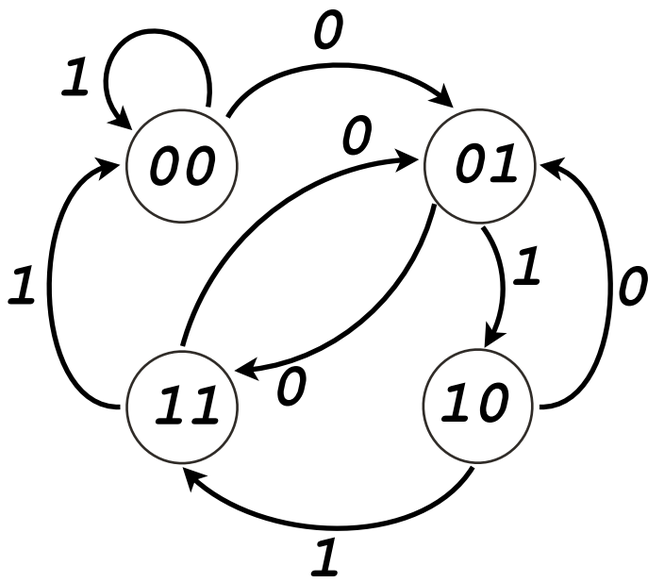
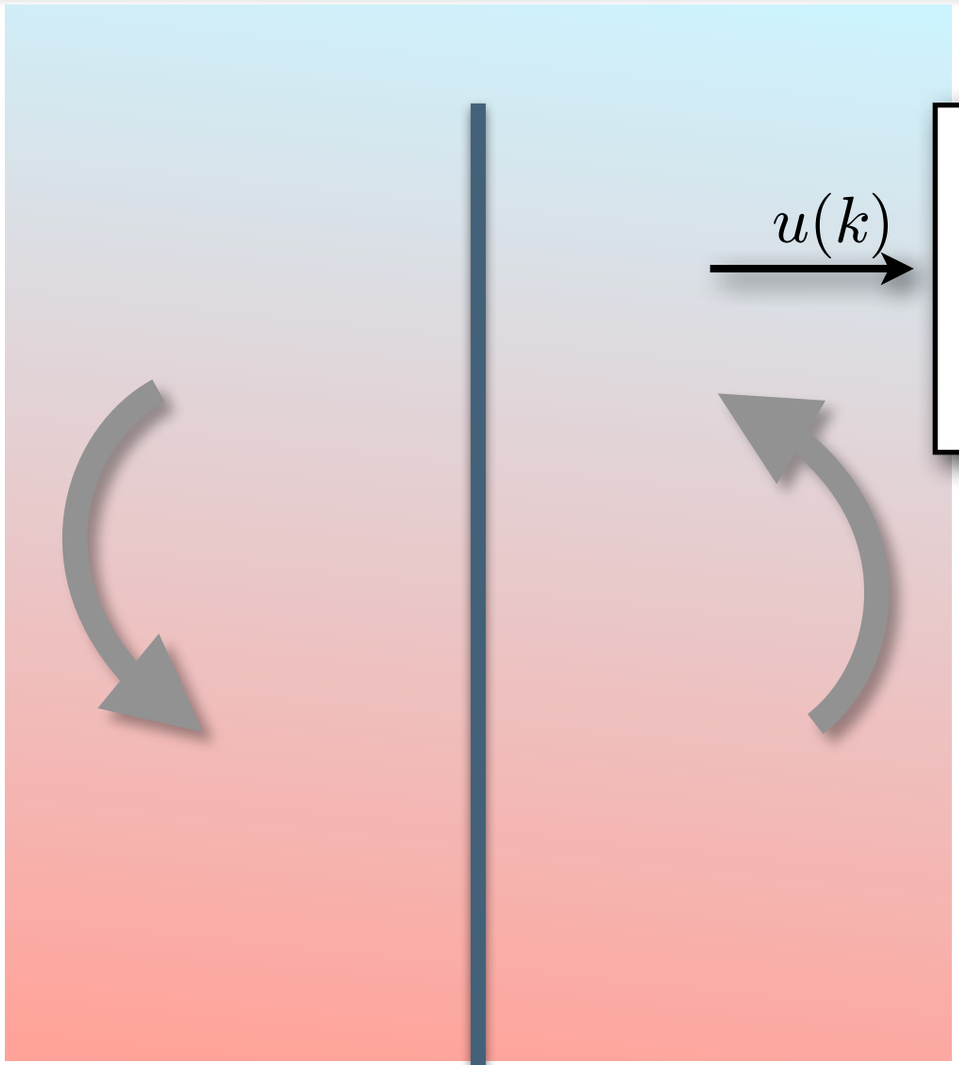
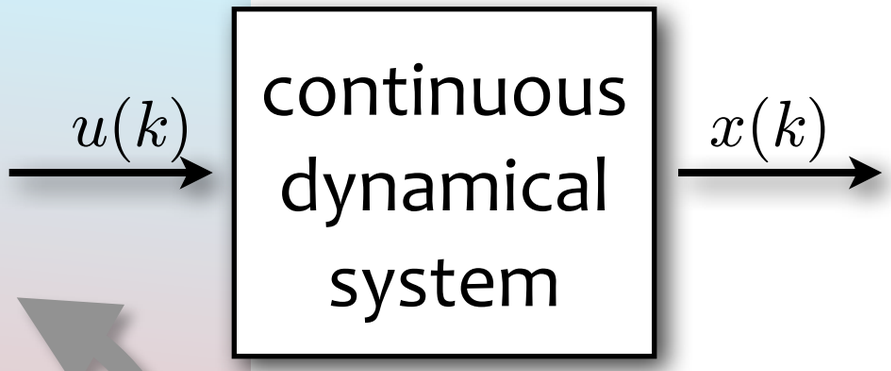
HYBRID SYSTEMS: MODELING

Alberto Bemporad - “Model Predictive Control” course - Academic year 2017/18

HYBRID DYNAMICAL SYSTEMS







hybrid dynamical system

- Variables are **discrete-valued**

$$x \in \{0,1\}^{n_b},\ u \in \{0,1\}^{m_b}$$

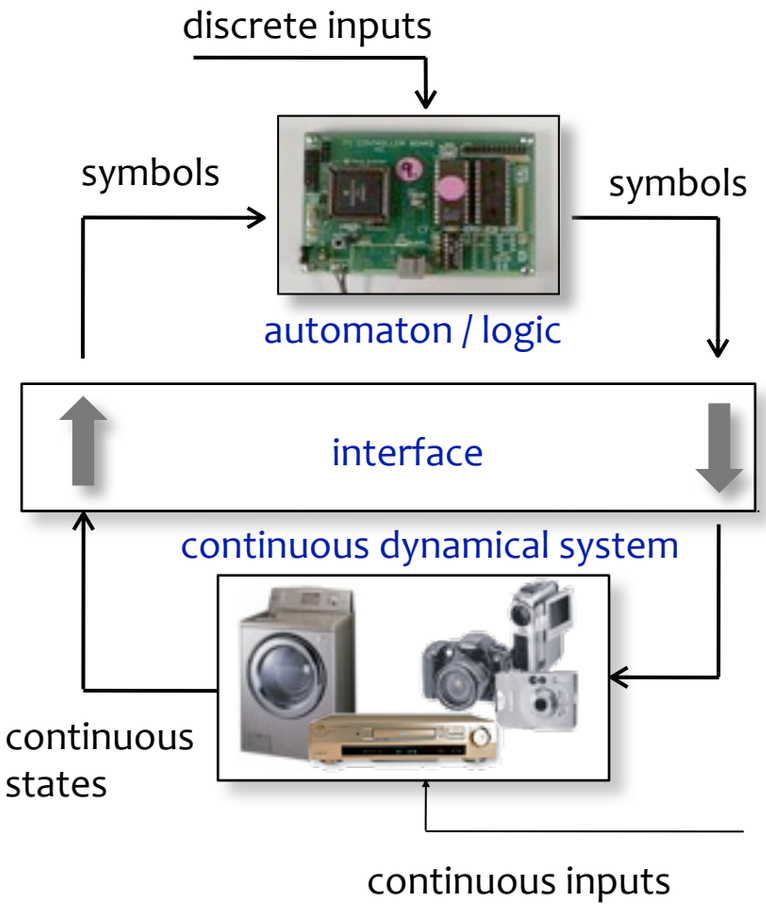
- Dynamics = **finite state machine**
- **Logic** constraints

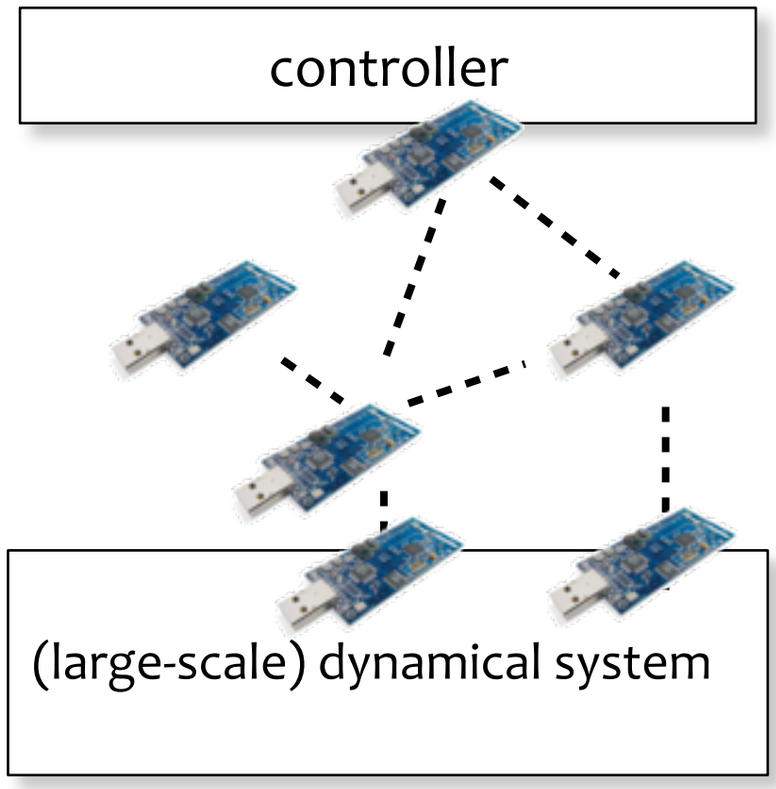
- Variables are **real-valued**

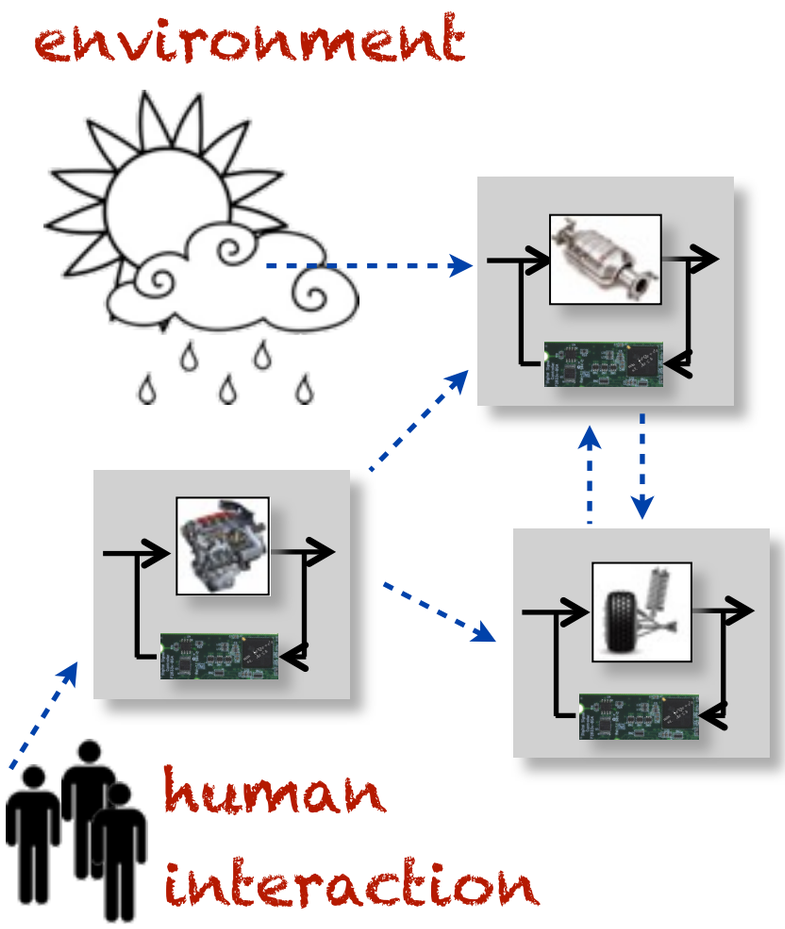
$$x \in \mathbb{R}^{n_c},\ u \in \mathbb{R}^{m_c}$$

- **Difference/differential equations**
- **Linear inequality** constraints

TECHNOLOGICAL PUSH FOR STUDYING HYBRID SYSTEMS







embedded systems

networked control systems

cyber-physical systems

>1995 >2005 >2010

AN EXAMPLE OF "INTRINSICALLY HYBRID" SYSTEM

- Gear-shift

continuous dynamical variables (speed, torque, ...)

continuous commands (brake & gas pedal)

+

discrete command (R,N,1,2,3,4,5)

KEY REQUIREMENTS FOR HYBRID MODELS

- **Descriptive** enough to capture the behavior of the system
  - continuous dynamics (physical laws)
  - logic components (switches, automata, software code)
  - interconnection between logic and dynamics

- **Simple** enough for solving *analysis* and *synthesis* problems

$$\begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$$

linear systems

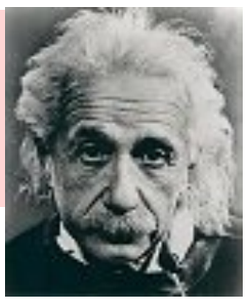
?

linear hybrid systems

$$\begin{cases} x' = f(x,u,t) \\ y = g(x,u,t) \end{cases}$$

nonlinear systems

"Make everything as simple as possible, but not simpler."
— Albert Einstein



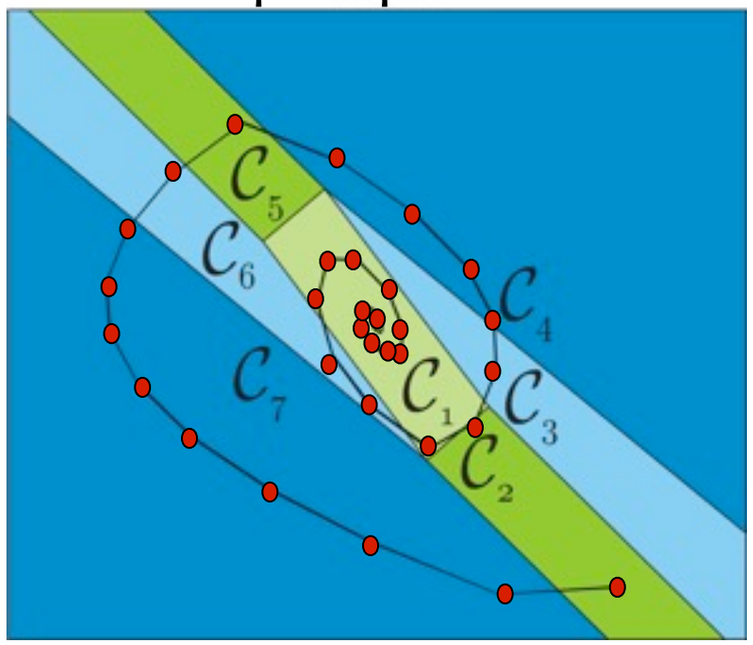
PIECEWISE AFFINE SYSTEMS

$$
\begin{aligned}
x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\
y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\
\\
i(k) \quad &\text{s.t.} \quad H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}
\end{aligned}
$$

$x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$

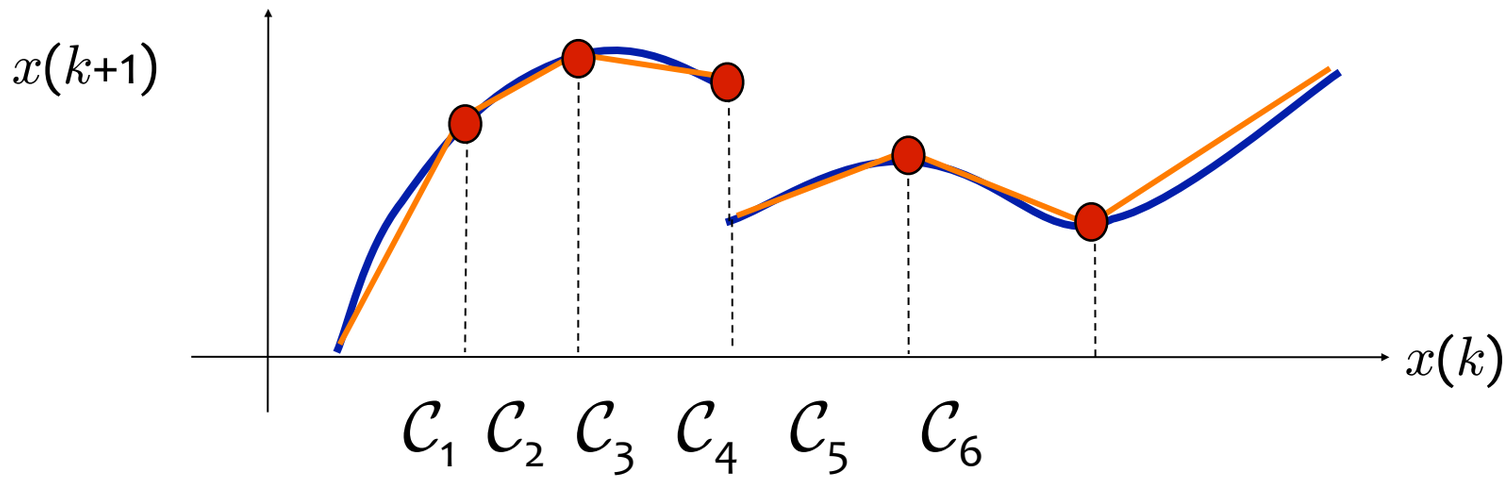
$i(k) \in \{1, \ldots, s\}$

state+input space



(Sontag 1981)

