PART III

PARAMETER IDENTIFICATION FROM RELAY TEST

3.1 FUNDAMENTALS OF RELAY FEEDBACK

Relay was mainly used as an amplifier in the fifties and was applied to adaptive control in the sixties. Astrom and co-workers successfully applied the relay feedback technique to the identify and auto-tuning of PID controllers for a class of common industrial processes. The relay feedback technique has several attractive features.

1. It facilitates simple push-button tuning since the scheme automatically extracts the process frequency response at an important frequency and the information is usually sufficient to tune the PID controller for many processes. The method is time-saving and easy to use.
2. The relay feedback test is carried out under closed-loop control so that with an appropriate choice of the relay parameters, the process can be kept close to the set point. This keeps the process in the linear region where the frequency response is of interest, which why the method works well on highly nonlinear processes.
3. Unlike other auto-tuning methods, the technique eliminates the need for a careful choice of the sampling rate from the a priori knowledge of the process. This is very useful in initializing a more sophisticated adaptive controller.
4. The relay feedback auto-tuning can be modified to cope effectively with disturbances and perturbations to the process.

To improve the standard relay technique, many works on modifying the relay feedback method have been reported. Improvements of the accuracy and efficiency have been proposed by reducing high-order harmonic terms or using the Fourier analysis instead of the describing function method. We will introduce these techniques in this part.

3.1.1 Generating Sustained Oscillation

A method to determine interesting points on the Nyquist curve is based on the observation that the appropriate oscillation can be generated by relay feedback. The system is thus connected as shown in the Figure 3.1.1. For many systems there will then be an oscillation where the control signal is a square wave and the process output is close to a sinusoid. Notice that the process input and output have opposite phase.

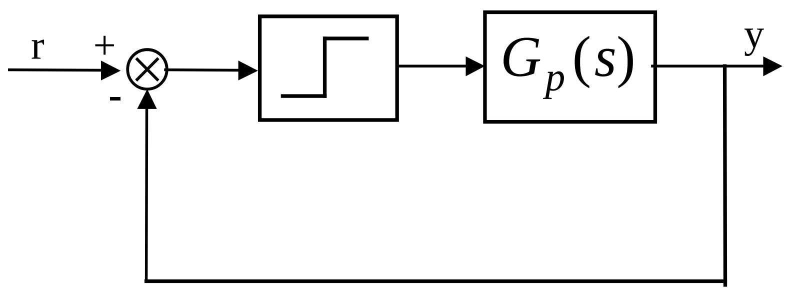


Fig. 3.1.1. Relay feedback system

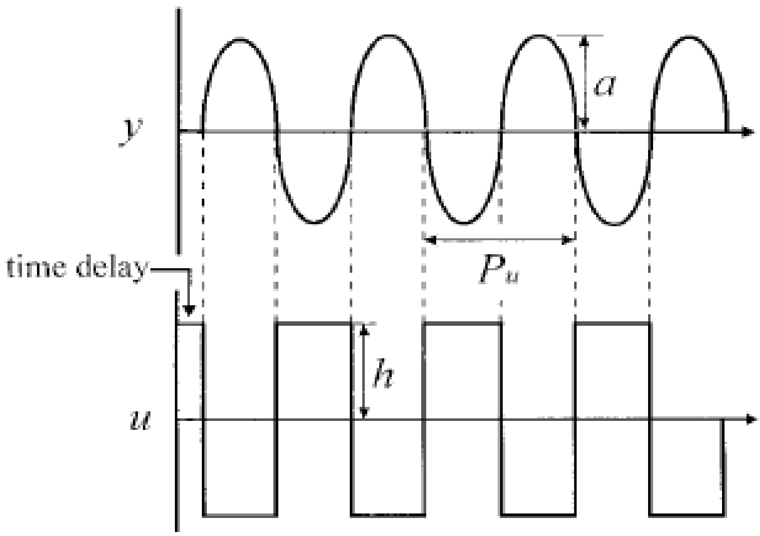


Fig. 3.1.2. Process input and output in the relay test

An on—off (ideal) relay is placed in the feedback loop. The relay feedback system is based on the observation: when the output lags behind the input by $-\pi$ radians, the closed-loop system may oscillate with a period P_u**.** Figure above illustrates how the relay feedback system works.

- A relay of magnitude h is inserted in the feedback loop, initially, the input u is increased by h,
- As the output y starts to increase (after a time delay L), the relay switches to the opposite position, $u = -h$.
- Since the phase lag is $-\pi$, a limit cycle with a period P_u results. The period of the limit cycle is the ultimate period. Therefore, the ultimate frequency from this relay feedback experiment is:

$$\omega_u = \frac{2\pi}{P_u}$$

- For the ideal relay, since $A_1 = 0$ and $B_1 = 4h/\pi$, we have:

$$N(a) = \frac{4h}{a\pi}$$

Since a sustained oscillation is generated from a relay feedback test (e.g., Figure 3.1.2), the frequency of oscillation corresponds to the limit of stability. That is:

$$1 + N(a)G(\omega_u) = 0$$

or the ultimate gain (K_u) becomes:

$$K_u = -\frac{1}{G(j\omega_u)} = N(a) = \frac{4h}{a\pi} \tag{3.1.1}$$

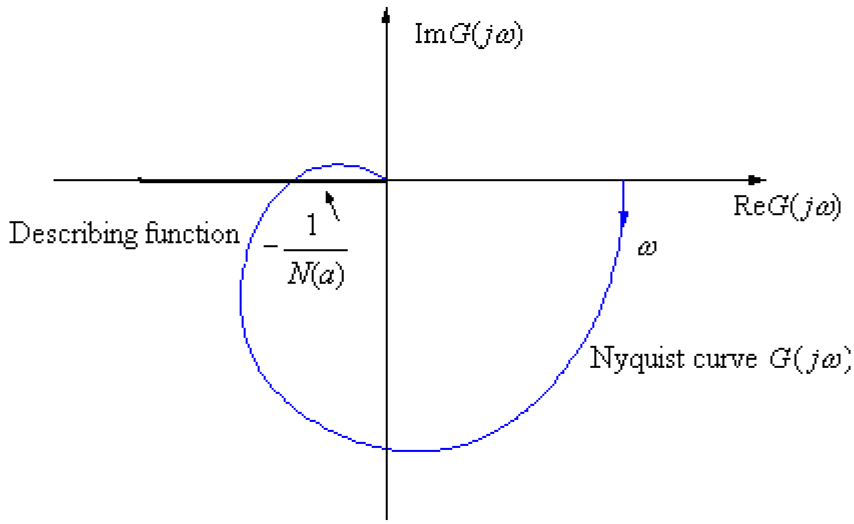


Fig. 3.1.3. Nyquist Plot and Describing Function

As shown in Figure 3.1.4, the relay feedback test can be carried out manually. The procedure requires the following steps

1. Bring the system to steady-state
2. Make a small (e.g., 5%) increase in the manipulated input. The magnitude of change depends on the process sensitivities and allowable deviations in the controlled output. Typical values are between 3— 10%.
3. As soon as the output crosses the set point, the manipulated input is switched to the opposite position (e.g., -5% change from the original value)
4. Repeat step 2 until sustained oscillation is observed (Figure 3.1.2).
5. Read off ultimate period P_u from the cycling and compute K_u from Equation (3.1.4).

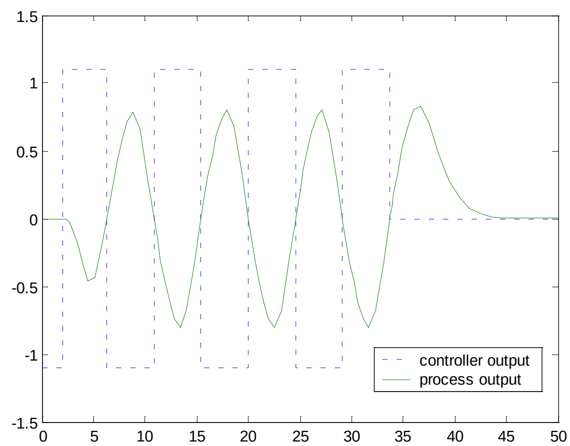


Fig. 3.1.4. Relay testing

This procedure is relatively simple and efficient. Physically, it implies you move the manipulated input against the process. Consider a system with a positive steady-state gain (Figure 3.1.2). When you increase the input (as in step 1), the output *y* tends to increase also. As a change in the output is observed, you switch the input to the opposite direction. It is meant to bring the output back down to the set point. However, as soon as the output comes down to the set point, you switch the input to the upper position. Consequently, a continuous cycling results but the amplitude of oscillation is under your control (by adjusting *h).* More importantly, in most cases, you obtain the information you need for tuning of the controller.

