

Stochastic Model Predictive Control

AN OVERVIEW AND PERSPECTIVES FOR FUTURE RESEARCH

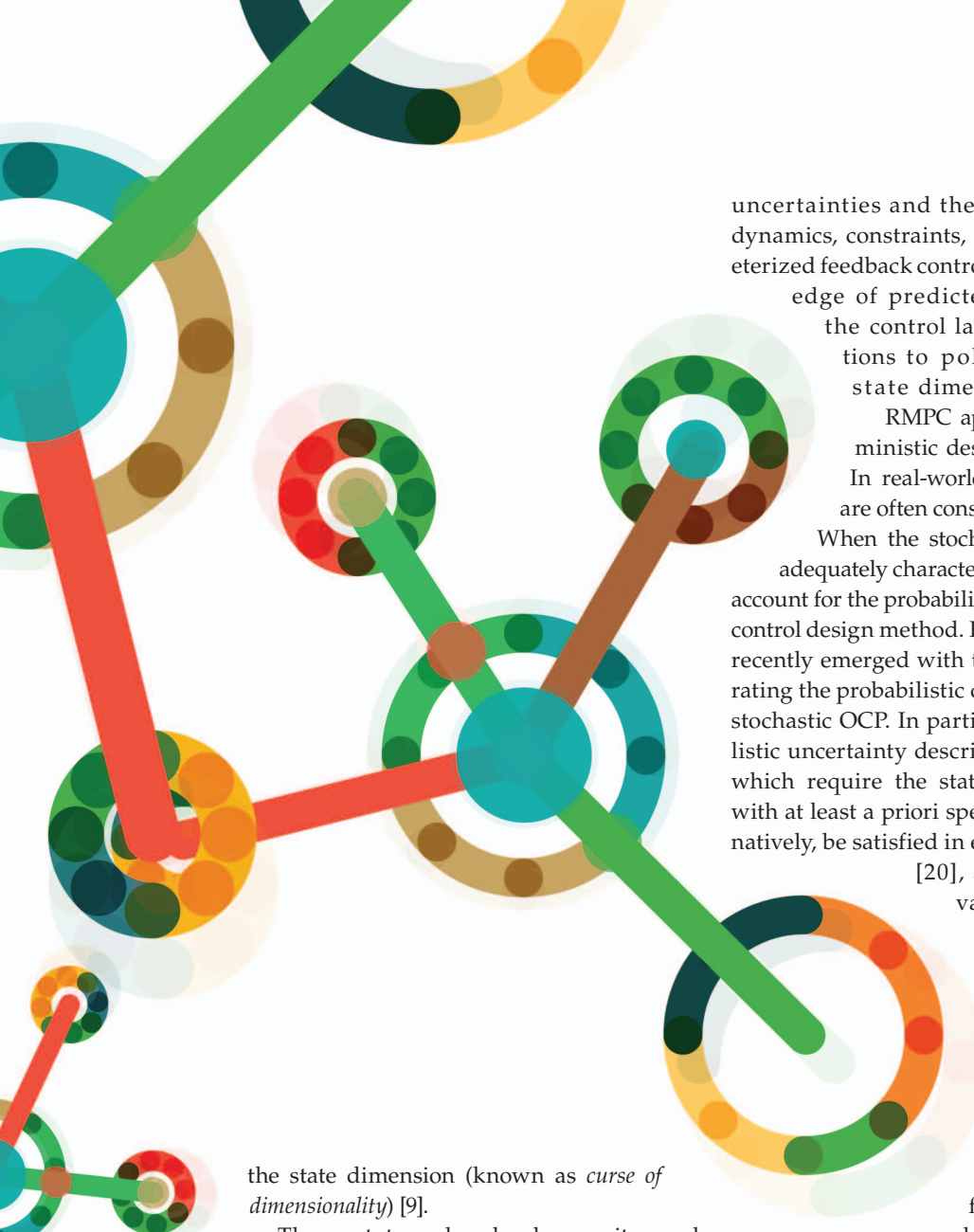
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Model predictive control (MPC) has demonstrated exceptional success for the high-performance control of complex systems [1], [2]. The conceptual simplicity of MPC as well as its ability to effectively cope with the complex dynamics of systems with multiple inputs and outputs, input and state/output constraints, and conflicting control objectives have made it an attractive multivariable constrained control approach [1]. MPC (a.k.a. receding-horizon control) solves an *open-loop* constrained optimal control problem (OCP) repeatedly in a receding-horizon manner [3]. The OCP is solved over a finite sequence of *control actions* $\{u_0, u_1, \dots, u_{N-1}\}$ at every sampling time instant that the current state of the system is measured. The first element of the sequence of optimal control actions is applied to the system, and the computations are then repeated at the next sampling time. Thus, MPC replaces a feedback control law $\pi(\cdot)$, which can have formidable offline computation, with the repeated solution of an open-loop OCP [2]. In fact, re-

peated solution of the OCP confers an “implicit” feedback action to MPC to cope with system uncertainties and disturbances. Alternatively, explicit MPC approaches circumvent the need to solve an OCP online by deriving relationships for the optimal control actions in terms of an “explicit” function of the state and reference vectors. However, explicit MPC is not typically intended to replace standard MPC but, rather, to extend its area of application [4]–[6].

Although MPC offers a certain degree of robustness to system uncertainties due to its receding-horizon implementation, its deterministic formulation typically renders it inherently inadequate for systematically dealing with uncertainties. Addressing the general OCP for uncertain systems would involve solving the dynamic programming (DP) problem over (arbitrary) feedback control laws $\pi(\cdot)$ [7], [8]. Solving the DP problem, however, is deemed to be impractical for real-world control applications since computational complexity of a DP problem increases exponentially with

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uncertainties and their interactions with the system dynamics, constraints, and performance criteria. Parameterized feedback control laws allow for using the knowledge of predicted uncertainties in computing the control law, while reducing the computations to polynomial dependence on the state dimension [16].

RMPC approaches rely on bounded, deterministic descriptions of system uncertainties. In real-world systems, however, uncertainties are often considered to be of probabilistic nature. When the stochastic system uncertainties can be adequately characterized, it is more natural to explicitly account for the probabilistic occurrence of uncertainties in a control design method. Hence, stochastic MPC (SMPC) has recently emerged with the aim of systematically incorporating the probabilistic descriptions of uncertainties into a stochastic OCP. In particular, SMPC exploits the probabilistic uncertainty descriptions to define *chance constraints*, which require the state/output constraints be satisfied with at least a priori specified probability level—or, alternatively, be satisfied in expectation (see, for example, [17]–[20], and the references therein for various approaches to chance-constrained optimization). Chance constraints enable the systematic use of the stochastic characterization of uncertainties to allow for an admissible level of closed-loop constraint violation in a probabilistic sense.

