Process Control: Part II- Model Predictive Control (EE6225, AY2019/20, S1)

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BASIC THEORY OF THE MODEL PREDICTIVE CONTROL (MPC): PART II

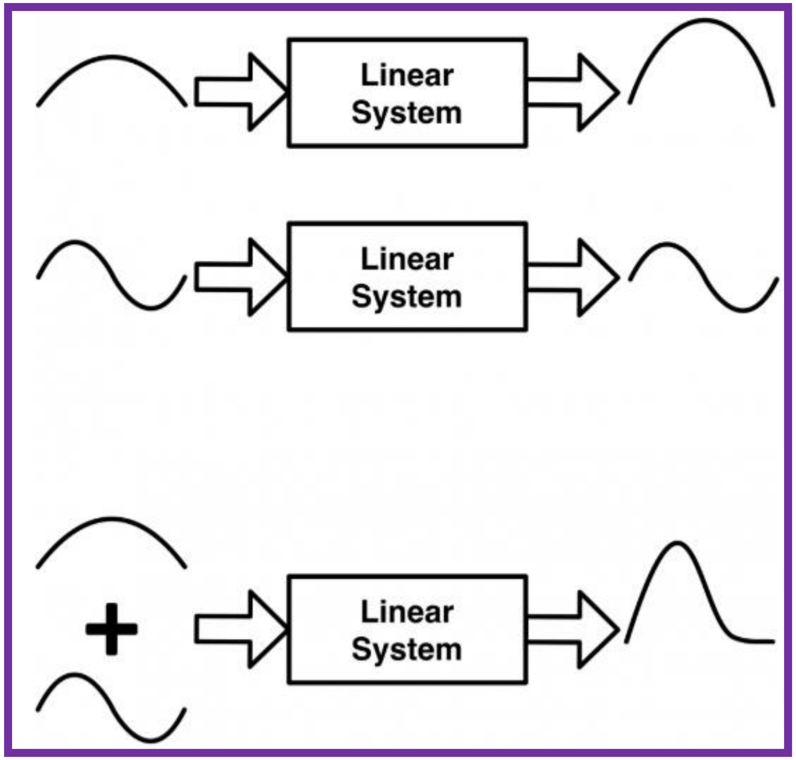
[24/10/2019]

- MPC with impulse/step response models
- MPC to ensure unbiased prediction
- Select performance indices for the MPC

Recall: MPC model

These slides do not discuss non-linear models

- Manipulation and algebra requires linear models as superposition can be used.
- Linear models are good enough for MPC.
- Typical linear models:
  - Transfer function
  - State-space
  - Step response models (subset of transfer functions).



BASIC THEORY OF THE MODEL PREDICTIVE CONTROL (MPC): PART II

[24/10/2019]

- MPC with impulse/step response models
- MPC to ensure unbiased prediction
- Select performance indices for the MPC

IMPULSE RESPONSE MODEL BASED MPC

- Transfer function model is also called as CARIMA model.

$$a(z)\Delta y_k = b(z)\Delta u_k + \cancel{T(z)\varsigma_k} \Rightarrow a(z)\Delta y_k = b(z)\Delta u_k$$

$$\Delta = 1 - Z^{-1}$$

$$a(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}$$

$$b(z) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}$$

$$\underbrace{\left(a(z)\Delta\right)}_{A(z)} y_k = b(z)\Delta u_k$$

$$A(z) = a(z)\Delta = 1 + A_1 z^{-1} + \cdots + A_n Z^{-n}$$

$$A(z) y_k = b(z)\Delta u_k$$

- Transfer function model is also called as CARIMA model.

$$\underbrace{\left(a(z)\Delta\right)}_{A(z)} y_k = b(z)\Delta u_k$$

$$a(z)\Delta y_k = b(z)\Delta u_k + \cancel{T(z)\varsigma_k} \Rightarrow a(z)\Delta y_k = b(z)\Delta u_k$$

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_A^{-1}C_b \cdot \Delta \underset{\rightarrow}{u}_k + \left(C_A^{-1}H_b \cdot \Delta \underset{\leftarrow}{u}_{k-1} - C_A^{-1}H_A \cdot \underset{\leftarrow}{y}_k\right)}$$

$$H = C_A^{-1}C_b \qquad P = C_A^{-1}H_b \qquad Q = C_A^{-1}H_A$$

$$\boxed{\underset{\rightarrow}{y}_{k+1} = H \cdot \Delta \underset{\rightarrow}{u}_k + \left(P \cdot \Delta \underset{\leftarrow}{u}_{k-1} - Q \cdot \underset{\leftarrow}{y}_k\right)}$$

Review: Parameters of C_A, C_b, H_b, H_A

$$\underbrace{\left(a(z)\Delta\right)}_{A(z)} y_k = b(z)\Delta u_k$$

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_A^{-1}C_b \cdot \Delta\underset{\rightarrow}{u}_k + \left(C_A^{-1}H_b \cdot \Delta\underset{\leftarrow}{u}_{k-1} - C_A^{-1}H_A \cdot \underset{\leftarrow}{y}_k\right)}$$

$$C_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_1 & 1 & 0 & 0 \\ A_2 & A_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$H_b = \begin{bmatrix} b_2 & b_3 & \cdots & b_{m-4} & b_{m-3} & b_{m-2} & b_{m-1} & b_m \\ b_3 & b_4 & \cdots & b_{m-3} & b_{m-2} & b_{m-1} & b_m & 0 \\ b_4 & b_5 & \cdots & b_{m-2} & b_{m-1} & b_m & 0 & 0 \\ b_5 & b_6 & \cdots & b_{m-1} & b_m & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

$$C_b = \begin{bmatrix} b_1 & 0 & 0 & \cdots \\ b_2 & b_1 & 0 & \cdots \\ b_3 & b_2 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$H_A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{n-4} & A_{n-3} & A_{n-2} & A_{n-1} & A_n \\ A_2 & A_3 & \cdots & A_{n-3} & A_{n-2} & A_{n-1} & A_n & 0 \\ A_3 & A_4 & \cdots & A_{n-2} & A_{n-1} & A_n & 0 & 0 \\ A_4 & A_5 & \cdots & A_{n-1} & A_n & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

Relationship: Carima & impulse response models

- Transfer function model (CARIMA model):

$$a(z)\Delta y_k = b(z)\Delta u_k + \cancel{T(z)\varsigma_k} \Rightarrow a(z)\Delta y_k = b(z)\Delta u_k$$

- Impulse response model:
  - $a(z)$ term is transferred to the right hand side :

$$\Delta y_k = \frac{b(z)}{a(z)}\Delta u_k = h(z)\Delta u_k$$

$$\boxed{h(z) = \frac{b(z)}{a(z)}}$$

$h(z)$ is the impulse response

Impulse response model based MPC

- CARIMA model: $a(z)\Delta y_k = b(z)\Delta u_k$
- Impulse response model: $1\cdot\Delta y_k = h(z)\Delta u_k$

Special

CARIMA model:

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_A^{-1}C_b\cdot\Delta\underset{\rightarrow}{u}_k + \left(C_A^{-1}H_b\cdot\Delta\underset{\leftarrow}{u}_{k-1} - C_A^{-1}H_A\cdot\underset{\leftarrow}{y}_k\right)}$$

Impulse response model:

$$\Downarrow \quad a(z)=1,\quad b(z)=h(z)$$

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_\Delta^{-1}C_h\cdot\Delta\underset{\rightarrow}{u}_k + \left(C_\Delta^{-1}H_h\cdot\Delta\underset{\leftarrow}{u}_{k-1} - C_\Delta^{-1}H_\Delta\cdot\underset{\leftarrow}{y}_k\right)}$$

$$\boxed{C_A} \rightarrow \boxed{C_\Delta} \qquad \boxed{C_b} \rightarrow \boxed{C_h} \qquad \boxed{H_b} \rightarrow \boxed{H_h} \qquad \boxed{H_A} \rightarrow \boxed{H_\Delta}$$

Derivation of C_Δ

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_\Delta^{-1} C_h \cdot \Delta \underset{\rightarrow}{u}_k + \left(C_\Delta^{-1} H_h \cdot \Delta \underset{\leftarrow}{u}_{k-1} - C_\Delta^{-1} H_\Delta \cdot \underset{\leftarrow}{y}_k \right)}$$

$$\boxed{C_A} \longrightarrow \boxed{C_\Delta}$$

$$C_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_1 & 1 & 0 & 0 \\ A_2 & A_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \xrightarrow[A_2 = A_3 = \ldots = 0]{A_1 = -1} C_\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\left.\begin{array}{l} A(\mathrm{z}) = a(z)\Delta \\ a(z) = 1 \\ \Delta = 1 - Z^{-1} \end{array}\right\} \Rightarrow A(\mathrm{z}) = 1 - 1 \cdot z^{-1} \Rightarrow A_1 = -1,\ A_2 = A_3 = \ldots = 0$$

Derivation of C_Δ & C_Δ^{-1}

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_\Delta^{-1} C_h \cdot \Delta \underset{\rightarrow}{u}_k + \left(C_\Delta^{-1} H_h \cdot \Delta \underset{\leftarrow}{u}_{k-1} - C_\Delta^{-1} H_\Delta \cdot \underset{\leftarrow}{y}_k \right)}$$

$$\boxed{C_A} \rightarrow \boxed{C_\Delta}$$

$$C_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_1 & 1 & 0 & 0 \\ A_2 & A_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$A_1 = -1$$

$$A_2 = A_3 = \ldots = 0$$

$$\boxed{C_\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}$$

$$\boxed{C_\Delta \cdot C_\Delta^{-1} = E}$$

$$\boxed{C_\Delta^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}$$

Derivation of C_h

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_\Delta^{-1} C_h \cdot \Delta \underset{\rightarrow}{u}_k + \left(C_\Delta^{-1} H_h \cdot \Delta \underset{\leftarrow}{u}_{k-1} - C_\Delta^{-1} H_\Delta \cdot \underset{\leftarrow}{y}_k \right)}$$

$$\boxed{C_b} \longrightarrow \boxed{C_h}$$

$$C_b = \begin{bmatrix} b_1 & 0 & 0 & \cdots \\ b_2 & b_1 & 0 & \cdots \\ b_3 & b_2 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \xrightarrow[\text{Replace } b \text{ by } h]{b(z) = h(z)} C_h = \begin{bmatrix} h_1 & 0 & 0 & \cdots \\ h_2 & h_1 & 0 & \cdots \\ h_3 & h_2 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Derivation of H_h

$$\boxed{\underset{\rightharpoondown}{y}_{k+1} = C_\Delta^{-1} C_h \cdot \Delta \underset{\rightharpoondown}{u}_k + \left(C_\Delta^{-1} H_h \cdot \Delta \underset{\leftharpoondown}{u}_{k-1} - C_\Delta^{-1} H_\Delta \cdot \underset{\leftharpoondown}{y}_k \right)}$$

$$\boxed{H_b} \longrightarrow \boxed{H_h}$$

$$H_b = \begin{bmatrix} b_2 & b_3 & \cdots & b_{m-4} & b_{m-3} & b_{m-2} & b_{m-1} & b_m \\ b_3 & b_4 & \cdots & b_{m-3} & b_{m-2} & b_{m-1} & b_m & 0 \\ b_4 & b_5 & \cdots & b_{m-2} & b_{m-1} & b_m & 0 & 0 \\ b_5 & b_6 & \cdots & b_{m-1} & b_m & 0 & 0 & 0 \end{bmatrix}$$

