PART I

FUNDEMENTALS OF EMPIRICAL MODELING & IDENTIFICATION

1.1 FUNDAMENTAL CONCEPTS

1.1.1 DEFINITIONS

1. Many processes are not well understood, the fundamentals of their dynamic behavior are poorly known. The theoretical approach is unlikely to be successful.
2. Enough is known about the processes for reasonable theoretical modeling. The resulting theoretical models are too complicated to be useful for controller design purposes.

In these cases: models can be obtained by process identification.

What is Process Identification?

When the "unknown" process is subjected to a forcing function, *u(t),* and the response, *y(t),* is measured. By correlating the measured input and output data to develop a model for the system; the unknown system is said to have been "identified".

Definition: *Process identification involves constructing a process model from experimentally obtained input/output data, with no recourse to any laws concerning the fundamental nature and properties of the system.*

In this case, no *a priori* knowledge about the process is necessary, the system is treated like a "black box," and the experimental information gathered from its response to external stimuli is used to "identify" what goes on within the "box."

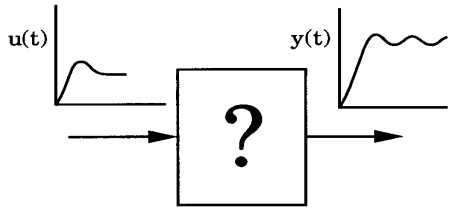


Fig. 1.1.1 Process Identification

Linear or Nonlinear Process Identification

For continuous systems, the result of input/output process identification is the linear transfer function models. However, it is also possible to obtain nonlinear models for nonlinear systems by process identification.

In process control practice, it is assumed that the process to be identified is approximately linear in some neighborhood of the operating steady-state; as such, the linear transfer function model identified for such a process is considered adequate. We will be concerned exclusively with *linear* process identification.

"Off-line" or "On-line" Identification

Process identification involves model construction from experimental data, it can be carried out "on-line," or "off-line."

On-line identification: Identification is done during normal process operation, using data obtained under these conditions.

Off-line identification: The identification is done by suspending "normal" process operation and perform experiments specifically designed for empirical model development.

We will be concerned with "off-line" identification, because it is more frequently practiced.

Open-loop and Closed-loop Identification

There are two ways to identify the parameters of a process for control application, i.e. open-loop and closed-loop ones. The open loop identification is to identify process parameter for system without control, while closed loop identification is to identify process parameter for system under control. In any case, an excitation of the process is needed to extract useful information on process dynamics. For open-loop transient response experiments, step or pulse excitation signals are commonly injected at the process inputs, and the responses are measured. The main advantage of these tests is that the testing procedure is simple and requires little prior knowledge. However, it is quite sensitive to non-linearity in the system. For closed-loop identification, a majority of existing techniques are in the frequency domain while the frequency range of interest for such applications is usually from zero up to the process critical frequency. Since the closed-loop testing causes less perturbation to the process, it is preferred to open-loop one in process control practice.

Continuous or Discrete Model Identification

Process measurements are almost always available as discrete data obtained at discrete points in time. Both continuous as well as discrete models can be identified from such data.

In "continuous" model identification, the data set is regarded appropriately as sampled information from an otherwise continuous signal: the task is to identify an appropriate *continuous* model generating a *continuous* signal that when sampled matches closely the observed *discrete* data.

Example: continuous system transfer function

$$E(s) = \frac{5}{s(s+1)(s+5)}$$

In "discrete" model identification, the discrete data set is taken at "face value" and utilized directly to identify a discrete empirical process model.

Example: discrete system transfer function

$$E(z) = \frac{z}{(z-1)} - \frac{1.25z}{(z-e^{-T})} + \frac{0.25z}{(z-e^{-5T})}$$

We will restrict to "continuous" model identification.

Theoretical Modeling versus Process Identification

Comparison between theoretical modeling versus process identification is made in the Table 1.1.1

Table 1.1.1 Comparison between theoretical modeling versus process identification

	THEORETICAL MODELING	PROCESS IDENTIFICATION
1.	Involves fewer measurement; requires experimentation only for the estimation of unknown model parameters.	Requires extensive measurements, because it relies entirely on experimentation for the model development.
2.	Provides information about the internal state of the process.	Provides information only about that portion of the process which can be influenced by control

	action and that portion can be measured.
3. Promotes fundamental understanding of the internal workings of the process.	Treats the process like a "black box."
4. Requires fairly accurate and complete process knowledge.	Requires no detailed knowledge; only that output data be obtainable in response to input changes.
5. Not particularly useful for poorly understood and/or complex processes.	Quite often proves to be the only alternative for modeling the behavior of poorly understood and/or complex processes.
6. Naturally produces both linear and nonlinear process models.	Requires special methods to produce nonlinear models.

1.1.2 PRINCIPLES OF EMPIRICAL MODELING

The basic principle for the development of empirical models by process identification may be given as follows:

> *Given u(t), the input to a dynamic process, and y(t), the process response to this input function, what is g(·), the process transfer function model?*

This is a converse of the process dynamics problem which may be stated: given *g(·)* and *u(t),* obtain *y(t).*

The process dynamics problem is more straightforward than the process identification problem; however, the fact that one is a converse of the other will be useful.

The process of constructing a working empirical model by process identification consists of the following stages:

1. Problem Definition
2. Model Formulation
3. Parameter Estimation
4. Model Validation

The difference lies in the specific issues involved at each stage.

Problem Definition

The scope of the problem we wish to solve is:

1. It is impossible to represent all aspect of the physical process, we can only hope that we can capture those aspects that most relevant to the problem at hand.
2. The behavior of a dynamic process can be interpreted mathematically in several different ways, resulting different models, all might attempt to capture the same aspect of the process, but from various angles, and to vary degree of complexity.
3. A process model is as useful as the tool available for obtaining solution to its equations, some modeling equation can be solved analytically, while some can only be numerical methods using a computer.

Before developing a mathematical model for a physical process, a number of questions should first be answered

1. What do we intend to use the model for?
2. How simple or complex, will the model have to be?
3. Which aspects of the process do we consider the most relevant and should be contained in such a process model?
4. To what extents are the fundamental principles underline the operation of this aspect of the process known?
5. How can we test the adequacy of the model?
6. How much time do we have for the modeling exercise?

Model Formulation

In process identification, this step involves which models are to be considered as possible candidates. In theoretical modeling, it involves the application of fundamental conservation laws, in process identification, it involves analysis of the input/output data to detect which model form is likely to be capable of explaining the observed behavior.

Process Data Files

Process data files provide the specific bases for model development of specific processes.

Since the model obtained by process identification is to be based entirely on experimental data, it is important to collect as much data in the proper range of operating conditions — as is necessary to "identify" those aspects of the process we consider important.

Example: to identify properly the steady-state gain of a process from *step-response* data, it is advantageous to collect data until the process is well established at the new steady state; if the data collection is stopped *before* the new steady state is achieved, the steady-state gain will be difficult to identify accurately.

Choice of Input Functions

As the useful process information content of the output data is dependent on the nature of the input function. It is important to choose an input function capable of providing output that is rich in useful process information and is easily extracted.

The typical input functions used in process identification are:

- Step#
- Impulse#
- Pulse (rectangular or arbitrary)#
- Sine waves#
- Relay#
- White noise
- Pseudo random binary sequences

The Role of Process Dynamics

In process dynamics studies, the fundamental problem is that of generating the responses of a

system to various types of inputs, using an available process model; by studying such responses we are then able to characterize these systems.

In process identification we have a reverse problem: that of identifying a process model from its response to certain input functions. It is therefore easy to see how results of process dynamics studies will be very useful.

A familiarity with the qualitative shape of the response of various classes of process systems will make it easy to propose candidate process models from input/output data records.