Can approximate nonlinear and/or discontinuous dynamics arbitrarily well



DISCRETE HYBRID AUTOMATON (DHA)

(Torrisi, Bemporad, 2004)

discrete

Event Generator

$$[\delta_e = 1] \leftrightarrow [Hx_c + Ku_c \leq W]$$

$\delta_e = 1$

$\delta_e = 0$

$\delta_e(k)$

$x_c(k)$

Finite State Machine

$$x_\ell(k+1) = f_B(x_\ell(k), u_\ell(k), \delta_e(k))$$

$u_\ell(k)$

discrete time counter

$u_c(k)$

Switched Affine System

$$x_c(k+1) = A_i x_c(k) + B_i u_c(k) + f_i$$

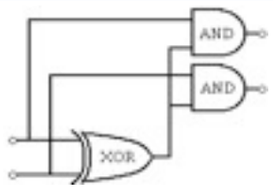
$$i = i(k)$$

$x_c(k)$

$x_\ell(k)$

mode $i(k)$

Mode Selector



$u_\ell(k)$

$\delta_e(k)$

$$i = f_{\mathsf{M}}(x_\ell, u_\ell, \delta_e)$$

continuous

$x_\ell \in \{0,1\}^{n_\ell}$ = binary state

$u_\ell \in \{0,1\}^{m_\ell}$ = binary input

$\delta_e \in \{0,1\}^{n_e}$ = event variable

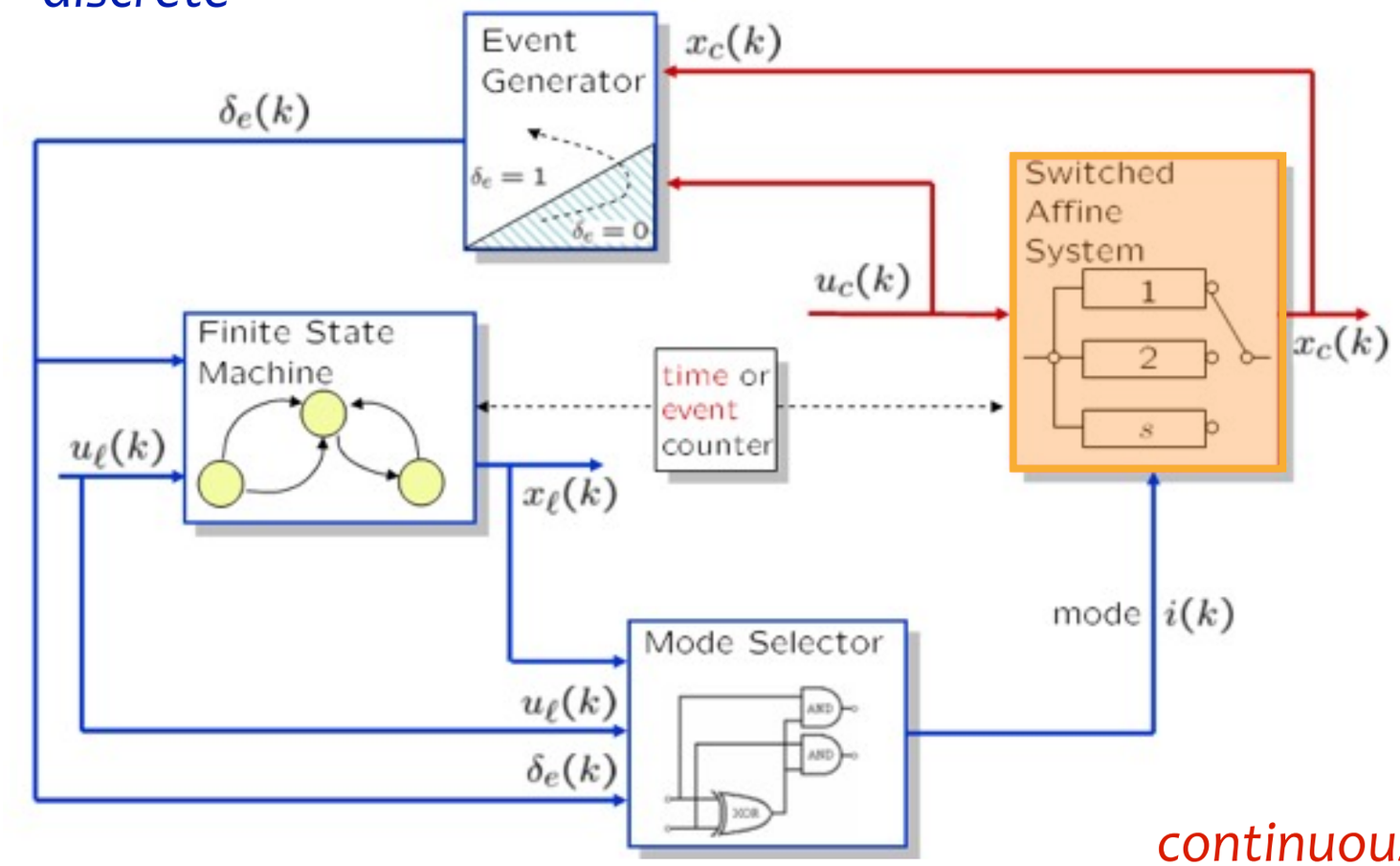
$x_c \in \mathbb{R}^{n_c}$ = real-valued state

$u_c \in \mathbb{R}^{m_c}$ = real-valued input

$i \in \{1, \ldots, s\}$ = current mode

SWITCHED AFFINE SYSTEM

discrete



continuous

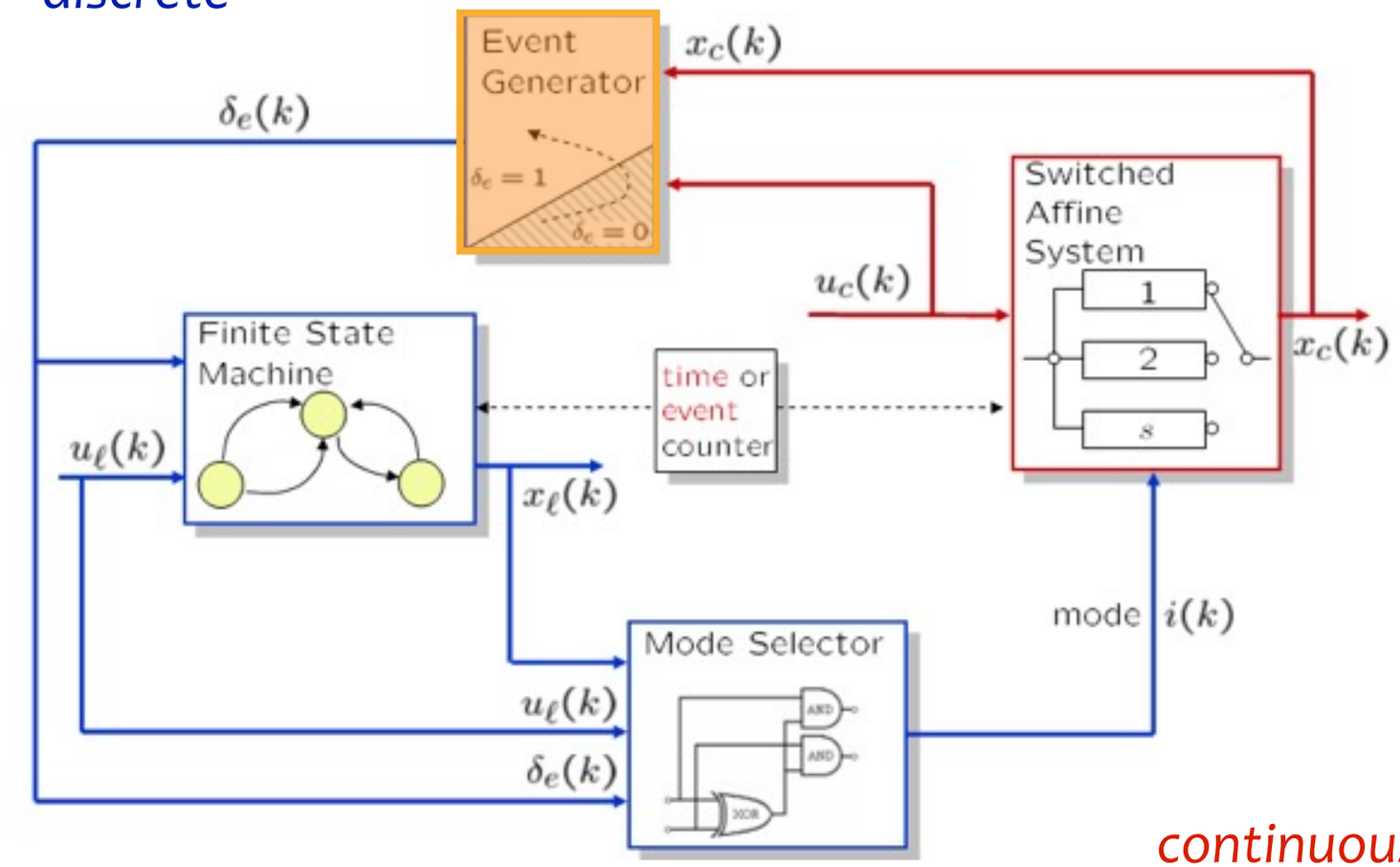
The affine dynamics depend on the current mode $i(k)$:

$$x_c(k+1) = A_{i(k)}x_c(k) + B_{i(k)}u_c(k) + f_{i(k)}$$

$x_c \in \mathbb{R}^{n_c}, \; u_c \in \mathbb{R}^{m_c}$

EVENT GENERATOR

discrete



continuous

Event variables are generated by linear threshold conditions over continuous states, continuous inputs, and time:

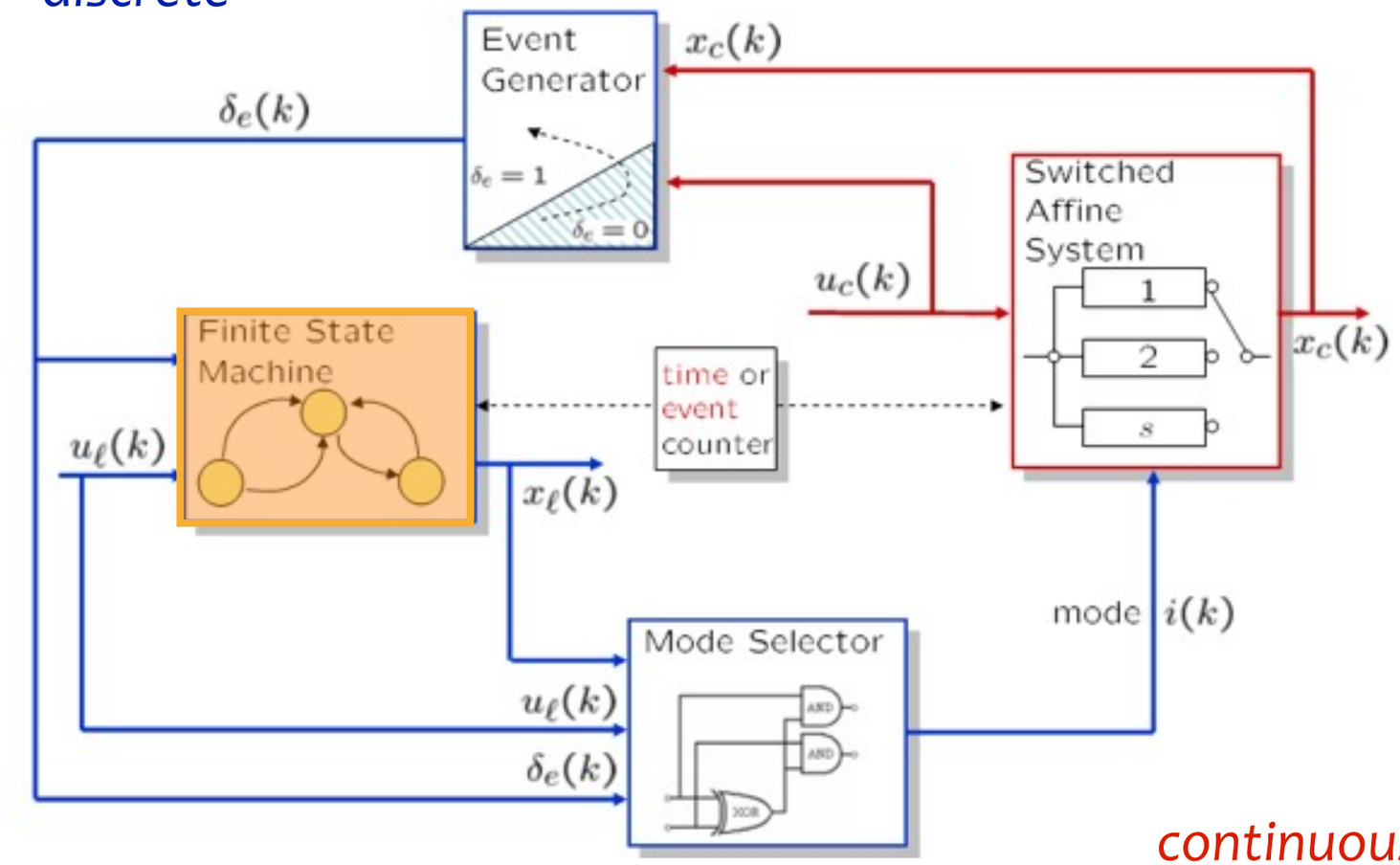
$$[\delta_e^i(k) = 1] \iff [H^i x_c(k) + K^i u_c(k) \leq W^i]$$

$$x_c \in \mathbb{R}^{n_c},\ u_c \in \mathbb{R}^{m_c},\ \delta_e \in \{0,1\}^{n_e}$$

Example: $[\delta(k) = 1] \leftrightarrow [x_c(k) \geq 0]$

FINITE STATE MACHINE

discrete



continuous

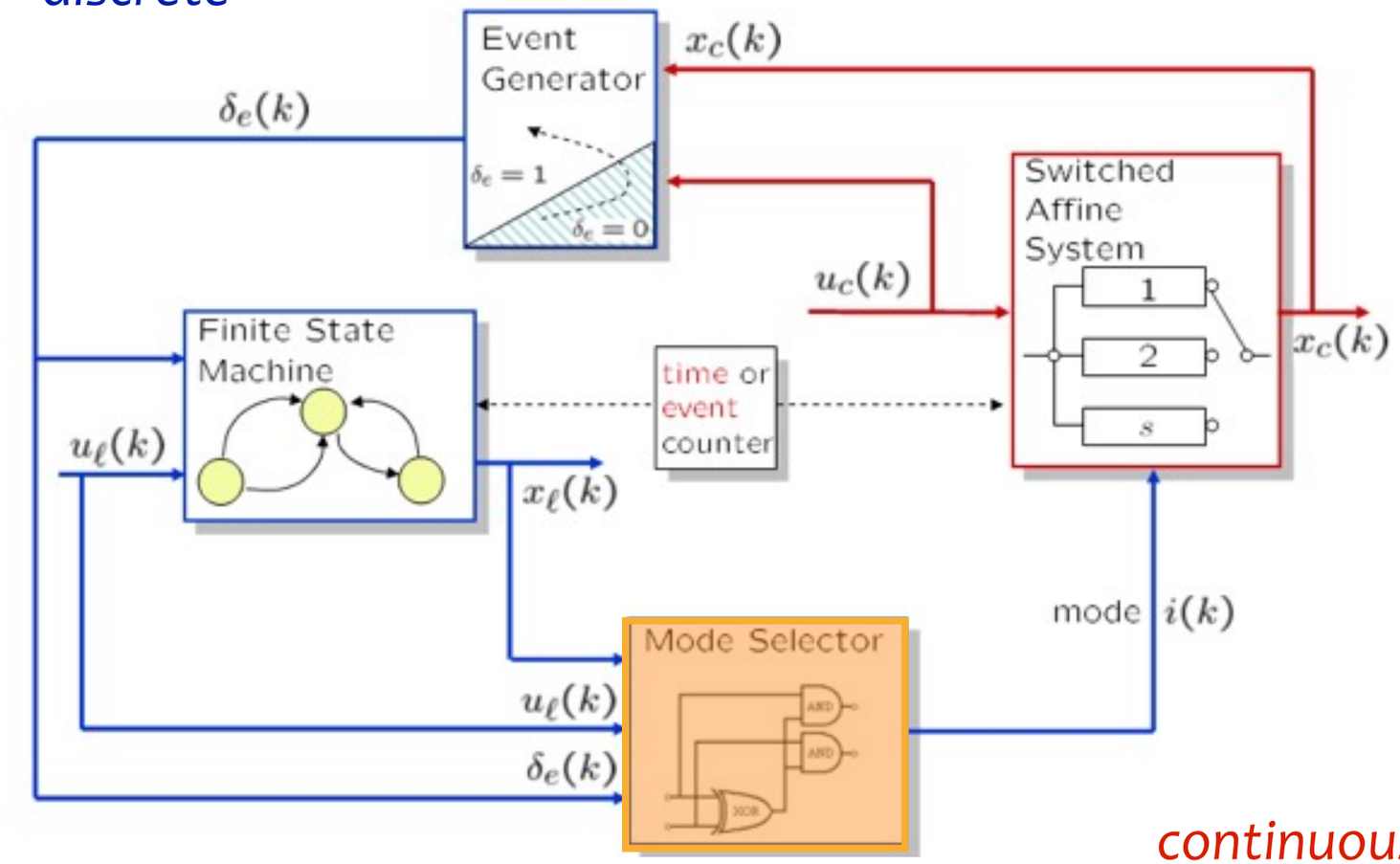
The binary state of the finite state machine evolves according to a Boolean state update function:

$$x_\ell(k+1) = f_{\mathsf{B}}(x_\ell(k), u_\ell(k), \delta_e(k)) \qquad x_\ell \in \{0,1\}^{n_\ell},\ u_\ell \in \{0,1\}^{m_\ell},\ \delta_e \in \{0,1\}^{n_e}$$