$b(z) = h(z)$ Replace b by h

$$H_h = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ h_4 & h_5 & h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Derivation of H_Δ

$$\boxed{\underset{\rightharpoondown}{y}_{k+1} = C_\Delta^{-1} C_h \cdot \Delta \underset{\rightharpoondown}{u}_k + \left(C_\Delta^{-1} H_h \cdot \Delta \underset{\leftharpoondown}{u}_{k-1} - C_\Delta^{-1} H_\Delta \cdot \underset{\leftharpoondown}{y}_k \right)}$$

$$\boxed{H_A} \longrightarrow \boxed{H_\Delta}$$

$$H_A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{n-4} & A_{n-3} & A_{n-2} & A_{n-1} & A_n \\ A_2 & A_3 & \cdots & A_{n-3} & A_{n-2} & A_{n-1} & A_n & 0 \\ A_3 & A_4 & \cdots & A_{n-2} & A_{n-1} & A_n & 0 & 0 \\ A_4 & A_5 & \cdots & A_{n-1} & A_n & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

$$A_1 = -1 \qquad A_2 = A_3 = \ldots = 0$$

$$H_\Delta = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Parameters of C_Δ^{-1}, C_h, H_h, H_Δ

$$\boxed{\underset{\rightarrow}{y}_{k+1} = C_\Delta^{-1} C_h \cdot \Delta \underset{\rightarrow}{u}_k + \left(C_\Delta^{-1} H_h \cdot \Delta \underset{\leftarrow}{u}_{k-1} - C_\Delta^{-1} H_\Delta \cdot \underset{\leftarrow}{y}_k \right)}$$

$$C_\Delta^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad C_h = \begin{bmatrix} h_1 & 0 & 0 & \cdots \\ h_2 & h_1 & 0 & \cdots \\ h_3 & h_2 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$H_h = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ h_4 & h_5 & h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad H_\Delta = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Impulse response model:

$$\Delta y_k = \frac{b(z)}{a(z)}\Delta u_k = h(z)\Delta u_k$$

$$h(z) = \frac{b(z)}{a(z)}$$

- Impulse response model based MPC:

$$\boxed{\underset{\rightarrow}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightarrow}{u}_k + \left(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftarrow}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftarrow}{y}_k \right)}$$

where:

$$\boxed{H = C_\Delta^{-1} C_h} \qquad \boxed{P = C_\Delta^{-1} H_h} \qquad \boxed{Q = -C_\Delta^{-1} H_\Delta}$$

$$H = C_\Delta^{-1} C_h$$

$$\underset{\rightharpoonup}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightharpoonup}{u}_k + \left(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftharpoonup}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftharpoonup}{y}_k \right)$$

$$\boxed{H = C_\Delta^{-1} C_h}$$

$$C_\Delta^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad C_h = \begin{bmatrix} h_1 & 0 & 0 & \cdots \\ h_2 & h_1 & 0 & \cdots \\ h_3 & h_2 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_1 + h_2 & h_1 & 0 & 0 \\ h_1 + h_2 + h_3 & h_1 + h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{C_{\Delta}^{-1} C_h}_{H} \cdot \Delta \underset{\rightarrow}{u}_k + \left(\underbrace{C_{\Delta}^{-1} H_h}_{P} \cdot \Delta \underset{\leftarrow}{u}_{k-1} \underbrace{- C_{\Delta}^{-1} H_{\Delta}}_{Q} \cdot \underset{\leftarrow}{y}_k \right)$$

$$\boxed{P = C_{\Delta}^{-1} H_h}$$

$$C_{\Delta}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad H_h = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ h_4 & h_5 & h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$P = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_2 + h_3 & h_3 + h_4 & h_4 + h_5 & \cdots \\ h_2 + h_3 + h_4 & h_3 + h_4 + h_5 & h_4 + h_5 + h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\boxed{\underset{\rightharpoonup}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightharpoonup}{u}_k + \left(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftharpoonup}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftharpoonup}{y}_k \right)}$$

$$\boxed{Q = -C_\Delta^{-1} H_\Delta}$$

$$C_\Delta^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad H_\Delta = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\boxed{Q = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = L}$$

A vector of ones!

Summary: impulse response model based MPC

- Impulse response model:

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{C_{\Delta}^{-1} C_h}_{H} \cdot \Delta \underset{\rightarrow}{u}_k + \left(\underbrace{C_{\Delta}^{-1} H_h}_{P} \cdot \Delta \underset{\leftarrow}{u}_{k-1} \underbrace{- C_{\Delta}^{-1} H_{\Delta}}_{Q} \cdot \underset{\leftarrow}{y}_k \right)$$

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_1 + h_2 & h_1 & 0 & 0 \\ h_1 + h_2 + h_3 & h_1 + h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = L$$

$$P = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_2 + h_3 & h_3 + h_4 & h_4 + h_5 & \cdots \\ h_2 + h_3 + h_4 & h_3 + h_4 + h_5 & h_4 + h_5 + h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

STEP RESPONSE MODEL BASED MPC

Step response & impulse response

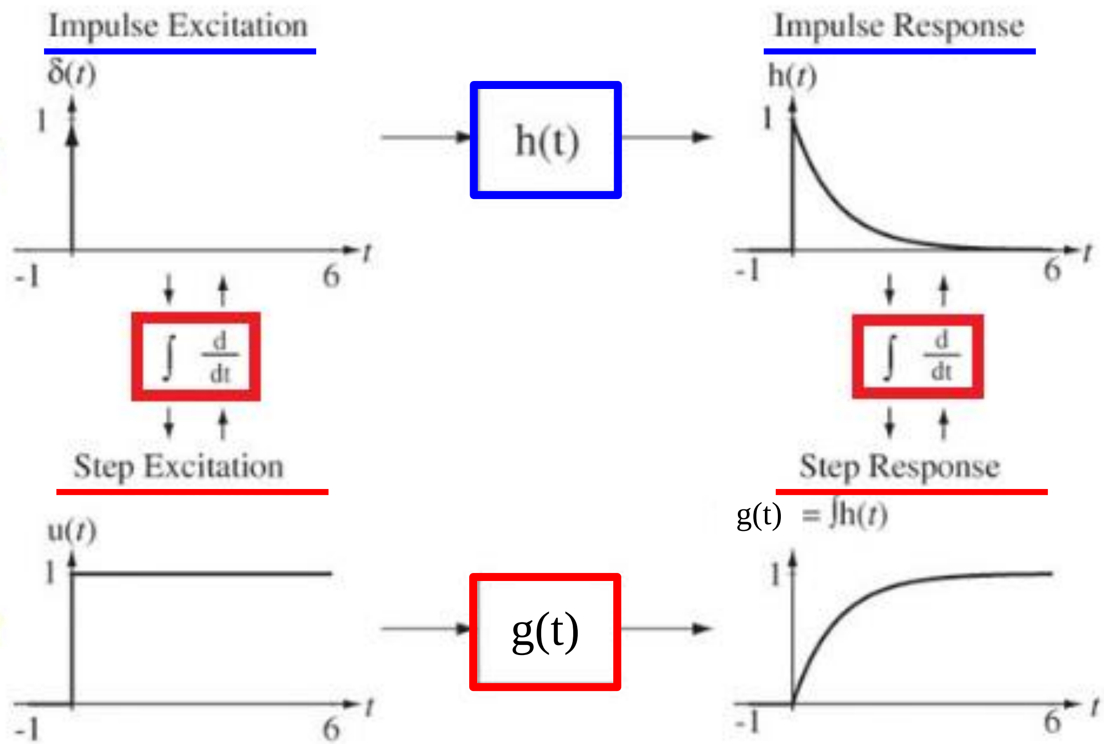
- Relationship: step response & impulse response:

Unit impulse

$$\Delta y_k = h(z)\Delta u_k$$

$$\Delta y_k = g(z)\Delta u_k$$

Unit step



$$\boxed{g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots = h_0 + \left[h_0 + h_1\right] z^{-1} + \left[h_0 + h_1 + h_2\right] z^{-2} + \cdots}$$

Impulse ⇦ $h(z)$ — Integration → $g(z)$ ⇨ Step

Step response based MPC

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightarrow}{u}_k + \left(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftarrow}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftarrow}{y}_k \right)$$

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_1 + h_2 & h_1 & 0 & 0 \\ h_1 + h_2 + h_3 & h_1 + h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = L$$

$$P = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_2 + h_3 & h_3 + h_4 & h_4 + h_5 & \cdots \\ h_2 + h_3 + h_4 & h_3 + h_4 + h_5 & h_4 + h_5 + h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$g_0 = h_0$

$g_1 = h_0 + h_1$

$g_2 = h_0 + h_1 + h_2$

$\vdots$

Parameters of the step response based MPC: H

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightarrow}{u}_k + \Big(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftarrow}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftarrow}{y}_k \Big)$$

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_1 + h_2 & h_1 & 0 & 0 \\ h_1 + h_2 + h_3 & h_1 + h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$g_0 = h_0$

$g_1 = h_0 + h_1$

$g_2 = h_0 + h_1 + h_2$

$\vdots$

$$H = \begin{bmatrix} g_1 - g_0 & 0 & 0 & 0 \\ g_2 - g_0 & g_1 - g_0 & 0 & 0 \\ g_3 - g_0 & g_2 - g_0 & g_1 - g_0 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

if $g_0 = h_0 = 0$

$$H = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ g_3 & g_2 & g_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Parameters of the step response based MPC: P

$$\underset{\rightharpoonup}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightharpoonup}{u}_k + \left(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftharpoonup}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftharpoonup}{y}_k \right)$$

$$P = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_2 + h_3 & h_3 + h_4 & h_4 + h_5 & \cdots \\ h_2 + h_3 + h_4 & h_3 + h_4 + h_5 & h_4 + h_5 + h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$g_0 = h_0$$

$$g_1 = h_0 + h_1$$

$$g_2 = h_0 + h_1 + h_2$$

$$\vdots$$

$$P = \begin{bmatrix} g_2 - g_1 & g_3 - g_2 & g_4 - g_3 & \cdots \\ g_3 - g_1 & g_4 - g_2 & g_5 - g_3 & \cdots \\ g_4 - g_1 & g_5 - g_2 & g_6 - g_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Parameters of the step response based MPC: Q

$$\boxed{\underset{\rightarrow}{y}_{k+1} = \underbrace{C_\Delta^{-1} C_h}_{H} \cdot \Delta \underset{\rightarrow}{u}_k + \left(\underbrace{C_\Delta^{-1} H_h}_{P} \cdot \Delta \underset{\leftarrow}{u}_{k-1} \underbrace{- C_\Delta^{-1} H_\Delta}_{Q} \cdot \underset{\leftarrow}{y}_k \right)}$$

$$\boxed{Q = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = L}$$

Summary: step response model based MPC

- Step response model based MPC:

$$\underset{\rightarrow}{y}_{k+1} = H \cdot \Delta \underset{\rightarrow}{u}_k + P \cdot \Delta \underset{\leftarrow}{u}_{k-1} + L \cdot \underset{\leftarrow}{y}_k$$

lower triangular of the step response parameters

Difference of the step response coefficients.

