Homework I - Advanced Game Theory

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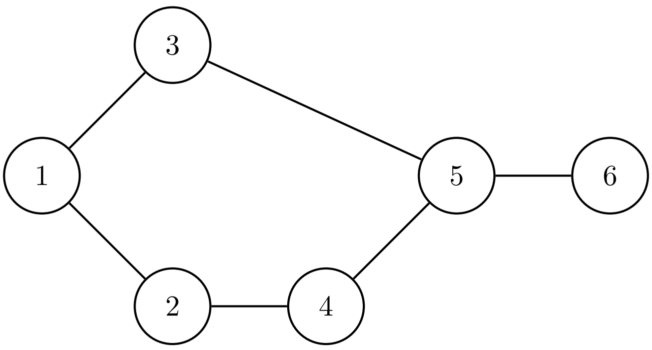
Exercise 1

The graph representation of (N, L) is,



(a)

The function v^L of the Myerson restricted game is,

$$v^L(S) = \sum_{T \in C_L(S)} v(T). \tag{1}$$

It is non-zero only for the component of the graphs that include both 1 and 6, namely,

$$v^L(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,5,6\}, \{1,2,3,5,6\}, \{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b)

We can define the communication game (N, v^L), using (2). Then we can compute the Harsanyi dividends based on this game. The non-zero dividends are,

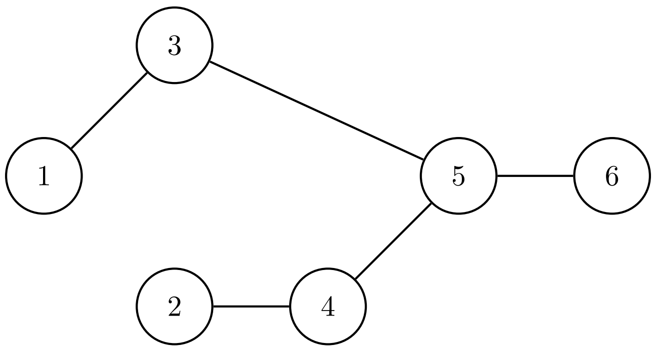
$$\begin{aligned} \Delta_{v^L}(\{1,3,5,6\}) &= 1 \\ \Delta_{v^L}(\{1,2,4,5,6\}) &= 1 \\ \Delta_{v^L}(\{1,2,3,4,5,6\}) &= -1 \end{aligned} \tag{3}$$

Then we can compute the Myerson value as the Shapley value of the new coordination game,

$$\begin{aligned} \mu_i(v, L) = f_i^S &= \sum_{T \in N(i)} \Delta_{v^L}(T) / |T| \\ \implies \mu(v, L) = f^S &= \begin{pmatrix} 17/60 & 1/30 & 1/12 & 1/30 & 17/60 & 17/60 \end{pmatrix} \end{aligned} \tag{4}$$

(c)

The new supply chain is represented by the graph,



We can hence define

$$v^{L'}(S) = \begin{cases} 0 & \text{if } S = \{1,2,4,5,6\} \\ v^L & \text{otherwise} \end{cases} \tag{5}$$

By repeating the procedure of (4), we obtain,

$$\mu(v, L') = \begin{pmatrix} 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \end{pmatrix}. \tag{6}$$

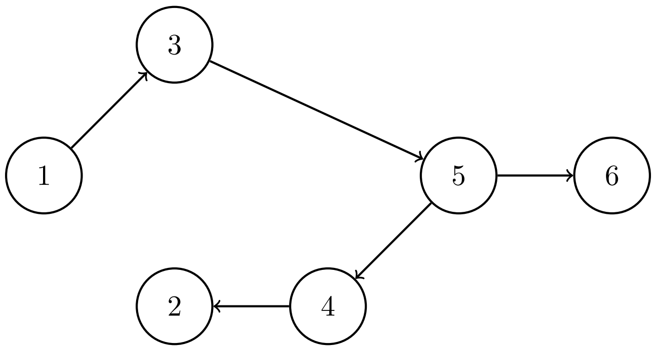
This result is expected since in the new supply chain 2 and 4 become null-players.

In order to determine fairness we can check that,

$$\begin{aligned} \mu_1(v, L) - \mu_1(v, L') &= \mu_2(v, L) - \mu_2(v, L') \\ 17/60 - 1/4 = 1/30 - 0 &\implies \text{ the fairness axiom is respected.} \end{aligned} \tag{7}$$

(d)

In order to compute the hierarchical outcome of node 1 we need to set 1 as root. The resulting directed graph (N, L^1) can be represented as,



Using the definition of followers and subordinates, we can compute the hierarchical outcomes as,

$$h^1(v, L') = \begin{pmatrix} v^{L'}(\{1,3,5,6,4,2\}) - v^{L'}(\{3,5,6,4,2\}) \\ v^{L'}(\{2\}) \\ v^{L'}(\{3,5,6,4,2\}) - v^{L'}(\{5,6,4,2\}) \\ v^{L'}(\{4,2\}) - v(\{2\}) \\ v^{L'}(\{5,6,4,2\}) - v^{L'}(\{4,2\}) - v^{L'}(\{6\}) \\ v^{L'}(6) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

In a similar manner,

$$h^4(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad h^6(v, L') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$