The results indicate that the ultimate gain *(K_u) is* estimated from the amplitude ratio of two sinusoidal waves at a given frequency (ω_u). Obviously, the output of the relay is a square wave instead of a sinusoidal wave. This leads to an erroneous result in the estimated ultimate gain. The truncation of the higher order terms (Fe., n = 3,5,7,…) affects the ultimate gain and ultimate frequency estimation. Mathematically, it is difficult to include the high order terms in a linear analysis. Instead of including higher order terms, a straightforward approach is to redesign the relay feedback experiment.

3.1.2 Approximate Transfer Functions

After the relay feedback experiment, the estimated ultimate gain *(K_u)*, ultimate frequency ω_u can be used directly to calculate controller parameters. Alternatively, it is possible to back-calculate the approximated process transfer functions. The other data useful in finding the transfer function are the time delay *(L)* and/or the steady-state gain *(K_p).*

In theory, the steady-state gain can be obtained from plant data. One simple way to find K_p is to compare the input and output values at two different steady-states. That is:

$$K_p = \frac{\Delta y}{\Delta u}$$

where Δy denotes the change in the controlled variable and Δu stands for the deviation in the manipulated input. However, precautions must be taken to make sure that the sizes of the changes in *u* are made small enough such that the gain truly represents the linearized gain. For highly nonlinear processes, these changes are typically as small as 10^{-3} to 10^{-6} percent of the full range (Luyben, 1987). Such small changes would only be feasible using a mathematical model, trying to obtain reliable steady-state gains from plant data is usually impractical.

The time delay *L* in the transfer function can be easily read off from the initial part of the relay feedback test. It is simply the time it takes *y* to start responding to the change in *u* (Figure 3.1.2). Therefore, it is more likely that we will have information on the time delay rather than the steady-state gain.

The transfer function model can be obtained from relay feedback experiment by the following steps:

1. If necessary, the time delay *L* can he read off from the initial response and the steady-state gain can be obtained from steady-state simulation.
2. The ultimate gain *(K_u)* and ultimate frequency *(ω_u)* are computed after the relay feedback experiment.
3. Different model structures are fitted to the data,

Simple Approach

Once the model is selected, we can back-calculate the model parameters from two equations of the ultimate gain and ultimate frequency as shown in Figure 3.1.5.

Gain condition: $K_u \left| G_p(j\omega_u) \right| = 1$

Phase condition: $\arg\left[G_p(j\omega_u) \right] = -\pi$

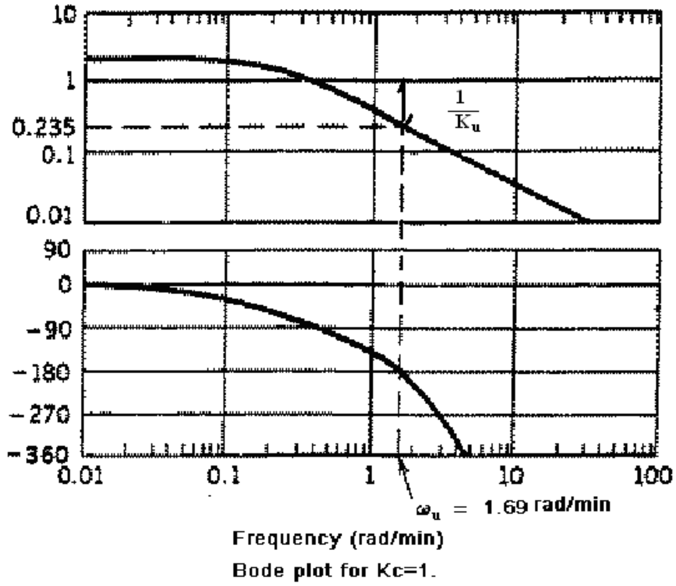


Figure 3.1.5 Bode Daigram

Model 1:

$$G_p(s) = \frac{K_p}{Ts+1} e^{-Ls}$$

Substitute the model into gain and phase equations, we have

$$K_u \left| \frac{K_p}{jT\omega_u + 1} e^{-jL\omega_u} \right| = 1 \Rightarrow K_u K_p = \sqrt{(T\omega_u)^2 + 1}$$

$$\mathrm{T} = \frac{\sqrt{(K_p K_u)^2 - 1}}{\omega_u}$$

and

$$-\omega_u L - \tan^{-1} T\omega_n = -\pi$$

$$T = \frac{\tan(\pi - L\omega_u)}{\omega_u}$$

For the model 1, either L or K_p is needed to solve for the time constant. For example, if the time delay is read off from the relay test, we can compute T, then K_p can be found.

Model 2:

$$G_p(s) = \frac{K}{(Ts+1)^2} e^{-Ls}$$

Solution for parameters:

$$T = \frac{\tan(\pi - L\omega_u)/2}{\omega_u}$$

$$T = \frac{\sqrt{K_p K_u - 1}}{\omega_u}$$

The equations describing model 2 are quite similar to those for model 1. Again, we need to know L or K_u before finding model parameters.

Model 3:

$$G_p(s) = \frac{Ke^{-Ls}}{(\tau_1 s+1)(\tau_2 s+1)}$$

Solution for parameters:

$$-\pi = -\omega_u L - \tan^{-1}(\omega_u T_1) - \tan^{-1}(\omega_u T_2)$$
$$\frac{1}{K_u} = \frac{K_p}{\sqrt{[1+(\omega_u T_1)^2][1+(\omega_u T_2)^2]}}$$

Since we have four parameters in the model 3, both K_u and L have to be known in order to solve for two time constants T_1 and T_2. This is the most complex model structure in our models and it is often sufficient for process control applications.

Example 3.1.1. Wood and Berry column

$$G_p(s) = \frac{12.8}{16.8s+1}e^{-s}$$

This is the transfer function between the top composition and reflux flow. From relay feedback test, we obtain the following ultimate gain and ultimate frequency: $K_u=1.71$ and $\omega_u=1.615$ which are the estimation of true value $K_u=2.1$ and $\omega_u=1.608$.

Parameters can be calculated for different model structures:

Model 1

$$G_p(s) = \frac{13.2}{14.8s+1}e^{-s}$$

Model 2: (assume $L=1$ is known)

$$G_p(s) = \frac{1.12}{(0.59s+1)^2}e^{-s}$$

Model 3: (assume $L=1$ and K_p are known)

$$G_p(s) = \frac{12.8e^{-s}}{(13.5s+1)(0.0009s+1)}$$

Despite varying model parameters, all these four models have the same ultimate gain and ultimate frequency. The models are correct around the ultimate frequency which is important for the controller design. However, this may not be correct for other frequencies. For example, the steady-state gain of model 2 is only 1.12 which is less than 10% of the true value. #

3.1.3 Fourier Transform Method

For the standard relay feedback system, the process input $u(t)$ and output $y(t)$ are recorded from the initial time until the system reaches a stationery oscillation. As $u(t)$ and $y(t)$ are not integrable since they do not die down in finite time. They cannot be directly transformed to frequency response. However, since $y(t)$ and $u(t)$ are piece wise continuous and periodic, the Laplace transform of $y(t)$ and $u(t)$ can be written as:

$$Y(s) = \frac{1}{1-e^{-Ps}} \int_0^P y(t)e^{-st}dt \tag{3.1.5a}$$

$$U(s) = \frac{1}{1-e^{-Ps}} \int_0^P u(t)e^{-st}dt \tag{3.1.5b}$$

and

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\int_0^P y(t)e^{-st}dt}{\int_0^P u(t)e^{-st}dt}$$

where, P=$2\pi/\omega$ and ω is the frequency of oscillations observed in the output. By substituting $s = j\omega$ and using Euler's expansion for $e^{-j\omega t}$, we get:

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{c_1 - jd_1}{c_2 - jd_2}$$

where

$$c_1 = \int_0^P y(t)\cos(\omega t)dt$$

$$d_1 = \int_0^P y(t)\sin(\omega t)dt$$

$$c_2 = \int_0^P u(t)\cos(\omega t)dt$$

$$d_2 = \int_0^P u(t)\sin(\omega t)dt$$

The method can identify accurate frequency response points as many as desired with one relay experiment. The required computations are more involved than the standard relay technique, especially if a large number of frequency response points are needed.

