Chapter 6 Interaction Analysis and Control Structure

6.1 Fundamental Concepts

6.1.1 Measure of Control Loop Interactions

Control system configuration problem is concerned with defining the control system structure, i.e., which of the available plant inputs is to be used to control each of the plant outputs. In any practical problem, there are (to a greater or lesser extent) interactions in the plant. This implies that even if the control system is decentralized, subsystems of the closed-loop system are not independent of each other. To the extent that the control system can be designed to make the closed-loop subsystems independent, the idealized characteristics outlined above can be realized.

When the process interactions are significant, the choice of a control system structure is far from trivial. For an $n\times n$ plant there are $n!$ possible SISO loop pairings. For plants beyond even a modest number of inputs and outputs, a brute force approach (to design controllers for every possible pairing and then select the design which provides the best closed-loop performance) is impractical. This complexity drives the need for analysis methods to determine achievable closed-loop system characteristics as a function of control system structure independent of controller design. With these tools, pairings with unacceptable closed-loop performance can be discarded before any controllers are designed.

To illustrate, let's consider a 2×2 system. When both loops are open, m_1 and m_2 can be manipulated independently and the effect of inputs on outputs is:

$$y_1(s) = g_{11}(s)m_1(s) + g_{12}(s)m_2(s)$$
$$y_2(s) = g_{21}(s)m_1(s) + g_{22}(s)\,m_2(s)$$

where each transfer function element consists of an unspecified dynamic portion, and a steady-state gain term K_{ij}.

Consider m_1 as a candidate input variable to pair with y_1, and perform two "experiments" on the system.

Experiment 1: Unit step change in m_1 with all loops open, as shown in Fig. 6.1.1

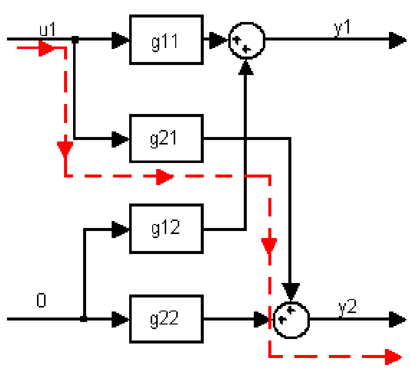


Fig. 6.1.1 TITO System with Both Loops are Open

In this case, the output variable y_1 will change, and so will y_2.Let the change in y_1 as a result of m_1 be Δy_{1m}; this is the *main effect* of m_1 on y_1. and

$$\Delta y_{1m} = g_{11}(s)$$

As no other input variable has been changed, and all the control loops are opened, there is no feedback control involved.

Experiment 2: Unit step change in m_1 with Loop 2 closed, as shown in Fig. 6.1.2

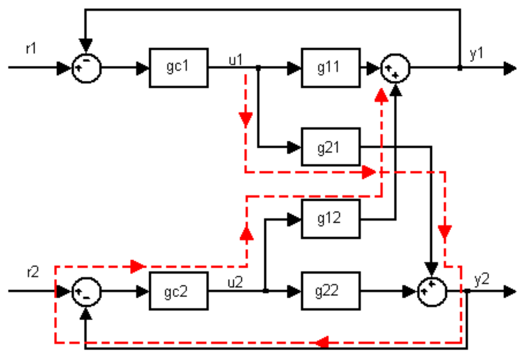


Fig. 6.1.2 TITO System with Loop 2 Closed and Loop 1 Open

In this case, the controller g_{c2} is charged with the responsibility of using the other input variable m_2 to ward off any upset in y_2 occurring as a result of this step change in m_1. m_1 has both a direct influence on y_1 and an indirect influence with the path given by the dotted line. The followings happen to the process due to the change in m_1:

1. y_1 changes because of g_{11}, but also because of interactions via the g_{21} element, so does y_2.
2. Under feedback control, Loop 2 wards off this interaction effect on y_2 by manipulating m_2 until y_2 is restored to its initial state before the occurrence of the "disturbance."
3. The changes in m_2 return to affect y_1 via the g_{12} transfer function element.

Thus, the changes observed in y_1 are from two different sources:

1. The direct influence of m_1 on y_1; Δy_{1m} and
2. The retaliatory action from the Loop 2 controller in warding off the interaction effect of m_1 on y_2 say, Δy_{1r}.

The net change in y_1, $\Delta\hat{y}_1$ given by:

$$\Delta\hat{y}_1 = \Delta y_{1m} + \Delta y_{1r} \tag{6.1.1}$$

The quantity $\Delta\hat{y}_1$ will be given by:

$$\Delta\hat{y}_1 = g_{11}(s)\left(1 - \frac{g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s)}\right) = g_{11}^*(s) \tag{6.1.2}$$

A measure of the process is controlled if m_1 is used for y_1 is:

$$\lambda_{11}(s) = \frac{\Delta y_{1m}}{\Delta\hat{y}_1} \tag{6.1.3a}$$

or

$$\lambda_{11}(s) = \frac{\Delta y_{1m}}{\Delta y_{1m} + \Delta y_{1r}} \tag{6.1.3b}$$

Thus the quantity, $\lambda_{11}(s)$, provides a measure of the extent of interaction in using m_1 to control y_1

while using m_2 to control y_2.

A similar set of “experiments” that using m_2 as the input for y_1 can be performed. By properly interpreting the measures, we can quantify the degree of steady-state interaction involved with each control configuration, and thus determine which configuration minimizes steady-state interaction.

The quantity λ_{11} is a measure of loop interaction.

1. ***$\lambda_{11}(s)=1$:*** Only if Δy_{1r} is zero which means that the main effect of m_1 on y_1 when all the loops are opened, and the total effect, when the other loop is closed are identical: m_1 does not affect y_2, or m_1 affect y_2, but control action from m_2 does not cause any changes in y_1. In this case, m_1 is the perfect input variable to use for y_1.

2. ***$\lambda_{11}(s)=0$:*** This indicates m_1 has no effect on y_1, Δy_{1m} be zero to a change in m_1. m_1 is the perfect input variable for controlling not y_1 but y_2:

3. ***$0<\lambda_{11}(s)<1$:*** For $\lambda_{11} > 0.5$, the main effect contributes more to the total effect. For $\lambda_{11} <0.5$, the contribution from interaction effect dominates. For $\lambda_{11} = 0.5$, the contributions from the two are exactly equal.

4. ***$\lambda_{11}(s) > 1$:*** When Δy_{1r}, is opposite in sign to Δy_{1m}, but smaller in value, the total effect, Δy_{1r}^*, is less than the main effect, Δy_{1m}, a larger m_1 is required to achieve a given change in y_1 in the closed loop than in the open loop. For λ_{11} very large and positive, the interactive effect almost cancels the main effect and closed-loop control of y_1 by m_1 will be very difficult to achieve.

5. ***$\lambda_{11}(s)<0$:*** When Δy_{1r} is not only opposite in sign to Δy_{1m}, but is larger in absolute value, the pairing of m_1 with y_1 is not desirable because the direction of the effect of m_1 on y_1 in the open loop is opposite to the direction in the closed 1oop.

6.1.2 Dynamic Relative Gain Array (DRGA)

Define $\lambda_{ij}(s)$, the relative gain between output variable y_i and input variable m_j, as the ratio of two dynamic gains:

$$\lambda_{ij}(s) = \frac{\left(\dfrac{\partial y_i}{\partial m_j}\right)_{all\ loop\ open}}{\left(\dfrac{\partial y_i}{\partial m_j}\right)_{all\ loop\ closed\ except\ m_j\ loop}} \tag{6.1.4}$$

$$= \left(\frac{open-loop\ gain}{closed-loop\ gain}\right)_{for\ loops\ closed\ except\ for\ the\ m_j\ loop}$$

When the relative gain is calculated for all the input/output combinations of a multivariable system, it results in an array of the form:

$$\Lambda(s) = \begin{bmatrix} \lambda_{11}(s) & \lambda_{12}(s) & \ldots & \lambda_{1n}(s) \\ \lambda_{21}(s) & \lambda_{22}(s) & \ldots & \lambda_{2n}(s) \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{n1}(s) & \lambda_{n2}(s) & \ldots & \lambda_{nn}(s) \end{bmatrix} \tag{6.1.5}$$

the result is the *dynamic relative gain array* (DRGA) and can be used to measure loop interactions.

Important properties of the RGA:

1. The elements of the RGA across any row, or down any column, sum up to 1, for $s \in (0, \infty)$, i.e.:

$$\sum_{i=1}^{n} \lambda_{ij}(s) = \sum_{j=1}^{n} \lambda_{ij}(s) = 1 \tag{6.1.6}$$

2. $\lambda_{ij}(s)$ is dimensionless, neither the units, nor the absolute values taken by the variables m_j or y_i affect it.

3. The value of $\lambda_{ij}(s)$ is a measure of the *dynamic* interaction expected in the *i*th loop if its output y_i is paired with m_j.

4. Let $\hat{g}_{ij}(s)$ represent the loop *i* dynamic gain when all the other loops are closed, whereas $g_{ij}(s)$ represents the normal, *open-loop* gain. By the definition of $\lambda_{ij}(s)$, that:

$$\hat{g}_{ij}(s) = \frac{1}{\lambda_{ij}} g_{ij}(s) \tag{6.1.7}$$

   $1/\lambda_{ij}(s)$ tells by what factor the *open-loop gain* between y_i and m_j will be altered when the other loops are closed.

5. When $\lambda_{ij}(s)$ is negative, it is indicative of a situation in which loop *i*, with all loops open, will produce a change in y_i in response to a change in m_j totally opposite in direction to that when the other loops are closed. Such input/output pairings are potentially unstable and should be avoided.

