Homework III - Advanced Game Theory

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Exercise 1

(a)

The benefit function of an allocation can be written as,

$$b(x) = b^T x, \text{ where } x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T \text{ and } b^T = \begin{pmatrix} 2 & 4 & 7 \end{pmatrix} \tag{1}$$

The water distribution problem, constraint by the water allocation vector $e = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ is,

$$x^* = \arg\max_x b^T x, \quad \text{s.t. } Ax \leq e, \tag{2}$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

Given the linearity of the problem it is easy to see that the optimal water allocation problem, restricted to a coalition S, yields,

$$x_i^* = \begin{cases} \sum_{i \in S} e_i & \text{if } i = \max_{j \in S} j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

So that all the resources available to a coalition goes to the furthest downstream node. Such that for example, $S = N \implies x = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$.

Using this optimal allocation we obtain the value function,

$$v(S) = \begin{cases} 2 & \text{if } S = \{1\} \\ 4 & \text{if } S = \{1,2\} \\ 2 & \text{if } S = \{1,3\} \\ 7 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b)

First we can compute the marginal vector $m^u(v)$, by looking at the permutation $u = (1, 2, 3)$. Then,

$$m^u(v) = \begin{pmatrix} v(1) \\ v(\{1,2\}) - v(1) \\ v(N) - v(\{1,2\}) \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \end{pmatrix}^T \tag{5}$$

Doing the same for $m^l(v)$, with permutation $l = (3, 2, 1)$, we obtain,

$$m^l(v) = \begin{pmatrix} 7 & 0 & 0 \end{pmatrix}^T \tag{6}$$

Then,

$$f^e(v) = \frac{1}{2} \cdot (m^l(v) + m^u(v)) = \begin{pmatrix} 4.5 & 1 & 1.5 \end{pmatrix}^T \tag{7}$$

Finally to compute the Shapley value of the game we can compute the Harsanyi dividends, which are trivially,

$$\Delta(S) = \begin{cases} 2 & \text{if } S = \{1\} \\ 2 & \text{if } S = \{1,2\} \\ 3 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

which yield a Shapley value of,

$$f^{Sh}(v) = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}^T. \tag{9}$$

(c)

If $e_3 > 1$, then the optimal allocation remains the one described in (3) since the constraint changes and ∇b remains constant.

Given this result, in case of the marginal vector m^l or m^u, the final node in the permutation would capture all the extra benefit of increased water supply in e_3. So that in m^u the allocation for 1 and 2 would not change. Likewise in m^l the allocation for 1 would be,

$$m_1^l = b^T \left(\begin{pmatrix} 0 \\ 0 \\ e_1 + e_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ e_3 \end{pmatrix} \right) = b_3 \cdot e_1 \tag{10}$$

so that 1's payoff does not change, since he only capture his contribution to the added benefit of the coalition. This implies that allocation of 1 and 2 in f^e does not change as well. In a similar manner the Shapley value does not yield a higher allocation for 1 and 2, since the Harsanyi dividends remain unchanged.

Intuitively, allocating all possible water within the coalition to 3 remains optimum hence all the benefits allocated *ex-ante* to 3 (i.e. $v(e_3)$) are never transferred to other agents, hence the marginal contribution of 1 and 2 in a coalition does not change.

(d)

In a line graph $[1, n]$, the hierarchical outcomes associated with roots 1 and n are,

$$\begin{aligned} h_k^n &= v[1, k] - v[1, k-1] = m_k^u \\ h_k^1 &= v[k, n] - v[k+1, n] = m_k^l \end{aligned} \tag{11}$$

since $F_k^1 = \{k+1\}$ and $F_{k+1}^n = \{k\}$. In our case, with $n = 3$, this allows us to use the values of part (b),

$$\begin{aligned} h^3 &= m^u = \begin{pmatrix} 2 & 2 & 3 \end{pmatrix}^T \\ h^1 &= m^l = \begin{pmatrix} 7 & 0 & 0 \end{pmatrix}^T. \end{aligned} \tag{12}$$

Using the directed graph $L^2 = \{(2, 1), (2, 3)\}$ to compute h^2, yields,

$$h^2 = \begin{pmatrix} v(1) \\ v(\{1, 2, 3\}) - v(1) - v(2) \\ v(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} \tag{13}$$

The average hierarchical outcome is then,

$$\bar{h} = \frac{h^1 + h^2 + h^3}{3} = \frac{1}{3} \cdot \begin{pmatrix} 11 \\ 7 \\ 3 \end{pmatrix} \tag{14}$$

(e)

For the outcomes to belong to the $Core(v)$ they need to satisfy, $\sum_{j \in N} h_j^i = v(N) = 1$ and $\sum_{j \in S} h_j^i \geq v(S)$.

