Chapter 5 Fundamentals of MIMO Control

5.1 Basic Concepts

When a process has only one input variable to be used in controlling one output variable, it constitutes a single-input, single-output (SISO) system. For such kinds of systems, the controller design problem can be handled, fairly conveniently. In many situations, however, the system under consideration has multiple inputs and multiple outputs, making it multivariable in nature. In actual fact, the most important industry processes are often multivariable ones. Many nontrivial issues, not encountered in SISO systems, are now raised when transfer functions need to be identified or controllers are to be designed for these multivariable systems.

A multivariable process is one with multiple inputs, u_1, u_2, u_3 … u_m and multiple outputs y_1, y_2, y_3, …y_n, where m is not necessarily equal to n. Depending the complexity of the industry plant, a multivariable process can be

- A single process
- Several process units of an entire plant,
- The entire plant itself.

The issues unique to multivariable systems are

1. Interaction between variables influences control stability and performance.
2. Feasibility of control depends on overall process, not just individual cause-effect relationships.
3. The source of the disturbance, not just the magnitude, must be considered in designing the control strategy.
4. The pairing of measured variables and final elements via control is a design decision.
5. Some processes have an unequal number of controlled and manipulated variables.
6. Some multivariable control designs are very sensitive to modelling errors.

5.1.1 Input/Output Pairing and Interactions

One of the consequences of having several input and output variables is that such a control system can be configured several different ways depending on which input variable is paired with which output variable. Each u_i ,to y_j, “pairing” constitutes a control configuration, and for a two-input, two-output system (2 x 2 system) we have two such configurations:

Configuration 1	Configuration 2
u_1—y_1	u_1—y_2
u_2—y_2	u_2—y_1

Example5.1.1: Control configuration for the stirred mixing tank

The stirred mixing tank shown in Figure 5.1.2 has two input variables, the cold stream flowrate, and the hot stream flowrate, to be used in controlling two output variables, the temperature of the liquid in the tank, and the liquid level. Enumerate the different ways the control system can be configured.

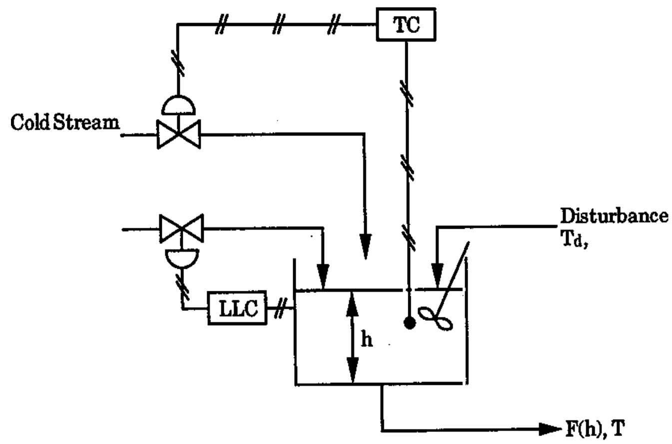

Figure 5.1.2 Stirred Mixing Tank

1. Use hot stream flowrate to control liquid level, and use cold stream to control liquid temperature.
2. Use cold stream flowrate to control liquid level, and use hot stream to control liquid temperature.

Note that these two configurations are *mutually exclusive*, i.e., only one of them can be used at any particular time. #

3 x 3 system: there are six such configurations;

4 x 4 system: there are twenty-four;

n x n system, there are *n!* possible input-output configurations.

Notation: The input (or manipulated) variables of a process in their free i.e., "unpaired" form as $m_1, m_2, \ldots, m_m$; the output variables will still be $y_1, y_2, y_3, \ldots y_n$. After the input variables have been paired with the output variables, the notation u_j will be used for the input variable paired with the jth output, y_j.

Example5.1.2: if third input variable m_3 is paired with y_1 then m_3 becomes u_1. The subscripts on the *m's* are used for counting purposes; the subscripts on *u's* indicate control loop assignment.#

Example5.1.3: For the stirred mixing tank, the input variables are:

The cold steam flowrate and the hot stream flowrate, respectively as m_1 and m_2, in their "unpaired state."

The output variables are: liquid level y_1 and liquid temperature y_2, respectively.

As configured in the Figure 5.1.2, the hot stream m_2 is paired with liquid level y_1 and the cold stream m_1 is paired with liquid temperature y_2. Thus, hot stream is u_1 and the cold stream is u_2.#

Another important consideration that did not feature in discussions involving SISO systems but that is of prime importance with multivariable systems is the following: after somehow arriving at the conclusion that input, m_i, is best used to control output y_j we must now answer the very pertinent question:

In addition to affecting y_j, its assigned output variable, will m_i affect any other output variable? That is, can m_i control output y_j *in isolation, or will the control effort of m_i influence other outputs apart from y_j?*

Unfortunately, in virtually all cases, a particular m_i will influence several outputs.

Example 5.1.4: Continue of 5.1.1, the hot stream flowrate is paired with the level in the stirred mixing process, changes in this input variable will affect both the liquid level and temperature.

This is the *interaction problem,* recognized as one of the main problems for the control of multivariable systems.

5.1.2 Operating Window

This issue is related the control system's range of attainable variable values. The term *operating window* is used for the range of possible (or feasible) steady-state process values. The operating window can be sketched using different variables as coordinates; such as the controlled variables are used to characterize the range of possible set points, with all inputs other than the manipulated variables (i.e., disturbances) constant.

Example 5.1.5: Blending is an important unit operation and is employed in a wide variety of industries, as in production of gasoline and cement. Typically, the controlled variables in a blending process are production rate and blended production composition. The blending process (shown in Fig. 5.1.3) is modelled with the following assumptions:

1. The inlet concentration are constant
2. Mixing where the flows merge is perfect.
3. The densities of the solvent and component A are equal.

