Pure Mathematics Core — Mock exam solutions

(A1) The denominator factorises as $(x+1)^2(x-1)$, so the general form is

$$\frac{4}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}.$$

Multiplying by $(x+1)^2(x-1)$ gives

$$\begin{aligned} 4 &= A(x+1)(x-1) + B(x-1) + C(x+1)^2 \\ &= Ax^2 - A + Bx - B + Cx^2 + 2Cx + 2C \\ &= (A+C)x^2 + (B+2C)x + (-A-B+C). \end{aligned}$$

We can now equate coefficients to see that

$$\begin{aligned} A + C &= 0 \\ B + 2C &= 0 \\ -A - B + C &= 4. \end{aligned}$$

The first two equations give $A = -C$ and $B = -2C$; substituting these into the third equation gives

$$4 = -(-C) - (-2C) + C = 4C,$$

so $C = 1$, $B = -2$ and $A = -1$. We conclude that

$$f(x) = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$$

so

$$\int f(x)\,dx = \log(x-1) - \log(x+1) + 2(x+1)^{-1}.$$

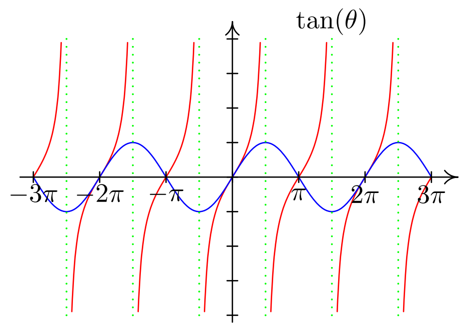
(A2) Put $y = f(x) = (5x+4)/(4x+5)$. Then $4xy + 5y = 5x + 4$, so $4xy - 5x = 4 - 5y$, so $(4y-5)x = 4 - 5y$, so $f^{-1}(y) = x = (4-5y)/(4y-5)$. Changing notation, we find that $f^{-1}(x) = (4-5x)/(4x-5)$.

(A3) If $x = f(y) = a + y$ then $f^{-1}(x) = y = x - a$. Similarly, if $x = g(y) = b - y$ then $g^{-1}(x) = y = b - x$. Finally, we have

$$(g \circ f \circ g)(x) = g(f(g(x))) = g(f(b-x)) = g(a+b-x) = b - (a+b-x) = x - a.$$

(A4) Note that $e^a e^b = e^{a+b}$ and $(e^c)^d = e^{cd}$, so $e^a e^b/((e^c)^d) = e^{a+b-cd}$. It follows that $\log(e^a e^b/(e^c)^d) = \log(e^{a+b-cd}) = a + b - cd$.

(A5)



We see from this that $\tan(5\pi/4) = 1$ and $\sin(5\pi/4) = -2^{-1/2} < 0$.

(A6) Put $u = e^x$, so $\cosh(x) = (u + u^{-1})/2$. Then

$$\begin{aligned} 8\cosh(x)^4 &= 8(u+u^{-1})^4/16 \\ &= (u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4})/2 \\ -8\cosh(u)^2 &= -8(u+u^{-1})^2/4 \\ &= (-4u^2 - 8 - 4u^{-2})/2, \end{aligned}$$

so

$$8\cosh(x)^4 - 8\cosh(x)^2 + 1 = (u^4 + u^{-4})/2 = \cosh(4x).$$

(A7) Put $u = (x-a)/(x-b)$ and $y = f(x) = u^n$. Then

$$\frac{du}{dx} = \frac{1.(x-b) - (x-a).1}{(x-b)^2} = \frac{a-b}{(x-b)^2},$$

so

$$f'(x) = \frac{dy}{dx} = nu^{n-1}\frac{du}{dx} = n(a-b)\left(\frac{x-a}{x-b}\right)^{n-1}(x-b)^{-2} = n(a-b)(x-a)^{n-1}(x-b)^{-n-1}.$$

(A8) By the chain rule, we have

$$\frac{d}{dx}\cos\left(\left(\frac{x+1}{2}\right)^2\right) = -\sin\left(\left(\frac{x+1}{2}\right)^2\right).\frac{d}{dx}\left(\frac{x+1}{2}\right)^2 = -\sin\left(\left(\frac{x+1}{2}\right)^2\right).\frac{x+1}{2}.$$

