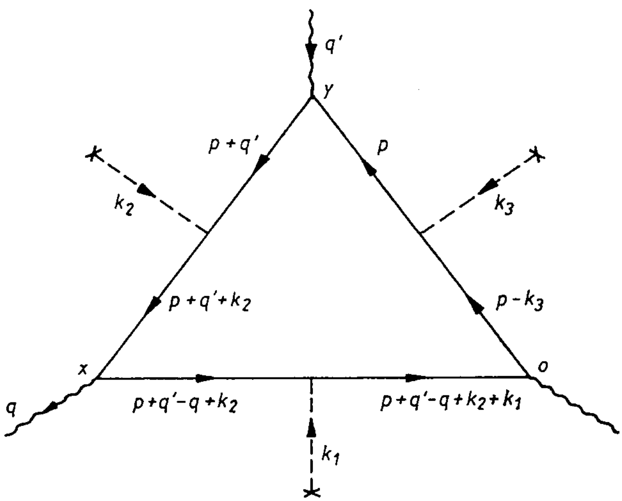


Fig. 6. One of the graphs determining G^3 correction to three-point function

Actually, the diagrams of fig. 4b represent an attempt to calculate quark condensate in terms of $\langle G^2 \rangle$, an attempt doomed to failure in the case at hand since $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ certainly do not reduce to a few rescatterings of a massless free quark. The following fact nicely demonstrates how far this model is from reality. According to eq. (2.23) the diagrams 4b give non-vanishing contribution to $\pi_{\mu\mu}(q)$ in the limit $m \to 0$. On the other hand, in terms of quark condensate the corresponding contribution is equal to $6m\langle \bar{\psi}\psi \rangle / Q^2$ (fig. 1). It tends to zero with $m \to 0$ provided that the vacuum expectation value $\langle \bar{\psi}\psi \rangle$ is non-singular in the chiral limit (and we known that this is indeed the case).

Thus, passing from heavy quarks to light ones it is impossible to mechanically put $m = 0$ in expressions for n-point functions. Such a procedure will generally speaking give a wrong answer for coefficients in front of the gluon operators. The defect can be eliminated by separating a piece associated with large distances and reparametrizing it in terms of quark operators.

This requires a few additional steps, actually quite superfluous, if one uses the virtues of the Fock-Schwinger gauge in full measure. The technique adequate for massless quarks, automatically yielding the correct operator expansion, is described below.

2.4. Massless Quarks. Gluon Operators

For $m \ll \mu$, when the quark mass can be treated perturbatively, the most economic procedure is as follows. First of all, practically up to the very last step there is no need in transition to the momentum space since the free quark propagator in the x representation looks just as simple as in the p representation,

$$S^{(0)}(x - y) = \frac{1}{2\pi^2} \frac{(\not x - \not y)}{\left((x - y)^2\right)^2}. \tag{2.28}$$

This simple observation essentially lowers the number of intermediate integrations. Moreover, there exists a compact expression for quark propagator in external field which, being once found, serves in all further computations.

To begin with we consider a problem most important in the practical respect — the G^2 correction — and shall return to discussion of other operators somewhat later.

The starting point is expansion (2.14) for quark propagator. The terms containing A^3, A^4 etc. will, evidently, give operators of higher dimension, 6 and higher. Therefore, concentrate on the first two non-trivial terms.

If we are interested only in the operator $G^a_{\mu\nu}G^a_{\mu\nu}$ then the external field (2.7) can be taken in the form

$$A_\mu(z) = \frac{1}{2} z_\varrho G_{\varrho\mu}(0). \tag{2.29}$$

Then iteration of this expression will give G^2. One can easily convince oneself that gluon operators with derivatives in eq. (2.7) cannot induce operator $G^a_{\mu\nu}G^a_{\mu\nu}$ in the correlation function of colourless currents.

Armed with this information we write

$$\begin{aligned} S(x, y) = S^{(0)}(x - y) &- \frac{1}{(2\pi^2)^2} \frac{1}{2} \int \frac{\not x - \not z}{(x - z)^4} z_\varrho G_{\varrho\mu}(0)\, \gamma_\mu \frac{\not z - \not y}{(z - y)^4} d^4z \\ &+ \frac{1}{(2\pi^2)^3} \frac{1}{2^2} \iint \frac{\not x - \not z'}{(x - z')^4} z_\varrho' G_{\varrho\mu}(0)\, \gamma_\mu \frac{\not z' - \not z}{(z' - z)^4} z_\alpha G_{\alpha\beta}(0) \\ &\times \gamma_\beta \frac{\not z - \not y}{(z - y)^4} d^4z\, d^4z' + \cdots, \end{aligned} \tag{2.30}$$

where the first integral is trivially calculable and generates G^2 in products of the type $S(x, y)\, S(y, x)$. The second integral contains G^2 from the very beginning and essentially simplifies after extracting the piece singlet with respect to Lorentz and colour indices (following eq. (2.21)). Just the singlet part determines corrections proportional to $\langle G^a_{\mu\nu}G^a_{\mu\nu}\rangle$.

It is easy to check that [23]

$$\begin{aligned} S(x, y) = \frac{1}{2\pi^2} \frac{\not r}{(r^2)^2} &- \frac{1}{8\pi^2} \frac{r_\alpha}{r^2} \tilde G_{\alpha\varphi}(0)\, \gamma_\varphi\gamma_5 \\ &+ \left\{\frac{i}{4\pi^2} \frac{\not r}{(r^2)^2} y_\varrho x_\mu G_{\varrho\mu}(0) - \frac{1}{192\pi^2} \frac{\not r}{(r^2)^2} \left(x^2y^2 - (xy)^2\right) G_{\varphi\chi}(0)\, G_{\varphi\chi}(0)\right\} \\ &+ \text{operators of higher dimension} \end{aligned} \tag{2.31}$$

where

$$r = x - y, \quad G_{\alpha\beta} \equiv \frac{g}{2} \lambda^a G^a_{\alpha\beta}, \quad \tilde{G}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} G_{\mu\nu}.$$

Notice that under the standard choice of the origin, $y = 0$, the expression in braces vanishes at all. Therefore, in two-point functions induced by currents with no derivatives[3]) the G^2 correction is determined entirely by the first two terms in eq. (2.31).

