Pure Mathematics Core — Exam solutions

(A1) The general form is

$$\frac{x^2+x+1}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2}.\mathbf{[2]}$$

Multiplying by $(x+1)^2$ gives

$$\begin{aligned} x^2+x+1 &= A(x+1)^2 + B(x+1) + C = Ax^2 + 2Ax + A + Bx + B + C \\ &= Ax^2 + (2A+B)x + (A+B+C), \mathbf{[1]} \end{aligned}$$

so $A = 1$ and $2A + B = 1$ and $A + B + C = 1$, which gives $B = -1$ and $C = 1$. **[1]**This means that

$$\frac{x^2+x+1}{(x+1)^2} = 1 - \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

$$\int \frac{x^2+x+1}{(x+1)^2}\,dx = x - \log(x+1) - \frac{1}{(x+1)}.\mathbf{[2]}$$

(A2) If $x = f(y) = 1/(1-e^{-y})$ **[1]**then $1/x = 1 - e^{-y}$ **[1]**, so $e^{-y} = 1 - 1/x = (x-1)/x$, so $e^y = x/(x-1)$ **[1]**, so $f^{-1}(x) = y = \log(x/(x-1))$ **[1]**.

(A3)

$$(\exp\circ f\circ\log)(x) = \exp(f(\log(x)))\mathbf{[1]} = \exp(2\log(x)+2)\mathbf{[1]} = (e^{\log(x)})^2e^2\mathbf{[1]} = e^2x^2\mathbf{[1]}.$$

(A4) We note that $16 = 2^4$ **[1]**, so $2 = 16^{1/4}$, so $1/2 = 16^{-1/4}$, so $\log_{16}(1/2) = -1/4$ **[1]**.

(A5) Note that $\sin(x)$ repeats with period 2π, so

$$\sin(-7\pi/3) = \sin(-7\pi/3 + 2\pi) = \sin(-\pi/3)\mathbf{[1]} = -\sin(\pi/3) = -\sqrt{3}/2\mathbf{[1]}.$$

(A6) Put $u = e^x$. Then

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(u-u^{-1})/2}{(u+u^{-1})/2} = \frac{u-u^{-1}}{u+u^{-1}}, \mathbf{[2]}$$

so

$$\begin{aligned} 1+\tanh(x)^2 &= 1 + \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 + \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}} \\ &= \frac{(u^2+2+u^{-2}) + (u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{2u^2+2u^{-2}}{u^2+2+u^{-2}}\mathbf{[2]} \\ 1-\tanh(x)^2 &= 1 - \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 - \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}} \\ &= \frac{(u^2+2+u^{-2}) - (u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{4}{u^2+2+u^{-2}}\mathbf{[1]} \end{aligned}$$

so

$$\frac{1+\tanh(x)^2}{1-\tanh(x)^2} = \frac{2u^2+2u^{-2}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x)\mathbf{[1]}.$$

(A7) Put $u = x^n + a$ and $y = f(x) = u^m$. Then $du/dx = nx^{n-1}$ and $dy/du = mu^{m-1}$, so

$$f'(x) = \frac{dy}{dx} = mu^{m-1}\frac{du}{dx} = mn(x^n+a)^{m-1}x^{n-1}.\mathbf{[2]}$$

(A8) Put $u = 1 + x + x^2 + x^3$ and $y = \log(u)$, so

$$y' = \frac{u'}{u} = \frac{1+2x+3x^2}{1+x+x^2+x^3}\mathbf{[2]}.$$

(A9) The quotient rule gives

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - x.\log'(x)}{\log(x)^2}[\mathbf{1}] = \frac{\log(x) - x.x^{-1}}{\log(x)^2}[\mathbf{1}]$$
$$= \frac{1}{\log(x)} - \frac{1}{\log(x)^2}[\mathbf{1}].$$