L is just a vector of ones.

$$H = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ g_3 & g_2 & g_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$P = \begin{bmatrix} g_2 - g_1 & g_3 - g_2 & g_4 - g_3 & \cdots \\ g_3 - g_1 & g_4 - g_2 & g_5 - g_3 & \cdots \\ g_4 - g_1 & g_5 - g_2 & g_6 - g_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

impulse/step response model based MPC

- Predictions with the impulse/step response coefficients.

$$\underset{\rightarrow}{y}_{k+1} = H \cdot \Delta \underset{\rightarrow}{u}_k + P \cdot \Delta \underset{\leftarrow}{u}_{k-1} + L \cdot \underset{\leftarrow}{y}_k$$

Impulse response:

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_1 + h_2 & h_1 & 0 & 0 \\ h_1 + h_2 + h_3 & h_1 + h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$P = \begin{bmatrix} h_2 & h_3 & h_4 & \cdots \\ h_2 + h_3 & h_3 + h_4 & h_4 + h_5 & \cdots \\ h_2 + h_3 + h_4 & h_3 + h_4 + h_5 & h_4 + h_5 + h_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Step response:

$$H = \begin{bmatrix} g_1 - g_0 & 0 & 0 & 0 \\ g_2 - g_0 & g_1 - g_0 & 0 & 0 \\ g_3 - g_0 & g_2 - g_0 & g_1 - g_0 & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$P = \begin{bmatrix} g_1 - g_0 & g_2 - g_1 & g_3 - g_2 & \cdots \\ g_2 - g_0 & g_3 - g_1 & g_4 - g_2 & \cdots \\ g_3 - g_0 & g_4 - g_1 & g_5 - g_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

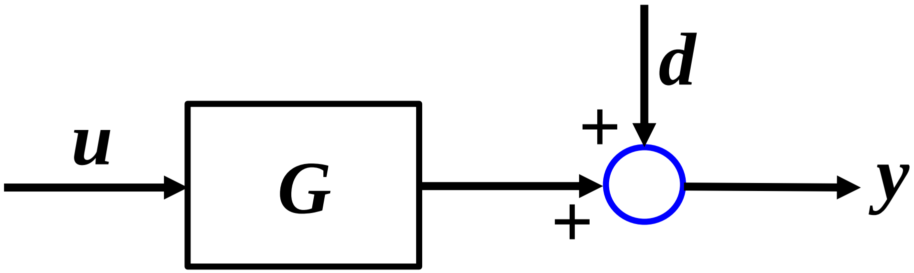
BASIC THEORY OF THE MODEL PREDICTIVE CONTROL (MPC): PART II

[24/10/2019]

- MPC with impulse/step response models
- MPC to ensure unbiased prediction
- Select performance indices for the MPC

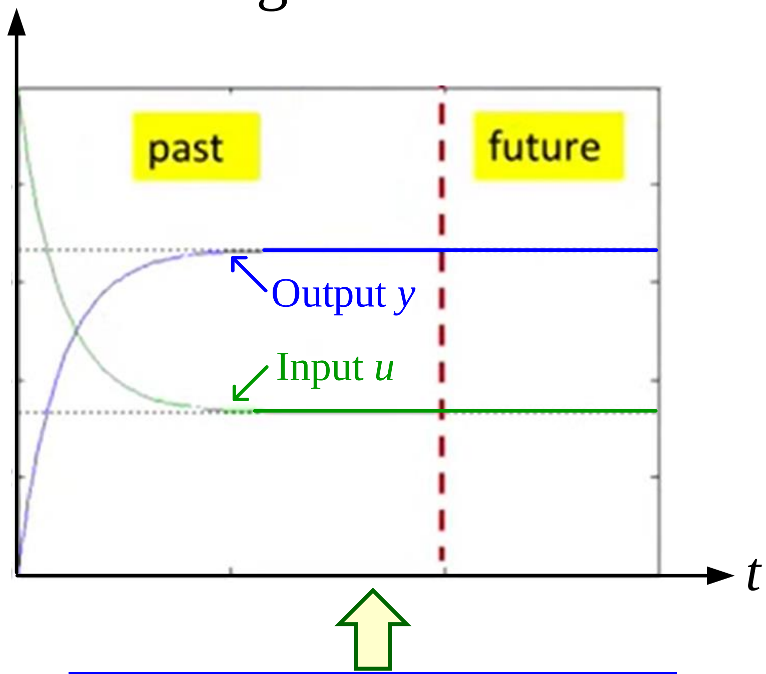
WHY NEED UNBIASED PREDICTION

Disturbance model



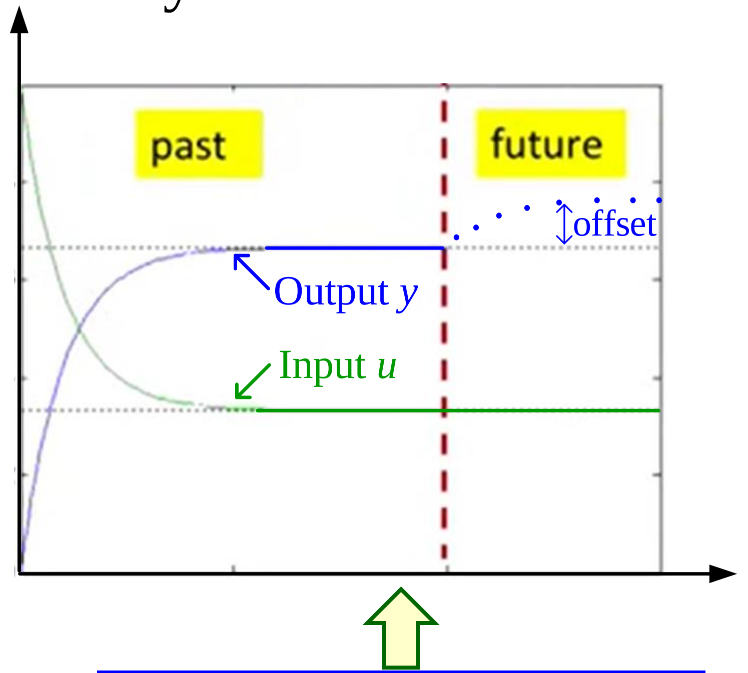
$$y = Gu + d$$

- Assuming d and u do not change



G does not have error

- Only know the estimated G

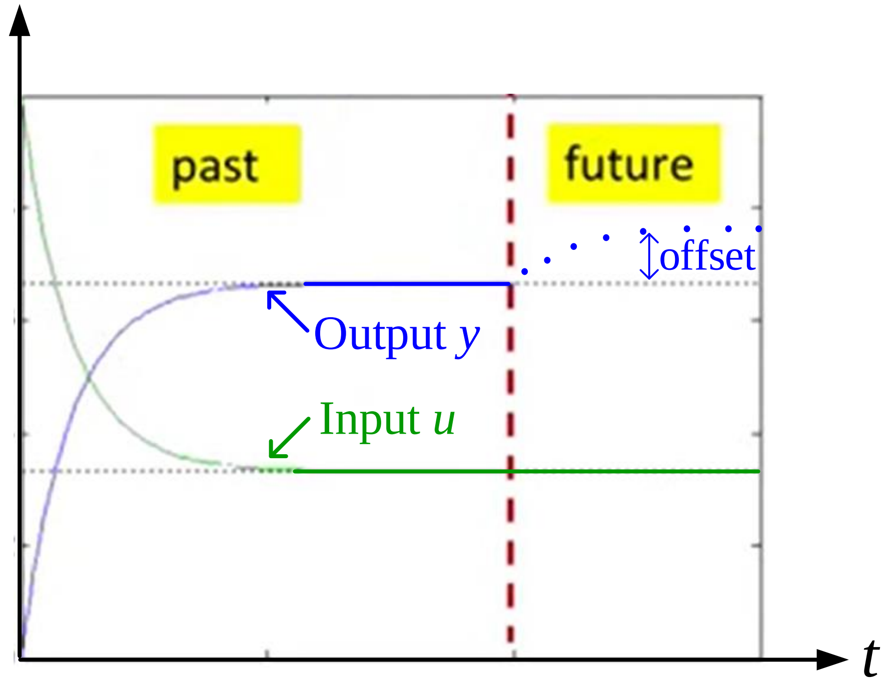


G has prediction error

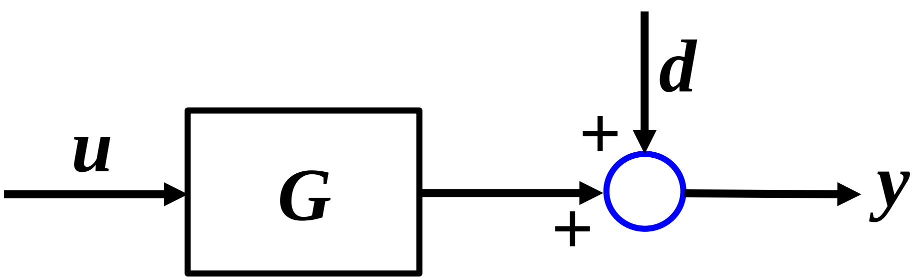
What is biased?

Future output prediction
$\neq$
Steady-state value

Predictions is biased



G has prediction error



$$\boxed{y = Gu + d}$$

Video: why need unbiased model

UNBIASED PREDICTION WITH STATE SPACE MODEL

- **Steady sate**: Output prediction = current steady-state value.

Example: One step prediction

$$y_k = Cx_k + d_k$$
$$x_{k+1} = Ax_k + Bu$$

$$d_k = d_{k+1} = d$$

$$y_{k+1} = Cx_{k+1} + d$$

If $y_k = y_{k+1}$ $\quad y_k = Cx_k + d$

$$x_{k+1} = x_k = x$$

$$x = Ax + Bu \Rightarrow y = Cx + d$$

Estimated output & plant output

$$y = Cx + d$$

- **Estimated output**:
  - $y=y_m$
  - $x=x_m$
  - $d=d_m$

$$y_m = Cx_m + d_m$$

- **Plant output**:
  - $y=y_p$
  - $x=x_p$
  - $d=d_p$

$$y_p = Cx_p + d_p$$

$$\Rightarrow d_m = y_p - Cx_m$$

Correct disturbance estimate is critical to ensure unbiased predictions.

m: Refer to estimate or model, i.e., predictive

p: refer to plant NOT prediction

Recall: Output prediction of the state space model based MPC

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}}_{P} \cdot x_k + \underbrace{\begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^{n-2}B & \cdots & CB \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} u_{k|k} \\ u_{k+1|k} \\ \vdots \\ u_{k+n\text{-}1|k} \end{bmatrix}}_{\underset{\rightarrow}{u}_k} + \underbrace{\begin{bmatrix} d_k \\ d_k \\ \vdots \\ d_k \end{bmatrix}}_{Ld_k}$$

$$\underset{\rightarrow}{y}_{k+1} = \left(P \cdot x_k + Ld_k \right) + H \cdot \underset{\rightarrow}{u}_k$$

Depends on **past**

Depends upon **decision variables**

Unbiased state space model based MPC

- $y_m = y_p$.

m: Estimate (Predictive) | p: Plant (process) | Estimate

$$\underset{\rightharpoondown}{y}_{k+1} = \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}}_{P} \cdot x_k + \underbrace{\begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^{n-2}B & \cdots & CB \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} u_{k|k} \\ u_{k+1|k} \\ \vdots \\ u_{k+n\text{-}1|k} \end{bmatrix}}_{\underset{\rightharpoondown}{u}_k} + \underbrace{\begin{bmatrix} d_k \\ d_k \\ \vdots \\ d_k \end{bmatrix}}_{Ld_k}$$

$$\underset{\rightharpoondown}{y} = y_{Plant}$$

$$d_k = d_m = y_p - Cx_k$$

$$x = Ax + Bu$$

ANOTHER UNBIASED PREDICTION WITH STEADY STATE ESTIMATES

Revisit steady state model

- The expected steady-state obeys:

$$y_{ss} = Cx_{ss} + d \qquad x_{ss} = Ax_{ss} + Bu_{ss}$$

$$\begin{bmatrix} y_{ss} - d \\ 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ A - I & B \end{bmatrix} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix}$$

$$\begin{bmatrix} C & 0 \\ A - I & B \end{bmatrix}^{-1} \begin{bmatrix} y_{ss} - d \\ 0 \end{bmatrix} = \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix}$$

A disturbance estimate (d) is required to estimate the steady-state states (x_{ss}) and inputs (u_{ss}) for a given steady output (y_{ss}).

