Pure Mathematics Core — Solutions for January 2002 exam

1 (i) Put $x = f(y) = (3-y)/(5-2y)$, so $f^{-1}(x) = y$. We have $(5-2y)x = 3-y$ so $5x - 2xy = 3 - y$ so $5x - 3 = 2xy - y = (2x-1)y$. Dividing by $2x-1$ gives $f^{-1}(x) = y = (5x-3)/(2x-1)$.

(ii) (a) By the product rule, we have

$$\frac{d}{dx}(e^{-x}\sin(x+1)) = \frac{d}{dx}(e^{-x})\sin(x+1) + e^{-x}\frac{d}{dx}\sin(x+1) = -e^{-x}\sin(x+1) + e^{-x}\cos(x+1) = e^{-x}(\cos(x+1) - \sin(x+1)).$$

(b) Put $u = x^2 + 3x + 1$ and $y = \cos(x^2+3x+1) = \cos(u)$. Then $du/dx = 2x+3$ and $dy/du = -\sin(u) = -\sin(x^2+3x+1)$ so the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -(2x+3)\sin(x^2+3x+1).$$

(c) Put $u = e^x$ and $v = 2 + \sin(x)$, so $du/dx = e^x$ and $dv/dx = \cos(x)$. Then

$$\frac{d}{dx}\left(\frac{e^x}{2+\sin(x)}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(du/dx)v - u(dv/dx)}{v^2} = \frac{e^x(2+\sin(x)) - e^x\cos(x)}{(2+\sin(x))^2} = \frac{2+\sin(x)-\cos(x)}{(2+\sin(x))^2}e^x.$$

(iii) (a) Put $y = \sinh^{-1}(x) = \operatorname{arcsinh}(x)$, so $x = \sinh(y)$. We then have $dx/dy = \cosh(y)$. Using the relation $\cosh(y)^2 - \sinh(y)^2 = 1$ we get $\cosh(y) = \sqrt{1+\sinh(y)^2} = \sqrt{1+x^2}$, so $dx/dy = \sqrt{1+x^2}$. It follows that

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sqrt{1+x^2}}.$$

(b) Put $y = x^{\sin(x)}$. Then $\log(y) = \sin(x)\log(x)$, so

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log(y) = \cos(x)\log(x) + \sin(x)x^{-1}.$$

It follows that

$$\frac{dy}{dx} = y(\cos(x)\log(x) + \sin(x)x^{-1}) = x^{\sin(x)}((\cos(x)\log(x) + \sin(x)x^{-1}).$$

2 (i) (a) Put $u = x^2$, so $du = 2x\,dx$. Then

$$\int x\sin(x^2)\,dx = \int x\sin(u)\frac{du}{2x} = \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u) = -\cos(x^2)/2.$$

(b) Put $v = x^2 + 4$, so $dv = 2x\,dx$. Then

$$\int \frac{3x}{x^2+4}\,dx = \frac{3}{2}\int \frac{dv}{v} = 3\log(v)/2 = 3\log(x^2+4)/2.$$

On the other hand, if we put $t = x/2$ we get $dt = dx/2$ and so

$$\int \frac{2}{x^2+4}dx = \int \frac{2}{4t^2+4}2dt = \int \frac{dt}{1+t^2} = \arctan(t) = \arctan(x/2).$$

Putting these together, we get

$$\int \frac{3x+2}{x^2+4}\,dx = \frac{3}{2}\log(x^2+4) + \arctan(x/2).$$

(ii) Put $u = \log(x)$, so $du = (dx)/x$. Note that when $x = 1$ we have $u = \log(1) = 0$, and when $x = e$ we have $u = \log(e) = 1$. We thus have

$$\int_{x=1}^{e} \frac{dx}{x(1+\log(x)^2)} = \int_{u=0}^{1} \frac{du}{1+u^2} = [\arctan(u)]_0^1 = \pi/4 - 0 = \pi/4.$$

