Pure Mathematics Core — Exam solutions

(A1) Consider the rational function $f(x) = \frac{x^2+x+1}{(x+1)^2}$. On the bottom we have $(x+1)^2$, indicating a pole of order 2 at $x = -1$. We therefore need multiples of $(x+1)^{-1}$ and $(x+1)^{-2}$ in the partial fraction form. The top and bottom of $f(x)$ are both quadratics, so they both have degree 2. When the top and the bottom have the same degree, we need to add in an extra constant. The general form is therefore

$$\frac{x^2+x+1}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

The rules for finding the general form are explained in Section 2.4 of the booklet, and further examples are given in the solutions to the AiM test on partial fractions and the solutions to past exams. Some mistakes to avoid:

- You should never have the same term repeated twice but with different arbitrary coefficients. Things like $A/(x+1) + B/(x+1)$ are never correct.
- In some cases one has terms like $Ax/(x^2+x+1)$ or $Cx/(x^2+1)^3$ with an x in the numerator. These only occur when the denominator is a quadratic with no real roots, or a power of such a quadratic. If the denominator is of the form $(x-a)$ or $(x-a)^n$ then the numerator should just be an arbitrary constant with no x. Terms like $A/(x+1)$ and $B/(x-3)^2$ are OK, but terms like $Cx/(x+1)$ or $(Dx+E)/(x-2)^3$ are not.

Multiplying the above equation by $(x+1)^2$ gives

$$\begin{aligned} x^2+x+1 &= A(x+1)^2 + B(x+1) + C = Ax^2 + 2Ax + A + Bx + B + C \\ &= Ax^2 + (2A+B)x + (A+B+C), \end{aligned}$$

so $A = 1$ and $2A + B = 1$ and $A + B + C = 1$, which gives $B = -1$ and $C = 1$. This means that

$$\frac{x^2+x+1}{(x+1)^2} = 1 - \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

$$\int \frac{x^2+x+1}{(x+1)^2}\,dx = x - \log(x+1) - \frac{1}{(x+1)}.$$

(A2) If $x = f(y) = 1/(1-e^{-y})$ then $1/x = 1 - e^{-y}$, so $e^{-y} = 1 - 1/x = (x-1)/x$, so $e^y = x/(x-1)$, so $f^{-1}(x) = y = \log(x/(x-1))$.

Some mistakes to avoid:

- $\log(1-e^{-y})$ is *not* the same as $\log(1) - \log(e^{-y})$. Similarly, $\log(y - ye^{-x})$ is not the same as $\log(y) - \log(ye^{-x})$. More generally, $\log(a+b)$ is almost never the same as $\log(a) + \log(b)$, so taking logs of sums is often not helpful. Yous hould manipulate your equations algebraically until you have e^y on its own first, and then take logs.
- Again, $\log(-e^{-y})$ is not the same as $\log(e^y) = y$; the signs do not "cancel out". In fact, $-e^{-y}$ is always negative (for real y) and so $\log(-e^{-y})$ is not even defined. If we allow complex numbers then

$$\log(-e^{-y}) = \log((-1).e^{-y}) = \log(-1) + \log(e^{-y}) = i\pi - y.$$

(A3)

$$(\exp\circ f\circ \log)(x) = \exp(f(\log(x))) = \exp(2\log(x)+2) = (e^{\log(x)})^2 e^2 = e^2x^2.$$

Here $f(t) = 2t+2$, so to apply f to something we double it and add 2. We start with x and work from right to left. We apply log to get $\log(x)$, then apply f to get $2\log(x)+2$, then apply exp to get $\exp(2\log(x)+2)$.

- The expression $\log(2x+2)$ is $\log(f(x))$ or $(\log\circ f)(x)$, not $(f\circ\log)(x)$. Thus, $\log(2x+2)$ should not appear in your calculation at all. Similarly, the expressions $(f\circ\exp)(x) = 2e^x + 2$ and $\exp(\log(2x+2)) = (\exp\circ\log\circ f)(x)$ should not appear.

