

Homework II - Advanced Game Theory

Andrea Tifton

January 26, 2021

Exercise 1

(a)

The benefit function of an allocation can be written as,

$$b(x) = b^T x, \text{ where } x = (x_1 \ x_2 \ x_3)^T \text{ and } b^T = (2 \ 4 \ 7) \quad (1)$$

The water distribution problem, constraint by the water allocation vector $e = (1 \ 0 \ 0)^T$ is,

$$x^* = \arg \max_x b^T x, \text{ s.t. } Ax \leq e, \quad (2)$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

Given the linearity of the problem it is easy to see that the optimal water allocation problem, restricted to a coalition S , yields,

$$x_i^* = \begin{cases} \sum_{i \in S} e_i & \text{if } i = \max_{j \in S} j \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

So that all the resources available to a coalition goes to the furthest downstream node. Such that for example, $S = N \implies x = (0 \ 0 \ 1)^T$.

Using this optimal allocation we obtain the value function,

$$v(S) = \begin{cases} 2 & \text{if } S = \{1\} \\ 4 & \text{if } S = \{1, 2\} \\ 7 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(b)

First we can compute the marginal vector $m^u(v)$, by looking at the permutation $u = (1, 2, 3)$. Then,

$$m^u(v) = \begin{pmatrix} v(1) \\ v(\{1, 2\}) - v(1) \\ v(N) - v(\{1, 2\}) \end{pmatrix} = (2 \quad 2 \quad 3)^T \quad (5)$$

Doing the same for $m^l(v)$, with permutation $l = (3, 2, 1)$, we obtain,

$$m^l(v) = (7 \quad 0 \quad 0)^T \quad (6)$$

Then,

$$f^e(v) = \frac{1}{2} \cdot (m^l(v) + m^u(v)) = (4.5 \quad 1 \quad 1.5)^T \quad (7)$$

Finally to compute the Shapley value of the game we can compute the Harsanyi dividends, which are trivially,

$$\Delta(S) = \begin{cases} 2 & \text{if } S = \{1\} \\ 2 & \text{if } S = \{1, 2\} \\ 3 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

which yield a Shapley value of,

$$f^{Sh}(v) = (4 \quad 2 \quad 1)^T. \quad (9)$$

(c)

If $e_3 > 1$, then the optimal allocation remains the one described in (3) since the constraint changes and ∇b remains constant.

Given this result, in case of the marginal vector m^l or m^u , the final node in the permutation would capture all the extra benefit of increased water supply in e_3 . So that in m^u

the allocation for 1 and 2 would not change. Likewise in m^l the allocation for 1 would be,

$$m_1^l = b^T \left(\begin{pmatrix} 0 \\ 0 \\ e_1 + e_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ e_3 \end{pmatrix} \right) = b_3 \cdot e_1 \quad (10)$$

so that 1's payoff does not change, since he only capture his contribution to the added benefit of the coalition. This implies that allocation of 1 and 2 in f^e does not change as well.

In a similar manner the Shapley value does not yield a higher allocation for 1 and 2, since their dividends Harsanyi dividends remain unchanged.

(d)

In a line graph $[1, n]$, the hierarchical outcomes associated with roots 1 and n are,

$$\begin{aligned} h_k^n &= v[1, k] - v[1, k-1] = m_k^u \\ h_k^1 &= v[k, n] - v[k+1, n] = m_k^l \end{aligned} \quad (11)$$

since $F_k^1 = \{k+1\}$ and $F_{k+1}^n = \{k\}$. In our case, with $n = 3$, this allows use the values of part (b),

$$\begin{aligned} h^3 &= m^u = (2 \ 2 \ 3)^T \\ h^1 &= m^l = (7 \ 0 \ 0)^T. \end{aligned} \quad (12)$$

In order to compute h^2 , using the directed graph $L^2 = \{(2, 1), (2, 3)\}$, yields,

$$h^2 = \begin{pmatrix} v(1) \\ v(\{1, 2, 3\}) - v(1) - v(2) \\ v(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} \quad (13)$$

(e)

In order for the outcomes to belong to the $Core(v)$ they need to satisfy, $\sum_{j \in N} h_j^i = v(N) = 1$ and $\sum_{j \in S} h_j^i \geq v(S)$.

