RINGS, MODULES AND LINEAR ALGEBRA — SHORT EXERCISES

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1. Introduction

2. Rings and Fields

Question 2.1. Why is $3\mathbb{Z}$ not a ring?

Answer: It does not contain the multiplicative identity element 1.

Question 2.2. Why is $\mathbb{Z}_6$ not an integral domain?

Answer: Because $\overline{2}$ and $\overline{3}$ are nonzero elements of $\mathbb{Z}_6$ whose product is zero.

Question 2.3. Which of the following is an element of $\mathbb{Z}[x]$: $x + x^{-1}$, $10x^{10}$, $x^5/5$.

Answer: Only $10x^{10}$ is an element of $\mathbb{Z}[x]$.

Question 2.4. Which of the following is an element of $\mathbb{Z}_{(3)}$: $3/5$, $5/3$, $1/12$, $1/13$.

Answer: Only $3/5$ and $1/13$ are elements of $\mathbb{Z}_{(3)}$

Question 2.5. What is a Gaussian integer?

Answer: A Gaussian integer is a complex number of the form $a + bi$ where a and b are integers.

3. Modules

Question 3.1. What is the other name for a module over $\mathbb{C}$?

Answer: A vector space over $\mathbb{C}$.

Question 3.2. What is the other name for a module over $\mathbb{Z}$?

Answer: An Abelian group.

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4. Modules over polynomial rings

Question 4.1. If $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and $f(x) = x^2 + 1$ then what is $f(A)$?

Answer: $f(A) = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 10 \end{smallmatrix}\right)$.

Question 4.2. If $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $f(x) = x^2 + 1$ then what is $f(A)$?

Answer: $f(A) = \left(\begin{smallmatrix} 2 & 2 \\ 0 & 2 \end{smallmatrix}\right)$.

Question 4.3. If $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{smallmatrix}\right)$ (considered as a matrix over $\mathbb{Q}$) then what is the dimension of M_A as a vector space over $\mathbb{Q}$?

Answer: The dimension is just the number of rows or columns in the matrix, which is 3.

Question 4.4. If we regard $C^\infty(\mathbb{R}, \mathbb{R})$ as a module over $\mathbb{R}[D]$ in the usual way, what is $(1 + D).t^2$?

Answer: $(1 + D).t^2 = t^2 + 2t$.

Question 4.5. A module over $\mathbb{C}[x]$ consists of a ________________ V over $\mathbb{C}$ together with ________________ .

Answer: A module over $\mathbb{C}[x]$ consists of a vector space V over $\mathbb{C}$ together with a $\mathbb{C}$-linear endomorphism of V.

5. General module theory

Question 5.1. What is another name for a submodule of an Abelian group (considered as a $\mathbb{Z}$-module)?

Answer: A subgroup.

Question 5.2. Let M be a module over a ring R. List two submodules of M that can be defined without knowing anything at all about R or M.

Answer: $\{0\}$ and M itself.

Question 5.3. Let V be a vector space over $\mathbb{R}$ equipped with an endomorphism ϕ, regarded as an $\mathbb{R}[x]$-module in the usual way. A vector subspace $W \leq V$ is an $\mathbb{R}[x]$-submodule if and only if ________________ .

Answer: A vector subspace $W \leq V$ is an $\mathbb{R}[x]$-submodule if and only if it is stable under ϕ.

Question 5.4. Why is $\mathbb{R}[t]$ an $\mathbb{R}[D]$-submodule of $C^\infty(\mathbb{R}, \mathbb{R})$?

Answer: Because the derivative of a polynomial is another polynomial, so $\mathbb{R}[t]$ is stable under ∂.

Question 5.5. If $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$ then what is the matrix $A \oplus B$?

Answer: $A \oplus B = \left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{smallmatrix}\right)$.

Question 5.6. Describe two nontrivial subgroups of $\mathbb{Z}_{12}$ such that $\mathbb{Z}_{12}$ is the internal direct sum of these subgroups.

Answer: $\mathbb{Z}_{12}$ is the internal direct sum of the subgroups $N_0 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$ and $N_1 = \{\overline{0}, \overline{4}, \overline{8}\}$.

Question 5.7. Put $W_0 = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f' = f\}$ and $W_1 = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f' = -f\}$. What is $W_0 \cap W_1$ and why?

Answer: $W_0 \cap W_1 = \{0\}$, because if $f \in W_0 \cap W_1$ then $f' = f$ and $f' = -f$ so $f = -f$ so $f = 0$.

Question 5.8. What is a cyclic module?

Answer: A module M over R is cyclic if there is an element $m \in M$ such that every other element of M is a multiple of m.

Question 5.9. Give an example of a cyclic module over $\mathbb{R}[D]$.

Answer: The space W_2 of polynomials of the form $at^2 + bt + c$ is a cyclic module over $\mathbb{R}[D]$, as discussed in the notes. Of course one can specify examples in other ways, for example $\mathbb{R}[D]/(D-5)$ or $\mathbb{R}[D]$ itself.

6. Homomorphisms

Question 6.1. Give an example of a homomorphism $\delta\colon R^2 \to R^3$ of R-modules.