SMPC allows for systematically seeking tradeoffs between fulfilling the control objectives and guaranteeing a probabilistic constraint satisfaction due to uncertainty. The ability to effectively handle constraints in a stochastic setting is particularly important for MPC of uncertain systems when high-performance operation is realized in the vicinity of constraints. In addition, the probabilistic framework of SMPC enables shaping the probability distribution of system states/outputs. The ability to regulate the probability distribution of system states/outputs is important for the safe and economic operation of complex systems when the control cost function is asymmetric, that is, when the probability distributions have long tails [21].

Stochastic optimal control is rooted in stochastic programming and chance-constrained optimization; see, for example, [22] and [23] for a historical perspective. The pioneering work on chance-constrained MPC includes [17], [18], [24], and [25]. In recent years, interest in SMPC has been growing from both the theoretical and application standpoints. SMPC has found applications in many different areas, including building climate control, power

the state dimension (known as *curse of dimensionality*) [9].

The past two decades have witnessed significant developments in the area of robust MPC (RMPC) with the aim to devise computationally affordable optimal control approaches that allow for the systematic handling of system uncertainties. RMPC approaches consider set-membership-type uncertainties—uncertainties are assumed to be deterministic and lie in a bounded set. Early work on RMPC was primarily based on min–max OCP formulations, where the control actions are designed with respect to worst-case evaluations of the cost function and the constraints of the OCP must be satisfied for all possible uncertainty realizations (see, for example, [10] and [11]). Min–max MPC approaches could not, however, contain the “spread” of state trajectories, rendering the optimal control actions overly conservative or, possibly, infeasible [12]. To address the shortcomings of min–max OCPs, tube-based MPC has recently been developed [12]–[15]. Tube-based MPC approaches use a partially separable feedback control law parameterization to allow for the direct handling of

TABLE 1 An overview of applications of stochastic model predictive control for linear (SMPC) and nonlinear systems (SNMPC).

	SMPC (Stochastic-Tube and Affine-Parameterization Approaches)	Stochastic Programming-Based SMPC	Sample-Based SNMPC	SNMPC
Air traffic control			[31], [32]	
Automotive applications	[133]	[28], [29]	[134]	[135]
Building climate control	[27], [84]			
Microgrids	[136]	[105]		
Networked control systems	[137], [138]	[139]		
Operation research and finance	[69], [140]–[142]	[30], [143]		
Process control	[17], [24], [54], [122], [138]			[33]–[35], [55], [110]
Robot and vehicle path planning	[89], [144], [145]	[93]		[64]
Telecommunication network control	[146]			
Wind turbine control	[26]			

generation and distribution, chemical processes, operation research, networked controlled systems, and vehicle path planning. Table 1 provides an overview of various emerging application areas for SMPC; this table by no means provides an exhaustive list of SMPC applications reported in the literature. The majority of reported SMPC approaches have been developed for linear systems (for example, algorithms based on the stochastic tube [26] or affine parameterization of control policy [27]). Several SMPC applications to linear and nonlinear systems have been reported based on stochastic programming-based approaches [28]–[30] and Monte Carlo sampling techniques [31], [32]. Stochastic nonlinear MPC (SNMPC) has received relatively little attention, with only a few applications reported mainly in the area of process control [33]–[35].

This article gives an overview of the main developments in the area of SMPC in the past decade and provides the reader with an impression of the different SMPC algorithms and the key theoretical challenges in stochastic predictive control without undue mathematical complexity. The general formulation of a stochastic OCP is first presented, followed by an overview of SMPC approaches for linear and nonlinear systems. Suggestions of some avenues for future research in this rapidly evolving field concludes the article.

NOTATION

\mathbb{R}^n denotes the n -dimensional Euclidean space. $\mathbb{R}_+ := [0, \infty)$. $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers. $\mathbb{N}_+ := \mathbb{N} \cup \{0\}$. $\mathbb{Z}_{[a,b]} := \{a, a+1, \dots, b\}$ is the set of integers from a to b . \mathbf{P}_x is the (multivariate) probability distribution of random variable(s) x . \mathbf{E} denotes expectation and $\mathbf{E}_x[\cdot] := \mathbf{E}[\cdot | x(0) = x]$ is the conditional expectation. \mathbf{Pr} denotes probability, and

$\mathbf{Pr}_x[\cdot] := \mathbf{Pr}[\cdot | x(0) = x]$ is the conditional probability. $\|x\|_A := \sqrt{x^\top A x}$ is the weighted two-norm of x , where A is a positive-definite matrix.

GENERAL FORMULATION OF SMPC

Consider a stochastic, discrete-time system

$$x_{t+1} = f(x_t, u_t, w_t), \quad (1a)$$

$$y_t = h(x_t, u_t, v_t), \quad (1b)$$

where $t \in \mathbb{N}_+$; $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{U} \subset \mathbb{R}^{n_u}$, and $y_t \in \mathbb{R}^{n_y}$ are the system states, inputs, and outputs, respectively; \mathbb{U} is a non-empty measurable set for the inputs; $w_t \in \mathbb{R}^{n_w}$ and $v_t \in \mathbb{R}^{n_v}$ are disturbances and measurement noise that are unknown at the current and future time instants but have known probability distributions \mathbf{P}_w and \mathbf{P}_v , respectively; and f and h are (possibly nonlinear) Borel-measurable functions that describe the system dynamics and outputs, respectively.

For simplicity, the formulation of SMPC is presented for the case of full state-feedback control, in which the system states are known at each sampling time instant. Let $N \in \mathbb{N}$ be the prediction horizon, and assume that the control horizon is equal to the prediction horizon. Define an N -stage feedback control policy as

$$\pi := \{\pi_0(\cdot), \pi_1(\cdot), \dots, \pi_{N-1}(\cdot)\}, \quad (2)$$

where the Borel-measurable function $\pi_i(\cdot): \mathbb{R}^{(i+1)n_x} \rightarrow \mathbb{U}$, for all $i = 0, \dots, N-1$ is a general (causal) state feedback control law. At the i th stage of the control policy, the control input u_i is selected as the feedback control law $\pi_i(\cdot)$, that is,

$u_i = \pi_i(\cdot)$. In SMPC, the value function of a stochastic OCP is commonly defined as

$$V_N(x_t, \pi) := \mathbf{E}_{x_t} \left[\sum_{i=0}^{N-1} J_c(\hat{x}_i, u_i) + J_f(\hat{x}_N) \right], \quad (3)$$

where $J_c: \mathbb{R}^{n_x} \times \mathbb{U} \rightarrow \mathbb{R}_+$ and $J_f: \mathbb{R}^{n_x} \rightarrow \mathbb{R}_+$ are the *cost-per-stage function* and the *final cost function*, respectively, and \hat{x}_i denotes the predicted states at time i given the initial states $\hat{x}_0 = x_t$, control laws $\{\pi_j(\cdot)\}_{j=0}^{i-1}$, and disturbance realizations $\{w_j\}_{j=0}^{i-1}$.