(iii) (a) Put $I_n = \int_0^3 x^n e^{3x}\,dx$. Write $u = x^n$ (so $du/dx = nx^{n-1}$) and $dv/dx = e^{3x}$ (so $v = e^{3x}/3$). We can then integrate by parts:

$$\begin{aligned} I_n &= \int u\frac{dv}{dx}\,dx = [uv]_0^3 - \int_0^3 \frac{du}{dx}v\,dx \\ &= \left[x^n e^{3x}/3\right]_0^3 - \int_0^3 nx^{n-1}e^{3x}/3\,dx \\ &= 3^n e^{3.3}/3 - nI_{n-1}/3 = 3^{n-1}e^9 - \frac{n}{3}I_{n-1}. \end{aligned}$$

(b) We have $I_0 = \int_0^3 e^{3x}\,dx = \left[e^{3x}/3\right]_0^3 = (e^9-1)/3$, so

$$\begin{aligned} \int_0^3 xe^{3x}\,dx &= I_1 = 3^{1-1}e^9 - I_0/3 = e^9 - e^9/9 + 1/9 = (8e^9+1)/9 \\ \int_0^3 x^2e^{3x}\,dx &= I_2 = 3e^9 - 2I_1/3 = (81e^9 - 16e^9 - 2)/27 = (65e^9-2)/27. \end{aligned}$$

3 (i) The given matrix can be row-reduced as follows:

$$\left[\begin{array}{ccccc} 1 & 2 & 0 & 2 & 7 \\ 0 & 3 & -3 & 4 & 5 \\ 2 & 1 & 3 & -1 & 10 \\ -1 & -3 & 1 & -5 & -7 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccccc} 1 & 2 & 0 & 2 & 7 \\ 0 & 3 & -3 & 4 & 5 \\ 0 & -3 & 3 & -5 & -4 \\ 0 & -1 & 1 & -3 & 0 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccccc} 1 & 2 & 0 & 2 & 7 \\ 0 & 3 & -3 & 4 & 5 \\ 0 & -3 & 3 & -5 & -4 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{3}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 2 & -4 & 7 \\ 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{4} \left[\begin{array}{ccccc} 1 & 0 & 2 & -4 & 7 \\ 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 3 \end{array}\right] \xrightarrow{6}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(In step 1 we subtracted $2R_1$ from R_3, and added R_1 to R_4. In step 2 we multiplied R_4 by -1. In step 3 we subtracted $2R_4$ from R_1, subtracted $3R_4$ from R_2, and added $3R_4$ to R_3. In step 4 we divided R_3 by 4. In step 5 we added $4R_3$ to R_1, added $5R_3$ to R_2, and subtracted $3R_3$ from R_4. Finally, in step 6 we reordered the rows.)
The initial matrix was the augmented matrix for the system of equations in the question, and the final matrix is the augmented matrix for the equations

$$\begin{aligned} x + 2z &= 3 \\ y - z &= 3 \\ w &= -1. \end{aligned}$$

It follows that z is independent, and the remaining variables are given by

$$\begin{aligned} x &= 3 - 2z \\ y &= 3 + z \\ w &= -1. \end{aligned}$$

(ii)

$$AB = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{array}\right]\left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} 2 & -2 & 3 \\ 2 & -1 & 2 \\ 0 & 2 & -1 \end{array}\right]$$

$$BA = \left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{array}\right] = \left[\begin{array}{ccc} -1 & 0 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 2 \end{array}\right]$$

$$AB - BA = \left[\begin{array}{ccc} 2 & -2 & 3 \\ 2 & -1 & 2 \\ 0 & 2 & -1 \end{array}\right] - \left[\begin{array}{ccc} 1 & 0 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{ccc} 3 & -2 & 2 \\ 4 & 0 & 2 \\ 0 & 2 & -3 \end{array}\right]$$