The overall and component A material balances at the point of mixing are

$$F = F_A + F_B \tag{5.1.1}$$

Component A Mass Balance:

$$FX = F_A X_A + F_B X_B \tag{5.1.2}$$

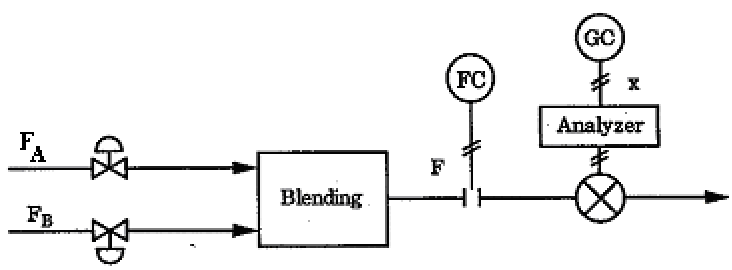

Fig. 5.1.3 Blending Process

with

F = Flow rate of mixed liquid (mass/time)

X_A = mass fraction of component A in pure A=1.0

X_B = mass fraction of component A in solvent = 0.0

X = mass fraction of component A in the mixed liquid

The component flow rates in the blending example can be adjusted continuously from zero to maximum rates, $F_{A,\max}$ and $F_{B,\max}$, Draw the operating window of attainable total flow rate and composition, assuming that the component compositions remain unchanged.

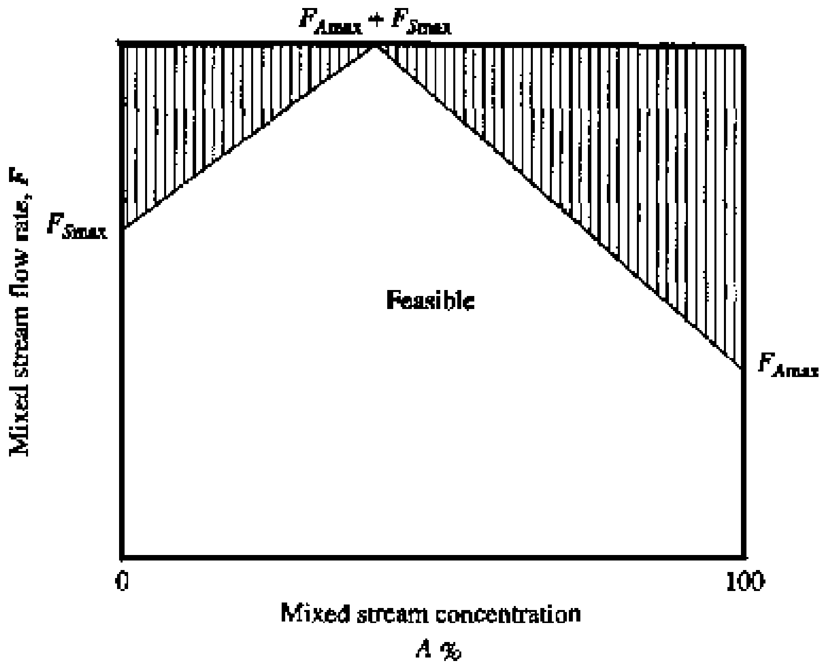

Fig. 5.1.4 Operating Window for Blending Process with Controlled Variables as Coordinates

The attainable total flow F and composition A are shown in Figure 5.1.4. The limiting values are easily determined by solving equations (5.1.1) and (5.1.2) for various values of one flow, with the other flow at its maximum value.

$$F_A = \frac{X - X_B}{X_A - X_B} F$$

$$F_B = \frac{X - X_A}{X_B - X_A} F$$

The interaction between variables is clear, because the value of one variable influences the range of the other variable. If the variables were independent and no interaction occurred, the operating window would be rectangular, which it clearly is not.

5.1.3 Multivariable Process Models

There are two representation forms for MIMO processes, i.e., state-space and transfer function matrix forms.

State Space Form

Linear: multivariable system is represented in state space as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \Gamma d(t) \\ y(t) &= Cx(t) \end{aligned} \tag{5.1.3}$$

where *x: l*-dimensional vector of state variables;

u: m-dimensional vector of control variables;

y: n-dimensional vector of output variables;

d: k-dimensional vector of disturbance variables;

A, B, C, and *Γ* as appropriately dimensioned system matrices.

Nonlinear: nonlinear differential equations, such as:

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x,u,d) \\ y(t) &= h(x(t)) \end{aligned}$$

where *f(x, u, d),* and *h(x(t))* are vectors of nonlinear functions.

Transfer Function Matrix Model Form

The vector of input variables to that of the output variables:

$$Y(s) = G(s)U(s) + G_d(s)d(s) \tag{5.1.4}$$

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1m}(s) \\ g_{21}(s) & g_{22}(s) & \cdots & g_{2m}(s) \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}(s) & g_{n2}(s) & \cdots & g_{nm}(s) \end{bmatrix}$$

for an $m \times n$ system, *G(s)* is an $m \times n$ transfer function matrix with elements and each $g_{ij}(s)$ is same as *SISO* transfer functions. $G_d(s)$ is an $n \times k$ transfer function matrix consisting of similar elements.

Interrelationships between Model Forms

Taking Laplace transforms to (5.1.3), we obtain:

$$\begin{aligned} sX(s) &= AX(s) + BU(s) + \Gamma D(s) \\ Y(s) &= CX(s) \end{aligned}$$

which can be rearranged to give:

$$Y(s) = \left[C(sI - A)^{-1}B\right]U(s) + \left[C(sI - A)^{-1}\Gamma\right]D(s) \tag{5.1.5}$$

Relationship between transfer function and state-space matrices:

$$\begin{aligned} G(s) &= \left[C(sI - A)^{-1}B\right] \\ G_d(s) &= \left[C(sI - A)^{-1}\Gamma\right] \end{aligned} \tag{5.1.6}$$

The reverse problem of deducing equivalent state-space model from transfer function model (realization) is not a simple problem

- No realization is *unique,* there are several equivalent sets of differential equations that, upon Laplace transformation, will yield the *same* transfer function matrix.
- There are different ways of obtaining these different, but equivalent, realizations. As a result, it is often necessary to utilize minimal realization method.

Example 5.1.6: Model for the two-tank system. The two tank system is illustrated in Figure 5.1.5.