Example: because the *ideal* step response of a first-order system is well known, it is possible to tell if a physical process may be approximately modeled as a first-order system or not, simply by examining its experimental step response. #

Typical Candidate Process Models

For most processes, the following is the collection of typical "continuous-time" candidate models:

1. Three parameter model
   - First-order-plus-time-delay

$$G_p(s) = \frac{K}{Ts+1} e^{-Ls}$$

   - Second-order-plus-time-delay with same poles

$$G_p(s) = \frac{K}{(Ts+1)^2} e^{-Ls}$$

2. Four parameter model
   - Second-order-plus-time-delay with two real poles

$$G_p(s) = \frac{Ke^{-Ls}}{(\tau_1 s+1)(\tau_2 s+1)}$$

   - Second-order-plus-time-delay with arbitrary poles

$$G_p(s) = \frac{e^{-sL}}{as^2+bs+c}$$

3. Five parameter model
   - Single zero, two real poles, plus time delay

$$G_p(s) = \frac{K(\xi s+1)e^{-Ls}}{(\tau_1 s+1)(\tau_2 s+1)}$$

   - Second-order-plus-time-delay with arbitrary poles and a zero

$$G_p(s) = \frac{(\xi s+1)e^{-sL}}{as^2+bs+c}$$

In many industrial situations, the process identification is often reduced to that of fitting the experimental data to one particular, pre-specified type of model; the model form is determined *a priori,* regardless of what the data might indicate.

Example: for controller tuning purposes, it is sometimes recommended to fit a first-order-plus-time-delay model to process data even though the response usually indicates a slightly nonlinear, possibly higher order behavior. In such cases, the process identification problem reduces to finding the first-order-plus-time-delay model that provides the "best" fit to the experimental data. #

Parameter Estimation

Once the candidate model has been selected, the next task is to estimate its unknown parameters. In classical process identification, it is of fitting the experimental data to a predetermined model form by finding the parameter values which provide the "best" fit.

Estimating unknown parameters in an empirical process model can be done in the time domain, or in the frequency domain.

Model Validation

As with theoretical modeling, the final step in process identification involves checking how the empirical model fits the data it is supposed to represent. Model validation is usually done by comparing model predictions with additional process data, and evaluating the fit. This can also be carried out either in the time domain, or in the frequency domain.

The overall steps in the process of Process Identification are illustrated in Figure 1.1.2.

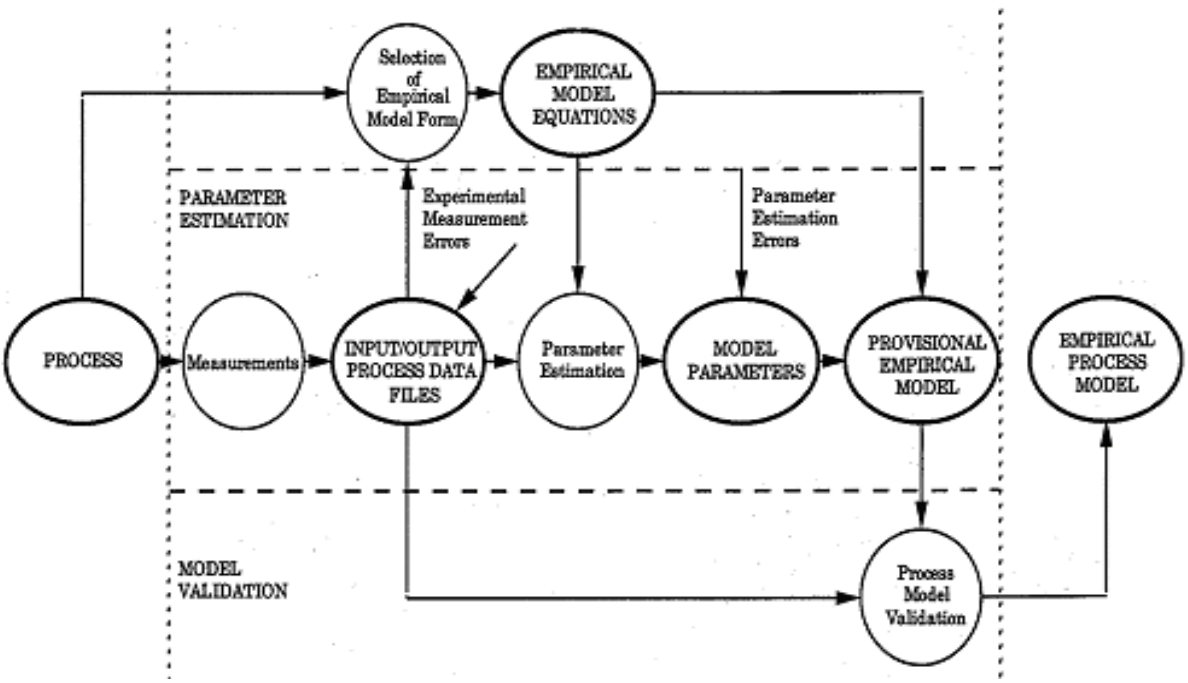


Fig. 1.1.2 Procedures of Process Identification

1.2 LEAST SQUARES METHODS

The fundamental concept of least squares methods is to estimate parameters of a model that predicts dependent variable outputs of a function, process or phenomenon based on observations of independent variable inputs. For a given data observation, independent variable inputs and dependent variable outputs are grouped into input and output vectors. The least squares optimality criterion minimizes the sum of squares of residuals between observed outputs and output values predicted from input observations.

1.2.1 LINEAR METHODS

The regression model:

$$y(t) = \varphi_1(t)\theta_1 + \cdots + \varphi_n(t)\theta_n = \varphi(t)^T\theta \tag{1.2.1}$$

where

y - observed data

θ_i - unknown parameters

φ_i - known regression variables

t - integer variable

$$\varphi(t) = \begin{bmatrix}\varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t)\end{bmatrix}^T$$
$$\theta = \begin{bmatrix}\theta_1 & \theta_2 & \cdots & \theta_n\end{bmatrix}$$

Assume that *N* measurements $\{y(t)\}$, $\varphi(t)$ } are available and define the equation error as

$$\varepsilon(t) = y(t) - \varphi(t)^T\theta \tag{1.2.2}$$

For all the measurements *t* = 1...*N* a linear set of equations can be obtained:

$$E = Y - \Phi\theta \tag{1.2.3}$$

where

$$Y = \begin{bmatrix}y(1) & y(2) & \cdots & y(N)\end{bmatrix}^T$$
$$\Phi = \begin{bmatrix}\varphi(1) & \varphi(2) & \cdots & \varphi(N)\end{bmatrix}^T$$
$$E = \begin{bmatrix}\varepsilon(1) & \varepsilon(2) & \cdots & \varepsilon(N)\end{bmatrix}^T$$

$\dim\theta = n\times 1,\ \dim\varphi = n\times 1,\ \dim Y = N\times 1,\ \dim\Phi = N\times n,\ \dim E = N\times 1$

The least squares estimate of θ is the vector θ which minimises:

$$J(\theta) = \frac{1}{2}\sum_{k=1}^{N}\varepsilon(k)^2 = \frac{1}{2}E^TE \tag{1.2.4}$$

The least-squares estimate of θ, can be found by minimizing J(θ).

$$\left.\frac{\partial J(\hat{\theta})}{\partial\hat{\theta}}\right|_{\theta=\hat{\theta}_{LS}} = 0$$

Before we proceed in determining the lest-squares estimate θ, the following lemma is required:

For any square matrix A, one has the following derivatives:

$$\frac{\partial}{\partial x}(x^T Az) = Az$$
$$\frac{\partial}{\partial x}(z^T Ax) = A^T z$$
$$\frac{\partial}{\partial x}(x^T Ax) = (A + A^T)x$$

Using the above lemma, we take the derivative of J w.r.t. θ and set it to zero

$$\frac{\partial J(\theta)}{\partial \theta} = E^T \frac{\partial E}{\partial \theta} = [Y - \Phi\theta]^T[-\Phi] = -Y^T\Phi + \theta^T(\Phi^T\Phi) = 0$$

which can be written as

$$(\Phi^T\Phi)\theta = \Phi^T Y$$

So, if matrix $\Phi^T\Phi$ is nonsingular the solution is given by

$$\theta = (\Phi^T\Phi)^{-1}\Phi^T Y \tag{1.2.5}$$

The above solution may be rewritten as:

$$\theta = \left[\sum_{t=1}^{N} \varphi(t)\varphi(t)^T\right]^{-1}\left[\sum_{t=1}^{N} \varphi(t)y(t)\right] \tag{1.2.6}$$

Note: The matrix $\Phi^T\Phi$ needs to be invertible. This is called an excitation condition. The input signal $\{u(t)\}$ needs to be such that the observed data is sufficiently rich in information to enable us to estimate the model parameters.

