

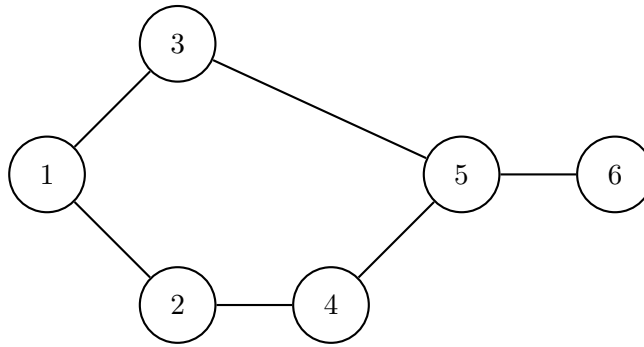
# Homework I - Advanced Game Theory

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## Exercise 1

The graph representation of  $(N, L)$  is,



**(a)**

The function  $v^L$  of the Myerson restricted game is,

$$v^L(S) = \sum_{T \in C_L(S)} v(T). \quad (1)$$

It is non-zero only for the component of the graphs that include both 1 and 6, namely,

$$v^L(S) = \begin{cases} 1 & \text{if } S \in \{\{1, 3, 5, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

**(b)**

We can define the communication game  $(N, v^L)$ , using (2). Then we can compute the Harsanyi dividends based on this game. The non-zero dividends are,

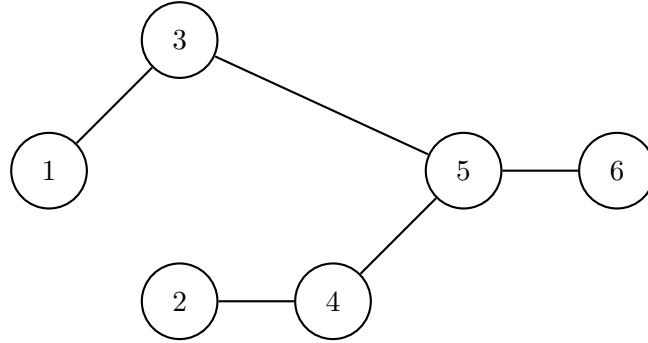
$$\begin{aligned}\Delta_{v^L}(\{1, 3, 5, 6\}) &= 1 \\ \Delta_{v^L}(\{1, 2, 4, 5, 6\}) &= 1 \\ \Delta_{v^L}(\{1, 2, 3, 4, 5, 6\}) &= -1\end{aligned}\tag{3}$$

Then we can compute the Myerson value as the Shapley value of the new coordination game,

$$\begin{aligned}\mu_i(v, L) &= f_i^S = \sum_{T \in N(i)} \Delta_{v^L}(T) / |T| \\ \implies \mu(v, L) &= f^S = (17/60 \quad 1/30 \quad 1/12 \quad 1/30 \quad 17/60 \quad 17/60)\end{aligned}\tag{4}$$

**(c)**

The new supply chain is represented by the graph,



We can hence define

$$v^{L'}(S) = \begin{cases} 0 & \text{if } S = \{1, 2, 4, 5, 6\} \\ v^L & \text{otherwise} \end{cases}\tag{5}$$

By repeating the procedure of (4), we obtain,

$$\mu(v, L') = (1/4 \quad 0 \quad 1/4 \quad 0 \quad 1/4 \quad 1/4). \quad (6)$$

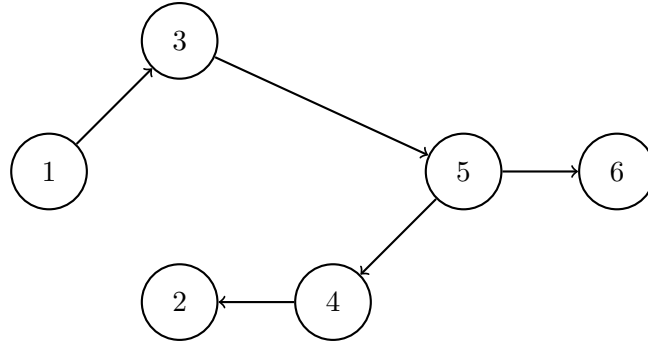
This result is expected since in the new supply chain 2 and 4 become null-players.