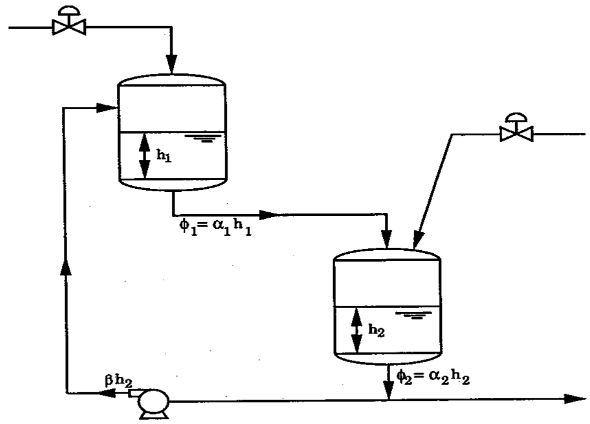

Fig. 5.1.5 Two-tank Process

Assuming uniform cross-sectional areas A_1, and A_2, respectively for Tanks 1 and 2, material balances over each of the tanks yield:

$$A_1 \frac{dh_1}{dt} = -\alpha_1 h_1 + \beta h_2 + F_1$$

$$A_2 \frac{dh_2}{dt} = -\alpha_2 h_2 + \alpha_1 h_2 + F_2$$

If let $\alpha_2 = \beta + \gamma$ and define deviation variables:

$$x_1 = h_1 - h_{1s} \qquad x_2 = h_2 - h_{2s}$$

$$u_1 = F_1 - F_{1s} \qquad u_2 = F_2 - F_{2s}$$

then these modeling equations become:

$$\frac{dx_1(t)}{dt} = -\frac{\alpha_1}{A_1} x_1 + \frac{\beta}{A_1} x_2 + \frac{1}{A_1} u_1$$

$$\frac{dx_2(t)}{dt} = \frac{\alpha_1}{A_2} x_1 - \frac{\beta + \gamma}{A_2} x_2 + \frac{1}{A_2} u_2$$

If we let,

$$y_1 = x_1 \qquad y_2 = x_2$$

with *both* are measured, the state-space vectors and matrices are:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}; \qquad \mathrm{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -\dfrac{\alpha_1}{A_1} & \dfrac{\beta}{A_1} \\ \dfrac{\alpha_1}{A_2} & -\dfrac{\beta+\gamma}{A_2} \end{bmatrix}; \qquad B = \begin{bmatrix} \dfrac{1}{A_1} & 0 \\ 0 & \dfrac{1}{A_2} \end{bmatrix}$$

with $C = I$, and $\Gamma=0$. This is a 2 x 2 system. #

Example 5.1.7: **Modelling for the Stirred Mixing Tank**

For the stirred mixing tank process of Figure 5.1.2, assuming uniform cross-sectional area A_c for the tank, constant liquid physical properties ρ and C_p and $F = K\sqrt{h}$, we obtain the following from material and energy balances over the tank:

$$A_c \frac{dh}{dt} = F_H + F_C + F_d - K\sqrt{h}$$

$$\rho C_p A_c \frac{d(hT)}{dt} = \rho C_p (F_H T_H + F_C T_C + F_d T_d - K\sqrt{h} T)$$

which are nonlinear because of the presence of the square root term, and the product functions of h and T. The nonlinear modelling equations can be linearized around some steady-state operating point, h_s, T_s. If in addition we define the following deviation variables:

$$\begin{aligned} x_1 &= h - h_s & x_2 &= T - T_s \\ u_1 &= F_H - F_{Hs}\,; & u_2 &= F_C - F_{Cs} \\ d_1 &= F_d - F_{ds} & d_2 &= T_d - T_{ds} \end{aligned}$$

then the linearized modelling equations become:

$$\frac{dx_1}{dt} = \frac{1}{A_c}\left(-\frac{K}{2\sqrt{h_s}} x_1 + u_1 + u_2 + d_1 \right)$$

$$\frac{dx2}{dt} = \frac{1}{A_c h_s}\left(-K\sqrt{h_s}\, x_2 + (T_H - T_s)u_1 + (T_C - T_s)u_2 + (T_{ds} - T_d)d_1 + F_{ds} d_2 \right)$$

These equations can now be expressed in the vector-matrix form; the resulting state-space matrices are:

$$A = \begin{bmatrix} -\dfrac{K}{2A_c\sqrt{h_s}} & 0 \\ 0 & -\dfrac{K}{A_c\sqrt{h_s}} \end{bmatrix}; \quad B = \begin{bmatrix} \dfrac{1}{A_c} & \dfrac{1}{A_c} \\ \dfrac{(T_H - T_s)}{A_c h_s} & \dfrac{(T_C - T_s)}{A_c h_s} \end{bmatrix}; \quad \Gamma = \begin{bmatrix} \dfrac{1}{A_c} & 0 \\ \dfrac{(T_{ds} - T_d)}{A_c h_s} & \dfrac{F_{ds}}{A_c h_s} \end{bmatrix}$$

If both level and temperature are measured, then $y = x$ and $C = I$, the identity matrix. From the approximate state-space model, we can use the relationships to deduce the corresponding transfer function matrices for this process. We see that $G(s)$ is given by:

$$G(s)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}-\frac{K}{2A_c\sqrt{h_s}} & 0\\ 0 & -\frac{K}{A_c\sqrt{h_s}}\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{A_c} & \frac{1}{A_c}\\ \frac{(T_H-T_s)}{A_ch_s} & \frac{(T_C-T_s)}{A_ch_s}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{A_c(s+\frac{K}{2A_c\sqrt{h_s}})} & \frac{1}{A_c(s+\frac{K}{2A_c\sqrt{h_s}})}\\ \frac{(T_H-T_s)}{A_ch_s(s+\frac{K}{A_c\sqrt{h_s}})} & \frac{(T_C-T_s)}{A_ch_s(s+\frac{K}{A_c\sqrt{h_s}})}\end{bmatrix}$$

and we also obtain, in similar fashion, that

$$G_d(s)=\begin{bmatrix}\frac{1}{A_c(s+\frac{K}{2A_c\sqrt{h_s}})} & 0\\ \frac{(T_{ds}-T_s)}{A_ch_s(s+\frac{K}{A_c\sqrt{h_s}})} & \frac{F_{ds}}{A_ch_s(s+\frac{K}{A_c\sqrt{h_s}})}\end{bmatrix}$$

State space models are usually obtained from theoretical modelling; they can be used to derive transfer function models when the modelling equations are linear. Often, when a multivariable process is modelled by experimental means — by correlating input/output data — the model is constructed in the transfer function matrix form.

5.1.4 Composite Transfer Functions for MIMO Systems

Cascade rule: For the cascade (series) interconnection of $G_1(s)$ and $G_2(s)$ in Figure 5.1.6, the overall transfer function matrix is $G(s)=G_2(s)G_1(s)$ (first $G_2(s)$ and then $G_1(s)$) which is the reverse of the order in which they appear in the block diagram of Figure 5.1.6 (first $G_1(s)$ and then $G_2(s)$).