Obtaining the DRGA by first-principles method: We illustrate the procedure by a 2×2 system, the model is:

$$\begin{aligned} y_1(s) &= g_{11}(s)m_1(s) + g_{12}(s)m_2(s) \\ y_2(s) &= g_{21}(s)m_1(s) + g_{22}(s)\, m_2(s) \end{aligned} \tag{6.1.8}$$

To obtain λ_{11} from the definition, we first need to evaluate the partial derivatives for open-loop conditions, the numerator partial derivative is obtained:

$$\left(\frac{\partial y_1}{\partial m_1}\right)_{all\ \ loops\ \ open} = g_{11}(s) \tag{6.1.9}$$

The second partial derivative is to close Loop 2, to obtain the second partial derivative, the value m_2 must take to keep $y_2 = 0$ in changes in m_1; Setting $y_2 = 0$ and solving for m_2 gives:

$$m_2 = -\frac{g_{21}(s)}{g_{22}(s)} m_1 \tag{6.1.10}$$

and substituting this into y_1 gives:

$$y_1 = g_{11}(s)m_1 - \frac{g_{12}(s)g_{21}(s)}{g_{22}(s)} m_1 \tag{6.1.11}$$

Having eliminated m_2 we may then differentiate, so that:

$$\left(\frac{\partial y_1}{\partial m_1}\right)_{loop\ 2\ closed} = g_{11}(s)\left(1-\frac{g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s)}\right) = \hat{g}_{11} \tag{6.1.12}$$

and we obtain:

$$\lambda_{11}(s) = \frac{1}{1-\zeta(s)} \tag{6.1.13}$$

where

$$\zeta(s) = \frac{g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s)}$$

As an exercise, to illustrate the procedure and confirm that the other relative gains, $\lambda_{12}, \lambda_{21}$, and λ_{22} are given by:

$$\lambda_{12}(s) = \lambda_{21}(s) = \frac{-\zeta(s)}{1-\zeta(s)} \qquad \lambda_{22}(s) = \lambda_{11}(s) = \frac{1}{1-\zeta(s)}$$

Thus, the RGA for the 2×2 system is given by:

$$\Lambda(s) = \begin{bmatrix} \frac{1}{1-\zeta(s)} & \frac{-\zeta(s)}{1-\zeta(s)} \\ \frac{-\zeta(s)}{1-\zeta(s)} & \frac{1}{1-\zeta(s)} \end{bmatrix} \tag{6.1.14a}$$

It is useful to observe that if we define:

$$\lambda(s) = \lambda_{11}(s) = \frac{1}{1-\zeta(s)}$$

for this 2×2 system, the RGA may be written:

$$\Lambda(s) = \begin{bmatrix} \lambda(s) & 1-\lambda(s) \\ 1-\lambda(s) & \lambda(s) \end{bmatrix} \tag{6.1.14b}$$

The single element *λ(s)* is sufficient to determine the RGA for a *2x2* system, it is called the relative gain parameter. The method will be too tedious to apply in cases where the system is of higher dimension than 2 x 2.

***Calculation of DRGA by Matrix operation**:* Consider an $n \times n$ plant *G(s).*

$$y(s) = G(s)m(s) \tag{6.1.15}$$

The open loop gain from input $m_j(s)$ to output $y_i(s)$ is $g_{ij}(s)$ when all other outputs are uncontrolled. Writing equation (6.1.15) as

$$m(s) = G^{-1}(s)y(s)\,, \tag{6.1.16}$$

it can be seen that the gain from $m_j(s)$ to $y_i(s)$ is $1/\left[G^{-1}(s)\right]_{ji}$ when all other outputs are perfectly controlled. The relative gain is the ratio of these "open-loop" and "closed-loop" gains. Thus a matrix of relative gains, the RGA matrix, can be computed using the formula

$$\Lambda(s) = G(s) \otimes G^{-T}(s) \tag{6.1.17}$$

where the $\otimes$ symbol denotes element by element multiplication (Hadamard or Schur product).

6.1.3 Interaction Analysis Based on Steady State Information

The inverse $G^{-T}(s)$ may be non-proper or non-causal, and a physical interpretation in terms of perfect control is not meaningful except at steady-state. This has restricted the applications of DRGA to plants with no RHP-zeros. To relax its restrictions, we consider $s = 0$, i. e. the process is at steady state. In this case, the DRGA become RGA and the calculations are as follows:

Calculate RGA by first-principles method: We illustrate the procedure by a 2×2 system. For the steady-state, the model is:

$$\begin{aligned} y_1 &= K_{11}m_1 + K_{12}m_2 \\ y_2 &= K_{21}m_1 + K_{22}m_2 \end{aligned} \tag{6.1.18}$$

To obtain λ_{11} from the definition, we first need to evaluate the partial derivatives for open-loop conditions, the numerator partial derivative is obtained:

$$\left(\frac{\partial y_1}{\partial m_1}\right)_{all\ loops\ open} = K_{11} \tag{6.1.19}$$

The second partial derivative is to close Loop 2, to obtain the second partial derivative, the value m_2 must take to keep $y_2 = 0$ in changes in m_1; Setting $y_2 = 0$ and solving for m_2 gives:

$$m_2 = -\frac{K_{21}}{K_{22}}m_1 \tag{6.1.20}$$

and substituting this into y_1 gives:

$$\hat{y}_1 = K_{11}m_1 - \frac{K_{12}K_{21}}{K_{22}}m_1 \tag{6.1.21}$$

Having eliminated m_2 we may then differentiate, so that:

$$\left(\frac{\partial \hat{y}_1}{\partial m_1}\right)_{loop\ 2\ closed} = K_{11}\left(1 - \frac{K_{12}K_{21}}{K_{11}K_{22}}\right) \tag{6.1.22}$$

and we obtain:

$$\lambda_{11} = \frac{1}{1-\zeta} \quad \text{with} \quad \zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}} \tag{6.1.23}$$

As an exercise, to illustrate the procedure and confirm that the other relative gains, $\lambda_{12}, \lambda_{21}$, and λ_{22} are given by:

$$\lambda_{12} = \lambda_{21} = \frac{-\zeta}{1-\zeta} \qquad \lambda_{22} = \lambda_{11} = \frac{1}{1-\zeta}$$

Thus the RGA for the 2×2 system is given by:

$$\Lambda = \begin{bmatrix} \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \\ \frac{-\zeta}{1-\zeta} & \frac{1}{1-\zeta} \end{bmatrix} \tag{6.1.24a}$$

It is useful to observe that if we define:

$$\lambda = \lambda_{11} = \frac{1}{1-\zeta}$$

for this 2×2 system, the RGA may be written:

$$\Lambda = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix} \tag{6.1.24b}$$

The single element λ is sufficient to determine the RGA for a 2×2 system, it is called the relative gain parameter. Using first-principles method of obtaining the RGA, it will be too tedious to apply in cases where the system is of higher dimension than 2×2.

Calculation of RGA by Matrix operation*:* Let K be the matrix of steady-state gains of $G(s)$, i.e.:

$$\lim_{s\to 0} G(s) = K \tag{6.1.25}$$

whose elements are K_{ij}; let R be the transpose of the inverse of K, i.e.:

$$R = (K^{-1})^T \tag{6.1.26}$$

with elements r_{ij}. Then, the elements of the RGA can be obtained from the elements of these two matrices according to:

$$\lambda_{ij} = K_{ij} \otimes r_{ij} \tag{6.1.27}$$

6.1.4 Interpreting the RGA Elements and loop pairing rules

1. λ_{ij} = 1, the open-loop gain and closed-loop gain between y_i and m_j are *identical*:

   Implication for loop interactions: Loop i will *not* be subject to retaliatory action from other loops when they are closed, m_j can control y_i without interference from the other control loops. If any of the g_{ik} elements in the transfer function matrix are nonzero, then the ith loop will experience some disturbances from control actions taken in the other loops, but they are not provoked by control actions in the ith loop.

   Pairing recommendation: pairing y_i with m_j.

2. λ_{ij} = 0, open-loop gain between y_i and m_j is *zero:*

   Implication for loop interactions: m_j has no direct influence on y_i, even though it may affect other output variables.

   Pairing recommendation: do *not* pair y_i with m_j.

3. 0< λ_{ij} <1, indicating that the open-loop gain between y_i and m_j is *smaller* than the closed-loop gain:

   Implication for loop interactions: As the closed-loop gain is the sum of the open-loop gain and the retaliatory effect from the other loops: The loops are definitely interacting, but the retaliatory effect from the other loops is in the same direction as the main effect of m_j on y_i. Thus the loop interactions "assist" m_j in controlling y_i. The extent is indicated by how close λ_{ij} is to 0.5:

   λ_{ij} = 0.5, the main effect of m_j on y_i is exactly identical to the retaliatory effect from other loops

$0.5 < \lambda_{ij} < 1$ the retaliatory effect from other loops is lower than the main effect of m_j on y_i;

$0 < \lambda_{ij} < 0.5$, the retaliatory effect is more substantial than the main effect.

Pairing recommendation: Avoid pairing m_j with y_i if $\lambda_{ij} \le 0.5$.

4. $\lambda_{ij} > 1$, indicating that the open-loop gain between y_i and m_j is *larger* than the closed-loop gain:

Implication for loop interactions: the loops interact, and the retaliatory effect from other loops acts in *opposition* to the main effect of m_j on y_i; main effect is still dominant. For large λ_{ij} values, the controller gain for loop *i* will have to be much larger than when other loops are open. This could cause loop *i* to become unstable when other loops are open.

Pairing recommendation: where possible, do not pair m_j with y_i if λ_{ij} takes a very high value.

5. $\lambda_{ij} < 0$; indicating that the open-loop and closed-loop gains between y_i and m_j have opposite signs:

Implication for loop interactions: the loops interact, and the retaliatory effect from the other loops is not only in *opposition* to the main effect of m_j on y_i, it is also the more dominant. This is a dangerous situation because opening the other loops will likely cause 1oop *i* to become unstable.

Pairing recommendation: avoid pairing m_j with y_i.

The ideal situation occurs when the RGA takes the form:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

It is obtained when the process transfer function matrix is diagonal, or triangular; the first indicates no interaction among the loops, while the latter indicates one-way interaction.

For *one-way* interaction case, consider the 2×2 system:

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{3}{3s+1} & \dfrac{4}{4s+1} \end{bmatrix}$$

Calculation of the RGA yields:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because element $g_{12}(s)$ is zero, the input m_2 has no effect on output y_1, any control action in Loop 2 has no effect on Loop 1. However, the input m_1 does influence the output y_2 through the nonzero $g_{21}(s)$ element. Upsets in Loop 1 requiring control action by m_1 would appear as a disturbance, which have to be handled by the feedback controller of Loop 2. Hence, even the RGA is ideal, Loop 2 would be at a disadvantage

In the non-ideal cases, the following pairing rule should be used:

Rule 1: *Pair input and output variables that have positive RGA elements that are closest to 1.0.*

$$\min \sum \left|\lambda_{ij}^{k} - 1.0\right| \qquad \forall\ k \tag{6.1.28}$$

where λ_{ij}^{k} denotes the paired RGA elements corresponding to the *k*th alternative.

