Homework II - Advanced Game Theory

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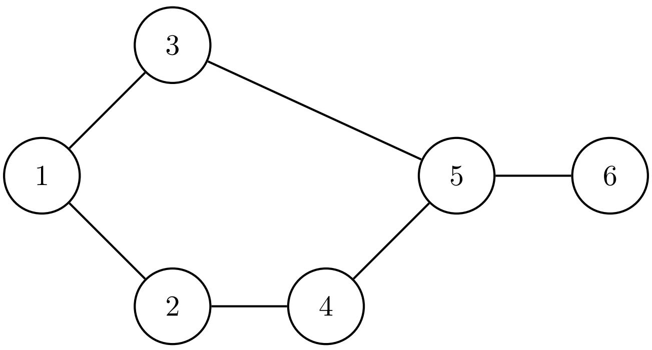
Exercise 1

(a)

The graph representation of (N, L) is,



The function v^L of the Myerson restricted game is,

$$v^L(S) = \sum_{T \in C_L(S)} v(T). \tag{1}$$

It is non-zero only for the component of the graphs that include both 1 and 6, namely,

$$v^L(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,5,6\}, \{1,2,3,5,6\}, \{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b)

We can define the communication game (N, v^L), using (2). Then we can compute the Harsanyi dividends based on this game. The non-zero dividends are,

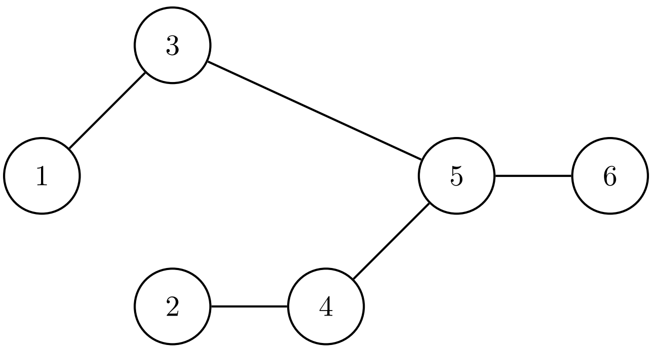
$$\begin{aligned}\Delta_{v^L}(\{1,3,5,6\}) &= 1\\ \Delta_{v^L}(\{1,2,4,5,6\}) &= 1\\ \Delta_{v^L}(\{1,2,3,4,5,6\}) &= -1\end{aligned} \tag{3}$$

Then we can compute the Myerson value as the Shapley value of the new coordination game,

$$\begin{aligned}\mu_i(v, L) = f_i^S &= \sum_{T \in N(i)} \Delta_{v^L}(T) / |T|\\ \implies \mu(v, L) = f^S &= \begin{pmatrix}17/60 & 1/30 & 1/12 & 1/30 & 17/60 & 17/60\end{pmatrix}\end{aligned} \tag{4}$$

(c)

The new supply chain is represented by the graph,



We can hence define

$$v^{L'}(S) = \begin{cases} 0 & \text{if } S = \{1,2,4,5,6\} \\ v^L & \text{otherwise} \end{cases} \tag{5}$$

By repeating the procedure of (4), we obtain,

$$\mu(v, L') = \begin{pmatrix} 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \end{pmatrix}. \tag{6}$$

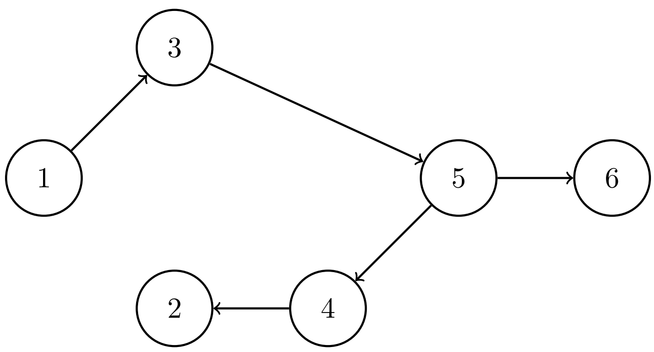
This result is expected since in the new supply chain 2 and 4 become null-players.

In order to determine fairness we can check that,

$$\begin{gathered} \mu_1(v, L) - \mu_1(v, L') = \mu_2(v, L) - \mu_2(v, L') \\ 17/60 - 1/4 = 1/30 - 0 \implies \text{ the fairness axiom is respected.} \end{gathered} \tag{7}$$

(d)

In order to compute the hierarchical outcome of node 1 we need to set 1 as root. The resulting directed graph (N, L^1) can be represented as,



Using the definition of followers and subordinates, we can compute the hierarchical outcomes as,

$$h^1(v, L') = \begin{pmatrix} v^{L'}(\{1,3,5,6,4,2\}) - v^{L'}(\{3,5,6,4,2\}) \\ v^{L'}(\{2\}) \\ v^{L'}(\{3,5,6,4,2\}) - v^{L'}(\{5,6,4,2\}) \\ v^{L'}(\{4,2\}) - v(\{2\}) \\ v^{L'}(\{5,6,4,2\}) - v^{L'}(\{4,2\}) - v^{L'}(\{6\}) \\ v^{L'}(6) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

In a similar manner,

$$h^4(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad h^6(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

(e)

The link game $(L', r^{L'})$ is given by the characteristic function,

$$\begin{aligned} \bar{E} &= \left\{ \begin{array}{l} \{(1,3),(3,5),(5,6)\} \\ \{(1,3),(4,2),(3,5),(5,6)\} \\ \{(1,3),(3,5),(5,4),(5,6)\} \\ \{(1,3),(4,2),(3,5),(5,4),(5,6)\} \end{array} \right\} \\ r^{L'}(E) &= \begin{cases} 1 & \text{if } E \in \bar{E} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{10}$$

