Homework I - Advanced Game Theory

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Exercise 1

(a)

We can compute the Harsanyi dividends by using the recursive definition,

$$\Delta_v(T) = v(T) - \sum_{S \subset T} \Delta_v(S). \tag{1}$$

This yields,

$$\begin{aligned}
\Delta_v(\{1\}) &= v(1) = 1 \\
\Delta_v(\{2\}) &= v(2) = 2 \\
\Delta_v(\{3\}) &= v(3) = 0 \\
\Delta_v(\{1,2\}) &= v(\{1,2\}) - \Delta_v(\{1\}) - \Delta_v(\{2\}) = 0 \\
\Delta_v(\{1,3\}) &= v(\{1,3\}) - \Delta_v(\{1\}) - \Delta_v(\{3\}) = 2 \\
\Delta_v(\{2,3\}) &= v(\{1,3\}) - \Delta_v(\{1\}) - \Delta_v(\{3\}) = 0 \\
\Delta_v(\{1,2,3\}) &= v(\{1,2,3\}) - \Delta_v(\{1,2\}) - \Delta_v(\{2,3\}) - \Delta_v(\{1,3\}) = -1
\end{aligned} \tag{2}$$

(b)

Let $N(i) \coloneqq \{T : T \subseteq N \wedge \ i \in T\}$. Using the definition of Shapley value with Harsanyi dividends, we can compute,

$$f_i^S = \sum_{T \in N(i)} \frac{1}{|T|} \cdot \Delta_v(T). \tag{3}$$

This yields,

$$
\begin{aligned}
f_1^S &= \frac{1}{3} \cdot \Delta_v(\{1,2,3\}) + \frac{1}{2} \cdot \Delta_v(\{1,2\}) + \frac{1}{2} \cdot \Delta_v(\{1,3\}) + \Delta_v(\{1\}) \\
&= -\frac{1}{3} + 0 + \frac{2}{2} + 1 = \frac{5}{3}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
f_2^S &= \frac{1}{3} \cdot \Delta_v(\{1,2,3\}) + \frac{1}{2} \cdot \Delta_v(\{1,2\}) + \frac{1}{2} \cdot \Delta_v(\{2,3\}) + \Delta_v(\{2\}) \\
&= -\frac{1}{3} + 0 + 0 + 2 = \frac{5}{3}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
f_3^S &= \frac{1}{3} \cdot \Delta_v(\{1,2,3\}) + \frac{2}{3} \cdot \Delta_v(\{1,2\}) + \frac{1}{2} \cdot \Delta_v(\{1,3\}) + \Delta_v(\{3\}) \\
&= -\frac{1}{3} + 0 + \frac{2}{2} + 0 = \frac{5}{3} = \frac{2}{3}
\end{aligned} \tag{6}
$$

hence,

$$
f^S = \begin{pmatrix} 5/3 & 5/3 & 2/3 \end{pmatrix}
$$

(c)

In order to verify that the core is empty we can use the definition of the core,

$$
\begin{aligned}
&x_i \in C(N,v) \implies \sum_{i \in S} x_i \geq v(S) \\
&\sum_{i \in N} x_i = v(N)
\end{aligned}
$$

Let $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ be a candidate allocation. Then the second property requires that,

$$
x_1 + x_2 + x_3 = v(N) = 4 \tag{7}
$$

The first property, on the other hand, requires

$$
\begin{aligned}
x_1 &\geq 1 \quad x_1 + x_2 \geq 3 \\
x_2 &\geq 2 \quad x_1 + x_3 \geq 3 \\
x_3 &\geq 0 \quad x_2 + x_3 \geq 2
\end{aligned} \tag{8}
$$

Combining (7) and (8) we know that the core allocation requires

$$x_3 = 0 \implies x_1 \geq 3 \implies x_2 = 0 \Rightarrow\Leftarrow x_2 \geq 2, \tag{9}$$

hence there is no allocation x that satisfies (7) and (8) which implies that $C(N, v) = \emptyset$.

(d)

Convexity fails for $S = \{1, 3\}$ and $T = \{2\}$. Since,

$$\begin{aligned} v\left(S \cup T\right) + v\left(S \cap T\right) &< v(S) + v(T) \\ v(\{1,2,3\}) + v(\emptyset) &< v(\{1,3\}) + v(2) \\ 4 &< 3 + 2 \end{aligned} \tag{10}$$

Therefore the game is not convex.

Exercise 2

The imputation set is defined as,

$$I(N, v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \wedge x_i \geq v(i) \ \forall i \in N \right\}. \tag{11}$$

First, we want to show that Shapley value has the first property of the elements in the imputation set,

$$\sum_{i \in N} f_i^S = v(N). \tag{12}$$

Let $N(i) \coloneqq \{T : T \subseteq N \wedge \ i \in T\}$ and interpret $S \subset T$ to mean every proper subset, $S \in 2^T \setminus T$.