The transfer function model parameters can be obtained using the method given in Part II.

Example 3.1. 2. Consider an oscillatory process given by

$$G(s) = \frac{1}{(s^2+s+1)(s+3)}e^{-s} .$$

Choosing the number of the frequency response points *M =10*, the frequency responses are computed, and then the second-order model is then calculated as

$$G'(s) = \frac{0.4072s + 0.3318}{1.3320s^2 + 1.6322s + 1}e^{-1.9455s}$$

The frequency response identified by the proposed method is shown in Fig. 3.1.6. (Solid line: Actual process, ×: Estimated frequency responses at 20% NSR, Dashed line: Estimated model).

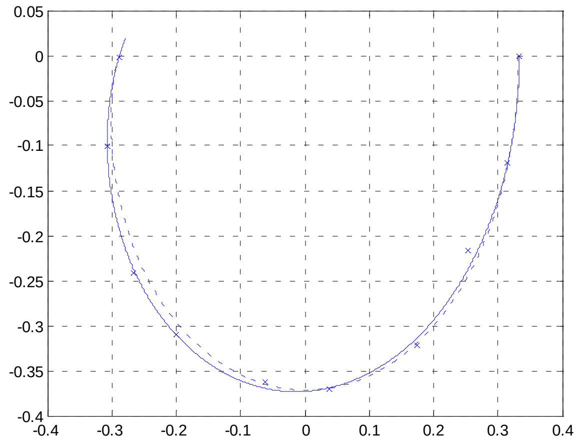


Fig. 3.1.6 Nyquist plot and estimated under noise

3.2 IMPROVED RELAY TEST

Even very popular in industry, there are several problems associated with the standard relay identification method:

- In theory, the steady-state gain can be obtained from plant data. In standard relay test, the K_p is determined by comparing the input and output values at two different steady-states. However, precautions must be taken to make sure that the sizes of the changes in u are made small enough such that the gain truly represents the linearized gain. Such small changes would only be feasible using a mathematical model, trying to obtain reliable steady-state gains from testing data is usually impractical.
- Since the sustained oscillation, the process information can only be available at the ω_u frequency; other information at important frequency is not available.
- Due to the adoption of describing function approximation, the estimation of the critical point is not accurate enough for some processes.

Consequently, the identified models may not accurate enough for controller design. In this section, we introduce several methods which can improve the accuracy of identified processes among many of them.

3.2.1 Relay feedback Plus Step

The process is first put under relay feedback test. After a couple of controlled stationary oscillations, a step-test is performed shown as Fig. 3.2.1. When the process enters a new steady-state again, the identification algorithm is triggered and a second-order plus dead-time process model is obtained as follows.

- ω_u is obtained as

$$\omega_u = \frac{2\pi}{P_u}$$

  where P_u is the period of the stationary oscillation.

- $G(j\omega_u)$ can be obtained by a simple digital integral

$$G(j\omega_u) = \frac{\int_0^{P_u} y(t)\cos\omega_u t dt - j\int_0^{P_u} y(t)\sin\omega_u t dt}{\int_0^{P_u} u(t)\cos\omega_u t dt - j\int_0^{P_u} u(t)\sin\omega_u t dt} \tag{3.2.1}$$

- The apparent dead-time L of the process can be read off from the output response in a relay test. For a positive static gain process, apparent dead-time of the process can be estimated as the time difference between negative-edge of the relay signal and the next peak of the process output.
- The static gain of the process is determined by

$$K = \frac{\Delta y(\infty)}{\Delta u(\infty)} ,$$

where $\Delta u(\infty)$ is the amplitude of step change in process input while $\Delta y(\infty)$ is the process output's new steady state value minus the original one.

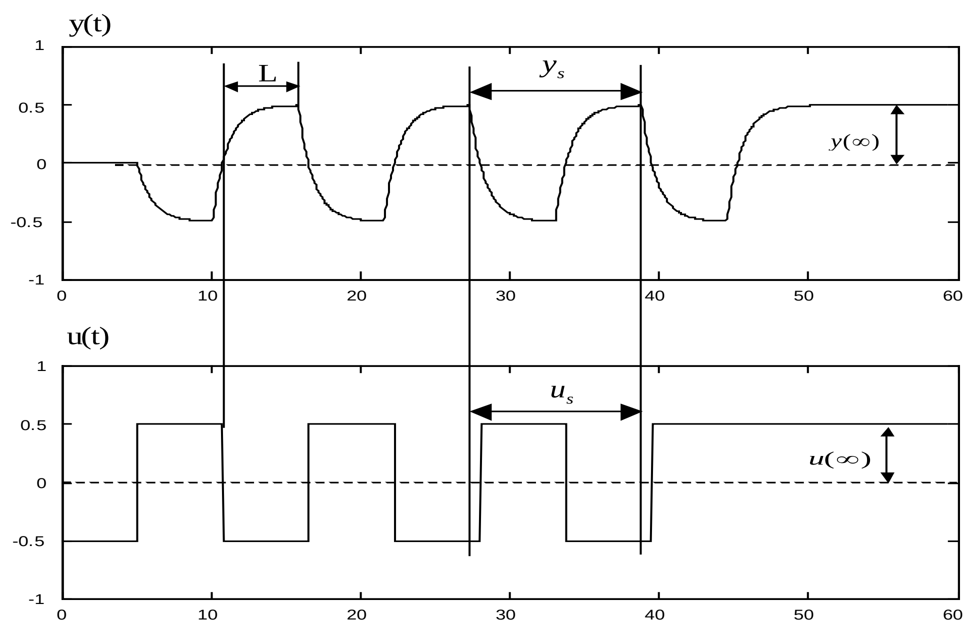


Fig. 3.2.1 Process input and output in a relay+step test

With the information, we now consider a second order plus dead-time model

$$G(s) = \frac{K}{s^2 + a_1 s + a_2} e^{-Ls} \tag{3.2.2}$$

rearrange (3.2.3)

$$s^2 + a_1 s + a_2 = \frac{K}{G(s)} e^{-Ls}$$

results

$$\left(a_2 - \omega_u{}^2\right) + ja_1\omega_u = \frac{Ke^{-j\omega_u L}}{G(j\omega_u)}$$

equalize imaginary and real parts on both side of the equation, we obtain

$$\begin{cases} c = 1/K, \\ a = [c - real(\dfrac{e^{-j\omega_u L}}{G(j\omega_u)})]/\omega_u^2 \\ b = imag(\dfrac{e^{-j\omega_u L}}{G(j\omega_u)})/\omega_u. \end{cases} \tag{3.2.3}$$

or simply but less accurately

$$G(j\omega_u) = \frac{1}{N(a)} = \frac{a\pi}{4h}$$

and substituting ω_u into the equation, we have

$$\left(a_2 - \omega_u{}^2\right) + ja_1\omega_u = \frac{4hK}{a\pi}\left(\cos\omega_u L - j\sin\omega_u L\right)$$

Comparing the real and image part on both side of the equation, we finally obtain

$$\begin{cases} a_1 = \dfrac{4hK}{a\pi\omega_u}\sin\omega_u L \\ a_2 = \dfrac{4hK}{a\pi}\cos\omega_u L + \omega_u{}^2 \end{cases} \tag{3.2.4}$$

Example 3.2.1. In the HVAC system, the supply air pressure is regulated by the speed of a supply air fan. Increasing the fan speed will increase the supply air pressure, and vice versa. The dynamics from the control signal feeding to the fan variable speed drive to the supply air pressure can be modeled as a second order plus dead-time process. A relay plus step test is used for process modeling. In the test, the sampling interval set to 0.5 s. The process response in the tuning test (t=11–38 s) is shown in Figure 3.2.2.