The minimization of the value function (3) is commonly performed subject to chance constraints on system outputs (or states). Let \hat{y}_i denote the predicted outputs at time i given the initial states $\hat{x}_0 = x_t$. In its most general form, a *joint chance constraint* over the prediction horizon is defined by [36], [37]

$$\Pr_{x_t}[g_j(\hat{y}_i) \leq 0, \text{ for all } j = 1, \dots, s] \geq \beta, \text{ for all } i = 1, \dots, N, \quad (4)$$

where $g_j: \mathbb{R}^{n_y} \rightarrow \mathbb{R}$ is a (possibly nonlinear) Borel-measurable function, s is the total number of inequality constraints, and $\beta \in (0, 1)$ denotes the lower bound for the probability that the inequality constraint $g_j(\hat{y}_i) \leq 0$ must be satisfied. The conditional probability \Pr_{x_t} in (4) indicates that the probability that $g_j(\hat{y}_i) \leq 0$, for all $j = 1, \dots, s$, for all $i = 0, \dots, N$ holds is dependent on the initial states $\hat{x}_0 = x_t$; note that the predicted outputs \hat{y}_i depend on disturbances $\{w_i\}_{i=0}^{i-1}$. A special case of (4) is a collection of *individual chance constraints* [38]

$$\Pr_{x_t}[g_j(\hat{y}_i) \leq 0] \geq \beta_j, \text{ for all } j = 1, \dots, s, \text{ } i = 1, \dots, N, \quad (5)$$

where different probability levels β_j are assigned for different inequality constraints. Expressions (4) and (5) can be simplified to define chance constraints pointwise in time or in terms of the expectation of the inequality constraints $g_j(\hat{y}_i) \leq 0$ (see, for example, [39]). In addition, state chance constraints can be handled through appropriate choice of the function h in (1b).

Using the value function (3) and joint chance constraint (4), the stochastic OCP for (1) is formulated as follows. Given the current system states x_t , the centerpiece of an SMPC algorithm with hard input constraint and joint chance constraint is the stochastic OCP

$$V_N^*(x_t) := \min_{\pi} V_N(x_t, \pi) \quad (6)$$

such that:

$$\begin{aligned} \hat{x}_{i+1} &= f(\hat{x}_i, \pi_i, w_i), & \text{for all } i \in \mathbb{Z}_{[0, N-1]}, \\ \hat{y}_i &= h(\hat{x}_i, \pi_i), & \text{for all } i \in \mathbb{Z}_{[0, N]}, \\ \pi_i(\cdot) &\in \mathbb{U}, & \text{for all } i \in \mathbb{Z}_{[0, N-1]}, \\ \Pr_{x_t}[g_j(\hat{y}_i) \leq 0, \text{ for all } j = 1, \dots, s] &\geq \beta, & \text{for all } i \in \mathbb{Z}_{[1, N]}, \\ w_i &\sim \mathbf{P}_{w_i}, & \text{for all } i \in \mathbb{Z}_{[0, N-1]}, \\ \hat{x}_0 &= x_t, \end{aligned}$$

where $V_N^*(x_t)$ denotes the optimal value function under the optimal control policy π^* . The receding-horizon implementation of the stochastic OCP (6) involves applying the first element of the sequence π^* to the true system at every time instant that the states x_t are measured, that is, $u_t = \pi_0^*(\cdot)$.

The key challenges in solving the stochastic OCP (6) include 1) the arbitrary form of the feedback control laws $\pi_i(\cdot)$, 2) the nonconvexity and general intractability of chance constraints [40], [41], and 3) the computational complexity associated with uncertainty propagation through complex system dynamics (for example, nonlinear systems). In addition, establishing theoretical properties, such as recursive feasibility and stability, of the stochastic OCP (6) poses a major challenge. Numerous SMPC approaches have been developed to obtain tractable surrogates for the stochastic OCP (6). Table 2 summarizes the key features based on which SMPC approaches can be broadly categorized. In subsequent sections, various SMPC formulations will be analyzed in light of the distinguishing features given in Table 2. Broadly, SMPC approaches can be categorized in terms of the type of system dynamics, that is, linear or nonlinear dynamics. SMPC approaches for linear systems are further categorized based on three main schools of thought: stochastic-tube approaches [42]–[46], approaches using affine parameterization of the control policy [47]–[56], and stochastic programming-based approaches [56]–[62]. There has been much less development in the area of SMPC for nonlinear systems. The main contributions in this area can be categorized in terms of their underlying uncertainty propagation methods, namely sample-based approaches [31], [32], [63], Gaussian-mixture approximations [64], generalized polynomial chaos (gPC) [33], [34], [65], and the Fokker–Planck equation [35], [66], [67]. It is worth nothing, however, that a unique way to classify the numerous SMPC approaches reported in the literature does not exist. It has been attempted throughout the discussion to contrast the various SMPC approaches in terms of the key features listed in Table 2.

SMPC FOR LINEAR SYSTEMS

Much of the literature on SMPC deals with stochastic linear systems. For linear systems with additive uncertainties, the general stochastic system (1) takes the form

$$x_{t+1} = Ax_t + Bu_t + Dw_t, \quad (7a)$$

$$y_t = Cx_t + Fv_t, \quad (7b)$$

where A, B, C, D , and F are the state-space system matrices, and the disturbance w and measurement noise v are often (but not always) assumed to be sequences of independent, identically distributed (i.i.d.) random variables. For linear systems with multiplicative uncertainties, the system matrices in (7a) consist of time-varying uncertain elements with known probability distributions

TABLE 2 An overview of the key distinguishing features of stochastic model predictive control (SMPC) approaches (gPC: generalized polynomial chaos, FP: Fokker–Planck, GM: Gaussian mixture, QP: Quadratic programming, SOCP: Second-order cone programming, SDP: Semidefinite programming).

	Linear		Nonlinear	
System dynamics	[42]–[59], [61], [62], [69], [73], [89], [91], [92], [147]		[31]–[35], [63]–[65], [110], [120]	
Type of uncertainties	Time-varying uncertainties		Time invariant uncertainties	Time-varying/invariant
	Additive [43]–[45], [47]–[53], [55], [56], [61], [73], [92], [147]	Multiplicative [57], [58], [69]	[33], [34], [65]	multiplicative uncertainties with additive disturbances [26], [42], [54], [59], [62], [89]
Uncertainty propagation	Stochastic tube [26], [42]–[46]	Scenario/sample based [31], [32], [56], [59]–[63], [92], [93]	gPC/FP/GM [33]–[35], [54], [64], [65], [89]	
Control input parameterization	Open-loop control actions [31]–[35], [57]–[59], [61]–[65], [89], [92], [93], [120]	Prestabilizing feedback control [26], [42]–[44]	Affine parameterization of control policy [46]–[56]	
Input constraint	Hard input constraints [33]–[35], [47], [65]		Probabilistic input constraints [48], [69], [73]	Saturated input constraints [50], [52], [55]
Chance constraint	Individual [26], [33], [42]–[45], [47], [58], [110]		Joint [34], [35], [55], [56], [61], [73], [89], [93]	Expectation type [52], [62], [65], [69]
Receding-horizon implementation	Full-state feedback (The majority of the reported work)		Output feedback [45], [52], [73], [77]	
Optimization algorithm	Convex QP/SOCP [43], [44], [47], [50]–[53], [55], [89], [147]	SDP [69]	Stochastic programming [57], [58], [91]	Nonconvex optimization [31]–[35], [63]–[65], [120]

$$x_{t+1} = Ax_t + Bu_t + \sum_{j=1}^q (\bar{A}_j x_t + \bar{B}_j u_t) w_{t,j}, \quad (8)$$

where $\{w_{t,j}\}_{j=1}^q$ is a sequence of zero-mean i.i.d. random variables with known variance (see, for example, [68] and [69]). An overview of the main SMPC approaches for linear systems is given below.