For instance, computation for the vector current of massless quarks, $j_\mu = \bar{q}\gamma_\mu q$, proceeds as follows. We start from the general expression (2.16) and substitute eq. (2.31) for the quark propagator. Then we obtain

$$\begin{aligned}
&\mathrm{Tr}\,\{\gamma_\mu S(x, 0)\,\gamma_\nu S(0, x)\} \\
&= \mathrm{Tr}\left\{\gamma_\mu \left(\frac{1}{2\pi^2} \frac{\not x}{x^4} - \frac{1}{8\pi^2} \frac{x_\alpha}{x^2} \tilde{G}_{\alpha\beta}\gamma_\beta\gamma_5\right) \gamma_\nu \left(-\frac{1}{2\pi^2} \frac{\not x}{x^4} + \frac{1}{8\pi^2} \frac{x_\gamma}{x^2} \tilde{G}_{\gamma\delta}\gamma_\delta\gamma_5\right)\right\} \\
&= -\frac{3}{\pi^4} \frac{2x_\mu x_\nu - x^2 g_{\mu\nu}}{x^8} - \frac{\langle g^2G^2\rangle}{384\pi^4} \frac{2x_\mu x_\nu + x^2 g_{\mu\nu}}{x^4},
\end{aligned} \tag{2.32}$$

where we have used also eq. (2.21). Having checked that the result (2.32) is transversal, as it should be, we contract indices μ and ν and go to the momentum space (the formulae collected in Appendix D will help us). Finally,

$$\Pi_{\mu\mu}(q)|_{\text{light quarks}, 1^-} = \frac{3}{4\pi^2} q^2 \ln Q^2 - \frac{1}{16\pi^2} \frac{1}{q^2} \langle g^2G^2\rangle, \tag{2.33}$$

which exactly coincides with the answer obtained first in ref. [1]. The whole computation takes no more than 10 minutes.

It is worth considering one more pedagogical example, only indirectly connected with the subject of the present review — the triangle anomaly in divergence of the axial current. Here the advantages of the proposed computational scheme are demonstrated from another point of view. If we regularize the axial current, following Schwinger, by virtue of ε splitting, then

$$\begin{aligned}
\partial_\mu j^{\text{reg}}_{\mu 5} &= \frac{\partial}{\partial x_\mu} \left\{\bar{\psi}(x + \varepsilon)\,\gamma_\mu\gamma_5 \left(\exp \int_{x-\varepsilon}^{x+\varepsilon} igA_\varrho(y)\,dy_\varrho\right) \psi(x - \varepsilon)\right\} \\
&= \bar{\psi}(x + \varepsilon)\,\{-ig\not{A}(x + \varepsilon)\,\gamma_5 - \gamma_5 ig\not{A}(x - \varepsilon) + ig\gamma_\mu\gamma_5\varepsilon_\beta G_{\mu\beta}(0)\}\;\psi(x - \varepsilon),
\end{aligned}$$

where the third term has emerged from differentiation of the exponential and, as usual, $A_\mu = 1/2\,\lambda^a A_\mu{}^a$.

This first step is more or less standard. (Notice, however, that we have used expansion (2.7) having substituted $A_\mu(y) = 1/2\,y_\varrho G_{\varrho\mu}(0)$). The further derivation is essentially shortened as compared to the normal route by use of expansion (2.31) for quark propagator in the Fock-Schwinger gauge. Only the second term of the expansion is operative. Namely,

$$\begin{aligned}
\partial_\mu j^{\text{reg}}_{\mu 5} &= -ig\,\mathrm{Tr}\,[-2i\varepsilon_\varrho G_{\varrho\mu}(0)\,\gamma_\mu\gamma_5 S(x - \varepsilon, x + \varepsilon)] \\
&= -\frac{g^2}{2} G^a_{\varrho\mu}(0)\,\tilde{G}^a_{\alpha\varphi}(0)\,\frac{\varepsilon_\varrho\varepsilon_\alpha}{\varepsilon^2}\,\frac{1}{8\pi^2}\,\mathrm{Tr}_L\,\gamma_\mu\gamma_5\gamma_\varphi\gamma_5 = \frac{\alpha_s}{4\pi} G^a_{\alpha\beta}\tilde{G}^a_{\alpha\beta}.
\end{aligned}$$

[3]) The expression in braces in eq. (2.31) is essential and cannot be omitted if one considers three-point functions provided that the current contain derivatives.

Of course, we have just rederived the well-known result but the whole business is the matter of a few minutes.

This short excursion seems to be sufficient for teaching the reader how he/she can calculate G^2 corrections in any correlation functions.

For currents with derivatives or in three-point Green functions, generally speaking, all four terms quoted in eq. (2.31) are operative. The procedure stays as simple and algorithmic as it was explained above; true, the number of elementary arithmetical operations somewhat increases.

There is no difficulty in extending the result to include gluon operators of higher dimension. Leaving aside all details we give here the complete expression for quark propagator (in p space) up to term $O(p^{-5})$ inclusively [6b]:

$$
\begin{aligned}
S(p) = \int e^{ipx}\, d^4x S(x,0) &= \frac{1}{\not p} - \frac{p_\alpha}{p^4}\, g\tilde{G}_{\alpha\beta}\gamma_\beta\gamma_5 \\
&+ \frac{2}{3}\, g\, \frac{1}{p^6}\left[p^2 D_\alpha G_{\alpha\beta}\gamma_\beta - \not p D_\alpha G_{\alpha\beta} p_\beta - p_\gamma D_\gamma p_\alpha G_{\alpha\beta}\gamma_\beta - 3ip_\gamma D_\gamma p_\alpha \tilde{G}_{\alpha\beta}\gamma_\beta\gamma_5\right] \\
&+ \frac{2}{p^8}\, g \Big[ip^2 p_\gamma D_\gamma D_\alpha G_{\alpha\beta}\gamma_\beta - i\not p p_\gamma D_\gamma D_\alpha G_{\alpha\beta} p_\beta - i(p_\gamma D_\gamma)^2\, p_\alpha G_{\alpha\beta}\gamma_\beta \\
&+ 2(p_\gamma D_\gamma)^2\, p_\alpha \tilde{G}_{\alpha\beta}\gamma_\beta\gamma_5 - \frac{1}{2}\, D_\gamma D_\gamma p^2 p_\alpha \tilde{G}_{\alpha\beta}\gamma_\beta\gamma_5 \Big] \\
&+ \frac{1}{p^8}\, g^2 [-2\not p p_\alpha G_{\alpha\beta} G_{\beta\gamma} p_\gamma + p^2 p_\alpha (G_{\alpha\beta} G_{\beta\delta} + G_{\beta\delta} G_{\alpha\beta})\, \gamma_\delta \\
&- p^2 p_\alpha (G_{\alpha\beta} G_{\beta\delta} - G_{\beta\delta} G_{\alpha\beta})\, \gamma_\delta] + \cdots. \qquad (2.34)
\end{aligned}
$$

It is assumed that all operators are taken at the origin, for instance, $G_{\alpha\beta} = 1/2\, \lambda^a G^a_{\alpha\beta}(0)$.