Example: $x_\ell(k+1) = \neg\delta_e(k) \vee (x_\ell(k) \wedge u_\ell(k))$

MODE SELECTOR

discrete



The mode selector can be seen as the output function of the discrete dynamics

continuous

The active mode $i(k)$ is selected by a Boolean function of the current binary states, binary inputs, and event variables:

$$i(k) = f_{\mathsf{M}}(x_\ell(k), u_\ell(k), \delta_e(k)) \qquad x_\ell \in \{0,1\}^{n_\ell},\ u_\ell \in \{0,1\}^{m_\ell},\ \delta_e\{0,1\}^{n_e}$$

Example:

$$i(k) = \begin{bmatrix} \neg u_\ell(k) \vee x_\ell(k) \\ u_\ell(k) \wedge x_\ell(k) \end{bmatrix}$$

u_ℓ / x_ℓ	0	1
0	$i = \begin{bmatrix}1\\0\end{bmatrix}$	$i = \begin{bmatrix}1\\0\end{bmatrix}$
1	$i = \begin{bmatrix}0\\0\end{bmatrix}$	$i = \begin{bmatrix}1\\1\end{bmatrix}$

the system has 3 modes

LOGIC -> INEQUALITIES

$$X_1 \vee X_2 = \text{TRUE} \quad \longrightarrow \quad \delta_1 + \delta_2 \geq 1, \qquad \delta_1, \delta_2 \in \{0,1\}$$

(Glover 1975, Williams 1977, Hooker 2000)

0. Given a Boolean statement

$$F(X_1, X_2, \ldots, X_n) = \text{TRUE}$$

1. Convert to Conjunctive Normal Form (CNF):

$$\bigwedge_{j=1}^{m} \left(\bigvee_{i \in P_j} X_i \bigvee_{i \in N_j} \bar{X}_i \right) = \text{TRUE}$$

$$P_j \cup N_j \subseteq \{1, n\}$$

2. Transform into inequalities:

$$\begin{aligned} \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) &\geq 1 \\ \vdots \quad \vdots \quad \vdots \\ \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) &\geq 1 \end{aligned}$$

polyhedron

$$A\delta \leq b, \quad \delta \in \{0,1\}^n$$

Any logic proposition can be translated into integer linear inequalities

LOGIC -> INEQUALITIES: SYMBOLIC APPROACH

Example: $F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$

1. Convert to Conjunctive Normal Form (CNF):

(see e.g. formal.cs.utah.edu:8080/pbl/PBL.php or just google "CNF + converter" ...)

$$(X_3 \vee \neg X_1 \vee \neg X_2) \wedge (X_1 \vee \neg X_3) \wedge (X_2 \vee \neg X_3)$$

2. Transform into inequalities:

$$\begin{cases} \delta_3 + (1 - \delta_1) + (1 - \delta_2) \geq 1 \\ \delta_1 + (1 - \delta_3) \geq 1 \\ \delta_2 + (1 - \delta_3) \geq 1 \end{cases}$$

LOGIC -> INEQUALITIES: GEOMETRIC APPROACH

Boolean statement

$$F(X_1, X_2, \ldots, X_n) = \text{TRUE}$$

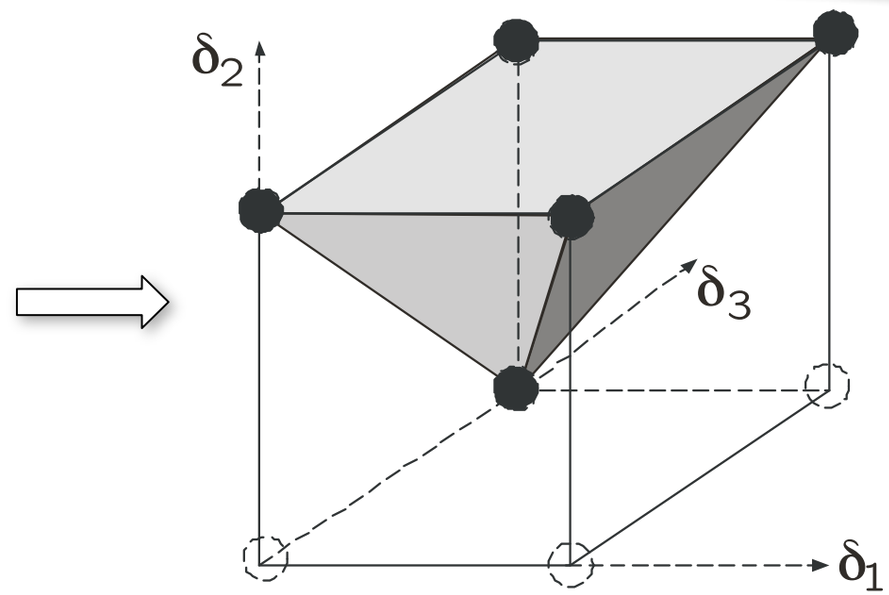
polyhedron

$$A\delta \leq b, \quad \delta \in \{0,1\}^n$$

The polytope $P = \{\delta : \ A\delta \leq b\}$ is the convex hull of the rows of the truth table T associated with formula $F(X_1, \ldots, X_N)$

T:

X_1	X_2	...	X_N
0	0	...	1
0	1	...	0
⋮	⋮	⋮	⋮
1	1	...	0



Convex hull algorithms: `cdd`, `lrs`, `qhull`, `chD`, `Hull`, `Porto`

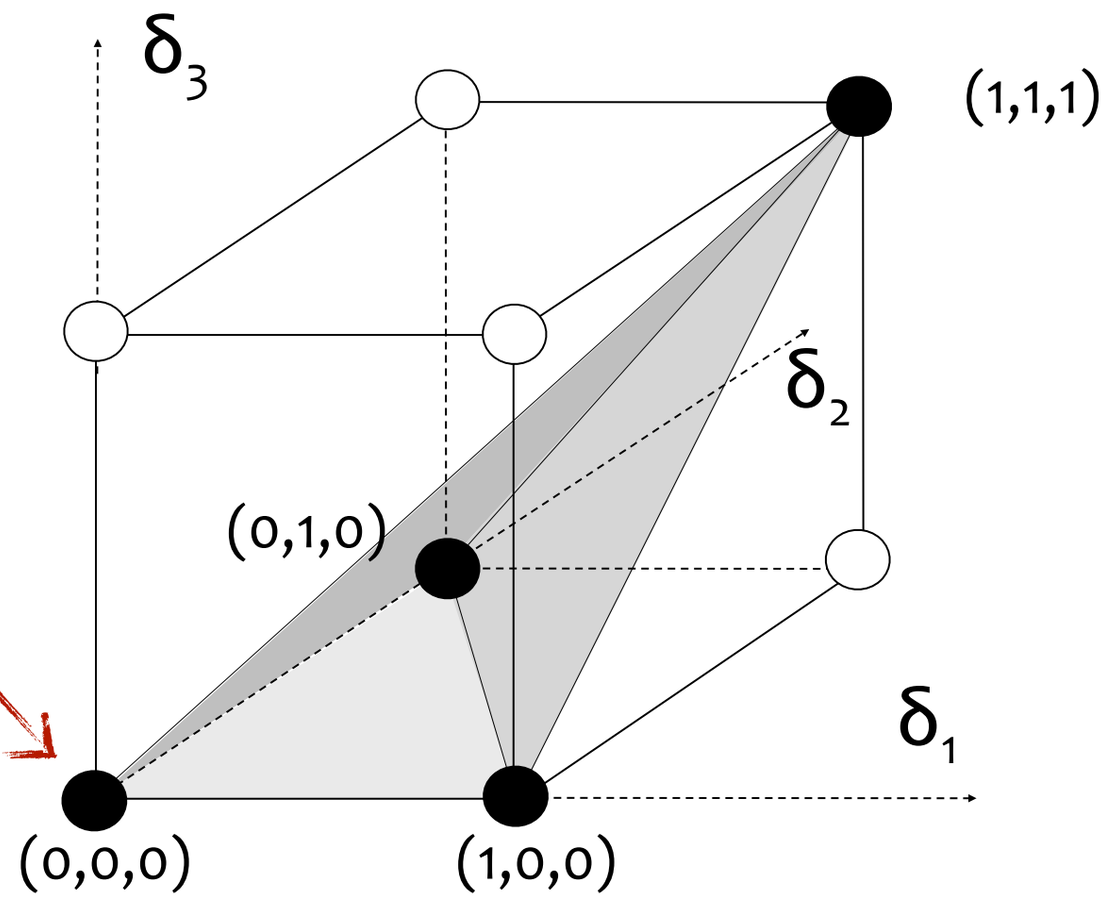
CDDMEX package included in the Hybrid Toolbox

LOGIC -> INEQUALITIES: GEOMETRIC APPROACH

Example: logic "AND"

$$F(X_1, X_2, X_3) = [X_3 \leftrightarrow X_1 \wedge X_2]$$

X_1	X_2	X_3
0	0	0
0	1	0
1	0	0
1	1	1



Key idea:

White points cannot be in the hull of black points

$$\text{conv}\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \left\{\delta \in \mathbb{R}^3 : \begin{array}{rcl} -\delta_1 + \delta_3 & \leq & 0 \\ -\delta_2 + \delta_3 & \leq & 0 \\ \delta_1 + \delta_2 - \delta_3 & \leq & 1 \end{array}\right\}$$

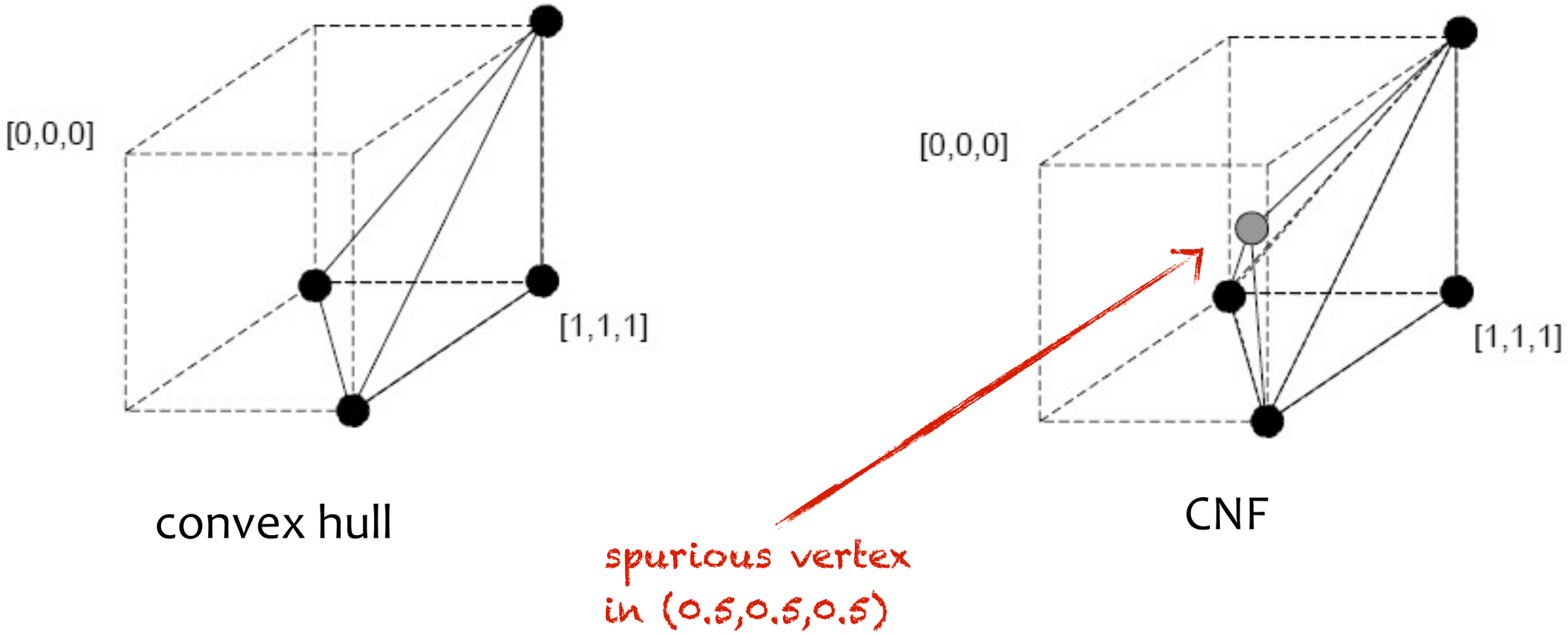
```
>> V=struct('V',[0 0 0;0 1 0;1 0 0;1 1 1]);
>> H=cddmex('hull',V);A=H.A,b=H.B
```

GEOMETRIC VS. SYMBOLIC APPROACH

- The polyhedron obtained via convex hull is the smallest one

- The one obtained via CNF may be larger. Example:

$$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) = \text{TRUE}$$



convex hull

CNF

Note: no other example with 3 vars but $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge (\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3)$

BIG-M TECHNIQUE (IFF)

- Consider the *if-and-only-if* condition

$$[\delta = 1] \leftrightarrow [a'x_c - b \leq 0]$$

$$x_c \in \mathcal{X} \subset \mathbb{R}^{n_c}$$
$$\delta \in \{0, 1\}$$

- Assume $\mathcal{X}$ bounded and let M and m such that

$$M > a'x_c - b, \ \forall x_c \in \mathcal{X}$$
$$m < a'x_c - b, \ \forall x_c \in \mathcal{X}$$

- The *if-and-only-if* condition is equivalent to

$$\begin{cases} a'x_c - b & \leq & M(1-\delta) \\ a'x_c - b & > & m\delta \end{cases}$$

- To avoid strict inequalities, replace by $a'x_c - b \geq \epsilon + (m-\epsilon)\delta$ where $\epsilon > 0$ is a small number (e.g., machine precision)

BIG-M TECHNIQUE (IF-THEN-ELSE)

- Consider the *if-then-else* condition

$$z = \begin{cases} a_1' x_c + f_1 & \text{if } \delta = 1 \\ a_2' x_c + f_2 & \text{otherwise} \end{cases}$$

$$x_c \in \mathcal{X} \subset \mathbb{R}^{n_c}$$
$$\delta \in \{0, 1\}$$
$$z \in \mathbb{R}$$

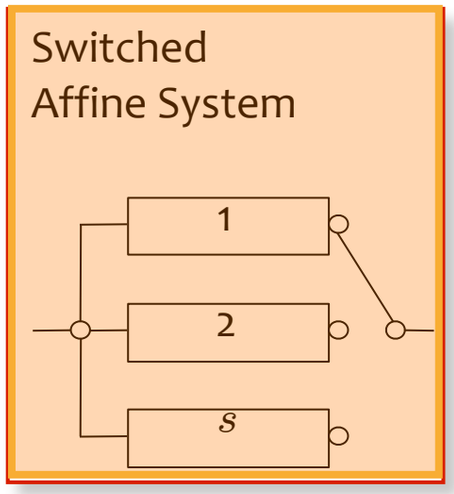
- Assume $\mathcal{X}$ bounded and let M_1, M_2 and m_1, m_2 such that

$$M_1 > a_1' x_c + f_1 > m_1, \ \forall x_c \in \mathcal{X}$$
$$M_2 > a_2' x_c + f_2 > m_2, \ \forall x_c \in \mathcal{X}$$

- The *if-then-else* condition is equivalent to

$$\begin{cases} (m_1 - M_2)(1-\delta) + z \leq a_1' x_c + f_1 \\ (m_2 - M_1)(1-\delta) - z \leq -a_1' x_c - f_1 \\ (m_2 - M_1)\delta + z \leq a_2' x_c + f_2 \\ (m_1 - M_2)\delta - z \leq -a_2' x_c - f_2 \end{cases}$$

SWITCHED AFFINE SYSTEM



The state-update equation can be rewritten as a difference equation + *if-then-else* conditions:

$$z_1(k) = \begin{cases} A_1 x_c(k) + B_1 u_c(k) + f_1 & \text{if } \delta_1(k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\vdots$$

$$z_s(k) = \begin{cases} A_s x_c(k) + B_s u_c(k) + f_s & \text{if } \delta_s(k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{i=1}^{s} \delta_i(k) = 1 \quad \Longrightarrow \quad \begin{cases} \sum_{i=1}^{s} \delta_i(k) \geq 1 \\ \sum_{i=1}^{s} \delta_i(k) \leq 1 \end{cases}$$

$$x_c(k+1) = \sum_{i=1}^{s} z_i(k)$$

where $z_i(k) \in \mathbb{R}^{n_c}$, $\delta_i(k) \in \{0,1\}$, $i = 1, \ldots, s$