Application Examples

Straight Line

The classic least squares problem is to fit a straight line model with a single input x and a single output y using parameters b and m.

$$y = b + mx = [1 \;\; x]\begin{Bmatrix} b \\ m \end{Bmatrix} \quad \text{for } n_{in} = 1, \;\; n_{out} = 1, \;\; n_{par} = 1$$

Unfortunately, individual data observations x_i and y_i may not fit the model perfectly due to experimental measurement error, process variation or insufficient model complexity.

$$y_i \approx [\,1 \;\; x_i \,]\begin{Bmatrix} b \\ m \end{Bmatrix}$$

Multiple data observations may be concatenated and represented in matrix form.

$$\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{nobs} \end{Bmatrix} \approx \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{nobs} \end{bmatrix} \begin{Bmatrix} b \\ m \end{Bmatrix}$$

$$Y \approx \Phi\theta \qquad \text{for} \quad \underset{N\times 1}{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix} \qquad \underset{N\times 2}{\Phi} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} \qquad \underset{2\,x\,1}{\theta} = \begin{Bmatrix} b \\ m \end{Bmatrix}$$

Even for optimal estimates of model parameters, each data observation will have some residual error e_i between the observed output y_i and the predicted model output ($b + mx_i$).

$$\begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix} - \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} \begin{Bmatrix} b \\ m \end{Bmatrix}$$

$$E = Y - \Phi\theta \qquad \text{for} \quad \underset{N\times 1}{E} = \begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{Bmatrix}$$

For given N observations, the Least Squares for the problem is given as

$$\underset{2\times 1}{\theta} = (\underset{2\times N}{\Phi^T}\ \underset{N\times 2}{\Phi})^{-1}\underset{2\times N}{\Phi^T}\ \underset{N\times 1}{Y}$$

Expanding this matrix solution, we obtain

$$\begin{Bmatrix} b \\ m \end{Bmatrix} = \begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix}^{-1} \begin{Bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i y_i \end{Bmatrix}$$

If the input observations x_i and the output observations y_i are both mean centered, then the offset term b is zero and the slope m is equal to the covariance of x_i and y_i divided by the variance of x_i. Consequently, mean centered data preclude the need to compute an offset term for many linear models.

$$\begin{Bmatrix} b \\ m \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & \sum_{i=1}^{N} x_i^2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ \sum_{i=1}^{N} x_i y_i \end{Bmatrix} \quad \text{for } x_0 = \tfrac{1}{N}\sum_{i=1}^{N} x_i = 0 \quad \text{and} \quad y_0 = \tfrac{1}{N}\sum_{i=1}^{N} y_i = 0$$

$$\begin{Bmatrix} b \\ m \end{Bmatrix} = \begin{Bmatrix} 0 \\ \sum_{i=1}^{N} x_i y_i \Big/ \sum_{i=1}^{N} x_i^2 \end{Bmatrix} \quad \text{for } x_0 = 0, \quad y_0 = 0$$

Cubic

A cubic model is presented below

$$y = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = [1 \;\; x_1 \;\; x_2 \;\; x_3] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \text{for } n_{in} = 1,\ n_{out} = 1,\ n_{par} = 4$$

$$\underset{N\,x\,1}{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix} \quad \underset{N\times 4}{\Phi} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & x_N^3 \end{bmatrix} \quad \underset{4\,x\,1}{\theta} = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\underset{4\times 1}{\theta} = (\underset{4\times N}{\Phi^T}\ \underset{N\times 4}{\Phi})^{-1} \underset{4\times N}{\Phi^T}\ \underset{N\times 1}{Y}$$

Harmonic

A linear harmonic model is presented below

$$y = [1 \;\; \cos\theta \;\; \sin\theta \;\; \cos 2\theta \;\; \sin 2\theta] \begin{Bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{Bmatrix} \quad \text{for } n_{in} = 1,\ n_{out} = 1,\ n_{par} = 5$$

$$\underset{N\times 1}{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix} \quad \underset{N\times 5}{\Phi} = \begin{bmatrix} 1 & \cos\theta_1 & \sin\theta_1 & \cos 2\theta_1 & \sin 2\theta_1 \\ 1 & \cos\theta_2 & \sin\theta_2 & \cos 2\theta_2 & \sin 2\theta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cos\theta_N & \sin\theta_N & \cos 2\theta_N & \sin 2\theta_N \end{bmatrix} \quad \underset{5\,x\,1}{\theta} = \begin{Bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{Bmatrix}$$

$$\underset{5\times 1}{\theta} = (\underset{5\times N}{\Phi^T}\ \underset{N\times 5}{\Phi})^{-1} \underset{5\times N}{\Phi^T}\ \underset{N\times 1}{Y}$$

Plane

A linear planar model is presented below.

$$z = a + bx + cy = [1 \;\; x \;\; y]\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \quad \text{for } n_{in} = 2,\ n_{out} = 1,\ n_{par} = 3$$

$$\underset{N\times 1}{Y} = \begin{Bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{Bmatrix} \quad \underset{N\times 3}{\Phi} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_N & y_N \end{bmatrix} \quad \underset{3\times 1}{\theta} = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\underset{3\times 1}{\theta} = (\underset{3\times N}{\Phi^T}\ \underset{N\times 3}{\Phi})^{-1}\underset{3\times N}{[X]^T}\ \underset{N\times 1}{Y}$$

1.2.2 LINEARIZED MODELS

It is often tempting to linearize models of nonlinear phenomena in an effort to arrive at estimates for parameters of truly nonlinear models. One should exercise caution however in realizing that linear least squares solutions for linearized models minimize the sum of squares of residuals for nonlinear forms of the dependent variables rather than residuals of those dependent variables themselves.

Newtonian Cooling

A classic example is to model the temperature T of an object as a function of time τ during exponential Newtonian cooling from an initial temperature T_0 toward ambient temperature T_∞ as

$$T = T_\infty + (T_0 - T_\infty)e^{-b\tau}$$

Taking the logarithm after a simple algebraic rearrangement provides the linearized model

$$\ln(T - T_\infty) \;=\; \ln(T_0 - T_\infty) - b\tau = [1 \quad \tau]\begin{Bmatrix} \ln(T_0 - T_\infty) \\ -b \end{Bmatrix}$$

If the ambient temperature T_∞ is known, measuring temperatures T_i of the object at times τ_i allows direct estimation of the Newtonian cooling rate b and indirect estimation the initial temperature difference above ambient from $ln(T_0 - T_\infty)$ with the linearized matrix model. Specifically, this approach minimizes residuals of the nonlinear function $ln(T - T_\infty)$ rather than residuals of the dependent variable temperature T.

$$\underset{N\times 1}{Y} = \begin{Bmatrix} \ln(T_1 - T_\infty) \\ \ln(T_2 - T_\infty) \\ \vdots \\ \ln(T_N - T_\infty) \end{Bmatrix} \quad \underset{N\times 2}{\Phi} = \begin{bmatrix} 1 & \tau_1 \\ 1 & \tau_2 \\ \vdots & \vdots \\ 1 & \tau_N \end{bmatrix} \quad \underset{2\times 1}{\theta} = \begin{Bmatrix} \ln(T_0 - T_\infty) \\ -b \end{Bmatrix}$$

$$\underset{2\times 1}{\theta} = (\underset{2\times N}{\Phi^T}\ \underset{N\times 2}{\Phi})^{-1}\underset{2\times N}{\Phi^T}\ \underset{N\times 1}{Y}$$