The coefficient of gluon operator of (normal) dimension d is proportional to $(p)^{-d+1}$. In closed loops only several first terms of expansion (2.34) are meaningful — namely, those which do not lead to infrared divergences under integration over p. If the integrals over p are saturated in the momentum domain $p^2 \sim q^2$ where q is a large external moment then the expansion in terms of gluon operators is legitimate. Passing to operators of higher dimension we, sooner or later, encounter with an integral with a power singularity at $p^2 = 0$. The corresponding quark line becomes soft and must be parametrized in terms of quark operators. In this case we refuse to integrate over p and say that the quark line is "cut" (fig. 1 and, in more detail, Sec. 2.5).

At what particular step of the expansion (2.34) does the infrared instability emerge? This depends on the form of the vertices. If there are no derivatives in the vertices the power infrared divergence appears for operators of dimension 6. Derivatives in the vertices shift the boundary towards higher dimensions. The general rule is universal — coefficient in the operator expansions are always determined by short distances. If the calculation becomes sensitive to large distances, this means that, instead of the coefficient, we are "calculating" the gluon matrix element of some operator missed at the previous step — the action contradicting our original intensions. Actually, a mistake. We should return back by a step and include the missed operator in the analysis.

And what if the divergence is logarithmic? In this case one should not hurry throwing away the result as irrelevant. The infrared logarithm does not mean a mistake. Log of q^2/μ^2 in the coefficient of the gluon operator simply signals that there is another (quark or gluon) operator of the same dimension present in the expansion, and the conventional logarithmic mixing between them takes place. Clearly, it must be accounted for. An example of such a situation is presented in fig. 7.

Concluding this Section we quote for the sake of reference the expression for gluon propagator [6b] up to terms $O(q^{-6})$:

$$
\begin{aligned}
D_{\mu\nu}(q) &= \int d^4x\, e^{iqx}\, D_{\mu\nu}(x, 0) \\
&= \frac{g_{\mu\nu}}{q^2} + g\,\frac{2}{q^4}\, G_{\mu\nu} + g\,\frac{4i}{q^6}\,(qD)\, G_{\mu\nu} - g\,\frac{2i}{3q^6}\, g_{\mu\nu} D_\alpha G_{\alpha\beta} q_\beta \\
&\quad + g\,\frac{2}{q^8}\,(qD)\, D_\alpha G_{\alpha\beta} q_\beta g_{\mu\nu} + g\,\frac{2}{q^8}\left(q^2 D^2 G_{\mu\nu} - 4(qD)^2\, G_{\mu\nu}\right) \\
&\quad + g^2\,\frac{1}{2q^8}\, g_{\mu\nu}\left(q^2 G^2_{\alpha\beta} - 4(q_\alpha G_{\alpha\beta})^2\right) + g^2\,\frac{4}{q^6}\, G_{\mu\alpha} G_{\alpha\nu},
\end{aligned}
\tag{2.35}
$$

where we use the following matrix notation

$$
G_{\mu\nu} \equiv G^{ab}_{\mu\nu}(0) = f^{acb} G^c_{\mu\nu}(0). \tag{2.36}
$$

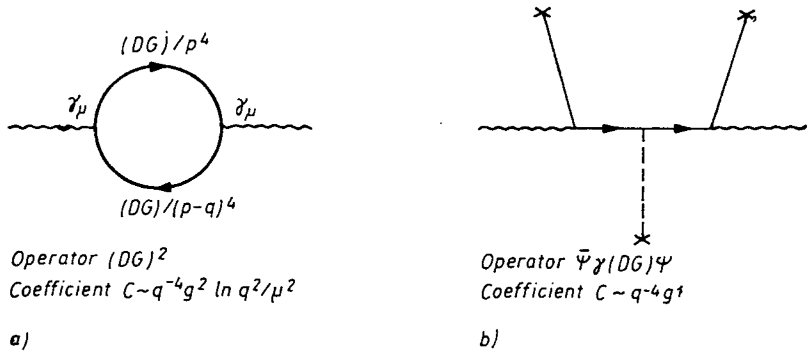


Fig. 7. Operators $(DG)^2$ and $\bar{\psi}\gamma(DG)\,\psi$ in the correlation function of vector currents. Diagram a gives an infrared logarithm reflecting mixing of the operators

Let us draw the reader's attention to one curious fact. In the quark propagator (2.34) it is impossible to separate the structure singlet with respect to Lorentz and colour indices, $G^a_{\mu\nu} G^a_{\mu\nu}$. Strictly speaking, this structure can be separated with zero coefficient as it is seen from eq. (2.31) with $y = 0$. At the same time, in gluon propagator it is explicitly present (see the last term in eq. (2.35). Not accidentally. Analytical properties of the propagators are such that $\overline{S(x, 0)}$ does not admit singularities of the type $\ln x^2$ while $\overline{D_{\mu\mu}(x, 0)}$ does admit it (here the bar denotes the Lorentz and colour averaging). One can easily prove the latter assertion following the line of reasoning of ref. [6a]. In this work a related problem was discussed: analytical properties of quark two-point functions in self-dual fields.

2.5. Quark Operators

Effects due to quark condensate are essential almost in all applications. They play the dominant role, for instance, in the sum rules for ϱ mesons, D mesons and, especially, baryons. As a rule, calculation of the corresponding coefficients is much simpler technically than for gluon operators. Almost always, after drawing the relevant graph, we immediately read off the answer. Still, even here there exist some useful devices which will be demonstrated below in a few examples.

As in other training exercises consider the correlation function (2.16) of vector currents, $j_\mu = \bar\psi\gamma_\mu\psi$, where ψ is the massless quark field. Let one of the quark line be soft. Then, cutting it (fig. 7), we get

$$\Pi_{\mu\nu}(q) = i\int e^{iqx}\, d^4x \langle T\{j_\mu(x)\, j_\nu(0)\}\rangle$$
$$\to -\int e^{iqx}\, d^4x\{\bar\psi(x)\,\gamma_\mu S(x,0)\,\gamma_\nu\psi(0) + \bar\psi(0)\,\gamma_\nu S(0,x)\,\gamma_\mu\psi(x)\}\,, \tag{2.37}$$

where the cut lines are represented by the Heisenberg operators $\psi(x)$ and $\psi(0)$.