Output equations $y_c(k)=C_i x_c(k)+D_i u_c(k)+g_i$ admit a similar transformation

LOGIC AND INEQUALITIES

$X_1 \vee X_2 = \text{TRUE} \longrightarrow \delta_1 + \delta_2 \geq 1, \quad \delta_1, \delta_2 \in \{0,1\}$ (Glover 1975, Williams 1977, Hooker 2000)

Any logic statement
$f(X) = \text{TRUE}$

$$\bigwedge_{j=1}^{m} \left(\vee_{i \in P_j} X_i \vee_{i \in N_j} \neg X_i \right) \quad \text{(CNF)}$$

$N_j, P_j \subseteq \{1, \ldots, n\}$

$$\begin{cases} 1 \leq \sum_{i \in P_1} \delta_i + \sum_{i \in N_1} (1 - \delta_i) \\ \vdots \\ 1 \leq \sum_{i \in P_m} \delta_i + \sum_{i \in N_m} (1 - \delta_i) \end{cases}$$

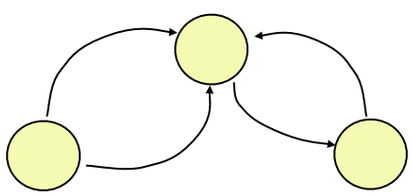
$[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) \leq W^i]$

$$\begin{cases} H^i x_c(k) - W^i \leq M^i (1 - \delta_e^i) \\ H^i x_c(k) - W^i > m^i \delta_e^i \end{cases}$$

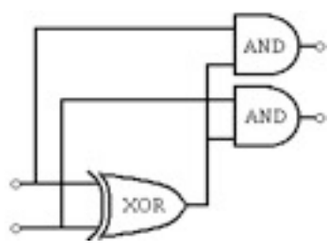
IF $[\delta = 1]$ THEN $z = a_1^T x + b_1^T u + f_1$
ELSE $z = a_2^T x + b_2^T u + f_2$

$$\begin{cases} (m_1 - M_2)(1 - \delta) + z \leq a_1 x + b_1 u + f_1 \\ (m_2 - M_1)(1 - \delta) - z \leq -a_1 x - b_1 u - f_1 \\ (m_2 - M_1)\delta + z \leq a_2 x + b_2 u + f_2 \\ (m_1 - M_2)\delta - z \leq -a_2 x - b_2 u - f_2 \end{cases}$$

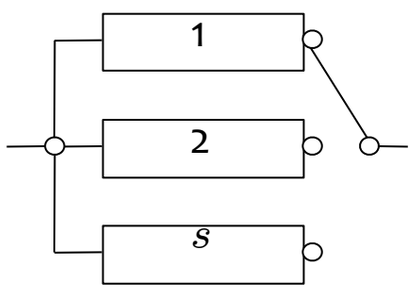
Finite State Machine



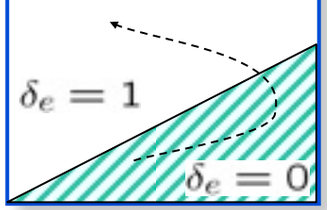
Mode Selector



Switched Affine System



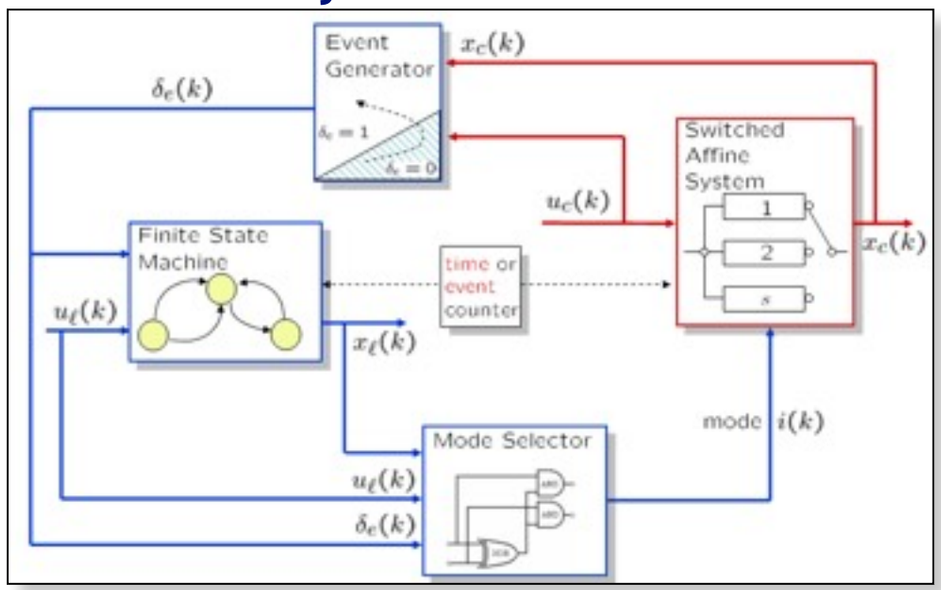
Event Generator



MIXED LOGICAL DYNAMICAL (MLD) SYSTEMS

(Bemporad, Morari 1999)

Discrete Hybrid Automaton



HYSDEL (Torrisi, Bemporad, 2004)

convert logic propositions to mixed-integer linear inequalities

Mixed Logical Dynamical (MLD) systems

$$\begin{cases} x_{k+1} = Ax_k + B_1u_k + B_2\delta_k + B_3z_k + B_5 \\ y_k = Cx_k + D_1u_k + D_2\delta_k + D_3z_k + D_5 \\ E_2\delta_k + E_3z_k \le E_4x_k + E_1u_k + E_5 \end{cases}$$

Continuous and binary variables $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b},\ u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$
$y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_b},\ \delta \in \{0,1\}^{r_b},\ z \in \mathbb{R}^{r_c}$

- Computationally oriented model (mixed-integer programming)
- Suitable for MPC control, verification, state estimation, fault detection, ...

A SIMPLE EXAMPLE

- System:

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if } x(k) \geq 0 \\ -0.8x(k) + u(k) & \text{if } x(k) < 0 \end{cases}$$

$$-10 \leq x(k) \leq 10$$

- Associate $[\delta(k) = 1] \leftrightarrow [x(k) \geq 0]$ and transform

$$\begin{aligned} x(k) &\geq m(1-\delta(k)) \qquad M = -m = 10 \\ x(k) &\leq -\epsilon + (M+\epsilon)\delta(k) \qquad \epsilon > 0 \text{ "small"} \end{aligned}$$

- Then $x(k+1) = 1.6\delta(k)x(k) - 0.8x(k) + u(k)$

$$z(k) = \delta(k)x(k)$$

$$\begin{aligned} z(k) &\leq M\delta(k) \\ z(k) &\geq m\delta(k) \\ z(k) &\leq x(k) - m(1-\delta(k)) \\ z(k) &\geq x(k) - M(1-\delta(k)) \end{aligned}$$

$$\delta(k) \in \{0, 1\}$$

- Rewrite as a linear equation

$$x(k+1) = 1.6z(k) - 0.8x(k) + u(k)$$

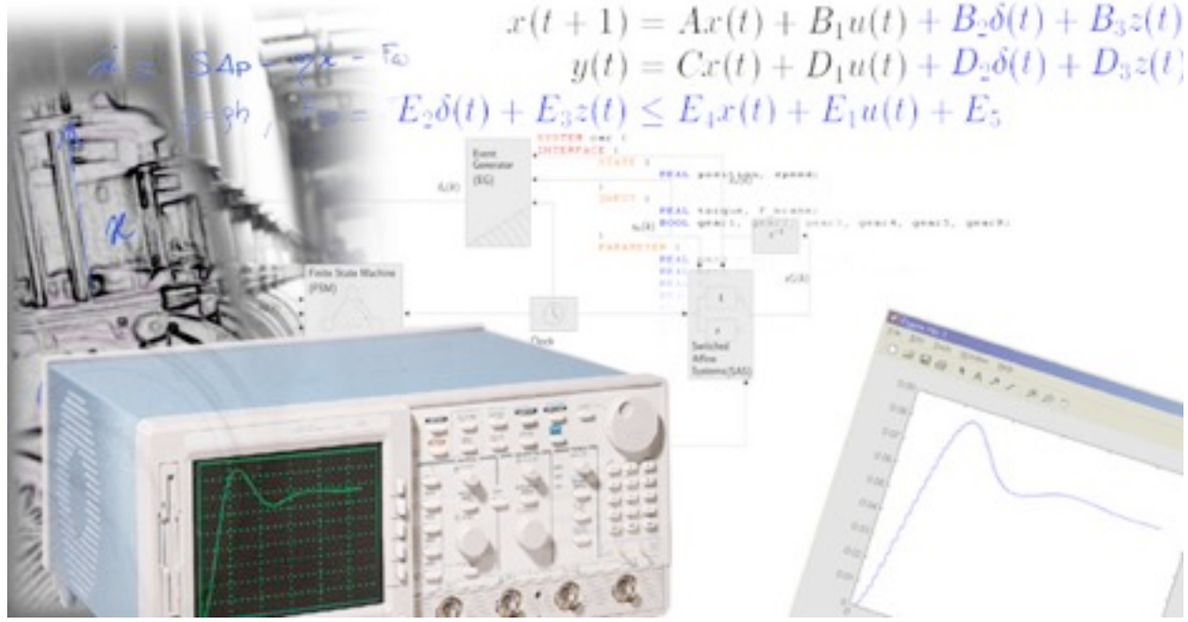
(**Note:** this is the nonlinear system $x(k+1) = 0.8|x(k)| + u(k)$)

HYSDEL

(HYbrid Systems DEscription Language)

- Describe *hybrid systems:*

  – Automata
  – Logic
  – Lin. Dynamics
  – Interfaces
  – Constraints



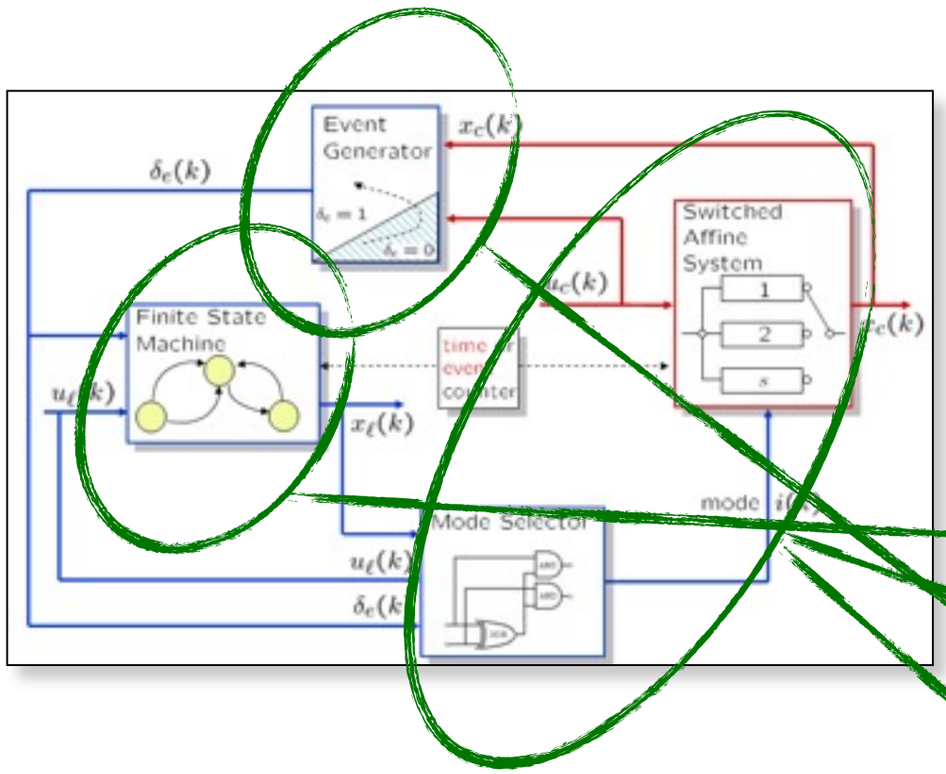
(Torrisi, Bemporad, 2004)

- Automatically generate MLD models in MATLAB

Download: **http://cse.lab.imtlucca.it/~bemporad/hybrid/toolbox**

Reference: **http://control.ethz.ch/~hybrid/hysdel**

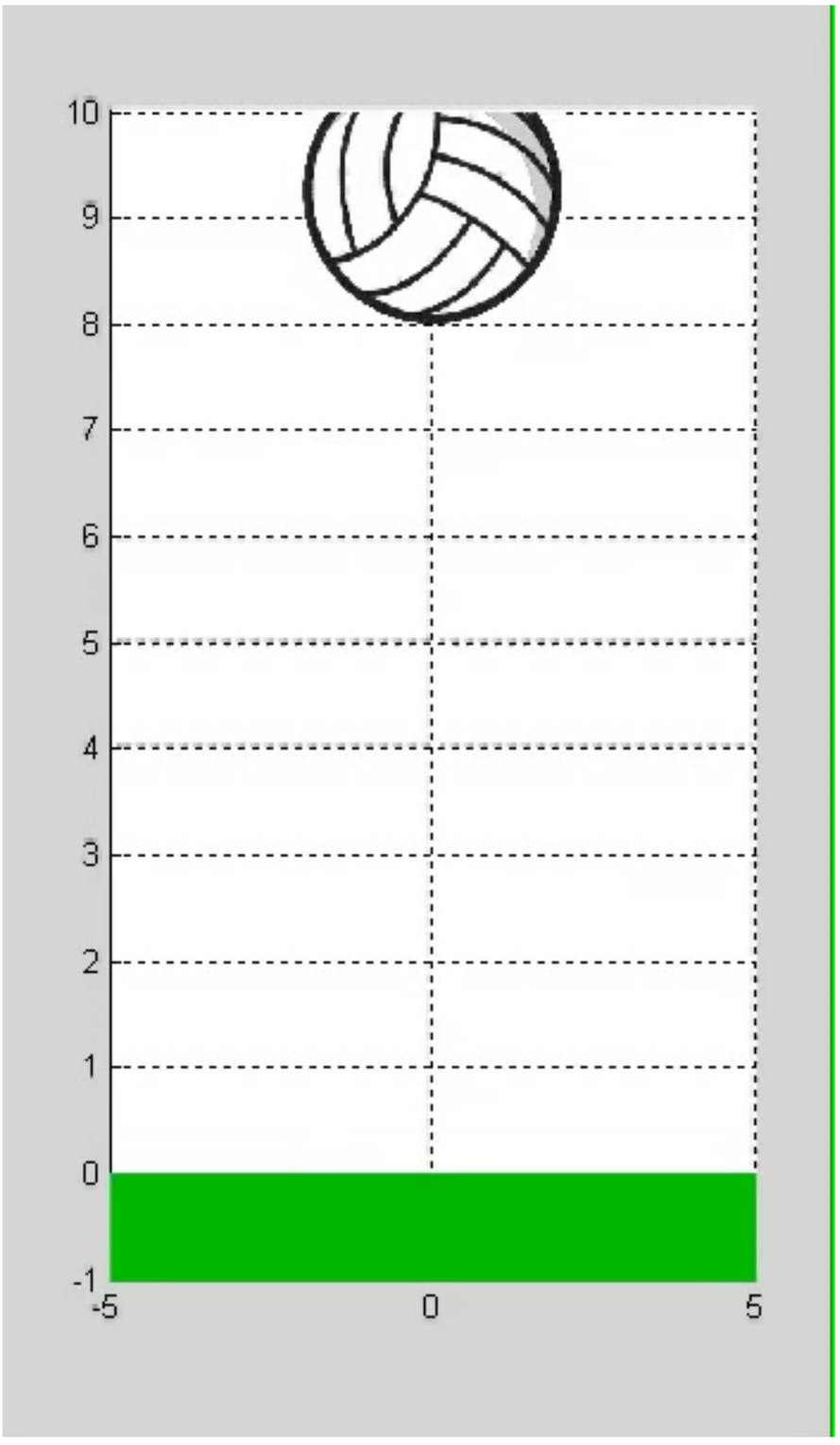
DHA AND HYSDEL MODELS




```
SYSTEM name {
    INTERFACE {
        STATE {
            REAL xc [xmin,xmax];
            BOOL xl; }
        INPUT {
            REAL uc [umin,umax];
            BOOL ul; }
        PARAMETER {
            REAL param1 = 1;}
    } /* end of interface */

    IMPLEMENTATION {
        AUX { BOOL d;
              REAL z; }

        AUTOMATA { xl = xl & ~ul; }

        DA { z = { IF d THEN 2*xc ELSE -xc };  }

        AD { d = xc - 1 <= 0; }

        CONTINUOUS {
            xc = z; }

        MUST {
            xc + uc <= 2;
            ~(xl & ul); }
    } /* end implementation */
} /* end system */
```

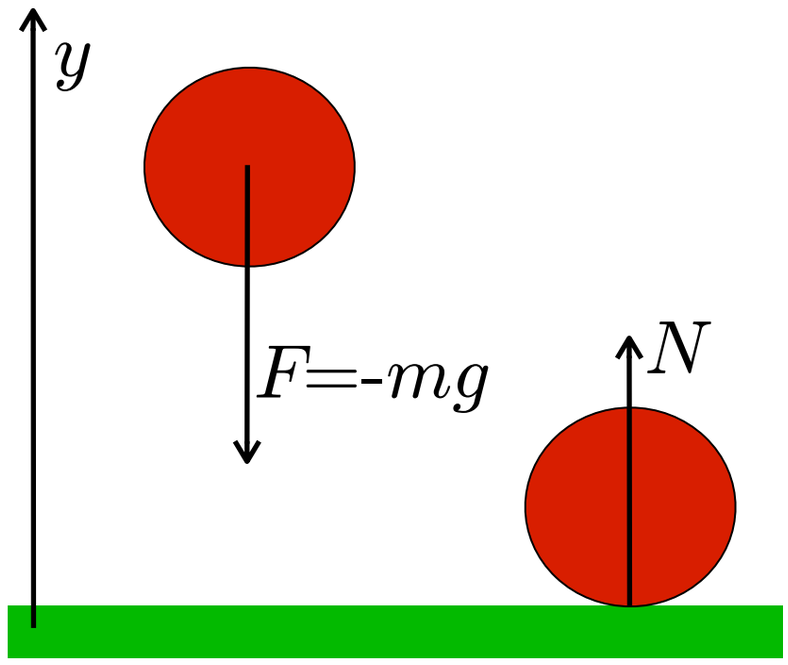
Additional relations constraining system's variables

BOUNCING BALL EXAMPLE



$$\ddot{y} = -g$$
$$y \le 0 \;\Rightarrow \dot{y}(t^+) = -(1-\alpha)\dot{y}(t^-)$$

$\alpha \in [0, 1]$

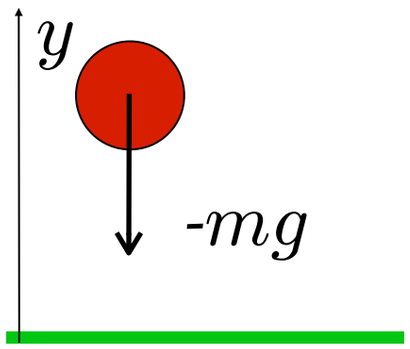


How to model the bouncing ball as a discrete-time hybrid system ?