(A10)

$$\frac{d}{dx}\left(\frac{3x+2}{4x+3}\right) = \frac{3(4x+3) - 4(3x+2)}{(4x+3)^2}[\mathbf{1}]$$
$$= \frac{12x + 9 - 12x - 8}{(4x+3)^2} = (4x+3)^{-2}[\mathbf{1}]$$

(A11) First put $u = -(x-a)^2/b$, so $du/dx = -2(x-a)/b$. Then put $v = \exp(u) = e^{-(x-a)^2/b}$, so the chain rule gives

$$\frac{dv}{dx} = -2(x-a)b^{-1}e^{-(x-a)^2/b}.[\mathbf{2}]$$

Finally, we apply the product rule:

$$\frac{d}{dx}\left(e^{-(x-a)^2/b}\sin(\omega x)\right) = -2(x-a)b^{-1}e^{-(x-a)^2/b}\sin(\omega x) + e^{-(x-a)^2/b}\omega\cos(\omega x)$$
$$= e^{-(x-a)^2/b}(\omega\cos(\omega x) - 2(x-a)b^{-1}\sin(\omega x))[\mathbf{2}].$$

(A12) We know that

$$\int x^2 e^x\,dx = (ax^2 + bx + c)e^x$$

for some constants a, b and c [**2**]. To find these, we differentiate to get

$$x^2e^x = \frac{d}{dx}((ax^2+bx+c)e^x) = (2ax+b)e^x + (ax^2+bx+c)e^x[\mathbf{1}]$$
$$= (ax^2 + (2a+b)x + (b+c))e^x.$$

We equate coefficients to see that $a = 1$ and $2a + b = b + c = 0$ [**1**], which gives $b = -2$ and $c = 2$. We conclude that

$$\int x^2e^x\,dx = (x^2 - 2x + 2)e^x.[\mathbf{1}]$$

(A13) We know that

$$\int e^{3x}\sin(4x)\,dx = e^{3x}(A\cos(4x) + B\sin(4x))$$

for some A and B [**2**]. To find these, we differentiate and equate coefficients:

$$e^{3x}\sin(4x) = \frac{d}{dx}\left(e^{3x}(A\cos(4x) + B\sin(4x))\right)$$
$$= 3e^{3x}(A\cos(4x) + B\sin(4x)) + e^{3x}(-4A\sin(4x) + 4B\cos(4x))$$
$$= e^{3x}((3A+4B)\cos(4x) + (3B-4A)\sin(4x))[\mathbf{1}],$$

so $3A + 4B = 0$ and $3B - 4A = 1$ [**1**]. This gives $A = -4B/3$ so $1 = 3B - 4A = 3B + 16B/3 = 25B/3$, so $B = 3/25$, so $A = -4B/3 = -4/25$. The conclusion is that

$$\int e^{3x}\sin(4x)\,dx = e^{3x}(3\sin(4x) - 4\cos(4x))/25.[\mathbf{1}]$$

(A14) The matrix of coefficients is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}[\mathbf{1}]$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \textbf{[2M 2A]}$$

There is no pivot in the last column, so the variable z is independent [**1**]. The final matrix corresponds to the equations $w - z = x + z = y + z = 0$, so $(w, x, y, z) = (z, -z, -z, z)$ [**1**].

(A15) We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] [\mathbf{1}] \to \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \textbf{[2M 2A]}$$

At the final stage, the left hand block is the identity, so the right hand block is the inverse of the original matrix, ie

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{1}].$$

(B1) Observe that $f(x) = (x+1)^2 + 2$ [**1**]. As x runs from -1 to 1 (excluding the endpoints), $x+1$ increases from 0 to 2 and so $(x+1)^2+2$ increases from $0^2+2 = 2$ to $2^2+2 = 6$ [**2**]. In all cases the endpoints are excluded, so the range of f is $(2, 6)$ [**1**].