For large q, to the leading order in q^{-1}, the field $\psi(x)$ can be substituted by $\psi(0)$. Accounting for the fact that $\langle\bar\psi_\alpha^i(0)\,\psi_{j\beta}(0)\rangle = 1/3\,\delta_j^i\; 1/4\,\delta_{\alpha\beta}$ we arrive at the following expression

$$\Pi_{\mu\nu}(q) = -\frac{1}{12}\langle\bar\psi\psi\rangle\,\mathrm{Tr}\,\{\gamma_\mu S(q)\,\gamma_\nu + \gamma_\nu\tilde S(-q)\,\gamma_\mu\}, \tag{2.38}$$

which vanishes if the quark mass is neglected. The zero result in the chiral limit is obvious beforehand since the structure $\bar\psi\psi$ is chirally non-invariant and appears in the correlation function (2.37) only being multiplied by the quark mass m. Thus, at least this general property is fulfilled and there is no error.

Feeling complete satisfaction from the internal selfconsistency we now put the problem of determining the coefficient in front of $m\bar\psi\psi$. For this problem the approximation $\psi(x) = \psi(0)$ in eq. (2.37) is not sufficient, and it is necessary to account for the next term of expansion in (2.9)

$$\psi(x) = \psi(0) + x_\varrho D_\varrho\psi(0)$$

bearing in mind that $D_\varrho\psi$ will eventually become $\not{D}\psi = -im\psi$.

It is no surprise that $x_\varrho D_\varrho\psi(0)$ induces a new term in $\Pi_{\mu\nu}(q)$, namely,

$$-\int e^{iqx}\, d^4x\{\bar\psi(0)\, x_\varrho\overleftarrow{D}_\varrho\gamma_\mu S(x,0)\,\gamma_\nu\psi(0) + \bar\psi(0)\,\gamma_\nu S(0,x)\,\gamma_\mu x_\varrho\overrightarrow{D}_\varrho\psi(0)\}\,.$$

Observing that

$$\langle\bar\psi_\alpha^i\overrightarrow{D}_\varrho\psi_{j\beta}\rangle = \frac{1}{3}\,\delta_j^i\,\frac{1}{16}\,(\gamma_\varrho)_{\beta\alpha}\langle\bar\psi\not{D}\psi\rangle = \frac{1}{48i}\,\delta_j^i(\gamma_\varrho)_{\beta\alpha}\, m\langle\bar\psi\psi\rangle$$

we reduce this extra term to

$$\frac{1}{48}\, m\langle\bar\psi\psi\rangle\,\frac{\partial}{\partial q_\varrho}\,\mathrm{Tr}\,\{-\gamma_\mu S(q)\,\gamma_\nu\gamma_\varrho + \gamma_\nu\tilde S(-q)\,\gamma_\mu\gamma_\varrho\}\,. \tag{2.39}$$

Combining now eqs. (2.38) and (2.39) we obtain

$$\Pi_{\mu\nu}(q) = 2m\langle\bar\psi\psi\rangle\left(\frac{q_\mu q_\nu}{q^4} - \frac{g_{\mu\nu}}{q^2}\right). \tag{2.40}$$

This result was exploited many times in ref. [1].

One more example referring to the operator of dimension 6,

$$\bar\psi\gamma_\nu\lambda^a\psi D_\mu G^a_{\mu\nu}\,,$$

appearing in eq. (2.16) already in the chiral limit. Correction to $\Pi_{\mu\mu}(q)$ associated with this operator is of order q^{-4}. The corresponding diagram is depicted in fig. 7b. The consistent procedure requires, generally speaking, expanding $\psi(x)$ up to terms $O(x^3)$, so that in $\Pi_{\mu\mu}(q)$ we deal with three different structures,

$$\Pi_{\mu\mu} \to -2 \int e^{iqx}\, d^4x \bar{\psi}(0) \{\gamma_\mu S^{(3)}(x, 0)\, \gamma_\mu + \tilde{D}_\varrho \gamma_\mu S^{(2)}(x, 0)\, \gamma_\mu x_\varrho + \frac{1}{6} \tilde{D}_\varrho \tilde{D}_\gamma \tilde{D}_\delta \gamma_\mu S^{(0)}(x, 0)\, \gamma_\mu x_\varrho x_\gamma x_\delta\}\, \psi(0)\,, \tag{2.41}$$

where it is taken into account that both terms in (2.37) give one and the same contribution, and the subscript (0, 2, 3) marks the dimension of gluon operator in $S(x, 0)$ (for instance, $S^{(0)} = \text{const}\, \not{x}/x^4$ is the free quark propagator, $S^{(2)} = \text{const}\, x_\varrho \tilde{G}_{\varrho\varphi} \gamma_\varphi \gamma_5$, etc.).

Averaging over the vacuum state — and this is equivalent to separation of Lorentz-singlet operators — we convince ourselves that the second and the third terms in (2.41) are independent of x at all. Hence their Fourier transformation is proportional to $\delta^{(4)}(q)$, and one can forget about these pieces. We are left with the first term,

$$\Pi_{\mu\mu} = -2\bar{\psi}(0)\, \gamma_\mu S^{(3)}(q)\, \gamma_\mu \psi(0)\,,$$

and will now again taste of the fruits of the work done beforehand. We bear in mind expansion (2.34) for $S(q)$ which implies

$$\Pi_{\mu\mu} = (-2)\, (-2)\, \frac{2}{3} \left(1 - \frac{1}{4} - \frac{1}{4}\right) g\langle \bar{\psi} (D_\alpha G_{\alpha\beta})\, \gamma_\beta \psi\rangle\, q^{-4} = \frac{2}{3}\, \frac{1}{q^4}\, g\langle \bar{\psi} \gamma_\beta \lambda^a \psi D_\alpha G^a_{\alpha\beta}\rangle\,. \tag{2.42}$$

The factor (-2) reflects the convolution of γ matrices, $\gamma_\mu(\ldots)\, \gamma_\mu$, the factor $(1 - 1/4 - 1/4)$ is a remant of three structures of dimension 3 which are averaged with respect to Lorentz indices in different ways.

