Exercises recipe

Evolutionary Game Theory

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February 22, 2021

1 Weibull dynamics

1.1 Steps to solve

You are given a matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ which represents the payoffs.

1.1.1 Utilities

Given a vector $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_3 \end{pmatrix}^T$ and $\boldsymbol{y}$ compute,

$$u(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}\boldsymbol{P}\boldsymbol{y} \tag{1}$$

Note that you can just set $\boldsymbol{x} \mapsto \boldsymbol{y}$ or vice-versa to get all of the utilities combinations.

1.1.2 Nash equilibria

Compute the Nash Equilibria of the symmetric game $(\boldsymbol{P}, \boldsymbol{P^T})$. Remember the mixed strategy. To find the mixed strategy you have to look at every combination of

$$\boldsymbol{x} \in \left\{ \begin{pmatrix} \alpha \\ \beta \\ 1-\alpha-\beta \end{pmatrix}, \begin{pmatrix} \alpha \\ 1-\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \\ 1-\alpha \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha \\ 1-\alpha \end{pmatrix} \right\} \tag{2}$$

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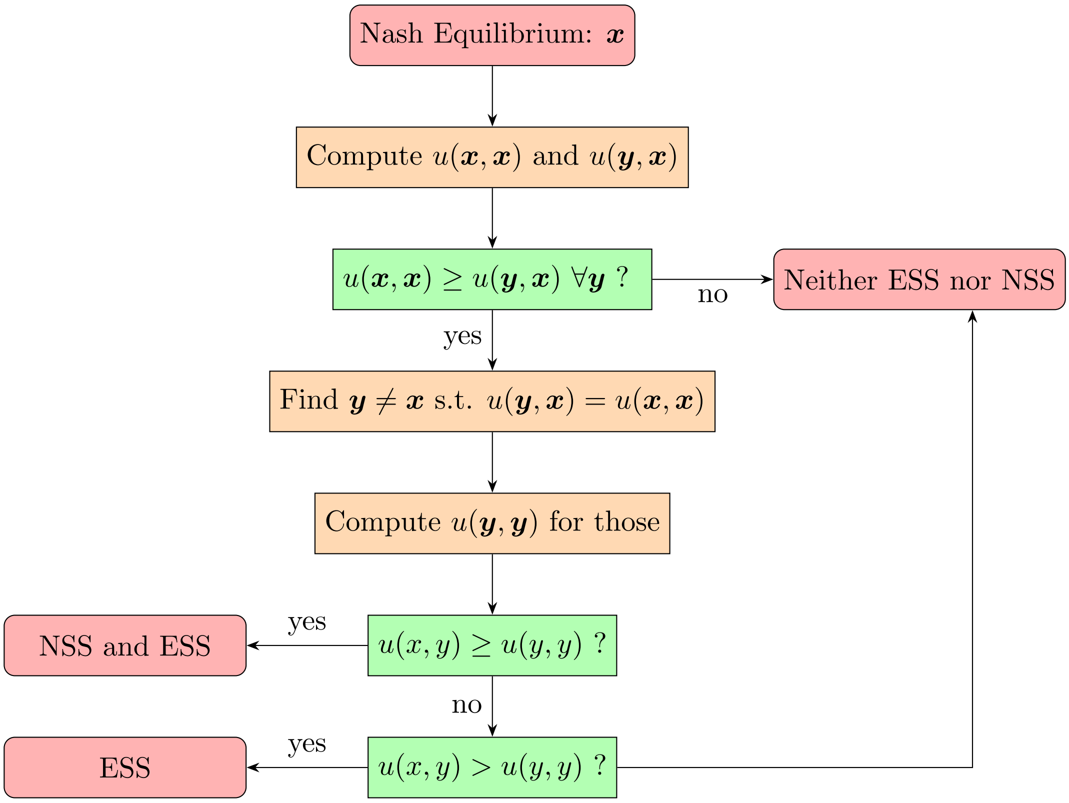
You can also use the online Irs solver by Avis and Fukuda. Make sure to