Thus, for a 2×2 system, if the RGA is:

$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

then it is recommended pair to pair y_1 with m_1, and y_2 with m_2. On the other hand, for the 2×2 system whose RGA is:

$$\Lambda = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}$$

the recommended pairing is 1-2/2-1 pairing

Fro the following RGA:

$$\Lambda = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

it recommends the 1-1/2-2 pairing to avoid pairing on a negative RGA element. Despite pairing on RGA values greater than 1 is not very desirable, to pair on a negative RGA value is worse still.

For the 3×3 distillation column with the RGA:

$$\Lambda = \begin{bmatrix} 1.95 & -0.65 & -0.3 \\ -0.66 & 1.88 & -0.22 \\ -0.29 & -0.23 & 1.52 \end{bmatrix}$$

1-1/2-2/3-3 pairing is recommended.

Niederlinski Theorem: Consider an n x n multivariable system whose input and output have been paired as follows: y_1 — u_1, y_2 — u_2, ... , y_n — u_n, resulting in a transfer function model of the form:

$$y(s) = G(s)u(s) \tag{6.1.29}$$

with $g_{ij}(s)$ be Rational, and Open-loop stable, and let *n* individual feedback controllers (with integral action) be designed for each loop so that each of the resulting n feedback control loops is stable when all the other *n-1* loops are open.

Under closed-loop conditions in all *n* loops, the system will be unstable for all possible values of controller parameters if the Niederlinski index NI defined below is negative, i.e.:

$$NI \equiv \frac{\left|G(0)\right|}{\prod_{i=1}^{n} g_{ii}(0)} < 0 \tag{6.1.30}$$

Note:

1. It necessary and sufficient only for 2×2 systems; for higher dimensional systems, it is only sufficient conditions: if it holds then the system is definitely unstable; if it does not hold,

the system may, or may not be unstable: the stability depend on the values taken by the controller parameters.

2. For 2 x 2 systems the Niederlinski index becomes:

$$NI = \zeta = \frac{K_{11}K_{22} - K_{12}K_{21}}{K_{11}K_{22}} = 1 - \frac{K_{12}K_{21}}{K_{11}K_{22}} = 1 - \zeta \qquad (6.1.31)$$

If $\zeta > 1$, the Niederlinski index is always negative; hence 2×2 systems paired with negative relative gains are always structurally unstable.

3. The theorem is for systems with rational transfer function elements, thereby excluding time-delay systems. However, since it depends only on steady-state gains, the results of this theorem provide useful information about time-delay systems also, but the analysis is no longer rigorous. It should be applied with caution when time delays are involved.

Rule 2: *Any loop pairing is unacceptable if it leads to a control system configuration for which the Niederlinski index is negative.*

The strategy for using the RGA and *Niederlinski* index for loop pairing may be summarized as follows:

1. Given $G(s)$, obtain steady-state gain matrix $K = G(0)$, obtain the RGA, Λ; obtain the determinant of K, and the product of the elements on its main diagonal.
2. Use Rule #1 to obtain tentative loop pairing suggestions from the RGA, by pairing on positive elements which are closest to 1.0.
3. Use Niederlinski's condition to verify the stability status of the control configuration resulting from 2; if the pairing is unacceptable, select another.
4. Variables should be paired in such a way that the resulting pairing corresponds to an *NI* closest to 1.0. The NI interaction rule is based on *empirical* observations of the definition of the *NI* and is justified largely from the relationship between the *size* of the RGA and that of the *NI*. The *NI* interaction rule has been found to be capable of avoiding ambiguities in using the RGA interaction rule.

6.1.5 Applications of the Loop Pairing Rules

Example 6.1.1: *RGA for the wood and binary distillation column*

Find the RGA for the Wood and Berry binary distillation column whose transfer function matrix is given as:

$$G(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

For this system, the steady-state gain matrix is obtained from the transfer function matrix by setting s=0, giving:

$$K = G(0) = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix}$$

its inverse and transpose of inverse are obtained as:

$$K^{-1} = \begin{bmatrix} 0.157 & -0.153 \\ 0.053 & -0.104 \end{bmatrix} \qquad R = (K^{-1})^T = \begin{bmatrix} 0.157 & 0.053 \\ -0.153 & -0.104 \end{bmatrix}$$

term-by-term multiplication of K_{ij} and r_{ij} gives the RGA as:

$$\Lambda = \begin{bmatrix} 2.0 & -1.0 \\ -1.0 & 2.0 \end{bmatrix}$$

Example 6.1.2: *Loop pairing for a 3×3 system*

Calculate RGA for system with steady-state gain matrix:

$$K = G(0) = \begin{bmatrix} \frac{5}{3} & 1 & 1 \\ 1 & \frac{1}{3} & 1 \\ 1 & 1 & \frac{1}{3} \end{bmatrix}$$

By taking the inverse, and calculating the RGA, we obtain:

$$\Lambda = \begin{bmatrix} 10 & -4.5 & -4.5 \\ -4.5 & 1 & 4.5 \\ -4.5 & 4.5 & 1 \end{bmatrix}$$

from which, by Rule #1, the *1-1/2-2/3-3* pairing is recommended.

According to this pairing, the steady-state gain matrix for the system has determinant given by:

$$|K| = -0.148$$

and the product of the diagonal elements of K is:

$$\prod_{i=1}^{3} K_{ii} = \left(\frac{5}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \left(\frac{5}{27}\right)$$

Thus, for this system, this pairing will lead to a negative Niederlinski index and an unstable configuration.

This is the case in which Rule #2 disqualifies a loop pairing suggested by Rule #1;

Example 6.1.3: an alternative loop pairing is considered for a possible pairing of 1-1/2-3/3-2 which has relative gain array:

$$\Lambda = \begin{bmatrix} 10 & -4.5 & -4.5 \\ -4.5 & 4.5 & 1 \\ -4.5 & 1 & 4.5 \end{bmatrix}$$

and new steady-state gain matrix:

$$K = G(0) = \begin{bmatrix} \frac{5}{3} & 1 & 1 \\ 1 & 1 & 1/3 \\ 1 & 1/3 & 1 \end{bmatrix}$$

with determinant $|K| = \dfrac{4}{27}$ and Niederlinski index:

$$N = \frac{4/27}{5/3} = \frac{4}{45}$$

with this pairing the system is no longer structurally unstable.

6.1.6 Semi-dynamic Loop Pairing

Assume that the process $\bar{g}_{ij}(s)$ is open-loop stable and its output $\bar{y}_i = \bar{g}_{ij}(s)u_j$ is initially rest at zero, then a unit step disturbance is applied at the process input u_j. Since most industrial processes are either non-oscillatory or even oscillatory but well damped as shown in Figure 6.2.5 ($\bar{A}_{ij}$ indicated by the shade area), the process output $\bar{y}_i$ will go to unity. We thus have

$$\bar{A}_{ij} = \int_0^{\infty} (\bar{y}_i(\infty) - \bar{y}_i(t))dt \,. \tag{6.1.32}$$

As a accumulation of the difference between the expected and the real outputs of process $\bar{g}_{ij}(s)$, $\bar{A}_{ij}$, in fact, is equal to the average residence time $\tau_{ar,ij}$ of $\bar{g}_{ij}(s)$, i.e., $\tau_{ar,ij} = \bar{A}_{ij}$. As a measure of the time from inject a step to input to the process response reaches 63.2% of its steady state value, the average residence time $\tau_{ar,ij}$ can effectively reflect the process dynamics of $\bar{g}_{ij}(s)$ as smaller $\tau_{ar,ij}$ indicates the transfer function has fast response to input disturbance, while larger $\tau_{ar,ij}$ indicates the open-loop process has slower process dynamics.

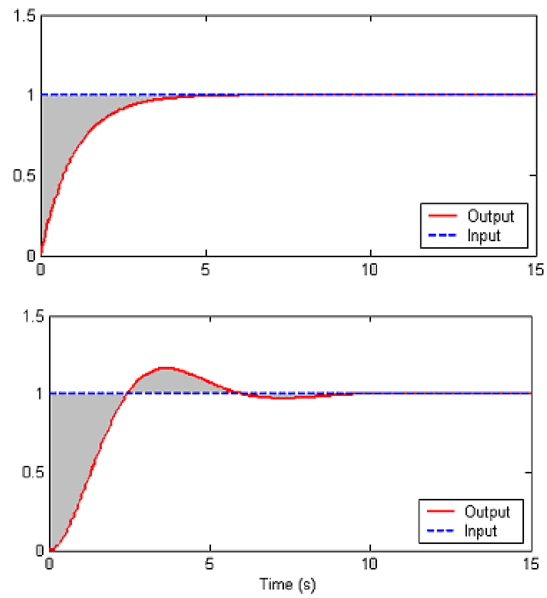


Figure 6.2.3 Typical waveforms of non-oscillatory (top) and oscillatory (bottom) processes

Thus, we obtained two important parameters for the process $g_{ij}(s)$

- Steady state gain $g_{ij}(j0)$: The steady state gain reflects the effect of the manipulated variable u_j to the controlled variable y_i;
- Average residence time $\tau_{ar,ij}$: The average residence time is accountable for the response speed of the controlled variable y_i to manipulated variable u_j.

In order to use above both parameters for interaction measure and loop pairing, we now define the normalized gain (NG) $k_{N,ij}$ for a particular transfer function $g_{ij}(s)$ as

$$k_{N,ij} = \frac{g_{ij}(j0)}{\tau_{ar,ij}} \tag{6.1.33}$$

Equation (6.1.33) indicates that a large value of $k_{N,ij}$ implies that the combination effect of the manipulated variable u_j to the controlled variable y_i and the response speed of the controlled variable y_i to manipulated variable u_j is large. Therefore, the loop pairing with large normalized gain $k_{N,ij}$ should be preferred.

Extend equation (6.1.33) to all elements of transfer function matrix $\mathbf{G}(s)$, one can obtain the normalized gain matrix $\mathbf{K}_N$ as

$$\mathbf{K}_N = [k_{N,ij}]_{n\times n} = \mathbf{G}(j0) \odot \mathbf{T}_{ar} \tag{6..34}$$

where $\mathbf{T}_{ar} = [\tau_{ar,ij}]_{n\times n}$ and $\odot$ indicates element-by-element division.

Since $k_{N,ij}$ indicates the control effectiveness from manipulated variable u_j to controlled variable y_i in terms of both steady state and process dynamics, the bigger the $k_{N,ij}$ value is, the more dominant the loop will be.

Even though more precise higher-order process models can be obtained by either physical model construction (following the mass and energy balance principles) or the classical parameter identification methods, from a practical point of view, the lower order process models are more convenient for control system design. The normalized gains of both first order plus delay time (FOPDT) and second order plus delay time (SOPDT) processes are given below.