The first condition is satisfied since $v(N) = 7 = \sum_{j \in N} h_j^i \; \forall i$. Let's check the second condition for every $S \subset N$. Note that it is sufficient to only check the connected coalitions with 1 in them, since 1 has the only source of water. If this is not the case for a coalition S, then $v(S) = 0 \leq \sum_j h_j^i$. The remaining coalitions are then $\{1,2\}$, $\{1,3\}$, and $\{1\}$. Then,

$$\begin{aligned} S = \{1\},\ (h_1^1, h_1^2, h_1^3) = (7,2,2) \geq v(S) = 2 \\ S = \{1,2\},\ \sum_{j \in S}(h_j^1, h_j^3, h_j^3) = (7,7,4) \geq v(S) = 4 \\ S = \{1,3\}, \sum_{j \in S}(h_j^1, h_j^3, h_j^3) = (7,2,5) \geq v(S) = 2 \end{aligned} \tag{15}$$

Hence the second condition is also satisfied and $h^i \in Core(v)\ \forall i$.

Exercise 2

The $Core(v^L)$ of the Myerson restricted game (N, v^L) where L is a line graph, is defined as,

$$Core(v^L) := \left\{ x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v^L(S) \wedge \sum_{i \in N} x_i = v^L(N) \right\}. \tag{16}$$

The upper equivalent marginal vector $m^u(v^L)$ is the marginal vector associated with the permutation of N, $u = (1, 2, \ldots, n)$. Hence we can write,

$$m_i^u(v^L) = v^L[1,i] - v^L[1,i-1]. \tag{17}$$

Hereafter we will denote the vector m_i^u and the value function v for simplicity.

Let S be a connected coalition, hence it can be represented as $S = [l, r]$. We can then rewrite,

$$
\begin{aligned}
\cup_{i\in S}[1,i] = [1, \max_{k\in S} k] = [1, \min_{j\in S} j] \cup \underbrace{[\min_{j\in S} j, \max_{k\in S} k]}_{S} \\
\implies [1,r] = [1, l-1] \cup S \\
\implies v[1,r] - v[1,l-1] \geq v(S) \text{ by superadditivity of } v
\end{aligned}
\tag{18}
$$

Using now equation (17).

$$
\begin{aligned}
\sum_{i\in S} m_i^u &= \sum_{i\in S=[l,r]} (v[1,i] - v[1,i-1]) \\
&= v[1,r] - v[1,r-1] + v[1,r-1] - v[1,r-2] \ldots + v[1,l] - v[1,l-1] \\
&= v[1,r] - v[1,l-1] \geq v(S) \text{ by (18).}
\end{aligned}
\tag{19}
$$

This result can easily be extended to a not connected coalition, since we can rewrite the coalition as a union of disjoint connected sets, namely, $S = [l, i_1] \cup [i_2, i_3] \cup \cdots \cup [i_n, r] = \cup_i I_i$ where $I_i \in \mathcal{I}$.This allow us to derive again equation (18) as,

$$
\begin{aligned}
[1,r] = [1,l-1] \cup \underbrace{\left(\bigcup_i I_i\right)}_{S} \cup ([1,r] \cap S)^c \\
v[1,r] - v[1,l-1] - v\left([1,r]\cap S\right)^c \geq v(S)
\end{aligned}
\tag{20}
$$

Then (19) yields,

$$
\begin{aligned}
\sum_{i\in S} m_i^u &= \sum_{i\in \cup_i I_i} (v[1,i] - v[1,i-1]) \\
&= v[1,r] - v[1,i_n-1] + v[1,i_{n-1}] - v[1,i_n-2] \ldots + v[1,i_1] - v[1,l-1] \\
&\geq v\left(\bigcup_i I_i\right) = v(S),
\end{aligned}
\tag{21}
$$

by noting that, via superadditivity,

$$
v\left([1,r]\cap S\right)^c \geq v[1,i_n-1] - v[1,i_{n-1}] + v[1,i_n-2] \ldots - v[1,i_1]. \tag{22}
$$

Now consider the case where $S = N$. Trivially $N \in \mathcal{I}$. Furthermore, using equation (19), and $[l,r] = [0,n]$, we obtain,

$$\begin{aligned}\sum_{i\in N} m_i^u &= \sum_{i\in N}(v[1,i]-v[1,i-1]) \\ &= v[1,n]-v(\emptyset)=v(N).\end{aligned} \tag{23}$$

The equations (21) and (23) imply that m^u is in the core of the Myerson restricted game.

