QCD Sum Rules for Parameters of the B-meson Distribution Amplitudes

Muslem Rahimi, Marcel Wald

University of Siegen

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Introduction

- Light-cone distribution amplitudes (LCDA's) are important to study exclusive B-meson decays such as $B \to \pi\pi$ or $B \to \pi K$.
- Parametrize matrix element of nonlocal heavy-light currents separated along the light-cone in heavy-quark effective field theory.
- LCDA's appear in factorization theorems such as QCD factorization:
  - E.g. $B \to \gamma \ell \nu$, where $E_\gamma \gg \Lambda_{\text{QCD}} \Rightarrow$ Probe light-cone structure of B-meson and study inverse moment λ_B.
- Three-particle LCDA's with local quark operators can be expressed with $\lambda_{E,H}^2$ [Grozin, Neubert; 96]:

$$\langle \omega^n \rangle = \int \mathrm{d}\omega \, \phi_\pm(\omega) \, \omega^n \,, \tag{1}$$

$$\langle \omega^2 \rangle_+ = 2\bar{\Lambda}^2 + \frac{2}{3}\lambda_E^2 + \frac{1}{3}\lambda_H^2 \,, \qquad \langle \omega^2 \rangle_- = \frac{2}{3}\bar{\Lambda} + \frac{1}{3}\lambda_H^2 \,. \tag{2}$$

Introduction

- Vacuum to meson matrix element:

$$\langle 0|\, g_s \bar{q}\; \vec{\alpha} \cdot \vec{E}\; \gamma_5 h_v \,|\bar{B}(v)\rangle = F(\mu)\, \lambda_E^2\,, \tag{3}$$

$$\langle 0|\, g_s \bar{q}\; \vec{\sigma} \cdot \vec{H}\; \gamma_5 h_v \,|\bar{B}(v)\rangle = iF(\mu)\, \lambda_H^2, \tag{4}$$

$$\langle 0|\, \bar{q}\gamma_\mu \gamma_5 h_v \,|\bar{B}(v)\rangle = iF(\mu) v_\mu. \tag{5}$$

- The HQET decay constant is related to the QCD decay constant with:

$$f_B\sqrt{m_B} = F(\mu)K(\mu) = F(\mu)\left[1 + \frac{C_F \alpha_s}{4\pi}\left(3 \cdot \ln\frac{m_b}{\mu} - 2\right) + ...\right] + \mathcal{O}\left(\frac{1}{m_b}\right). \tag{6}$$

Motivation

- First estimates of $\lambda_{E,H}^2$ done by Grozin and Neubert in 1996.
  - Considered contributions up to mass dimension five
  - Unstable sum rules with large uncertainty
- Estimates of Nishikawa and Tanaka in 2011:
  - Consider all contribution up to mass dimension six and a consistent treatment of all $\mathcal{O}(\alpha_s)$ corrections.
  - Better sum rules compared to previous analysis but the values of the parameters are smaller by a factor of three
- Our goal: Estimates of $\lambda_{E,H}^2$ with new alternative correlation function

Correlation function

Grozin and Neubert starting point:

$$\Pi_{\mathrm{GN}} = i \int \mathrm{d}^d x \, e^{-i\omega v\cdot x} \langle 0| \, T\{\bar{q}(0)\Gamma_1^{\mu\nu} g_s G_{\mu\nu}(0) h_v(0) \bar{h}_v(x) \gamma_5 q(x)\} \, |0\rangle \; . \tag{7}$$

Our starting point:

$$\Pi_{\mathrm{diag}} = i \int \mathrm{d}^d x \; e^{-i\omega v\cdot x} \langle 0| \, T\{\bar{q}(0)\Gamma_1^{\mu\nu} g_s G_{\mu\nu}(0) h_v(0) \bar{h}_v(x) \Gamma_2^{\rho\sigma} g_s G_{\rho\sigma}(x) q(x)\} \, |0\rangle \; . \tag{8}$$

$$\begin{aligned} \frac{1}{\pi}\mathrm{Im}\Pi_{\mathrm{diag}}(\omega) = \; & \delta(\omega - \bar{\Lambda})\Theta(\omega^0) \langle 0| \, \bar{q}(0)\Gamma_1 g_s G_{\mu\nu}(0) h_v(0) \, |\bar{B}\rangle \\ & \times \langle \bar{B}| \, \bar{h}_v(x)\Gamma_2 g_s G_{\rho\sigma}(x) q(x) \, |0\rangle + \rho^{\mathrm{hadr.}}(\omega)\Theta(\omega - s^{th}) \end{aligned} \tag{9}$$