After the test, the model for the supply air pressure loop can be obtained as

$$G(s) = \frac{1}{0.12s^2 + 1.33s + 1.42}e^{-3s}$$

With the gain and phase PID design formula, a PID controller can be computed as

$$G_c(s) = 0.34 + \frac{0.31}{s} + 0.03s$$

The PID controller is implemented to the supply-air pressure loop from t=38 s to drive the air pressure to the value before the tuning test (solid line). A step set-point change to 9 Pa was introduced at t=54 s and the response (solid line) is shown in Figure 3.2.2.

With the same testing data, a PID was tuned using modified Ziegler and Nichols method.

$$G_c(s) = 0.65 + \frac{0.15}{s} + 0.71s$$

The process response under this PID controller is shown in Figure 3.2.2 (dashed line). The proposed PID auto-tuner gives a quicker response.

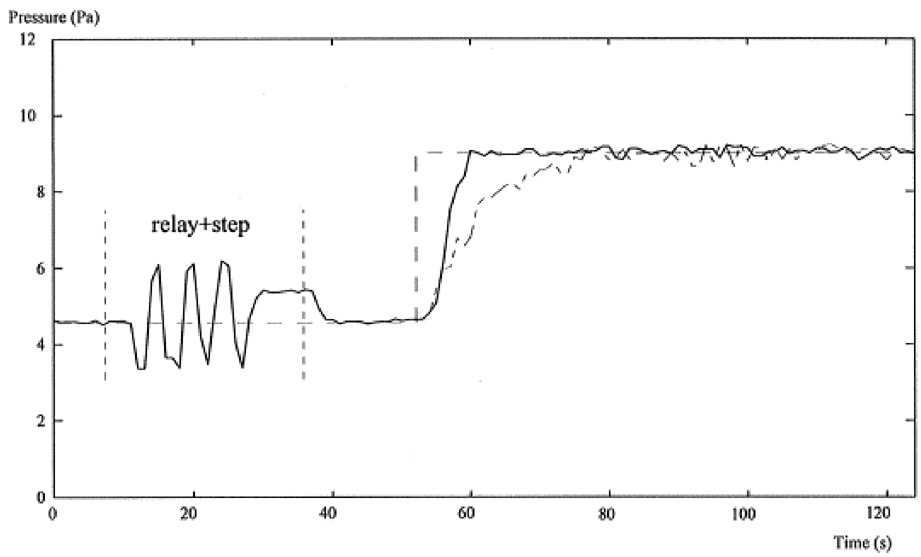


Fig. 3.2.2. Control performance for supply air pressure loop: (—) GP method, (– –) Z–N method.

Example 3.2.2 In cleanroom applications, the room pressure needs to be controlled by regulating the position of the relief air damper. The wider the relief damper is opened, the lower will be the room pressure. The a relay plus step test is introduced for the control loop, as shown in Figure 3.2.3, the process model is estimated as

$$G(s) = \frac{-1}{181.9s^2 + 169.1s + 9.7} e^{-2s}$$

A GP based PID controller can be computed as

$$G_c(s) = -\left(42.8 + \frac{-2.46}{s} + 46.05s\right)$$

The resultant control performance is shown in Figure 3.2.3.

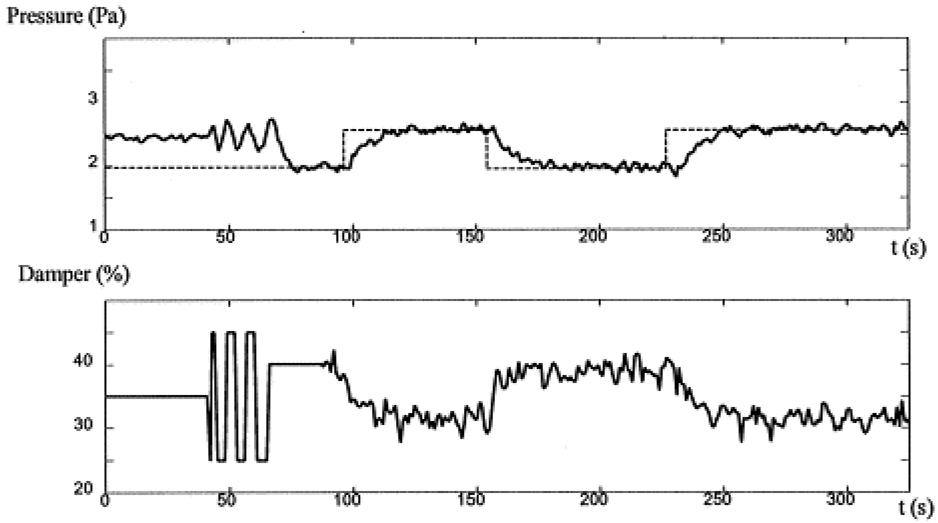


Fig. 3.2.3. Auto-tuning room pressure loop.

The control system performance is satisfactory, implies that the parameter of the identified model is correct. Why?

3.2.2 Relay with Hysteresis

A pure relay with amplitude *d* gives the information on the critical point of the process Nyquist (precisely, with a phase lag of -180°). If other points on the process Nyquist are of interest, a relay with adjustable hysteresis ε can be used, thus making possible the identification of any point in the third quadrant (see *Figure* 3.2.4), with a phase lag between -90° and -180°.

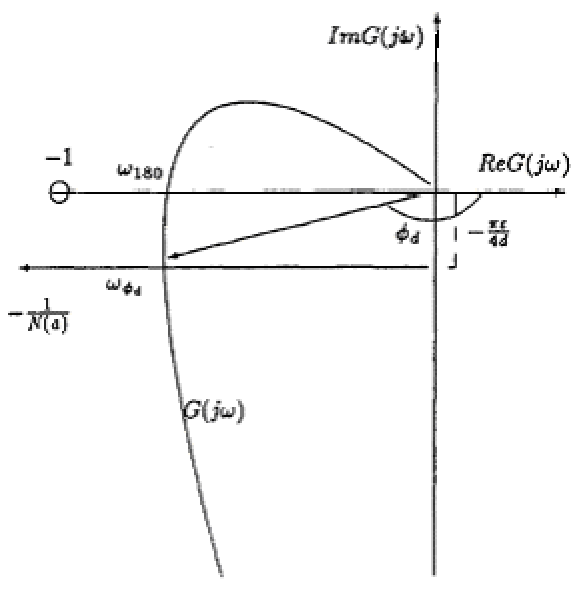


Fig. 3.2.4 Nyquist diagram of $G(j\omega)$ and negative inverse of the relay describing function *N(a)*

For some classes of processes, whose dynamic is characterized by two 'time scales'; small time constants and large time constants. This equivalent small time constant can be identified by measuring the oscillation corresponding at the phase lag -135°, induced by the relay with hysteresis, in feedback with the process.

For determining a point (gain and frequency) on the Nyquist diagram of the plant, in the third quadrant, a relay with hysteresis ε can be used, in order to generate oscillation (see *Figure* 3.2.5).

The control scheme thus obtained is composed of a linear element (the process) and a static nonlinearity (the relay). The control signal *u(t)* is a squarewave, of amplitude *d* and the process output *y(t)* is close to a sinusoid, with amplitude *a.*

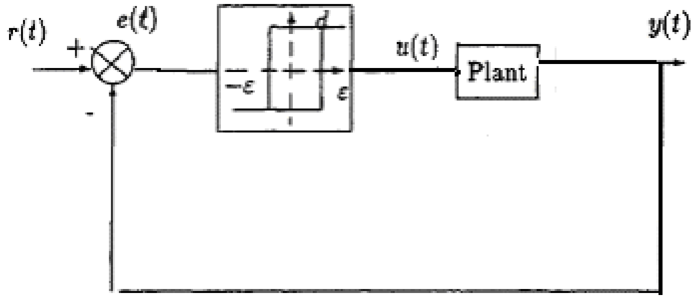


Fig. 3.2.5 Process connected to relay with hysteresis

The nonlinear block is described by a describing function, *N(a),* which depends on signal amplitude *a* at the nonlinearity input. The oscillation condition of such a scheme is given by the existence of an intersection between the frequency diagram of the plant $G(j\omega)$ and the negative inverse of the describing function of the relay, $-1/N(a)$.

$$G(j\omega) = \frac{-1}{N(a)}$$

Now considering the describing function of the relay with hysteresis *N(a)*, it follows:

$$\frac{-1}{N(a)} = -\frac{\pi}{4d}\left(\sqrt{a^2 - \varepsilon^2} - j\varepsilon\right), \quad a > \varepsilon \tag{3.2.5}$$

This oscillation condition is represented graphically in Figure 3.2.4.