Stochastic Tube Approaches

Stochastic tube approaches to MPC of linear systems with additive, bounded disturbances are presented in [43] and [44] with the objective of minimizing the infinite-horizon value function

$$V_\infty(x_t, \pi) = \mathbb{E}_{x_t} \left[\sum_{i=0}^{\infty} \|\hat{x}_i\|_Q^2 + \|u_i\|_R^2 \right], \quad (9)$$

subject to state chance constraints. Stochastic tubes are deployed to provide guarantees for recursive feasibility and, as a result, to ensure closed-loop stability and constraint satisfaction. Like in tube-based MPC [12], [13], [15],

[70], the states x_t are expressed in terms of the sum of a deterministic component z_t and a random component e_t

$$x_t = z_t + e_t, \quad z_{t+1} = \Phi z_t + Bv_t, \quad e_{t+1} = \Phi e_t + Dw_t,$$

where $u_t = Kx_t + v_t$ in (7a) with $\{v_0, \dots, v_{N-1}\}$ is the decision variables in the stochastic OCP and K is a constant feedback gain, and $\Phi = A + BK$ is assumed to be strictly stable. When the disturbances w_t are Gaussian, the representation $x_t = z_t + e_t$ allows for replacing linear chance constraints of the form $\Pr[cx_i \leq d] \geq \beta$ with tightened hard constraints $cx_i \leq d' \leq d$, so that $cx_i \leq d'$ implies $\Pr[cx_i \leq d] \geq \beta$, for all $i = 0, \dots, N-1$ [43]. Hence, stochastic tubes enable replacing the state chance constraints with linear constraints defined in terms of the nominal predictions z_i . This leads to a significant reduction in the number of decision variables in the online optimization. Tightened constraints can also be determined (with more complex offline computations) when w_t has an arbitrary, but known, distribution [44].

To obtain a computationally efficient algorithm, stochastic tubes with fixed ellipsoidal cross sections are used

in [43] such that the centers and scalings of the cross sections could vary with time [13], [15]. These stochastic tubes can be computed offline with respect to the states that guarantee satisfaction of chance constraints and recursive feasibility. The offline computation of stochastic tubes significantly improves the computational efficiency of the algorithm compared to stochastic tube approaches that use nested ellipsoidal sets [26] or nested layered tubes with variable polytopic cross sections [42] where the probability of transition between tubes and the probability of constraint violation within each tube are constrained. The use of variable polytopic cross sections as well as several tube layers in [26] and [42] leads to a large number of variables and linear inequalities that could restrict the application of the SMPC algorithm to low-dimensional systems with short prediction horizons. Offline computation of the stochastic tubes in [43] enables using many layered tubes and, therefore, considerably reduces the performance conservatism of [26] and [42]. However, the ellipsoidal nature of the tube cross sections in [43] precludes the possibility of using information on the direction of disturbances, leading to some degree of conservatism. The latter shortcoming is addressed in [44] by constructing recursively feasible stochastic tubes directly using the probabilistically constrained states.

In stochastic tube approaches, the *dual mode prediction* paradigm [3] is often used to minimize the value function (9). The dual mode prediction scheme involves the control parameterization $u_t = Kx_t + v_t$ over the first N steps of the prediction horizon ($\{v_0, \dots, v_{N-1}\}$ being the decision variables in the online optimization problem) and the prestabilizing state feedback control law $u_t = Kx_t$ over the subsequent infinite prediction horizon. The prestabilizing feedback control law $u_t = Kx_t$ provides mean-square stability for the system $x_{t+1} = \Phi x_t + Dw_t$ in the absence of constraints, that is, the limit of $x_t x_t^\top$ remains finite as $t \rightarrow \infty$. The control law $u_t = Kx_t$ is used to define a finite-horizon value function. The monotonically nonincreasing property of the value function guarantees that $\{v_0, \dots, v_{N-1}\}$ will tend to zero so that the control law reduces to $u_t = Kx_t$ at some time instant in the prediction horizon. The control law $u_t = Kx_t$ will steer the system to a terminal invariant set that is constructed based on the stochastic tubes to ensure satisfaction of constraints over all future prediction instants [71].

Stochastic tube approaches typically consist of two steps [43], [44]: 1) computing the tube scalings and terminal invariant sets offline and 2) solving a convex quadratic programming problem online to minimize the function $V_\infty(x, \pi) - I_{ss}$ subject to the deterministic surrogates for chance constraints and the terminal invariant set. I_{ss} is the finite nonzero limit of the cost-per-stage function $\mathbb{E}_{x_t}[\|\hat{x}_t\|_Q^2 + \|u_t\|_R^2]$. The approaches presented in [43] and [44] are based on full state-feedback control, assuming that the states are measurable. The stochastic tube algorithm in [44] is extended in [45] to consider probabilistic information on bounded measurement noise and uncertain state estimates, where recursive

feasibility and (a form of) quadratic stability of the output feedback control scheme are also established.

Stochastic tube approaches have been used for the control of systems with multiplicative bounded disturbances [see (8)] [71], [72]. Unlike the case of additive disturbances, however, a closed-form expression does not exist for the deterministic surrogates of the chance constraints. The use of stochastic tubes has been extended for the affine-disturbance parameterization of feedback control laws with a striped structure [46]. The latter approach can potentially lead to domains of attraction that are larger than those obtained in [43] and [44]. The larger domains of attraction, however, come at the cost of a weaker notion of stability (input-to-state stability) as opposed to guaranteeing convergence using a prestabilizing feedback control law.

In summary, stochastic tube approaches consider additive and/or multiplicative bounded disturbances. They commonly use a state-feedback control law with a constant feedback gain to minimize an infinite-horizon value function subject to individual chance constraints. The prestabilizing state-feedback control law allows for guaranteeing recursive feasibility as well as closed-loop stability in a mean-square sense. However, stochastic tube approaches cannot handle hard input constraints since the prestabilizing state-feedback controller is computed offline. The computational efficiency of stochastic tube approaches largely depends on how the cross sections of stochastic tubes are defined.

Approaches Based on Affine Parameterization of the Control Policy

Solving the stochastic OCP (6) using arbitrary feedback control laws $\pi_i(\cdot)$ is intractable. To achieve a computationally tractable formulation, the stochastic tube approaches use a feedback control law with a prestabilizing feedback gain. Hence, the online optimization would become limited to only a sequence of open-loop control actions $\{v_0, \dots, v_{N-1}\}$, which are offsets to the prestabilizing feedback controller. Alternatively, the stochastic OCP (6) can be solved over the feedback control gains $\{K_0, \dots, K_{N-1}\}$ as well as the open-loop control actions $\{v_0, \dots, v_{N-1}\}$ to have a larger set of decision variables. Using feedback gains and control actions as decision variables will, however, result in a nonconvex SMPC algorithm (see, for example, [73]).

