Pure Mathematics Core — Exam solutions

(A1) The general form is

$$\frac{x^2}{(x+2)^2} = A + \frac{B}{x+2} + \frac{C}{(x+2)^2}.\mathbf{[2]}$$

Multiplying by $(x+2)^2$ gives

$$\begin{aligned} x^2 &= A(x+2)^2 + B(x+2) + C = Ax^2 + 4Ax + 4A + Bx + 2B + C \\ &= Ax^2 + (4A+B)x + (4A+2B+C), \mathbf{[1]} \end{aligned}$$

so $A = 1$ and $4A+B = 0$ and $4A+2B+C = 0$, which gives $B = -4$ and $C = 4$. **[1]**This means that

$$\frac{x^2}{(x+2)^2} = 1 - \frac{4}{x+2} + \frac{4}{(x+2)^2}$$

$$\int \frac{x^2}{(x+2)^2}\,dx = x - 4\log(x+2) - \frac{4}{(x+2)}.\mathbf{[2]}$$

(A2) If $x = f(y) = \log(1+y^2)$ **[1]**then $e^x = 1+y^2$ **[1]**, so $e^x - 1 = y^2$, so $f^{-1}(x) = y = \sqrt{e^x - 1}$ **[1]**.

(A3)

$$(\log \circ f \circ \exp)(x) = \log(f(\exp(x)))\mathbf{[1]} = \log(2(e^x)^3)\mathbf{[1]} = \log(2) + 3\log(e^x)\mathbf{[1]} = \log(2) + 3x\mathbf{[1]}.$$

(A4) We note that $1000 = 10^3 = \sqrt{10}^6$ **[1]**, so $\sqrt{10} = (1000)^{1/6}$, so $\log_{1000}(\sqrt{10}) = 1/6$ **[1]**.

(A5) Note that $\tan(x)$ repeats with period π **[1]**, so

$$\tan(9999\pi/4) = \tan(9999\pi/4 - 2500\pi) = \tan(-\pi/4)\mathbf{[1]} = -1\mathbf{[1]}.$$

(A6) Put $u = e^x$. Then

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(u-u^{-1})/2}{(u+u^{-1})/2} = \frac{u-u^{-1}}{u+u^{-1}}, \mathbf{[2]}$$

so

$$\begin{aligned} 1+\tanh(x)^2 &= 1 + \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 + \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}} \\ &= \frac{(u^2+2+u^{-2}) + (u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{2u^2+2u^{-2}}{u^2+2+u^{-2}}\mathbf{[2]} \\ 1-\tanh(x)^2 &= 1 - \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 - \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}} \\ &= \frac{(u^2+2+u^{-2}) - (u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{4}{u^2+2+u^{-2}}\mathbf{[1]} \end{aligned}$$

so

$$\frac{1+\tanh(x)^2}{1-\tanh(x)^2} = \frac{2u^2+2u^{-2}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x)\mathbf{[2]}.$$

(A7) Put $u = x^p - x^q$, so $y = u^{1/pq}$ **[1]**. Then

$$du/dx = px^{p-1} - qx^{q-1} = x^{-1}(px^p - qx^q)\mathbf{[1]}$$

and

$$\frac{dy}{du} = \frac{1}{pq}u^{1/pq-1} = \frac{1}{pq}(x^p - y^q)^{1/pq-1}\mathbf{[1]}.$$

We therefore have

$$\begin{aligned} x(x^p - x^q)\frac{dy}{dx} &= xu\frac{dy}{du}\frac{du}{dx} \\ &= x(x^p - x^q)\frac{1}{pq}(x^p - x^q)^{1/pq-1}x^{-1}(px^p - qx^q) \\ &= (x^p - x^q)^{1/pq}(px^p - qx^q)/(pq) \\ &= (x^p - x^q)^{1/pq}(x^p/q - x^q/p)\mathbf{[3]}. \end{aligned}$$

(A8) Put $u = x + 2x^2 + 3x^3 + 4x^4$ and $y = \log(u)$, so

$$y' = \frac{u'}{u} = \frac{1 + 4x + 9x^2 + 16x^3}{x + 2x^2 + 3x^3 + 4x^4}\mathbf{[2]}.$$

(A9) The quotient rule gives

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2}{\log(x)}\right) &= \frac{2x.\log(x) - x^2.\log'(x)}{\log(x)^2}\mathbf{[2]} = \frac{2x\log(x) - x^2.x^{-1}}{\log(x)^2}\mathbf{[1]} \\ &= \frac{2x}{\log(x)} - \frac{x}{\log(x)^2}\mathbf{[1]}. \end{aligned}$$