Circle

A second example is to estimate the coordinates a and b for the center of a circle of radius r by measuring the coordinates of points x and y on the circle as shown in the familiar nonlinear model.

$$(x-a)^2+(y-b)^2=r^2$$

Expanding and rearranging provides a linearized model

$$(x^2\ +\ y^2)=[1\ \ x\ \ y]\begin{Bmatrix} r^2-a^2-b^2 \\ 2a \\ 2b \end{Bmatrix}$$

Measuring coordinates x_i and y_i for points on the circle allows direct estimation of the center coordinates a and b and indirect estimation of the radius from $(r^2-a^2-b^2)$ with the linearized model. This approach minimizes residuals of the nonlinear function (x^2+y^2) rather than residuals for the independent variables x and y. Additionally, there is no actual dependent variable for this model and other models that fit coordinate data to geometric structures.

$$\underset{N\times 1}{Y}=\begin{Bmatrix} (x_i^2+y_i^2) \\ (x_2^2+y_2^2) \\ \vdots \\ (x_N^2+y_N^2) \end{Bmatrix} \quad \underset{N\times 3}{\Phi}=\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_N & y_N \end{bmatrix} \quad \underset{3x1}{\theta}=\begin{Bmatrix} r^2-a^2-b^2 \\ 2a \\ 2b \end{Bmatrix}$$

$$\underset{3\times 1}{\theta}=(\underset{3\times N}{\Phi^T}\ \underset{N\times 3}{\Phi})^{-1}\underset{3\times N}{\Phi^T}\ \underset{N\times 1}{Y}$$

1.2.3 WEIGHTED LEAST SQUARES

The weighted least squares solution is given by

$$\hat{\theta}=(\Phi^T W\Phi)^{-1}\Phi^T Wy \tag{1.2.7}$$

where W is positive definite and symmetric ($m\times m$). W may be used to assign different costs to each of the errors $(y-\Phi\hat{\theta})_i$. If $W = I$, it is equal weighting. If one desires to match certain data points more closely than others, a weighting coefficient can be assigned to each term in the sum to be minimized. It can be determined by minimizing the quadratic cost function

$$J(\hat{\theta})=\frac{1}{2}(y-\Phi\hat{\theta})^T W(y-\Phi\hat{\theta})$$

1.2.4 RECURSIVE LEAST SQUARES

The least squares estimate for a SISO system is, given a sequence of {1....t} observations:

$$\hat{\theta}=\left[\sum_{k=1}^{t}\varphi(k)\varphi(k)^T\right]^{-1}\left[\sum_{k=1}^{t}\varphi(k)y(k)\right] \tag{1.2.8}$$

Define $P(t)$ as:

$$P(t)=\left[\sum_{k=1}^{t}\varphi(k)\varphi(k)^T\right]^{-1} \tag{1.2.9}$$

Then

$$P(t)^{-1} = \sum_{k=1}^{t} \varphi(k)\varphi(k)^T = \sum_{k=1}^{t-1} \varphi(k)\varphi(k)^T + \varphi(t)\varphi(t)^T = P(t-1)^{-1} + \varphi(t)\varphi(t)^T \tag{1.2.10}$$

rewritten Equation (1.2.8) as

$$\hat{\theta} = P(t)\left[\sum_{k=1}^{t} \varphi(k)y(k)\right] = P(t)\left[\sum_{k=1}^{t-1} \varphi(k)y(k) + \varphi(t)y(t)\right] \tag{1.2.11}$$

But we have the following:

$$\sum_{k=1}^{t-1} \varphi(k)y(k) = P(t-1)^{-1}\hat{\theta}(t-1) = \left[P(t)^{-1} - \varphi(t)\varphi(t)^T\right]\hat{\theta}(t-1) \tag{1.2.12}$$

Substituting (1.2.12) into (1.2.11), $\hat{\theta}$ can be rewritten as

$$\begin{aligned}
\hat{\theta} &= P(t)\left[P(t)^{-1}\hat{\theta}(t-1) - \varphi(t)\varphi(t)^T\hat{\theta}(t-1) + \varphi(t)y(t)\right] \\
&= \hat{\theta}(t-1) - P(t)\varphi(t)\varphi(t)^T\hat{\theta}(t-1) + P(t)\varphi(t)y(t) \\
&= \hat{\theta}(t-1) - P(t)\varphi(t)\left[y(t) - \varphi(t)^T\hat{\theta}(t-1)\right] \\
&= \hat{\theta}(t-1) + K(t)\varepsilon(t)
\end{aligned} \tag{1.2.13}$$

where

$$K(t) = P(t)\varphi(t), \qquad \varepsilon(t) = y(t) - \varphi(t)^T\hat{\theta}(t-1)$$

Now introducing the Matrix Inversion Lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

and apply it on the right hand side of Equation (1.2.10), with $A = P(t)^{-1}$, $B = \varphi(t)$, $C = I$ and $D = \varphi(t)^T$, we obtain

$$P(t) = P(t-1) - P(t-1)\varphi(t)\left[I + \varphi(t)^T P(t-1)\varphi(t)\right]\varphi(t)P(t-1) \tag{1.2.14}$$

Which gives:

$$K(t) = P(t)\varphi(t) = P(t-1)\varphi(t)\left[I + \varphi(t)^T P(t-1)\varphi(t)\right]^{-1} \tag{1.2.15}$$

Summary - Recursive Least Squares

$$\begin{aligned}
\hat{\theta} &= \hat{\theta}(t-1) + K(t)\varepsilon(t) \\
\varepsilon(t) &= y(t) - \varphi(t)^T\hat{\theta}(t-1) \\
K(t) &= P(t-1)\varphi(t)\left[I + \varphi(t)^T P(t-1)\varphi(t)\right]^{-1} \\
P(t) &= \left[I - K(t)\varphi(t)^T\right]P(t-1)
\end{aligned} \tag{1.2.16}$$

Initial values $\theta(0) = \theta_0$, $P(0) = rI$, where *r* should be large.

The Equations (1.2.16) have a strong intuitive appeal. The current parameter estimate $\theta(t)$ is

obtained by adding a correction to the previous estimate $\theta(t-1)$. The correction is proportional to the one-step-ahead prediction error $\sum(t)$, which is based on the previous parameter estimate.

The matrix $P(t)$ may be interpreted as the covariance of the parameter vector. Its magnitude gives a measure of the uncertainty of the parameter values.

1.2.5 EXPONENTIAL FORGETTING LEAST SQUARES METHOD

In the case of systems with slowly varying parameters it is useful to give more emphasis to recently observed data than to older data.

One approach to do this is to modify the LS criterion:

$$J(\theta,t)=\frac{1}{2}\sum_{k=1}^{t}\lambda^{t-k}\left(Y(k)-\Phi(k)^T\theta\right)^2 \tag{1.2.17}$$

where λ $(0<\lambda<1)$ is called the forgetting factor. The recursive least squares estimate becomes:

$$\begin{aligned}\hat{\theta}&=\hat{\theta}(t-1)+K(t)\varepsilon(t)\\ \varepsilon(t)&=y(t)-\varphi(t)^T\hat{\theta}(t-1)\\ K(t)&=P(t-1)\varphi(t)\left[\lambda I+\varphi(t)^T P(t-1)\varphi(t)\right]^{-1}\\ P(t)&=\left[I-K(t)\varphi(t)^T\right]P(t-1)/\lambda\end{aligned} \tag{1.2.18}$$

The 'memory' of the estimator is given by:

$$T_0=\frac{2h}{1-\lambda}$$

where h is the sampling interval. Typical values of the forgetting factor are λ = 0.95 ... 0.98

1.3 FOURIER THEORY

Linear transforms, especially Fourier and Laplace transforms, are widely used in solving problems in science and engineering. The Fourier transform is used in linear systems analysis, and process modeling etc. It becomes a mathematical or physical tool to alter a problem into one that can be more easily solved.