1. The transfer function for FOPDT process is given as

$$g_{ij}(s) = \frac{k_{ij}}{\tau_{ij}s+1}e^{-\theta_{ij}s}.$$

The normalized transfer function and its step response in time domain are thus obtained respectively as:

$$g_{ij}(s) = \frac{1}{\tau_{ij}s+1}e^{-\theta_{ij}s},$$

and

$$\bar{y}_i(t) = 1 - e^{-(t-\theta_{ij})/\tau_{ij}}.$$

Subsequently, the average residence time $\tau_{ar,ij}$ can be obtained as

$$\begin{aligned}\tau_{ar,ij} &= \bar{A}_{ij} = \int_0^\infty \left[\bar{y}_i(\infty) - \bar{y}_i(t)\right]dt \\ &= \int_0^\infty [1-(1-e^{-(t-\theta_{ij})/\tau_{ij}})]dt \\ &= \int_0^\infty e^{-(t-\theta_{ij})/\tau_{ij}}dt = \tau_{ij} + \theta_{ij}\end{aligned}$$

Hence, the normalized gain of $g_{ij}(s)$ is obtained as

$$k_{N,ij} = \frac{k_{ij}}{\tau_{ar,ij}} = \frac{k_{ij}}{\tau_{ij}+\theta_{ij}} \tag{6.1.34}$$

2. The transfer function for SOPDT process is given as

$$g_{ij}(s) = k_{ij} \times \frac{\omega_{n,ij}^2}{s^2 + 2\zeta_{ij}\omega_{n,ij}s + \omega_{n,ij}^2} e^{-\theta_{ij}s} \tag{6.1.35}$$

Two cases should be considered:

(i) When $0 < \xi_{ij} < 1$, the transient function and its step response in time domain are thus obtained respectively as

$$\bar{g}_{ij}(s) = \frac{\omega_{n,ij}^2}{s^2 + 2\zeta_{ij}\omega_{n,ij}s + \omega_{n,ij}^2} e^{-\theta_{ij}s} \tag{6.1.36}$$

and

$$\bar{y}_i(t) = \begin{cases} 0 & t < \theta_{ij} \\ 1 - \dfrac{e^{-\zeta_{ij}\omega_{n,ij}(t-\theta_{ij})}}{\sqrt{1-\zeta_{ij}^2}} \sin\left[\omega_{n,ij}\sqrt{1-\zeta_{ij}^2}\left(t-\theta_{ij}\right) + \tan^{-1}\dfrac{\sqrt{1-\zeta_{ij}^2}}{\zeta_{ij}}\right] & t \ge \theta_{ij} \end{cases}$$

Subsequently, the average residence time $\tau_{ar,ij}$ can be obtained as

$$\begin{aligned} \tau_{ar,ij} &= \bar{A}_{ij} = \int_0^\infty \left[\bar{y}_i(\infty) - \bar{y}_i(t)\right]dt \\ &= \int_0^{\theta_{ij}} 1dt + \int_{\theta_{ij}}^{\infty} \frac{e^{-\zeta_{ij}\omega_{n,ij}(t-\theta_{ij})}}{\sqrt{1-\zeta_{ij}^2}} \sin\left[\omega_{n,ij}\sqrt{1-\zeta_{ij}^2}\left(t-\theta_{ij}\right) + \tan^{-1}\frac{\sqrt{1-\zeta_{ij}^2}}{\zeta_{ij}}\right]dt \\ &= \frac{2\zeta_{ij}}{\omega_{n,ij}} + \theta_{ij} \end{aligned}$$

(ii) When $1 < \xi_{ij} < \infty$, the transfer function given in equation (6.1.35) can be re-written as

$$\bar{g}_{ij}(s) = \frac{1}{\left(\tau_{1,ij}s+1\right)\left(\tau_{2,ij}s+1\right)} e^{-\theta_{ij}s}, \tag{6.1.37}$$

with

$$\tau_{1,ij} = \frac{1}{\omega_{n,ij}\left(\zeta_{ij} + \sqrt{\zeta_{ij}^2 - 1}\right)} \text{ and } \tau_{2,ij} = \frac{1}{\omega_{n,ij}\left(\zeta_{ij} - \sqrt{\zeta_{ij}^2 - 1}\right)}.$$

The step response in time domain is thus obtained as

$$\bar{y}_i(t) = \begin{cases} 0 & t < \theta_{ij} \\ 1 + \dfrac{1}{\tau_{2,ij} - \tau_{1,ij}}\left(\tau_{1,ij}e^{-\frac{t-\theta_{ij}}{\tau_{1,ij}}} - \tau_{2,ij}e^{-\frac{t-\theta_{ij}}{\tau_{2,ij}}}\right) & t \ge \theta_{ij} \end{cases}.$$

Subsequently, the average residence time $\tau_{ar,ij}$ can be obtained as

$$\tau_{ar,ij} = \bar{A}_{ij} = \int_0^\infty \left[\bar{y}_i(\infty) - \bar{y}_i(t)\right] dt$$
$$= \int_0^{\theta_{ij}} 1dt + \int_{\theta_{ij}}^{\infty} \frac{-1}{\tau_{2,ij} - \tau_{1,ij}} \left(\tau_{1,ij} e^{-\frac{t-\theta_{ij}}{\tau_{1,ij}}} - \tau_{2,ij} e^{-\frac{t-\theta_{ij}}{\tau_{2,ij}}} \right) dt$$
$$= \tau_{1,ij} + \tau_{2,ij} + \theta_{ij} = \frac{2\zeta_{ij}}{\omega_{n,ij}} + \theta_{ij}$$

Hence, the normalized gain of $g_{ij}(s)$ in both cases is obtained as

$$k_{N,ij} = \frac{k_{ij}}{\tau_{ar,ij}} = \frac{k_{ij}}{\dfrac{2\zeta_{ij}}{\omega_{n,ij}} + \theta_{ij}} \tag{6.1.38}$$

Let the relative energy between output variable $y_{N,i}$ and input variable $u_{N,j}$, $\lambda_{N,ij}$, be the ratio of two transmission energies:

$$\lambda_{N,ij} = \frac{k_{N,ij}}{\hat{k}_{N,ij}} \tag{6.1.39}$$

where $\hat{k}_{N,ij}$ is the transmission energy between output variable $y_{n,i}$ and input variable $u_{e,j}$ when all other loops are closed. When the relative energies are calculated for all the input/output combinations of the energy complementary system, it results in an array of the form similar to RGA, we call it as Relative Normalized Gain Array (RNGA):

$$\Lambda_N = \begin{bmatrix} \lambda_{N,11} & \lambda_{N,12} & \cdots & \lambda_{N,1n} \\ \lambda_{N,21} & \lambda_{N,22} & \cdots & \lambda_{N,2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{N,n1} & \lambda_{N,n2} & \cdots & \lambda_{N,nn} \end{bmatrix} \tag{6.1.40}$$

which can be calculated by

$$\Lambda_N = K_N \otimes K_N^{-T} \tag{6.1.41}$$

Consequently, the properties of RGA can be direct extended to RNGA:

1. The value of $\lambda_{N,ij}$ is a measure of the interaction expected in the i^{th} loop if its output y_i is paired with u_j.
2. The elements of the RNGA across any row, or down any column, sum up to 1, i.e.

$$\sum_{i=1}^{n} \lambda_{N,ij} = \sum_{j=1}^{n} \lambda_{N,ij} = 1.$$

3. Let $\hat{k}_{N,ij}$ represent the loop i energy when all the other loops are closed, whereas $k_{e,ij}$

represents the normal, *open loop* energy, then:

$$\hat{k}_{N,ij} = \frac{1}{\lambda_{N,ij}} k_{N,ij} \tag{6.1.42}$$

4. When $\lambda_{N,ij}$ is negative, with other loops open, produces a energy change in y_i in response to a change in u_j totally opposite in direction to that when the other loops are closed.

As RGA and NI tools are based on steady state information and can provide sufficient conditions for the structurally unstable control configurations, they are adopted here to eliminate those structures with unstable pairing options. Thus, the RGA-NI-RNGA based control configuration rules are developed as: Manipulated and controlled variables in a decentralized control system should be paired in the following way that:

(i) All paired RGA elements are positive,

(ii) NI is positive,

(iii) The paired RNGA elements are closest to 1.0.

(iv) Large RNGA elements should be avoided.

Example 6.1.4: Consider a process with transfer function matrix

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{5e^{-40s}}{100s+1} & \dfrac{e^{-4s}}{10s+1} \\ \dfrac{-5e^{-4s}}{10s+1} & \dfrac{5e^{-40s}}{100s+1} \end{bmatrix}$$

The RGA is

$$\mathbf{\Lambda}(\mathbf{G}(j0)) = \begin{pmatrix} 0.8333 & 0.1667 \\ 0.1667 & 0.8333 \end{pmatrix}.$$

Obviously, both diagonal and off-diagonal pairings have positive RGA elements, and it is easy to verify that they also have positive NIs (diagonal pairing: NI = 1.2 and off-diagonal pairing: NI = 6.0, respectively). Hence, the RGA based loop pairing rule definitely suggests the diagonal pairing because their relative gains are close to unity. However, the pairing criterion based on RNGA makes a very different selection.

According to Appendix, the normalized gain matrix is obtained as

$$\mathbf{K}_N = \begin{bmatrix} 0.0357 & 0.0714 \\ -0.3571 & 0.0357 \end{bmatrix}, \qquad \Lambda_N = \mathbf{K}_N \otimes \mathbf{K}_N^{-T} = \begin{bmatrix} 0.0476 & 0.9524 \\ 0.9524 & 0.0476 \end{bmatrix}$$

Apparently, the off-diagonal pairing is the best one with the smallest interactions between control loops, and should be selected.

Example 6.1.5: Consider a 3×3 process given by

$$\mathbf{G}(s)=\begin{bmatrix} \dfrac{-2e^{-s}}{10s+1} & \dfrac{1.5e^{-s}}{s+1} & \dfrac{e^{-s}}{s+1} \\ \dfrac{1.5e^{-s}}{s+1} & \dfrac{e^{-s}}{s+1} & \dfrac{-2e^{-s}}{10s+1} \\ \dfrac{e^{-s}}{s+1} & \dfrac{-2e^{-s}}{10s+1} & \dfrac{1.5e^{-s}}{s+1} \end{bmatrix}.$$

The RGA of the system is

$$\mathbf{\Lambda}(\mathbf{G}(j0))=\begin{pmatrix} -0.9302 & 1.1860 & 0.7442 \\ 1.1860 & 0.7442 & -0.9302 \\ 0.7442 & -0.9302 & 1.1860 \end{pmatrix}.$$

From the RGA based loop pairing rules, the desirability of pairing $y_1-u_2/y_2-u_1/y_3-u_3$ is close to that of $y_1-u_3/y_2-u_2/y_3-u_1$ pairing, and it cannot be determined whether pairing $y_1-u_2/y_2-u_1/y_3-u_3$ outperforms pairing $y_1-u_3/y_2-u_2/y_3-u_1$ or not. Without calculating the corresponding NI, closed-loop instability of the latter pairing $y_1-u_3/y_2-u_2/y_3-u_1$ is not reflected from the RGA.