One step prediction with deviation variables

- State space model is linear and thus superposition holds.

$$y_k = Cx_k + d_k \qquad x_{k+1} = Ax_k + Bu_k$$

$$y_{ss} = Cx_{ss} + d_k \qquad x_{ss} = Ax_{ss} + Bu_{ss}$$

$$\hat{x}_k = x_k - x_{ss}$$
$$\hat{y}_k = y_k - y_{ss}$$
$$\hat{u}_k = u_k - u_{ss}$$

$$\hat{y}_k = C\hat{x}_k$$

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k$$

$$a(z)\Delta y_k = b(z)\Delta u_k \ ?$$

Model in terms of the deviation variable
no longer **needs** the disturbance term
as it has been absorbed in the estimation of the correct steady-state.

n step prediction with deviation variables

$$\hat{y}_{k+n|k} = CA^n\hat{x}_k + C\left(A^{n-1}B\hat{u}_{k|k} + A^{n-2}B\hat{u}_{k+1|k} + \ldots + AB\hat{u}_{k+n\text{-}2|k} + B\hat{u}_{k+n-1|k}\right)$$

Known based on the current and past measurement

Unknown as based on the future input choices which remain to be decided

$$\hat{x}_k = x_k \text{-} x_{ss}$$

$$\hat{y}_k = y_k \text{-} y_{ss}$$

$$\hat{u}_k = u_k \text{-} u_{ss}$$

Compact prediction notation

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}}_{P_x} \cdot \hat{x}_k + \underbrace{\begin{bmatrix} CB & 0 & \dots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^{n-2}B & \dots & CB \end{bmatrix}}_{H_x} \cdot \underbrace{\begin{bmatrix} \hat{u}_{k|k} \\ \hat{u}_{k+1|k} \\ \vdots \\ \hat{u}_{k+n-1|k} \end{bmatrix}}_{\underset{\rightarrow}{\hat{u}}_k} + \underbrace{\begin{bmatrix} y_{ss} \\ y_{ss} \\ \vdots \\ y_{ss} \end{bmatrix}}_{Ly_{ss}}$$

$$\begin{bmatrix} y_{k+1|k} \\ y_{k+2|k} \\ \vdots \\ y_{k+n|k} \end{bmatrix} = \underset{\rightarrow}{y}_{k+1}$$

$$\underset{\rightarrow}{\hat{y}}_{k+1}$$

$$\underset{\rightarrow}{y}_{k+1} = \underbrace{\left(P_x \cdot \hat{x}_k + H_x \cdot \underset{\rightarrow}{\hat{u}}_k \right)}_{\Delta \underset{\rightarrow}{y}_{k+1}} + Ly_{ss}$$

$$\underset{\rightarrow}{y}_{k+1} = \left(P_x \cdot \hat{x}_k + Ly_{ss} \right) + H_x \cdot \underset{\rightarrow}{\hat{u}}_k$$

Depends on past

Depends on decision variables

UNBIASED PREDICTION WITH TRANSFER FUNCTION MODEL

Unbiased condition for the transfer model

- If the system is in steady-state, $y_m = y_p$:

m: Estimate (Predictive)

$$y_m = d_m + G_m(0)u$$

Modelled transfer function

p: Plant (process)

$$y_p = d_p + G_p(0)u$$

Actual transfer function

$$y_m = y_p$$

$$d_p + G_p(0)u = d_m + G_m(0)u$$

Find a suitable d_m to meet the above equation

Obtain the right disturbance

$$d_p + G_p(0)u = d_m + G_m(0)u$$

p: Plant (process)

m: Estimate (Predictive)

Predicted

$$y_p(k) = d_p + G_p(0)u$$

$$y_m(k) = d_m + G_m(0)u$$

$$y_p = y_m$$

Known

Can be measured directly in practice

Controlled by yourself

$$d_m = y_p(k) - G_m(0)u$$

Measured

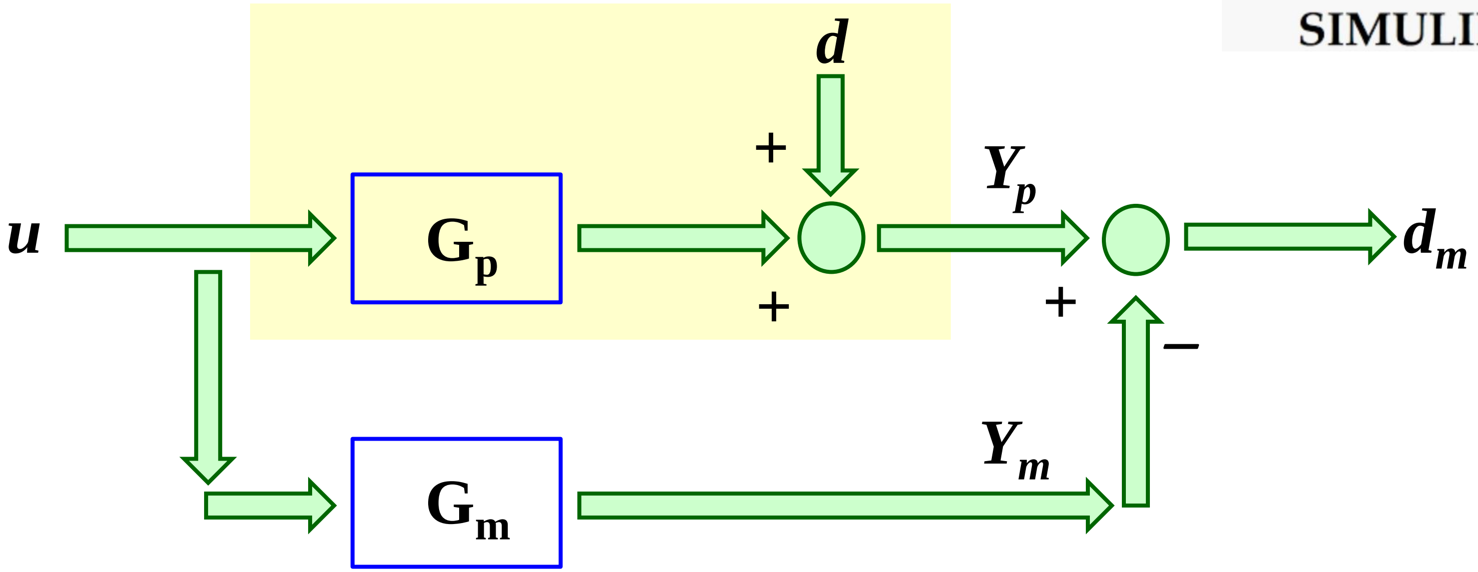
Modelled by yourself

Control block to obtain the right disturbance

- Run a simple model with the actual process

$$d_m = y_p(k) - G_m(0)u$$

MATLAB SIMULINK®



p: Plant (process)

m: Estimate (Predictive)

UNBIASED PREDICTION WITH CARIMA MODEL

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Recall: CARIMA model

- Transfer function model with MPC is so called CARIMA Model.

Input

Disturbance

$$a(z)y_k = b(z)u_k + T(z)\frac{\varsigma_k}{\Delta}$$

Zero-mean random variable

Output

$$\Delta = 1 - Z^{-1}$$

- *Uncertainty* is included
- *Slowly varying disturbances* is considered
- $T(z)$ is treated as a design parameters

Recall: CARIMA model with input of Δu_k

$$a(z)y_k = b(z)u_k + T(z)\frac{\varsigma_k}{\Delta}$$

$$\Downarrow$$

$$\Delta a(z)y_k = \Delta b(z)u_k + \cancel{T(z)\varsigma_k}$$

Zero-mean random variable

$$\Downarrow$$

$$\Delta a(z)y_k = \Delta b(z)u_k$$

$$\Downarrow$$

$$\boxed{A(z) = a(z)\Delta}$$

$$[a(z)\Delta]y_k = b(z)(\Delta u_k) \Rightarrow A(z)y_k = b(z)\Delta u_k$$

Combine a(z) and Delta

Use input increments

One step predication with simplified model

- One-step ahead prediction models: Given data at sample 'k', Determine data at sample '$k+1$'.

$$A(z)y_k = b(z)\Delta u_k$$

$$A(z) = a(z)\Delta$$

$$A(z) = 1 + A_1 z^{-1} + \cdots + A_n Z^{-n}$$

$$b(z) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}$$

$$y_{k+1} + A_1 y_k + \cdots + A_n y_{k-n+1} = b_1 \Delta u_k + b_2 \Delta u_{k-1} + \cdots + b_m \Delta u_{k-m+1}$$

$$y_{k+1} = b_1 \Delta u_k + b_2 \Delta u_{k-1} + \cdots + b_m \Delta u_{k-m+1} - A_1 y_k - \cdots - A_n y_{k-n+1}$$

No need for a **disturbance estimate** in this prediction model as within the use of **increments**

Prediction from the steady-state

- Steady-state means:

$$y_k = y_{k-1} = \cdots = y_{k-n+1} \qquad \Delta u_{k-1} = \Delta u_{k-2} = \cdots = \Delta u_{k-m+1} = 0$$

$$\Downarrow \quad A(z) y_k = b(z) \Delta u_k$$

$$A(z) y_k = b(z) \Delta u_k = 0$$

$$\Downarrow$$

$$A(z) = a(z)\Delta = 0 \Rightarrow 1 + A_1 + \cdots + A_n = 0 \rightarrow A_1 + \cdots + A_n = -1$$

- One step prediction:

$$y_{k+1} + A_1 y_k + \cdots + A_n y_{k-n+1} = b_1 \Delta u_k + b_2 \Delta u_{k-1} + \cdots + b_m \Delta u_{k-m+1}$$

$$\Downarrow$$

$$y_{k+1} + [A_1 + \cdots + A_n] y_k = [b_1 + b_2 + \cdots + b_m] \cdot 0$$

$$\Downarrow \quad A_1 + \cdots + A_n = -1$$

$$y_{k+1} \equiv y_k \Leftarrow y_{k+1} - y_k = 0$$

CARIMA model predictions is <u>unbiased naturally</u>

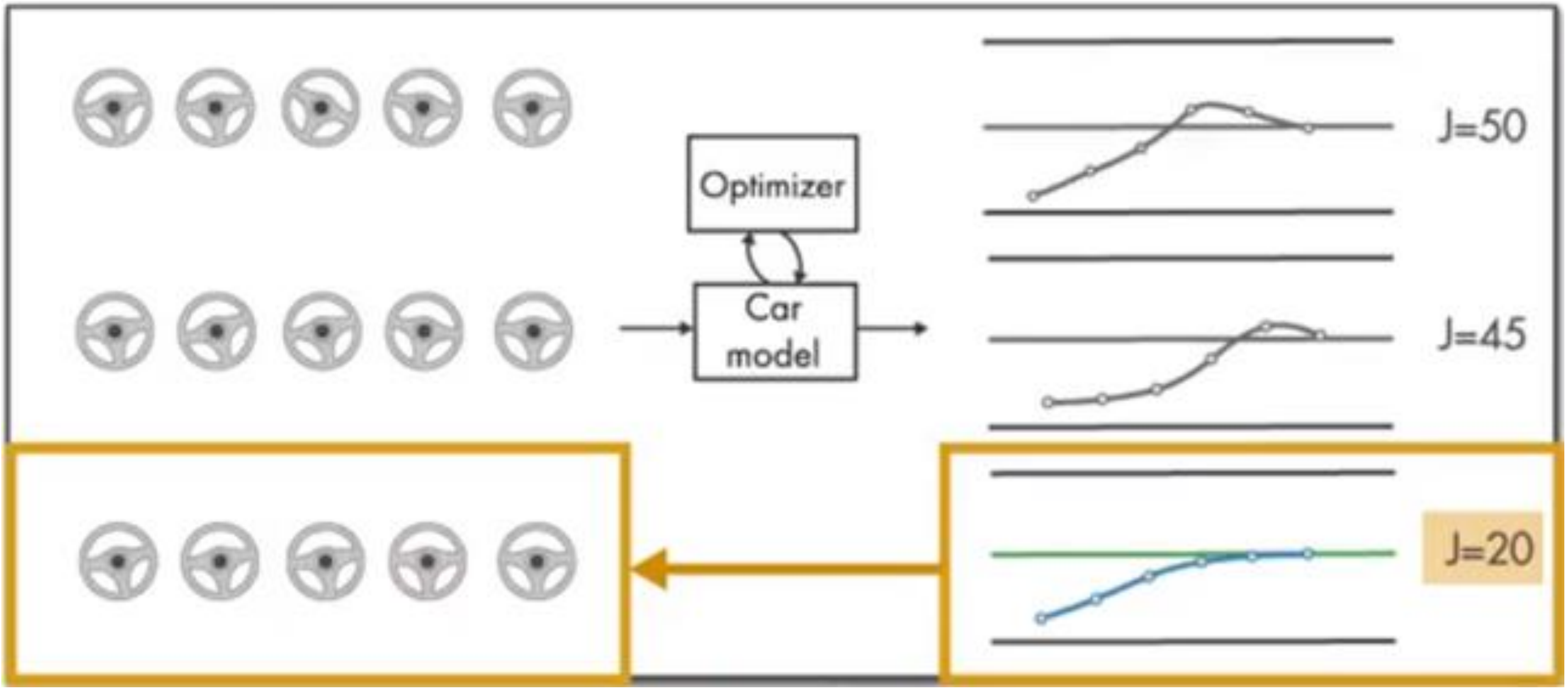
BASIC THEORY OF THE MODEL PREDICTIVE CONTROL (MPC): PART II

[24/10/2019]