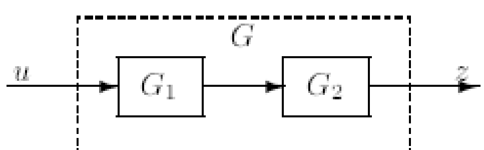

Figure 5.1.6: Block diagrams for the cascade system

Feedback rule: With reference to the positive feedback system in Figure 5.1.7, we have

$$v=\left(I-L(s)\right)^{-1}u \tag{5.1.7}$$

where $L(s)=G_2(s)G_1(s)$ is the transfer function around the loop.

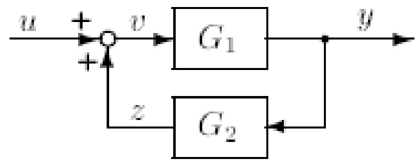

Figure 5.1.7: Block diagrams for positive feedback system

Push-through rule:

$$G_1(s)\left(I - G_2(s)G_1(s)\right)^{-1} = \left(I - G_1(s)G_2(s)\right)^{-1} G_1(s) \tag{5.1.8}$$

The push-through rule can be obtained by premultiplying both sides by $\left(I - G_1(s)G_2(s)\right)$ and postmultiplying both sides by $\left(I - G_2(s)G_1(s)\right)$ of equation (5.1.8), resulting the same equation.

The cascade and feedback rules can be combined into the following MIMO rule for evaluating closed-loop transfer functions from block diagrams.

MIMO rule: Start from the output and write down the blocks as you meet them when moving backwards (against the signal flow) towards the input. If you exit from a feedback loop then include a term $\left(I - L(s)\right)^{-1}$ for positive feedback (or $\left(I + L(s)\right)^{-1}$ for negative feedback) where (is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).Parallel branches should be treated independently and their contributions added together.

Care should be taken when applying this rule to systems with nested loops. For such systems it is probably safer to write down the signal equations and eliminate internal variables to get the transfer function of interest. The rule is best understood by considering an example.

Example 5.1.7: The transfer function for the block diagram in Figure 5.1.8 is given by

$$z = \left(G_{11}(s) + G_{12}(s)K\left(I - G_{22}(s)K\right)^{-1} G_{21}(s)\right)w \tag{5.1.9}$$

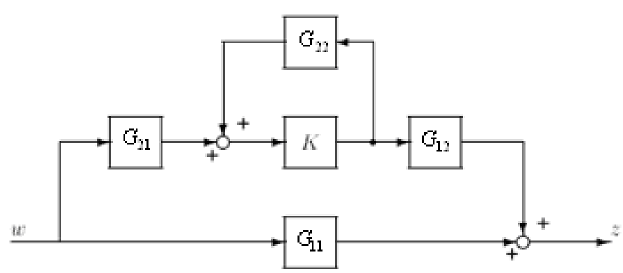

Figure 5.1.8: Block diagram

To derive this from the MIMO rule above we start at the output z and move backwards towards w. There are two branches, one of which gives the term $G_{11}(s)$ directly. In the other branch we move backwards and meet $G_{12}(s)$ and then K. We then exit from a feedback loop and get a term $\left(I - L(s)\right)^{-1}$ (positive feedback) with $L(s) = G_{22}(s)K$, and finally we meet $G_{21}(s)$.

5.1.5 Multivariable Block Diagrams

For an m-input, n-output (k-disturbance) system, the block diagram of the closed-loop control system can be expressed as in Figure 5.1.9:

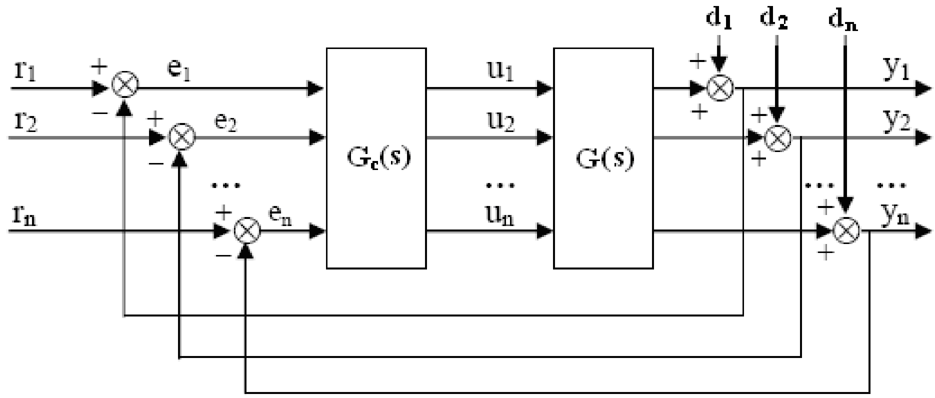

Fig. 5.1.9 M-input, N-output (K-disturbance) system

This block diagram can often be “simplified” to transfer function matrices, and the signals are designated as vectors as shown in Figure 5.1.10.

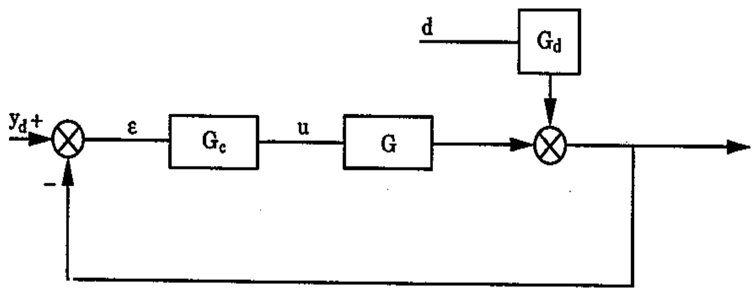

Fig. 5.1.10 Transfer Function Matrices Form of M-input, N-output (K-disturbance) system

For low-dimensional processes, it can also be represented as in Figure 5.1.11 to show the interactive effects of u_2 on y_1 and of u_1 on y_2. This block diagram is useful for interaction analysis and the design of interaction compensators.