BOUNCING BALL – TIME DISCRETIZATION

$\underline{y(t) > 0}$

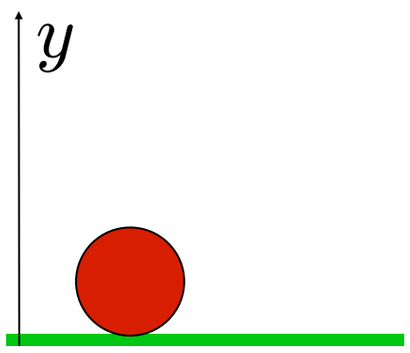
$$v(t) \approx \frac{y(t) - y(t-1)}{T_s} \qquad -g = \ddot{y}(t) \approx \frac{v(t) - v(t-1)}{T_s}$$



$$\Rightarrow \begin{cases} v(t+1) &= v(t) - T_s g \\ y(t+1) &= y(t) + T_s v(t+1) \\ &= y(t) + T_s v(t) - T_s^2 g \end{cases}$$

$\underline{y(t) \leq 0}$

$$v(t) = -(1-\alpha)v(t-1) \qquad \begin{aligned} y(t+1) &= y(t-1) \\ &= y(t) - T_s v(t) \end{aligned}$$



$$\Rightarrow \begin{cases} v(t+1) &= -(1-\alpha)v(t) \\ y(t+1) &= y(t) - T_s v(t) \end{cases}$$

go to demo **/demos/hybrid/bball.m**

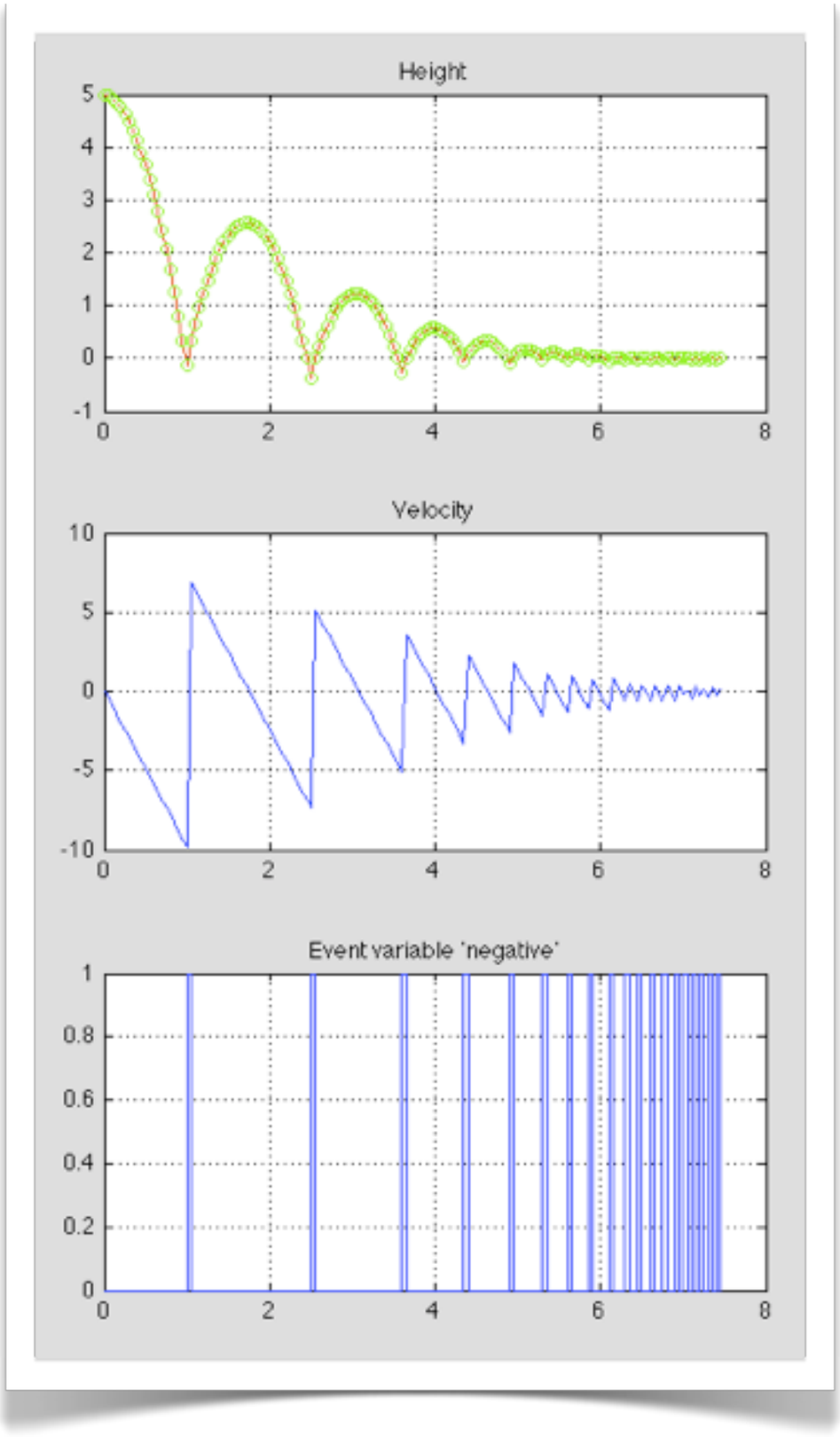
BOUNCING BALL - HYSDEL MODEL

```
SYSTEM bouncing_ball {
INTERFACE {
/* Description of variables and constants */
        STATE { REAL height [-10,10];
                REAL velocity [-100,100];    }


        PARAMETER {
                REAL g;
                REAL alpha;  /* 0=elastic, 1=completely anelastic */
                REAL Ts; }
}
IMPLEMENTATION {
        AUX {    BOOL negative;
                 REAL hnext;
                 REAL vnext;    }


        AD {     negative = height <= 0; }


        DA {     hnext = {  IF negative THEN height-Ts*velocity
                            ELSE height+Ts*velocity-Ts*Ts*g};
                 vnext = {  IF negative THEN -(1-alpha)*velocity
                            ELSE velocity-Ts*g};  }
        CONTINUOUS {
                 height   = hnext;
                 velocity = vnext;}
}}
```

go to demo **/demos/hybrid/bball.m**

BOUNCING BALL - SIMULATION

```
>>Ts=0.05;
>>g=9.8;
>>alpha=0.3;

>>S=mld('bouncing_ball',Ts);

>>N=150;
>>U=zeros(N,0);
>>x0=[5 0]';

>>[X,T,D]=sim(S,x0,U);
```

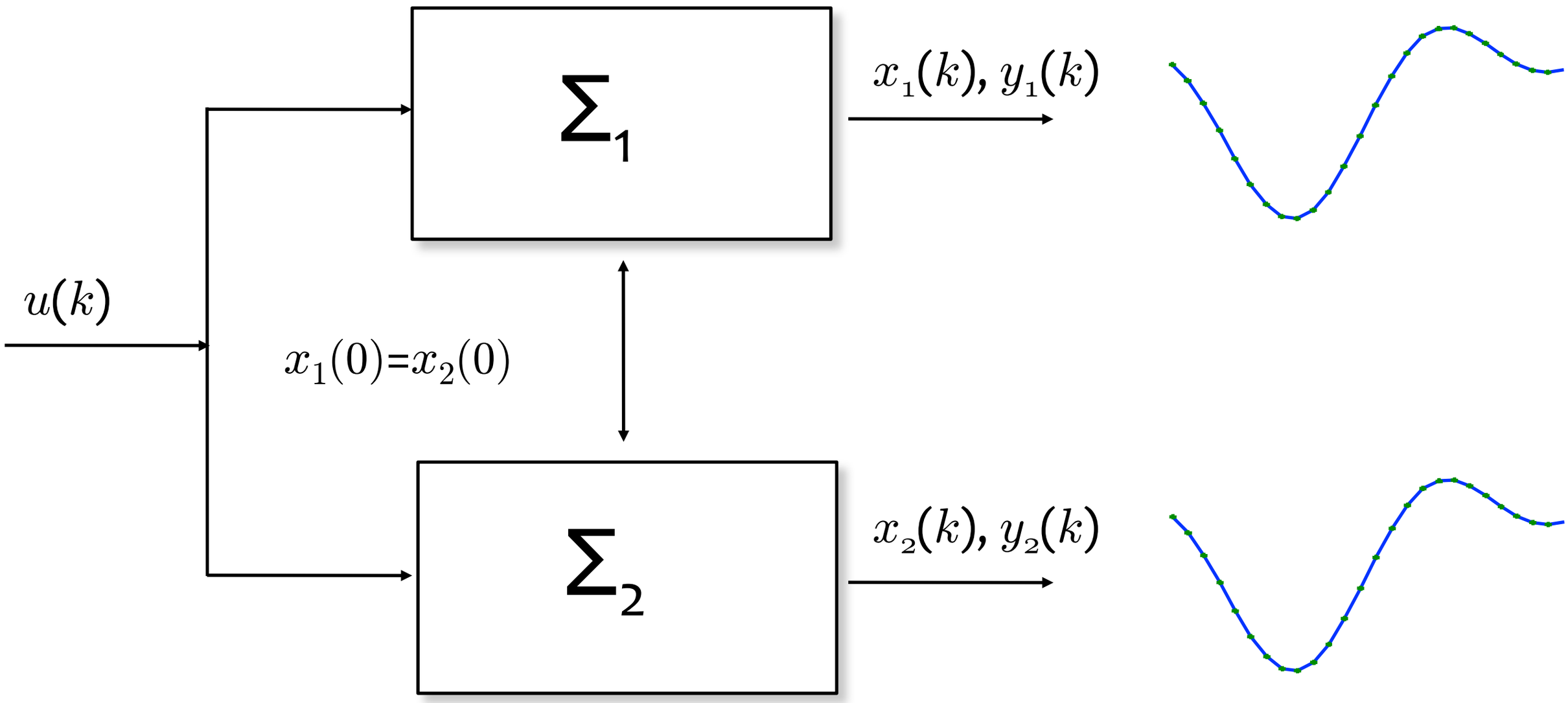


Note: no Zeno effect in discrete time !

REALIZATION AND TRANSFORMATIONS (STATE-SPACE HYBRID MODELS)

EQUIVALENCES OF HYBRID MODELS

Definition 1 *Two hybrid systems* Σ_1, Σ_2 *are* equivalent *if for all initial conditions* $x_1(0) = x_2(0)$ *and input* $\{u_1(k)\}_{k\in\mathbb{Z}_+} = \{u_2(k)\}_{k\in\mathbb{Z}_+}$ *then* $x_1(k) = x_2(k)$ *and* $y_1(k) = y_2(k)$, *for all* $k \in \mathbb{Z}_+$.



EQUIVALENCE OF HYBRID MODELS

- **MLD** systems and **PWA** systems are equivalent

(Bemporad, Ferrari-Trecate, Morari, *IEEE TAC, 2000*)

- Efficient algorithms for **converting** MLD models into PWA form exist

(Bemporad, *IEEE TAC, 2004*) (Geyer, Torrisi, Morari, HSCC, 2003)

- **Further equivalences** exist with other classes of hybrid dynamical systems, such as Linear Complementarity (**LC**) systems

(Heemels, De Schutter, Bemporad, *Automatica,* 2001)

MODELING A PWA SYSTEM IN MLD FORM

- Given the PWA system with bounded states and inputs

$$\begin{aligned} x(t+1) &= A_{i(t)}x(t) + B_{i(t)}u(t) + f_{i(t)} \\ y(t) &= C_{i(t)}x(t) + D_{i(t)}u(t) + g_{i(t)} \\ i(t) &= i \in \{1,\dots,s\} : \ H_i x(t) + J_i u(t) \le K_i \end{aligned}$$

$$C_i = \{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] : \ H_i x + J_i u \le K_i \}$$

$$\mathring{C}_i \cap \mathring{C}_j = \emptyset, \ \forall i \neq j$$ ← **polyhedral partition**

- Introduce s binary variables $\delta_i, \ i = 1,\dots,s$, subject to the constraints

$$[\delta_i = 1] \rightarrow [H_i x + J_i u \le K_i] \quad \longrightarrow \quad H_i x + J_i u \le K_i + M_i(1-\delta_i)$$

$$i = 1,\dots,s$$

$$\oplus_{i=1}^{s} [\delta_i = 1] = \text{TRUE} \quad \longrightarrow \quad \sum_{i=1}^{s} \delta_i = 1$$

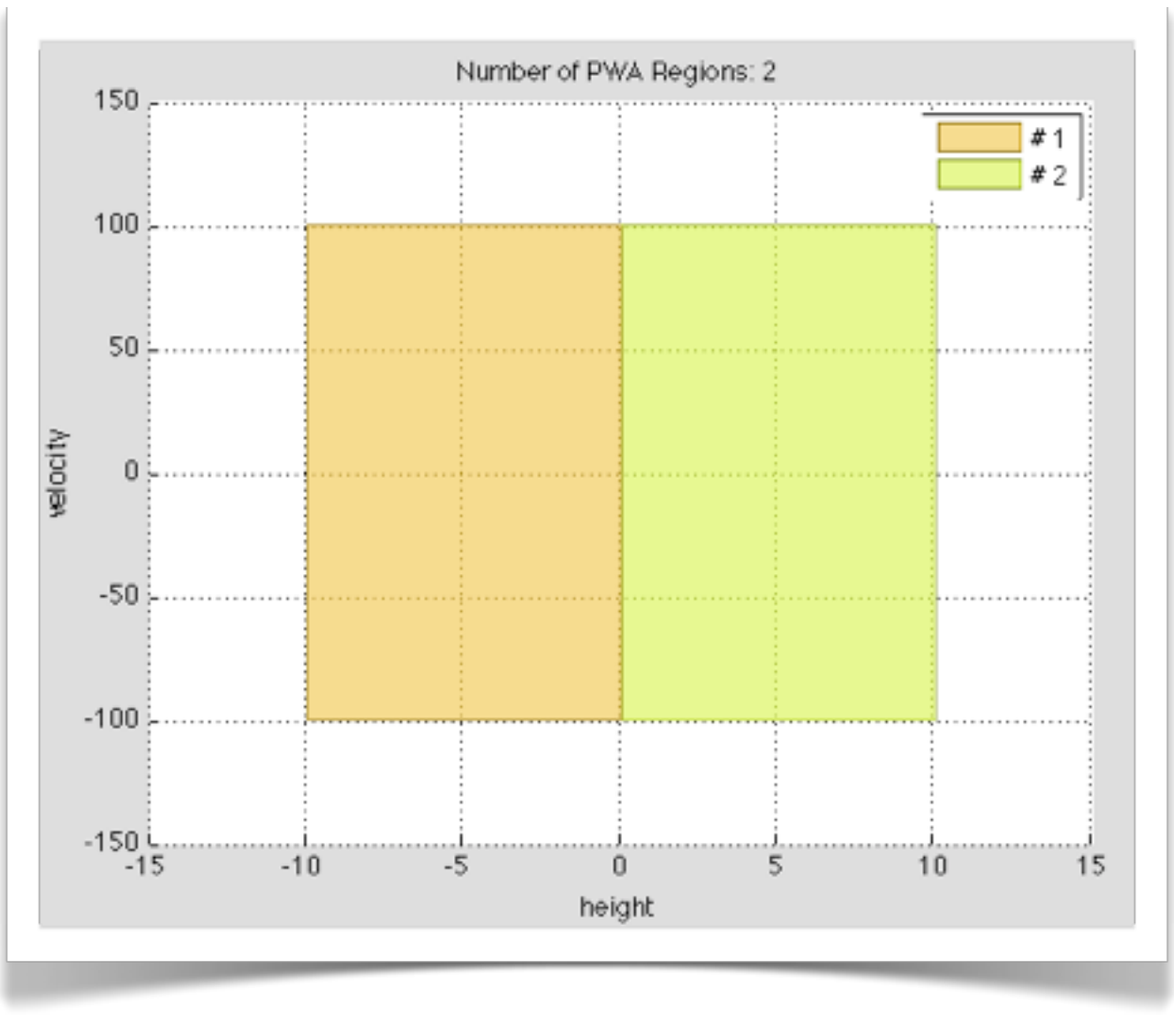
were the vector M_i of upper-bounds can be computed, e.g., via LP