The first condition is satisfied since $v(N) = 7 = \sum_{j \in N} h_j^i \ \forall i$. Let's check the second condition for every $S \subset N$. Note that it is sufficient to only check the connected coalitions

with 1 in them, since 1 has the only source of water. If this is not the case for a coalition S , then $v(S) = 0 \leq \sum_j h_j^i$. The remaining coalitions are then $\{1, 2\}$ and $\{1\}$. Then,

$$\begin{aligned} S = \{1\}, \quad (h_1^1, h_1^2, h_1^3) &= (7, 2, 2) \geq v(S) = 2 \\ S = \{1, 2\}, \quad \sum_{j \in S} (h_j^1, h_j^2, h_j^3) &= (7, 7, 4) \geq v(S) = 4 \end{aligned} \quad (14)$$

Hence the second condition is also satisfied and $h^i \in \text{Core}(v) \forall i$.

Exercise 2

The $\text{Core}(v^L)$ of the Myerson restricted game (N, v^L) where L is a line graph, is defined as,

$$\text{Core}(v^L) := \left\{ x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v^L(S) \wedge \sum_{i \in N} x_i = v^L(N) \right\}. \quad (15)$$

The upper equivalent marginal vector $m^u(v^L)$ is the marginal vector associated with the permutation of N , $u = (1, 2, \dots, n)$. Hence we can write,

$$m_i^u(v^L) = v^L[1, i] - v^L[1, i-1]. \quad (16)$$

Hereafter we will denote the vector m_i^u and the value function v for simplicity.

Let S be a connected coalition, hence it can be represented as $S = [l, r]$. We can then rewrite,

$$\begin{aligned} \cup_{i \in S} [1, i] &= [1, \max_{k \in S} k] = [1, \min_{j \in S} j] \cup \underbrace{[\min_{j \in S} j, \max_{k \in S} k]}_S \\ \implies [1, r] &= [1, l-1] \cup S \\ \implies v[1, r] - v[1, l-1] &\geq v(S) \text{ by superadditivity of } v \end{aligned} \quad (17)$$

Using now equation (16).

$$\begin{aligned} \sum_{i \in S} m_i^u &= \sum_{i \in S=[l, r]} (v[1, i] - v[1, i-1]) \\ &= v[1, r] - v[1, r-1] + v[1, r-1] - v[1, r-2] \dots + v[1, l] - v[1, l-1] \\ &= v[1, r] - v[1, l-1] \geq v(S) \text{ by (17).} \end{aligned} \quad (18)$$

This result can easily be extended to a not connected coalition, since we can rewrite the coalition as a union of disjoint connected sets, namely, $S = [l, i_1] \cup [i_2, i_3] \cup \dots \cup [i_n, r] = \cup_i I_i$ where $I_i \in \mathcal{I}$. This allow us to derive again equation (17) as,

$$\begin{aligned} [1, r] &= [1, l-1] \cup \underbrace{\left(\bigcup_i I_i \right)}_S \cup ([1, r] \cap S)^c \\ v[1, r] - v[1, l-1] - v([1, r] \cap S)^c &\geq v(S) \end{aligned} \quad (19)$$

Then (18) yields,

$$\begin{aligned} \sum_{i \in S} m_i^u &= \sum_{i \in \cup_i I_i} (v[1, i] - v[1, i-1]) \\ &= v[1, r] - v[1, i_n-1] + v[1, i_{n-1}] - v[1, i_n-2] \dots + v[1, i_1] - v[1, l-1] \\ &\geq v\left(\bigcup_i I_i\right) = v(S), \end{aligned} \quad (20)$$

by noting that, via superadditivity,

$$v([1, r] \cap S)^c \geq v[1, i_n-1] - v[1, i_{n-1}] + v[1, i_n-2] \dots - v[1, i_1]. \quad (21)$$

Now consider the case where $S = N$. Trivially $N \in \mathcal{I}$. Furthermore, using equation (18), and $[l, r] = [0, n]$, we obtain,

$$\begin{aligned} \sum_{i \in N} m_i^u &= \sum_{i \in N} (v[1, i] - v[1, i-1]) \\ &= v[1, n] - v(\emptyset) = v(N). \end{aligned} \quad (22)$$

The equations 20 and 22 imply that m^u is in the core of the Myerson restricted game.