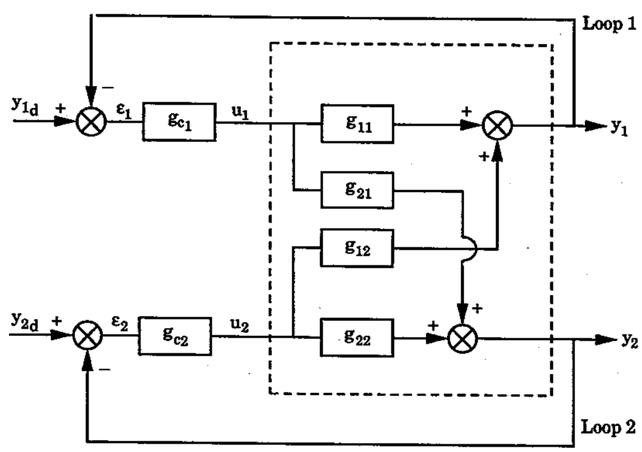

Fig. 5.1.11 Block Diagram shows Interactive Effects for TITO Process

From the block diagram of Figure 5.1.10, we have:

$$Y(s) = G(s)U(s) + G_d(s)d(s) \tag{5.1.10a}$$

but

$$U(s) = G_c(s)\varepsilon(s) \tag{5. 1.10b}$$

and

$$\varepsilon(s) = y_d - y(s) \tag{5. 1.10c}$$

so it becomes:

$$y(s) = GG_c[y_d - y(s)] + G_d d \tag{5.1.11a}$$

We may now rearrange this to give:

$$(I + GG_c)y(s) = GG_c y_d + G_d d \tag{5.1.11b}$$

by premultiplying with the inverse matrix $(I+GG_c)$, we obtain:

$$y(s) = (I + GG_c)^{-1} GG_c y_d + (I + GG_c)^{-1} G_d d \tag{5.1.12}$$

Note:

1. Compared with the SISO system, the only difference is that it involves multiplication by the inverse of $(I+GG_c)$. This matrix $(I+GG_c)$ is referred to as the *return difference matrix.*
2. If a measurement device $H(s)$ is introduced in the feedback path, the closed-loop transfer function matrix will then be:

$$y(s) = (I + GG_c H)^{-1} GG_c y_d + (I + GG_c H)^{-1} G_d d$$

From the definition of a matrix inverse, $y(s)$ can be rewritten as:

$$y(s) = \frac{Adj(I + GG_c)GG_c}{|(I + GG_c)|} y_d + \frac{Adj(I + GG_c)G_d}{|(I + GG_c)|} d \tag{5.1.13}$$

and each element of each of the indicated transfer function matrices will have *identical* denominator polynomials corresponding to the determinant of the return difference matrix, these will determine the nature of the closed-loop responses.

The sensitivity and complementary sensitivity are then defined as

$$S \triangleq (I + L)^{-1} = \frac{Adj(I + GG_c)G_d}{|(I + GG_c)|}$$

$$T \triangleq I - S = L(I + L)^{-1} = \frac{Adj(I + GG_c)GG_c}{|(I + GG_c)|}$$

In Figure 5.1.8, T is the transfer function from r to y, and S is the transfer function from d to y.

Equation (5.1.13) can be written as

$$Y(s) = T(s)R(s) + S(s)D(s) \tag{5.1.14}$$

5.2 MIMO System Analysis

5.2.1 Transfer Function Analysis

The transfer function of multivariable system:

$$Y(s) = G(s)U(s) \tag{5.2.1a}$$

For the SISO system whose transfer function is given by:

$$G(s) = \frac{N(s)}{D(s)} \tag{5.2.1b}$$

the poles are the roots of the denominator polynomial, *D(s)*, while the zeros are the roots of the numerator polynomial, *N(s)*.

Multivariable System Poles

The poles are defined as the collection of *all* the poles of the individual transfer function elements.

Example 5.2.1: *the transfer function matrix*

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+3} \\ \frac{1}{s+4} & \frac{1}{s+2} \end{bmatrix}$$

can be rearranged to give:

$$G(s) = \frac{\begin{bmatrix} (s+2)(s+3)(s+4) & (s+1)(s+2)(s+4) \\ (s+1)(s+2)(s+3) & (s+1)(s+3)(s+4) \end{bmatrix}}{(s+1)(s+2)(s+3)(s+4)}$$

The common denominator polynomial is made up of the totality of the poles of all the individual transfer function elements.

In the transfer function matrix, the individual elements have single poles at *s = -1, -2, -3,* and *–4*

#

If the transfer function matrix was obtained from the state-space form by Laplace transformation, we have

$$G(s) = \frac{[C\ Adj(sI - A)B]}{|(sI - A)|}$$

The poles of the multivariable system are the roots of:

$$|(sI - A)| = 0 \tag{5.2.2}$$

This leads us to the following result:

The poles of a transfer function matrix, and the eigenvalues of the equivalent system matrix A in the state-space form, are one and the same.

Therefore,

A MIMO system is stable if all the poles of the transfer function matrix lie in the left-half plane (LHP); otherwise it is unstable.

Stability requires the roots of the equation all lie in the LHP. It is identical whether we determine the stability of a multivariable system in terms of its transfer function matrix or its state-space model.

Multivariable System Zeros

The definition of multivariable system zeros is obtained by extension of the SISO system zeros: as the *poles* of *1/g(s)*:

$$\frac{1}{g(s)} = \frac{D(s)}{N(s)}$$

For a multivariable system, it is defined as the *poles* of the inverse of the transfer function. The inverse of the transfer function matrix *G(s)*, if the matrix is square and nonsingular, is given by:

$$G^{-1}(s) = \frac{Adj[G(s)]}{|G(s)|}$$

Thus the roots of the equation:

$$|G(s)| = 0 \tag{5.2.3}$$

are the *zeros* of the transfer function matrix *G(s).*

The zeros of a (square) multivariable system with transfer function matrix G(s) are the zeros of the determinant of this transfer function matrix, i.e., |G(s)|=0.

When the transfer function matrix is not square, the more general definition of the zeros of a multivariable system:

Those values of s for which the rank of the transfer function matrix G(s) is reduced are the zeros of G(s).

Example 5.2.2: Find the zero(s) of the multivariable system whose transfer function is given as:

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+3} \\ \frac{1}{s+4} & \frac{1}{s+2} \end{bmatrix}$$

The determinant of this transfer function matrix is obtained as:

$$|G(s)| = \frac{1}{(s+1)(s+2)} - \frac{1}{(s+3)(s+4)}$$

which simplifies to:

$$|G(s)| = \frac{4s+10}{(s+1)(s+2)(s+3)(s+4)}$$

the multivariable system has a single zero at $s = -10/4$.