Answer: The example in the notes is $\delta(u, v) = (u, v - u, -v)$, although of course there are infinitely many other examples.

Question 6.2. Is there a homomorphism $\alpha\colon \mathbb{Z}_6 \to \mathbb{Z}_{15}$ of Abelian groups with $\alpha(\overline{m}) = \overline{10m}$ for all $m \in \mathbb{Z}$?

Question 6.3. How can we describe the R-module homomorphisms from R^4 to R^5?

Answer: They are essentially the same as the 4×5 matrices over R.

Question 6.4. Suppose we have K-vector spaces V and W considered as $K[x]$ modules using endomorphisms $\phi\colon V \to V$ and $\psi\colon W \to W$. Suppose that $\alpha\colon V \to W$ is a K-linear map. When is α a homomorphism of $K[x]$-modules?

Answer: This holds if and only if $\alpha\phi = \psi\alpha$.

Question 6.5. Let A be a 3×3 diagonalisable matrix over $\mathbb{C}$ with eigenvalues 5, 6 and 7. Describe another module over $\mathbb{C}[x]$ that M_A is isomorphic to.

Answer: $M_A \simeq M_5 \oplus M_6 \oplus M_7$.

Question 6.6. Define $\alpha\colon \mathbb{Z} \to \mathbb{Z}_{12}$ by $\alpha(n) = \overline{4n}$. What are image$(\alpha)$ and $\ker(\alpha)$?

Answer: image$(\alpha) = \{\overline{0}, \overline{4}, \overline{8}\}$ and $\ker(\alpha) = \{n \in \mathbb{Z} \mid n = 0 \pmod 3\} = 3\mathbb{Z}$.

Question 6.7. Suppose we have modules L, M and N and homomorphisms $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$. What does it mean to say that this sequence is exact?

Answer: The sequence is exact if and only if the image of α is the same as the kernel of β.

Question 6.8. Suppose that $\alpha\colon M \to N$ is a homomorphism. Give a criterion in terms of kernels and images for α to be an isomorphism.

Answer: α is an isomorphism if and only if $\ker(\alpha) = \{0\}$ and image$(\alpha) = N$.

7. Factor modules

8. Ideals and factor rings

Question 8.1. Give an example of an ideal in $\mathbb{Z}$.

Answer: The set of even integers is an ideal in $\mathbb{Z}$.

Question 8.2. Put $I = \{f \in \mathbb{Z}[x] \mid f(2) = 0\}$. Find a nonzero element of I.

Answer: The polynomial $x - 2$ is the simplest answer, although there are of course many others.

Question 8.3. What is the definition of the standard homomorphism $\pi\colon R \to R/I$?

Answer: It is defined by $\pi(a) = a + I$.

Question 8.4. Which well-known field is isomorphic to $\mathbb{R}[x]/(x^2 + 1)$?

Answer: $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

Question 8.5. Let M be an Abelian group such that $5m = 0$ for all $m \in M$. Over which factor ring can we regard M as a module?

Answer: M can be regarded as a module over $\mathbb{Z}/5$.

9. Euclidean domains

Question 9.1. If we use the standard Euclidean valuations defined in the notes, what are the valuations of the elements $-7 \in \mathbb{Z}$, $2 + 2i \in \mathbb{Z}[i]$, $1 + x + x^2 \in \mathbb{Q}[x]$ and $18/7 \in \mathbb{Z}_{(3)}$?

Answer:

$$
\begin{aligned}
\nu_{\mathbb{Z}}(-7) &= |-7| = 7 \\
\nu_{\mathbb{Z}[i]}(2 + 2i) &= 2^2 + 2^2 = 8 \\
\nu_{\mathbb{Q}[x]}(1 + x + x^2) &= 2 \\
\nu_{\mathbb{Z}_{(3)}}(18/7) &= \nu_{\mathbb{Z}_{(3)}}(3^2 \times 2/7) = 2.
\end{aligned}
$$

Question 9.2. What is the main theorem about ideals in Euclidean domains?

Answer: Every ideal in a Euclidean domain is principal.

Question 9.3. Give a condition in terms of ideals that is satisfied if and only if d is a gcd of a and b.

Answer: d is a gcd of a and b if and only if $Rd = Ra + Rb$.

10. Factorisation in Euclidean domains

Question 10.1. Write down a unit, an irreducible element, and a reducible element of $\mathbb{Z}$.

Answer: -1 is a unit, 3 is an irreducible element, 4 is a reducible element.

Question 10.2. Write down a unit, an irreducible element, and a reducible element of $\mathbb{C}[x]$.

Answer: 7 is a unit, $x-3$ is an irreducible element, and $x^2-1=(x-1)(x+1)$ is a reducible element.

Question 10.3. Write down a complete set of irreducibles in $\mathbb{C}[x]$.

Answer: The set $\{x-\alpha \mid \alpha \in \mathbb{C}\}$ is a complete set of irreducibles in $\mathbb{C}[x]$.

Question 10.4. What special property does the ring R/p have when R is a Euclidean domain and p is irreducible?

Answer: R/p is a field.