If the control designer is interested in identifying the gain and frequency of a point with a desired phase lag on the Nyquist diagram of the plant ϕ, one way is to adjust the relay hysteresis ε. This modifies

$$\operatorname{Im} G(j\omega_u) = \operatorname{Im}\frac{-1}{N(a)} = -\frac{\pi\varepsilon}{4d} \tag{3.2.6}$$

and makes it intersect $G(j\omega_u)$ at ϕ. To "determine the adjustment rule of ε, it can be seen from *Figure* 3.2.4 and Equation (3.2.6) that:

$$\frac{\pi\varepsilon}{4d} = \left|G(j\omega_u)\right|\sin\phi \tag{3.2.7}$$

The adjustment of ε is realized in an iterative manner, by measuring the amplitude of the output oscillations *a*, and imposing:

$$\varepsilon = a\sin\phi \tag{3.2.8}$$

the phase ϕ can be determined.

3.2.4 Realizing Hysteresis by a Delay

The effect of using a positive hysteresis ε in the relay, is to move the oscillation versus lower phase lags compared with the pure relay (see Figure 3.2.4), that is, to make $-1/N(a)$ intersect $G(j\omega)$ in the third quadrant instead on the negative real axis.

Qualitatively, an equivalent effect can be realized by using a time delay L instead of the hysteresis ε. A point of desired phase lag $-180^0 \le \phi \le -90^0$ is identified by making oscillate the system, using a pure relay and a variable time delay τ (see *Figure* 3.2.6).

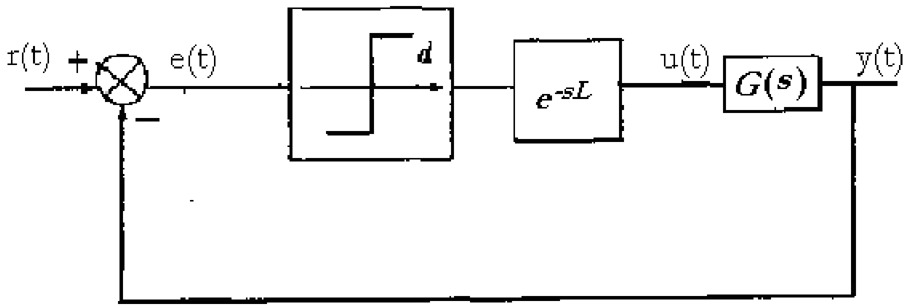


Fig. 3.2.6 The relay with time delay

The time delay or added to the system is adjusted so as the stabilised oscillation corresponds to the point of desired phase lag ϕ.

The system can be interpreted as an equivalent process with an additional time delay

$$G'(j\omega) = e^{-Ls}G(j\omega) \tag{3.2.9}$$

which is tested by a pure relay. The influence of the time delay or on the frequency diagram of the plant is shown in Figure 3.2.7, consisting of moving the point (a) from the $G(j\omega)$ to point (b) on Equation (3.2.9). In other words, a point (a) with a phase lag of ϕ_p of the real plant $G(j\omega)$ corresponds to a point (b) with the same magnitude and a greater phase lag ($\phi_e = \phi_p + \phi_d$), on the equivalent delayed plant $G'(j\omega)$.

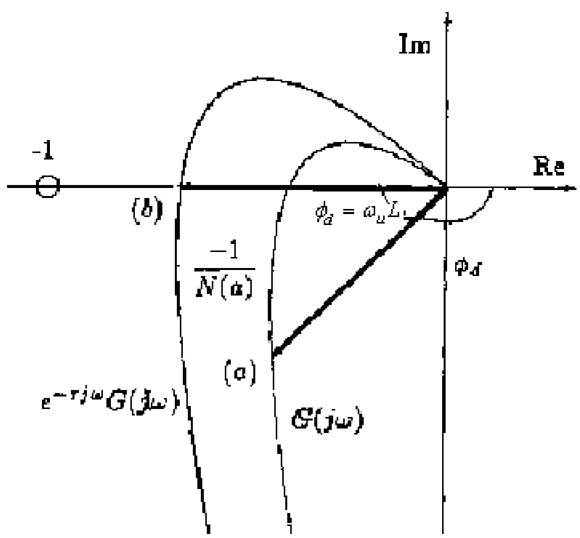


Figure 3.2.7. Nyquist diagram of the process $G(j\omega)$ and $G'(j\omega)$

The notation is: ϕ_p for the process $G(j\omega)$ phase lag, ϕ_e for the phase lag of the equivalent process $G'(j\omega)$, ϕ_d being the phase lag introduced by the delay L.

The oscillation condition of the global system becomes:

$$G'(j\omega) = \frac{-1}{N(a)}$$

where now

$$\operatorname{Im}\left(\frac{-1}{N(a)}\right) \approx 0$$

The phase lag ϕ_e of the equivalent process $G'(j\omega)$ is then always $\phi_e \approx -180^0$ when the system oscillates and

$$\phi_e = \phi_p + \phi_d = \phi_p - \omega_u L \qquad (3.2.10a)$$

The purpose is to make the system oscillate for $\phi_e = \phi$ (desired phase lag):

$$\phi = \phi_e + \omega_u L \qquad (3.\ 2.10b)$$

As long as this condition is not satisfied, the time delay τ is adjusted as follows:

$$L = \frac{\phi - \phi_e}{\omega_u} \qquad (3.\ 2.10c)$$

where ϕ and $\phi_e = -180^0$ are fixed values and ω_u is the oscillation frequency, which is measured on the system.

Open Problem

By the relation

$$G'(j\omega_u) = e^{-j\omega_u L} G(j\omega_u) = \frac{-1}{N(a)} \Rightarrow G(j\omega_u) = \frac{-e^{j\omega_u L}}{N(a)} \qquad (3.\ 2.11a)$$

$$\phi_e = \phi_p + \phi_d = \phi_p - \omega_u L = -180^0 \Rightarrow \phi_p = -180^0 + \omega_u L \qquad (3.\ 2.11b)$$

Can you use this relation and multiple delay to calculate model?

3.2.5 Asymmetrical Relay Feedback

Due to the adoption of describing function approximation, the estimation of the critical point is not accurate enough for some processes. To improve the estimation accuracy, we can use an asymmetrical relay.

Consider a relay feedback system where *G(s)* is the process transfer function, *y* is the output, y_r is the set point, *e* is the error and *u* is the manipulated variable. An asymmetrical relay is placed in the feedback loop. Depending on the error and the process, the relay switches between $+\gamma h$ and $-h$, where $\gamma \neq 1$.

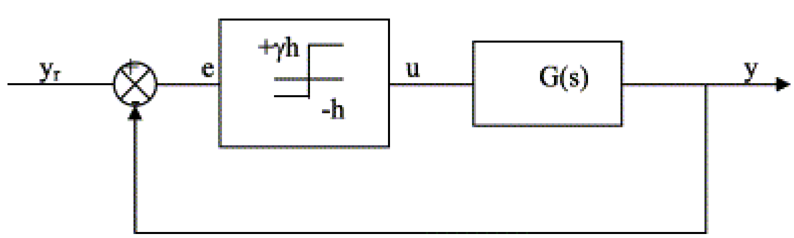


Fig 3.2.8 Block diagram for the asymmetric relay feedback system.