This example illustrates:

1. Finding the zeros of a multivariable system is not straightforward. By inspection alone, it is difficult to find zero values or whether or not the system even has a zero.
2. If the system has no time-delay, the determinant of transfer function matrix is a ratio of two polynomials; the roots of the numerator polynomial are the zeros, and the roots of the denominator polynomial are the poles.
3. It is possible for a multivariable system to have poles and zeros in the same location. It is very important to ensure that there has been no pole-zero cancellation when forming the determinant.

In terms of the system matrices of the state-space model, the system zeros are the roots of the equation:

$$[C \times adj(sI - A)B] = 0 \tag{5.2.4}$$

Thus, system zeros can be computed from the state-space model formulation as well.

A unique problem for decentralized control of MIMO processes is the zero crossing, where stable (left half plane) or unstable (right half plane) zeros might be introduced into a particular control loop when other loops are closed. If an unstable zero is introduced, it will result phase shift to the left in the frequency domain which reduced the stability margin. The effects of unstable zeros must be accurately estimated in each control loop, in order to guarantee the entire system stability by the designed controllers.

5.2.2 Controllability

In the world of control engineering, the task of a control engineer is to design controller and compensator units to interact with these pre-existing systems. However, some systems simply cannot be controlled. The concept of **controllability** refers to the ability of a controller to arbitrarily alter the functionality of the system plant

Controllability: *A system is controllable if the controlled variables can be maintained at their set points, in the steady state, in spite of disturbances entering the system.*

Controllability Matrix: For LTI systems, a system is controllable if and only if it's controllability matrix, ζ, has a full row rank of p, where p is the dimension of the matrix A, and $p \times q$ is the dimension of matrix B.

$$\zeta = \begin{bmatrix} B & AB & A^2B & \cdots & A^{p-1}B \end{bmatrix} \in R^{p \times pq}$$

A system is controllable or "Controllable to the origin" when any state x_1 can be driven to the zero state $x = 0$ in a finite number of steps.

A system is controllable when the rank of the system matrix A is p, and the rank of the controllability matrix is equal to:

$$Rank(\zeta) = Rank(A^{-1}\zeta) = p$$

If the second equation is not satisfied, the system is not. If

$$Rank(A) < p$$

Example 5.2.3: To better understand the concept, considering a two-tank process of Example

5.1.7, liquid streams flow into Tanks 1 and 2 at volumetric rates F_1, and F_2, the outflow from each tank is proportional to the respective liquid levels h_1 and h_2 in each tank. The liquid leaving Tank 2 is split into two with a fraction F, exiting, and the remainder R pumped back to the first tank. This is a 2 x 2 system - the flowrates of the two inlet streams as inputs, the liquid level in each tank as the output variables.

We now wish to investigate (for now only in a qualitative sense) how this process will operate under the following conditions:

1. As shown in Figure 5.1.5, with both inlet streams, and the recycle pump in operation;
2. The recycle pump is turned off, and only stream F_1 is operational; stream F_2 is set at constant flow which *cannot* be changed;
3. The recycle pump is turned off, but this time stream F_1 is now set at constant flow that *cannot* be changed; only stream F_2 is operational.

In particular, we are interested in answering the following question:

With whatever input variables we have available, can we simultaneously influence the liquid level in both tanks? Or, put more formally, can we "drive" the liquid level in both tanks from a given initial value to any other arbitrarily set desired value, in finite time?

Upon some reflection, we can answer this question for each of the three situations noted above as follows:

1. Under Condition 1, since both inlet streams are available, we can clearly influence the level in both tanks simultaneously.
2. Under Condition 2, observe that with stream F_2 no longer available for control, we have lost our source of *direct* influence on h_2 however, we still have *indirect influence* on it trough the outflow from Tank 1. Thus, even though we do not have as much control over the system variables (particularly h_2) as we would under Condition 1, we have not lost total control; h_2 can still be influenced by the only available input variable, the stream F_1 flowrate (you can control either h_1 or h_2 but not both).
3. Under Condition 3, it is clear that we have *absolutely* no control over h_1 the stream F_1 flowrate cannot be changed, and the recycle stream is no longer in operation; thus we can only control h_2. (Note that if the recycle stream were still in operation, through it we would have indirect influence on the level in Tank 1, but this is not the case.)

In term of controllability, we have the following conclusion:

1. Under Conditions 1, the system is said to be *controllable* because, we can influence each of the two process output variables with whatever input variables we have available;
2. Under Conditions 2, the system is said to be conditional *controllable* because, we can only influence one of the two process output variables;
3. under Condition 3, the system is said to be *uncontrollable* in the sense that we have no control over one of the process output variables.

#

Assume that the system begins at steady state. The definition of controllability will be met if the controlled variables can be maintained at their set points, so that their deviation variables are zero, by adjusting the specified manipulated variables in the presence of step like disturbances, which achieve a constant value. The behavior of the system at steady state can be determined through the final value theorem.

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} K_{11} & \dots & K_{1m} \\ \vdots & \ddots & \vdots \\ K_{m1} & \dots & K_{mm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} K_{d1} \\ \vdots \\ K_{dm} \end{bmatrix} D \tag{5.2.5}$$

with $K_{ij} = \underset{s\to 0}{Lim}\, G_{ij}(s)$.

The system is controllable if there is a solution for this set of linear algebraic equation for arbitrary nonzero values of K_{di} , *D's*, (all possible input variables as disturbances).

A solution exists for a square system of linear equations when an inverse to the matrix of feedback process gains (*K*) exists; thus, the system is controllable if the determinant of the gain matrix is nonzero.

Not all multivariable systems are ***controllable***, it is an essential in designing effective controllers for MIMO systems. It is defined for a selected set of manipulated and controlled variables, it may be controllable for one selection and uncontrollable for another.

1.2.3 Observability

The state-variable of a system, *x*, which represents the internal workings of the system that can be separate from the regular input-output relationship of the system also needs to be measured or *observed*. The term **observability** describes whether the internal state variables of the system can be externally measured.

