Homework I - Advanced Game Theory

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Exercise 1

(a)

We can compute the Harsanyi dividends recursively as,

$$
\begin{aligned}
\Delta_v(\{1\}) &= v(1) = 1 \\
\Delta_v(\{2\}) &= v(2) = 2 \\
\Delta_v(\{3\}) &= v(3) = 0 \\
\Delta_v(\{1,2\}) &= v(\{1,2\}) - \Delta_v(\{1\}) - \Delta_v(\{2\}) = 0 \\
\Delta_v(\{1,3\}) &= v(\{1,3\}) - \Delta_v(\{1\}) - \Delta_v(\{3\}) = 3 \\
\Delta_v(\{2,3\}) &= v(\{1,3\}) - \Delta_v(\{1\}) - \Delta_v(\{3\}) = 2 \\
\Delta_v(\{1,2,3\}) &= v(\{1,2,3\}) - \Delta_v(\{1,2\}) - \Delta_v(\{2,3\}) - \Delta_v(\{1,3\}) = -1
\end{aligned}
\tag{1}
$$

(b)

Using the definition of Shapley value with Harsanyi dividends, we can compute,

$$
f_i^S = \sum_{T \in N(i)} \frac{1}{|T|} \cdot \Delta_v(T) \tag{2}
$$

this yields,

$$
f^S = \begin{pmatrix} 5/3 & 5/3 & 2/3 \end{pmatrix}
$$

(c)

In order to verify that the core is empty we can use the definition of the core,

$$x_i \in C(N,v) \implies \sum_{i\in S} x_i \geq v(S)$$
$$\sum_{i\in N} x_i = v(N)$$

Let $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ be a candidate allocation. Then the second property requires that,

$$x_1 + x_2 + x_3 = v(N) = 4 \tag{3}$$

The first property, on the other hand, requires

$$\begin{aligned} x_1 &\geq 1 \quad x_1 + x_2 \geq 3 \\ x_2 &\geq 2 \quad x_1 + x_3 \geq 3 \\ x_3 &\geq 0 \quad x_2 + x_3 \geq 2 \end{aligned} \tag{4}$$

Combining (3) and (4) we know that the core allocation requires

$$x_3 = 0 \implies x_1 \geq 3 \implies x_2 = 0 \Rightarrow\Leftarrow x_2 \geq 2, \tag{5}$$

hence there is no allocation x that satisfies (3) and (4) which implies that $C(N,v) = \emptyset$.

(d)

Convexity fails for $S = \{1,3\}$ and $T = \{2\}$. Since,

$$\begin{aligned} v\left(S \cup T\right) + v\left(S \cap T\right) &< v(S) + v(T) \\ v(\{1,2,3\}) + v(\emptyset) &< v(\{1,3\}) + v(2) \\ 4 &< 3 + 2 \end{aligned} \tag{6}$$

Therefore the game is not convex.

Exercise 2

The imputation set is defined as,

$$I(N,v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \wedge x_i \geq v(i) \ \forall i \in N \right\}. \tag{7}$$

First, notice that, starting from the definition with Harsanyi dividends, we can rewrite the Shapley, for a single player as the $v(i)$ and a residual term.

Let $N(i) = \{T : T \subseteq N \wedge \ i \in T\}$ and note that $\{i\} \subseteq N(i)$. Then we can write the Shapley value for i as,

$$\begin{aligned} f_i^S &= \sum_{T \in N(i)} \frac{1}{|T|} \cdot \Delta_v(T) \\ &= \sum_{T \in N(i)} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) \\ &= v(i) + \sum_{T \in N(i) \setminus \{i\}} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) \end{aligned} \tag{8}$$

where $S \subset T$ implies every proper subset $S \in 2^T \setminus T$.

Consider now, for a given T, the term,

$$v(T) - \sum_{S \subset T} \Delta_v(S). \tag{9}$$

In this case $|T| \neq 1$, since the only singleton in $N(i)$ is $\{i\}$ by construction. Now consider a set $T = \{i, j\} \in N(i), \ |T| = 2$, then, by super additivity,

$$v(T) = v(\{i\} \cup \{j\}) \geq v(i) + v(j) \implies v(T) - \sum_{S \subset T} \Delta_v(S) \geq 0 \tag{10}$$

By induction we can construct any bigger set T as union of smaller sets and show that (9) is positive. If $v(T) - \sum_{S \subset T} \Delta_v(S) \geq 0 \ \forall T \in N(i)$, then

$$f_i^S = v(i) + \sum_{T \in N(i) \setminus \{i\}} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) \geq v(i). \tag{11}$$

Exercise 3

(a)

The function v is the mapping, $v(\emptyset) = 0$, $v(\{1\}) = 0$, $v(\{2\}) = 5$, $v(\{3\}) = 0$, $v(\{1,2\}) = 15$, $v(\{1,3\}) = 5$, $v(\{2,3\}) = 10$, $v(\{1,2,3\}) = 20$.

(b)

Using (2) we obtain,

$$f^S = \begin{pmatrix} 35/6 & 65/6 & 10/3 \end{pmatrix} \tag{12}$$

(d)

We can check every combination of $S, T \in 2^N$, for the condition,

$$v\left(S \cup T\right) + v\left(S \cap T\right) \geq v(S) + v(T). \tag{13}$$

This is true for all sets hence the G is convex. Furthermore, convexity implies superadditivity, by taking S, T such that $S \cap T = \emptyset$, hence the game is also superadditive.