Inspired by [16], [74], and [75], an SMPC approach is presented in [47] for linear systems with Gaussian additive disturbances [see (7a)], where the feedback control law $\pi_i(\cdot)$ is defined in terms of an affine function of past disturbances

$$\pi_i(x_t, \mathbf{w}) = \eta_i + \sum_{j=1}^{i-1} M_{i,j} w_j,$$

with $\eta_i \in \mathbb{R}^{n_u}$, $M_{i,j} \in \mathbb{R}^{n_u \times n_w}$ and $\mathbf{w} := \{w_1, \dots, w_{i-1}\}$. The notion of affine-disturbance parameterization of feedback control laws originates from the fact that disturbance realizations and system states will be known at the future time instants. Therefore, the controller will have the disturbance

information when determining the future control inputs over the control horizon. There exists a one-to-one (nonlinear) mapping between affine-disturbance feedback control policies and affine-state feedback control policies since both parameterizations would lead to the same control inputs [16]. An affine parameterization of the feedback control law $\pi_i(\cdot)$ allows for obtaining a stochastic OCP that is convex in decision variables η_i and $M_{i,j}$.

In [47], the value function is defined in terms of a linear function in disturbance-free states and control inputs, while polytopic constraints on inputs and state chance constraints are included in the stochastic OCP. The Gaussian disturbances are assumed to be bounded in a polytopic set, which enables handling hard input bounds and establishing input-to-state stability for the closed-loop system. In the presence of unbounded additive disturbances, it is impractical to guarantee any bounds on the control inputs using a linear state feedback controller; the additive nature of disturbances will surely drive the states outside any bounded set at some time. To enable dealing with unbounded disturbances, the hard input bounds considered in [47] are relaxed in [48] so that the inputs are restricted to lie in a subset with a prescribed probability level. For the case of bounded disturbances, the recursive feasibility of an SMPC algorithm under an affine-disturbance feedback policy is established in [49] by using the concept of robust invariant sets (see [76]). Similar SMPC algorithms have also been proposed for output feedback control of linear systems with Gaussian additive disturbances and measurement noise [77], and for a stochastic linear setup in the absence of input bounds [78]. The SMPC algorithms [77], [78], however, do not provide guarantees on recursive feasibility and closed-loop stability.

MPC of stochastic linear systems has been investigated in a series of articles [50]–[53]. A discrete-time system subject to unbounded disturbances with bounded variance is considered in [50]. An affine-disturbance feedback control policy is used to minimize a finite-horizon value function that is defined in terms of expectation of the sum of cost-per-stage functions (quadratic in the state and control inputs). The algorithm considers hard input bounds but not state chance constraints. To enable handling hard input bounds in the presence of unbounded disturbances, the disturbance measurements are saturated before being used for computing the control inputs. Thus, the affine-disturbance feedback control policy is defined by

$$\pi_i(x_t, \mathbf{w}) = \eta_i + \sum_{j=1}^{i-1} M_{i,j} \Psi_j(w_j), \quad \Psi_j(z) = \{\Psi_{j,1}(z_1), \dots, \Psi_{j,p}(z_p)\},$$

where $\Psi_{j,i}: \mathbb{R} \rightarrow \mathbb{R}$ is a measurable (nonlinear) function such that $\sup_{s \in \mathbb{R}} |\Psi_{j,i}| \leq \Psi_{\max} < \infty$ for some $\Psi_{\max} > 0$. The use of a saturation function renders the feedback control law nonlinear. The nonlinear feedback control law circumvents the need for relaxing the hard input bounds to soft

constraints when the additive disturbances are unbounded. In addition, the above nonlinear feedback control law facilitates establishing recursive feasibility in an unbounded disturbance setting. In [50], the mean-square boundedness of the closed-loop states is guaranteed by using the classical Foster–Lyapunov conditions [79] under the assumption that the unexcited (zero-input and zero-noise) system is asymptotically stable. The latter assumption implies that the system matrix A is Schur stable, that is, all eigenvalues of A are contained in the interior of the unit disc. In [50], the notion of closed-loop stability is relaxed to the mean-square boundedness of states, and additional conditions are imposed on the system matrix A since it is impossible to globally asymptotically stabilize the closed-loop system with bounded inputs when A has unstable eigenvalues.

The stability results of [50] are extended in [51] to the case where the system matrix A is Lyapunov stable, the pair (A, B) is stabilizable, and the stochastic disturbances have a bounded fourth moment. Lyapunov stability of the matrix A implies that all eigenvalues of A lie in the closed unit disc, and those with magnitude one have equal algebraic and geometric multiplicities (see [80] for a detailed analysis on mean-square boundedness of stochastic linear systems). In [51], the stochastic OCP is augmented with a negative drift condition defined in terms of a *stability constraint* to render the states of the closed-loop system mean-square bounded. The same stochastic OCP is addressed in [53] using a vector-space approach. As in [50] and [51], feedback policies are defined to be affine in bounded functions of the past disturbances. The stochastic OCP is lifted onto general vector spaces of candidate control functions, from which the controller can be selected algorithmically by solving a convex optimization problem. The most general treatment of the SMPC problem for linear systems with (additive) unbounded stochastic disturbances, imperfect state information, and hard input bounds is provided in [52] where the SMPC algorithm of [51] is generalized to the case of output feedback control, while providing guarantees on recursive feasibility and stability. In [52], the stochastic OCP is approximated by a globally feasible second-order cone program. Inspired by stability properties of Markov processes, an alternative approach for establishing closed-loop stability via appropriate selection of the value function in the stochastic OCP is presented in [81]. However, numerical tractability of this approach in terms of convexity of the value function has not been explored.

A stochastic linear setup similar to [52] is adopted in [73]. Hard input bounds, however, are relaxed to input chance constraints. The input and state chance constraints are reformulated as deterministic, convex constraints using the Chebyshev–Cantelli inequality [82]. This SMPC algorithm does not use an affine parameterization of the control policy. Hence, the feedback control gains $\{K_0, \dots, K_{N-1}\}$ and the control actions $\{\nu_0, \dots, \nu_{N-1}\}$ are the decision variables in the online optimization problem. In [73], the cost

SMPC allows for systematically seeking tradeoffs between fulfilling the control objectives and guaranteeing a probabilistic constraint satisfaction due to uncertainty.