(iii) The matrix of coefficients for the given system of equations is

$$C = \begin{bmatrix} 1 & a & 0 \\ 2 & 1 & -1 \\ -a & 2 & 3 \end{bmatrix}$$

We find that

$$\begin{aligned} \det(C) &= \det \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - a \det \begin{bmatrix} 2 & -1 \\ -a & 3 \end{bmatrix} + 0 \det \begin{bmatrix} 2 & 1 \\ -a & 2 \end{bmatrix} \\ &= 5 - a(6-a) = a^2 - 6a + 5 = (a-1)(a-5). \end{aligned}$$

The equations have a unique solution unless $\det(C) = 0$, which happens when $a = 1$ or $a = 5$.

4 (i) (a) We have

$$\frac{7x^2 - 2x + 13}{(x+2)(x^2 - x + 3)} = \frac{Ax + B}{x^2 - x + 3} + \frac{C}{x+2}$$

(for all x) if and only if

$$\begin{aligned} 7x^2 - 2x + 13 &= (Ax + B)(x+2) + C(x^2 - x + 3) \\ &= Ax^2 + 2Ax + Bx + 2B + Cx^2 - Cx + 3C \\ &= (A + C)x^2 + (2A + B - C)x + (2B + 3C). \end{aligned}$$

Putting $x = -2$ gives

$$7.(-2)^2 - 2.(-2) + 13 = C((-2)^2 - (-2) + 3),$$

or $45 = 9C$, so $C = 5$. Comparing coefficients in the previous equation gives $A + C = 7$ and $2B + 3C = 13$, so $A = 7 - C = 2$ and $B = (13 - 3C)/2 = -1$. We thus have

$$\frac{7x^2 - 2x + 13}{(x+2)(x^2 - x + 3)} = \frac{2x - 1}{x^2 - x + 3} + \frac{5}{x+2}.$$

(b) We now integrate the above equation. For the first term, put $u = x^2 - x + 3$, so $du = (2x - 1)dx$, so

$$\int \frac{2x-1}{x^2 - x + 3} dx = \int \frac{du}{u} = \log(u) = \log(x^2 - x + 3).$$

We also have $\int (x+2)^{-1} dx = \log(x+2)$, so

$$\int \frac{7x^2 - 2x + 13}{(x+2)(x^2 - x + 3)} dx = \log(x^2 - x + 3) + 5\log(x+2).$$

$$\begin{aligned} \int_1^2 \frac{7x^2 - 2x + 13}{(x+2)(x^2 - x + 3)} dx &= \left[\log(x^2 - x + 3) + 5\log(x+2)\right]_1^2 \\ &= \log(5) + 5\log(4) - \log(3) - 5\log(3) = \log(5) + 10\log(2) - 6\log(3). \end{aligned}$$

(ii) (a) We have $|x - 2| < 2$ iff the distance from x to 2 is less than 2. This happens when $0 < x < 4$, so $\{x \in \mathbb{R} : |x - 2| < 2\} = (0, 4)$

(b) We have $x^2 - 5x + 4 = (x-1)(x-4)$. When $x \leq 1$, both factors are less than or equal to zero so the product is nonnegative. When $1 < x < 4$ we see that $x - 4 < 0 < x - 1$ so $(x-4)(x-1) < 0$. When $x \geq 4$ the product becomes nonnegative again. Thus

$$\{x \in \mathbb{R} \mid x^2 - 5x + 4 \geq 0\} = (-\infty, 1] \cup [4, \infty).$$

(c) We have $x^4 - 5x^2 + 4 = (x^2)^2 - 5(x^2) + 4$, and part (b) tells us that this is nonnegative if and only if $x^2 \in (-\infty, 1] \cup [4, \infty)$, which means $-1 \leq x \leq 1$ or $x \leq -2$ or $x \geq 2$. Thus

$$\{x \in \mathbb{R} \mid x^4 - 5x^2 + 4 \geq 0\} = (-\infty, -2] \cup [-1, 1] \cup [2, \infty).$$

(iii) omitted.