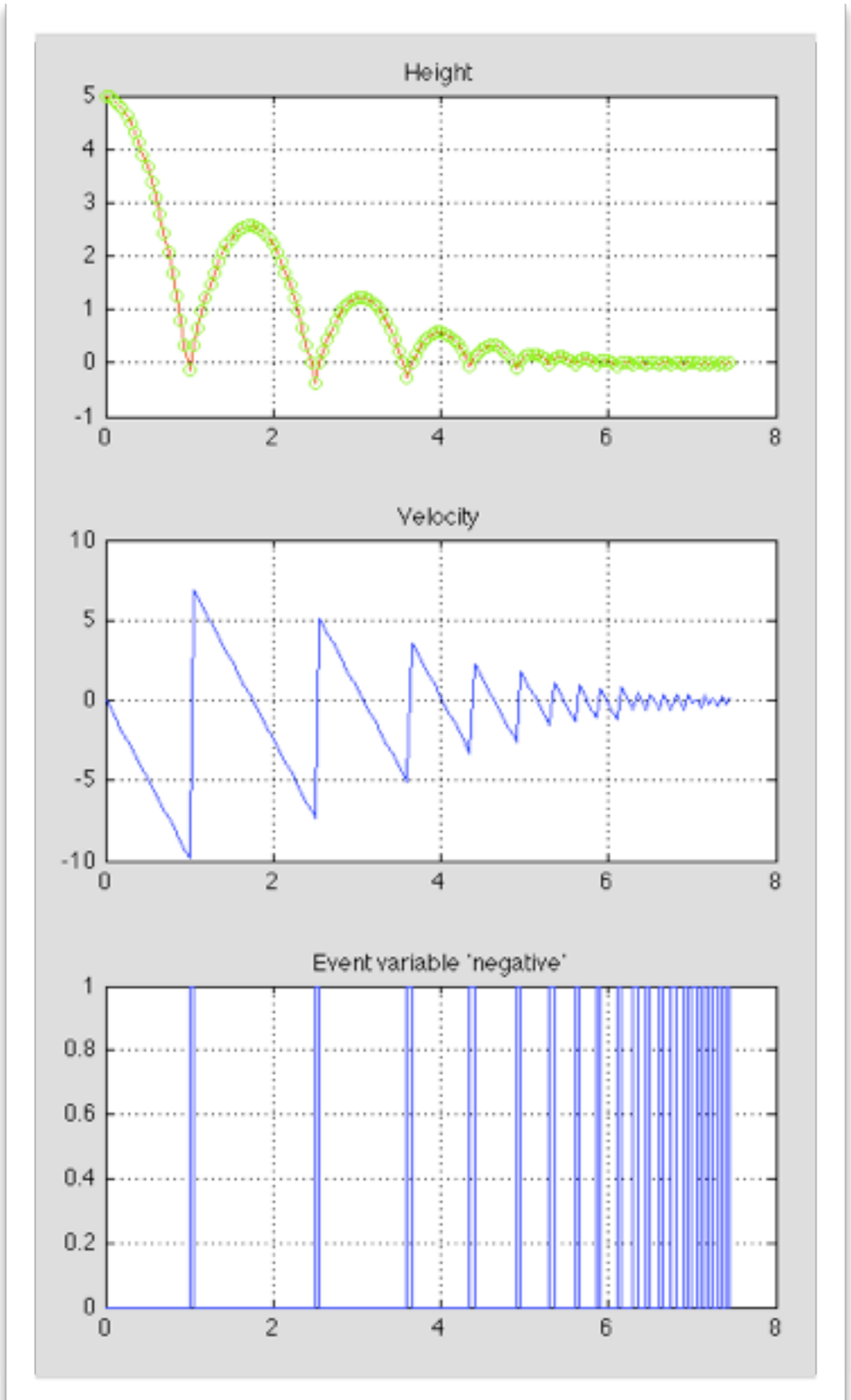
- Introduce also auxiliary continuous vectors z_i, w_i defined by if-then-else rules

$$\begin{aligned} z_i &= A_i x + B_i u + f_i \text{ if } \delta_i = 1, \ 0 \text{ otherwise} \\ w_i &= C_i x + D_i u + g_i \text{ if } \delta_i = 1, \ 0 \text{ otherwise} \end{aligned} \longrightarrow x(t+1) = \sum_{i=1}^{s} z_i(t), \ y(t) = \sum_{i=1}^{s} w_i(t)$$

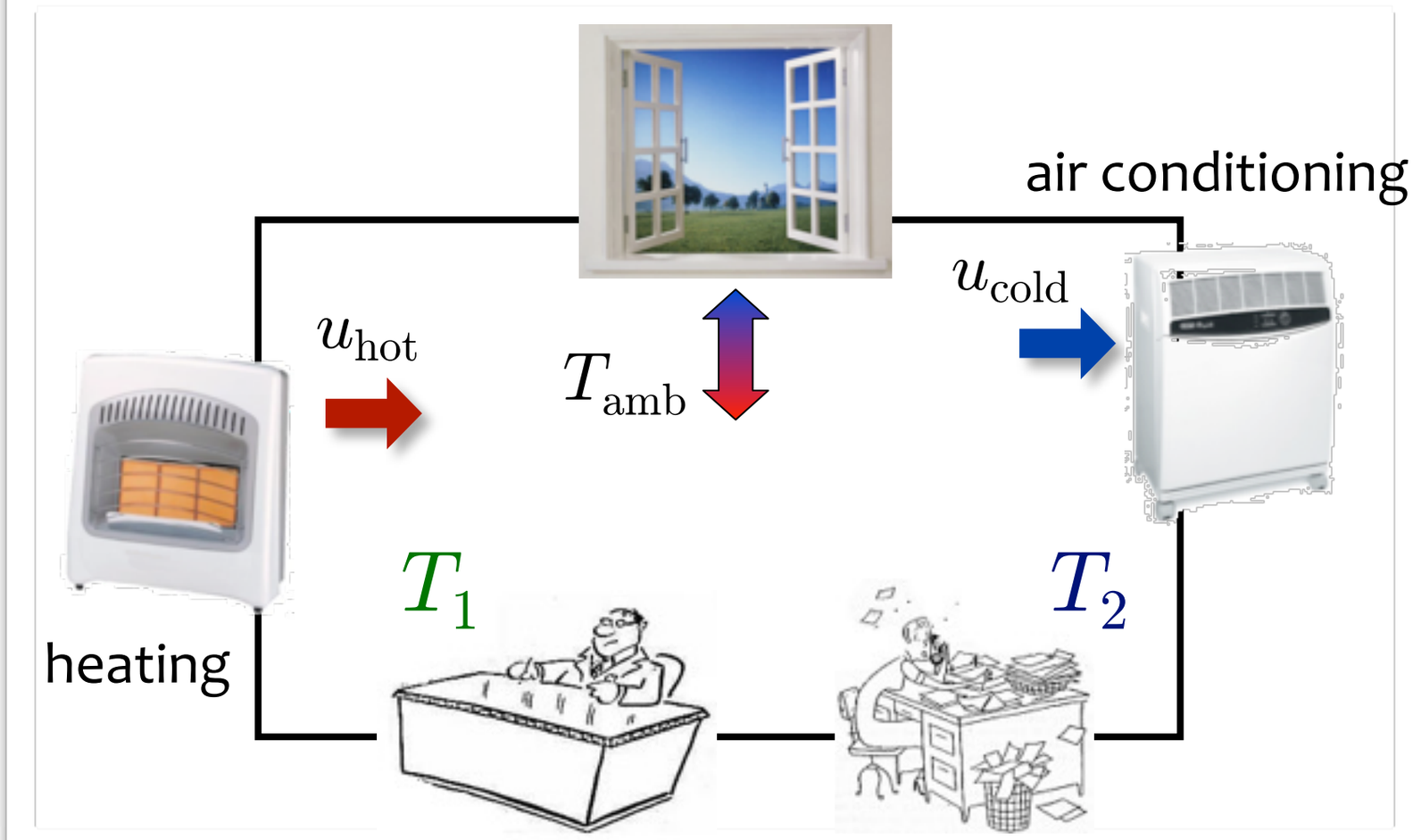
BOUNCING BALL - PWA EQUIVALENT

```
>>P=pwa(S);
>>plot(P)

>>[X,T,I]=sim(P,x0,U);
```





EXAMPLE: ROOM TEMPERATURE CONTROL



discrete dynamics

- #1=cold → heater=on
- #2=cold → heater=on **unless** #1=hot
- A/C activation has similar rules

continuous dynamics

$$\frac{dT_i}{dt} = -\alpha_i(T_i - T_{\text{amb}}) + k_i(u_{\text{hot}} - u_{\text{cold}})$$

$$i = 1, 2$$

go to demo **/demos/hybrid/heatcool.m**

HYSDEL MODEL

```
SYSTEM heatcool {

INTERFACE {
    STATE { REAL T1 [-10,50];
            REAL T2 [-10,50];
          }
    INPUT { REAL Tamb [-10,50];
          }
    PARAMETER {
        REAL Ts, alpha1, alpha2, k1, k2;
        REAL Thot1, Tcold1, Thot2, Tcold2, Uc, Uh;
        }
}
IMPLEMENTATION {
        AUX { REAL uhot, ucold;
              BOOL hot1, hot2, cold1, cold2;
        }
        AD  { hot1 = T1>=Thot1;
              hot2 = T2>=Thot2;
              cold1 = T1<=Tcold1;
              cold2 = T2<=Tcold2;
        }
        DA  { uhot = (IF cold1 | (cold2 & ~hot1) THEN Uh ELSE 0);
              ucold = (IF hot1 | (hot2 & ~cold1) THEN Uc ELSE 0);
        }
        CONTINUOUS { T1 = T1+Ts*(-alpha1*(T1-Tamb)+k1*(uhot-ucold));
                     T2 = T2+Ts*(-alpha2*(T2-Tamb)+k2*(uhot-ucold));
        }
    }
}
```

```
>>S=mld('heatcoolmodel',Ts)
```

get the MLD model in MATLAB

```
>>[XX,TT]=sim(S,x0,U);
```

simulate the MLD model

HYBRID MLD MODEL

- MLD model

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) \\ y(k) &= Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) \\ E_2 \delta(k) + E_3 z(k) &\leq E_1 u(k) + E_4 x(k) + E_5 \end{aligned}$$

- 2 continuous states: (temperatures T_1,T_2)
- 1 continuous input: (room temperature T_{amb})
- 2 auxiliary continuous vars: (power flows u_{hot}, u_{cold})
- 6 auxiliary binary vars: (4 thresholds + 2 for OR condition)
- 20 mixed-integer inequalities

Possible combination of integer variables: $2^6 = 64$

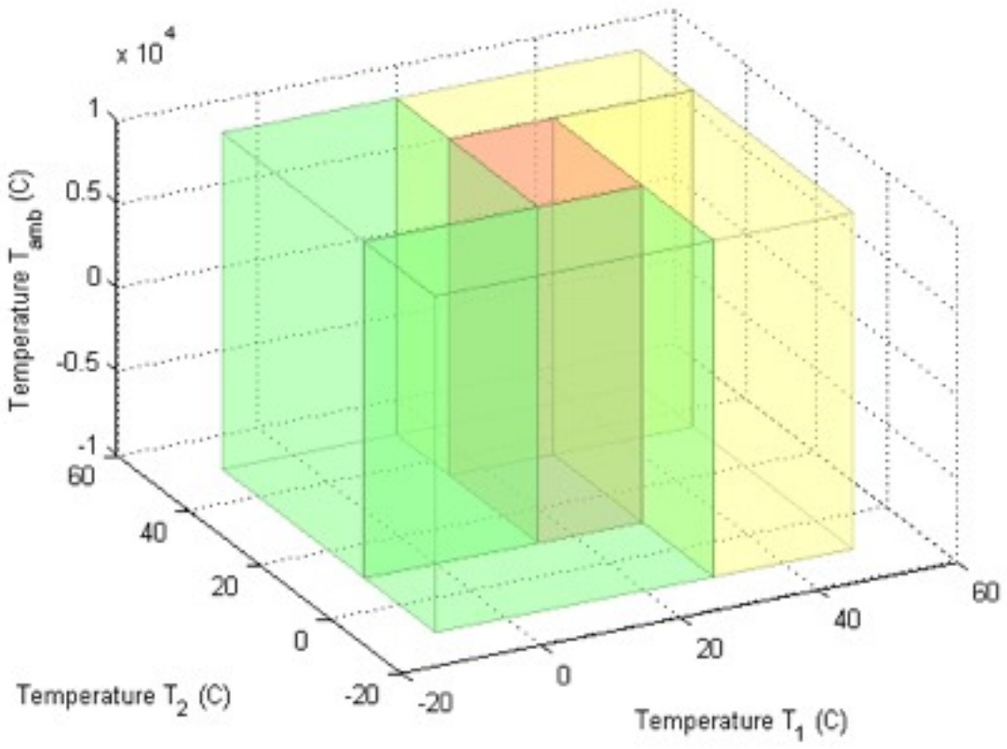
HYBRID PWA MODEL

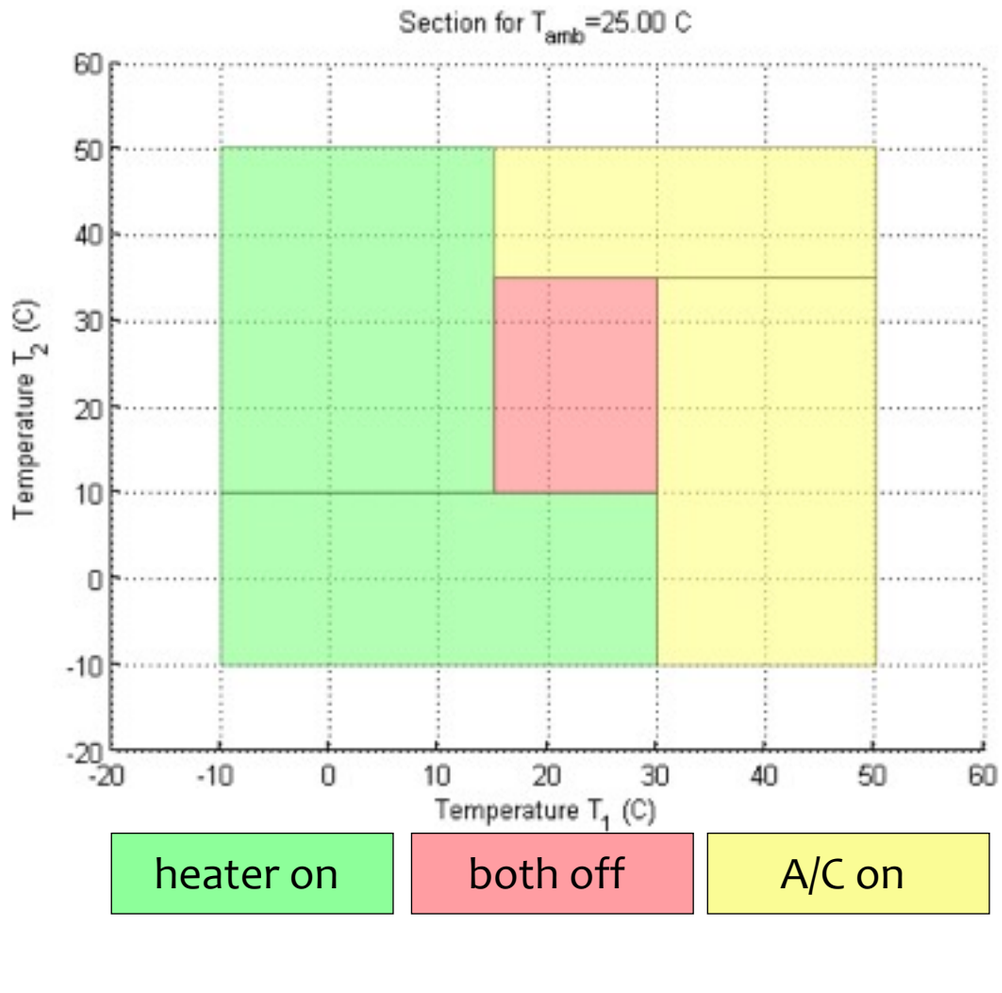
- PWA model

$$\begin{aligned} x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ \\ i(k) \quad &\text{s.t.} \quad H_{i(k)}x(k) + J_{i(k)}u(k) \le K_{i(k)} \end{aligned}$$

```
>>P=pwa(S);
```