per stage and the final cost functions are defined in terms of the expected values of future nominal states and the variances of future errors in the estimated states. The control actions $\{\nu_0, \dots, \nu_{N-1}\}$ and the feedback gains $\{K_0, \dots, K_{N-1}\}$ are used to drive the mean of the states to zero and minimize the variance of states, respectively. Terminal constraints are included in the stochastic OCP to guarantee recursive feasibility [3]. To handle unbounded disturbances that can lead to infeasibility of the problem, an initialization constraint is included in the OCP to allow the optimizer to select the initial conditions based on the feasibility and optimality requirements of the control algorithm. The main shortcomings of the SMPC algorithm presented in [73] are 1) an inability to consider saturation functions in the control policy to enable handling hard input bounds (as in [52]), 2) the conservatism associated with the Chebyshev–Cantelli inequality used for chance constraint approximation, and 3) nonconvexity of the algorithm. Using a problem setup similar to that in [52] and [73], an SMPC approach is presented in [55] that can handle joint state chance constraints and hard input constraints under closed-loop prediction in the presence of arbitrary (possibly unbounded) additive disturbances. The SMPC approach uses risk allocation [83], [84] in combination with the Cantelli–Chebyshev inequality to obtain computationally tractable surrogates for the joint state-chance constraints when the first two moments of the arbitrary disturbance distributions are known. An algorithm is then presented for solving two convex optimization problems iteratively to determine the best feedback gain and optimal risk allocation. Simulation studies show that optimizing the risk allocation in [55] can lead to improved control performance as compared to an MPC algorithm with fixed risk allocation.

Reference [54] addresses the problem of predictive control of linear systems with arbitrary time-invariant probabilistic uncertainties and additive Gaussian disturbances

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t + Dw_t, \quad (10)$$

with δ denoting the time-invariant uncertain system parameters with known (finite-variance) probability distributions and w_t being zero-mean Gaussian disturbance. In contrast to the above SMPC approaches (see, for example, [47] and [52]), the algorithm in [54] handles system descriptions in which the system matrices are nonlinearly dependent on stochastic uncertainties. To obtain a

computationally tractable algorithm, the gPC framework [85], [86] with Galerkin projection [87] is used for propagating the time-invariant uncertainties through the system dynamics (the gPC framework was first adopted in [88] to address the problem of linear-quadratic regulator design). An affine-state parameterization is used to minimize the expectation of a quadratic cost-per-stage function over a finite prediction horizon. Under the assumptions of perfect state information and stabilizability of the pair (A, B) for all uncertainty realizations, the closed-loop stability of the SMPC algorithm is established for the unconstrained case by bounding the value function. For a linear system with a similar setup, a polynomial chaos-based SMPC algorithm is also reported in [89]. This SMPC algorithm, however, solves the stochastic OCP merely over the control actions ν (without considering any feedback control parameterization) and does not provide guarantees on closed-loop stability. An SMPC algorithm for linear systems where the matrices A and B consist of multiplicative, unbounded disturbances [see (8)] is presented in [69]. The algorithm in [69] considers expectation-type constraints on inputs and states and establishes the closed-loop stability and constraint satisfaction properties of the control algorithm by defining a terminal constraint. This algorithm uses the feedback control law $u_t = K_t x_t + \nu_t$ to transform the stochastic OCP into a semidefinite programming (SDP) problem, in which the feedback gains k_t and control actions ν_t are computed online. The SDP formulation can render the SMPC algorithm computationally involved for high-dimensional systems.

To summarize, affine-disturbance and affine-state parameterizations of a feedback control policy have been widely used to obtain convex SMPC algorithms. In the presence of unbounded disturbances, dealing with hard input bounds and establishing closed-loop stability pose key challenges for these algorithms. An effective approach to address these challenges is to introduce a (nonlinear) saturation function to the affine parameterization of a feedback control policy. The saturated feedback control laws not only enable handling of hard input bounds for systems with unbounded disturbances but also facilitate ensuring mean-square boundedness of states when the system is Lyapunov stable. Closed-loop stability of these SMPC algorithms is commonly guaranteed by defining a negative drift condition via either a stability constraint or appropriate selection of the value function. Affine control policies have also been

used for deriving convex SMPC algorithms for linear systems that are nonlinearly dependent on time-invariant stochastic uncertainties.

Stochastic Programming-Based Approaches

A natural approach to solving the stochastic OCP (6) is to consider stochastic programming techniques [38], [90] as, for example, reported in [57] and [91]. Ideas from multi-stage stochastic optimization are adopted in [57] to develop an SMPC algorithm for a linear system with multiplicative uncertainties. In this algorithm, an optimization tree is designed by using a maximum-likelihood approach for scenario generation. The algorithm consists of two separate steps: 1) the offline step, where a stochastic Lyapunov function is derived to ensure recursive feasibility and exponential stability of the control algorithm in a mean-square sense, and 2) the online step in which a time-varying optimization tree is constructed based on the most recent system information and a quadratically constrained quadratic problem is solved to obtain the control actions. This SMPC approach is extended in [58] to handle chance constraints. However, imposing state constraints on every node of the optimization tree may become computationally intractable.

To obtain tractable solutions for stochastic programming problems, various sample-based approaches have been considered for approximating a stochastic optimization problem. The fundamental idea in sample-based MPC is to characterize the stochastic system dynamics using a finite set of random realizations of uncertainties, which are used to solve the OCP. Sample-based MPC algorithms readily consider the stochastic nature of the OCP (6). Unlike the SMPC approaches discussed in the previous sections (except [54] and [89]), the sample-based approaches typically do not require convexity assumptions with respect to uncertainties. An MPC algorithm based on Monte Carlo sampling is presented in [92] for unconstrained linear systems with stochastic disturbances. Chance-constrained predictive control of linear and jump-Markov linear systems with arbitrary (nonconvex or multimodal) disturbance distributions is considered in [93]. In [93], samples are generated using importance sampling techniques. Samples allow for approximating the chance-constrained stochastic OCP as a deterministic OCP with the property that the approximation becomes exact as the number of samples approaches infinity. The approximated control problem is then solved to global optimality using mixed-integer linear programming. This algorithm, however, provides no guidance for choosing the number of samples required for adequate approximation of chance constraints. The algorithm in [93] is also applicable to systems with Markovian-jump linear dynamics. The problem of stochastic optimal control of Markovian-jump linear systems as well as the more general case of Markovian-switching systems has been addressed in [58], [94], and [95].

A potential drawback of sample-based approaches to MPC is their computational cost due to the large sample size often required. The high computational cost can render sample-based MPC approaches prohibitive for practical applications. A significant development in the area of stochastic optimization is the *scenario approach* [96]–[98]; see also [99] for a tutorial overview of this approach. Appropriate sampling of constraints enables a stochastic optimization problem to be approximated by a standard convex optimization problem, whose solution is *approximately feasible* for the original problem [96]. An explicit bound is derived for the number of scenarios required to obtain a solution to the convex optimization problem that guarantees constraint satisfaction with a prespecified probability level. The main feature of this result is that the probability of constraint violation rapidly decreases to zero as the number of scenarios grows. Similar results on explicit bounds for the required number of scenarios/samples have also been developed using randomized algorithms [100]–[103].