Observability refers to the ability or inability to uniquely reconstruct (estimate) the state of a system in finite time given output and input measurements. If the state can be determined uniquely in finite time, then we say that the system is observable. If not, we say that the system is unobservable.The state-variables of a system might not be able to be measured for any of the following reasons:

1. The location of the particular state variable might not be physically accessible (a capacitor or a spring, for instance).
2. There are no appropriate instruments to measure the state variable, or the state-variable might be measured in units for which there does not exist any measurement device.
3. The state-variable is a derived "dummy" variable that has no physical meaning.

If things cannot be directly observed, for any of the reasons above, it can be necessary to calculate or **estimate** the values of the internal state variables, using only the input/output relation of the system, and the output history of the system from the starting time. In other words, we must ask whether or not it is possible to determine what the inside of the system (the internal system states) is like, by only observing the outside performance of the system (input and output)? We can provide the following formal definition of mathematical observability

Example 5.2.4: To illustrate the closely related issue of *observability*, let us imagine that with the same two-tank system of Fig. 5.1.5, we can now only measure the liquid level in *one* tank; along with this single-level measurement restriction, the recycle loop is also taken out of operation.

The observability of the system is dependant only on the system states and the system output, so we can simplify our state equations to remove the input terms:

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned} \tag{5.2.6}$$

If the only measurement available is h_2, which give C matrix

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Observe that since its value is *influenced* by h_1 though the Tank 1 outflow, we can, in principle, extract information about the value h_1 given only h_2, and a fairly accurate process model.

If we turn the situation around, and consider the situation in which the only available measurement is h_1, that is

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

it is clear from the physics of the process that *without the recycle loop in operation*, there is *absolutely* no way we can infer anything about the value of h_2 given only information about h_1.

In the former case, the process is said to be *observable*, while it is *unobservable* in the latter case. #

Observability: A system with an initial state, $x(t_0)$ is observable if and only if the value of the initial state can be determined from the system output y(t) that has been observed through the time interval $t_0 < t < t_f$. If the initial state cannot be so determined, the system is unobservable.

Observability Matrix: The observability of the system is dependant only on the coefficient matrices A and C. We can show precisely how to determine whether a system is observable, using only these two matrices. If we have the **observability matrix** Q:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.2.7}$$

we can show that the system is observable if and only if the Q matrix has a rank of n. Notice that the Q matrix has the dimensions $nm \times n$.

5.2.4 Stability

Bounded-Input, Bounded-Output (BIBO) stability is defined as:

> *A MIMO process in the state space form is open-loop stable if and only if all the eigenvalues of the matrix A have negative real parts; otherwise it is unstable.*

It is important to note the following points:

1. The open-loop stability depends only on the matrix A.
2. The stability conditions hold for both responses in the *state* vector, x, or in the *output* vector, y.

3. The eigenvalues of any matrix A are determined by:

$$\left|A-\lambda I\right| = 0 \tag{5.2.8a}$$

   and engenvectors is determined by

$$\begin{aligned}(A-\lambda_j I)X_j &= 0\\ M &= [X_1 \vdots X_2 \vdots \cdots \vdots X_n]\end{aligned} \tag{5.2.8b}$$

Closed-Loop Stability: The poles of the closed-loop transfer function matrices are the roots of the polynomial of the return difference matrix:

$$|(I+GG_c)| = 0 \tag{5.2.9}$$

these determine whether the multivariable system is stable or not.

Example 5.2.5: *Stability of a 2 x 2 System under decentralized control*

The general *2 x 2* system whose transfer function is given as:

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

is to be controlled by two independent feedback controllers, so that the controller transfer function matrix is:

$$G_c(s) = \begin{bmatrix} g_{c1}(s) & 0 \\ 0 & g_{c2}(s) \end{bmatrix}$$

Obtain the characteristic equations to be used in investigating the closed-loop stability under the following three conditions:

1. Only Loop 1 is closed; Loop 2 is open.
2. Only Loop 2 is closed; Loop 1 is open.
3. Both loops are closed.

Under Conditions 1 and 2, we have the system operating as a SISO system since only one loop is closed in each case.

The characteristic equations for Condition 1 and 2 are:

$$1 + g_{11}g_{c1} = 0$$

$$1 + g_{22}g_{c2} = 0$$

respectively.

For the complete multivariable system, with both loops closed (Condition 3), the characteristic equation is obtained from

$$\begin{bmatrix} 1+g_{11}g_{c1} & g_{12}g_{c2} \\ g_{21}g_{c1} & 1+g_{22}g_{c2} \end{bmatrix} = 0$$

and upon evaluating the determinant, we obtain:

$$(1+g_{11}g_{c1})(1+g_{22}g_{c2}) - g_{12}g_{21}g_{c1}g_{c2} = 0$$ #

The conditions, which will guarantee stability for each individual loop, are completely different from the conditions that will guarantee stability when both loops are closed. However, one should note that

1. If *either* g_{11} or g_{22} is zero, then the conditions are identical and the individual loops can be considered independently in analysing stability.
2. If either g_{12} or g_{21} (but not both) are zero, there are still interactions between the loops, but these do not influence the stability condition. In this case (called *one-way interaction),* the interaction appears as a disturbance in the affected loop.

#

The implications of interactions in multivariable systems is that guaranteeing stability for individual loops operating by themselves provides no guarantee when all loops are closed, and are operating simultaneously, the overall system will be stable.

It is not always possible to design a control system for a multivariable system by treating it like several, independent SISO systems.

5.2.5 Singularity — Singular Values

When the determinant of a matrix is zero, the matrix is said to be singular. How close a transfer function matrix is to being singular provides information about how easy, or difficult to control such a system. Evaluating the determinant of a matrix and inspecting how close it is to zero is one way of finding out how near to being singular the matrix is, however, this approach is not always very reliable.

Example 5.2.6: *the following matrix*

$$A(\varepsilon) = \begin{bmatrix} 1 & \varepsilon \\ \frac{1}{\varepsilon} & 1 \end{bmatrix}$$

is singular, regardless of the value of ε. However, consider the situation in which ε takes on very small values, and if we neglect the small ε value at the top right hand corner, we obtain:

$$A_1(\varepsilon) = \begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix}$$

In this case, whereas A is singular, the determinant $|A_1|$ is always equal to 1. Thus, even though we know, by construction, that as ε gets smaller, the matrix A_1 approaches singularity, its determinant gives no such indication. #

The measure of the near singularity of a matrix can be found from the *singular values* of the matrix. For any general, possibly complex, n x m matrix A, the singular values are defined as:

$$\sigma_i = (\lambda_i(A^* A))^{1/2} \qquad i = 1, 2, \ldots, n \tag{5.2.10}$$

where A^* is the transpose of the complex conjugate and $\lambda_i(A^*A)$ represents the ith eigenvalue of the matrix (A^*A).