Correlation Function

$$\langle 0|\, \bar{q}(0)\Gamma_1 g_s G_{\mu\nu}(0) h_v(0)\, |\bar{B}\rangle = \frac{-i}{6} F(\mu)\{\lambda_H^2(\mu)\cdot \mathrm{Tr}[\Gamma_1 P_+ \gamma_5 \sigma_{\mu\nu}] + [\lambda_H^2(\mu) - \lambda_E^2(\mu)]\cdot \mathrm{Tr}[\Gamma_1 P_+ \gamma_5 (iv_\mu \gamma_\nu - iv_\nu \gamma_\mu)]\} \tag{10}$$

- Through the usage of dispersion relation we obtain:

$$\Pi_{E,H}(\omega) = F(\mu)^2 \cdot \lambda_{E,H}^4 \cdot \frac{1}{\bar{\Lambda} - \omega - i0^+} + \int_{s^{th}}^{\infty} \mathrm{d}s \frac{\rho_{E,H}^{\text{hadr.}}(s)}{s - \omega - i0^+} \tag{11}$$

$$\Pi_{HE}(\omega) = F(\mu)^2 \cdot (\lambda_H^2 + \lambda_E^2)^2 \cdot \frac{1}{\bar{\Lambda} - \omega - i0^+} + \int_{s^{\text{th}}}^{\infty} \mathrm{d}s \frac{\rho_{HE}^{\text{hadr.}}(s)}{s - \omega - i0^+} \tag{12}$$

Correlation Function

- By using quark-hadron duality and applying the Borel transformation on Eq. (8), (9) one obtains:

$$F(\mu)^2 \cdot \lambda_{E,H}^4 \cdot e^{-\bar{\Lambda}/M} = \int_0^{\omega_{\text{th}}} \mathrm{d}\omega \, \frac{1}{\pi} \mathrm{Im} \Pi_{E,H}^{\text{OPE}}(\omega) \, e^{-\omega/M} \tag{13}$$

$$F(\mu)^2 \cdot (\lambda_H^2 + \lambda_E^2)^2 \cdot e^{-\bar{\Lambda}/M} = \int_0^{\omega_{\text{th}}} \mathrm{d}\omega \, \frac{1}{\pi} \mathrm{Im} \Pi_{HE}^{\text{OPE}}(\omega) \, e^{-\omega/M} \tag{14}$$

- Spectral function in OPE:

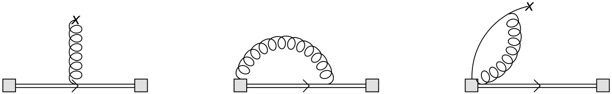
$$\begin{aligned} \Pi_{\text{X}}^{\text{OPE}}(\omega) = {}& C_{\text{pert}}^{\text{X}}(\omega) + C_{\bar{q}q}^{\text{X}} \langle \bar{q}q \rangle + C_{G^2}^{\text{X}} \langle \frac{\alpha_s}{\pi} G^2 \rangle + C_{\bar{q}Gq}^{\text{X}} \langle \bar{q} g_s \sigma \cdot G q \rangle \\ & + C_{G^3}^{\text{X}} \langle g_s^3 f^{abc} G^a G^b G^c \rangle + C_{\bar{q}qG^2}^{\text{X}} \langle \bar{q}q \rangle \langle \frac{\alpha_s}{\pi} G^2 \rangle + \ldots \end{aligned} \tag{15}$$

- We use Fock-Schwinger gauge with reference point $x_0 = 0$:

$$x_\mu \, A^\mu(x) = 0 \qquad \text{and} \qquad A_\mu(x) = \int_0^1 \mathrm{d}u \; u x^\nu \, G_{\nu\mu}(ux). \tag{16}$$

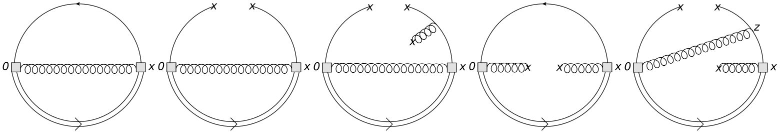
Diagrams

- Vanishing diagrams:

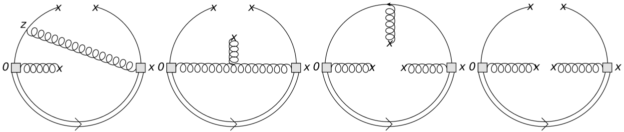


(a) (b) (c)

- Non-vanishing diagrams:

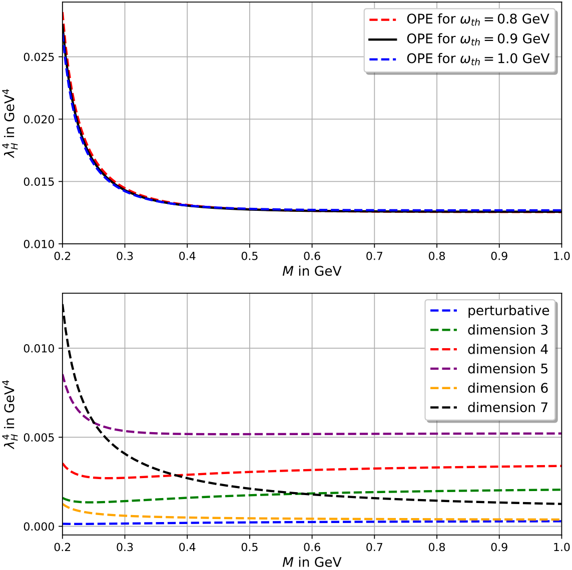


(a) (b) (c) (d) (e)

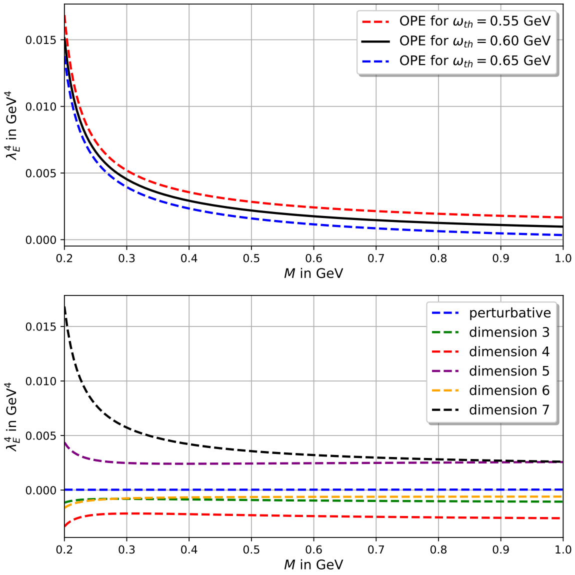


(f) (g) (h) (i)