- Even if you think that $\log(2x+2)$ is relevant, you should not write equations like $\log(x) = \log(2x+2)$ to indicate this. The symbol "=" means equality, and $\log(x)$ is obviously not the same function as $\log(2x+2)$.
- The expression $(2x+2)\log(x)$ is $f(x)\log(x)$, not $f(\log(x))$. Thus, $(2x+2)\log(x)$ should not appear in your calculation at all.
- The rule is that $\exp(a+b) = \exp(a)\exp(b)$, not $\exp(a)+\exp(b)$ or $\exp(a)+b$. Thus $\exp(2\log(x)+2)$ is $\exp(2\log(x)).\exp(2)$ or $e^{2\log(x)}.e^2$, which simplifies further to x^2e^2. We do not have $\exp(2\log(x)+2) = \exp(2\log(x)) + \exp(2)$ (which would simplify to x^2+e^2) or $\exp(2\log(x)+2) = \exp(2\log(x)) + 2$ (which would simplify to x^2+2).
- The rule is that $\exp(2a) = \exp(a)^2$, not $\exp(2a) = 2\exp(a)$. Thus $\exp(2\log(x))$ is $\exp(\log(x))^2$ (which simplifies to x^2) not $2\exp(\log(x))$ (which simplifies to $2x$).

(A4) We must find $\log_{16}(1/2)$, which is the number t such that $16^t = 1/2$. We note that $16 = 2^4$, so $2 = 16^{1/4}$, so $1/2 = 16^{-1/4}$, so $\log_{16}(1/2) = -1/4$.

(A5) Note that $\sin(x)$ repeats with period 2π, so

$$\sin(-7\pi/3) = \sin(-7\pi/3 + 2\pi) = \sin(-\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2.$$

(A6) Put $u = e^x$. Then

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(u-u^{-1})/2}{(u+u^{-1})/2} = \frac{u-u^{-1}}{u+u^{-1}},$$

so

$$\begin{aligned}
1+\tanh(x)^2 &= 1 + \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 + \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}} \\
&= \frac{(u^2+2+u^{-2}) + (u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{2u^2+2u^{-2}}{u^2+2+u^{-2}} \\
1-\tanh(x)^2 &= 1 - \left(\frac{u-u^{-1}}{u+u^{-1}}\right)^2 = 1 - \frac{u^2-2+u^{-2}}{u^2+2+u^{-2}} \\
&= \frac{(u^2+2+u^{-2}) - (u^2-2+u^{-2})}{u^2+2+u^{-2}} = \frac{4}{u^2+2+u^{-2}}
\end{aligned}$$

so

$$\frac{1+\tanh(x)^2}{1-\tanh(x)^2} = \frac{2u^2+2u^{-2}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x).$$

Alternatively,

$$\frac{1+\tanh(x)^2}{1-\tanh(x)^2} = \frac{1+\frac{\sinh(x)^2}{\cosh(x)^2}}{1-\frac{\sinh(x)^2}{\cosh(x)^2}} = \frac{\cosh(x)^2+\sinh(x)^2}{\cosh(x)^2-\sinh(x)^2} = \frac{\cosh(2x)}{1}.$$

This is shorter, but you need to remember the identities $\cosh(2x) = \cosh(x)^2 + \sinh(x)^2$ and $\cosh(x)^2 - \sinh(x)^2 = 1$, and you need to get the signs right. The first method is more systematic and will work for all possible identities.

(A7) Put $u = x^n + a$ and $y = f(x) = u^m$. Then $du/dx = nx^{n-1}$ and $dy/du = mu^{m-1}$, so

$$f'(x) = \frac{dy}{dx} = mu^{m-1}\frac{du}{dx} = mn(x^n+a)^{m-1}x^{n-1}.$$

(A8) Put $u = 1+x+x^2+x^3$ and $y = \log(u)$, so $dy/du = 1/u$ and

$$\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du} = (1+2x+3x^2)u^{-1} = \frac{1+2x+3x^2}{1+x+x^2+x^3}.$$

The most common wrong answers were $(1+x+x^2+x^3)^{-1}$ and $(1+2x+3x^2)\log(1+x+x^2+x^3).x^{-1}$. Make sure you understand why these are not correct.