The sum over elements of the Shapley value is then,

$$\begin{aligned} \sum_{i \in N} f_i^S &= \sum_{i \in N} \sum_{T \in N(i)} \frac{1}{|T|} \cdot \Delta_v(T) \\ &= \sum_{i \in N} \frac{v(N) - \sum_{S \subset N} \Delta_v(S)}{|N|} + \sum_{i \in N} \sum_{T \in N(i) \setminus N} \frac{1}{|T|} \cdot \Delta_v(T) \\ &= v(N) - \sum_{S \subset N} \Delta_v(S) + \sum_{i \in N} \sum_{T \in N(i) \setminus N} \frac{1}{|T|} \cdot \Delta_v(T) \end{aligned} \tag{13}$$

In the third term, $\sum_{i\in N}\sum_{T\in N(i)\setminus N}\Delta_v(T)/\left|T\right|$, each entry $\Delta_v(T)$ appears whenever $i\in T$. There are $|T|$ such terms, hence we can write,

$$\sum_{i\in N}\sum_{T\in N(i)\setminus N}\frac{1}{|T|}\cdot\Delta_v(T)=\sum_{T\subset N}|T|\cdot\frac{\Delta_v(T)}{|T|}=\sum_{S\subset N}\Delta_v(S). \tag{14}$$

This implies in (13) that, $\sum_{i\in N} f_i^S=v(N)$.

Next we want to show that the Shapley value has the second property,

$$f_i^S\geq v(i). \tag{15}$$

The aim is to rewrite the Shapley value for an individual player as $v(i)$ and a positive term. To do so, note that $\{i\}\in N(i)$. Then the Shapley value for i is,

$$\begin{aligned}
f_i^S&=\sum_{T\in N(i)}\frac{1}{|T|}\cdot\Delta_v(T)\\
&=\sum_{T\in N(i)}\frac{1}{|T|}\cdot\left(v(T)-\sum_{S\subset T}\Delta_v(S)\right)\\
&=v(i)+\sum_{T\in N(i)\setminus\{i\}}\frac{1}{|T|}\cdot\left(v(T)-\sum_{S\subset T}\Delta_v(S)\right).
\end{aligned} \tag{16}$$

We want now to show that

$$v(T)-\sum_{S\subset T}\Delta_v(S), \tag{17}$$

is positive for all $T\in N(i)\setminus\{i\}$.

Consider a set $D=\{i,j\}\in N(i),\ |D|=2$, then, by super additivity,

$$\begin{aligned}
v(D)=v\left(\{i\}\cup\{j\}\right)&\geq v(i)+v(j)\\
v\left(\{i\}\cup\{j\}\right)-\Delta_v(i)-\Delta_v(j)&\geq 0\\
\Delta_v(D)=v(D)-\sum_{S\subset D}\Delta_v(S)&\geq 0
\end{aligned} \tag{18}$$

Next take the extension set $M=D\cup\{m\}\in N(i)$. By super additivity and using (18),

$$
\begin{aligned}
&v(M) = v(D \cup \{m\}) \geq v(D) + v(m) = \Delta_v(D) + \Delta_v(i) + \Delta_v(j) + \Delta_v(m) \\
&\implies \Delta_v(M) = v(M) - \sum_{S \subset M} \Delta_v(S) \geq 0
\end{aligned} \tag{19}
$$

By induction, we can construct any bigger set $T \in N(i)$ as done in (19) for (18). Hence for every $T \in N(i)$, (17) is positive.

If $v(T) - \sum_{S \subset T} \Delta_v(S) \geq 0$ for every $T \in N(i)$, then

$$
f_i^S = v(i) + \sum_{T \in N(i) \setminus \{i\}} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) \geq v(i). \tag{20}
$$

Exercise 3

(a)

The function v is the mapping, $v(\emptyset) = 0$, $v(\{1\}) = 0$, $v(\{2\}) = 5$, $v(\{3\}) = 0$, $v(\{1,2\}) = 15$, $v(\{1,3\}) = 5$, $v(\{2,3\}) = 10$, $v(\{1,2,3\}) = 20$.

(b)

We first compute the Harsanyi dividends,

$$
\begin{aligned}
\Delta_v(\{1\}) &= 0, \\
\Delta_v(\{2\}) &= 5, \\
\Delta_v(\{3\}) &= 0, \\
\Delta_v(\{1,2\}) &= 10, \\
\Delta_v(\{1,3\}) &= 5, \\
\Delta_v(\{2,3\}) &= 5, \\
\Delta_v(\{1,2,3\}) &= -5
\end{aligned} \tag{21}
$$

Using (3) and the same procedure as Exercise 1.b we obtain,

$$
f^S = \begin{pmatrix} 35/6 & 65/6 & 20/6 \end{pmatrix} \tag{22}
$$

(c)

To find the core of the game we need to find the set such that

$$C(N,v)=\left\{\begin{array}{l} x\in\mathbb{R}^3: \\ x_1\geq 0,\ x_2\geq 5,\ x_3\geq 0 \\ x_1+x_2\geq 15,\ x_1+x_3\geq 5,\ x_2+x_3\geq 10 \\ x_1+x_2+x_3=20 \end{array}\right\} \tag{23}$$