In order to determine fairness we can check that,

$$\begin{aligned} \mu_1(v, L) - \mu_1(v, L') &= \mu_2(v, L) - \mu_2(v, L') \\ 17/60 - 1/4 &= 1/30 - 0 \implies \text{the fairness axiom is respected.} \end{aligned} \quad (7)$$

**(d)**

In order to compute the hierarchical outcome of node 1 we need to set 1 as root. The resulting directed graph  $(N, L^1)$  can be represented as,



Using the definition of followers and subordinates, we can compute the hierarchical outcomes as,

$$h^1(v, L') = \begin{pmatrix} v^{L'}(\{1, 3, 5, 6, 4, 2\}) - v^{L'}(\{3, 5, 6, 4, 2\}) \\ v^{L'}(\{2\}) \\ v^{L'}(\{3, 5, 6, 4, 2\}) - v^{L'}(\{5, 6, 4, 2\}) \\ v^{L'}(\{4, 2\}) - v(\{2\}) \\ v^{L'}(\{5, 6, 4, 2\}) - v^{L'}(\{4, 2\}) - v^{L'}(\{6\}) \\ v^{L'}(6) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (8)$$

In a similar manner,

$$h^4(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad h^6(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (9)$$

**(e)**

The link game  $(L', r^{L'})$  is given by the characteristic function,

$$\bar{E} = \left\{ \begin{array}{l} \{(1, 3), (3, 5), (5, 6)\} \\ \{(1, 3), (4, 2), (3, 5), (5, 6)\} \\ \{(1, 3), (3, 5), (5, 4), (5, 6)\} \\ \{(1, 3), (4, 2), (3, 5), (5, 4), (5, 6)\} \end{array} \right\} \quad (10)$$

$$r^{L'}(E) = \begin{cases} 1 & \text{if } E \in \bar{E} \\ 0 & \text{otherwise} \end{cases}$$

The Shapley value of the link game is,

$$f^S(L', r^{L'}) = (1/3 \quad 0 \quad 1/3 \quad 0 \quad 1/3) \quad (11)$$

The associated position value is.

$$\pi(v, L') = (1/6 \quad 0 \quad 1/3 \quad 0 \quad 1/3 \quad 1/6) \quad (12)$$

## Exercise 2

A hierarchical outcome  $h^i$  is in the  $Core(N, v_L)$  if and only if,

$$\sum_{j \in S} h_j^i \geq v(S) \quad \text{and} \quad \sum_{j \in N} h_j^i = v(N).$$

First let the subordinates of  $j$  be

$$S_j^i := \{h : \exists(i_1, \dots, i_t) \text{ s.t. } i_1 = j \wedge i_t = h \wedge (i_k, i_{k+1}) \in L^i \forall k \in \{1, \dots, t\}\} \quad (13)$$

Entry  $j$  in the hierarchical outcome can be written as,

$$h_j^i = v(\hat{F}_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i). \quad (14)$$

By superadditivity,

$$\hat{F}_j^i = \{j\} \cup S_j^i \implies v(\hat{F}_j^i) \geq v(j) + v(S_j^i). \quad (15)$$

Furthermore,

$$\hat{F}_h^i \subseteq S_j^i \quad \forall h \in F_j^i \implies \bigcup_{h \in F_j^i} \hat{F}_h^i \subseteq S_j^i \implies \sum_{h \in F_j^i} v(\hat{F}_h^i) \leq v(S_j^i). \quad (16)$$

Combining these results, we can use (14), and show that,

$$h_j^i = v(\hat{F}_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i) \geq v(j) + v(S_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i) \geq v(j). \quad (17)$$

r Given that  $h_j^i \geq v(j)$ , by superadditivity we can show that,

$$\sum_{j \in S} h_j^i \geq \sum_{j \in S} v(j). \quad (18)$$

## Exercise 4

### (a)

Union stability requires that,

$$A, B \in \mathcal{F} \wedge A \cap B = \emptyset \implies A \cup B \in \mathcal{F} \quad (19)$$

Let  $B \subset M$  and  $A \subseteq H$ . Then,

$$\begin{aligned} A, B \in \mathcal{F} \wedge A \cap B, \\ A \cup B \not\subset H \wedge A \cup B \not\subset M \wedge M \not\subset A \cup B \implies A \cup B \notin \mathcal{F}. \end{aligned} \quad (20)$$

Hence  $\mathcal{F}$  does not satisfy union stability.

### (b)