(A9) The quotient rule gives

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - x.\log'(x)}{\log(x)^2} = \frac{\log(x) - x.x^{-1}}{\log(x)^2}$$
$$= \frac{1}{\log(x)} - \frac{1}{\log(x)^2}.$$

Note here that $\log(x)^2$ is not the same as $\log(x^2)$. In fact we have $\log(x^2) = 2\log(x)$, but $\log(x)^2 \neq 2\log(x)$.

(A10)

$$\frac{d}{dx}\left(\frac{3x+2}{4x+3}\right) = \frac{3(4x+3) - 4(3x+2)}{(4x+3)^2}$$
$$= \frac{12x + 9 - 12x - 8}{(4x+3)^2} = (4x+3)^{-2}$$

(A11) First put $u = -(x-a)^2/b$, so $du/dx = -2(x-a)/b$. Then put $v = \exp(u) = e^{-(x-a)^2/b}$, so the chain rule gives

$$\frac{dv}{dx} = -2(x-a)b^{-1}e^{-(x-a)^2/b}.$$

Now put $w = \sin(\omega x)$, so $dw/dx = \omega\cos(\omega x)$. Finally, put $y = vw = e^{-(x-a)^2/b}\sin(\omega x)$ and apply the product rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dv}{dx}w + v\frac{dw}{dx} \\ &= -2(x-a)b^{-1}e^{-(x-a)^2/b}\sin(\omega x) + e^{-(x-a)^2/b}\omega\cos(\omega x) \\ &= e^{-(x-a)^2/b}(\omega\cos(\omega x) - 2(x-a)b^{-1}\sin(\omega x)).\end{aligned}$$

With practice you can leave out some of these steps, but it is always safest to write them all out carefully.

(A12) We know that

$$\int x^2 e^x\, dx = (ax^2 + bx + c)e^x$$

for some constants a, b and c . To find these, we differentiate to get

$$\begin{aligned}x^2e^x &= \frac{d}{dx}((ax^2 + bx + c)e^x) = (2ax + b)e^x + (ax^2 + bx + c)e^x \\ &= (ax^2 + (2a+b)x + (b+c))e^x.\end{aligned}$$

We equate coefficients to see that $a = 1$ and $2a + b = b + c = 0$, which gives $b = -2$ and $c = 2$. We conclude that

$$\int x^2e^x\, dx = (x^2 - 2x + 2)e^x.$$

Alternatively, we can integrate twice by parts. For the first step, put $u = x^2$ (so $du/dx = 2x$) and $dv/dx = e^x$ (so $v = e^x$ as well). We then have

$$\int x^2e^x\, dx = \int u\frac{dv}{dx}\, dx = uv - \int \frac{du}{dx}v\, dx = x^2e^x - 2\int x\, e^x\, dx.$$

For the second step, put $u = x$ (so $du/dx = 1$) and $dv/dx = e^x$ (so $v = e^x$ as well). We then have

$$\int x\, e^x\, dx = \int u\frac{dv}{dx}\, dx = uv - \int \frac{du}{dx}v\, dx = xe^x - \int e^x\, dx = xe^x - e^x.$$