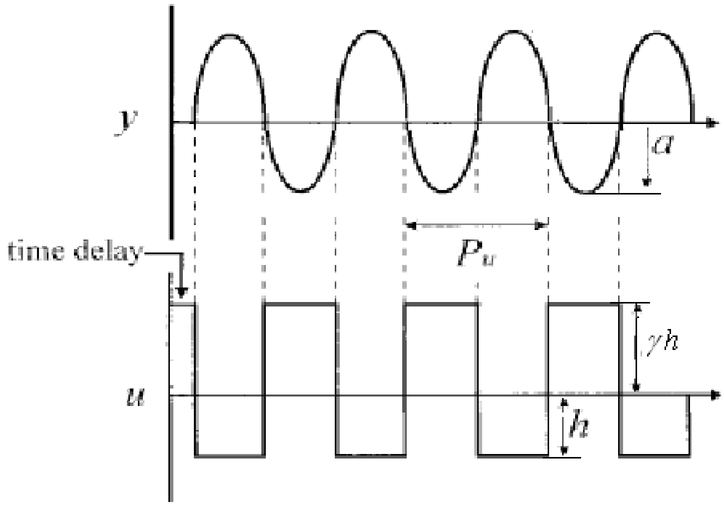


Fig. 3.2.9. Process input and output in the asymmetric relay test

The key advantage of this method is that the steady state gain can be more accurately estimated which can be found by integrating the system input–output response.

$$K_p = \frac{\int_{t_0}^{t_0+P} y(t)dt}{\int_{t_0}^{t_0+P} u(t)dt} \tag{3.2.12}$$

where t_o is the starting time for integration and P is the period of oscillations.

Fro a FOPTD model given as:

$$G(s) = \frac{K_p e^{-Ls}}{Ts+1} \tag{3.2.13}$$

Direct using the Fourier transformation method given in Equation (3.2.1), we can write $G(j\omega)$ for this model as:

$$a + jb = k_p \left[\frac{\cos(L\omega_u) - j\sin(L\omega_u)}{1 + jT\omega_u} \right] \tag{3.2.14}$$

On cross multiplying and equating the real part and imaginary part separately to zero for Equation (3.2.14), we get:

$$\begin{aligned} a - bT\omega_u - k_p \cos(L\omega_u) &= 0 \\ aT\omega_u + b + k_p \sin(L\omega_u) &= 0 \end{aligned} \tag{3.2.15}$$

Let

$$\begin{aligned} x_1 &= T\omega_u \\ x_2 &= L\omega_u \end{aligned} \tag{3.2.16}$$

Multiplying first by *a* and second by *b* and then adding the resulting equations we get:

$$bk_p \sin(x_2) + dk_p \cos(x_2) = q$$

where $d = -a,\ q = -(a^2 + b^2)$.

The above equation can be written as

$$M\sin(x_2 + \phi) = q \tag{3.2.17}$$

where

$$\begin{cases} \phi = \tan^{-1}\left(\dfrac{b}{d}\right) \\ M = k_p^2(d^2 + b^2) \end{cases}$$

It gives

$$\begin{aligned} x_2 &= \sin^{-1}\left(\frac{q}{M}\right) - \phi \\ x_1 &= \frac{\left[a - k_p\cos(x_2)\right]}{b} \end{aligned} \tag{3.2.18}$$

Substituting (3.2.18) into (3.2.16), results

$$\begin{aligned} T &= \frac{x_1}{\omega_u} \\ L &= \frac{x_2}{\omega_u} \end{aligned} \tag{3.2.19}$$

Here ω is the known from the closed loop oscillation.

Another method is to use least square method given in Part II

Open Problem

For the four parameters of a SOPTD model in the form of

$$G(s) = \frac{K_p e^{-Ls}}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

For this model, we can write $G(j\omega)$ as:

$$G(j\omega) = \frac{K_p e^{-j\omega L}}{(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

Equating the real and imaginary parts and after simplifications we get:

$$a(1 - \tau_1\tau_2\omega_u{}^2) - b\omega(\tau_1 + \tau_2) - K_p\cos(\omega_u L) = 0$$
$$b(1 - \tau_1\tau_2\omega_u{}^2) - a\omega(\tau_1 + \tau_2) - K_p\sin(\omega_u L) = 0$$

From the relay test, the values for ω_u and P are obtained. Using this, K_p can be solved. Substituting ω_u for ω in above equations, we can evaluate c_1, c_2, d_1, and d_2 and hence a and b. With K_p, a and b known, the three unknown parameters: L, τ_1 and τ_2 can be estimated, need one more equation.

3.2.6 Assymetrical Relay with Hesteresis

For the first order plus dead-time model

$$G(s) = \frac{Ke^{-Ls}}{Ts+1}$$

For these kinds of processes, using a biased relay feedback test and derived the formulae that could precisely yield the critical point and the static gain simultaneously with a single relay test. The biased relay is shown as in Figure 3.2.10.

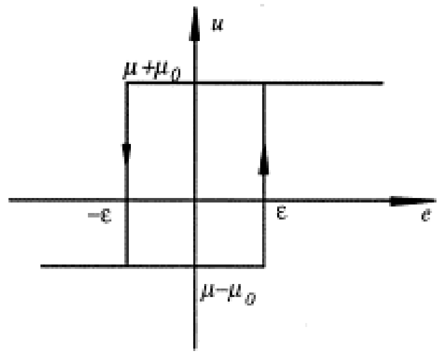


Fig. 3.2.10 The biased relay.

Under the biased relay feedback, the process input *u* and the process output *y* is shown as in Fig. 3.2.11.

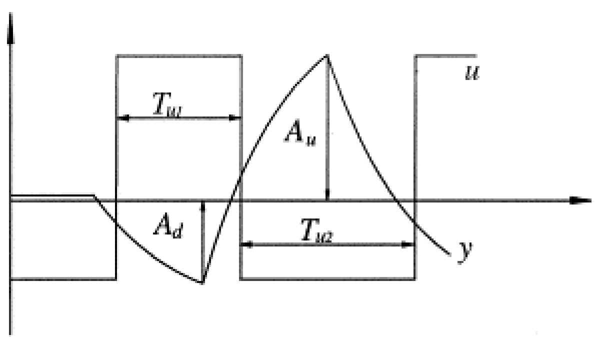


Fig. 3.2.11 Oscillatory waveforms.

It is shown that for the first order plus dead-time model, the output *y* converges to the stationary oscillation in one period $T_{u1} + T_{u2}$, and the oscillation is characterized by

$$A_u = (\mu_0 + \mu)K(1 - e^{-L/T}) + \varepsilon e^{-L/T} \qquad (3.2.20a)$$

$$A_d = (\mu_0 - \mu)K(1 - e^{-L/T}) + \varepsilon e^{-L/T} \qquad (3.2.20b)$$

$$T_{u1} = T \ln \frac{2\mu K e^{L/T} + \mu_0 K - \mu K + \varepsilon}{\mu_0 K + \mu K - \varepsilon} \qquad (3.2.20c)$$

and

$$T_{u2} = T \ln \frac{2\mu K e^{L/T} - \mu_0 K - \mu K + \varepsilon}{-\mu_0 K + \mu K - \varepsilon} \qquad (3.2.20d)$$

The above four equations are the accurate expressions for the period and the amplitude of the limit cycle oscillation of the first order plus dead-time. By measuring any three of A_u, A_d, T_{u1} and T_{u2}, the parameters of the model K, T and L can be calculated. Solving these four equations is not an easy task. To simplify the computation, K may be alternatively computed as the ratio of DC components in the output and input:

$$K = \frac{\int_0^{T_{u1}+T_{u2}} y(t)dt}{\int_0^{T_{u1}+T_{u2}} u(t)dt} \tag{3.2.21}$$

The normalized dead time of the process $\theta = L/T$ is obtained as

$$\theta = \ln \frac{(\mu_0 + \mu)K - \varepsilon}{(\mu_0 + \mu)K - A_u} \tag{3.2.22a}$$

or

$$\theta = \ln \frac{(\mu_0 - \mu)K - \varepsilon}{(\mu_0 - \mu)K - A_d} \tag{3.2.22b}$$

It then follows that

$$T = T_{u1}\left(\ln \frac{2\mu K e^{\theta} + \mu_0 K - \mu K + \varepsilon}{\mu_0 K + \mu K - \varepsilon}\right)^{-1} \tag{3.2.23a}$$

or

$$T = T_{u2}\left(\ln \frac{2\mu K e^{\theta} - \mu_0 K - \mu K + \varepsilon}{-\mu_0 K + \mu K - \varepsilon}\right)^{-1} \tag{3.2.23b}$$

The dead time is thus

$$L = T\theta \tag{3.2.24}$$

The method produces two accurate process frequency points in just one relay test. The two points can then be used to design the controller. The method is simple for implementation and suitable for the processes that can be characterized by the first order plus dead-time model.