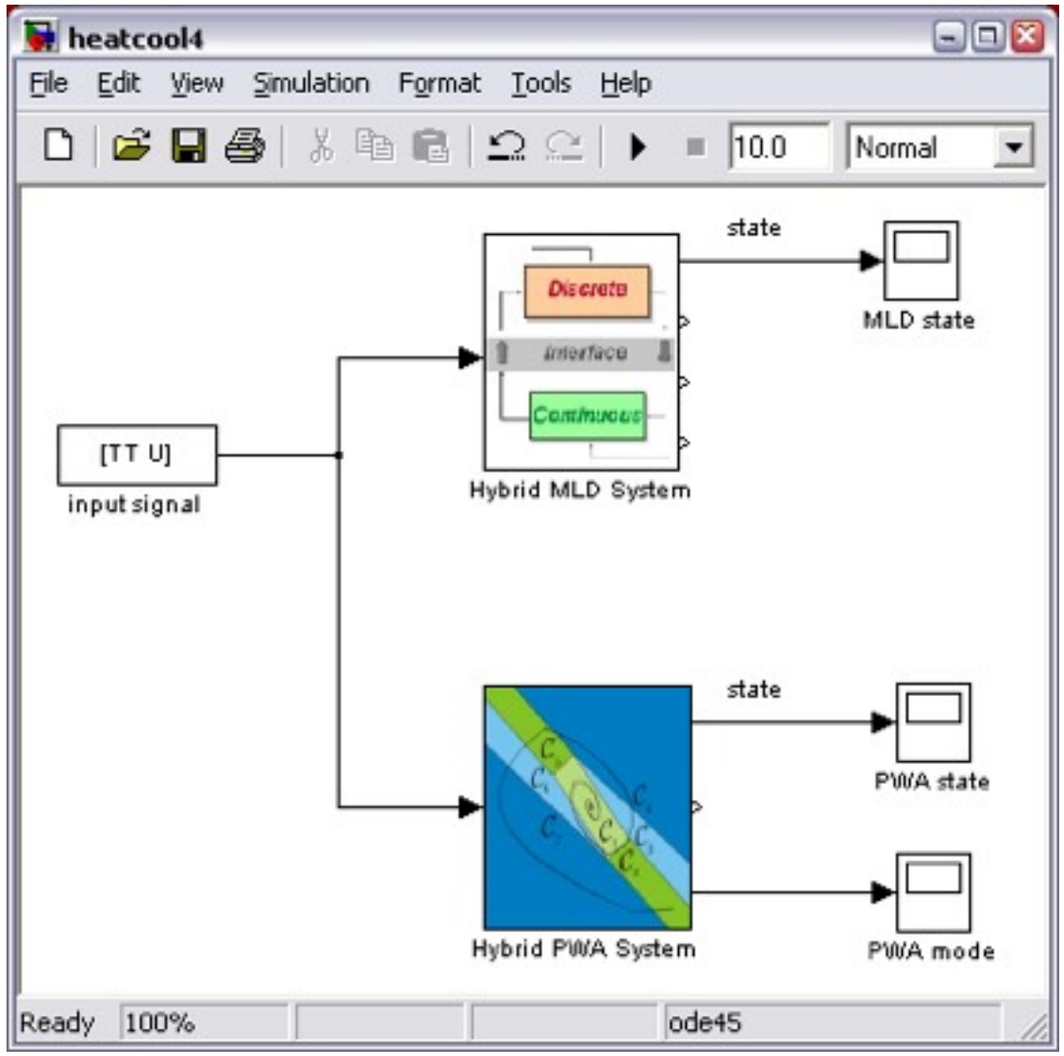
- 2 continuous states:
  (temperatures T_1,T_2)
- 1 continuous input:
  (room temperature T_{amb})

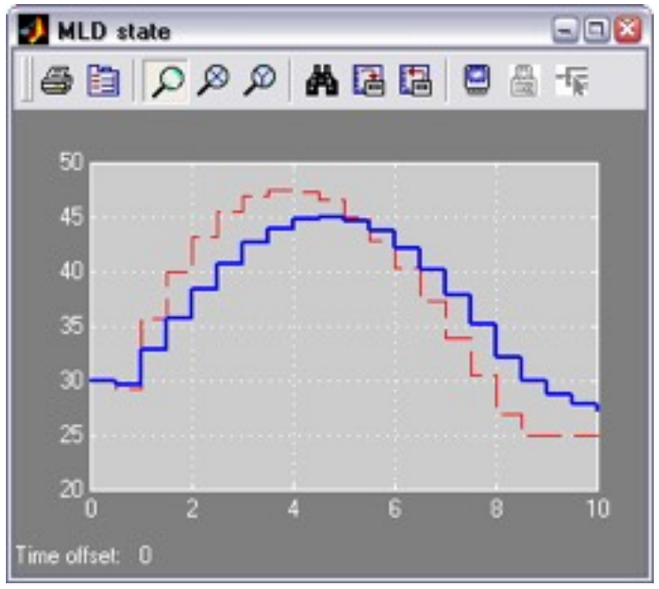


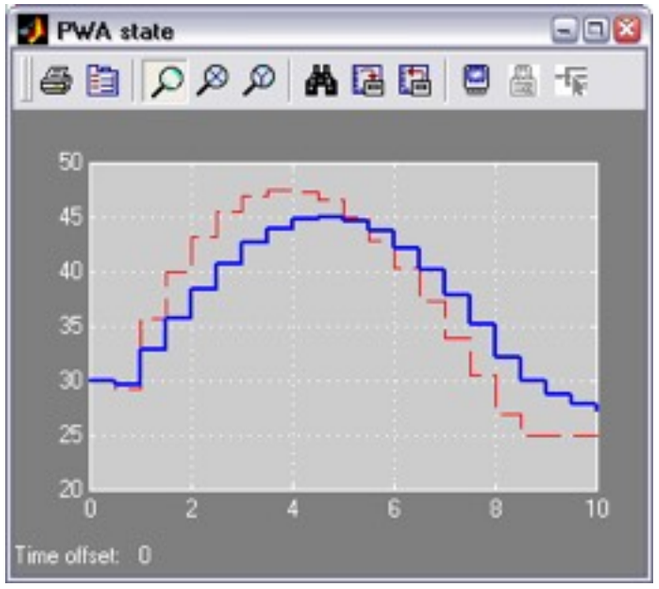


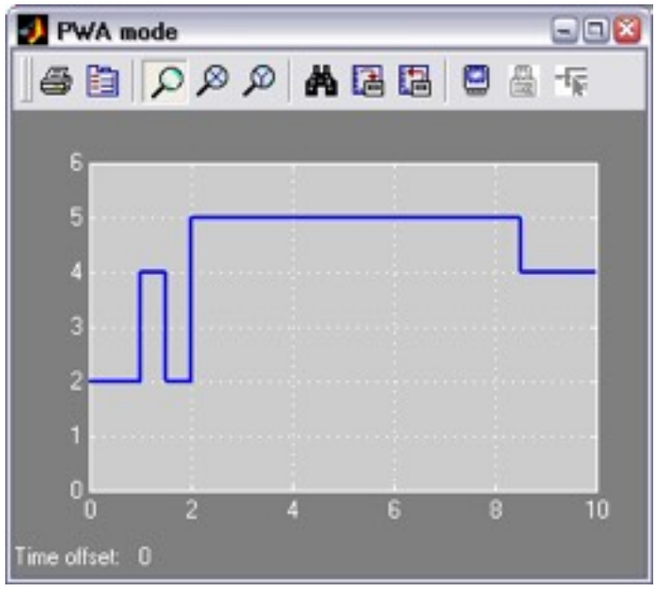
- 5 polyhedral regions
  (partition does not depend on input)

SIMULATION IN SIMULINK









MLD and PWA models are equivalent

USING MLD TO PWA FOR MODEL CHECKING

- Assume plant + controller can be modeled as DHA:

  - **Plant** = PWA approximation (e.g.: nonlinear switched model)

  - **Controller** = switched linear controller (e.g: a combination of threshold conditions, logic, linear feedback laws, ...)

- Write HYSDEL model, convert to MLD, then to PWA

- The resulting PWA map tells how the closed-loop system behaves in different regions of the state-space

IDENTIFICATION OF HYBRID SYSTEMS

HYBRID SYSTEM IDENTIFICATION

- A *hybrid model* of the process may not be available from physical principles

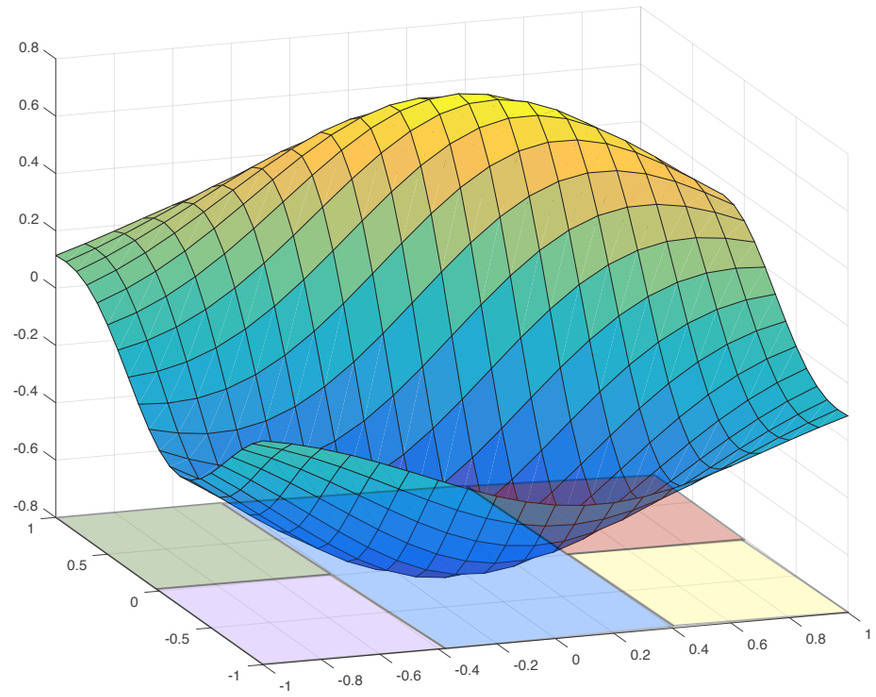
- Therefore, a model must be either
  - Estimated from data (model unknown)
  - or *hybridized* (model known but nonlinear)

- If one linear model is enough: easy problem (see Ljung's SYS-ID TBX)

- If switching sequence known: easy, just identify one linear model per mode

- If modes & dynamics must be identified simultaneously, we need **hybrid system identification**

In industrial MPC applications, most of the effort is spent in **identifying (multiple) linear** prediction models from data



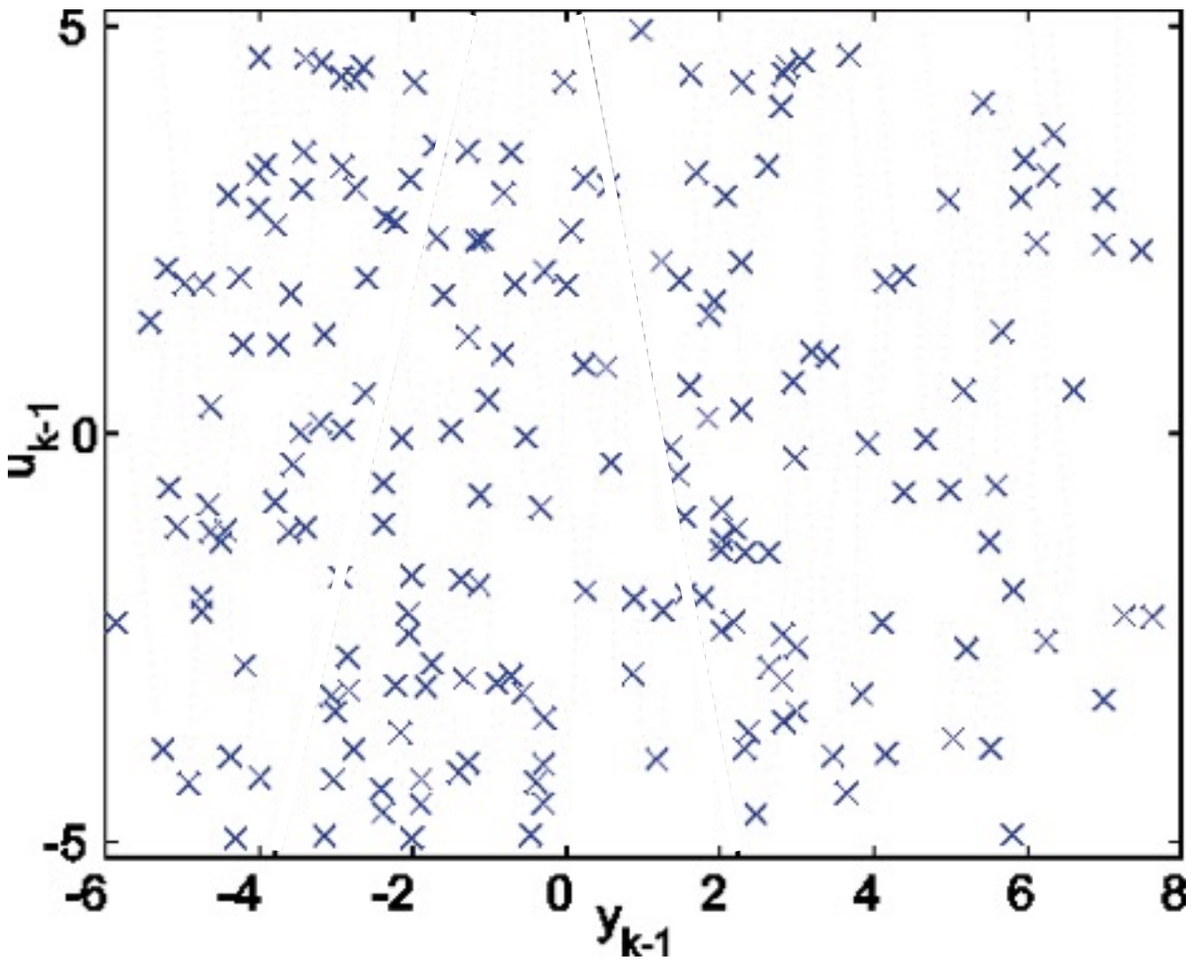
PWA IDENTIFICATION PROBLEM

Estimate from data **both** the **parameters** of the affine submodels and the **partition** of the PWA map

Example Let the data be generated by the PWARX system

$$y_k = \begin{cases} \begin{bmatrix} -0.4 & 1 & 1.5 \end{bmatrix} \varphi_k + \varepsilon_k \\ \quad \text{if } \begin{bmatrix} 4 & -1 & 10 \end{bmatrix} \varphi_k < 0 \\ \\ \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} \varphi_k + \varepsilon_k \\ \quad \text{if } \begin{bmatrix} -4 & 1 & -10 \\ 5 & 1 & -6 \end{bmatrix} \varphi_k \leq 0 \\ \\ \begin{bmatrix} -0.3 & 0.5 & -1.7 \end{bmatrix} \varphi_k + \varepsilon_k \\ \quad \text{if } \begin{bmatrix} -5 & -1 & 6 \end{bmatrix} \varphi_k < 0 \end{cases}$$

with $\varphi_k = [y_{k-1} \; u_{k-1} \; 1]'$, $|u_k| \leq 5$
and $|\varepsilon_k| \leq 0.1$



APPROACHES TO PWA IDENTIFICATION

- Mixed-integer linear or quadratic programming (Roll, Bemporad, Ljung, 2004)
- Partition of infeasible set of inequalities (Bemporad, Garulli, Paoletti, Vicino, 2005)
- K-means clustering in a feature space (Ferrari-Trecate, Muselli, Liberati, Morari, 2003)
- Bayesian approach (Juloski, Wieland, Heemels, 2004)
- Kernel-based approaches (Pillonetto, 2016)
- Hyperplane clustering in data space (Münz, Krebs, 2002)
- Recursive multiple least squares & PWL separation (Breschi, Piga, Bemporad, 2016)

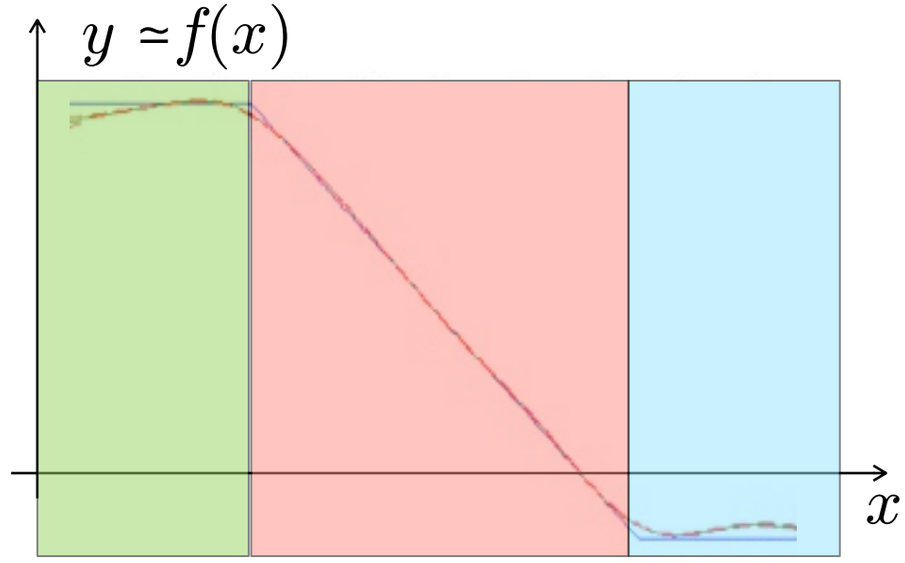
PWA REGRESSION PROBLEM

- **Problem:** given input/output pairs $\{x(k),y(k)\},\ k=1,\ldots,N$ and number s of models, compute an approximation $y \simeq f(x)$

$$f(x) = \begin{cases} F_1 x + g_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \\ F_s x + g_s & \text{if } H_s x \leq K_s \end{cases}$$

