UNIVERSITY OF SHEFFIELD PMA101R

PURE MATHEMATICS

Autumn Semester 2004–2005 2 **hours**

Pure Mathematics Core

Attempt ***all*** *the questions. The allocation of marks is shown in brackets; Section A is worth 66 marks in total, and Section B is worth 34 marks.*

A1 Convert the function $x^2/(x+2)^2$ to partial fraction form, and thus find $\displaystyle\int \frac{x^2}{(x+2)^2}\,dx$. ***(6 marks)***

A2 Let $f\colon (0,\infty) \to (0,\infty)$ be given by $f(x) = \log(1+x^2)$. Find a formula for $f^{-1}(x)$. ***(3 marks)***

A3 If $f(x) = 2x^3$, what is $(\log \circ f \circ \exp)(x)$? Simplify your answer as much as possible. ***(4 marks)***

A4 Find $\log_{1000}(\sqrt{10})$. ***(2 marks)***

A5 Find $\tan(9999\pi/4)$,giving a brief justification for your answer. ***(3 marks)***

A6 Show that $\dfrac{1+\tanh(x)^2}{1-\tanh(x)^2} = \cosh(2x)$. ***(7 marks)***

A7 Let p and q be nonzero constants, and put $y = (x^p - x^q)^{1/pq}$. Simplify $x(x^p - x^q)\dfrac{dy}{dx}$. ***(6 marks)***

A8 Find $\dfrac{d}{dx}\log(x + 2x^2 + 3x^3 + 4x^4)$. ***(2 marks)***

A9 Find $\dfrac{d}{dx}\left(\dfrac{x^2}{\log(x)}\right)$. ***(4 marks)***

A10 Let a be a constant. Find $f'(x)$, where $f(x) = x^2\, e^{-1/(x+a)}$. ***(4 marks)***

A11 Find $\int (x^2 - x + 1)e^{-x}\,dx$ ***(6 marks)***

A12 Find $\int e^{-3x}\cos(4x)\,dx$ ***(6 marks)***

A13 Find the general solution of the following system of equations:

$$\begin{aligned} w + x + y - z &= -2 \\ w + x - y - z &= 0 \\ w - x - y - z &= 2. \end{aligned}$$

(5 marks)

A14 Find the determinant and inverse of the matrix

$$A = \begin{bmatrix} -3 & 3 & 4 \\ 4 & 0 & 3 \\ 3 & 3 & -4 \end{bmatrix}.$$

(Hint: You will find it easiest to use the cofactor method.) ***(8 marks)***

B1 Define $f\colon [-2, 1] \to \mathbb{R}$ by $f(x) = x^2 + 2x + 3$. Find the range of f. ***(4 marks)***

B2 Find $\int \sin(x)^2 \cos(x)^2\,dx$. (Hint: what are the formulae for $\sin(2\theta)$ and $\cos(2\theta)$?) ***(8 marks)***

B3 By making a suitable substitution, find $\int \cos(x)\log(\sin(x))\,dx$. ***(6 marks)***

B4 You may assume that $\int x^3 \log(x)^2\,dx = x^4(a\log(x)^2 + b\log(x) + c)$ for some constants a, b and c. Find these constants, and thus evaluate $\int_1^e x^3 \log(x)^2\,dx$. ***(9 marks)***

B5 Find the determinant of the following matrix:

$$\begin{bmatrix} -a & a & 1 \\ 1 & 0 & -a \\ a & a & -1 \end{bmatrix}.$$

For which values of a is the matrix invertible? ***(7 marks)***

End of Question Paper