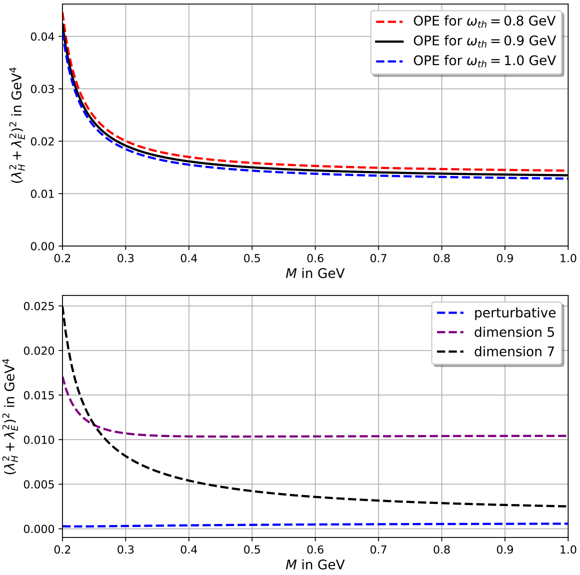
Sum Rule: Plot for λ_H^4



Sum Rule: Plot for λ_E^4

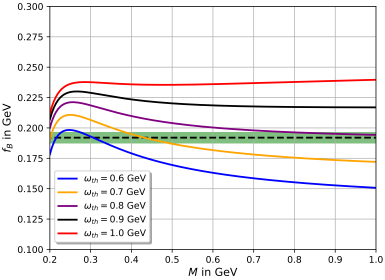


Sum Rule: Plot for $(\lambda_H^2 + \lambda_E^2)^2$

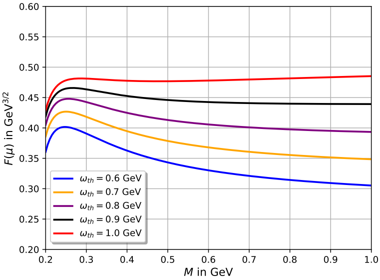


Sum Rule: Finding the threshold window

- Standard way: Derivative on both sides $\partial/\partial(-1/M)$ of the sum rules and vary ω_{th} to compute $\bar{\Lambda}$:
  - True ✓: λ_H^4 and $(\lambda_H^2 + \lambda_E^2)^2$ sum rule
  - False ×: λ_E^4 sum rule
- Alternative way: Compute f_B by varying ω_{th} in $F(\mu)$ sum rule
  - True ✓: $\lambda_{E,H}^4$ and $(\lambda_H^2 + \lambda_E^2)^2$ sum rule



(a)



(b)

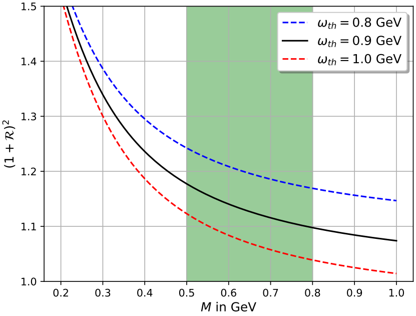
Sum Rule: Finding the Borel window

- Lower bound: $\frac{\text{dim. 7 contribution}}{\text{total OPE}} < 40\%$
  - True ✓: λ_H^4 and $(\lambda_H^2 + \lambda_E^2)^2$ sum rule
  - False ×: λ_E^4 sum rule
- Upper bound: $R_{\text{cont.}} = 1 - \frac{\int_0^{\omega_{th}} \mathrm{d}\omega \frac{1}{\pi} \mathrm{Im}\Pi_X^{\text{OPE}}(\omega) e^{-\omega/M}}{\int_0^{\infty} \mathrm{d}\omega \frac{1}{\pi} \mathrm{Im}\Pi_X^{\text{OPE}}(\omega) e^{-\omega/M}} \leq 50\%$
  - False ×: $\lambda_{E,H}^4$ and $(\lambda_H^2 + \lambda_E^2)^2$ sum rule
- Solution: Use the following combination to resolve the continuum problem

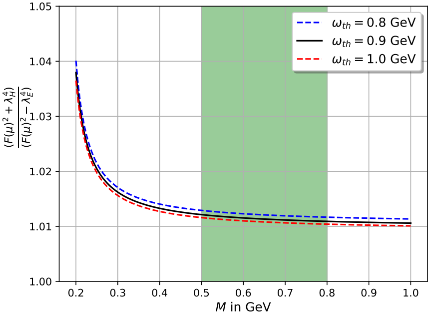
$$\frac{(\lambda_H^2 + \lambda_E^2)^2}{\lambda_H^4} = (1 + \mathcal{R})^2 \quad \text{and} \quad \frac{F(\mu)^2 + \lambda_H^4}{F(\mu)^2 - \lambda_E^4} \tag{17}$$

Sum rule	Borel window	threshold window
$(1 + \mathcal{R})^2$	$0.5\ \mathrm{GeV} \leq M \leq 0.8\ \mathrm{GeV}$	$0.8\ \mathrm{GeV} \leq \omega_{th} \leq 1.0\ \mathrm{GeV}$
$(F(\mu)^2 + \lambda_H^4)/(F(\mu)^2 - \lambda_E^4)$	$0.5\ \mathrm{GeV} \leq M \leq 0.8\ \mathrm{GeV}$	$0.8\ \mathrm{GeV} \leq \omega_{th} \leq 1.0\ \mathrm{GeV}$

Sum Rule: Plot for Combinations



(a)



(b)

Sum Rule: Result

- Result:

Parameters	Grozin and Neubert	Nishikawa and Tanaka	this work
$\mathcal{R}(1\text{ GeV})$	(0.6 ± 0.4)	(0.5 ± 0.4)	(0.1 ± 0.1)
$\lambda_H^2(1\text{ GeV})$	$(0.18 \pm 0.07)\text{ GeV}^2$	$(0.06 \pm 0.03)\text{ GeV}^2$	$(0.11 \pm 0.02)\text{ GeV}^2$
$\lambda_E^2(1\text{ GeV})$	$(0.11 \pm 0.06)\text{ GeV}^2$	$(0.03 \pm 0.02)\text{ GeV}^2$	$(0.01 \pm 0.01)\text{ GeV}^2$

