# Homework I - Advanced Game Theory

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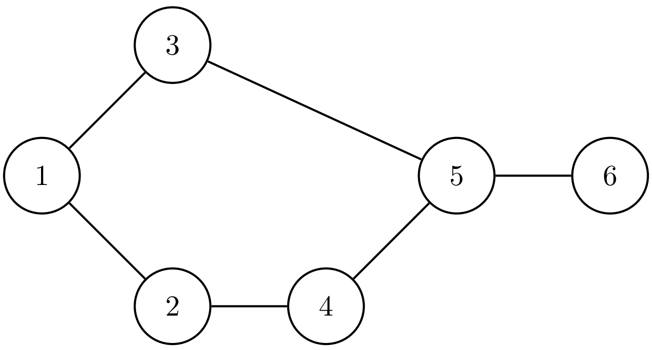
## Exercise 1

The graph representation of $(N, L)$ is,




### (a)

The function $v^L$ of the Myerson restricted game is,

$$v^L(S) = \sum_{T \in C_L(S)} v(T). \tag{1}$$

It is non-zero only for the component of the graphs that include both 1 and 6, namely,

$$v^L(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,5,6\}, \{1,2,3,5,6\}, \{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

## (b)

We can define the communication game $(N, v^L)$, using (2). Then we can compute the Harsanyi dividends based on this game. The non-zero dividends are,

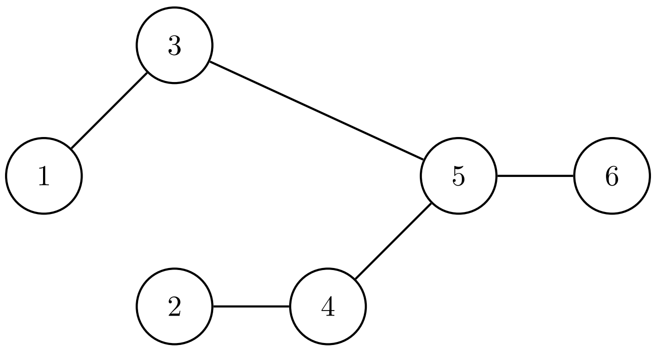
$$\begin{aligned} \Delta_{v^L}(\{1,3,5,6\}) &= 1 \\ \Delta_{v^L}(\{1,2,4,5,6\}) &= 1 \\ \Delta_{v^L}(\{1,2,3,4,5,6\}) &= -1 \end{aligned} \tag{3}$$

Then we can compute the Myerson value as the Shapley value of the new coordination game,

$$\begin{aligned} \mu_i(v, L) = f_i^S &= \sum_{T \in N(i)} \Delta_{v^L}(T) / |T| \\ \implies \mu(v, L) = f^S &= \begin{pmatrix} 17/60 & 1/30 & 1/12 & 1/30 & 17/60 & 17/60 \end{pmatrix} \end{aligned} \tag{4}$$

## (c)

The new supply chain is represented by the graph,




We can hence define

$$v^{L'}(S) = \begin{cases} 0 & \text{if } S = \{1,2,4,5,6\} \\ v^L & \text{otherwise} \end{cases} \tag{5}$$

By repeating the procedure of (4), we obtain,

$$\mu(v, L') = \begin{pmatrix} 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \end{pmatrix}. \tag{6}$$

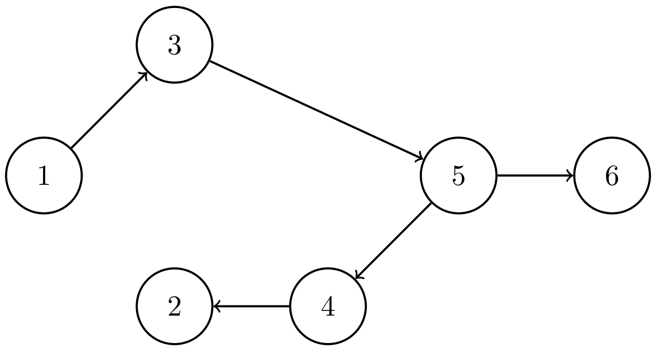
This result is expected since in the new supply chain 2 and 4 become null-players.