- MPC with impulse/step response models
- MPC to ensure unbiased prediction
- Select performance indices for the MPC

SELECT PERFORMANCE INDICES

The function of performance indices



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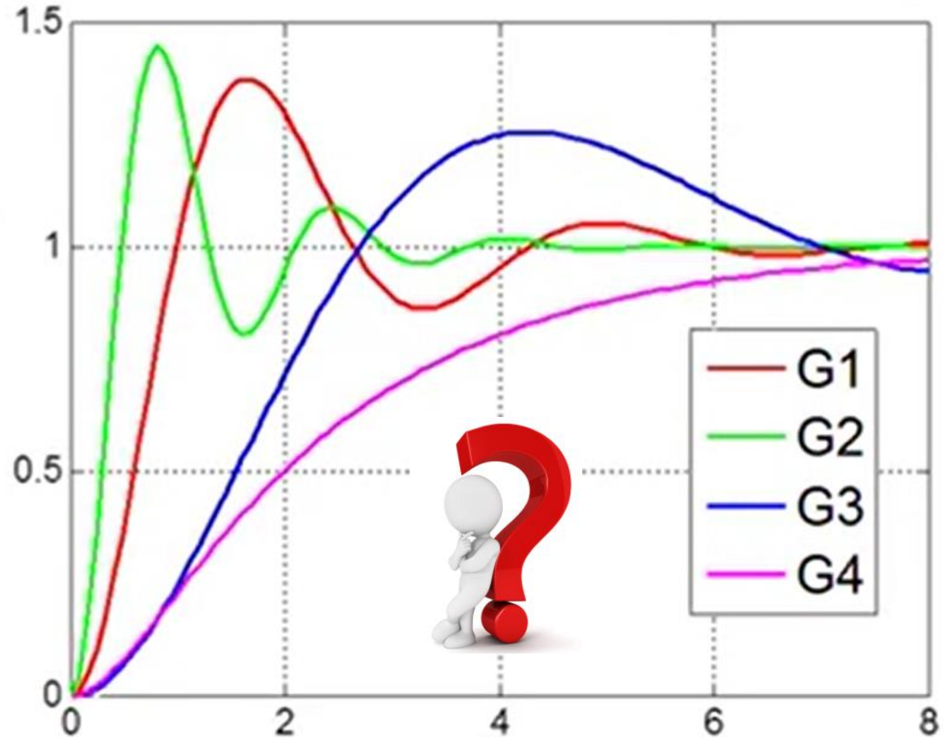
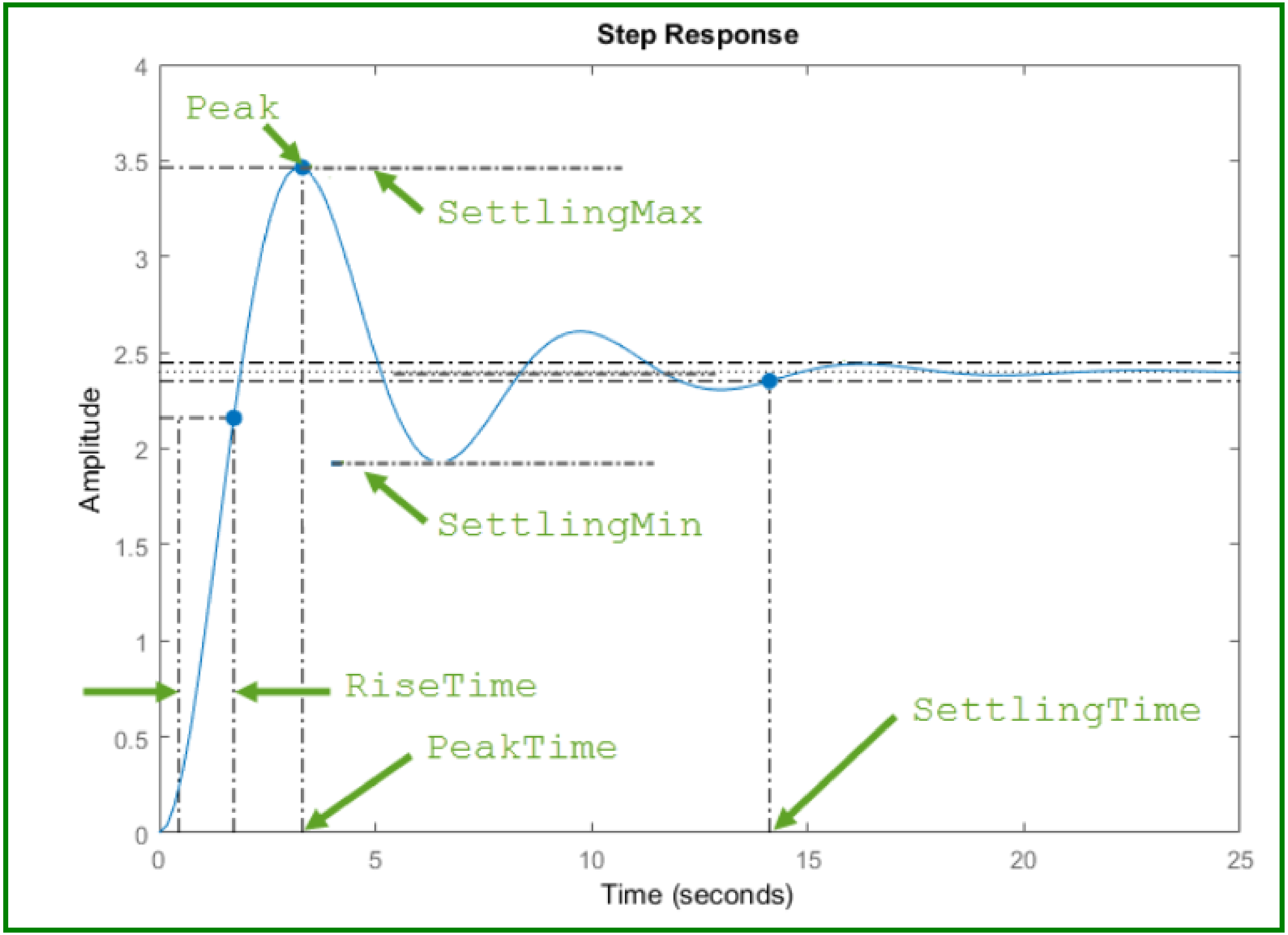
Example: Which response is the best and why?

Rise-time **Setting time** **Overshoot**

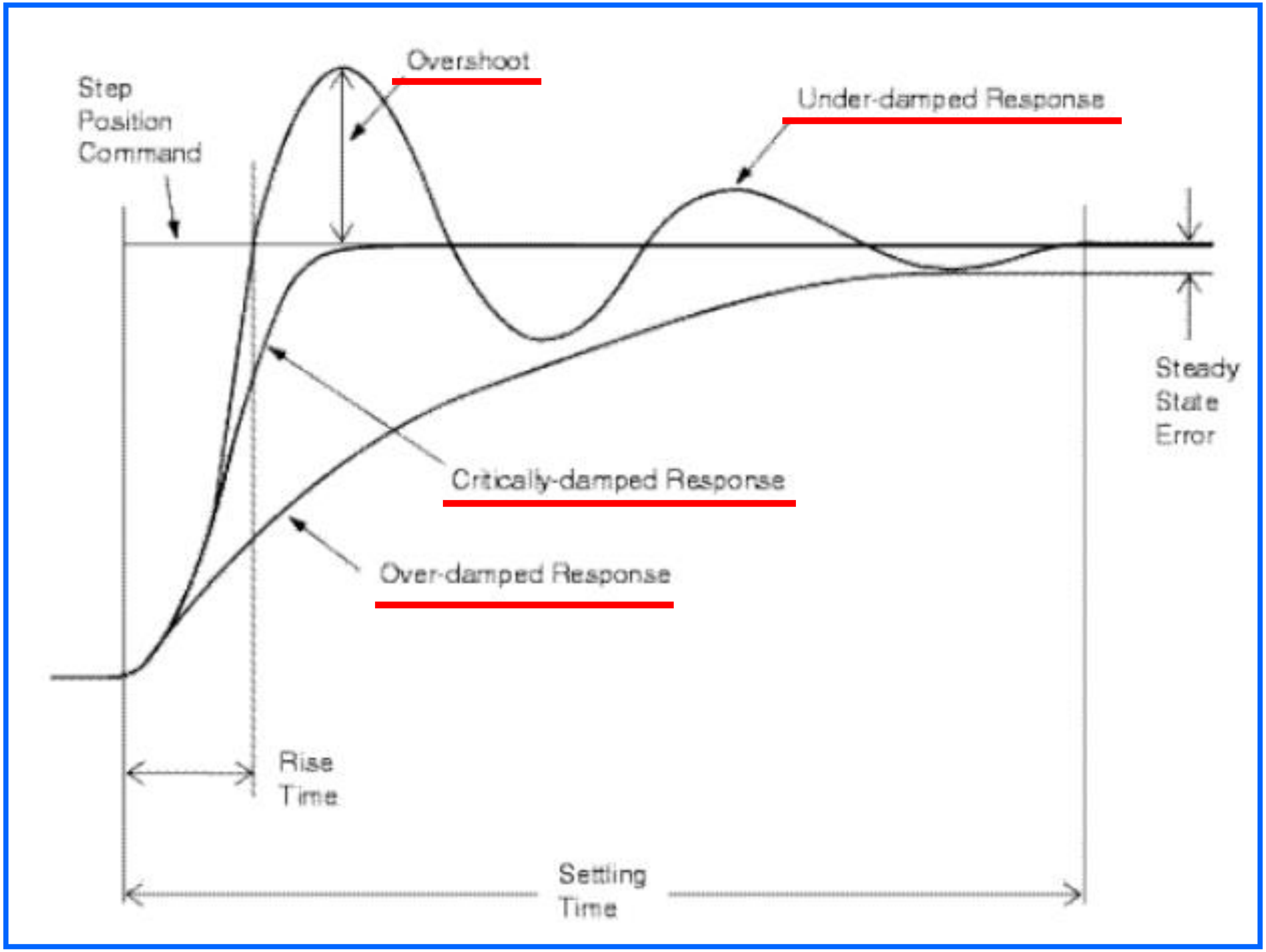


- Humans use rather **vague** performance indices
- Trade-off among rise-time, overshoot and setting time, etc.