3.3 RELAY FEEDBACK TEST FOR DIFFICULT DYNAMICS

3.3.1 Review of Basic Concepts

The presence of any of difficult characteristics in the dynamic behavior of a process can be identified below:

- Time delay
- Inverse response
- Open loop instability

A process with *time delay* violates the first condition noted above normal dynamic behavior: it does not respond instantaneously to input change.

A process with *inverse response* violates the second condition for normal dynamic behavior. The response, even though its step response eventually ends up heading in the direction of the new steady state, it starts out initially heading in the opposite direction, away from the new steady state, changing direction somewhere during the course of time.

A process for which the step response is unbounded, i.e., the output increases (or decreases) indefinitely with time, is said to be open loop unstable. An open loop unstable process violates the third condition noted above; its output fails to settle to a new steady-state value in response to a step change in the input.

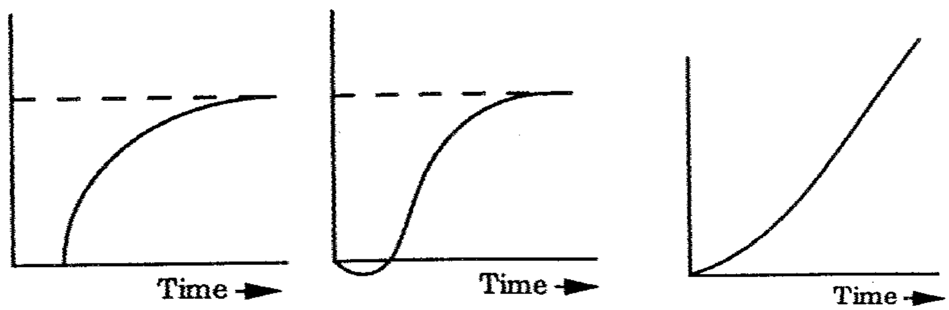


Fig. 3.3.1 typical difficult dynamic processes

3.3.2 Process with Time Delay

Systems with time delay only. Obviously, this is the typical transfer function for industry processes. As an example, consider the transfer function between x_D and R for the WB column.

Example: First order plus dead time system

$$G(s) = \frac{12.8e^{-s}}{16.8s+1}$$

Assume that we know the sign of $G(s)$, $G(0)$=12.8 (>0).

- One is expecting an increase in x_D when a positive change in R is made as shown is Fig. 3.3.2. A sustained oscillation is observed when we continue the relay feedback test.
- If we have a misconception about the sign of the process gain and the sign is thought to be \`negative'. Under this circumstance, if an initial \`positive' change in R is made, one is expecting x_D becoming negative. Clearly, this will not happen since the sign of the process is \`positive' (Fig. 3.3.3).

Therefore, the relay never switches and the experiment fails.

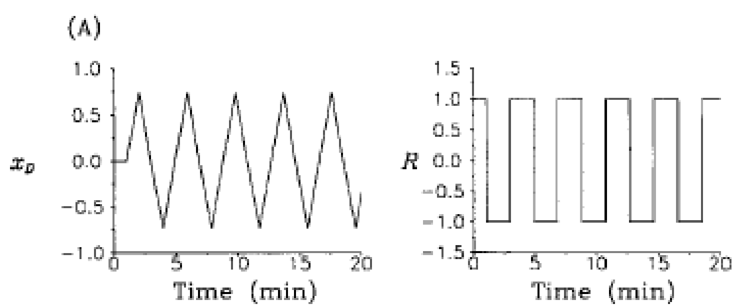


Fig. 3.3.2 Relay-feedback test for the Example with correct prior knowledge (positive sign)

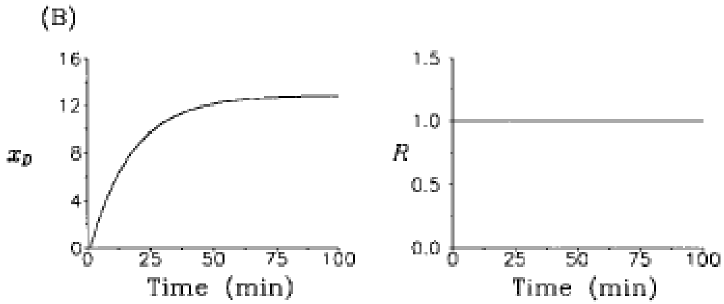


Fig. 3.3.3 Relay-feedback test for the Example with incorrect prior knowledge (negative sign)

This simple example indicates that the relay feedback test may fail if one has incorrect knowledge about the system. One way to overcome this possible failure is: let the relay switches when the output starts moving (ignoring the knowledge about of sign of the process).

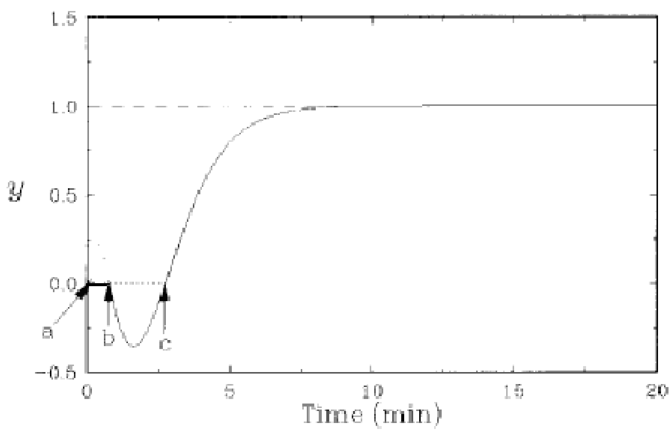
3.3.3 Processes with RHP Zeros

Consider the following system with one RHP zero.

Example: System with one RHP zero.

$$G(s)=\frac{(-3s+1)e^{-0.6s}}{(5s+1)(s+1)}$$

This system shows an inverse response for a step change in u.



Following the simple minded procedure (ignoring the knowledge about the `sign'), the relay switches as soon as the output starts to move and the responses is shown in Fig. 3.3.4.

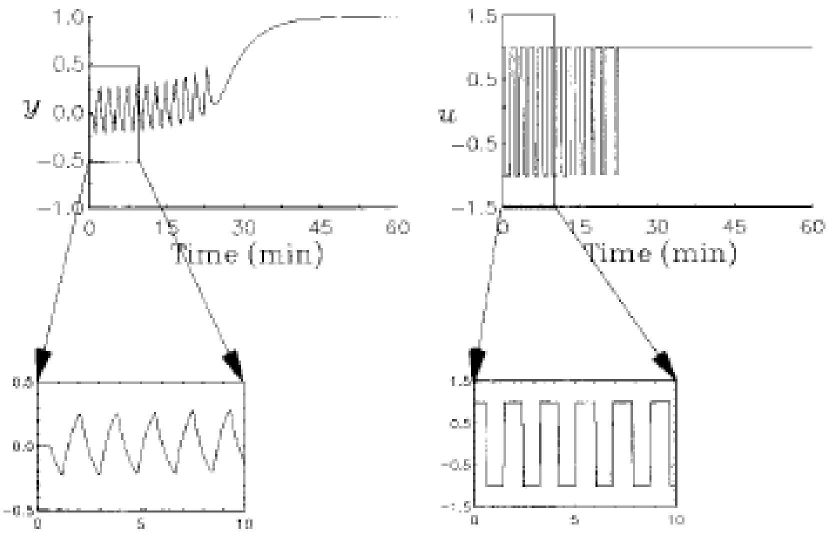


Fig. 3.3.4 Relay feedback tests for the Example with no prior knowledge (sign unknown)

The results show that the relay feedback test does not generate a stable limit cycle. More importantly, this experiment looks successful for the first 20 min (the operator or the computer is very likely to stop the experiment before t=20 min).

By ignoring any information about the sign (taking the averaging height between the 2nd and 4th cycling):

$$K_u=4h/\pi a=3.90 \quad \text{and} \quad P_u=2.44$$

The controller designed based on above information gives the Ziegler-Nichols (ZN) tuning for a PI controller as

$$K_c=1.77 \quad \text{and} \quad \tau_I=2.0$$

This set of tuning constants will produce an unstable closed loop response, since the true ultimate gain and period are:

$$K_u=1.40 \text{ and } P_u=13.5.$$

If the knowledge about the sign of the system is utilized, a positive change in u one will result an increase in y and the relay switches accordingly. This results a successful relay feedback test as shown in Fig. 3.3.5. Again, the ZN setting becomes:

$$K_c=0.636 \text{ and } \tau_I=11.25$$

This gives stable set point responses.