- Uncertainty analysis:

$$\begin{aligned}
\mathcal{R}(1\,\text{GeV}) &= 0.1 + \begin{pmatrix} +0.03 \\ -0.03 \end{pmatrix}_{\omega_{th}} + \begin{pmatrix} +0.01 \\ -0.02 \end{pmatrix}_{M} + \begin{pmatrix} +0.01 \\ -0.01 \end{pmatrix}_{\alpha_s} + \begin{pmatrix} +0.01 \\ -0.01 \end{pmatrix}_{\langle \bar{q}q \rangle} \\
&\quad + \begin{pmatrix} +0.02 \\ -0.03 \end{pmatrix}_{\langle \frac{\alpha_s}{\pi} G^2 \rangle} + \begin{pmatrix} +0.05 \\ -0.04 \end{pmatrix}_{\langle \bar{q}gG\cdot\sigma q \rangle} + \begin{pmatrix} +0.01 \\ -0.01 \end{pmatrix}_{\langle g_s^3 f^{abc} G^a G^b G^c \rangle} \\
&= 0.1 \pm 0.07 \qquad (18)
\end{aligned}$$

$$\begin{aligned}
\lambda_H^2(1\,\text{GeV}) &= \left[0.110 + \begin{pmatrix} +0.002 \\ -0.003 \end{pmatrix}_{\omega_{th}} + \begin{pmatrix} +0.002 \\ -0.004 \end{pmatrix}_{M} + \begin{pmatrix} +0.001 \\ -0.001 \end{pmatrix}_{\langle \frac{\alpha_s}{\pi} G^2 \rangle} + \begin{pmatrix} +0.001 \\ -0.001 \end{pmatrix}_{\langle \bar{q}gG\cdot\sigma q \rangle} \right] \text{GeV}^2 \\
&= (0.110 \pm 0.005)\,\text{GeV}^2 \qquad (19)
\end{aligned}$$

$$\begin{aligned}
\lambda_E^2(1\,\text{GeV}) &= \left[0.01 + \begin{pmatrix} +0.003 \\ -0.004 \end{pmatrix}_{\omega_{th}} + \begin{pmatrix} +0.002 \\ -0.001 \end{pmatrix}_{M} + \begin{pmatrix} +0.001 \\ -0.001 \end{pmatrix}_{\alpha_s} + \begin{pmatrix} +0.003 \\ -0.003 \end{pmatrix}_{\langle \bar{q}q \rangle} \right. \\
&\quad \left. + \begin{pmatrix} +0.003 \\ -0.003 \end{pmatrix}_{\langle \frac{\alpha_s}{\pi} G^2 \rangle} + \begin{pmatrix} +0.005 \\ -0.004 \end{pmatrix}_{\langle \bar{q}gG\cdot\sigma q \rangle} + \begin{pmatrix} +0.001 \\ -0.001 \end{pmatrix}_{\langle g_s^3 f^{abc} G^a G^b G^c \rangle} \right] \text{GeV}^2 \\
&= (0.01 \pm 0.008)\,\text{GeV}^2 \,. \qquad (20)
\end{aligned}$$

Summary

- We obtain independent values for the parameters $\lambda^2_{E,H}$ and their ratio $\mathcal{R} = \lambda^2_E/\lambda^2_H$ by using a diagonal correlation function.
  - Advantage ✓: positive definite which results in better quark-hadron duality compared to previous analysis.
  - Disadvantage ×: Higher mass dimension of the correlation function $\rightarrow$ huge contributions of higher resonances and continuum contributions.
  - Solution: Consider combinations of the sum rules to extract the parameters to fulfill the continuum condition
- Observation:
  - OPE converges well for λ^2_H sum rule
  - OPE does not converge for λ^2_E sum rule
- Future improvement:
  - Compute $\mathcal{O}(\alpha_s^2)$ corrections to the OPE