1.3.1 THE FOURIER TRANSFORM

The *Fourier transform*, in essence, decomposes or separates a waveform or function into sinusoids of different frequency which sum to the original waveform. It identifies or distinguishes the different frequency sinusoids and their respective amplitudes. The Fourier transform of *f(x)* is defined as

$$F(s)=\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-i2\pi xs}\,dx \tag{1.3.1a}$$

Applying the same transform to *F(s)* gives

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{i2\pi xs}\, ds \tag{1.3.1b}$$

so that

$$f(x) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}dx\right]e^{i2\pi xs}ds \tag{1.3.1c}$$

as long as the integral exists and any discontinuities, usually represented by multiple integrals of the form ½[*f(x+) + f(x-)*], are finite. The transform quantity *F(s)* is often represented as $\tilde{f}(s)$ and the Fourier transform is often represented by the operator $\mathfrak{F}$.

Usually functions or waveforms can be split into even and odd parts as follows

$$f(x) = E(x) + O(x) \tag{1.3.2a}$$

where

$$E(x) = \frac{[f(x) + f(-x)]}{2}$$
$$O(x) = \frac{[f(x) - f(-x)]}{2} \tag{1.3.2b}$$

and *E(x), O(x)* are, in general, complex. In this representation, the Fourier transform of *f(x)* reduces to

$$F(s) = 2\int_0^{\infty} E(x)\cos(2\pi xs)dx - 2i\int_0^{\infty} O(x)\sin(2\pi xs)dx$$

It follows then that an even function has an even transform and that an odd function has an odd transform. Additional symmetry properties are shown in Table 1.3.1.

Table 1.3.1: Symmetry Properties of the Fourier Transform

Function	Transform
Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Complex and even	Complex and even
Complex and odd	Complex and odd
Real and asymmetrical	Complex and asymmetrical
Imaginary and asymmetrical	Complex and asymmetrical
Real even plus imaginary odd	Real
Real odd plus imaginary even	Imaginary
Even	Even
Odd	Odd

The *cosine transform* of a function *f(x)* is defined as

$$F_c(s) = 2\int_0^{\infty} f(x)\cos 2\pi sx dx \tag{1.3.3a}$$

This is equivalent to the Fourier transform if *f(x)* is an even function. In general, the even part of the Fourier transform of *f(x)* is the cosine transform of the even part of *f(x)*. The cosine transform has a reverse transform given by

$$f(x) = 2\int_0^\infty F_c(s)\cos 2\pi sxds \tag{1.3.3b}$$

Likewise, the *sine transform* of *f(x)* is defined by

$$\begin{aligned} F_s(s) &= 2\int_0^\infty f(x)\sin 2\pi sxdx \\ f(x) &= 2\int_0^\infty F_s(s)\sin 2\pi sxds \end{aligned} \tag{1.3.4}$$

As a result, *i* times the odd part of the Fourier transform of *f(x)* is the sine transform of the odd part of *f(x)*. Combining the sine and cosine transforms of the even and odd parts of *f(x)* leads to the Fourier transform of the whole of *f(x)*:

$$\mathfrak{F}\, f(x) = \mathfrak{F}_C E(x) - i\mathfrak{F}_S O(x)$$

where $\mathfrak{F}$, $\mathfrak{F}_C$, and $\mathfrak{F}_S$ stand for -*i* times the Fourier transform, the cosine transform, and the sine transform respectively, or

$$F(s) = \frac{F_c(s) - iF_s(s)}{2}$$

Since the Fourier transform *F(s)* is a frequency domain representation of a function *f(x)*, the *s* characterizes the frequency of the decomposed cosinusoids and sinusoids and is equal to the number of cycles per unit of *x*. If a function or waveform is not periodic, then the Fourier transform of the function will be a continuous function of frequency.

1.3.2 FOURIER TRANSFORM PROPERTIES

Scaling Property

If $\mathfrak{F}\{f(x)\} = F(s)$ and *a* is a real, nonzero constant, then

$$\begin{aligned} \mathfrak{F}\{f(ax)\} &= \int_{-\infty}^{\infty} f(ax)\,\mathrm{e}^{-i2\pi sx}\,dx \\ &= |a|^{-1}\int_{-\infty}^{\infty} f(\beta)\,\mathrm{e}^{-i2\pi s\beta/a}\,d\beta \\ &= |a|^{-1} F(s/a) \end{aligned} \tag{1.3.5a}$$

From this time scaling property, it is evident that if the width of a function is decreased while its height is kept constant, then its Fourier transform becomes wider and shorter. If its width is increased, its transform becomes narrower and taller.

A similar frequency scaling property is given by

$$\mathfrak{F}\{|a|^{-1} f(x/a)\} = F(as) \tag{1.3.5b}$$

Shifting Property

If $\mathfrak{F}\{f(x)\} = F(s)$ and x_0 is a real constant, then

$$\begin{aligned}\mathcal{F}\{f(x-x_0)\} &= \int_{-\infty}^{\infty} f(x-x0)e^{-i2\pi sx}dx \\ &= \int_{-\infty}^{\infty} f(\beta)e^{-i2\pi s(\beta+x_0)}d\beta \\ &= e^{-i2\pi sx_0}\int_{-\infty}^{\infty} f(\beta)e^{-i2\pi s\beta}d\beta \\ &= F(s)e^{-i2\pi sx_0}\end{aligned} \tag{1.3.6a}$$

This *time shifting property* states that the Fourier transform of a shifted function is just the transform of the unshifted function multiplied by an exponential factor having a linear phase.

Likewise, the frequency shifting property states that if *F(s)* is shifted by a constant s_0, its inverse transform is multiplied by $\exp(i2\pi s_0 x)$

$$\mathcal{F}\{f(x)e^{i2xs_0}\} = F(s-s_0) \tag{1.3.6b}$$

Convolution Theorem

We now derive the aforementioned *time convolution theorem*. Suppose that , $g(x) = f(x) \otimes h(x)$. Then, given that $\mathcal{F}\{g(x)\} = G(s)$, $\mathcal{F}\{f(x)\} = F(s)$, and $\mathcal{F}\{h(x)\} = H(s)$

$$\begin{aligned}G(s) &= \mathcal{F}\{f(x) \otimes h(x)\} \\ &= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(\beta)h(x-\beta)d\beta\right\} \\ &= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\beta)h(x-\beta)d\beta\right\}e^{-i2\pi sx}dx \\ &= \int_{-\infty}^{\infty} f(\beta)\left\{\int_{-\infty}^{\infty} h(x-\beta)e^{-i2\pi sx}dx\right\}d\beta \\ &= H(s)\int_{-\infty}^{\infty} f(\beta)e^{-i2\pi s\beta}d\beta \\ &= H(s)F(s)\end{aligned}$$

This extremely powerful result demonstrates that the Fourier transform of a convolution is simply given by the product of the individual transforms, that is

$$\mathcal{F}\{f(x) \otimes g(x)\} = F(s)G(s) \tag{1.3.7a}$$

Using a similar derivation, it can be shown that the Fourier transform of a product is given by the convolution of the individual transforms, that is

$$\mathcal{F}\{f(x)g(x)\} = F(s) \otimes G(s) \tag{1.3.7b}$$

This is the statement of the *frequency convolution theorem*.

Correlation Theorem

The correlation integral, like the convolution integral, is important in theoretical and practical applications. The correction integral is defined as

$$h(x) = \int_{-\infty}^{\infty} f(u)g(x+u)du$$

and like the convolution integral, it forms a Fourier transform pair given by

$$\mathcal{F}\{h(x)\} = F(s)\, G^*(s) \tag{1.3.8}$$

This is the statement of the correlation theorem. If f(x) and g(x) are the same function, the integral above is normally called the autocorrelation function, and the crosscorrelation if they differ. The Fourier transform pair for the autocorrelation is simply

$$\mathcal{F}\{\int_{-\infty}^{\infty} f(u)\, f(x+u)du\} = |F|^2 \tag{1.3.9}$$

Parseval's Theorem

Parseval's Theorem states that the power of a signal represented by a function *h(t)* is the same whether computed in signal space or frequency (transform) space; that is,

$$\int_{-\infty}^{\infty} h^2(t)dt = \int_{-\infty}^{\infty} H^2(f)df \tag{1.3.10}$$

The *power spectrum, P(f),* is given by

$$P(f) = |H(f)|^2, \quad -\infty < f < \infty \tag{1.3.11}$$

Sampling Theorem

Bandlimited signal: *f(t),* has no spectral components beyond a frequency *B* Hz; that is, $F(s) = 0$, for $|s| > 2\pi B$.