(A10) First put $u = -1/(x+a)$, so $du/dx = 1/(x+a)^2 = (x+a)^{-2}$ **[1]**. Then put $v = \exp(u) = e^{-1/(x+a)}$, so the chain rule gives

$$\frac{dv}{dx} = (x+a)^{-2}e^{-1/(x+a)}\mathbf{[1]}.$$

Finally, we apply the product rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2\,v) = 2x.v + x^2\frac{dv}{dx} \\ &= (2x + x^2(x+a)^{-2})e^{-1/(x+a)}\mathbf{[2]} \\ &= (2x^2 + (4a+1)x + 2a^2)x(x+a)^{-2}e^{-1/(x+a)}. \end{aligned}$$

(A11) We know that

$$\int (x^2 - x + 1)e^{-x}\,dx = (ax^2 + bx + c)e^{-x}$$

for some constants a, b and c **[2]**. To find these, we differentiate to get

$$\begin{aligned} (x^2 - x + 1)e^{-x} &= \frac{d}{dx}((ax^2 + bx + c)e^{-x}) = (2ax + b)e^{-x} - (ax^2 + bx + c)e^{-x}\mathbf{[1]} \\ &= (-ax^2 + (2a - b)x + (b - c))e^{-x}.\mathbf{[1]} \end{aligned}$$

We equate coefficients to see that $-a = 1$ and $2a - b = -1$ and $b - c = 1$ **[1]**, which gives $a = -1$ and $b = -1$ and $c = -2$. We conclude that

$$\int (x^2 - x + 1)e^{x}\,dx = -(x^2 + x + 2)e^{-x}.\mathbf{[1]}$$

Alternatively, one can integrate by parts:

$$\begin{aligned} \int (x^2 - x + 1)e^{-x}\,dx &= (x^2 - x + 1)(-e^{-x}) - \int (2x - 1)(-e^{-x})\,dx \\ &= (-x^2 + x - 1)e^{-x} + \int (2x - 1)e^{-x}\,dx \\ &= (-x^2 + x - 1)e^{-x} + (2x - 1)(-e^{-x}) - \int 2(-e^{-x})\,dx \\ &= (-x^2 + x - 1)e^{-x} + (-2x + 1)e^{-x} + \int 2e^{-x}\,dx \\ &= (-x^2 + x - 1)e^{-x} + (-2x + 1)e^{-x} - 2e^{-x} \\ &= -(x^2 + x + 2)e^{-x}. \end{aligned}$$

(A12) We know that

$$\int e^{-3x}\cos(4x)\,dx = e^{-3x}(A\cos(4x)+B\sin(4x))$$

for some A and B **[2]**. To find these, we differentiate and equate coefficients:

$$\begin{aligned} e^{-3x}\cos(4x) &= \frac{d}{dx}\left(e^{-3x}(A\cos(4x)+B\sin(4x))\right) \\ &= -3e^{-3x}(A\cos(4x)+B\sin(4x)) + e^{-3x}(-4A\sin(4x)+4B\cos(4x)) \\ &= e^{3x}((-3A+4B)\cos(4x)+(-3B-4A)\sin(4x))\mathbf{[1]}, \end{aligned}$$

so $-3A+4B=1$ and $-3B-4A=0$ **[1]**. These equations give $A=-3/25$ and $B=4/25$ **[1]**, so

$$\int e^{-3x}\cos(4x)\,dx = e^{-3x}(-3\cos(4x)+4\sin(4x))/25.\mathbf{[1]}$$

(A13) The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & -1 & 2 \end{array}\right] \mathbf{[1]}$$

This can be row-reduced as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 0 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -2 & -2 & 0 & 4 \\ 0 & 0 & -2 & 0 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right].\mathbf{[2]}$$

There is no pivot in the last column, so the variable z is independent **[1]**. The final matrix corresponds to the equations $w-z=0$ and $x=y=-1$, so $(w,x,y,z)=(z,-1,-1,z)$ **[1]**.

Alternatively (and more efficiently), one can manipulate the equations directly. Subtracting adjacent equations gives $x=y=-1$, and the rest is then easy.

(A14) To get the adjugate, we write down the matrix of minors, transpose it, and then multiply by the associated signs:

$$\begin{bmatrix} -9 & -25 & 12 \\ -24 & 0 & -18 \\ 9 & -25 & -12 \end{bmatrix} \to \begin{bmatrix} -9 & -24 & 9 \\ -25 & 0 & -25 \\ 12 & -18 & -12 \end{bmatrix} \mathbf{[2]} \to \begin{bmatrix} -9 & 24 & 9 \\ 25 & 0 & 25 \\ 12 & 18 & -12 \end{bmatrix} \mathbf{[2]}.$$

The determinant is the dot product of the first row of A with the first column of $\mathrm{adj}(A)$, which is

$$\det(A) = (-3,3,4).(-9,25,12) = 150.\mathbf{[2]}$$

This gives

$$A^{-1} = \frac{1}{150}\begin{bmatrix} -9 & 24 & 9 \\ 25 & 0 & 25 \\ 12 & 18 & -12 \end{bmatrix} = \begin{bmatrix} -3/50 & 4/25 & 3/50 \\ 1/6 & 0 & 1/6 \\ 2/25 & 3/25 & -2/25 \end{bmatrix}.\mathbf{[2]}$$