Question 10.5. If ν is a Euclidean valuation on a ring R and $a \in R$ satisfies $\nu(a)=0$, what can you say about a?

Answer: a is a unit.

11. Finite free modules over a Euclidean domain

Question 11.1. Put $N=\{(w,x,y,z)\in\mathbb{Z}^4 \mid w=x \text{ and } y=z\}$. Give a basis for N.

Answer: If $n_1=(1,1,0,0)$ and $n_2=(0,0,1,1)$ then $\{n_1,n_2\}$ is a basis for N.

Question 11.2. Why is $\mathbb{Z}_{12}$ not free as a $\mathbb{Z}$-module?

Answer: The element $\overline{1}\in\mathbb{Z}_{12}$ satisfies $\overline{1}\neq 0$ but $12.\overline{1}=0$ so $\mathbb{Z}_{12}$ is not torsion-free so it is not free.

Question 11.3. List two torsion elements of $C^\infty(\mathbb{R},\mathbb{R})$.

Answer: The functions $f(t)=e^t$ and $g(t)=t$ are torsion elements, because $(D-1)f=D^2g=0$.

Question 11.4. Explain why no submodule of $\mathbb{Z}^{12}$ can be isomorphic to $\mathbb{Z}_{12}$.

Answer: Every submodule of $\mathbb{Z}^{12}$ is free, but $\mathbb{Z}_{12}$ is not free.

12. Row and column operations

Question 12.1. Which of the following matrices over $\mathbb{Z}$ is in normal form? $\begin{pmatrix}2&0&0&0&0\\0&4&0&0&0\\0&0&8&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$, $\begin{pmatrix}8&0&0\\0&4&0\\0&0&2\end{pmatrix}$, $\begin{pmatrix}2&0&0\\0&4&0\\0&0&8\end{pmatrix}$, $\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}2&0\\0&22\end{pmatrix}$.

Answer: The matrices $\begin{pmatrix}2&0&0&0&0\\0&4&0&0&0\\0&0&8&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$, $\begin{pmatrix}2&0&0\\0&4&0\\0&0&8\end{pmatrix}$, $\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$ and $\begin{pmatrix}2&0\\0&22\end{pmatrix}$ are in normal form.

Question 12.2. Which of the following matrices over $\mathbb{Z}$ is in prenormal form? $\begin{pmatrix}2&1&4\\1&3&4\\1&5&6\end{pmatrix}$, $\begin{pmatrix}2&0&0\\0&3&4\\0&5&6\end{pmatrix}$, $\begin{pmatrix}2&0&0\\0&4&4\\0&6&6\end{pmatrix}$.

Answer: Only the matrix $\begin{pmatrix}2&0&0\\0&4&4\\0&6&6\end{pmatrix}$ is in prenormal form.

13. Finite subgroups of fields

Question 13.1. Let K be a field with precisely 27 elements. What can you say about the group $K^\times$?

Answer: It is isomorphic to C_{26}.

Question 13.2. Give a generator of the group $\mathbb{Z}_{11}^\times$.

Answer: $\overline{2}$ generates $\mathbb{Z}_{11}^\times$.

Question 13.3. Give a generator of the group $\mathbb{Z}_7^\times$.

Answer: $\overline{3}$ generates $\mathbb{Z}_7^\times$.

14. Primary decomposition

Question 14.1. List three different basic $\mathbb{Z}$-modules.

Answer: $\mathbb{Z}/5$, $\mathbb{Z}/5^2 = \mathbb{Z}/25$ and $\mathbb{Z}/7$ are basic $\mathbb{Z}$-modules.

Question 14.2. Note that $42 = 6 \times 7$ and $(6, 7) = 1$. What can you deduce about $\mathbb{Z}_{42}$?

Answer: It is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_7$.

Question 14.3. How many different groups of order 36 are there (up to isomorphism)?

Answer: There are four possibilities, listed below:

$$\begin{aligned} M &\simeq \mathbb{Z}_4 \oplus \mathbb{Z}_9 \\ M &\simeq \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \\ M &\simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \\ M &\simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3. \end{aligned}$$

Question 14.4. What is the definition of $F_p^k(M)$?

Answer: $F_p^k(M) = \{m \in p^{k-1}M \mid pm = 0\}$.

Question 14.5. What are $g_3^5(\mathbb{Z}/2^5)$, $g_3^5(\mathbb{Z}/3^6)$ and $g_3^5(\mathbb{Z}/3^5)$?

Answer: 0, 0 and 1.

Question 14.6. Let A be a 3×3 diagonal matrix with eigenvalues 5, 6 and 7. Write M_A in terms of basic $\mathbb{C}[x]$-modules.

Answer: $M_A \simeq B(5, 1) \oplus B(6, 1) \oplus B(7, 1)$.

15. Canonical forms for square matrices

Question 15.1. Write down the Jordan block of size 3 and eigenvalue -1.

Answer: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$.

Question 15.2. What is the characteristic polynomial of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$?

Answer: $\det \begin{pmatrix} t-2 & 0 \\ 0 & t-1 \end{pmatrix} = (t-2)(t-1) = t^2 - 3t + 2$.