Sampling theorem: a real signal, *f(t),* which is bandlimited to *B* Hz can be reconstructed without error from samples taken uniformly at a rate *R > 2B* samples per second.

This minimum sampling frequency, $F_s = 2B\ Hz$, is called the ***Nyquist rate*** or the ***Nyquist frequency***.

The corresponding sampling interval, $T = 1/2B$ (where *t = nT*), is called the ***Nyquist interval***.

A signal bandlimited to *B* Hz which is sampled at less than the Nyquist frequency of *2B, i.e.,* which was sampled at an interval $T > 1/2B$, is said to be ***undersampled***.

Aliasing

A number of practical difficulties are encountered in reconstructing a signal from its samples. The sampling theorem assumes that a signal is bandlimited. In practice, however, signals are timelimited, not bandlimited. As a result, determining an adequate sampling frequency which does not lose desired information can be difficult. When a signal is undersampled, its spectrum has overlapping tails; that is *F(s)* no longer has complete information about the spectrum and it is no longer possible to recover *f(t)* from the sampled signal. In this case, the tailing spectrum does not go to zero, but is folded back onto the apparent spectrum. This inversion of the tail is called *spectral folding* or *aliasing*.

Example: Figure 1.3.1 shows a unit gaussian curve sampled at three different rates. The FFT (or Fast Fourier Transform) of the undersampled gaussian appears flattened and its tails do not reach zero. This is a result of aliasing. Additional spectral components have been folded back onto the ends of the spectrum or added to the edges to produce this curve.

The FFT of the oversampled gaussian reaches zero very quickly. Much of its spectrum is zero and is not needed to reconstruct the original gaussian.

Finally, the FFT of the critically-sampled gaussian has tails which reach zero at their ends. The data in the spectrum of the critically-sampled gaussian is just sufficient to reconstruct the original. This gaussian was sampled at the Nyquist frequency.

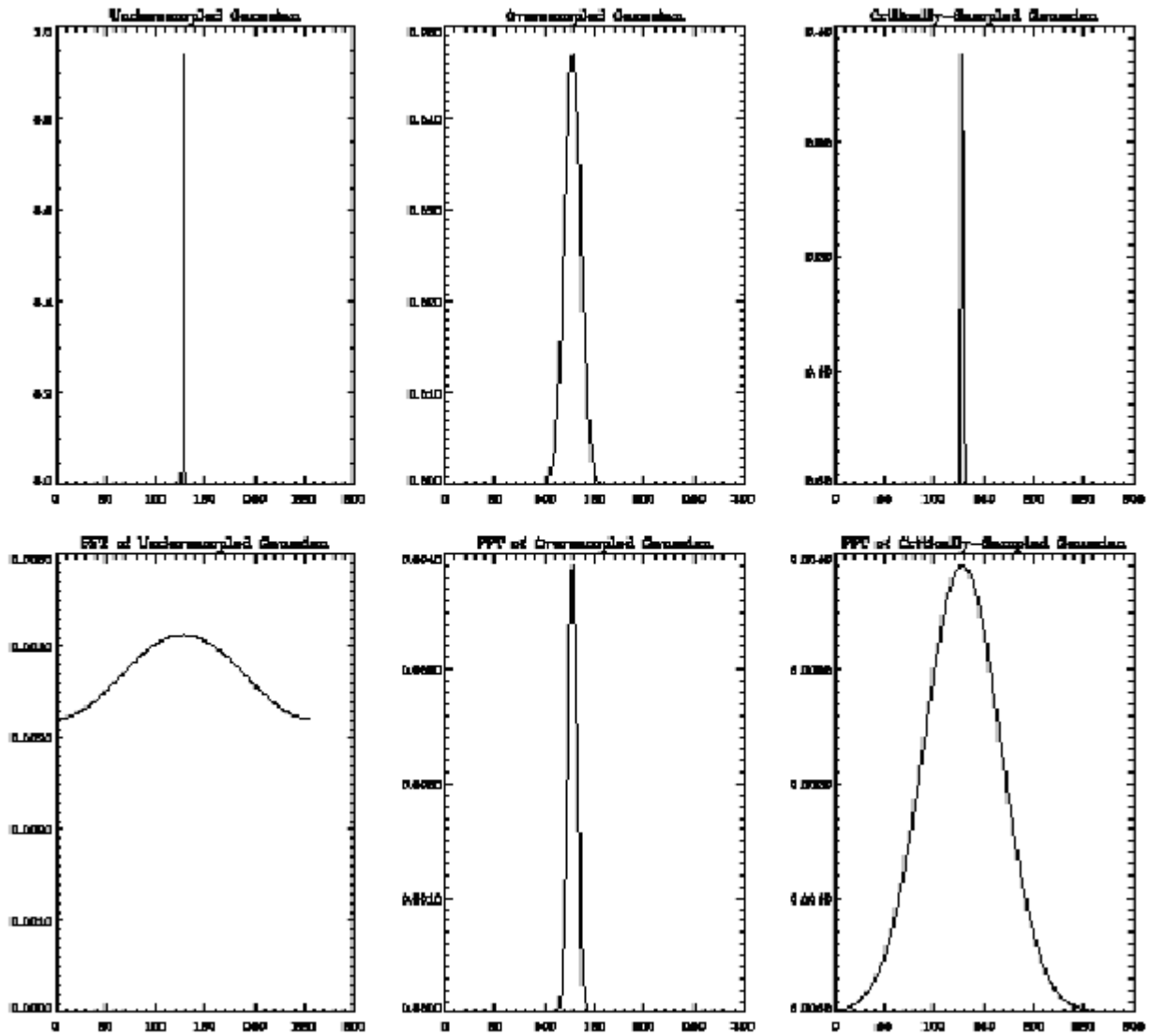


Fig. 1.3.1 Undersampled, oversampled, and critically-sampled unit area gaussian curves.

1.3.3 DISCRETE FOURIER TRANSFORM (DFT)

Because a digital computer works only with discrete data, numerical computation of the Fourier

transform of *f(t)* requires discrete sample values of *f(t)*, which we will call f_k. In addition, a computer can compute the transform *F(s)* only at discrete values of *s*, that is, it can only provide discrete samples of the transform, F_r. If *f(kT)* and $F(rs_0)$ are the *k*th and *r*th samples of *f(t)* and *F(s)*, respectively, and N_0 is the number of samples in the signal in one period T_0, then

$$f_k = Tf(kT) = \frac{T_0}{N_0} f(kT)$$

and

$$F_r = F(rs_0)$$

where

$$s_0 = 2\pi F_0 = \frac{2\pi}{T_0}$$

The *discrete Fourier transform (DFT)* is defined as

$$F_r = \sum_{r=0}^{N_0-1} f_k \, e^{-ir\Omega_0 k} \tag{1.3.12a}$$

where $\Omega_0 = 2\pi/N_0$. Its inverse is

$$f_k = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r \, e^{ir\Omega_0 k} \tag{1.3.12b}$$

These equations can be used to compute transforms and inverse transforms of appropriately sampled data.

1.3.4 FAST FOURIER TRANSFORM (FFT)

The *Fast Fourier Transform (FFT)* is a DFT algorithm developed in 1965 which reduces the number of computations from something on the order of N_0^2 to $N_0 \log N_0$. There are basically two types of Tukey-Cooley FFT algorithms in use: decimation-in-time and decimation-in-frequency. The algorithm is simplified if N_0 is chosen to be a power of 2, but it is not a requirement.