Applying the RNGA approach on the process,

$$\mathbf{K}_N=\begin{bmatrix} -0.1818 & 0.7500 & 0.5000 \\ 0.7500 & 0.5000 & -0.1818 \\ 0.5000 & -0.1818 & 0.7500 \end{bmatrix}$$

$$\Lambda_N=\mathbf{K}_N\otimes\mathbf{K}_N^{-T}=\begin{bmatrix} 0.0834 & 0.6574 & 0.2592 \\ 0.6574 & 0.2592 & 0.0834 \\ 0.2592 & 0.0834 & 0.6574 \end{bmatrix}$$

which indicates that the pairing $y_1-u_2/y_2-u_1/y_3-u_3$ ($\mathrm{NI}=0.2593>0$) should be preferred for decentralized control. The same loop pairing decision was obtained by the methodology of generalized dynamic relative gain array. Comparatively, however, the RNGA-based methodology is much simpler.

6.2 Control System Configuration

6.2.1 Forms of Control

Generally, a multivariable processes can be controlled by either decentralized, decoupling or sparse control schemes, the choice of the appropriate control scheme depends on the process characteristics, the economic value of improved control, and the available computing and process control resources. The pros and cons as well as the bases for the selection of different control schemes may be stated as:

Decentralized control: A diagonal controller structure

$$G_c(s) = \begin{bmatrix} g_{c1}(s) & 0 & \dots & 0 \\ 0 & g_{c2}(s) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g_{cn}(s) \end{bmatrix}, \tag{6.2.1}$$

which is used for mildly interacted processes. The control structure simplicity, easier to design and implement, and loop failure tolerance, i.e., without readjustment to the other parts of the control system, stability can be preserved in the case of any sensor failure and/or actuator failure, are the main advantages.

Centralized control: A full controller structure

$$G_c(s) = \begin{bmatrix} g_{c,11}(s) & g_{c,12}(s) & \dots & g_{c,1n}(s) \\ g_{c,21}(s) & g_{c,22}(s) & \dots & g_{c,2n}(s) \\ \dots & \dots & \dots & \dots \\ g_{c,n1}(s) & g_{c,n2}(s) & \dots & g_{c,nn}(s) \end{bmatrix}, \tag{6.2.2}$$

which often adopted for two dimensional closely coupled processes or the system requires higher control quality. By eliminating the undesirable cross-couplings which are responsible for the control loop interactions through decoupling controller, the control system performance and robustness can be improved. The drawbacks may be the sacrifice of structure simplicity and control system integrity.

Sparse Control: A sparse controller structure

$$G_c(s) = \begin{bmatrix} g_{c,11}(s) & \kappa_{12} & \dots & \kappa_{1n} \\ \kappa_{21} & g_{c,22}(s) & \dots & \kappa_{2n} \\ \dots & \dots & \dots & \dots \\ \kappa_{n1} & \kappa_{n2} & \dots & g_{c,nn}(s) \end{bmatrix}, \tag{6.2.3}$$

where κ_{ij} is the off-diagonal controller index:

$$\kappa_{ij} = \begin{cases} 1 & g_{c,ij}(s) \\ 0 & \text{no controller} \end{cases} \quad \text{for } i \neq j,\ i, j = 1, 2, \cdots n \tag{6.2.4}$$

The advantages of sparse control lay that the control performance can be improved with limited structure complexity while control system integrity can be preserved.

In this Chapter, we will discussion some fundamental concepts which will be used in the control system design.

6.2.2 Normalized Interaction Index

Based on RNGA, it is assumed that the decentralized control structure has been determined and the selected pairs are been put in the diagonal position. Then, rationalize the matrix by making all diagonal elements equal to one which can be calculated through defining an “Normalized Interaction Index (NII)” , $\overline{\mathrm{X}} = [\chi_{ij}]_{n\times n}$, where

$$\chi_{ij} = \left|\frac{\lambda_{N,ij}}{\lambda_{N,ii}}\right| = \left|\frac{\gamma_{ii}}{\lambda_{ii}} \times \frac{\lambda_{ij}}{\gamma_{ij}}\right| = \left|\frac{\lambda_{ij}}{\lambda_{ii}} \times \frac{\gamma_{ii}}{\gamma_{ij}}\right| \tag{6.2.5}$$

To analysis the effects of χ_{ij} to the selection of κ_{ij}, let's consider two extreme cases:

1. χ_{ij}, $j \neq i$ is very small, which implies either $\lambda_{ij}/\lambda_{ii}$ is very small or γ_{ij}/γ_{ii} is very big.
    - $\lambda_{ij}/\lambda_{ii}$ is very small means k_{ij} is very small compared with k_{ii}. The input u_j has very little influence on y_i.
    - γ_{ij}/γ_{ii} is very big means σ_{ij} very big compared with σ_{ii}, such that the loop react very slow, the effect of the slow loop appears as a constant disturbance which can be effectively rejected by the paired loop controllers.
2. χ_{ij}, $j \neq i$ is very big, implies either $\lambda_{ij}/\lambda_{ii}$ is very big or γ_{ij}/γ_{ii} is very small.
    - $\lambda_{ij}/\lambda_{ii}$ is very big, with this loop included the steady state gain matrix is near singular. The system is very sensitive to modeling errors, a small modeling errors will be magnified into very large errors in y_i and a small change in controller output u_j will also result in large errors in y_i. Control will be difficult to achieve for such a loop and it will also be very sensitive to modeling errors.
    - γ_{ij}/γ_{ii} is very small means σ_{ij} very small compared with σ_{ii}, *i-j* loop react very fast, the effect of the fast loop appears as a high frequency disturbance which can be effectively filter out by the relatively slow paired control loop.

In order to uniformly describe the interaction strength, let NII be, B, with its element, χ_{ij}, determined by

$$\mathrm{X} \overset{\Delta}{=} \begin{cases} \chi_{ij} = \chi_{ij} & \chi_{ij} \leq 1 \\ \chi_{ij} \square 1/\chi_{ij} & \chi_{ij} > 1 \end{cases} \qquad i = 1,\ 2, \ldots,\ n \tag{6.2.6}$$

To provide a feasible method for performance based control structure selection, define a relative control performance index (RCPI), ε, between 0 and 1 which indicates the confidence level on system overall control performance with respect to the promising control structures. By control structure selection criterion, it leads to a dominant model on which the control system will be designed.

$$G_0 \overset{\Delta}{=} g_{ik} = \begin{cases} g_{ik} & \text{If } |\chi_{ij}| \geq \varepsilon \\ 0 & \text{otherwise} \end{cases} \qquad i,k = 1,2,\ldots.,n \tag{6.2.7}$$

Proper control structure selection is to find a fine balance between the control system complexity and the performance. If tight control system performance is required or the interaction is severe among the loops, a more complex control structure would be necessary. Otherwise, if the system does not require very tight control or the interaction is not severe, a simpler control structure may be preferred. Based on the process characteristics analysis, the economic value of improved control, and the reasonable computing and process control resources, it is proposed that the following control selection criterion for κ_{ij} is $\varepsilon = 0.15$ as rule of thumb for engineering applications.

The specification of ε is a designer's choice, it follows that:

(i) if the best overall control system performance is desired, $\varepsilon = 0$, so that all interactions among control loops are deemed to be severe and should be counted for. Hence, centralized or full decoupling control structure is preferred regardless of complexity of the processes;

(ii) if selecting $\varepsilon = 1$, the simplest controller structure is called for rather than best overall control system performance so that the interactions to an individual control loop from the others are deemed to be small and can be ignored, hence, decentralized controller would be preferred;

(iii) for $0<\varepsilon<1$, the smaller value of ε is, the better overall control system performance will be achievable and the more interactions among control loop are deemed to be severe and should be considered, hence, block diagonal controller or sparse controller would be preferred, and vice versa.

(iv) if all off-diagonal elements of X are less than 0.5, the main loops are dominant and good designed decentralized control can generally result in satisfactory performances. Otherwise, if all or some off-diagonal elements of X are close to 1, the interactions are considered to be severe and block diagonal controller or sparse controller would be preferred.

6.2.3 Examples

Example 6.2.1 Consider the Ogunnaike and Ray (OR) System with the transfer function matrix given by

$$G(s) = \begin{bmatrix} \dfrac{0.66}{6.7s+1}e^{-2.6s} & \dfrac{-0.61}{8.64s+1}e^{-3.5s} & \dfrac{-0.0049}{9.06s+1}e^{-s} \\ \dfrac{1.11}{3.25s+1}e^{-6.5s} & \dfrac{-2.36}{5s+1}e^{-3s} & \dfrac{-0.01}{7.09s+1}e^{-1.2s} \\ \dfrac{-34.68}{8.15s+1}e^{-9.2s} & \dfrac{46.2}{10.9s+1}e^{-9.4s} & \dfrac{0.87(11.61s+1)}{(3.89s+1)(18.8s+1)}e^{-s} \end{bmatrix}$$

It is easy to find that the diagonal pairing is the best loop pairing for decentralized control, and both RGA and RNGA are obtained as

$$\Lambda = \begin{bmatrix} 2.0084 & -0.7220 & -0.2864 \\ -0.6460 & 1.8246 & -0.1786 \\ -0.3624 & -0.1026 & 1.4650 \end{bmatrix} \quad K_N = \begin{bmatrix} 0.0710 & -0.0502 & -0.0005 \\ 0.1138 & -0.2950 & -0.0012 \\ -1.9988 & 2.2759 & 0.1189 \end{bmatrix}$$

$$\Lambda_N = \begin{bmatrix} 1.4573 & -0.3551 & -0.1022 \\ -0.3519 & 1.3986 & -0.0468 \\ -0.1054 & -0.0436 & 1.1490 \end{bmatrix}$$

And based on Interaction Index method, the index matrix B is calculated as

$$X = \begin{bmatrix} 1 & 0.2395 & 0.0952 \\ 0.2434 & 1 & 0.0460 \\ 0.1211 & 0.0501 & 1 \end{bmatrix}$$

According to structure selection criterion $\varepsilon = 0.15$, the sparse control should have the following structure

$$G_{C\_sparse1}(s) = \begin{bmatrix} g_{c,11} & g_{c,12} & 0 \\ g_{c,21} & g_{c,22} & 0 \\ g_{c,31} & 0 & g_{c,33} \end{bmatrix}$$

6.3 Equivalent Transfer Function

6.3.1 Equivalent Transfer Function Based on RNGA Approximation

To reveal the transfer function changes when all other loops are closed, we further define the relative average residence time, γ_{ij}, as the ratio of loop $y_i - u_j$ average residence time between when other loops are closed and when other loops are open, i.e.

$$\gamma_{ij} \triangleq \frac{\hat{\sigma}_{ij}}{\sigma_{ij}} \quad i, j = 1, 2, \cdots n \tag{6.3.1}$$

where $\hat{\sigma}_{ij}$ is the average residence time of loop *i-j* when other loops are closed. Using the definition and rewrite RNGA as

$$\hat{k}_{ij} \times \frac{1}{\hat{\sigma}_{ij}} = \frac{k_{ij} \times \frac{1}{\sigma_{ij}}}{\lambda_{N,ij}} \quad i, j = 1, 2, \cdots n \tag{6.3.2}$$

Equation (6.3.2) provides both gain and average residence time change information when all other loops are closed.