(B2) Note that $\sin(x)\cos(x) = \sin(2x)/2$ [**1**], so

$$\sin(x)^2\cos(x)^2 = \sin(2x)^2/4[\mathbf{1}] = (1-\cos(4x))/8[\mathbf{2}].$$

Thus

$$\begin{aligned}\int \sin(x)^2\cos(x)^2\,dx &= \tfrac{1}{8}\int 1-\cos(4x)\,dx[\mathbf{1}] \\ &= \frac{x}{8} - \frac{\sin(4x)}{32} = \frac{4x - \sin(4x)}{32}[\mathbf{2}].\end{aligned}$$

(B3) Put $u = \cos(x)$, so $du = -\sin(x)\,dx$ [**2**]. Then

$$\begin{aligned}\int \sin(x)\log(\cos(x))\,dx &= -\int \log(u)\,du[\mathbf{1}] = -(u\log(u) - u)[\mathbf{2}] = u(1-\log(u)) \\ &= \cos(x)(1-\log(\cos(x)))[\mathbf{1}].\end{aligned}$$

(B4) We first note that

$$\begin{aligned}\frac{d}{dx}\left(x^3(a\log(x)^2 + b\log(x) + c)\right) &= 3x^2(a\log(x)^2 + b\log(x) + c) + x^3(2a\log(x)/x + b/x)[\mathbf{1}] \\ &= x^2(3a\log(x)^2 + (3b+2a)\log(x) + (3c+b)).[\mathbf{1}]\end{aligned}$$

This must also be equal to $x^2\log(x)^2$ for all x, [**1**]so we must have

$$\begin{aligned}3a &= 1 \\ 3b + 2a &= 0 \\ 3c + b &= 0, [\mathbf{1}]\end{aligned}$$

so $a = 1/3$ and $b = -2/9$ and $c = 2/27$, [**1**]giving

$$\int x^2\log(x)^2\,dx = x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27).$$

It follows that

$$\begin{aligned}\int_1^e x^2\log(x)^2\,dx &= \left[x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27)\right]_1^e \,[\mathbf{1}] \\ &= e^3(1/3 - 2/9 + 2/27) - 1^3(0/3 - 0/9 + 2/27) \\ &= (5e^3 - 2)/27.[\mathbf{1}]\end{aligned}$$

(B5) Put

$$A = \begin{bmatrix} 1 & a & 0 & 0 \\ a & 1 & b & 0 \\ 0 & b & 1 & c \\ 0 & 0 & c & 1 \end{bmatrix}.$$

The direct approach is as follows:

$$\begin{aligned}\det(A) &= \det\begin{bmatrix} 1 & b & 0 \\ b & 1 & c \\ 0 & c & 1 \end{bmatrix} - a\det\begin{bmatrix} a & b & 0 \\ 0 & 1 & c \\ 0 & c & 1 \end{bmatrix} [\mathbf{2}] \\ &= \left(\det\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} - b\det\begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix}\right) - a\left(a\det\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} - b\det\begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix}\right) [\mathbf{2}] \\ &= (1 - c^2 - b(b-0)) - a(a(1-c^2) - b.0)[\mathbf{1}] \\ &= 1 - a^2 - b^2 - c^2 + a^2c^2.[\mathbf{1}]\end{aligned}$$

Alternatively, if we subtract a times the first row from the second row, and subtract c times the fourth row from the third row, we obtain the matrix

$$B = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1-a^2 & b & 0 \\ 0 & b & 1-c^2 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

with $\det(A) = \det(B)$. We can expand down the first column to see that

$$\det(A) = \det(B) = \det\begin{bmatrix} 1-a^2 & b & 0 \\ b & 1-c^2 & 0 \\ 0 & c & 1 \end{bmatrix},$$

and then expand this down the last column to get

$$\det(A) = \det\begin{bmatrix} 1-a^2 & b \\ b & 1-c^2 \end{bmatrix} = (1-a^2)(1-c^2) - b^2 = 1 - a^2 - b^2 - c^2 + a^2c^2.$$