The condition $x_1 + x_2 = 20 - x_3$ yields $x_3 \in [0, 5]$. By then combining with the other conditions we obtain $x_1 \in [0, 10]$ and $x_2 \in [5, 15]$. Hence the core is,

$$C(N,v)=\left\{\begin{array}{l} x\in\mathbb{R}^3: x_1+x_2+x_3=20 \\ x_1\in[0,10] \\ x_2\in[5,15] \\ x_3\in[0,5] \end{array}\right\} \tag{24}$$

(d)

We can check every combination of $S, T \in 2^N$, for the condition,

$$v\left(S\cup T\right)+v\left(S\cap T\right)\geq v(S)+v(T). \tag{25}$$

Every combination of S and T yields,

$$
\begin{aligned}
S = \{1\}, T = \emptyset &: \; 0 + 0 \geq 0 + 0 \\
S = \{2\}, T = \emptyset &: \; 5 + 0 \geq 5 + 0 \\
S = \{3\}, T = \emptyset &: \; 0 + 0 \geq 0 + 0 \\
S = \{1,2\}, T = \emptyset &: \; 15 + 0 \geq 15 + 0 \\
S = \{1,3\}, T = \emptyset &: \; 5 + 0 \geq 5 + 0 \\
S = \{2,3\}, T = \emptyset &: \; 10 + 0 \geq 10 + 0 \\
S = \{1,2,3\}, T = \emptyset &: \; 20 + 0 \geq 20 + 0 \\
S = \{2\}, T = \{1\} &: \; 15 + 0 \geq 5 + 0 \\
S = \{3\}, T = \{1\} &: \; 5 + 0 \geq 0 + 0 \\
S = \{1,2\}, T = \{1\} &: \; 15 + 0 \geq 15 + 0 \\
S = \{1,3\}, T = \{1\} &: \; 5 + 0 \geq 5 + 0 \\
S = \{2,3\}, T = \{1\} &: \; 20 + 0 \geq 10 + 0 \\
S = \{1,2,3\}, T = \{1\} &: \; 20 + 0 \geq 20 + 0 \\
S = \{3\}, T = \{2\} &: \; 10 + 0 \geq 0 + 5 \\
S = \{1,2\}, T = \{2\} &: \; 15 + 5 \geq 15 + 5 \\
S = \{1,3\}, T = \{2\} &: \; 20 + 0 \geq 5 + 5 \\
S = \{2,3\}, T = \{2\} &: \; 10 + 5 \geq 10 + 5 \\
S = \{1,2,3\}, T = \{2\} &: \; 20 + 5 \geq 20 + 5 \\
S = \{1,2\}, T = \{3\} &: \; 20 + 0 \geq 15 + 0 \\
S = \{1,3\}, T = \{3\} &: \; 5 + 0 \geq 5 + 0 \\
S = \{2,3\}, T = \{3\} &: \; 10 + 0 \geq 10 + 0 \\
S = \{1,2,3\}, T = \{3\} &: \; 20 + 0 \geq 20 + 0 \\
S = \{1,3\}, T = \{1,2\} &: \; 20 + 0 \geq 5 + 15 \\
S = \{2,3\}, T = \{1,2\} &: \; 20 + 5 \geq 10 + 15 \\
S = \{1,2,3\}, T = \{1,2\} &: \; 20 + 15 \geq 20 + 15 \\
S = \{2,3\}, T = \{1,3\} &: \; 20 + 0 \geq 10 + 5 \\
S = \{1,2,3\}, T = \{1,3\} &: \; 20 + 5 \geq 20 + 5 \\
S = \{1,2,3\}, T = \{2,3\} &: \; 20 + 10 \geq 20 + 10
\end{aligned}
$$

Since the condition is true for all sets, the game is convex. Furthermore, convexity implies superadditivity, by taking S, T such that $S \cap T = \emptyset$, hence the game is also superadditive.

Exercise 4

In order to show that cooperative games are a generalization of hypergraphs, we will provide a construction that maps any cooperative game to an hypergraph and any hypergraph to some cooperative game.

For the former, take the arbitrary cooperative game (N, v) where N is a finite set and $v : 2^N \mapsto \mathbb{R}$ with $v(\emptyset) = 0$. Now define $X \coloneqq N$ and

$$E \coloneqq \left\{S : S \in 2^N \wedge v(S) > 0\right\}. \tag{26}$$

Note that (X, E) is an hypergraph since $E \subseteq 2^N \setminus \emptyset$. Hence any hypergraph can be constructed by a cooperative game.

For the latter, take the arbitrary hypergraph (X, E). Let $N \coloneqq X$ and

$$v : 2^N \mapsto \{0, 1\}, \quad v(S) \coloneqq \begin{cases} 1 & \text{if } S \in E \\ 0 & \text{if } S \notin E \end{cases}. \tag{27}$$

Note then that (N, v) is a cooperative game with a value function with image restricted to 0 or 1. Hence, not any cooperative game can be constructed from an hypergraph.