Alternatively, we can write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|ccc} -3 & 3 & 4 & 1 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 & 0 \\ 3 & 3 & -4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & -4/3 & -1/3 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 & 0 \\ 3 & 3 & -4 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -4/3 & -1/3 & 0 & 0 \\ 0 & 4 & 25/3 & 4/3 & 1 & 0 \\ 0 & 6 & 0 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & -4/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 1/6 & 0 & 1/6 \\ 0 & 4 & 25/3 & 4/3 & 1 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4/3 & -1/6 & 0 & 1/6 \\ 0 & 1 & 0 & 1/6 & 0 & 1/6 \\ 0 & 0 & 25/3 & 2/3 & 1 & -2/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -4/3 & -1/6 & 0 & 1/6 \\ 0 & 1 & 0 & 1/6 & 0 & 1/6 \\ 0 & 0 & 1 & 2/25 & 3/25 & -2/25 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/50 & 4/25 & 3/50 \\ 0 & 1 & 0 & 1/6 & 0 & 1/6 \\ 0 & 0 & 1 & 2/25 & 3/25 & -2/25 \end{array}\right]$$

At the final stage, the left hand block is the identity, so the right hand block is the inverse of the original matrix.

(B1) Observe that $f(x) = (x+1)^2 + 2$. As x runs from -2 to 1 (including the endpoints), $x+1$ increases from -1 to 2. Thus $(x+1)^2 + 2$ decreases from 3 to 2 and then increases again to 6. The range is thus $[2, 6]$. Alternatively, one can read this off from the graph. **[4]**

(B2) Note that $\sin(x)\cos(x) = \sin(2x)/2$ **[2]**, so

$$\sin(x)^2 \cos(x)^2 = \sin(2x)^2/4 = (1 - \cos(4x))/8[\mathbf{2}].$$

Thus

$$\begin{aligned} \int \sin(x)^2 \cos(x)^2 \, dx &= \tfrac{1}{8} \int 1 - \cos(4x) \, dx[\mathbf{2}] \\ &= \frac{x}{8} - \frac{\sin(4x)}{32} = \frac{4x - \sin(4x)}{32}[\mathbf{2}]. \end{aligned}$$

(B3) Put $u = \sin(x)$, so $du = \cos(x)\, dx$ **[2]**. Then

$$\begin{aligned} \int \cos(x) \log(\sin(x)) \, dx &= \int \log(u) \, du = u \log(u) - u[\mathbf{3}] = \\ &= \sin(x)(\log(\sin(x)) - 1)[\mathbf{1}]. \end{aligned}$$

(B4) We first note that

$$\begin{aligned} \frac{d}{dx}\left(x^4(a\log(x)^2 + b\log(x) + c)\right) &= 4x^3(a\log(x)^2 + b\log(x) + c) + x^4(2a\log(x)/x + b/x)[\mathbf{2}] \\ &= x^3(4a\log(x)^2 + (4b + 2a)\log(x) + (4c + b)).[\mathbf{1}] \end{aligned}$$

This must also be equal to $x^3 \log(x)^2$ for all x, **[1]**so we must have

$$\begin{aligned} 4a &= 1 \\ 4b + 2a &= 0 \\ 4c + b &= 0, [\mathbf{1}] \end{aligned}$$

so $a = 1/4$ and $b = -1/8$ and $c = 1/32$, **[1]**giving

$$\int x^3 \log(x)^2 \, dx = x^4(\log(x)^2/4 - \log(x)/8 + 1/32) = x^4(8\log(x)^2 - 4\log(x) + 1)/32.[\mathbf{1}]$$

It follows that

$$\begin{aligned} \int_1^e x^3 \log(x)^2 \, dx &= \left[x^4(\log(x)^2/4 - \log(x)/8 + 1/32)\right]_1^e [\mathbf{1}] \\ &= e^4(1/4 - 1/8 + 1/32) - 1^3(0/4 - 0/8 + 1/32) \\ &= (5e^4 - 1)/32.[\mathbf{1}] \end{aligned}$$

(B5) Put

$$A = \begin{bmatrix} -a & a & 1 \\ 1 & 0 & -a \\ a & a & -1 \end{bmatrix}.$$

Then

$$\begin{aligned}\det(A) &= -a\det\begin{bmatrix} 0 & -a \\ a & -1 \end{bmatrix} - a\det\begin{bmatrix} 1 & -a \\ a & -1 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 \\ a & a \end{bmatrix} \textbf{[3]} \\ &= -a(0-(-a^2)) - a(-1-(-a^2)) + (a-0) = -a^3 + a - a^3 + a \\ &= 2a(1-a^2).\textbf{[1]}\end{aligned}$$

The matrix is invertible for all a where $2a(1-a^2) \neq 0$, or in other words all a except $a = 0$ or $a = \pm 1$. **[3]**