Exercise 3

(a)

I believe modelling *Absolute Territorial Sovereignty* via *core-stability* is a sensible choice. In particular core stability implies for an allocation x that $\sum_{i \in S} x_i \geq v(S)$. If S is a single country, this trivially requires that the allocation a country receives is greater

than its own individual benefit, namely $x_i \geq v(\{i\})$. In the river game $v(\{i\}) = v(e_i)$: a country's benefit from not participating in a coalition (one can say, its outside option) is using all of its own water, consistent with *ATS*. *Core-stability* extends this concept to any coalition, $\sum_{i \in S} x_i \geq v(S)$, namely an allocation needs to make the coalition better off than using the water of its own members.

(b)

α -*TIBS fairness* is satisfied if the gains of cooperating between two coalitions are divided among all members proportionally to a weight vector α . This notion is a correct formalization of the notion of *Territorial Integration of all Basin States*. In particular, the latter requires that the ownership over the watercourse is shared among all basin states independently of the entry of the water course. This implies that an allocation which satisfies α -*TIBS fairness* and where the weight vector α is independent of the water course entry vector e respects the *TIBS* principle.

Exercise 4

(a)

First notice that $\frac{2|N|}{3=6}$. Hence,

$$v^L(S) = \begin{cases} 1 & \text{if } S \text{ is connected and } |S| \geq 6 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

This implies that in the marginal vector m^u the only marginal contribution is given by the sixth node, which is the first in the u permutation to yield $|\{1, \dots, i\}| \geq 6$. Hence the $m_i^u = 1$ if $i = 6$ and 0 otherwise. Likewise, m_i^l is 1 if $n - i = 6, i = 4$ and 0 otherwise since, $|\{4, \dots, n\}| = 6$. Then f^e is simply the average of the two vectors, namely,

$$\begin{aligned} m^u &= (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0) \\ m^l &= (0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0) \\ f^e &= (0 \ 0 \ 0 \ 0.5 \ 0 \ 0.5 \ 0 \ 0 \ 0) \end{aligned} \quad (24)$$

As before, we need to check that $\sum_{i \in N} m^u = \sum_{i \in N} m^l = \sum_{i \in N} f^e = v(N)$. This is satisfied since all allocations sum to 1. It is also necessary that,

$$\sum_{i \in S} x_i \geq v(S), \ x \in \{m^u, m^l, f^e\}. \quad (25)$$

First note that for any S such that $V(S) = 0$, the condition is trivially satisfied since $x_i \geq 0 \forall i$. Let the coalitions that are connected and have order 6 be $M = \{[1, 6], [2, 7], [3, 8], [4, 9]\}$. All of the coalitions in M have value of 1, and a coalition has value of 1 if and only if it is a superset of a coalition in M .

Given that $x_i = 1 \implies i \in \{4, 6\} \implies i \in I \forall I \in M$, it holds that,

$$\sum_{j \in S} x_j = 1 \implies v(S) = 1 \implies \sum_{j \in S} x_j \geq v(S). \quad (26)$$

Therefore the three allocations m^u, m^l, f^e are in the $Core(v^L)$.

(b)

As proven by van den Nouweland and Borm (1991), if v is convex and L is cycle-complete, $f^{Sh}(v^L) = \mu(v, L) \in Core(v^L)$.

In our case L has no cycles since it is a line-game, hence L is trivially cycle-complete.

Now we want to prove that v is convex, namely, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. Take S and T , arbitrary coalitions in N . This is not convex, $S = [1, 6], T = [4, 9]$

$$\begin{aligned} v[1, 9] + v[4, 6] &< v[1, 6] + v[4, 9] \\ 1 &< 1 + 1 \end{aligned} \quad (27)$$

References

- van den Nouweland, A. & Borm, P. (1991). On the Convexity of Communication Games. *International Journal of Game Theory*, 19(4), 421–430. Retrieved from <https://ideas.repec.org/a/spr/jogath/v19y1991i4p421-30.html>