(A9) By the logarithmic rule, we have

$$\frac{d}{dx}\log(\cos(x)) = \frac{\cos'(x)}{\cos(x)} = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

(A10) Put $u = ax + b/x$ and $v = cx + d/x$ and $y = u/v$; we must find y'. Note that

$$\begin{aligned}
u' &= a - b/x^2 \\
v' &= c - d/x^2 \\
u'v - uv' &= (a - b/x^2)(cx + d/x) - (ax + b/x)(c - d/x^2) \\
&= acx + ad/x - bc/x - bd/x^3 - acx + ad/x - bc/x + bd/x^3 \\
&= 2(ad - bc)/x, \\
y' &= \frac{u'v - uv'}{v^2} = \frac{2(ad-bc)}{x(cx + d/x)^2}.
\end{aligned}$$

(A11) Put $y = \sqrt{2\pi}x^{x-1/2}e^{-x}$, so

$$\log(y) = \log(\sqrt{2\pi}) + (x - 1/2)\log(x) - x,$$

so

$$\begin{aligned}
\frac{y'}{y} &= \log(y)' \\
&= 0 + 1.\log(x) + (x - 1/2).\log'(x) - 1 \\
&= \log(x) + (x - 1/2)/x - 1 = \log(x) + 1 - 1/(2x) - 1 \\
&= \log(x) - 1/(2x).
\end{aligned}$$

(A12) Put $u = \log(x)$, so $du = x^{-1}dx$. Then

$$\begin{aligned}
\int \frac{(1+\log(x))^2}{x}\,dx &= \int (1+u)^2 du = (1+u)^3/3 \\
&= (1+\log(x))^3/3.
\end{aligned}$$

(A13) The general form is

$$\int (4x^2 + 2x + 1)e^{2x}\,dx = (Ax^2 + Bx + C)e^{2x}$$

for some constants A, B and C. Differentiating, we find that

$$\begin{aligned}
(4x^2 + 2x + 1)e^{2x} &= \frac{d}{dx}((Ax^2 + Bx + C)e^{2x}) \\
&= (2Ax + B)e^{2x} + (Ax^2 + Bx + C).2e^{2x} \\
&= (2Ax^2 + (2A + 2B)x + (B + 2C))e^{2x},
\end{aligned}$$

so $2A = 4$ and $2A + 2B = 2$ and $B + 2C = 1$. It follows that $A = 2$ and $B = -1$ and $C = 1$, so

$$\int (4x^2 + 2x + 1)e^{2x}\,dx = (2x^2 - x + 1)e^{2x}.$$

(A14) The augmented matrix for this problem is

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ -1 & 0 & 3 & 2 \\ -2 & -3 & 0 & 1 \end{array}\right].$$

This can be row-reduced as follows:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ -1 & 0 & 3 & 2 \\ -2 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ -2 & -3 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -6 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$$

There is no pivot in the third column, so the variable z is independent. The final matrix corresponds to the equations $x = 3z - 2$ and $y = -2z + 1$, which give the general solution.

(A15)

$$\begin{aligned} \det\begin{pmatrix} t & a & b \\ a & t & c \\ b & c & t \end{pmatrix} &= t\det\begin{pmatrix} t & c \\ c & t \end{pmatrix} - a\det\begin{pmatrix} a & c \\ b & t \end{pmatrix} + b\det\begin{pmatrix} a & t \\ b & c \end{pmatrix} \\ &= t(t^2 - c^2) - a(at - bc) + b(ac - bt) \\ &= t^3 - c^2t - a^2t + abc + abc - b^2t \\ &= t^3 - (a^2 + b^2 + c^2)t + 2abc. \end{aligned}$$

(B1) As x increases from zero to infinity, e^{-x} decreases from 1 to 0 , without ever reaching 0 . This means that $\pi e^{-x}/2$ decreases from $\pi/2$ to 0 (without ever reaching 0) , and thus that $\sin(\pi e^{-x}/2)$ decreases from $\sin(\pi/2) = 1$ to 0 (without ever reaching 0) . This means that the range of g is $(0, 1]$.