To separate these two changes, we first use the definition of RGA

$$\hat{k}_{ij} = \frac{k_{ij}}{\lambda_{ij}} \quad i, j = 1, 2, \cdots n. \tag{6.3.3}$$

Then substitute equation (6.3.3) into (6.3.2) and rearrange, we obtain a formula for calculating γ_{ij}

$$\gamma_{ij} = \frac{\hat{\sigma}_{ij}}{\sigma_{ij}} = \frac{\lambda_{N,ij}}{\lambda_{ij}} \quad i, j = 1, 2, \cdots n, \tag{6.3.4}$$

When the relative average residence times are calculated for all the input/output combinations of a multivariable process, it results in an array of the form, i.e., Relative Average Residence Time Array (RARTA) defined as

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{bmatrix},$$

and its elements can be calculated by

$$\begin{aligned} \Gamma &\triangleq \Lambda_N \odot \Lambda \\ &= \begin{bmatrix} \lambda_{N,11} & \lambda_{N,12} & \cdots & \lambda_{N,1n} \\ \lambda_{N,21} & \lambda_{N,22} & \cdots & \lambda_{N,2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{N,n1} & \lambda_{N,n2} & \cdots & \lambda_{N,nn} \end{bmatrix} \odot \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} \end{aligned} \tag{6.3.5}$$

where the operator $\odot$ is the hadamard division (element by element division).

Since the relative average residence time is the ratio of the average residence times between when other loops are closed and when other loops are open, γ_{ij} represent the dynamic changes of the transfer function $g_{ij}(s)$ when other loops closed. By the definition of RARTA, we can write

$$\hat{\sigma}_{ij} = \gamma_{ij} \times \sigma_{ij} = \gamma_{ij} \times \tau_{ij} + \gamma_{ij} \times \theta_{ij} \quad \text{for } i, j = 1, 2, \cdots n \tag{6.3.6}$$

The average resident time of loop i-j when other loops are closed is the open loop average resident time scaled by a factor γ_{ij}.

In process control, steady state gain, time constant and time delay are the three model parameters which are uppermost for control system design. By using RGA and RARTA information, gain and phase changes of a transfer function element when other loops closed can be uniquely determined. *That is, a transfer function element of a MIMO process when other loops are closed can be approximated by a transfer function element having the same form as the open-loop transfer function element, but the steady state gain, time constant and time delay are scaled RGA and RARTA, respectively,*

For first order plus time delay process, we have.

$$\hat{g}_{ij}(s) = \hat{k}_{ij} \times \frac{1}{\hat{\tau}_{ij}s+1} e^{-\hat{\theta}_{ij}s} = \frac{k_{ij}}{\lambda_{ij}} \times \frac{1}{\gamma_{ij}\tau_{ij}s+1} e^{-\gamma_{ij}\theta_{ij}s} \quad \text{for } i, j = 1, 2, \cdots n \tag{6.3.7}$$

$\hat{g}_{ij}(s)$ in Equation (6.3.7) is optimal ETF of loop *i-j* when other loops are closed under IE criterion. Therefore, $\hat{g}_{ij}(s)$ should resample the dynamic response of corresponding true transfer function element when other loops are closed.

For SOPDT process, we also considered two cases

(iii) When $0 < \xi_{ij} < 1$, its step response in time domain are thus obtained as

$$\bar{y}_i(t) = \begin{cases} 0 & t < \theta_{ij} \\ 1 - \dfrac{e^{-\zeta_{ij}\omega_{n,ij}(t-\theta_{ij})}}{\sqrt{1-\zeta_{ij}^2}} \sin\left[\omega_{n,ij}\sqrt{1-\zeta_{ij}^2}\left(t-\theta_{ij}\right) + \tan^{-1}\dfrac{\sqrt{1-\zeta_{ij}^2}}{\zeta_{ij}}\right] & t \ge \theta_{ij} \end{cases}$$

Its equivalent time constant becomes

$$T_{ij} = 1/\zeta_{ij}\omega_{n,ij}$$

(iv) When $1 < \xi_{ij} < \infty$, the step response in time domain is thus obtained as

$$\bar{y}_i(t) = \begin{cases} 0 & t < \theta_{ij} \\ 1 + \dfrac{1}{\tau_{2,ij} - \tau_{1,ij}}\left(\tau_{1,ij} e^{-\frac{t-\theta_{ij}}{\tau_{1,ij}}} - \tau_{2,ij} e^{-\frac{t-\theta_{ij}}{\tau_{2,ij}}}\right) & t \ge \theta_{ij} \end{cases}.$$

where

$$\tau_{1,ij} = \frac{1}{\omega_{n,ij}\left(\zeta_{ij} + \sqrt{\zeta_{ij}^2 - 1}\right)} \text{ and } \tau_{2,ij} = \frac{1}{\omega_{n,ij}\left(\zeta_{ij} - \sqrt{\zeta_{ij}^2 - 1}\right)}$$

Subsequently, the equivalent time can be approximated by

$$\begin{aligned} T_{ij} &= \sqrt{\tau_{1,ij} \times \tau_{2,ij}} \\ &= \frac{1}{\zeta_{ij}\omega_{n,ij}} \end{aligned}$$

Hence,

$$\begin{aligned} \hat{g}_{ij}(s) &= \hat{k}_{ij} \times \frac{\omega_{n,ij}^2}{s^2 + 2\hat{\zeta}_{ij}\omega_{n,ij}s + \omega_{n,ij}^2} e^{-\hat{\theta}_{ij}s} \\ &= \frac{k_{ij}}{\lambda_{ij}} \times \frac{\omega_{n,ij}^2}{s^2 + 2\gamma_{ij}\zeta_{ij}\omega_{n,ij}s + \omega_{n,ij}^2} e^{-\gamma_{ij}\theta_{ij}s} \quad \text{for } i, j = 1, 2, \cdots n \end{aligned} \tag{6.3.8}$$

6.3.2 Examples

Example 6.3.1 Consider a temperature control problem of the three neighboring rooms for a HVAC system; the transfer function matrix is obtained using empirical modeling techniques:

$$G(s) = \begin{bmatrix} \dfrac{-0.098}{122s+1}e^{-17s} & \dfrac{-0.036}{149s+1}e^{-27s} & \dfrac{-0.014}{158s+1}e^{-32s} \\ \dfrac{-0.043}{147s+1}e^{-25s} & \dfrac{-0.092}{130s+1}e^{-16s} & \dfrac{-0.011}{156s+1}e^{-33s} \\ \dfrac{-0.012}{153s+1}e^{-31s} & \dfrac{-0.016}{151s+1}e^{-34s} & \dfrac{-0.102}{118s+1}e^{-16s} \end{bmatrix}$$

The process steady state gain and normalized gain are

$$K = \begin{bmatrix} -0.098 & -0.036 & -0.014 \\ -0.043 & -0.092 & -0.011 \\ -0.012 & -0.016 & -0.102 \end{bmatrix}, \qquad K_N = \begin{bmatrix} -0.0007 & -0.0002 & -0.0001 \\ -0.0002 & -0.0006 & -0.0001 \\ -0.0001 & -0.0001 & -0.0007 \end{bmatrix}$$

And simple calculation gives RGA, RNGA and RARTA

$$\Lambda = \begin{bmatrix} 1.2138 & -0.2060 & -0.0078 \\ -0.1994 & 1.2162 & -0.0168 \\ -0.0143 & -0.0102 & 1.0246 \end{bmatrix},$$

$$\Lambda_N = \begin{bmatrix} 1.1341 & -0.1293 & -0.0049 \\ -0.1265 & 1.1359 & -0.0094 \\ -0.0076 & -0.0066 & 1.0142 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 0.9344 & 0.6276 & 0.6208 \\ 0.6344 & 0.9339 & 0.5590 \\ 0.5300 & 0.6470 & 0.9898 \end{bmatrix},$$

The ETF parameters are then

$$\hat{K} = \begin{bmatrix} -0.0807 & 0.1747 & 1.7870 \\ 0.2156 & -0.0756 & 0.6544 \\ 0.8334 & 1.5618 & -0.0995 \end{bmatrix}$$

$$\hat{T} = \begin{bmatrix} 113.9921 & 93.5207 & 98.0884 \\ 93.2618 & 121.4149 & 87.2103 \\ 81.0973 & 97.7009 & 116.8043 \end{bmatrix},$$

$$\hat{L} = \begin{bmatrix} 15.8841 & 16.9467 & 19.8660 \\ 15.8608 & 14.9433 & 18.4483 \\ 16.4314 & 21.9988 & 15.8378 \end{bmatrix}$$

which results

$$\hat{g}_{11} = \frac{-0.0807}{113.9921s+1} e^{-15.8841s} \qquad \hat{g}_{12} = \frac{0.2156}{93.2618s+1} e^{-15.8608s} \qquad \hat{g}_{13} = \frac{0.8334}{81.0973s+1} e^{-16.4314s}$$

$$\hat{g}_{21} = \frac{0.1747}{93.5207s+1} e^{-16.9467s} \qquad \hat{g}_{22} = \frac{-0.0756}{121.4149s+1} e^{-14.9433s} \qquad \hat{g}_{23} = \frac{97.7009s+1}{1.5618} e^{21.9988s}$$

$$\hat{g}_{31} = \frac{1.7870}{98.0884s+1} e^{-19.8660s} \qquad \hat{g}_{32} = \frac{0.6544}{87.2103s+1} e^{-18.4483s} \qquad \hat{g}_{33} = \frac{-0.0995}{116.8043s+1} e^{-15.8378s}$$

6.4 Relations between Open-loop and closed-loop Transfer Functions

6.4.1 DRGA Based Determination

The ultimate goals of closed-loop control system are to achieve good performances for tracking setpoint changes, rejecting disturbances and robustness to parameter variations. To accomplish these goals, interactions between all possible control loops are to be measured and most dominant loops are to be identified to results best control structure.