2.6. Correlation Functions of Gluon Currents

We proceed in the same vein as with massless quarks. There is some subtlety, though, due to the fact that the four-potential A_μ plays a two-fold role,

$$A_\mu{}^a = (A_\mu{}^a)_{\text{ext}} + a_\mu{}^a, \tag{2.43}$$

where $(A_\mu{}^a)_{\text{ext}}$ is the vacuum field while $a_\mu{}^a$ describes deviations from the vacuum field. In other words, $a_\mu{}^a$ is a quantum fluctuation; gluon propagator in external field is defined in the following way:

$$D^{ab}_{\mu\nu} = \langle T\{a_\mu{}^a(x),\, a_\nu{}^b(0)\}\rangle\,. \tag{2.44}$$

The Fock-Schwinger gauge condition (2.2) is nonlocal in the momentum space, and one should work very hardly in order to write Green function $D^a_{\mu\nu}$ satisfying this condition. Even having overcome all difficulties one would certainly get an expression useless from the practical point of view. However, there is absolutely no need in this work. Indeed, let us assume that the Fock-Schwinger condition is imposed on external field

$$x_\mu \big(A_\mu{}^a(x)\big)_{\text{ext}} = 0\,. \tag{2.45}$$

Substituting eq. (2.43) into standard QCD Lagrangian we get

$$L = -\frac{1}{4}(G^a_{\mu\nu}G^a_{\mu\nu})_{\text{ext}} - \frac{1}{2}(D_\mu{}^{\text{ext}}a_\nu{}^a)^2 + \frac{1}{2}(D_\mu{}^{\text{ext}}a_\nu{}^a)(D_\nu{}^{\text{ext}}a_\mu{}^a) + \frac{1}{2}g a_\mu{}^a(G^b_{\mu\nu})_{\text{ext}}\, a_\nu{}^c f^{abc} + \cdots,$$
$$D_\mu{}^{\text{ext}}a_\nu{}^a = \partial_\mu a_\nu{}^a + g f^{abc}(A_\mu{}^b)_{\text{ext}}\, a_\nu{}^c\,. \tag{2.46}$$

This Lagrangian describes dynamics of field $a_\mu{}^a$ in the background field $(A_\mu{}^a)_{\text{ext}}$ and still possesses gauge freedom; namely it is invariant under the transformations

$$\big(A_\mu(x)\big)_{\text{ext}} \to (A_\mu{}')_{\text{ext}} = U^{-1}(x)\,\big(A_\mu(x)\big)_{\text{ext}}\, U(x)\,,$$
$$a_\mu(x) \to a_\mu{}' = U^{-1}(x)\, a_\mu(x)\, U(x) + \frac{i}{g}\, U^{-1}(x)\, \partial_\mu U(x)\,,$$

with arbitrary unitary matrix $U(x)$. It is evident that the first of these transformations does not violate condition (2.45): $x_\mu\big(A_\mu{}'(x)\big)_{\text{ext}}$ automatically vanishes provided $x_\mu\big(A_\mu(x)\big)_{\text{ext}} = 0$.

To extract the maximal profit from this gauge freedom we fix the gauge of a_μ as follows. We add to Lagrangian (2.46) the term

$$-\frac{1}{2}(D_\mu{}^{\text{ext}}a_\mu{}^a)^2 \tag{2.47}$$

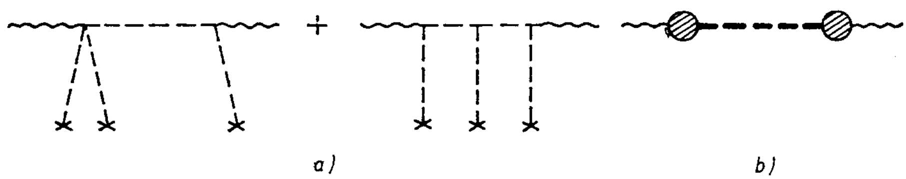


Fig. 8. Operator G^3 in the two-point function of gluon currents. Crosses mark soft lines annihilated by the vacuum (a); b — symbolic picture of the same two-point function in the external field. Gluon propagator (▬ ▬ ▬ ▬) and vertices (circles) are taken in the external field

and, certainly, the ghosts according to standard rules. This is the so called background gauge extensively exploited by de Witt [24], 't Hooft [18] and in many recent works, both purely theoretical, and of more pragmatic character [25—27]. Gluon propagator, eq. (2.35), is given just in this gauge.

The exercise we concentrate on in this Section is the calculation of the G^3 correction in the two-point function of scalar gluonic currents,

$$j_s = \alpha_s G^a_{\mu\nu} G^a_{\mu\nu}\,. \tag{2.48}$$

The coefficient of G^3 to the leading in α_s approximation is determined by the Born graphs of fig. 8a. The same diagrams in a symbolical form are depicted in fig. 8b; the external vacuum field is not shown here but it is implied that the gluon transferring the large momentum q propagates in a background field.

The next step is rewriting current (2.47) in terms of the background and quantum fields; as it is seen from fig. 8b it is sufficient to keep only linear in a_μ^a terms,

$$j_s(x) = 4\alpha_s (G^a_{\mu\nu})_{\text{ext}} \left(D_\mu^{\text{ext}} a_\nu^a(x)\right). \tag{2.49}$$

$O(G^3)$ terms arise in the current correlation function from two sources: from expansion of $G_{\mu\nu}(x)$ and from correction to the gluon propagator:

$$\begin{aligned} &i \int d^4x e^{iqx} \langle T\{j_s(x), j_s(0)\}\rangle \\ &\to 16\alpha_s^2 i \int d^4x e^{iqx} \left(G^a_{\mu\nu}(0) \frac{1}{2} \bar{D}_\alpha \bar{D}_\beta x_\alpha x_\beta \langle T\{\partial_\mu a_\nu(x), \partial_\gamma a_\delta(0)\}\rangle^{(0)}_{ab} G^b_{\gamma\delta}(0)\right. \\ &\quad \left. + G^a_{\mu\nu}(0) \langle T\{D_\mu a_\nu(x), D_\alpha a_\beta(0)\}\rangle^{(2)}_{ab} G^b_{\alpha\beta}(0)\right) \end{aligned} \tag{2.50}$$

where subscripts (0, 2) refer to dimension of gluon operators and subscript (ext) is omitted. In particular, (0) marks the free propagator

$$\langle T\{\partial_\mu a_\nu(x), \partial_\gamma a_\delta(0)\rangle^{(0)}_{ab} = \frac{1}{4\pi^2} \frac{\partial}{\partial x_\mu} \left(-\frac{\partial}{\partial x_\gamma}\right) \frac{1}{x^2} g_{\nu\delta} \delta_{ab}\,.$$

One can easily convince oneself that after separating Lorentz-singlet structure the first term in eq. (2.50) vanishes. The second term, in turn, reduces to

$$\begin{aligned} &16\alpha_s^2 i \int d^4x e^{iqx} G^a_{\mu\nu}(0) \langle T\{(\partial_\mu + gA_\mu) a_\nu(x), \partial_\alpha a_\beta(0)\}\rangle^{(2)}_{ab} G^b_{\alpha\beta}(0) \\ &= 16\alpha_s^2 i \int d^4x e^{iqx} G^a_{\mu\nu}(0) \left\langle T\left\{\left(-iq_\mu + g \frac{1}{2} x_\varrho G_{\varrho\mu}(0)\right) a_\nu(x), \partial_\alpha a_\beta(0)\right\}\right\rangle^{(2)}_{ab} G^b_{\alpha\beta}(0)\,, \end{aligned} \tag{2.51}$$

where, as in eq. (2.35), we use the matrix notation $A_\mu = f^{abc} A_\mu^b$ and account for the fact that $A_\mu(0) = 0$.