(B2) First note that $8x\sin(x)\cos(x) = 4x\sin(2x)$, so

$$\begin{aligned} \int 8x\sin(x)\cos(x)\,dx &= \int 4x\sin(2x)\,dx \\ &= -2x\cos(2x) + \int 2\cos(2x)\,dx \\ &= -2x\cos(2x) + \sin(2x). \end{aligned}$$

(B3) Put $u = x^n$, so $du = nx^{n-1}\,dx$, so $dx = du/(nx^{n-1})$. The integral becomes

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^{-2n} - 1}} &= \int \frac{du}{nx^{n-1}.x\sqrt{x^{-2n} - 1}} = \frac{1}{n}\int \frac{du}{x^n\sqrt{x^{-2n} - 1}} \\ &= \frac{1}{n}\int \frac{du}{u\sqrt{u^{-2} - 1}} = \frac{1}{n}\int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u)/n \\ &= \arcsin(x^n)/n. \end{aligned}$$

(B4) The efficient method is as follows: we have $x^5 - 1 = (x^4 + x^3 + x^2 + x + 1)(x - 1)$, so

$$\frac{x^5 - 1}{x^2(x - 1)} = \frac{x^4 + x^3 + x^2 + x + 1}{x^2} = x^2 + x + 1 + x^{-1} + x^{-2},$$

so

$$\int \frac{x^5 - 1}{x^2(x - 1)} = x^3/3 + x^2/2 + x + \log(x) - x^{-1}.$$

If we follow the partial fraction method more mechanically, the solution is as follows:

$$\frac{x^5 - 1}{x^2(x - 1)} = Ax^2 + Bx + C + \frac{D}{x} + \frac{E}{x^2} + \frac{F}{x - 1}$$

for some constants $A,\dots,F$. Multiplying by $x^2(x-1)$ gives

$$\begin{aligned}x^5-1 &= (Ax^2+Bx+C)x^2(x-1)+Dx(x-1)+E(x-1)+Fx^2\\ &= Ax^5+Bx^4+Cx^3-Ax^4-Bx^3-Cx^2+Dx^2-Dx+Ex-E+Fx^2\\ &= Ax^5+(B-A)x^4+(C-B)x^3+(D+F-C)x^2+(E-D)x-E,\end{aligned}$$

so

$$\begin{aligned}A&=1\\ B-A&=0\\ C-B&=0\\ D+F-C&=0\\ E-D&=0\\ -E&=-1.\end{aligned}$$

The first three equations give $A=B=C=1$, the last two give $D=E=1$, and the remaining equation then gives $F=0$. The conclusion is that

$$\frac{x^5-1}{x^2(x-1)}=x^2+x+1+x^{-1}+x^{-2},$$

so

$$\int\frac{x^5-1}{x^2(x-1)}=x^3/3+x^2/2+x+\log(x)-x^{-1}.$$

as before.

(B5) We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{cccc|cccc}0&1&1&1&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&1&0&0&1&0\\1&1&1&1&0&0&0&1\end{array}\right]\xrightarrow{1}\left[\begin{array}{cccc|cccc}1&1&1&1&0&0&0&1\\0&1&1&1&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&1&1&0&0&1&0\end{array}\right]\xrightarrow{2}\left[\begin{array}{cccc|cccc}1&1&1&1&0&0&0&1\\0&1&1&1&1&0&0&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&1&0&0\end{array}\right]\xrightarrow{3}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&-1&0&0&1\\0&1&1&1&1&0&0&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&1&0&0\end{array}\right]\xrightarrow{4}\left[\begin{array}{cccc|cccc}1&0&0&0&-1&0&0&1\\0&1&0&0&1&0&-1&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&1&0&0\end{array}\right]\xrightarrow{5}\left[\begin{array}{cccc|cccc}1&0&0&0&-1&0&0&1\\0&1&0&0&1&0&-1&0\\0&0&1&0&0&-1&1&0\\0&0&0&1&0&1&0&0\end{array}\right]$$

In step 1 we reorder the rows by bringing R_4 to the top, then in step 2 we exchange R_3 and R_4. The next three steps are $R_1\mapsto R_1-R_2$, $R_2\mapsto R_2-R_3$ and $R_3\mapsto R_3-R_4$. At this stage the left hand block is the identity, so the right hand block is the inverse of our original matrix, or in other words

$$\begin{pmatrix}0&1&1&1\\0&0&0&1\\0&0&1&1\\1&1&1&1\end{pmatrix}^{-1}=\begin{pmatrix}-1&0&0&1\\1&0&-1&0\\0&-1&1&0\\0&1&0&0\end{pmatrix}.$$