To find the relationship between open and closed-loop transfer function matrices, recall that the basic definition of DRGA ($s \neq 0$)

$$\lambda_{ij} = \frac{\left[\int_0^{\omega_{c,ij}} \left(\partial y_i / \partial u_j\right) d\omega\right]_{all\ loops\ open}}{\left[\int_0^{\omega_{c,ij}} \left(\partial y_i / \partial u_j\right) d\omega\right]_{all\ other\ loops\ close\ except\ \ for\ loop\ y_i - u_j}} \tag{6.4.1}$$

Using the definition for each element in $\Lambda(s)$, we have

$$\Lambda(s) = \begin{bmatrix} g_{11}(s)/\hat{g}_{11}(s) & g_{12}(s)/\hat{g}_{12}(s) & \dots & g_{1n}(s)/\hat{g}_{1n}(s) \\ g_{21}(s)/\hat{g}_{21}(s) & g_{22}(s)/\hat{g}_{22}(s) & \dots & g_{2n}(s)/\hat{g}_{2n}(s) \\ \dots & \dots & \dots & \dots \\ g_{n1}(s)/\hat{g}_{n1}(s) & g_{n2}(s)/\hat{g}_{n2}(s) & \dots & g_{nn}(s)/\hat{g}_{nn}(s) \end{bmatrix},$$

which is

$$\begin{aligned} \Lambda(s) &= G(s) \otimes \hat{G}(s) \\ &= \begin{bmatrix} g_{11}(s) & g_{12}(s) & \dots & g_{1n}(s) \\ g_{21}(s) & g_{22}(s) & \dots & g_{2n}(s) \\ \dots & \dots & \dots & \dots \\ g_{n1}(s) & g_{n2}(s) & \dots & g_{nn}(s) \end{bmatrix} \otimes \begin{bmatrix} 1/\hat{g}_{11}(s) & 1/\hat{g}_{12}(s) & \dots & 1/\hat{g}_{1n}(s) \\ 1/\hat{g}_{21}(s) & 1/\hat{g}_{22}(s) & \dots & 1/\hat{g}_{2n}(s) \\ \dots & \dots & \dots & \dots \\ 1/\hat{g}_{n1}(s) & 1/\hat{g}_{n2}(s) & \dots & 1/\hat{g}_{nn}(s) \end{bmatrix}. \end{aligned} \tag{6.4.2}$$

On the other hand, we can obtain the dynamic RGA matrix by matrix calculation

$$\begin{aligned} \Lambda(s) &= G(s) \otimes G^{-T}(s) \\ &= \begin{bmatrix} g_{11}(s) & g_{12}(s) & \dots & g_{1n}(s) \\ g_{21}(s) & g_{22}(s) & \dots & g_{2n}(s) \\ \dots & \dots & \dots & \dots \\ g_{n1}(s) & g_{n2}(s) & \dots & g_{nn}(s) \end{bmatrix} \otimes \begin{bmatrix} g_{11}(s) & g_{12}(s) & \dots & g_{1n}(s) \\ g_{21}(s) & g_{22}(s) & \dots & g_{2n}(s) \\ \dots & \dots & \dots & \dots \\ g_{n1}(s) & g_{n2}(s) & \dots & g_{nn}(s) \end{bmatrix}^{-T}. \end{aligned} \tag{6.4.3}$$

Comparing equations (6.4.3)and (6.4.2), we have

$$\begin{bmatrix} g_{11}(s) & g_{12}(s) & \dots & g_{1n}(s) \\ g_{21}(s) & g_{22}(s) & \dots & g_{2n}(s) \\ \dots & \dots & \dots & \dots \\ g_{n1}(s) & g_{n2}(s) & \dots & g_{nn}(s) \end{bmatrix}^{-T} = \begin{bmatrix} 1/\hat{g}_{11}(s) & 1/\hat{g}_{12}(s) & \dots & 1/\hat{g}_{1n}(s) \\ 1/\hat{g}_{21}(s) & 1/\hat{g}_{22}(s) & \dots & 1/\hat{g}_{2n}(s) \\ \dots & \dots & \dots & \dots \\ 1/\hat{g}_{n1}(s) & 1/\hat{g}_{n2}(s) & \dots & 1/\hat{g}_{nn}(s) \end{bmatrix},$$

and take transpose on both sides gives

$$G^{-1}(s) = \hat{G}^T(s)\,. \tag{6.4.4}$$

Since $G^{-1}(s)G(s) = I$, the elements in the inverse $G^{-1}(s)$ may be non-proper or non-causal, as shown in the expression $\hat{G}^T(s)$. If $g_{ij}(s)$ is proper and causal, the $\hat{g}_{ij}(s)$ is also proper and causal. Then, $1/\hat{g}_{ij}(s)$ may be non-proper or non-causal.

6.4.2 Performance Limitation for Decentralized Control Systems

For a MIMO control system shown in Figure 6.4.1.

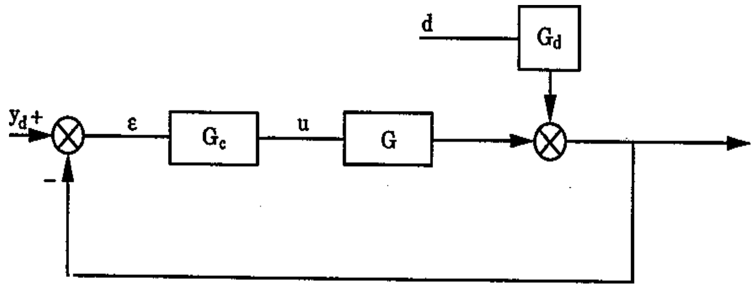


Fig. 6.4.1 Block Diagram of MIMO Control System

The closed-loop transfer function is

$$y(s) = (I + G(s)G_c(s))^{-1}G(s)G_c(s)y_d(s) + (I + G(s)G_c(s))^{-1}G_d(s)d(s) \tag{6.4.5}$$

Define the sensitivity and complementary sensitivity functions as

$$S(s) \triangleq (I + G(s)G_c(s))^{-1} \tag{6.4.6a}$$

and

$$T(s) \triangleq (I + G(s)G_c(s))^{-1}G(s)G_c(s) \tag{6.4.6b}$$

respectively, where, $T(s)$ is the closed loop transfer function from the reference inputs, *r*, to the system outputs, *y*, and $S(s)$ is the closed loop transfer function from the output disturbances, *d*, to the system outputs, *y*.

Using the sensitivity and complementary sensitivity functions, the closed loop transfer function of Figure 6.4.1 can be written as

$$Y(s) = T(s)R(s) + S(s)D(s) \tag{6.4.7}$$

The ultimate goals of closed loop control system are to achieve good performances for tracking setpoint changes, rejecting disturbances and robustness to parameter variations. For the controller to result in physically realizable manipulated variable responses, $G_c(s)$ must be selected such that all poles of $T(s)$ and $S(s)$ lie in the open *left half plane* (LHP). Furthermore, for the feedback control system leading to no offset for setpoint or disturbance changes, we have

$$\lim_{s\to 0} T(s) = I \ \text{ and } \lim_{s\to 0} S(s) = [0].$$

For a decentralized controller

$$G_c(s) = diag\{g_{c1}(s),\ g_{c2}(s), \ldots, g_{cn}(s)\}$$

is to be designed for the system

$$\bar{G}(s) = diag\ \{g_{11}(s), g_{22}(s), \ldots, g_{nn}(s)\}$$

such that the block diagonal closed-loop system with the transfer matrix

$$\bar{T}(s) = \bar{G}(s)G_C(s)(1 + \bar{G}(s)G_C(s))^{-1}$$

is stable. The interaction measure expresses the constraints imposed on the choice of the closed-loop complementary sensitivity function

$$\bar{T}(s) = diag\{\bar{T}_1(s), \bar{T}_2(s), \ldots, \bar{T}_n(s)\}$$

for the block diagonal system, which guarantee that the full closed-loop system

$$T(s) = G(s)G_C(s)(1 + G(s)G_C(s))^{-1}$$

is stable. The SSV interaction measure gives the following constraint for the stability of the closed-loop system *T*:

$$\bar{\sigma}(\bar{T}_i(j\omega)) < \sigma^{-1}(L_T(j\omega)), \text{ for i=1,2 . . . . . n}, \ \ \omega \geq 0$$

where $\bar{\sigma}$ denotes the maximum structural singular value and L_T is the relative error matrix

$$L_T(s) = (G(s) - \bar{G}(s))\bar{G}^{-1}(s)$$

It is assumed that *G(s)* and $\bar{G}(s)$ have the same RHP poles and that $\bar{T}(s)$ is stable.

The nominal sensitivity $\bar{S}(s)$

$$\bar{S}(s) = (1 + \bar{G}(s)G_C(s))^{-1}$$

and the achieved sensitivity *S(s)* are related by

$$S(s) = \bar{S}(s)[1 + H_T(s)]$$

where the element (*i*, *j*) of the matrix $H_T(s)$ is given by

$$[H_T(s)]_{ij} = \begin{cases} \dfrac{g_{ij}(s)}{g_{jj}(s)}\bar{T}_j(s) & i = j \\ 0 & i \neq j \end{cases}$$

$\bar{T}_j(s)$ is the nominal complementary sensitivity of the *j*th loop. If the frequency responses of the off-diagonal terms, $[H_T(s)]_{ij}$, are very small, then the resulting loops are well decoupled and almost behave like p independent SISO control loops. This is achieved if *G(s)* is strongly (column) diagonally dominant.