Evaluation indices of the step response



Evaluation of the step response



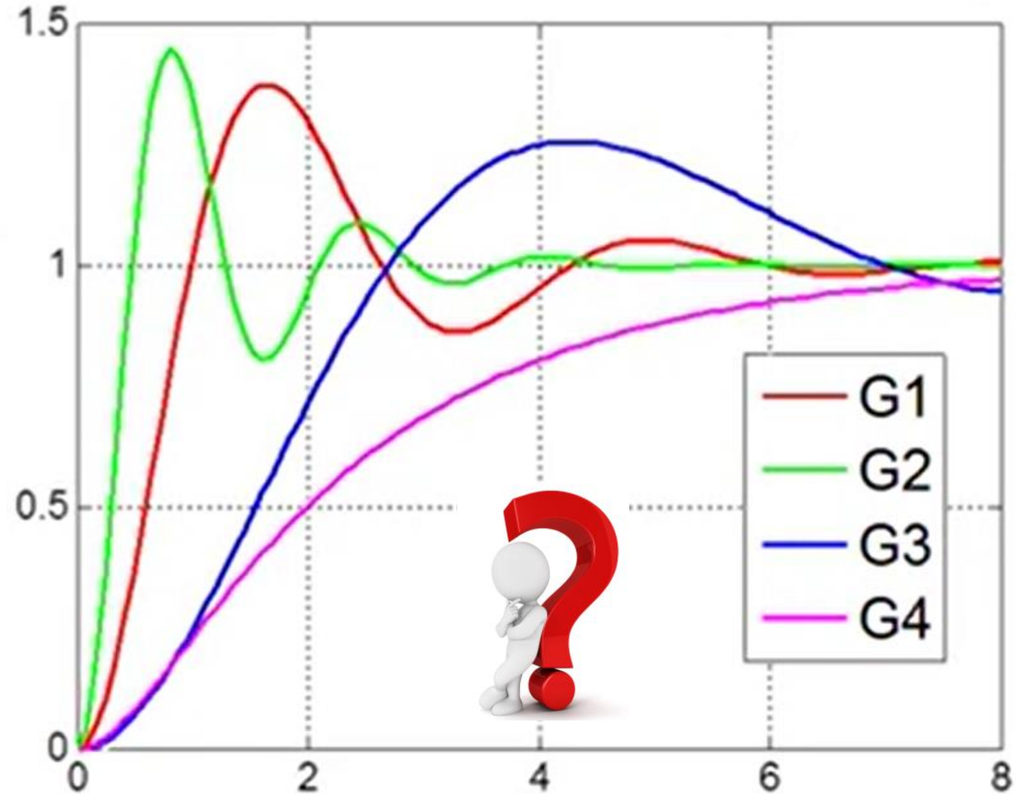
Performance index: requirement for designer

- Designer must have some requirements in their mind.

Q: How do we know if control law 1 is better than control law 2?

A: it meets the requirements better in some sense.

Rise-time Setting time Overshoot

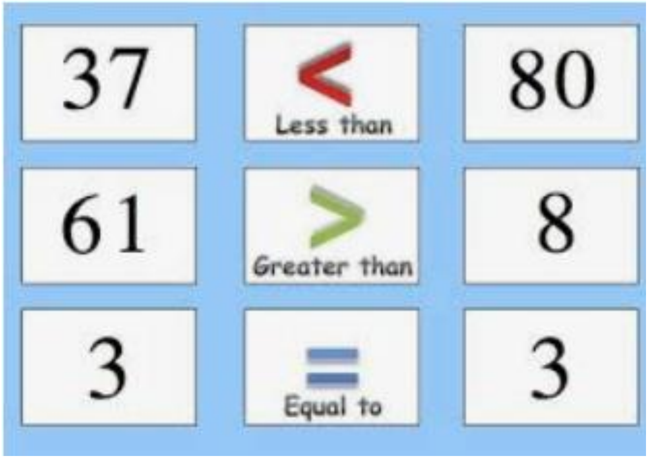


Performance index: precise definition is needed

- MPC is based on a precise numerical optimum
- Precise definition of ‘optimum’ performance is required.

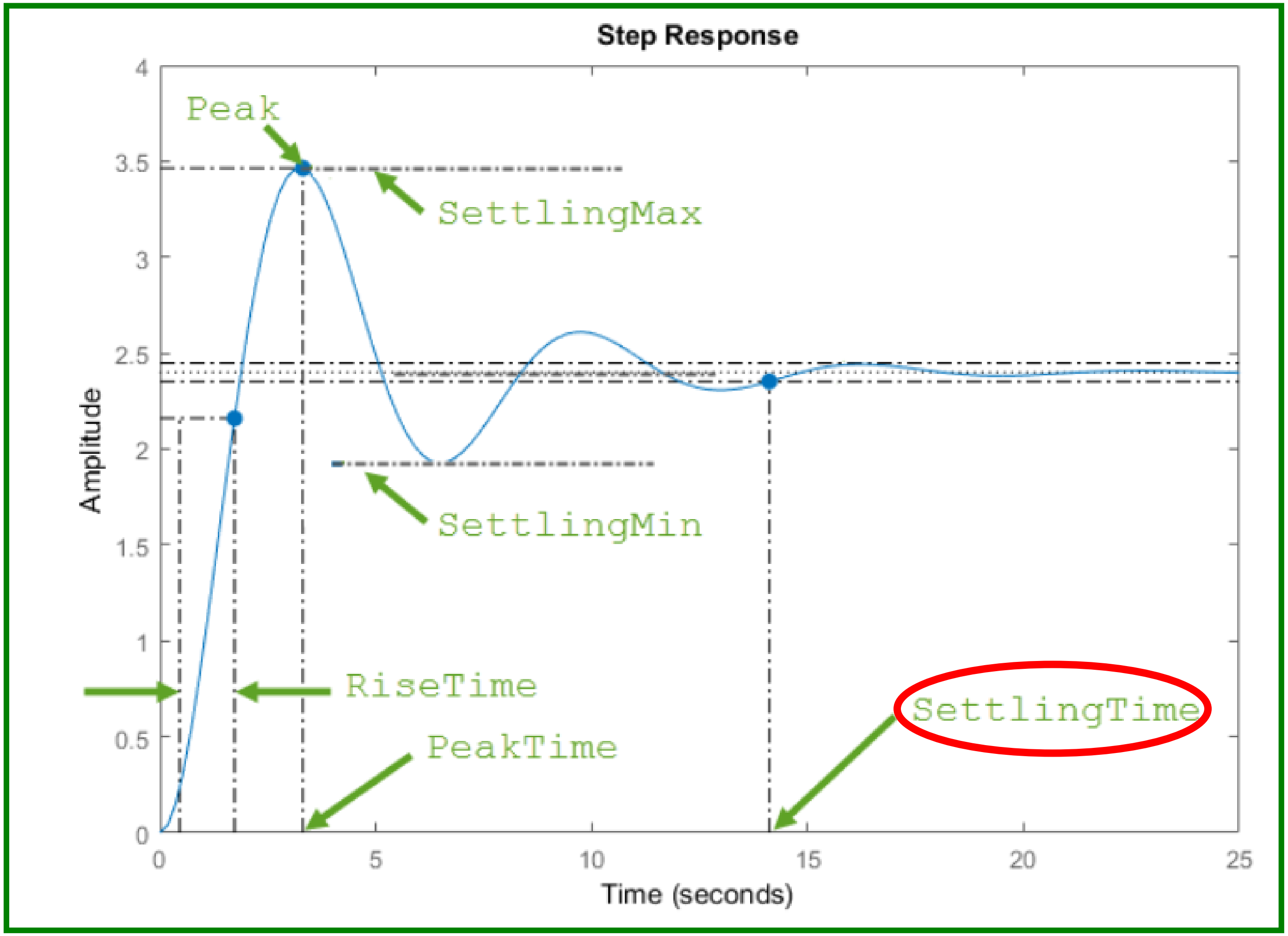
Number = feeling

Optimum means larger or smaller



Example 1 of different performance index

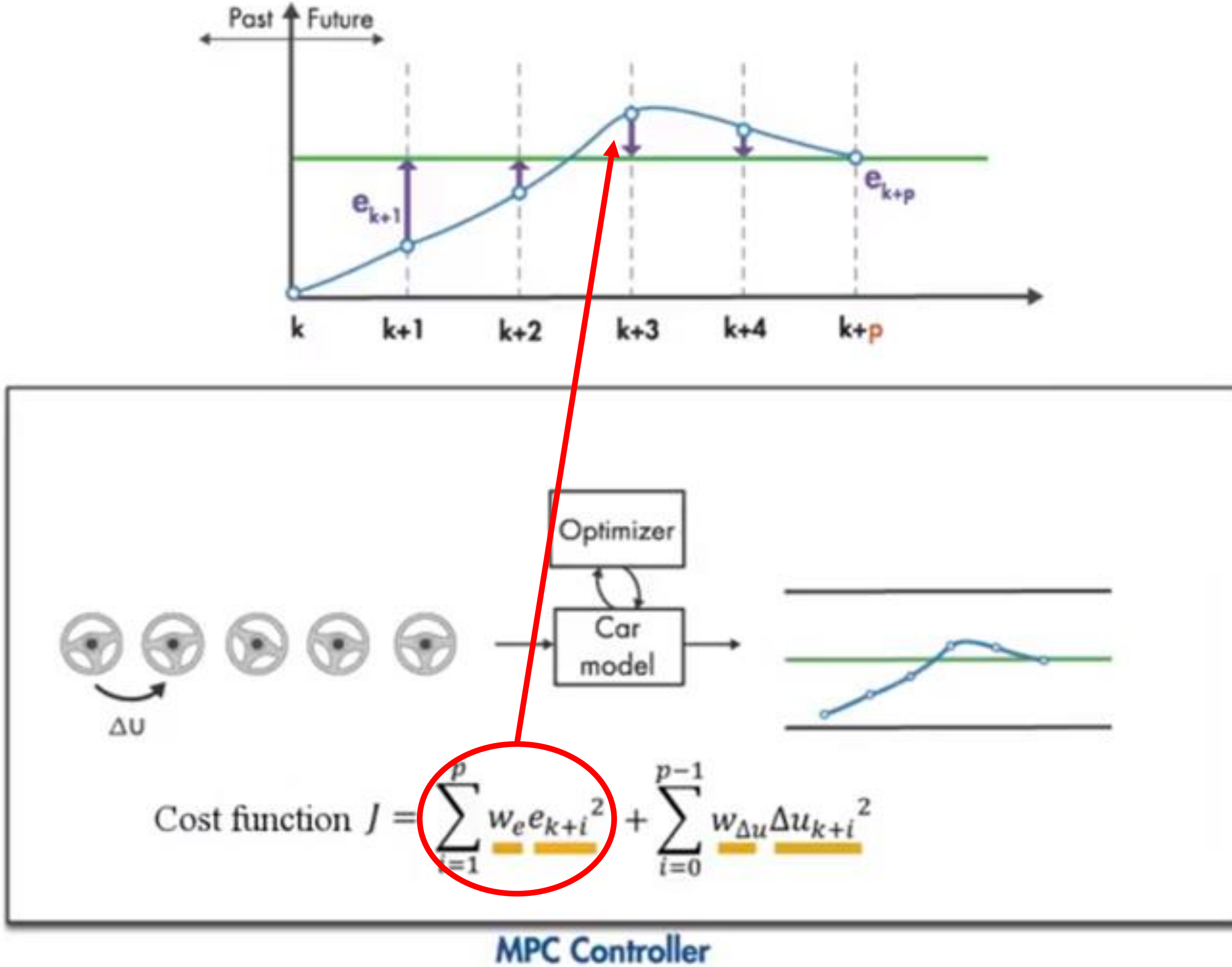
- Fastest setting time (to within say 5%).



Example 2 of different performance index

- Smallest error on average.

$$\int_0^t |e(t)|dt$$



- **Smallest actuation energy** (minimise control energy).