The scenario approach is adopted in [59] to develop an MPC algorithm for (10) with stochastic multiplicative uncertainties δ_t and disturbances w_t ; the uncertainties δ_t and w_t are considered to be bounded. In [59], the cost-per-stage function is defined in terms of the maximum distance between the states x_t and a terminal constraint set that is derived based on a prestabilizing feedback controller $u_t = Kx_t$. The cost-per-stage function is computed over all independent scenarios. The online convex optimization problem then uses the control actions to minimize the evaluated cost-per-stage functions subject to input bounds and state constraints (evaluated for all scenarios) as well as a terminal constraint that is robustly positively invariant under $u_t = Kx_t$. The number of scenarios is chosen according to the scheme given in [96], so that probabilistic guarantees on constraint satisfaction can be achieved. The hard input and state constraints are transformed to soft constraints by introducing a slack variable that quantifies the extent of (possible) constraint violations. The scenario-based MPC approach ensures that either the states converge asymptotically to the terminal set or reach the terminal set in finite time with a prespecified probability level.

In the scenario approach, the theoretical bound for the number of scenarios may lead to a larger sample size than what is actually required [60]; this is commonly referred to as the *conservatism* of the scenario approach. In addition, the scenario-based MPC approach essentially treats a RMPC problem, rather than a chance-constrained stochastic OCP. Scenario-based approximation of the stochastic OCP has been addressed by replacing chance constraints with several hard constraints computed on the basis of the chosen scenarios [56], [61]. In these approaches, scenario-reduction methods can be used to significantly reduce the number of scenarios [104]. To obtain an adequate representation of the probability distributions, most scenario-based stochastic control approaches use branches of forecast scenarios, called

A key challenge in SMPC of nonlinear systems is the efficient propagation of stochastic uncertainties through the system dynamics.

scenario fans, which consist of bunches of independent and equiprobable scenarios. However, *scenario trees*, in which scenarios are not equiprobable, provide a causal representation of the uncertainties acting on the system, leading to more compact scenario representations [105].

A technique for fast scenario removal based on mixed-integer quadratic optimization is proposed in [61]. On the other hand, a scenario-based SMPC algorithm is introduced in [62] in which chance constraints are replaced with time-averaged, instead of pointwise-in-time, constraints to reduce the theoretically required number of scenarios. It is shown in [62] that the sample size is independent of the state dimension and, in fact, the sample size is dictated by the *support rank* of state constraints. This implies that a small sample size would be sufficient even for a system with high state dimension as long as the support rank of state constraints is low. The algorithm in [62] also allows for handling multiple state-chance constraints (see [106]).

Despite the extensive work done in the area of scenario-based MPC, the primary challenge still lies in identifying the appropriate number of scenarios that not only guarantees an admissible level of constraint satisfaction but also makes the computational requirements of the algorithm manageable for practical control applications. Hence, recent work on scenario-based MPC has mainly attempted to reduce the computational complexity of these algorithms [107]–[109]. Further development in this area is crucial to facilitate more practical applications of scenario-based control approaches. Another challenging problem in scenario-based MPC arises from establishing the theoretical properties such as recursive feasibility and closed-loop stability, in particular for the case of unbounded uncertainties.

STOCHASTIC MODEL PREDICTIVE CONTROL FOR NONLINEAR SYSTEMS

MPC of stochastic nonlinear systems has received relatively little attention. A pioneering work in this area is the SNMPC approach presented in [110]. The algorithm relies on the notion of optimizing a deterministic feedforward trajectory for constraint handling and optimizing a linear time-varying feedback controller for minimizing the closed-loop variance around the reference trajectory ($u_t = K_t x_t + v_t$). To explicitly account for back-off with respect to the reference trajectory, the nonlinear system dynamics are linearized around the reference trajectory. The control algorithm is rendered convex through the Youla–Kučera parameterization, and the chance constraints are reduced to second-order cone constraints using an ellipsoidal relaxation. The cone constraints lead to a

sequential conic-programming algorithm that is computationally expensive for high-dimensional systems with many state constraints and long prediction horizons. To tackle the computational complexity of the algorithm, the problem is decomposed using the linear-quadratic Gaussian decomposition, where the feedback and feedforward control problems are solved separately.

In [31] and [63], the use of a Markov-chain Monte Carlo (MCMC) technique [111] is proposed for solving constrained nonlinear stochastic optimization problems. A well-established theoretical framework (including general convergence results and central limit theorems) exists for MCMC approaches as well as *sequential* Monte Carlo approaches under weak assumptions (see, for example, [112]). The algorithms in [31] and [63] do not rely on convexity assumptions. In addition, they guarantee convergence (in probability) to a near-optimal solution under mild conditions related to the convergence of a homogeneous Markov chain. However, the effectiveness of these stochastic optimal control algorithms has not been demonstrated when implemented in a receding-horizon manner. A sequential Monte Carlo technique is adopted in [32] to reduce the computational complexity associated with MCMC approaches in [31] and [63].

A key challenge in SMPC of nonlinear systems is the efficient propagation of stochastic uncertainties through the system dynamics. For discrete-time nonlinear systems with additive disturbances, the Gaussian-mixture approximation [113] is used in [64] to describe the transition probability distributions of states in terms of weighted sums of a predetermined number of Gaussian distributions. Due to its universal approximation property, the method of Gaussian-mixture approximation provides a flexible framework for constructing the probability distributions of stochastic variables and, therefore, for uncertainty propagation. The knowledge of complete probability distributions allows for defining the value function of the stochastic OCP in terms of the complete distributions of stochastic states, instead of merely some moments of the probability distributions. However, online adaptation of the weights of the Gaussian distributions, which is essential for describing the time evolution of the probability distributions, poses a computational challenge in Gaussian-mixture approximations.

The gPC framework [85], [86] is used in [33] to develop an SNMPC approach for a general class of nonlinear systems that are subject to arbitrary time-invariant stochastic uncertainties in system parameters and initial conditions. The gPC framework replaces the implicit mappings between uncertain variables/parameters and states (defined in terms

of nonlinear differential equations) with expansions of orthogonal polynomial basis functions; see [114] for a recent review on polynomial chaos. The orthogonality property of the basis functions allows for readily computing the statistical moments of stochastic variables from the expansion coefficients. Thus, polynomial chaos expansions provide an efficient machinery for predicting the time evolution of the moments of probability distributions of stochastic states. Alternatively, polynomial chaos expansions can also be used as a surrogate for the nonlinear system model to significantly accelerate sample-based construction of probability distributions using Monte Carlo techniques (see [115]). In [33], the value function of the stochastic OCP is defined in terms of the moments of stochastic states. The statistical moments of states are also used for converting individual state chance constraints into convex second-order cone expressions. To facilitate receding-horizon implementation of the control algorithm, a sample-based collocation method [21], [116] is adopted in [33] to recursively adapt the coefficients of polynomial chaos expansions based on the most recent state information. A similar SNMPC algorithm (with expectation-type constraints) is proposed in [65], where the coefficients of polynomial chaos expansions are determined via weighted l_2 -norm regularization in the collocation method. A polynomial chaos-based SNMPC algorithm is presented in [34] that uses Galerkin projection [87] for determining the coefficients of polynomial chaos expansions. The Galerkin projection method yields a set of closed-form ordinary differential equations for the coefficients. However, Galerkin projection can only be employed for nonlinear, polynomial-type systems. In [34], the polynomial chaos expansions are used to efficiently construct the probability distributions of states through Monte Carlo simulations to approximate the chance constraints.