Exercise 3

(a)

I believe modelling *Absolute Territorial Sovereignty* via *core-stability* is a sensible choice. In particular core stability implies for an allocation x that $\sum_{i\in S} x_i \geq v(S)$. If S is a single country, this trivially requires that the allocation a country receives is greater than its own individual benefit, namely $x_i \geq v(\{i\})$. In the river game $v(\{i\}) = v(e_i)$: a country's benefit from not participating in a coalition (one can say, its outside option) is using all of its own water, consistent with *ATS*. *Core-stability* extends this concept to any coalition, $\sum_{i\in S} x_i \geq v(S)$, namely an allocation needs to make the coalition better off than using the water of its own members.

(b)

α-TIBS fairness is satisfied if the gains of cooperating between to coalitions are divided among all members proportionally to a weight vector α. This notion is a correct formalization of the notion of *Territorial Integration of all Basin States*. In particular, the latter requires that the ownership over the watercourse is shared among all basin states independently of the entry of the water course. This implies that an allocation which satisfies *α-TIBS fairness* and where the weight vector α is independent of the water course entry vector e respects the *TIBS* principle. Furthermore, the characterization in *TIBS* of equitable and fair allocation leaves room for interpretation but any interpretation can be achieved by some vector α.

Exercise 4

(a)

First notice that $\frac{2}{3}\cdot|N| = 6$. Hence,

$$v^L(S) = \begin{cases} 1 \text{ if } S \text{ is connected and } |S|\geq 6 \\ 0 \text{ otherwise} \end{cases} \tag{24}$$

This implies that in the marginal vector m^u the only marginal contribution is given by the sixth node, which is the first in the u permutation to yield $|\{1, \dots i\}| \geq 6$. Hence the $m_i^u = 1$ if $i = 6$ and 0 otherwise. Likewise, m_i^l is 1 if $n - i = 6, i = 4$ and 0 otherwise since, $|\{4, \dots n\}| = 6$. Then f^e is simply the average of the two vectors, namely,

$$\begin{aligned} m^u &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ m^l &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ f^e &= \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad . \tag{25}$$

As before, we need to check that $\sum_{i \in N} m^u = \sum_{i \in N} m^l = \sum_{i \in N} f^e = v(N)$. This is satisfied since all allocations sum to 1. It is also necessary that,

$$\sum_{i \in S} x_i \geq v(S), \; x \in \left\{ m^u, m^l, f^e \right\}. \tag{26}$$

First note that for any S such that $V(S) = 0$, the condition is trivially satisfied since $x_i \geq 0 \; \forall i$. Let the coalitions that are connected and have order 6 be $M = \{[1, 6], [2, 7], [3, 8], [4, 9]\}$. All of the coalitions in M have value of 1, and a coalition has value of 1 if and only if it is a superset of a coalition in M.

Given that $x_i = 1 \implies i \in \{4, 6\} \implies i \in I \; \forall \; I \in M$, it holds that,

$$\sum_{j \in S} x_j = 1 \implies v(S) = 1 \implies \sum_{j \in S} x_j \geq v(S). \tag{27}$$

Therefore the three allocations m^u, m^l, f^e are in the $Core(v^L)$.

(b)

First note that the Myerson value is the mean of the marginal vectors for every permutation of of N. For an arbitrary permutation the marginal vector needs to be strictly positive since for any permutation there is an element i such that, $v(\{1, \dots, j, i\}) - v(\{1, \dots, j\}) = 1$ and $v(\{1, \dots, j, i, \dots\}) - v(\{1, \dots, j, i\}) = 0$. As an average of positive vectors, the Myerson value is strictly positive, namely,

$$\mu_i > 0 \; \forall i \tag{28}$$

Assume that the Myerson value satisfies the first property of the $Core(v^L)$, namely,

$$\sum_{i \in N} \mu_i = v(N) = 1. \tag{29}$$

Now take $S = [1, 6]$ and $S^c = [7, 9]$. Assume that μ satisfies the second property. This implies,

$$\sum_{i \in S} \mu_i \geq V(S) = V(N) = 1. \tag{30}$$

But $\sum_{i \in S} \mu_i \geq 1$ and $\sum_{i \in N} \mu_i = 1$, imply that $\sum_{i \in S^c} \mu_i = 0$, which contradicts equation (28). Hence μ cannot satisfy the second property of the $Core(v, L)$ if it satisfies the first, hence $\mu \notin Core(v, L)$