Example 5.2.7: for the matrix, A_1:

$$A_1^T A_1 = \begin{bmatrix} 1+\dfrac{1}{\varepsilon^2} & \dfrac{1}{\varepsilon} \\ \dfrac{1}{\varepsilon} & 1 \end{bmatrix}$$

and the eigenvalues may be obtained as the two roots of:

$$\lambda^2 - \left(2 + \frac{1}{\varepsilon^2}\right)\lambda + 1 = 0$$

i.e.:

$$\lambda_1 = 1 + \frac{1}{2\varepsilon^2} + \frac{\sqrt{1+4\varepsilon^2}}{2\varepsilon^2}$$

$$\lambda_2 = 1 + \frac{1}{2\varepsilon^2} - \frac{\sqrt{1+4\varepsilon^2}}{2\varepsilon^2}$$

The singular values are therefore given by:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{1 + \frac{1}{2\varepsilon^2} + \frac{\sqrt{1+4\varepsilon^2}}{2\varepsilon^2}}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1 + \frac{1}{2\varepsilon^2} - \frac{\sqrt{1+4\varepsilon^2}}{2\varepsilon^2}}$$

Observe that as $\varepsilon \longrightarrow 0$, $\sigma_1 \longrightarrow \infty$, and $\sigma_2 \longrightarrow 1$, this widening "separation" between the two singular values provides a reliable indication of the onset of singularity, or "ill-conditioning." The ratio of the largest and smallest singular value:

$$k = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{5.2.11}$$

is known as the "condition number" of a matrix. Thus for our example, $k \to \infty$ smoothly as $\varepsilon \to 0$ and provides a good measure of the progress towards singularity.

In general, given the transfer function matrix $G(s)$, the singular values are obtained, as a function of frequency, by substituting $s = j\omega$, and evaluating the square root of the eigenvalues of the matrix $S(j\omega)$ given by:

$$S(j\omega) = G^T(-j\omega)G(j\omega) \tag{5.2.12}$$

since the complex conjugate of $G(j\omega)$ is $G(-j\omega)$. Plots of these singular values and the condition number k as a function of frequency provide useful design information for multivariable systems.

Singular value analysis has proved to be a useful tool in the analysis and design of multivariable control systems.

5.3 Dynamic Solution in Transfer Function Form

Consider the general $n \times m$ multivariable system:

$$y(s) = G(s)u(s)$$

it can also be written as:

$$\begin{aligned} y_1(s) &= g_{11}(s)u_1(s) + g_{12}(s)u_2(s) + \cdots + g_{1m}(s)u_m(s) \\ y_2(s) &= g_{21}(s)u_1(s) + g_{22}(s)u_2(s) + \cdots + g_{2m}(s)u_m(s) \\ &\cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ y_n(s) &= g_{n1}(s)u_1(s) + g_{n2}(s)u_2(s) + \cdots + g_{nm}(s)u_m(s) \end{aligned} \tag{5.3.6}$$

Define x_{ij} as the product of the *ijth* transfer function element and the *jth* input, *i.e.:*

$$x_{ij}(s) = g_{ij}(s)u_j(s) \tag{5.3.7a}$$

If each transfer function element g_{ij} (s) has p_{ij} poles, inverting to time domain, we will obtain:

$$x_{ij}(t) = \sum_{q=1}^{p_{ij}} A_{ijq} e^{r_{ijq}t} \tag{5.3.7b}$$

where A_{ijq} are the coefficients, and r_{jjq}, q = 1, 2, ..., are the roots of the denominator polynomial of g_{ij} (s).

The multivariable system outputs are obtained by taking the appropriate sums of these x_{ij} responses, i.e.:

$$y_i == \sum_{j=1}^{m} x_{ij}(t); \quad \text{or} \quad y_i = \sum_{j=1}^{m}\sum_{q=1}^{p_{ij}} (A_{ijq} e^{r_{ijq}t})$$

Example 5.3.2: *Step Responses of the Binary Distillation Column*

The distillation column used in separating methanol and water are with two output, two input, one disturbance variable defined as:

Y_1: overhead mole fraction methanol;*Y_2:* bottoms mole fraction methanol; *U_1:* overhead reflux flowrate; *U_2:* bottoms steam flowrate; *d:* column feed flowrate. The transfer function matrices

$$G(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \quad \text{and} \quad G_d(s) = \begin{bmatrix} \dfrac{3.8e^{-8.1s}}{14.9s+1} \\ \dfrac{4.9e^{-3.4s}}{13.2s+1} \end{bmatrix}$$

Obtain the response to a unit step change in u_1, and u_2.

Response to unit step change in u_1: Given that $u_1 = 1$ and $u_2 = 0$, then

$$y_1(s) = \frac{12.8e^{-s}}{16.7s+1}\frac{1}{s}; \qquad y_2(s) = \frac{6.6e^{-7s}}{10.9s+1}\frac{1}{s}$$

we obtain:

$$y_1(t) = \begin{cases} 0 & t < 1 \\ 12.8(1 - e^{-(t-1)/16.7}) & t \geq 1 \end{cases}$$

and

$$y_2(t) = \begin{cases} 0 & t < 7 \\ 6.6(1 - e^{-(t-7)/10.9}) & t \geq 7 \end{cases}$$

Note that the “interaction” effect of u_1 on y_2 is significant with a net change of 6.6 in y_2 at steady state.

Response to unit step change in u_2: Setting $u_2 = 1$ and $u_1 = 0$, the transient responses

$$y_1(s) = \frac{-18.9e^{-3s}}{20.1s+1}\frac{1}{s}; \qquad y_2(s) = \frac{-19.4e^{-3s}}{14.4s+1}\frac{1}{s}$$

which result in the unit step responses:

$$y_1(t) = \begin{cases} 0 & t < 3 \\ -18.9(1 - e^{-(t-3)/21.0}) & t \geq 3 \end{cases}$$

and

$$y_2(t) = \begin{cases} 0 & t < 3 \\ -19.4(1 - e^{-(t-3)/14.4}) & t \geq 3 \end{cases}$$

There is also a significant “interaction effect” of u_2 on y_1 with a steady-state change in y_1.