The above generalized polynomial chaos-based SNMPC approaches have two shortcomings: 1) the algorithms may not efficiently handle time-varying disturbances since the gPC framework requires a large number of expansion terms to describe time-varying uncertainties, and 2) reconstructing a complete probability distribution from its statistical moments can be a computationally formidable task. A gPC-based SNMPC approach that can handle additive stochastic disturbances is presented in [117]. The stochastic disturbances are mapped to the space of coefficients of polynomial chaos expansions. The probability distribution of states is then computed by integrating the conditional probability distribution of the polynomial-chaos-approximated states over the probability distribution of disturbances. A gPC-based histogram filter [118] is also used to recursively update the uncertainty description of parameters when the system states are measured. For continuous-time stochastic nonlinear systems, a Lyapunov-based SNMPC approach is proposed in [35] for shaping the probability distribution of states. The Fokker–Planck equation [119] is used to describe the dynamic evolution of the probability distributions.

Complete characterization of probability distributions allows for shaping the distributions of states, as well as direct computation of joint chance constraints without conservative approximations. In [35], closed-loop stability is ensured by designing a stability constraint in terms of a stochastic-control Lyapunov function that explicitly characterizes stability in a probabilistic sense.

In summary, the developments in the area of MPC of stochastic nonlinear systems are limited due to the computational complexities associated with uncertainty propagation in nonlinear systems. The algorithms discussed above that use efficient uncertainty propagation approaches have shown promise for SNMPC. Establishing the theoretical properties of SNMPC algorithms such as closed-loop stability and constraint satisfaction in the closed-loop sense poses a great challenge. Preliminary results on stability of MPC of stochastic nonlinear systems have recently been reported in [120] and [121].

FUTURE RESEARCH DIRECTIONS

In recent years, the field of SMPC has significantly matured, particularly for linear systems. A key feature of SMPC is the ability to incorporate chance constraints into the stochastic OCP. The probabilistic framework of SMPC facilitates seeking tradeoffs between the attainable control performance and robustness to stochastic uncertainties. However, various approximations are commonly made in most SMPC approaches (for example, approximations in uncertainty descriptions or the handling of chance constraints) to obtain tractable algorithms. It is paramount to critically assess whether such approximations would degrade the effectiveness of SMPC.

In light of real-world systems with complex dynamics, some open research challenges in the area of SMPC are presented below. Theoretical advances in these research directions will likely further the development of more comprehensive frameworks for stochastic predictive control of practical applications.

- » *Efficient uncertainty propagation approaches:* Stochastic predictive control of nonlinear systems hinges on the tractability of probabilistic uncertainty analysis in online computations. The development of efficient uncertainty propagation approaches that can deal with general probabilistic uncertainty descriptions would facilitate advances in SNMPC. In addition, uncertainty propagation approaches that characterize the complete probability distribution would allow for direct evaluation of chance constraints without the need for conservative approximations [35].
- » *Chance-constraint evaluation:* Most SMPC algorithms use tractable convex expressions or, alternatively, sample-based approaches to approximate chance constraints. However, such approximations tend to be conservative, which may, in turn, reduce the effectiveness of SMPC. Quantifying the conservatism associated

with chance-constraint approximations and developing less conservative, but tractable, approaches for chance-constraint handling are essential for effectively using the probabilistic framework of SMPC.

- » *Tailored algorithms for high-dimensional systems:* Many complex dynamical systems are distributed, the dynamics of which are described by partial differential equations. The finite-dimensional approximation of partial differential equations typically results in high-dimensional state-space models with large state dimension. SMPC of high-dimensional systems requires the development of tailored algorithms that can exploit the special characteristics of such systems (for example, typically having a limited number of inputs and outputs [122]) to achieve tractable stochastic control formulations.
- » *Output-feedback control:* Most SMPC approaches are developed for the case of full state feedback, whereas full states cannot be measured in many practical applications. A few SMPC algorithms that include state estimation are presented in [45], [52], and [73]. Moving-horizon estimation (MHE) [123], which is particularly attractive for use with MPC, remains an open problem for stochastic systems. Establishing the stability of output-feedback control based on SMPC and MHE poses a major challenge (see [124] for recent results on this problem on the basis of worst-case uncertainty analysis).
- » *Distributed predictive control:* Complex systems are increasingly becoming highly integrated. Stochastic predictive control of interacting systems gives rise to several open theoretical issues related to system-wide stability and control performance in the presence of probabilistic uncertainties. In addition, there is a need for systematic approaches for efficient and reliable uncertainty propagation through networked systems to address challenges associated with computational complexity of SMPC of integrated systems.
- » *Adaptive (dual) control:* MPC integrated with persistent excitation has recently gained interest with the aim of ensuring uniform quality of system models in MPC applications [125], [126]. Combining SMPC with input design to consistently adapt model structure and/or parameters in the face of stochastic system uncertainties remains an open research challenge that lends itself to several theoretical issues (see [127] and [148] for recent results on SMPC integrated with input design).
- » *Explicit stochastic predictive control:* The benefits of explicit MPC include efficient online computations and verifiability of the control policy. In a stochastic setting, explicit predictive control is likely to be even more beneficial since the verifiability becomes more critical. Stochastic extensions of explicit MPC can potentially find many practical applications, in

particular in safety-critical applications [128], [129]. A key challenge in explicit SMPC lies in addressing the problem of deriving an explicit control law when the disturbances are not normally distributed, and the inverse cumulative distribution is not known in an explicit form.

- » *Risk-averse predictive control:* The discussed stochastic formulations of the OCP are based on risk-neutral expected values of performance measures. However, the assessment of future stochastic outcomes of a system through a risk-neutral expectation may not be suitable when the system must be protected from large deviations. Inclusion of risk aversion, originally considered in operations research, into MPC has shown promise to balance conservatism in “decision making” with robustness to uncertainties [130]. Theoretical properties of risk-averse MPC (for example, stability when states and inputs are constrained) are, however, poorly understood.
- » *Stochastic economic MPC:* Despite developments in economic MPC, probabilistic approaches to robust economic MPC have received little attention [131]. Controlling periodic state trajectories typically observed in these control algorithms as well as establishing the closed-loop stability properties of stochastic economic MPC algorithms remain interesting open research problems.
- » *GPU-based computing:* Graphics processing units (GPUs) are widely used for performing Monte Carlo simulations. Tailoring GPU-based computing algorithms for solving stochastic OCPs can create opportunities for SMPC (for example, sample-based SMPC approaches) to be used in new application areas [132].
- » *Applications:* Several promising application areas have emerged for SMPC. However, further research in the field of stochastic predictive control will benefit greatly from close interaction between theory and practice. Close collaboration between researchers and MPC practitioners is crucial to avoid the risk of developing elegant SMPC algorithms with limited applicability to practical problems.

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