Unfortunately, it is impossible to substitute simply propagator (2.35) in eq. (2.51) since expansion (2.35) refers actually to $D_{\mu\nu}(x, 0)$. At first, we must write $D_{\mu\nu}(x, y)$, differentiate over y and only then put $y = 0$. In gluon propagator we are interested only in terms proportional to G since eq. (2.51) already contains G^2 explicitly. To this order modifications which are needed in (2.35) are trivial,

$$\begin{aligned} \int e^{iqx}\, dx D_{\mu\nu}(x, y) &= \int i e^{iqx}\, d^4x \langle T\{a_\mu^a(x)\, a_\nu^b(y)\}\rangle \\ &= \frac{g_{\mu\nu}}{q^2} e^{iqy} + \frac{2g}{q^4} G_{\mu\nu}(0)\, e^{iqy} - giy_\varphi G_{\varphi\varrho}(0) \frac{q_\varrho}{q^4} e^{iqy} g_{\mu\nu} + \cdots. \end{aligned} \tag{2.52}$$

The reader will easily derive this formula himself (herself) starting from Lagrangian (2.46).

Combining eqs. (2.52) and (2.51) we obtain

$$i \int e^{iqx}\, d^4x \langle T\{j_s(x)\, j_s(0)\}\rangle = -8\alpha_s^2 \frac{g}{q^2} \langle G^a_{\mu\nu} G^b_{\nu\beta} G^c_{\beta\mu} f^{abc}\rangle + \text{other operators}. \tag{2.53}$$

We have done above a series of exercises in operator expansion. The coefficients of several operators have been found. The material is organized in such a way that the main emphasis is put on routine computations. Performing them from the very beginning up to the very end we do not try to hide the theoretical "kitchen". On the other hand, we do not seek to carry the problems through, up to physical applications and concentrate on purely computational aspects, i.e. those aspects which are presented in original papers only by a few lines (if at all).

The reasons which compelled us to deviate from the settled tradition reading "the way of deriving formulae is a question of no importance" is as follows. First of all, it is impossible to find technical details elsewhere. At best, there are short comments explaining the points with peculiarities, deviations from the "standard". Here we clarify the standard itself, demonstrate the consistent procedure which proves to be extremely effective in all known cases.

Experience shows that the reader who has not got used to the language of operator expansion usually encounters with some difficulties in attempts to obtain this or that coefficient, although, in principle, no additional knowledge is needed except for the Feynman diagram technique. True, the diagrams look somewhat unusually — some lines are cut and "inserted" in vacuum — and this fact seems to discourage. Certainly, not for a long time. As a rule one quickly succeeds in adapting the Feynman diagrammar. Unfortunately, this language is not the most economic one in the case at hand. The Schwinger approach, much more convenient in this range of problems, is much less known, and, therefore, the main aim which we pursued in this review is pedagogical. In correspondence with this are the most typical examples chosen above, and they are supplied with rather lengthy explanations.

Appendix A

In this appendix the results for a few successive commutators are collected for the sake of reference:

$$[P^2, P_\mu] = -2igG_{\mu\alpha}P_\alpha;$$

$$\big[P^2[P^2, P_\mu]\big] = -4g^2G_{\mu\beta}G_{\beta\alpha}P_\alpha + 2igD^2G_{\mu\alpha}P_\alpha - 4gD_\beta G_{\alpha\mu}P_\beta P_\alpha;$$

$$\begin{aligned}\big[P^2[P^2[P^2, P_\mu]]\big] = {} & -8igD_\gamma D_\beta G_{\alpha\mu}P_\gamma P_\beta P_\alpha \\ & + \big(-8ig^2D_\beta(G_{\mu\gamma}G_{\gamma\alpha}) - 4gD_\beta D^2G_{\mu\alpha} + 4gD^2D_\beta G_{\mu\alpha} \\ & + 8ig^2D_\gamma G_{\alpha\mu}G_{\gamma\beta} + 8ig^2(D_\beta G_{\gamma\mu})\,G_{\gamma\alpha}\big)\,P_\beta P_\alpha + \big(4g^2D^2(G_{\mu\beta}G_{\beta\alpha}) \\ & + 8ig^3G_{\mu\beta}G_{\beta\gamma}G_{\gamma\alpha} - 2igD^4G_{\mu\alpha} + 4g^2(D^2G_{\mu\gamma})\,G_{\gamma\alpha} \\ & - 8g^2D_\beta G_{\gamma\mu}D_\beta G_{\gamma\alpha}\big)\,P_\alpha.\end{aligned}$$

In the following two commutators we omit terms containing the field operators of dimension higher than 6:

$$\begin{aligned}\big[P^2[P^2[P^2[P^2, P_\mu]]]\big] = {} & 16gD_\delta D_\gamma D_\beta G_{\alpha\mu}P_\delta P_\gamma P_\beta P_\alpha \\ & + 8igD^2D_\gamma D_\beta G_{\alpha\mu}P_\gamma P_\beta P_\alpha - 16g^2D_\gamma D_\beta G_{\alpha\mu}(G_{\gamma\delta}P_\delta P_\beta P_\alpha \\ & + P_\gamma G_{\beta\delta}P_\delta P_\alpha + P_\gamma P_\beta G_{\alpha\delta}P_\delta) + \big(16g^2D_\delta D_\beta(G_{\mu\gamma}G_{\gamma\alpha}) \\ & - 8igD_\delta D_\beta D^2G_{\mu\alpha} + 8igD_\delta D^2D_\beta G_{\alpha\mu} - 16g^2D_\delta(D_\gamma G_{\alpha\mu}G_{\gamma\mu}) \\ & - 16g^2D_\delta\big((D_\beta G_{\gamma\mu})\,G_{\gamma\alpha}\big)\big)\,P_\delta P_\beta P_\alpha + \cdots;\end{aligned}$$

$$\Big[P^2\big[P^2[P^2[P^2[P^2, P_\mu]]]\big]\Big] = 32igD_\varrho D_\delta D_\gamma D_\beta G_{\alpha\mu}P_\varrho P_\delta P_\beta P_\alpha + \cdots.$$