1.4 DESCRIBING FUNCTIONS

1.4.1 BASIC CONCEPTS

The Describing Function looks specifically at nonlinearities that are easily decoupled from an otherwise linear system by split the system up into a nonlinear element followed by a linear system as shown in Fig. 1.4.1. Here, only sinusoidal inputs to the NL are considered. The output of the nonlinearity, *u*, is in general, not to be sinusoidal, but can be represented as a Fourier series: ie a fundamental frequency plus harmonics. The DF is the amplitude ratio of the fundamental component of *u* to the input *e*.

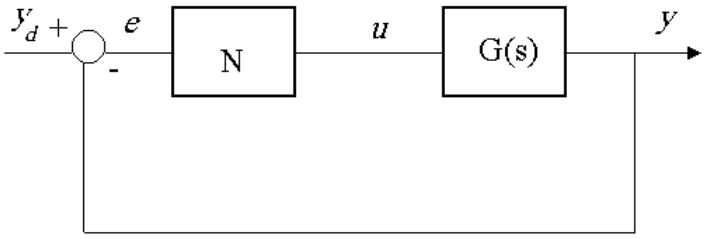


Fig. 1.4.1. Nonlinear system

The limitations/assumptions of this method are

- Only a single nonlinearity can be considered
- The NL cannot vary with time.

The technique is an extension of linear techniques, it is very effective if

- Nonlinearity is not 'too' severe
- There is a linear part of plant works as a LPF.

1.4.2 DESCRIBING FUNCTION EVALUATION

Suppose the NL input is given by

$$e(t) = E_m \sin(\omega t) \tag{1.4.1}$$

The output of the NL element is given by its Fourier series representation

$$u(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \tag{1.4.2}$$

where

$$A_n = \frac{2}{T} \int_{t_0}^{t_0+T} u(t) \cos n\omega t dt$$

$$B_n = \frac{2}{T} \int_{t_0}^{t_0+T} u(t) \sin n\omega t dt$$

Because of our assumption that the NL element is an odd function we can immediately set $A_0 = 0$, another assumption is that the linear system will filter out the higher harmonics allows us to concentrate only on the fundamental frequency of the system, i.e $A_n = B_n = 0$ for all n >1, since these harmonics will be negligible at the output anyway.

Hence,

$$u_1(t) = A_1 \cos \omega t + B_1 \sin \omega t = U_1 \sin(\omega t + \phi) \tag{1.4.3a}$$

with

$$\begin{aligned} \phi &= \tan^{-1} A_1 / B_1 \\ U_1 &= \sqrt{A_1^2 + B_1^2} \end{aligned} \tag{1.4.3b}$$

The describing function is then given as the input over the output of the NL element (similar to transfer function)

$$N(E_m,\omega)=\frac{u(t)}{e(t)}\approx\frac{U_1\sin(\omega t+\phi)}{E_m\sin\omega t}=|N|e^{j\phi} \tag{1.4.4}$$

1.4.3 APPLICATION EXAMPLES

Ideal relay

$f(y) = h\times\text{sgn}(e)$, where we assume $e(t)=a\sin(\omega t)$

Set up and evaluate the integral for the first Fourier coefficient divided by *a* as follows:

$$N_s(a)=\frac{1}{\pi a}\int_0^{2\pi} f(a\sin(x))\sin(x)dx$$

$$\frac{4h}{\pi a}\int_0^{\pi/2}\sin(x)dx \qquad \text{(by symmetry)} \tag{1.4.5a}$$

$$=\frac{4h}{\pi a}$$

$$\phi=\tan^{-1}A_1/B_1=180^0 \tag{1.4.5b}$$

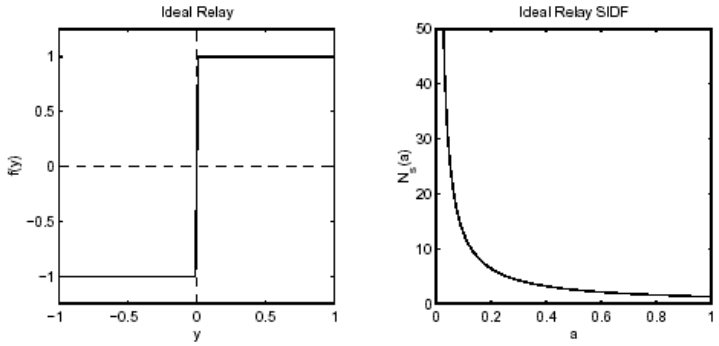


Fig 1.4.2 Ideal Ralay

Saturation function

Lets first consider a saturation function nonlinearity where we assume saturation is occurring, i.e

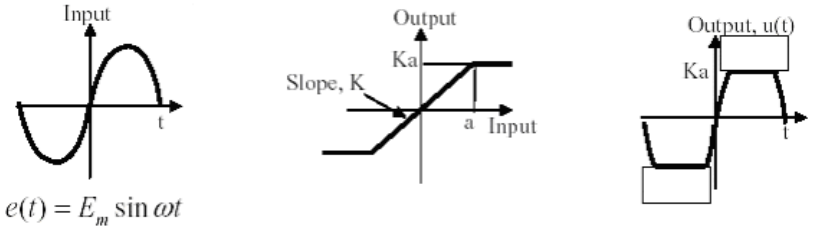
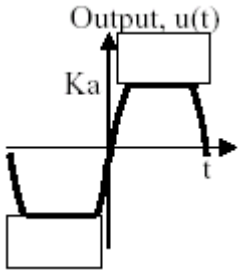


Fig 1.4.2 Saturation Function

There is no phase shift between the input and output $\phi=0\rightarrow A_1=0$, so we only have to find $B_1\ldots$

$$B_1 = \frac{2}{T}\int_0^T u(t)\sin\omega t dt = \frac{8}{T}\int_0^{T/4} u(t)\sin\omega t dt \tag{1.4.6}$$

Since *u(t)* is in phase with $\sin n\omega t$. Now find a description for u(t):



$$u(t) = \begin{cases} KE_m \sin\omega t & t < t_s \\ K_a & t \geq t_s \end{cases} \tag{1.4.7}$$

where

$$\sin n\omega t_s = \frac{a}{E_m}$$

So

$$B_1 = \frac{8}{T}\int_0^{t_s} KE_m \sin^2\omega t dt + \frac{8}{T}\int_{t_s}^{T/4} K_a \sin\omega t dt$$

$$= \frac{8K}{T}\left\{E_m\left[0.5t - \frac{0.25}{\omega}\sin 2\omega t\right]_0^{t_s} - a\left[\frac{\cos\omega t}{\omega}\right]_{t_s}^{T/4}\right\} = \frac{2K}{\pi}\left\{E_m\omega t_s + a\cos\omega t_s\right\}$$

Substituting the saturation onset time into this equation:

$$B_1 = \frac{2K}{\pi}\left\{E_m \sin^{-1}\frac{a}{E_m} + a\sqrt{1-\left(\frac{a}{E_m}\right)^2}\right\} \tag{1.4.8}$$

Thus the describing function is:

$$N = \frac{B_1}{E_m} = \frac{2K}{\pi}\left\{\sin^{-1}\frac{a}{E_m} + \frac{a}{E_m}\sqrt{1-\left(\frac{a}{E_m}\right)^2}\right\} \tag{1.4.9}$$

Relay with Hesteresis

Now consider a relay with hesteresis, where to switch output the input must pass some distance ‘a’ past the zero point

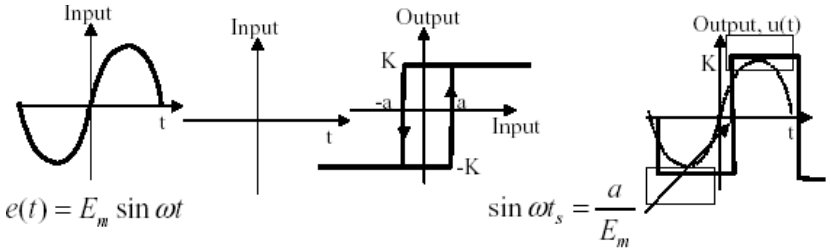


Fig. 1.4.4 Relay with Hesteresis

First of all we note that if *Em* < a there will be no sine wave output, otherwise, this time there is clearly a phase shift for *a* not equal to zero, so

$$\begin{aligned} A_1 &= \frac{2}{T}\int_0^T u(t)\cos\omega t dt = \frac{2}{T}\cdot 2\left\{-K\int_0^{t_s}\cos\omega t dt + K\int_{t_s}^{T/2}\cos\omega t dt\right\} \\ &= \frac{4}{T}\left\{K\int_{t_s}^{T/4}\cos\omega t dt - K\int_{T/4}^{(T/2)+t_s}\cos\omega t dt\right\} \\ &= \frac{4}{T}\left\{\left[\frac{\sin\omega t}{\omega}\right]_{t_s}^{T/4} - \left[\frac{\sin\omega t}{\omega}\right]_{T/4}^{(T/2)+t_s}\right\} \\ &= \frac{-8K}{T\omega}\sin\omega t_s = \frac{-4aK}{\pi E_m} \end{aligned} \tag{1.4.10a}$$