Putting this together, we get

$$\int x^2e^x\, dx = x^2e^x - 2(xe^x - e^x) = (x^2 - 2x + 2)e^x.$$

(A13) We know that

$$\int e^{3x}\sin(4x)\,dx = e^{3x}(A\cos(4x)+B\sin(4x))$$

for some A and B . To find these, we differentiate and equate coefficients:

$$\begin{aligned}
e^{3x}\sin(4x) &= \frac{d}{dx}\left(e^{3x}(A\cos(4x)+B\sin(4x))\right)\\
&= 3e^{3x}(A\cos(4x)+B\sin(4x)) + e^{3x}(-4A\sin(4x)+4B\cos(4x))\\
&= e^{3x}((3A+4B)\cos(4x)+(3B-4A)\sin(4x)),
\end{aligned}$$

so $3A+4B=0$ and $3B-4A=1$. This gives $A=-4B/3$ so $1=3B-4A=3B+16B/3=25B/3$, so $B=3/25$, so $A=-4B/3=-4/25$. The conclusion is that

$$\int e^{3x}\sin(4x)\,dx = e^{3x}(3\sin(4x)-4\cos(4x))/25.$$

Alternatively, we can integrate twice by parts. Put $I=\int e^{3x}\sin(4x)\,dx$ and $J=\int e^{3x}\cos(4x)\,dx$. Put $du/dx=e^{3x}$ (so $u=e^{3x}/3$) and $v=\sin(4x)$ (so $dv/dx=4\cos(4x)$). This gives

$$I=\int\frac{du}{dx}v\,dx = uv-\int u\frac{dv}{dx}\,dx = e^{3x}\sin(4x)/3-\int 4e^{3x}\cos(4x)/3\,dx = e^{3x}\sin(4x)/3-4J/3.$$

Now put $w=\cos(4x)$, so $dw/dx=-4\sin(4x)$. This gives

$$J=\int\frac{du}{dx}w\,dx = uw-\int u\frac{dw}{dx}\,dx = e^{3x}\cos(4x)/3-\int 4e^{3x}(-\sin(4x))/3\,dx = e^{3x}\cos(4x)/3+4I/3.$$

Putting this together gives

$$\begin{aligned}
I &= \frac{1}{3}e^{3x}\sin(4x)-\frac{4}{3}\left(\frac{1}{3}e^{3x}\cos(4x)+\frac{4}{3}I\right)\\
&= e^{3x}\left(\frac{1}{3}\sin(4x)-\frac{4}{9}\cos(4x)\right)-\frac{16}{9}I.
\end{aligned}$$

We can rearrange this to get

$$\begin{aligned}
\frac{25}{9}I &= e^{3x}\left(\frac{1}{3}\sin(4x)-\frac{4}{9}\cos(4x)\right)\\
I &= \frac{9}{25}e^{3x}\left(\frac{1}{3}\sin(4x)-\frac{4}{9}\cos(4x)\right)\\
&= e^{3x}(3\sin(4x)-4\cos(4x))/25.
\end{aligned}$$

(A14) The matrix of coefficients is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

There is no pivot in the last column, so the variable z is independent (you need to say this explicitly). The final matrix corresponds to the equations $w-z=x+z=y+z=0$, so $(w,x,y,z)=(z,-z,-z,z)$.

(A15) We write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

At the final stage, the left hand block is the identity, so the right hand block is the inverse of the original matrix, ie

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

Alternatively, we can use the cofactor method. The determinants of the minors are as follows:

$$\begin{bmatrix} \det\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} a & b \\ 1 & c \end{bmatrix} & \det\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} & \det\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac-b & c & 1 \end{bmatrix}.$$

We multiply by the associated signs and take the transpose to get

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}^T.$$

The determinant is the dot product of the first row of A (which is $(1,a,b)$) with the first column of $\operatorname{adj}(A)$ (which is $(1,0,0)$). Thus $\det(A) = 1.1 + 0.a + 0.b = 1$ and $A^{-1} = \operatorname{adj}(A)/\det(A) = \operatorname{adj}(A)$.

(B1) Observe that $f(x) = (x+1)^2 + 2$. As x runs from -1 to 1 (excluding the endpoints), $x+1$ increases from 0 to 2 and so $(x+1)^2+2$ increases from $0^2+2 = 2$ to $2^2+2 = 6$. In all cases the endpoints are excluded, so the range of f is $(2,6)$.