Backup: Sum Rule

$$F(\mu)^2 \cdot (\lambda_H^2 + \lambda_E^2)^2\, e^{-\bar{\Lambda}/M} = \frac{\alpha_s C_A C_F}{\pi^3} \cdot 24 M^7 \cdot G_6\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F C_A}{4\pi} \cdot \langle \bar{q} g_s \sigma \cdot G q\rangle \cdot M^2 \cdot G_1\left(\frac{\omega_{th}}{M}\right) - \frac{3\alpha_s C_F}{2\pi} \cdot \langle \bar{q} g_s \sigma \cdot G q\rangle \cdot M^2 \cdot G_1\left(\frac{\omega_{th}}{M}\right) - \frac{\pi^2}{2N_c} \langle \bar{q} q\rangle \langle \frac{\alpha_s}{\pi} G^2\rangle\,, \tag{21}$$

$$\begin{aligned} F(\mu)^2 \cdot \lambda_H^4\, e^{-\bar{\Lambda}/M} &= \frac{\alpha_s C_A C_F}{\pi^3} \cdot 12 M^7 \cdot G_6\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F}{\pi} \langle \bar{q} q\rangle \cdot 6 \cdot M^4 \cdot G_3\left(\frac{\omega_{th}}{M}\right) \\ &+ \frac{1}{2} \langle \frac{\alpha_s}{\pi} G^2\rangle \cdot M^3 \cdot G_2\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F C_A}{8\pi} \cdot \langle \bar{q} g_s \sigma \cdot G q\rangle \cdot M^2 \cdot G_1\left(\frac{\omega_{th}}{M}\right) \\ &- \frac{3\alpha_s C_F}{4\pi} \cdot \langle \bar{q} g_s \sigma \cdot G q\rangle \cdot M^2 \cdot G_1\left(\frac{\omega_{th}}{M}\right) + \frac{\langle g_s^3 f^{abc} G^a G^b G^c\rangle}{64\pi^2} \cdot M \cdot \\ &G_0\left(\frac{\omega_{th}}{M}\right) - \frac{\pi^2}{4N_c} \langle \bar{q} q\rangle \langle \frac{\alpha_s}{\pi} G^2\rangle\,, \end{aligned} \tag{22}$$

$$\begin{aligned} F(\mu)^2 \cdot \lambda_E^4\, e^{-\bar{\Lambda}/M} &= \frac{\alpha_s N_c C_F}{\pi^3} \cdot 12 M^7 \cdot G_6\left(\frac{\omega_{th}}{M}\right) + \frac{\alpha_s C_F}{\pi} \langle \bar{q} q\rangle \cdot 6 \cdot M^4 \cdot G_3\left(\frac{\omega_{th}}{M}\right) \\ &- \frac{1}{2} \langle \frac{\alpha_s}{\pi} G^2\rangle \cdot M^3 \cdot G_2\left(\frac{\omega_{th}}{M}\right) - \frac{\alpha_s C_F}{2\pi} \cdot \langle \bar{q} g_s \sigma \cdot G q\rangle \cdot M^2 \cdot G_1\left(\frac{\omega_{th}}{M}\right) \\ &- \frac{\langle g_s^3 f^{abc} G^a G^b G^c\rangle}{64\pi^2} \cdot M \cdot G_0\left(\frac{\omega_{th}}{M}\right) - \frac{\pi^2}{4N_c} \langle \bar{q} q\rangle \langle \frac{\alpha_s}{\pi} G^2\rangle\,. \end{aligned} \tag{23}$$

Backup: Numerical Inputs

Parameters	Value
$\alpha_s(1\ \mathrm{GeV})$	0.471
$\langle \bar{q}q \rangle$	$(-0.242 \pm 0.015)^3\ \mathrm{GeV}^3$
$\langle \frac{\alpha_s}{\pi} G^2 \rangle$	$(0.012 \pm 0.004)\ \ \mathrm{GeV}^4$
$\langle \bar{q} g G \cdot \sigma q \rangle \, / \, \langle \bar{q} q \rangle$	$(0.8 \pm 0.2)\ \mathrm{GeV}^2$
$\langle g_s^3 f^{abc} G^a G^b G^c \rangle$	$(0.045 \pm 0.01)\ \mathrm{GeV}^6$
$\bar{\Lambda}$	$(0.55 \pm 0.06)\ \mathrm{GeV}$

Table: List of the numerical inputs, which will be used in our analysis. The vacuum condensates are normalized at the point $\mu = 1$ GeV. For the strong coupling constant we use the two-loop expression with $\Lambda^{(4)}_{\mathrm{QCD}} = 0.31$ GeV.