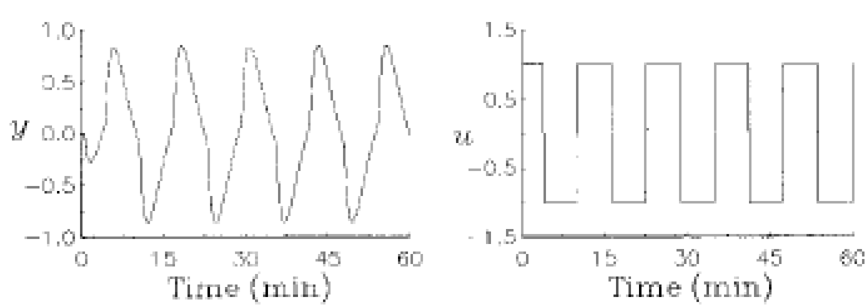


Fig. 3.3.5 Relay feedback tests for the Example with correct prior knowledge (positive sign)

3.3.4 Open-loop Unstable Systems

Open-loop unstable systems are often encountered in chemical and biochemical industries. Systems with one RHP poles are classified according to the shape of the Nyquist plot.

1. The Nyquist plot starts from the negative real axis and moves toward the third quadrant

$$\lim_{\omega \to 0} \frac{d\,\mathrm{Im}(G)}{d\omega} < 0$$

where Im(.) stands for the imaginary part of a complex number. A relay feedback test gives sustained oscillation regardless of the knowledge about the process gain (positive or negative sign).

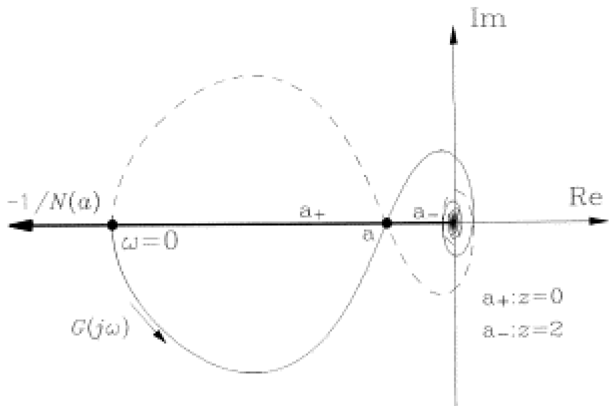
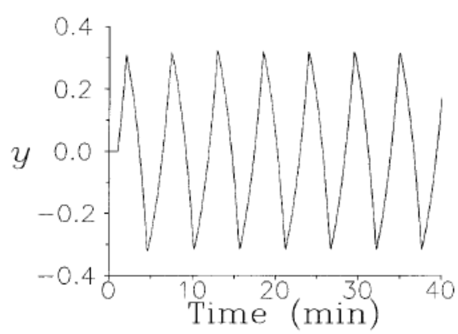


Fig. 3.3.6. Nyquist plot and $-1/N$ loci for open-loop unstable system with $\lim_{\omega \to 0} \frac{d\,\mathrm{Im}(G)}{d\omega} < 0$

Example: Open-loop unstable system with

$$G(s) = \frac{(3s+1)e^{-s}}{(3s-1)(4s+1)}$$

Fig. 3.3.7 shows a relay feedback test gives sustained oscillation regardless of the knowledge about the process gain (positive or negative sign).



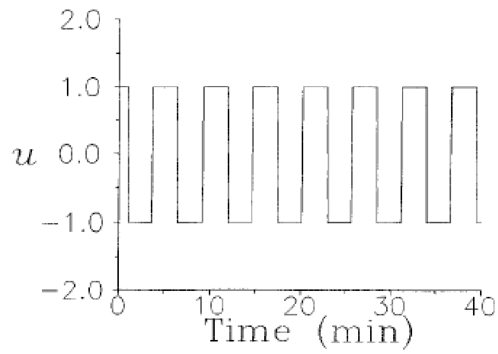


Fig. 3.3.7 Relay feedback test for open-loop unstable system

2. The Nyquist plot satisfies

$$\lim_{\omega \to 0} \frac{d\,\mathrm{Im}(G)}{d\omega} > 0$$

In such case, a relay feedback test may or may not give sustained oscillation regardless of the knowledge about the process gain (positive or negative sign).

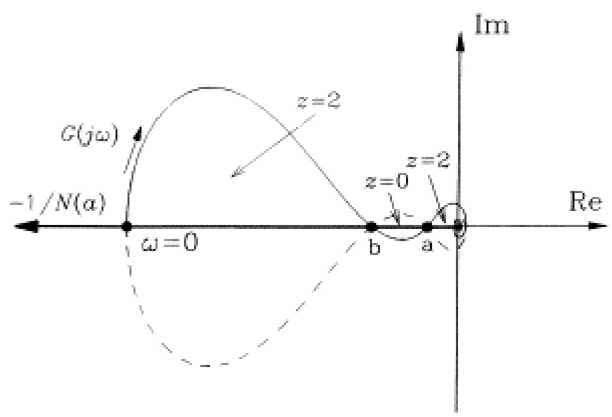


Fig. 3.3.8. Nyquist plot and $-1/N$ loci for open-loop unstable system with $\lim_{\omega \to 0} \frac{d\,\mathrm{Im}(G)}{d\omega} > 0$

Example: Open-loop unstable system with

$$G(s) = \frac{(s+1)e^{-s}}{(2s-1)(10s+1)}$$

The Nyquist plot of this Example shows a different characteristic where the *G(jω)* moves toward the second quadrant initially and further crosses the imaginary axis into the first quadrant. Furthermore, it is not possible to generate a stable limit cycle regardless of how the relay switches (Fig. 3.3.9).

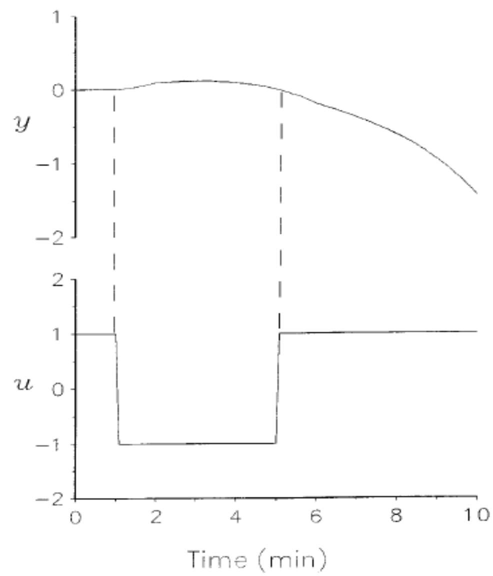


Fig. 3.3.9 Relay feedback test for open-loop unstable system

The above Example clearly indicates the relay feedback test may fail to generate continuous cycling for open-loop unstable systems and the criterion, $\lim_{\omega\to 0}\frac{d\,\mathrm{Im}(G)}{d\omega} > 0$, provides the necessary condition for this failure. However, this is not a sufficient condition.

From the on-going analyses, heuristics can be found for the success/failure of a relay feedback provided with different levels of process understanding. For open-loop stable systems, the relay feedback can generate a stable limit cycle if the knowledge of the `sign' of the process gain is utilized in the experiment (Table below). However, for open-loop unstable system, the results are not quite as clear. For some systems, it is simply not possible to generate a stable limit cycle (e.g. Example above) for an ideal relay. Table 3.3.1 summarizes these heuristics. A simple rule of thumb immediately follows: utilize the knowledge of process sign throughout the relay feedback experiment.

Table 3.3.1 Summary of success/failure for relay feedback test with different degree of prior knowledge

Sign of SS Gain	**Open loop stable system**			**Open loop unstable system**	
	No RHP zero	RHP zero		$\lim_{\omega\to 0}\frac{d\,\mathrm{Im}(G(j\omega))}{d\omega} < 0$	$\lim_{\omega\to 0}\frac{d\,\mathrm{Im}(G(j\omega))}{d\omega} > 0$
		Odd	Even		
Know	success	success	success	success	unknown
Unknown	success	fail	success	success	unknown