1. Put 3 3 as dimensions on top
2. Click **Symmetric** under **type of game**
3. Input matrix $\boldsymbol{P}$
4. Only pick the symmetric Nash equilibria

1.1.3 ESS and NSS

Once you found the Nash equilibrium, for each one do,



1.1.4 Replicator dynamics

Let $\boldsymbol{e}^1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. The replicator dynamics are

$$\dot{\boldsymbol{x}}_i = \left[u(\boldsymbol{e}^i,\boldsymbol{x}) - u(\boldsymbol{x},\boldsymbol{x})\right] \cdot \boldsymbol{x}_i \tag{3}$$

The main steps to find the dynamics are,

1. Find fixed points $u(\boldsymbol{e}^i,\boldsymbol{x}) = u(\boldsymbol{x},\boldsymbol{x})$
2. Find dynamics on the boundaries, so set $\boldsymbol{x_j} = 0$ for all j
3. Find dynamics on the bisector, like $\boldsymbol{x} = \begin{pmatrix} \alpha & \alpha & 1 - 2\cdot\alpha \end{pmatrix}$ with $\alpha < 1/2$

1.2 Example, ex. 14

Given

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4}$$

1.2.1 Utilities

$$\begin{aligned} u(\boldsymbol{x}, \boldsymbol{y}) &= x_1 \cdot y_1 + x_2 \cdot \overbrace{(y_2 + y_3)}^{1-y_1} + \overbrace{(1 - x_1 - x_2)}^{x_3} \cdot y_2 \\ (\boldsymbol{y} \mapsto \boldsymbol{x}),\ u(\boldsymbol{x}, \boldsymbol{x}) &= x_1^2 + x_2 \cdot (1 - x_1) + x_3 \cdot x_2 \end{aligned} \tag{5}$$

1.2.2 Nash equilibria

We get

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Rational Output

  EE  1  P1:  (1)  1/2  1/2    0  EP=  1/2  P2:  (1)  1/2  1/2    0  EP=  1/2
  EE  2  P1:  (1)  1/2  1/2    0  EP=  1/2  P2:  (2)  1/2    0  1/2  EP=  1/2
  EE  3  P1:  (2)    0    1    0  EP=    1  P2:  (3)    0    1    0  EP=    1
  EE  4  P1:  (2)    0    1    0  EP=    1  P2:  (4)    0    0    1  EP=    1
  EE  5  P1:  (3)    1    0    0  EP=    1  P2:  (5)    1    0    0  EP=    1
  EE  6  P1:  (4)  1/2    0  1/2  EP=  1/2  P2:  (1)  1/2  1/2    0  EP=  1/2
  EE  7  P1:  (5)    0    0    1  EP=    1  P2:  (3)    0    1    0  EP=    1
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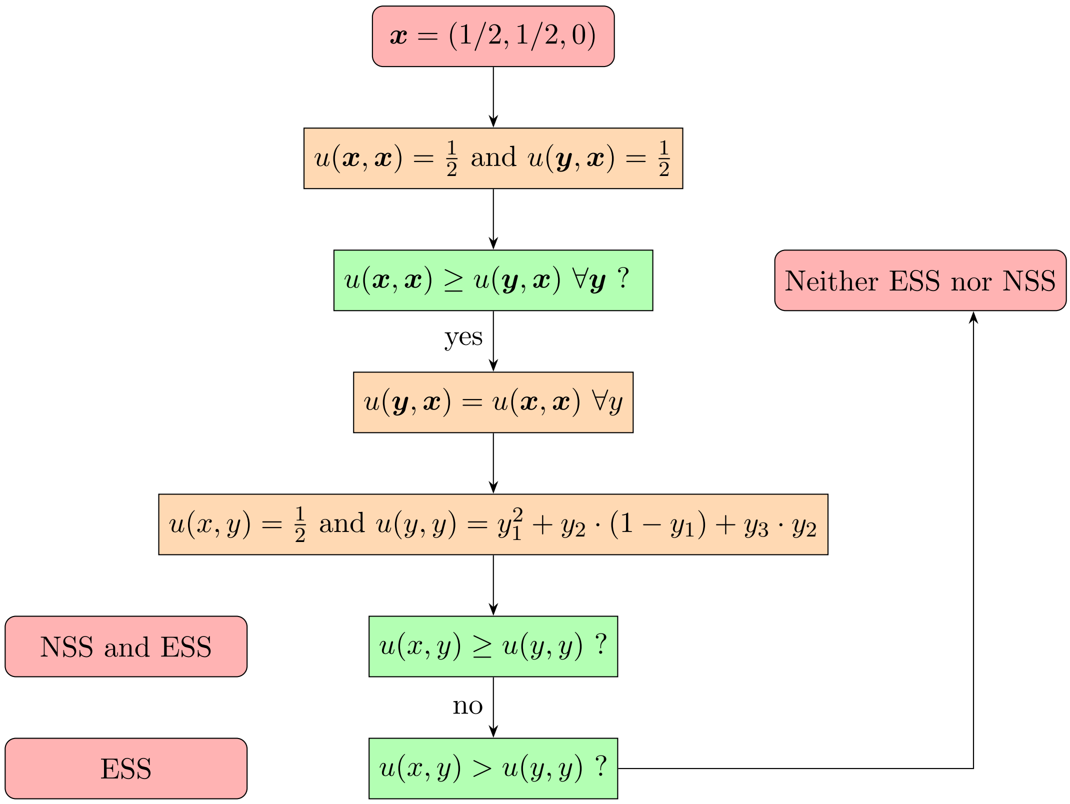
We only want the symmetric ones so rows 1, 3, and 5, namely,