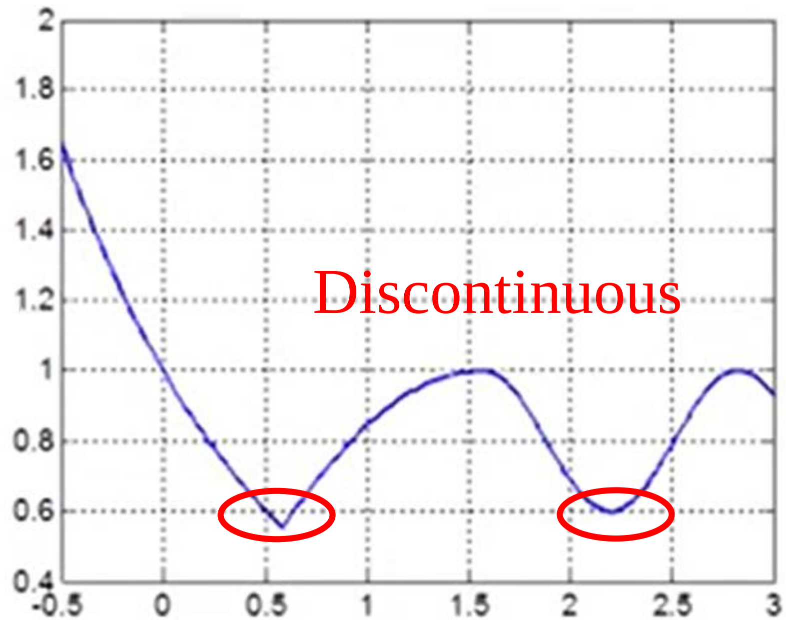
Tractable performance index

Tractable index needs:

- Linear (or simple) dependence on any available parameters.
- Continuous function with continuous derivatives.

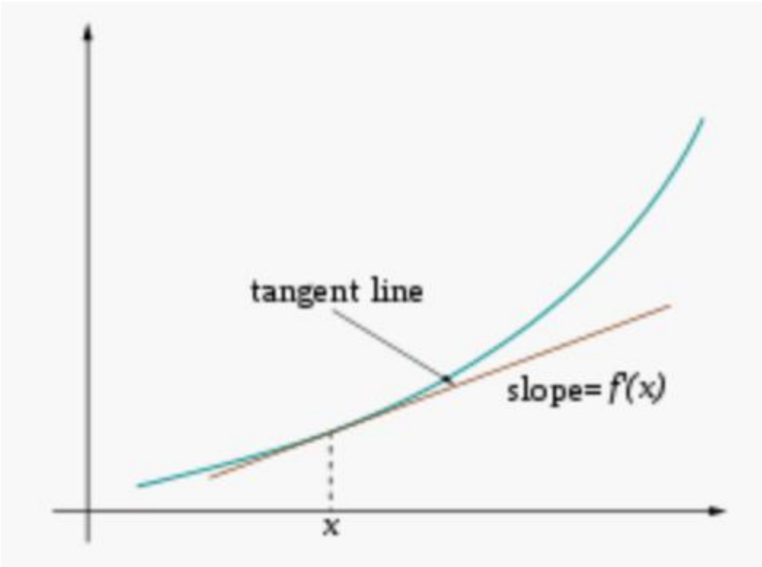
Non-continuity makes any optimisation much more challenging.

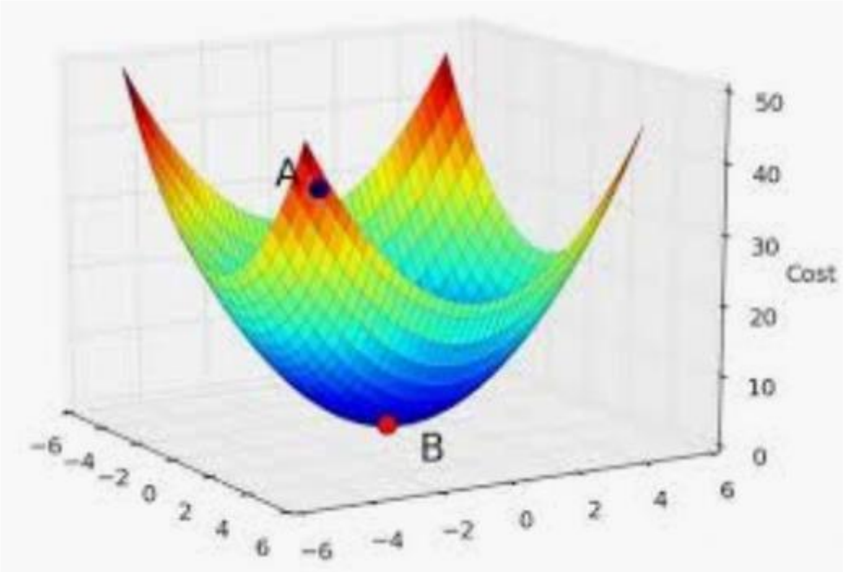
$$\min_x \frac{e^{x^3-2x}}{|x+1|\left(x-\sin x\right)}$$



Requirements of performance indices

- Have continuous derivatives.
- Have an unique minimum.
- Do not contain non-linearities
  – (e.g., |a| requires knowledge of when a<0).
- Be positive not negative
  – Negative cost could be misleading (i.e., ±error).





Tips to realize the performance indices

- The un-suitable performance indices. 😟

1. Sinusoids do not have an unique minimum
2. Exponentials have no stationary points
3. $F = ax + b$ can arrive $-\infty$ and thus is meaningless

- The suitable performance indices. 🙂

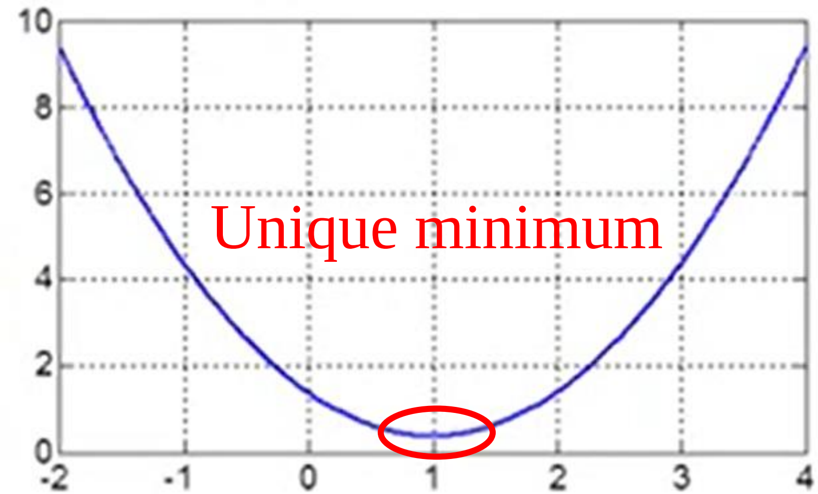
1. Polynomials broadly meet the requirement.
   - Cubic and higher order have multiple stationary point
   - **Quadratics only remain. (simpler)**

Quadratics applied for performance indices

- Quadratics: sensible engineering and simple optimization.

$$\boxed{\min_x x^2 + ax + b}$$

$$\boxed{x^2 + ax + b \equiv (x\text{-}\alpha)^2 + \beta}$$



- Energy terms often link to the square of a state
  - Assessing x^2 is logical.

- x^2 terms are always positive for any 'x'.

- Quadratic penalises larger deviations heavily, but not small ones
  - makes good sense.

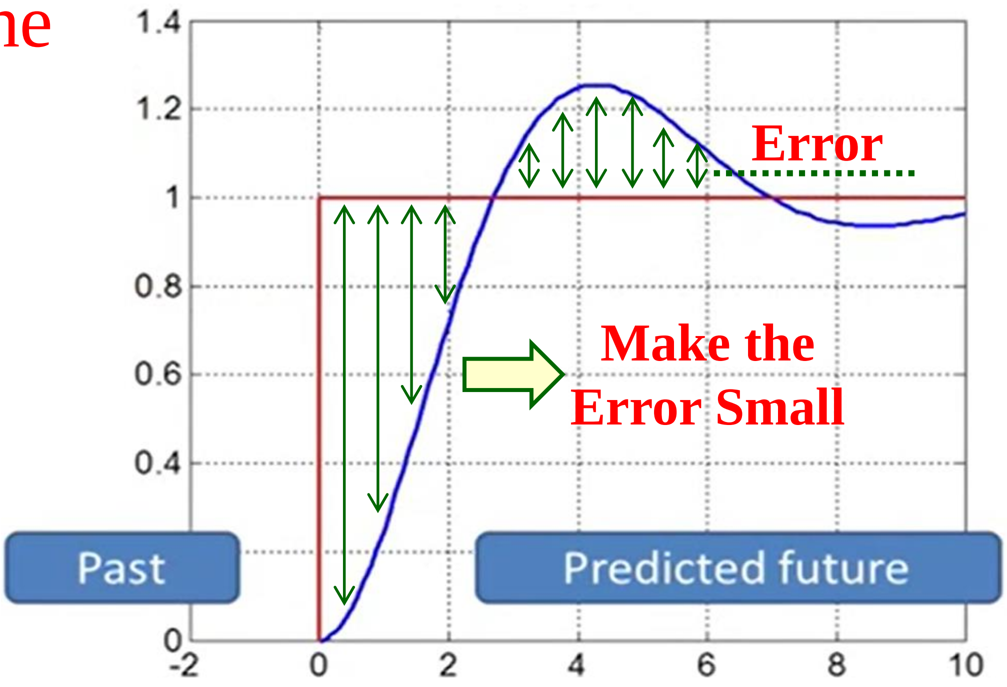
Predictions, errors & performance index

- Errors: between the predicted output and the target at a number of sampling instants.

- Performance: linked to the predicted output errors.

$$\boxed{J = \sum_{k=1}^{n} e_k^2}$$

Needs to determine an appropriate ‘n’



- Smaller J, better performance.

$J=\sum_{k=1}^{n} e_k^2$ maybe inappropriate: control limitation1

$$\boxed{J=\sum_{k=1}^{n} e_k^2}$$

- The above J can not lead to sensible control decisions

System G:

$$G=\frac{z^{-1}-1.2z^{-2}}{1-0.64z^{-1}+0.8z^{-2}}$$

If define target is r:

$$\boxed{J=\sum_{k=1}^{n} e_k^2}$$

$$\boxed{r=Gu}$$

$$\boxed{J=\sum_{k=1}^{n} e_k^2=0}$$

$$\boxed{u=\frac{1}{G}r}$$

Find u to Minimize J to make error to zero

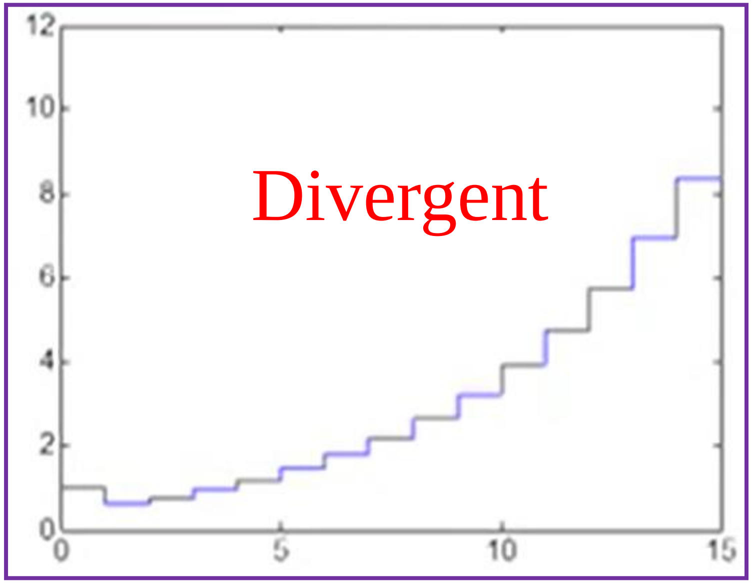
$J=\sum_{k=1}^{n} e_k^2$ maybe inappropriate: control limitation1

