Homework I - Advanced Game Theory

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January 7, 2021

Exercise 2

The imputation set is defined as,

$$I(N, v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \wedge x_i \geq v(i) \ \forall i \in N \right\}. \tag{1}$$

First, notice that, starting from the definition with Harsanyi dividends, we can rewrite the Shapley, for a single player as the $v(i)$ and a residual term.

Let $N(i) = \{T : T \subseteq N \wedge \ i \in T\}$ and note that $\{i\} \subseteq N(i)$. Then we can write the Shapley value for i as,

$$\begin{aligned} f_i^S &= \sum_{T \in N(i)} \frac{1}{|T|} \cdot \Delta_v(T) \\ &= \sum_{T \in N(i)} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) \\ &= v(i) + \sum_{T \in N(i) \setminus \{i\}} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) \end{aligned} \tag{2}$$

where $S \subset T$ implies every proper subset $S \in 2^T \setminus T$.

For a given $T \in N(i) \setminus \{i\}$, we can expand recursively the expression above in order to rewrite the expression in terms of v applied to proper subsets of T,

$$\begin{aligned} v(T) - \sum_{S \subset T} \Delta_v(S) &= v(T) - \sum_{S \subset T} \left(v(S) - \sum_{G \subset S} \Delta_v(G) \right) \\ &= v(T) - \sum_{S \subset T} \left(v(S) - \sum_{G \subset S} (v(G) - \dots) \right) \end{aligned} \tag{3}$$

Given this recursive definition, notice that innermost difference will be constructed with a finite number of singletons sets P of the form,

$$v\left(\bigcup P \right) - \sum_P v(P) \geq 0 \text{ by superadditivity.} \tag{4}$$

By induction then, any difference is positive, which implies that,

$$
\begin{aligned}
&v(T) - \sum_{S \subset T} \Delta_v(S) \geq 0 \ \forall T \\
&\implies v(i) + \sum_{T \in N(i) \setminus \{i\}} \frac{1}{|T|} \cdot \left(v(T) - \sum_{S \subset T} \Delta_v(S) \right) = f_i^S \geq v(i)
\end{aligned} \tag{5}
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