(B2) The general method that works for all trigonometric functions is to rewrite them as sums of terms like $\sin(nx)$ or $\cos(mx)$. (In special cases other methods may work or may be easier, but this is not one of them.)

Note that $\sin(x)\cos(x) = \sin(2x)/2$, so

$$\sin(x)^2\cos(x)^2 = \sin(2x)^2/4 = (1-\cos(4x))/8.$$

Thus

$$\begin{aligned} \int \sin(x)^2\cos(x)^2\,dx &= \tfrac{1}{8}\int 1-\cos(4x)\,dx \\ &= \frac{x}{8} - \frac{\sin(4x)}{32} = \frac{4x-\sin(4x)}{32}. \end{aligned}$$

(B3) Put $u = \cos(x)$, so $du = -\sin(x)\,dx$. Then

$$\begin{aligned} \int \sin(x)\log(\cos(x))\,dx &= -\int \log(u)\,du = -(u\log(u) - u) = u(1-\log(u)) \\ &= \cos(x)(1-\log(\cos(x))). \end{aligned}$$

(B4) We first note that

$$\begin{aligned} \frac{d}{dx}\left(x^3(a\log(x)^2 + b\log(x) + c)\right) &= 3x^2(a\log(x)^2 + b\log(x) + c) + x^3(2a\log(x)/x + b/x) \\ &= x^2(3a\log(x)^2 + (3b+2a)\log(x) + (3c+b)). \end{aligned}$$

This must also be equal to $x^2\log(x)^2$ for all x, so we must have

$$\begin{aligned} 3a &= 1 \\ 3b + 2a &= 0 \\ 3c + b &= 0, \end{aligned}$$

so $a = 1/3$ and $b = -2/9$ and $c = 2/27$, giving

$$\int x^2\log(x)^2\,dx = x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27).$$

It follows that

$$\begin{aligned} \int_1^e x^2\log(x)^2\,dx &= \left[x^3(\log(x)^2/3 - 2\log(x)/9 + 2/27)\right]_1^e \\ &= e^3(1/3 - 2/9 + 2/27) - 1^3(0/3 - 0/9 + 2/27) \\ &= (5e^3 - 2)/27. \end{aligned}$$

(B5) Put

$$A = \begin{bmatrix} 1 & a & 0 & 0 \\ a & 1 & b & 0 \\ 0 & b & 1 & c \\ 0 & 0 & c & 1 \end{bmatrix}.$$

The direct approach is as follows:

$$\begin{aligned} \det(A) &= \det\begin{bmatrix} 1 & b & 0 \\ b & 1 & c \\ 0 & c & 1 \end{bmatrix} - a\det\begin{bmatrix} a & b & 0 \\ 0 & 1 & c \\ 0 & c & 1 \end{bmatrix} \\ &= \left(\det\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} - b\det\begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix}\right) - a\left(a\det\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} - b\det\begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix}\right) \\ &= (1 - c^2 - b(b-0)) - a(a(1-c^2) - b.0) \\ &= 1 - a^2 - b^2 - c^2 + a^2c^2. \end{aligned}$$

Alternatively, if we subtract a times the first row from the second row, and subtract c times the fourth row from the third row, we obtain the matrix

$$B = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1-a^2 & b & 0 \\ 0 & b & 1-c^2 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

with $\det(A) = \det(B)$. We can expand down the first column to see that

$$\det(A) = \det(B) = \det\begin{bmatrix} 1-a^2 & b & 0 \\ b & 1-c^2 & 0 \\ 0 & c & 1 \end{bmatrix},$$

and then expand this down the last column to get

$$\det(A) = \det\begin{bmatrix} 1-a^2 & b \\ b & 1-c^2 \end{bmatrix} = (1-a^2)(1-c^2) - b^2 = 1 - a^2 - b^2 - c^2 + a^2c^2.$$