Let us give also one more set of commutators

$$[P^2, \sigma G] = -D^2\sigma G + 2iD_\alpha\sigma GP_\alpha;$$

$$[P^2, [P^2, \sigma G]] = D^4\sigma G - 2iD_\alpha D^2\sigma GP_\alpha + 4gD_\beta\sigma GG_{\beta\alpha}P_\alpha - 2iD^2D_\alpha\sigma GP_\alpha - 4D_\beta D_\alpha\sigma GP_\beta P_\alpha;$$

$$[P^2[P^2[P^2, \sigma G]]] = -8iD_\gamma D_\beta D_\alpha\sigma GP_\gamma P_\beta P_\alpha + \cdots.$$

Appendix B

In this Appendix we present the expression for the effective Lagrangian in the instanton background field

$$G^a_{\mu\nu} = -\frac{4}{g}\frac{\eta_{a\mu\nu}\varrho^2}{[(x-x_0)^2+\varrho^2]^2},$$

where x_0 stands for the instanton center, ϱ is its size and $\eta_{a\mu\nu}$ are the 'T HOOFT symbols [18]. Straightforward calculations lead to the following expressions for invariants of the field:

$$\mathrm{Tr}\, G_{\mu\nu}G_{\mu\nu} = \left(\frac{4}{g}\right)^2\pi^2;$$

$$\mathrm{Tr}\, G_{\mu\alpha}G_{\alpha\beta}G_{\beta\mu} = \frac{3i}{2}\left(\frac{4}{g}\right)^3\frac{\pi^2}{5}\frac{1}{\varrho^2};$$

$$\mathrm{Tr}\,(G_{\mu\nu}G_{\mu\nu})^2 = 3\left(\frac{4}{g}\right)^4\frac{\pi^2}{7}\frac{1}{\varrho^4};$$

$$\mathrm{Tr}\,\{G_{\mu\nu}G_{\nu\alpha}\}_+{}^2 = 3\left(\frac{4}{g}\right)^4\frac{\pi^2}{7}\frac{1}{\varrho^4};$$

$$\mathrm{Tr}\,[G_{\mu\nu}G_{\nu\alpha}]_-{}^2 = -4\left(\frac{4}{g}\right)^4\frac{\pi^2}{7}\frac{1}{\varrho^4};$$

$$\mathrm{Tr}\,[G_{\mu\nu}G_{\alpha\beta}]_-{}^2 = -8\left(\frac{4}{g}\right)^4\frac{\pi^2}{7}\frac{1}{\varrho^4},$$

where integration over d^4x is included in Tr. As a result, the effective lagrangian turns out to be

$$L_{\mathrm{eff}}(\varrho, x_0) = \frac{d\varrho}{\varrho^5}\, d(\varrho)\left\{-\frac{1}{3}\ln\frac{M_R{}^2}{m^2} - \frac{4}{75}\frac{1}{(m\varrho)^2} - \frac{4\cdot 17}{3\cdot 5\cdot 7^2}\frac{1}{(m\varrho)^4} + \cdots\right\}$$

where $d(\varrho)$ is the density of instantons with size ϱ.

Appendix C

The Borel Transformation

Physically observed quantities are connected with n-point functions in Euclidean domain by dispersion relations. After the Borel transformation the latter convert into the sum rules with exponential weight which much more effectively fix parameters of low-lying resonances. Here we give some useful definitions and relations.

Consider an arbitrary function $f(x)$. The Borel transformation of $f(x)$ looks as follows:

$$\tilde{f}(\lambda) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} e^{\lambda/x} f(x)\, x d(1/x), \tag{C.1}$$

where the integration contour runs to the right of all singularities of $f(x)$. The inverse transformation is

$$f(x) = \int\limits_0^\infty \tilde{f}(\lambda)\, e^{-\lambda/x}\, d\lambda/x. \tag{C.2}$$

The meaning of the procedure is simple. If $f(x)$ is given in the form of a series in x (may be, asymptotic one),

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots \tag{C.3}$$

then the corresponding expansion for $\tilde{f}(\lambda)$

$$\tilde{f}(\lambda) = a_0 + \frac{a_1\lambda}{1!} + \frac{a_2\lambda^2}{2!} + \cdots + \frac{a_k\lambda^k}{k!} + \cdots$$

contains coefficients factorially suppressed with respect to (C.3). Convergence of the series improves. In order to emphasize the fact we often refer to the procedure as to the Borel improvement.

The general definition (C.1) — the integral transformation in the complex plane — is rather inconvenient when one deals with n-point functions known in the form of (asymptotic) expansion in Q^{-2}. In ref. [1] an equivalent differential operator was introduced. Now we denote it by symbol $\hat{B}$. For analytical functions with normal dispersion representation $\hat{B}$ looks as follows

$$\hat{B}\Pi(Q^2) = \lim_{\substack{Q^2, n\to\infty \\ Q^2/n = M^2 = \text{const}}} \frac{1}{(n-1)!} (Q^2)^n \left(-\frac{d}{dQ^2}\right)^n \Pi(Q^2), \tag{C.4}$$

where we have taken a two-point function $\Pi(Q^2)$ as an example. In principle, the definition can be generalized to include n-point functions (see [28]).

If $\Pi(Q^2)$ possesses normal analytical properties $\Pi(Q^2) = 1/\pi \int \operatorname{Im} \Pi(s)\, ds/(s + Q^2)$ + possible subtractions then functions $\tilde{\Pi}(M^2) = \hat{B}\Pi(Q^2)$ and $\{Q^2(-d/dQ^2)\, \Pi(Q^2)\}$ are connected with each other by the Borel transformation (C.1), and variable $1/Q^2$ plays the role of x [1].

It is instructive to give here a table of examples showing the action of operator $\hat{B}$ on different functions encountered with in practice. These examples make explicit the factorial suppression of series in $1/Q^2$.