In order to determine fairness we can check that,

$$\begin{gathered} \mu_1(v, L) - \mu_1(v, L') = \mu_2(v, L) - \mu_2(v, L') \\ 17/60 - 1/4 = 1/30 - 0 \implies \text{ the fairness axiom is respected.} \end{gathered} \tag{7}$$

## (d)

In order to compute the hierarchical outcome of node 1 we need to set 1 as root. The resulting directed graph $(N, L^1)$ can be represented as,



Using the definition of followers and subordinates, we can compute the hierarchical outcomes as,

$$h^1(v, L') = \begin{pmatrix} v^{L'}(\{1,3,5,6,4,2\}) - v^{L'}(\{3,5,6,4,2\}) \\ v^{L'}(\{2\}) \\ v^{L'}(\{3,5,6,4,2\}) - v^{L'}(\{5,6,4,2\}) \\ v^{L'}(\{4,2\}) - v(\{2\}) \\ v^{L'}(\{5,6,4,2\}) - v^{L'}(\{4,2\}) - v^{L'}(\{6\}) \\ v^{L'}(6) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

In a similar manner,

$$h^4(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad h^6(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

### (e)

The link game $(L', r^{L'})$ is given by the characteristic function,

$$\begin{aligned} \bar{E} &= \left\{ \begin{array}{l} \{(1,3),(3,5),(5,6)\} \\ \{(1,3),(4,2),(3,5),(5,6)\} \\ \{(1,3),(3,5),(5,4),(5,6)\} \\ \{(1,3),(4,2),(3,5),(5,4),(5,6)\} \end{array} \right\} \\ r^{L'}(E) &= \begin{cases} 1 & \text{if } E \in \bar{E} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{10}$$

The Shapley value of the link game is,

$$f^S(L', r^{L'}) = \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix} \tag{11}$$

The associated position value is.

$$\pi(v, L') = \begin{pmatrix} 1/6 & 0 & 1/3 & 0 & 1/3 & 1/6 \end{pmatrix} \tag{12}$$

## Exercise 2

A hierarchical outcome $h^i$ is in the $Core(N, v_L)$ if and only if,

$$\sum_{j \in S} h^i_j \geq v(S) \quad \text{and} \quad \sum_{j \in N} h^i_j = v(N).$$

First let the subordinates of $j$ be

$$S^i_j \coloneqq \left\{ h :\ \exists (i_1, \ldots, i_t)\ s.t. i_1 = j \wedge i_t = h \wedge (i_k, i_{k+1}) \in L^i\ \forall\ k \in \{1, \ldots, t\} \right\} \tag{13}$$

Entry $j$ in the hierarchical outcome can be written as,

$$h_j^i = v(\hat{F}_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i). \tag{14}$$

By superadditivity,

$$\hat{F}_j^i = \{j\} \cup S_j^i \implies v(\hat{F}_j^i) \geq v(j) + v(S_j^i). \tag{15}$$

Furthermore,

$$\hat{F}_h^i \subseteq S_j^i \; \forall h \in F_j^i \implies \bigcup_{h \in F_j^i} \hat{F}_h^i \subseteq S_j^i \implies \sum_{h \in F_j^i} v(\hat{F}_h^i) \leq v(S_j^i). \tag{16}$$

Combining these results, we can use (14), and show that,

$$h_j^i = v(\hat{F}_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i) \geq v(j) + v(S_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i) \geq v(j). \tag{17}$$

Given that $h_j^i \geq v(j)$, by superadditivity we can show that,

$$\sum_{j \in S} h_j^i \geq \sum_{j \in S} v(j). \tag{18}$$