PWA model (PieceWise Affine)

- Need to learn **both** the parameters (F_i,g_i) of the affine submodels **and** the partition (H_i,K_i) of the PWA map from data (*off-line* learning)



- Possibly need to update model and partition as new data are collected (*on-line* learning)

PWA REGRESSION PROBLEM

- **Special case #1:** hybrid (PWARX) model

$$\begin{aligned} x(k) = \; & [y'(k-1) \;\; y'(k-2) \;\; \cdots \;\; y'(k-n_a) \\ & \;\; u'(k-1) \;\; u'(k-2) \;\; \cdots \;\; u'(k-n_b)]' \end{aligned}$$

$$y(k) = \; f(x(k))$$

f = piecewise affine function of regressor x

- **Special case #2:** switched parameter-dependent ARX model

$$y(k) = \sum_{i=1}^{n_a} a_i(p(k))y(k-i) + \sum_{j=0}^{n_b} b_j(p(k))u(k-j) + e(k)$$

(a_i, b_j) = piecewise affine functions of scheduling parameter p

PWA REGRESSION ALGORITHM (1/2)

(Breschi, Piga, Bemporad, 2016)

- Two-stage algorithm for PWA regression

- **Stage 1: Cluster** the regressors $x(k)$ and **simultaneously estimate** the parameter matrices (F_i,g_i) **recursively** (=one sample at the time)

$$i(k) \leftarrow \arg\min_{i=1,\dots,s} e_i(k)'\Lambda_e^{-1}e_i(k) + (x(k)-c_i)'R_i^{-1}(x(k)-c_i)$$

one-step prediction error of model #i

proximity to centroid of cluster #i

- Only update $(F_{i(k)},g_{i(k)})$ using recursive least squares based on inverse QR decomposition

(Alexander, Ghirnikar, 1993)

- Iterate the procedure M times for improved results

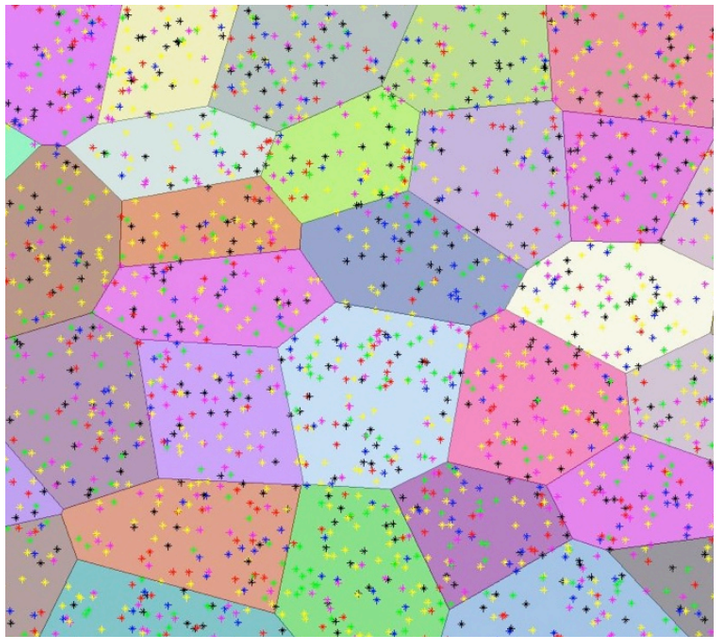
PWA REGRESSION ALGORITHM (2/2)

(Breschi, Piga, Bemporad, 2016)

- **Stage 2:** Compute a polyhedral partition (H_i,K_i) of the regressor space via multi-category linear separation (batch or incrementally):

$$\phi(x) = \max_{i=1,\dots,s} \{w_i'x - \gamma_i\}$$

$$X_i = \left\{x \in \mathbb{R}^n : \phi(x) = w_i'x - \gamma_i\right\}$$



- Alternative ways to compute (w_i,γ_i):

→ **Robust Linear Programming *(batch)*** (Bennett, Mangasarian, 1994)

→ **Piecewise-smooth Newton method *(batch)*** (Bemporad, Bernardini, Patrinos, 2015)

→ **Averaged stochastic gradient descent *(online)*** (Bottou, 2012)

PWA REGRESSION EXAMPLES

(Breschi, Piga, Bemporad, 2016)

- **Identification** of **piecewise-affine ARX** model

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -0.83 & 0.20 \\ 0.30 & -0.52 \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} + \begin{bmatrix} -0.34 & 0.45 \\ -0.30 & 0.24 \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix}$$
$$+ \begin{bmatrix} 0.20 \\ 0.15 \end{bmatrix} + \max \left\{ \begin{bmatrix} 0.20 & -0.90 \\ 0.10 & -0.42 \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 0.42 & 0.20 \\ 0.50 & 0.64 \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + \begin{bmatrix} 0.40 \\ 0.30 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} + e_{\mathrm{o}}(k),$$

Results: ***quality of fit***

		$N = 4000$	$N = 20000$	$N = 100000$
BFR_1	(Off-line) RLP [8]	96.0 %	96.5 %	99.0 %
	(Off-line) RPSN	96.2 %	96.4 %	98.9 %
	(On-line) ASGD	86.7 %	95.0 %	96.7 %
BFR_2	(Off-line) RLP [8]	96.2 %	96.9 %	99.0 %
	(Off-line) RPSN	96.3 %	96.8 %	99.0 %
	(On-line) ASGD	87.4 %	95.2 %	96.4 %

RLP = robust linear programming
RPSN = piecewise-smooth Newton
ASGD = (one-pass) averaged stochastic gradient

$$\mathrm{BFR}_i = \max \left\{ 1 - \frac{\|y_{\mathrm{o},i} - \hat{y}_i\|_2}{\|y_{\mathrm{o},i} - \bar{y}_{\mathrm{o},i}\|_2}, 0 \right\}$$

(Best Fit Rate)

CPU time for computing the partition

	$N = 4000$	$N = 20000$	$N = 100000$
(Off-line) RLP [8]	0.308 s	3.227 s	112.435 s
(Off-line) RPSN	0.016 s	0.086 s	0.365 s
(On-line) ASGD	0.013 s	0.023 s	0.067 s

PWA REGRESSION EXAMPLES

(Breschi, Piga, Bemporad, 2016)

- **Identification** of **piecewise-linear LPV-ARX** model

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} \bar{a}_{1,1}(p(k)) & \bar{a}_{1,2}(p(k)) \\ \bar{a}_{2,1}(p(k)) & \bar{a}_{2,2}(p(k)) \end{bmatrix} \begin{bmatrix} y_1(k-1) \\ y_2(k-1) \end{bmatrix} + \begin{bmatrix} \bar{b}_{1,1}(p(k)) & \bar{b}_{1,2}(p(k)) \\ \bar{b}_{2,1}(p(k)) & \bar{b}_{2,2}(p(k)) \end{bmatrix} \begin{bmatrix} u_1(k-1) \\ u_2(k-1) \end{bmatrix} + e_o(k)$$

$$\bar{a}_{1,1}(p(k)) = \begin{cases} -0.3 & \text{if} \quad 0.4\,(p_1(k)+p_2(k)) \le -0.3, \\ 0.3 & \text{if} \quad 0.4\,(p_1(k)+p_2(k)) \ge 0.3, \\ 0.4\,(p_1(k)+p_2(k)) & \text{otherwise}, \end{cases}$$

$$\bar{a}_{1,2}(p(k)) = 0.5\,(|p_1(k)|+|p_2(k)|)\,, \quad \bar{a}_{2,1}(p(k)) = p_1(k)-p_2(k)$$

$$\bar{a}_{2,2}(p(k)) = \begin{cases} 0.5 & \text{if} \quad p_1(k) < 0, \\ 0 & \text{if} \quad p_1(k) = 0, \\ -0.5 & \text{if} \quad p_1(k) > 0, \end{cases}$$

$$\bar{b}_{1,1}(p(k)) = 3p_1(k) + p_2(k),$$

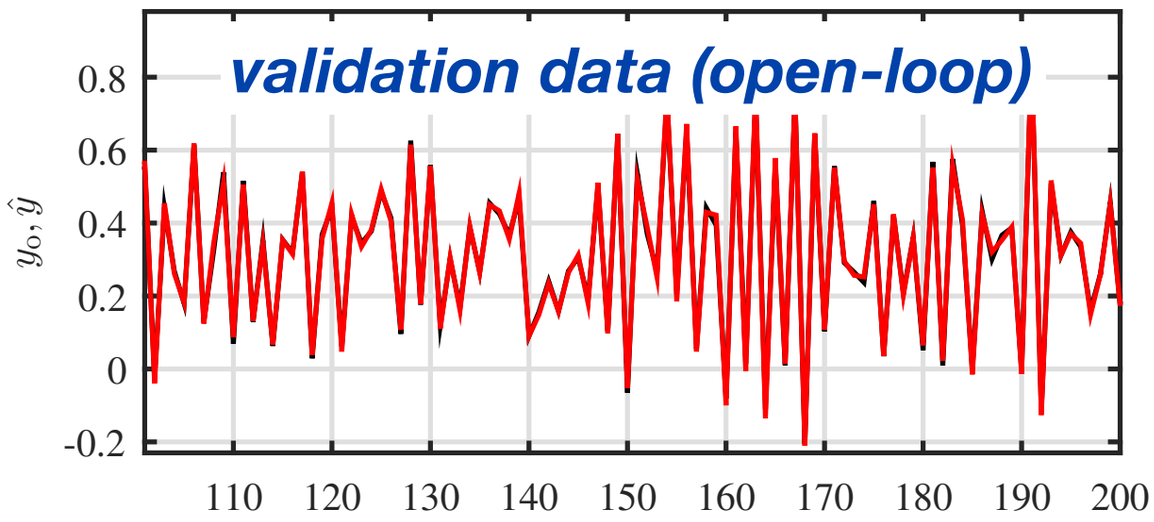
$$\bar{b}_{1,2}(p(k)) = \begin{cases} 0.5 & \text{if} \quad 2\left(p_1^2(k)+p_2^2(k)\right) \ge 0.5, \\ 2\left(p_1^2(k)+p_2^2(k)\right) & \text{otherwise}, \end{cases}$$

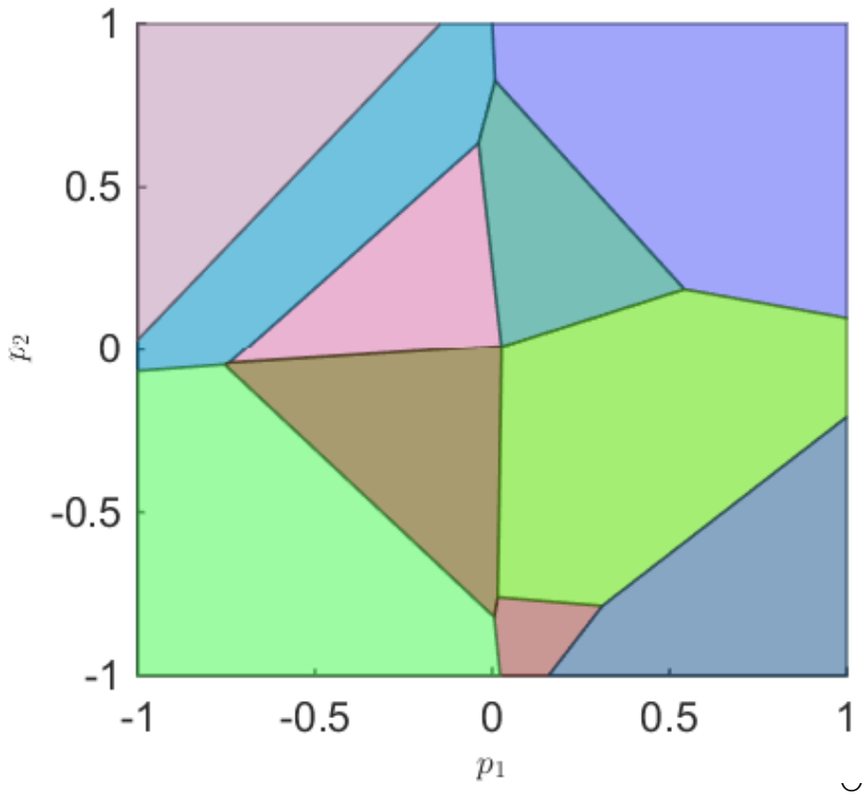
$$\bar{b}_{2,1}(p(k)) = 2\sin\{p_1(k)-p_2(k)\}\,, \quad \bar{b}_{2,2}(p(k)) = 0.$$

Results: ***quality of fit***

	BFR_1	BFR_2
PWA regression	87 %	84 %
parametric LPV [3]	80 %	70 %

[3] = Bamieh, Giarré (2002)

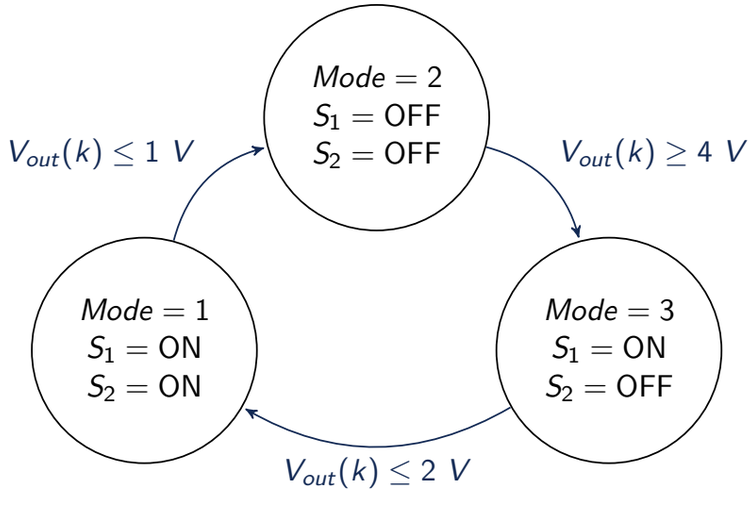


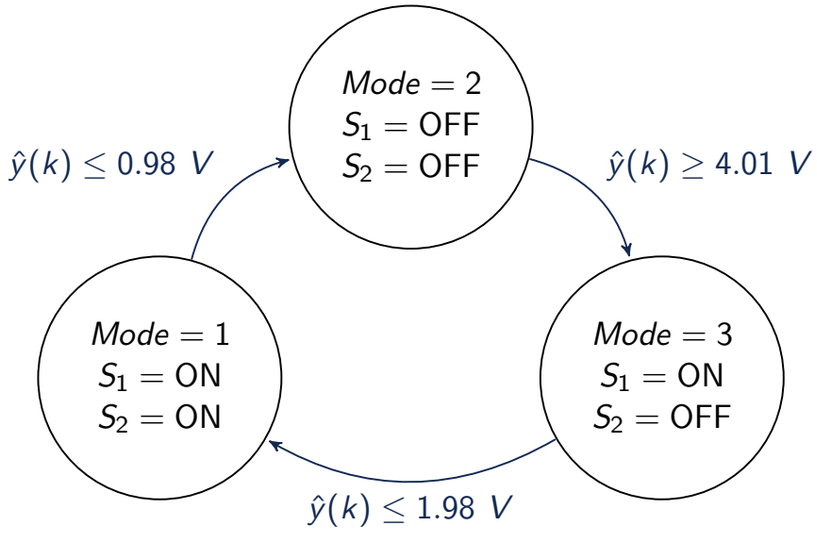


IDENTIFICATION OF HYBRID MODELS WITH LOGIC STATES

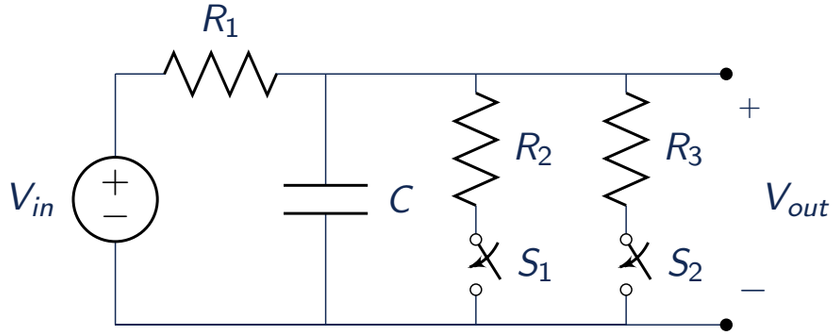
(Breschi, Piga, Bemporad, CDC 2016)

- **Identification** of **hybrid** models from data

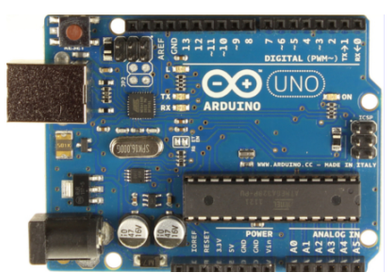




identified system



true system



CPU time to compute the partitions: **0.033 s**
CPU required to identify the DHA: **0.078 s***

Validation set of $N_V = 2000$ samples:
$\mathrm{BFR} = 98.64\ \%$