It is easy to check that

$$\hat{B}\left(\frac{1}{Q^2}\right)^k = \frac{1}{(k-1)!}\left(\frac{1}{M^2}\right)^k. \tag{C.5}$$

Moreover, for positive integer k

$$\hat{B}(Q^2)^k = 0.$$

One more transformation figuring in the sum rules refers to expressions of the type $(Q^2)^k \ln Q^2$. Applying (C.4) we get

$$\hat{B}(Q^2)^k \ln Q^2 = (-1)^{k+1} \Gamma(k+1) (M^2)^k. \tag{C.6}$$

A logarithmic Q^2 dependence sometimes appears due to anomalous dimensions of operators, and then the coefficients in the operator expansion are typically as follows

$$(1/Q^2)^k (\ln Q^2/\mu^2)^{-\varepsilon} \tag{C.7}$$

where ε, generally speaking, is a non-integer number. What is the Borel transform of (C.7)? To the leading order in $1/(\ln Q^2/\mu^2)$ one can easily answer this question if one uses the following auxiliary relation:

$$\left(\frac{1}{Q^2}\right)^k (\ln Q^2/\mu^2)^{-\varepsilon} = \frac{1}{\varepsilon\Gamma(\varepsilon)} \frac{1}{(Q^2)^k} \int\limits_0^\infty dz \exp\left(-z^{1/\varepsilon} \ln \frac{Q^2}{\mu^2}\right). \tag{C.8}$$

Applying operator $\hat{B}$ and using (C.8) we get

$$\hat{B}\left[\left(\frac{1}{Q^2}\right)^k \left(\ln \frac{Q^2}{\mu^2}\right)^{-\varepsilon}\right] = \frac{1}{\Gamma(k)} \left(\frac{1}{M^2}\right)^k \left(\ln \frac{M^2}{\mu^2}\right)^{-\varepsilon} \left[1 + O\left(\left(\ln \frac{M^2}{\mu^2}\right)^{-1}\right)\right]. \tag{C.9}$$

As a rule there is no need in better accuracy in applications since the coefficients of operator expansion are usually calculated to the leading order in α_s. The exact Borel transform of (C.7) is known in the literature, though (see [29]).

The authors of several works [30, 31] deal with hadrons contaning heavy quarks. It turned out that the original sum rules [1] have a parallel in nonrelativistic limit. In particular, the non-relativistic analogue of the operator (C.4) is

$$(\hat{B})_{\text{non-rel}} = \lim_{\substack{E\to\infty, n\to\infty \\ E/n=\varepsilon=\text{const}}} \frac{1}{(n-1)!} E^n \left(-\frac{d}{dE}\right)^n, \tag{C.10}$$

where is the energy measured from the threshold. The physical meaning of the Borel transformation here is even more transparant than within the relativistic approach. Proceeding from E to ε we pass, in essence, from Green function at a given energy to the time Green function (or, which is the same, the evolution operator) and $(-i/\varepsilon)$ plays the role of time [30, 32].

Appendix D

The Fourier Transformation

Here we give a collection of formulae allowing one to proceed quickly from the coordinate to momentum space in the correlation function of currents.

Throughout the review we adopt the following standard definition of the Fourier transformation

$$\begin{aligned} F(p) &= \int d^4x e^{ipx} F(x), \\ F(x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} F(p). \end{aligned} \tag{D.1}$$

The most frequently encountered transformations are well-known:

$$\frac{1}{p^2} \leftrightarrow \frac{i}{4\pi^2}\frac{1}{x^2}, \quad \frac{p_\mu}{p^2} \leftrightarrow \frac{1}{2\pi^2}\frac{x_\mu}{x^4}, \quad \frac{p_\mu}{p^4} \leftrightarrow \frac{1}{8\pi^2}\frac{x_\mu}{x^2}. \tag{D.2}$$

In analysing the sum rules we deal with more general functions

$$\int \frac{d^4x}{(x^2)^n} e^{ipx}, \tag{D.3}$$

where n are integer numbers, $n \geqq 4$. Integrals of this type are divergent in the ultraviolet domain and have the structure const $(p^2)^{n-2} \ln(-p^2) + P_{n-2}(p^2)$ where P_{n-2} is a polynomial of power $n-2$. The coefficients of this polynomial are ill-defined (infinite). However, *the constant in front of the logarithmic term is finite and fixed unambiguously*. The presence of the divergent polynomial never affects analysis of the sum rules. It has no singularities in p^2 and disappears after borelization (see Appendix C). In all subsequent formulae we write out only logarithms and consistently suppress subtraction polynomials.

Integral (D.3) can be done by virtue of analytic continuation in dimensionality of space — this method is most economic.

Start from

$$A_D(p^2) = \int d^Dx e^{ipx} \frac{1}{(x^2)^n} \xrightarrow{\text{Eucl.}} (-i)(-1)^n \int d^Dx e^{-ipx} \frac{1}{(x^2)^n},$$

where we have proceeded to the Euclidean space. Integrating over angles yields

$$\begin{aligned} A_D(p^2) &= -i(-1)^n (2)^{D/2} (\pi)^{D/2} |p|^{1-D/2} \int |x|^{(D/2)-2n} d|x| \, J_{(D/2)-1}(|p|\,|x|) \\ &= (-i)(-1)^n 2^{D-2n} \pi^{D/2} |p|^{2n-D} \frac{\Gamma\left(\frac{D}{2} - n\right)}{\Gamma(n)}. \end{aligned} \tag{D.4}$$

If $D \to 4$ this expression is singular — it has a pole — the corresponding residue, however, is proportional to $(p^2)^{n-2}$. As was already noted, any polynomial in p^2 is inessential and can be omitted. The non-singular part of eq. (D.4) evidently contains $\ln p^2$. Separating the nonsingular part and returning to the Minkowski space we have

$$\int \frac{d^4x}{(x^2)^n} e^{ipx} = \frac{i(-1)^n 2^{4-2n}\pi^2}{\Gamma(n-1)\,\Gamma(n)} (p^2)^{n-2} \ln(-p^2). \tag{D.5}$$

This expression generates a series of other useful relations, for instance

$$\begin{aligned} I_{n,l} &\equiv \int d^4x e^{ipx} \frac{(x\xi)^n (x\eta)^n}{(x^2)^{2n+l}} \\ &= \frac{i\pi^2 2^{4-3n-2l} n!}{(2n+l-1)!\,(n+l-2)!} (\xi\eta)^n (-p^2)^{n+l-2} \ln(-p^2), \end{aligned} \tag{D.6}$$

where ξ, η are auxiliary vectors,

$$\xi^2 = \eta^2 = 0, \qquad \xi p = \eta p = 0, \qquad \xi\eta \neq 0.$$

We encounter with transformation (D.6) in the problem of mesons with spin n. Its derivation is given in ref. [23].

The key observation on which the derivation is based is the following

$$I_{n,l} = \frac{(n!)^2}{(2n)!} \int d^4x e^{ipx} \frac{\left(x(\xi + \eta)\right)^{2n}}{(x^2)^{2n+l}},$$

while integral

$$\int d^4x e^{i(p+\xi+\eta)x} \frac{1}{(x^2)^{2n+l}}$$

is already known.

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