$$G=\frac{z^{-1}-1.2z^{-2}}{1-0.64z^{-1}+0.8z^{-2}}$$

$$u=\frac{1}{G}r$$

If $r=\frac{1}{1-z^{-1}}$

$$u=\frac{1-0.64z^{-1}+0.8z^{-2}}{z^{-1}-1.2z^{-2}}\frac{1}{1-z^{-1}}$$



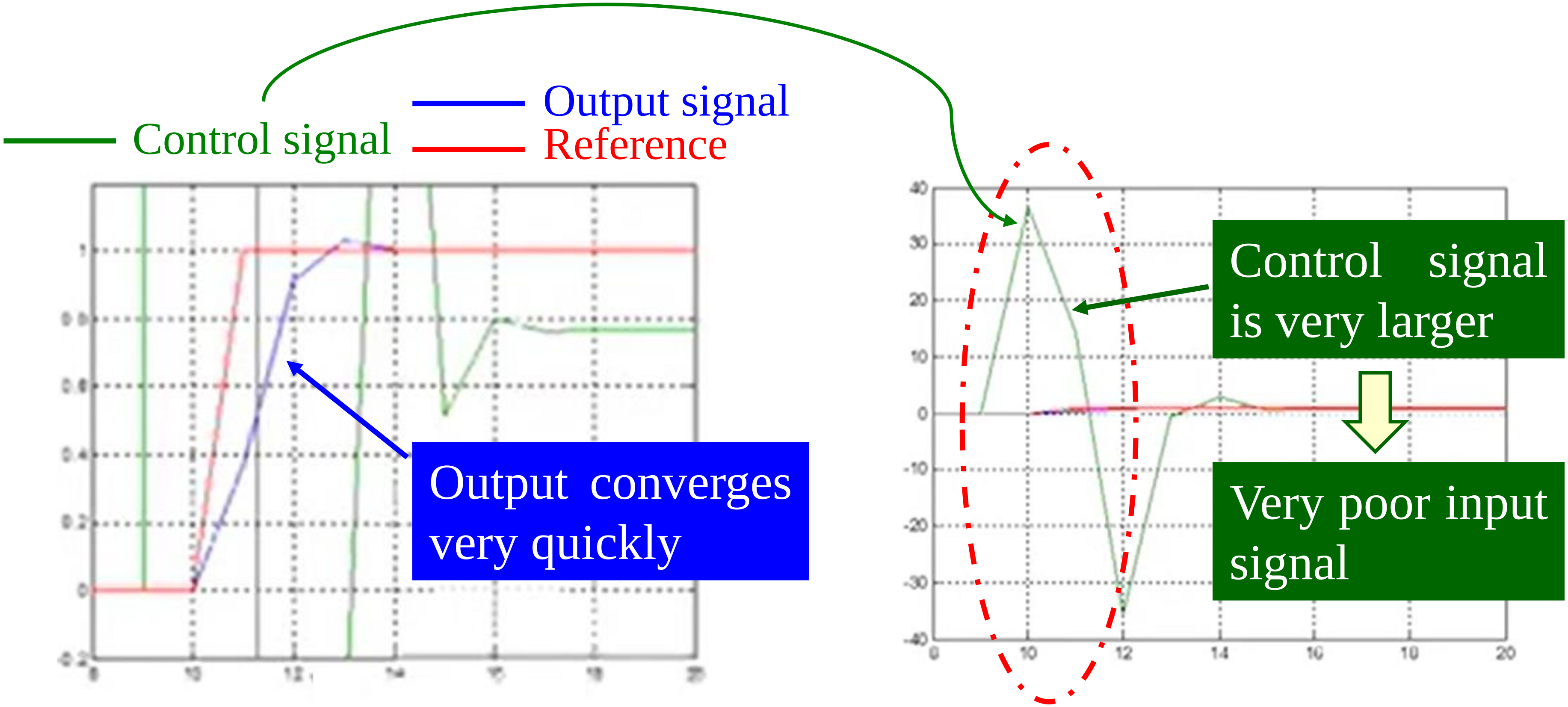
- Divergent control signal

$$J=\sum_{k=1}^{n} e_k^2$$

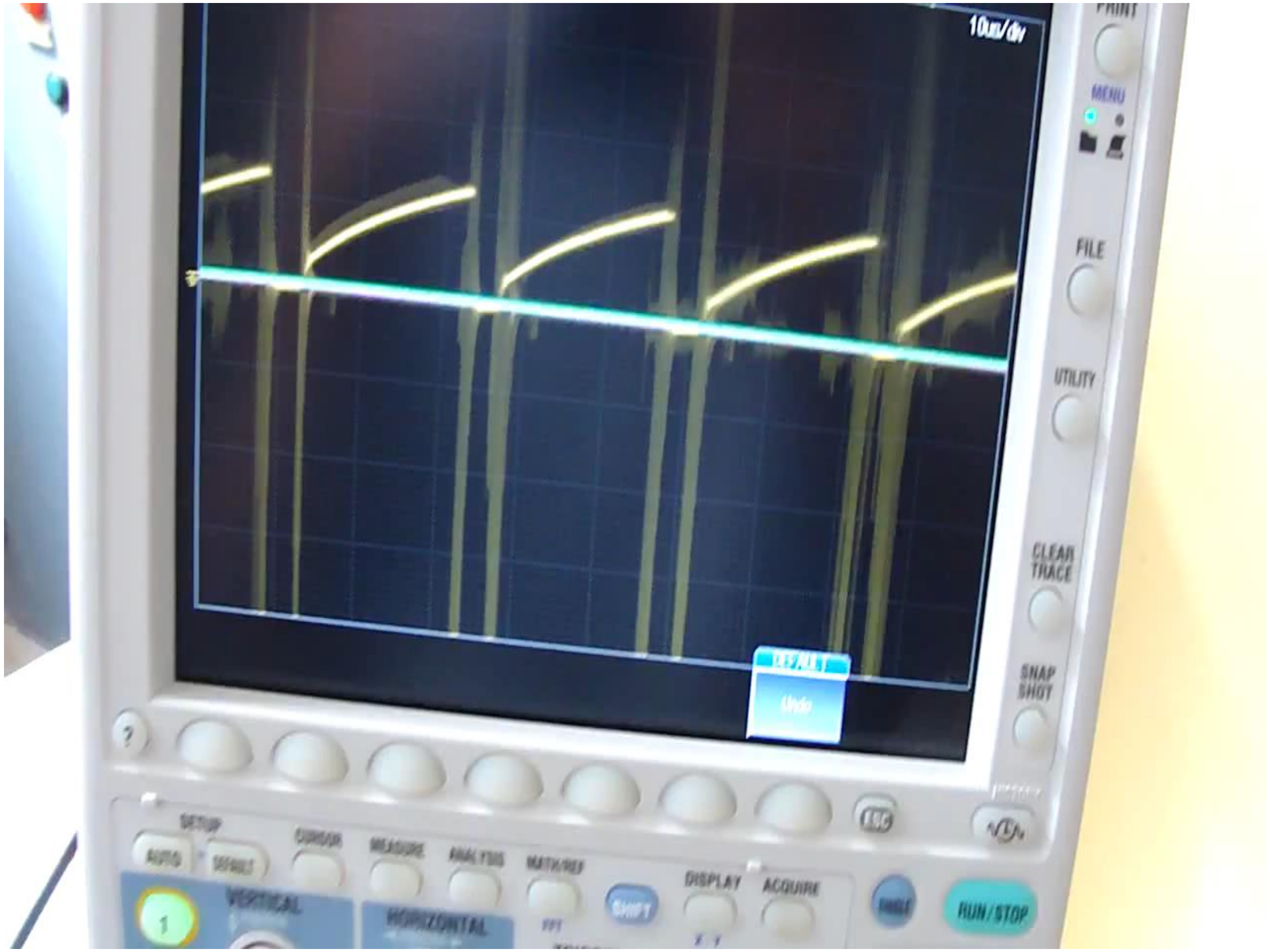
Cannot use a cost just based on tracking errors in general

$J=\sum_{k=1}^{n} e_k^2$ maybe inappropriate: control limitation 2

- The control input should also be limited:



Simple J maybe inappropriate: Example



First try to improve the simple J

- Besides the predicted output errors, we also care about actuation.
- Too much actuation causes fatigue
- Should penalise the input control signal.

$$\boxed{J = \sum_{k=1}^{n}\left(e_k^2 + \lambda u_k^2\right)}$$

Okay or not?

Still wrong

Cannot drive both error and the input to zero simultaneously

$$J = \sum_{k=1}^{n} \left(e_k^2 + \lambda u_k^2 \right)$$

$$y_{ss} = G u_{ss}$$

$$\min = \sum_{k=1}^{n} \left(e_{ss}^2 + \lambda u_{ss}^2 \right) \equiv \left(\left(r - G u_{ss} \right)^2 + \underbrace{\lambda u_{ss}^2}_{\neq 0} \right)$$

$$r - G u_{ss} \neq 0 \Rightarrow u_{ss} \neq \frac{r}{G}$$

$$y_{ss} \neq r$$

Cannot converge to correct target.

Summary

- Quadratic performance indices is the common choice results in smooth & robust control designs.

- Square of tracking errors alone cannot be utilized due to it cannot result in the desired control actions.

- Just add the square of input actions will result in steady-state offset → the performance index is ill-posed.

UNBIASED PERFORMANCE INDEX

Recall: Limitation of the performance index

- Tracking errors and input activity should be penalised
  - However, a simplistic realisation does not work

$$J = \sum_{k=1}^{n}\left(e_k^2 + \lambda u_k^2\right)$$

$$y_{ss} = G u_{ss}$$

Cannot make both error and control to 0 simultaneously

$$\Rightarrow \min = \sum_{k=1}^{n}\left(e_{ss}^2 + \lambda u_{ss}^2\right) \equiv \left(\left(r - G u_{ss}\right)^2 + \lambda u_{ss}^2\right)$$

$$\Rightarrow r - G u_{ss} \neq 0 \Rightarrow u_{ss} \neq \frac{r}{G} \Rightarrow y_{ss} \neq r$$

Cannot converge to correct target.

How to get right performance index?

- Penalise the deviations & drive these deviations to zero
  - Make the output to the desired value

$$\underline{\hat{y}}_{k+1} = P_x \hat{x}_k + H_x \underline{\hat{u}}_k$$

$$\hat{y} = y - r$$

$$J = \sum_{k=1}^{n} \left(\hat{y}_k^2 + \lambda \hat{u}_k^2 \right)$$

$$\hat{u} = u - u_{ss}$$

Unbiased predictions

$$\hat{y}_k^2 = 0 \Rightarrow \hat{y}_k = 0 \Rightarrow \hat{y} = y - r = 0 \rightarrow y = r$$

$$\hat{u}_k^2 = 0 \Rightarrow \hat{u}_k = 0 \Rightarrow \hat{u} = u - u_{ss} = 0 \rightarrow u = u_{ss}$$

- Penalises distance of input from the steady-state

$$J=\sum_{k=1}^{n}\left(e_{k+1}^{2}+\lambda\left(u_{k}-u_{ss}\right)^{2}\right)$$

- Penalises the rate of charge of the input

$$J=\sum_{k=1}^{n}\left(e_{k+1}^{2}+\mu\left(\Delta u_{k}\right)^{2}\right)$$

- Generic performance indices

$$J=\sum_{k=1}^{n}\left(e_{k+1}^{2}+\lambda\left(u_{k}-u_{ss}\right)^{2}+\mu\left(\Delta u_{k}\right)^{2}\right)$$

More generic performance indices

- Using the same horizons for the inputs and outputs and scalar weights, a more generic form is as follows.

$$\boxed{J = \sum_{k=1}^{n_y} \left\| W_y e_k \right\|_2^2 + \sum_{k=1}^{n_u} \left\| W_u \left(u_k - u_{ss} \right) \right\|_2^2 + \left\| R_u \Delta u_k \right\|_2^2}$$

- Matrix weights on each term (W_y, W_u, R_u).
- Different horizons for inputs (n_y) and outputs (n_u).

Review of today's topic

- Understand the MPC with impulse/step response model
- Understand unbiased MPC
- Understand how to select performance indices for MPC

Thank you!

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