The Shapley value of the link game is,

$$f^S(L', r^{L'}) = \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix} \tag{11}$$

The associated position value is.

$$\pi(v, L') = \begin{pmatrix} 1/6 & 0 & 1/3 & 0 & 1/3 & 1/6 \end{pmatrix} \tag{12}$$

Exercise 2

A hierarchical outcome h^i is in the $Core(N, v_L)$ if and only if,

$$\sum_{j \in S} h^i_j \geq v(S) \quad \text{and} \quad \sum_{j \in N} h^i_j = v(N).$$

First let the subordinates of j be

$$S^i_j \coloneqq \left\{ h : \; \exists (i_1, \ldots, i_t) \; s.t. i_1 = j \wedge i_t = h \wedge (i_k, i_{k+1}) \in L^i \; \forall \; k \in \{1, \ldots, t\} \right\} \tag{13}$$

Entry j in the hierarchical outcome can be written as,

$$h_j^i = v(\hat{F}_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i). \tag{14}$$

By superadditivity,

$$\hat{F}_j^i = \{j\} \cup S_j^i \implies v(\hat{F}_j^i) \geq v(j) + v(S_j^i). \tag{15}$$

Furthermore,

$$\hat{F}_h^i \subseteq S_j^i \; \forall h \in F_j^i \implies \bigcup_{h \in F_j^i} \hat{F}_h^i \subseteq S_j^i \implies \sum_{h \in F_j^i} v(\hat{F}_h^i) \leq v(S_j^i). \tag{16}$$

Combining these results, we can use (14), and show that,

$$h_j^i = v(\hat{F}_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i) \geq v(j) + v(S_j^i) - \sum_{h \in F_j^i} v(\hat{F}_h^i) \geq v(j). \tag{17}$$

Given that $h_j^i \geq v(j)$, by superadditivity we can show that,

$$\sum_{j \in S} h_j^i \geq \sum_{j \in S} v(j). \tag{18}$$

Exercise 3

Using the degree centrality as the Harsanyi degree solution allows for the dividends to be distributed proportionally to an agent weights within the graph. By computing the solution for the game (N, L) from Exercise 1.a, we obtain,

$$\phi^d(v, L) = \frac{1}{24} \cdot \begin{pmatrix} 3 & 2 & 4 & 2 & 8 & 5 \end{pmatrix}. \tag{19}$$

Such a result clearly highlights the usefulness of the measure since player 5 is heavily rewarded for being necessary to connect 1 and 6. Another good property is that node 3 is rewarded as much as 2 and 4 since it serves the same purpose within the graph. This property was not obtained with the Myerson value, in equation (4).

Exercise 4

(a)

Let $A, B \in \mathcal{F}$ and $A \cap B \neq \emptyset$. We need to show that $A \cup B \in \mathcal{F}$. Note that $B \in \mathcal{F}$ if and only if $B \subseteq M$ or $B \subseteq H$ or $B = (B \cap H) \cup M$.

The simple case where both sets are contained in either H or M is straight forward. Let $A, B \subseteq E \in \{H, M\}$, then

$$A \cup B \subseteq E \in \mathcal{F}. \tag{20}$$

Consider now the case in which, without loss of generality, $B = (B \cap H) \cup M$. Using set algebra we can rewrite,

$$\begin{aligned} A \cup B &= A \cup ((B \cap H) \cup M) \text{ by commutativity} \\ &= A \cup (B \cap H) \cup M \text{ by distributivity} \\ &= (A \cup B) \cap (A \cup H) \cup M \end{aligned} \tag{21}$$

Now if $A \subseteq M$, then $A \cup B = B \in \mathcal{F}$. If $A \subseteq H$, we can rewrite (21) as,

$$A \cup B = ((A \cup B) \cap H) \cup M \in \mathcal{F}. \tag{22}$$

Lastly, if $A = (A \cap H) \cup M$, then,

$$\begin{aligned} A \cup B &= ((A \cap H) \cup \underbrace{M \cup B}_{B}) \cap \underbrace{((A \cap H) \cup M \cup H))}_{N} \cup M \\ &= ((A \cap H) \cup B) \cup M \\ &= ((A \cup B) \cap H) \cup M \in \mathcal{F} \end{aligned} \tag{23}$$

Hence, we have shown that,

$$A, B \in \mathcal{F} \wedge A \cap B \neq \emptyset \implies A \cup B \in \mathcal{F} \tag{24}$$

(b)

By van den Brink (2012), a set of feasible coalition $\mathcal{F} \subseteq 2^N$ is the set of of connected coalitions in some (undirected) communication graph if and only if it contains the empty set, it satisfies union stability, 2-accessibility, and normality. In our case the set contains the empty set and satisfies union stability (as proven in **(a)**).

In our particular case $\mathcal{F}$ fails to satisfy 2-accessibility. Take $E = \{1, 3, 4, 5\}$. Clearly $|E| \geq 2$. But,

$$\begin{aligned} E \setminus \{i\} \in \mathcal{F} &\iff i = 1 \\ &\implies \nexists\, i, j \in E \land i \neq j :\ E \setminus \{i\}, E \setminus \{j\} \in \mathcal{F}.w \end{aligned} \tag{25}$$

References

van den Brink, R. (2012). On hierarchies and communication. *Social Choice and Welfare*, *39*(4), 721–735. doi:10.1007/s00355-011-0557-y