Now find a description for B_1:

$$\begin{aligned} B_1 &= \frac{2}{T}\cdot 2\left\{-K\int_0^{t_s}\sin\omega t dt + K\int_{t_s}^{T/2}\sin\omega t dt\right\} \\ &= \frac{4}{T}\left\{\left[\frac{-\cos\omega t}{\omega}\right]_{t_s}^{t_s} - \left[\frac{-\cos\omega t}{\omega}\right]_{t_s}^{(T/2)+t_s}\right\} = \frac{4K}{\pi}\sqrt{1-\left(\frac{a}{E_m}\right)^2} \end{aligned} \tag{1.4.10b}$$

Hence the describing function is given by

$$N(E_m,\omega) = \frac{u(t)}{e(t)} \approx \frac{U_1\sin(\omega t+\phi)}{E_m\sin\omega t} = |N|e^{j\phi}$$

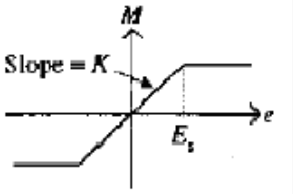
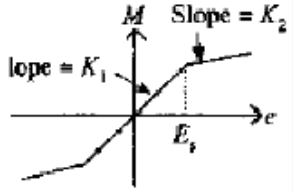
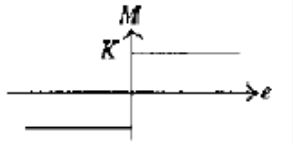
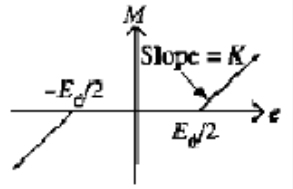
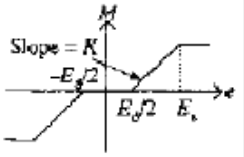
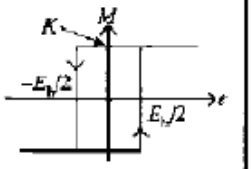
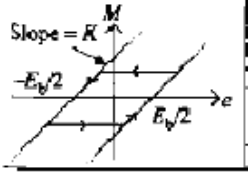
$$N(E_m,\omega,a,k) = |N|e^{j\phi}$$

where

$$\begin{aligned} |N(E_m,\omega,a,k)| &= \frac{\sqrt{A_1^2+B_1^2}}{E_m} = \frac{4K}{\pi E_m} \\ \phi &= \tan^{-1}A_1/B_1 = -\sin^{-1}\frac{a}{K} \end{aligned} \tag{1.4.11}$$

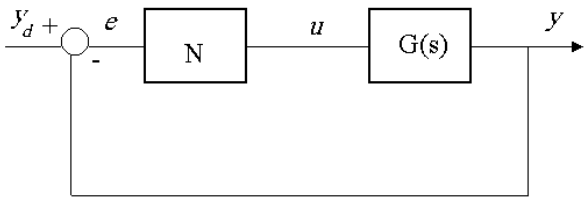
Table 1. provide describing functions for some common nonlinear functions

Table Some common describing functions for an input signal $e(t) = E_m \sin \omega t$.

Input–output characteristics	Describing function
	$N = K, \quad E_m \le E_s$ $N = \frac{2K}{\pi}\left(\sin^{-1}\frac{E_s}{E_m} + \frac{E_s}{E_m}\sqrt{1-\left(\frac{E_s}{E_m}\right)^2}\right), \quad E_m > E_s$
	$N = K_1, \quad E_m \le E_s$ $N = \frac{2K_2}{\pi}\left\{\frac{\pi}{2} + \left(\frac{K_1}{K_2} - 1\right)\left[\sin^{-1}\frac{E_s}{E_m} + \frac{E_s}{E_m}\sqrt{1-\left(\frac{E_s}{E_m}\right)^2}\right]\right\}, \quad E_m > E_s$
	$N = \frac{4K}{\pi E_m}$
	$N = 0, \quad E_m \le E_d/2$ $N = \frac{2K}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\frac{E_d}{2E_m} - \frac{E_d}{2E_m}\sqrt{1-\left(\frac{E_d}{2E_m}\right)^2}\right], \quad E_m > E_d/2$
	$N = 0, \quad E_m \le E_d/2$; N = as for deadzone, $E_d/2 < E_m \le E_s$; otherwise $N = \frac{2K}{\pi}\left[\sin^{-1}\frac{E_s}{E_m} - \sin^{-1}\frac{E_d}{2E_m} + \frac{E_s}{E_m}\sqrt{1-\left(\frac{E_s}{E_m}\right)^2} - \frac{E_d}{2E_m}\sqrt{1-\left(\frac{E_d}{2E_m}\right)^2}\right]$
	$N = 0, \quad E_m \le E_b/2$; otherwise $N = \frac{\sqrt{A_1^2 + B_1^2}}{E_m}, \quad \phi = \tan^{-1}\frac{A_1}{B_1}$ with: $A_1 = -\frac{2E_bK}{\pi E_m}, \quad B_1 = \frac{4K}{\pi}\sqrt{1-\left(\frac{E_b}{2E_m}\right)^2}$
	$N = 0, \quad E_m \le E_b/2$; otherwise N and ϕ as for hysteresis, with: $A_1 = -\frac{E_bK}{\pi}\left(2 - \frac{E_b}{E_m}\right), B_1 = \frac{KE_m}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(1-\frac{E_b}{E_m}\right) + \frac{E_b}{E_m}\left(1-\frac{E_b}{E_m}\right)\sqrt{\frac{2E_m}{E_b} - 1}\right]$

1.4.4 LIMIT CYCLES

The describing functions (approximately) tell the transfer function of the NL element at a given frequency. However we know that at some frequencies, we will have sustained oscillation in a nonlinear system when the output is connected in feedback: ie an initial disturbance creates a constant frequency output (limit cycle behaviour). This can be investigated using describing functions concept



By assuming the input to the nonlinearity is sinusoidal with magnitude E_m and frequency ω, ie

$$e(t) = E_m \sin(\omega t)$$

Now suppose the nonlinearity at the given input conditions (frequency, amplitude, nonlinear factor) is described by

$$N(E_m, \omega) = |N| e^{j\phi}$$

Then the output of the nonlinearity is given by

$$u = |N| e^{j\phi} E_m e^{j\omega t}$$

Hence the output of the linear system is given by

$$y = G(j\omega)u = G(j\omega)|N| e^{j\phi} E_m e^{j\omega t} \tag{1.4.12}$$

Following the feedback loop around, we see $e(t^+) = -y$, so we will have sustained oscillations if $e(t^+) = e(t)$, ie

$$E_m e^{j\omega t} = -G(j\omega)|N| e^{j\phi} E_m e^{j\omega t} \Rightarrow G(j\omega) = \frac{1}{|N| e^{j\phi}} \tag{1.4.13}$$

Limit cycle analysis

If w N is purely a gain term, ie $N = |N(E_m, \omega)|$ and $\phi = 0$, then we can rewrite this equation as

$$1 + NG(j\omega) = 0 \tag{1.4.14}$$

The condition $G(j\omega) \cdot N = -1$ (or $G(j\omega) = -1/N$) is easily investigated on a Nyquist plot