6.4.3 Performance Limitations in General Control Structure

To derive direct relationship between the controllers and ETFs, we first note that, ideally, the forward transfer function should have the form

$$G(s)G_c(s) \approx \frac{I}{s}, \tag{6.4.8}$$

such that the complementary sensitivity function

$$\lim_{s \to 0} T(s) = G(s)G_c(s)(I + G(s)G_c(s))^{-1} \approx I \,. \tag{6.4.9}$$

Since $G(s)G^{-1}(s) = I$, we have the relation

$$sG(s)G_c(s) \approx G(s)G^{-1}(s) = I$$

Now, the design target is to find $G_c(s)$ such that for perfect control

$$sG_c(s) \approx G^{-1}(s)\,, \tag{6.4.10}$$

hold.

Compare equations (6.4.4) and (6.4.10), the design of perfect control problem is now can be transformed to find the parameter of $G_c(s)$, such that

$$sG_c(s) \approx \hat{G}^T(s)\,. \tag{6.4.11}$$

For control structure either in the form of decentralized, decoupled or sparse, the control objective is to find $G_c(s)$ such that equation (6.4.11) hold.

Define an error function as

$$\mathrm{E}(s) = sG_c(s) - \hat{G}^T(s)\,.$$

To minimize the error function, let's define an objective function

$$J = \min\left|\mathrm{E}(s)\right| = \min\sum_{i=1}^{n}\sum_{j=1}^{n}\left|sG_c(s) - \hat{G}^T(s)\right|_{ij}\,, \tag{6.4.12}$$

where the subscript *ij* indicating the *i*th row and *j*th column element in [•].

Since

$$\left|G_c(s) - \hat{G}^T(s)\right|_{ij} \approx \begin{cases} \left|sg_{c,ij}(s) - 1/\hat{g}_{ji}(s)\right| & g_{c,ij}(s) \neq 0 \\ \left|-1/\hat{g}_{ji}(s)\right| & g_{c,ij}(s) = 0 \end{cases}, \tag{6.4.13}$$

the minimization of *J* requires that $g_{c,ij}(s)$ is determined by

$$sg_{c,ij}(s) - \frac{1}{\hat{g}_{ji}(s)} = 0 \Rightarrow g_{c,ij}(s)\hat{g}_{ji}(s) = \frac{1}{s}\,. \tag{6.4.14}$$

As perfect control is impossible, PI/PID controller can be used for $g_{c,ij}(s)$ such that the closed-loop of $g_{c,ij}(s)\hat{g}_{ji}(s)$ has good dynamic properties. Under Gain and phase margin design method, the loop forward transfer function is usually expressed by

$$g_{c,ij}(s)\hat{g}_{ji}(s) = \frac{k_{p,ij}}{s}e^{-L_{ij}s}$$

For given positive $k_{p,ij}$ and L_{ij}

$$\lim_{s\to 0}\left|\frac{k_{p,ij}}{s}e^{-L_{ij}s}\left(1+\frac{k_{p,ij}}{s}e^{-L_{ij}s}\right)^{-1} - \frac{1}{s}\left(1+\frac{1}{s}\right)^{-1}\right| = 0 \tag{6.4.15}$$

the control system performance asymptotically approaches the perfect control. Therefore, equation (6.4.15) provides the fundamental relationship between open and closed-loop transfer function matrices.

Furthermore, Equations (6.4.13) and (6.4.14) reveals:

1. The control structure limit the control performance, the more controller be used, the better control performance could be possibly achieved.
2. Using Equation (6.4.14), the complicated counter interaction measure when other loops closed is avoided and the detuning of the designed single controllers is no longer required. The controllers can be directly designed based on the ETFs and by using existing single loop controller design techniques.

Perfect control is only theoretically possible for full decoupling control, while other schemes are impossible to achieve perfect control due to the structure limitations

6.4.4 Examples

Example 6.4.1 For the VL column system with its transfer function and equivalent transfer function matrices

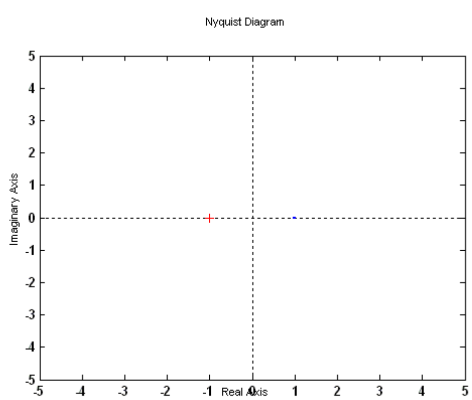
$$G(s)=\begin{bmatrix}\dfrac{-2.2e^{-s}}{7s+1} & \dfrac{1.3e^{-0.3s}}{7s+1}\\ \dfrac{-2.8e^{-1.8s}}{9.5s+1} & \dfrac{4.3e^{-0.35s}}{9.2s+1}\end{bmatrix}$$

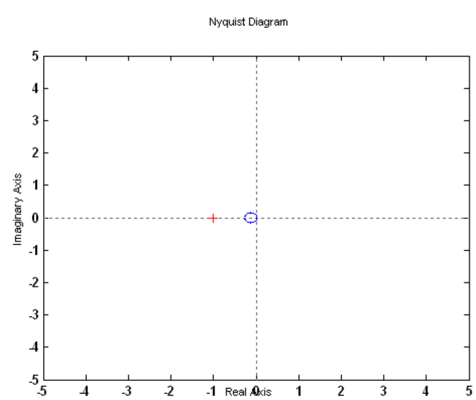
$$\hat{G}^T(s)=\begin{bmatrix}\dfrac{6.6910\ s+1}{-1.3534}e^{0.9558s} & \dfrac{8.4103\ s+1}{4.4769}e^{1.5935s}\\ \dfrac{6.1970\ s+1}{-2.0785}e^{0.2655s} & \dfrac{8.7939\ s+1}{2.6454}e^{0.3345s}\end{bmatrix}$$

The relation of $G(s)\hat{G}^T(s)=I$ is verified by both frequency domain and time domain methods:

- The Nequest plot $G(s)\hat{G}^T(s)$ is shown in Figures 6.4.1;
- The step response of $\tilde{y}(s)=G(s)\hat{G}^T(s)\tilde{u}(s)$ is shown in Figures 6.4.2.

The simulation result clearly demonstrated that, even though the results have some small deviations from that of identity matrix's, they are close enough to consider that the relation of Equation (6.4.4) hold.





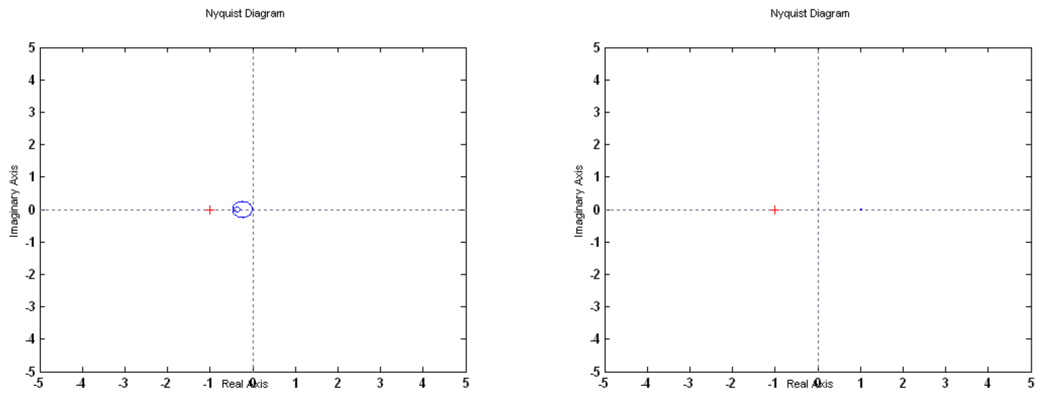


Figure 6.4.1 Niquest Plot of $G(s)\hat{G}^T(s)$

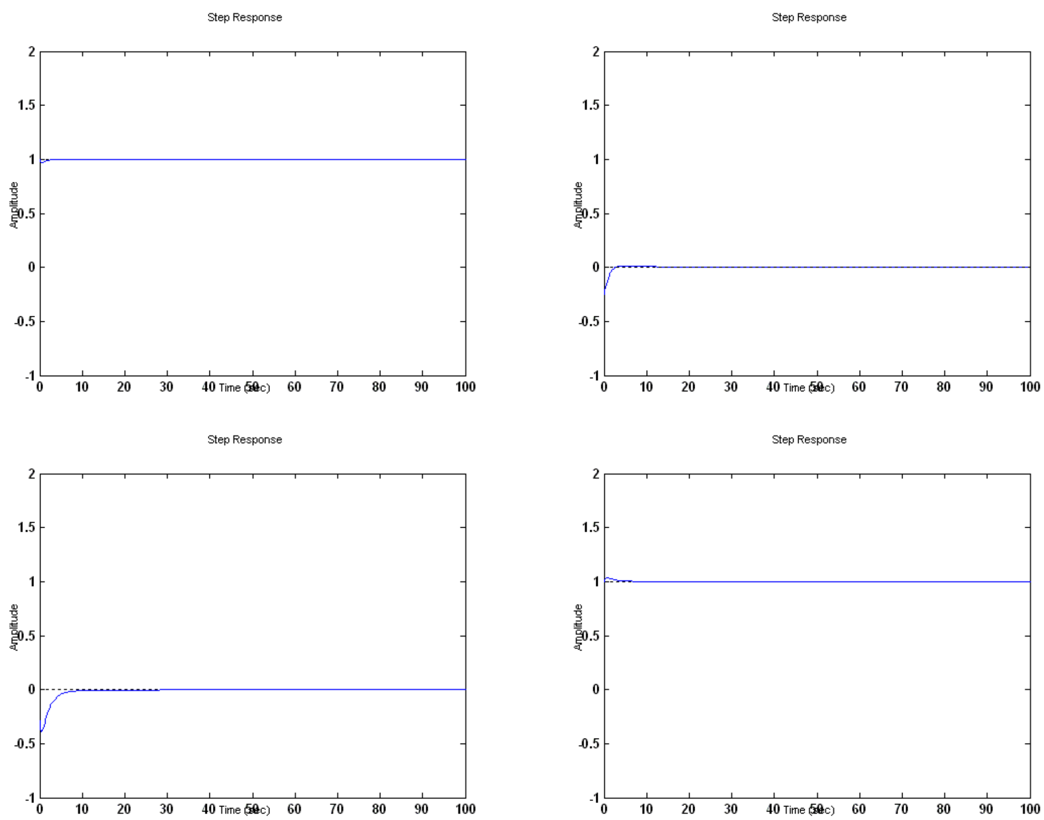


Figure 6.4.2 Step response of $\tilde{y}(s) = G(s)\hat{G}^T(s)\tilde{u}(s)$