$$NE = \{(1, 0, 0), (0, 1, 0), (1/2, 1/2, 0)\} \tag{6}$$

1.2.3 ESS and NSS

The third one,



1.2.4 Replicator dynamics

Corner utilities are,

$$u(\boldsymbol{e}^1,\boldsymbol{x}) = x_1,\ \ u(\boldsymbol{e}^2,\boldsymbol{x}) = (1-x_1),\ \ u(\boldsymbol{e}^3,\boldsymbol{x}) = x_2 \tag{7}$$

1. Fixed points

The fixed points here are too complex, probably not there

2. Boundaries

Remember that $\sum_i \dot{x}_i = 0$ so on the boundaries one only needs to find one x_i.

- $\boldsymbol{x} = (\alpha, 0, 1-\alpha)$ yields, $\dot{x}_1 = 1-\alpha > 0$, we move towards (1)
- $\boldsymbol{x} = (0, \alpha, 1-\alpha)$ yields $1 - 2\alpha + \alpha^2 > 0$, we move towards (2)
- $\boldsymbol{x} = (\alpha, 1-\alpha, 0)$ yields $2\alpha - 1 > 0 \iff \alpha > \frac{1}{2}$, we move away from the mid point.

3. Bisector

The only interesting bisector is the one were 1 and 2 are constant since the whole system moves towards 1 and 2. That is $\boldsymbol{x} = (\alpha, \alpha, 1-2\alpha)$.

Then we can compute,

$$
\begin{aligned}
\dot{x}_1 &= \left(\alpha - \alpha^2 - \alpha \cdot (1-\alpha) - \alpha + 2\alpha^2\right)\alpha \\
&= \alpha^2 \cdot (2\alpha - 1) < 0 \\
\dot{x}_2 &= \alpha \cdot (1-\alpha) \cdot (1 - 2\alpha) > 0 \\
\dot{x}_3 &= -(\dot{x}_1 + \dot{x}_2) = -\alpha \cdot (1-2\alpha)^2 < 0
\end{aligned}
\tag{8}
$$

When we are the bisector apart from the two edge points $\alpha = 0$ and $\alpha = \frac{1}{2}$ we are moving towards 2. This implies that the manifold is towards 1.