Leading Order Gravitational Wave Frequencies from Short-Term Post-Newtonian Orbits via Fourth-Order Symplectic Integration

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Road Map

Introduction

1. Why should we care? $\leftarrow$ **We are here**
2. Where does this investigation fit in with previous?

Methods

1. Physical Theory
    1. Inspiral & Gravitational Radiation
    2. Weak-Field Approximations
2. Simulation
    1. Integration Method
    2. Error Analysis

Discussion

1. Results
2. Conclusion & Future Work

Why should we care?

1. Math is hard (right)[1].
2. Start the state and let it go, is easy...er.
3. Most systems need numerical approximation anyway.
4. Sun-Mercury system is an excellent source of weak-field data.

$$g_{ab} = \eta_{ab} + h_{ab} \tag{1}$$
$$\Box \bar{h}_{ab} = -16\pi T_{ab} \tag{2}$$
$$h_{ij}^{\mathrm{TT}} \simeq \frac{2}{d} \ddot{\mathcal{I}}_{ij}^{\mathrm{TT}}(t-d) \tag{3}$$
$$\mathcal{I}_{ij}^{\mathrm{TT}} = \left(P_i{}^j P_i{}^j - \frac{1}{2} P_{ij} P^{kl} \right) \mathcal{I}_{kl} \tag{4}$$
$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3} \eta_{ij} I_i{}^i \tag{5}$$
$$\frac{dE}{dt} = -\frac{1}{5} \left\langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \right\rangle \tag{6}$$

and more!

[1] T. W. Baumgarte and S. L. Shapiro (2021). *Numerical Relativity: Starting from Scratch*. Cambridge University Press. DOI: 10.1017/9781108933445

(1) Metric perturbation
(2) Linear Field Equation
(3) Perturbation Magnitude
(4) Transverse-Traceless RQMT
(5) Reduced Quadrupole Moment Tensor (RQMT)
(6) Rate of Energy Loss $\propto$ Time Average

Where does this investigation fit in with previous?

1. Numerical validation of theoretical prediction[2]
2. Provide support for observation like LIGO[3]
3. Simulation of other planets' perihelion shift[4]
4. Simulation of Mercury's perihelion shift with Euler integration[5]
5. Simulation of Mercury's perihelion shift with Verlet integration[6]
6. Simulation of Mercury's gravitational wave emission ($\leftarrow$ That's us!)

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[2] A. Einstein (1916). "Die Grundlage der allgemeinen Relativitätstheorie". In: *Annalen der Physik* 354.7, pp. 769–822. DOI: https://doi.org/10.1002/andp.19163540702. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.19163540702

[3] B. P. Abbott et al. (Feb. 2016). "Observation of Gravitational Waves from a Binary Black Hole Merger". In: *Phys. Rev. Lett.* 116 (6), p. 061102. DOI: 10.1103/PhysRevLett.116.061102

[4] A. Biswas and K. Mani (2008). In: *Open Physics* 6.3, pp. 754–758. DOI: doi:10.2478/s11534-008-0081-6

[5] C. Körber, I. Hammer, J.-L. Wynen, J. Heuer, C. Müller, and C. Hanhart (June 2018). "A primer to numerical simulations: the perihelion motion of Mercury". In: *Physics Education* 53.5, p. 055007. DOI: 10.1088/1361-6552/aac487

[6] N. Chapman (June 2019). "Effects of Eccentricity on the Precession of Orbital Periapsides due to General Relativity in the Weak Field Limit". In: eprint: https://github.com/NonDairyNeutrino/ug_computation/blob/1e8f14936b1d7f5ddd2cd29961eee56ba7820a30/computational_physics/Weak_Field_GR_Eccentricity_Effects/Weak_Field_GR_Eccentricity_Effects.pdf

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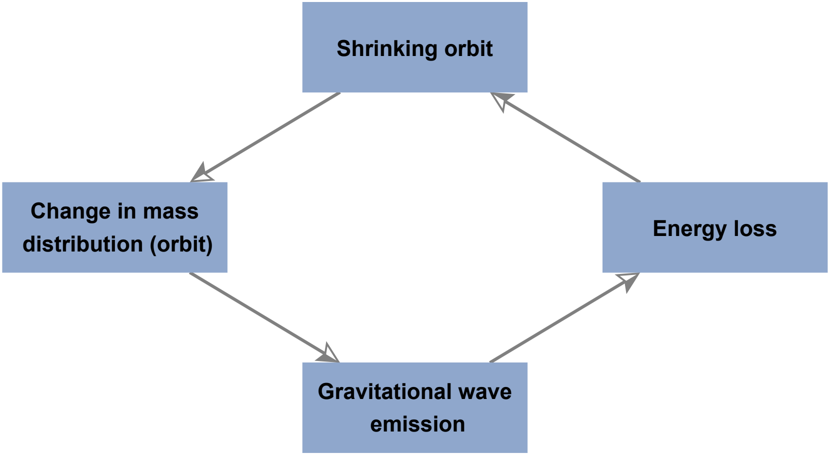
Physical Theory - Inspiral & Gravitational Radiation



Physical Theory - Weak-Field Approximations

1. Small mass & low velocity $\Longrightarrow$ Small amplitude gravitational waves
2. Small amplitude gravitational waves $\Longrightarrow$ Post-Newtonian Expansion[7]

Newtonian

$$\ddot{\vec{r}} = -\frac{c^2}{2}\frac{r_S}{r^2}\frac{\vec{r}}{r}$$

Relativistic Correction

$$\ddot{\vec{r}} = -\frac{c^2}{2}\frac{r_S}{r^2}\underbrace{\left(1 + 3\frac{r_L^2}{r^2}\right)}_{\text{Relativity}}\frac{\vec{r}}{r}$$

3. Adiabatic approximation - breaks down when orbital radius no longer shrinks on timescales much longer than the orbital period
4. Leading order gravitational wave frequencies are double the orbital frequency[8]

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[7] C. Körber, I. Hammer, J.-L. Wynen, J. Heuer, C. Müller, and C. Hanhart (June 2018). "A primer to numerical simulations: the perihelion motion of Mercury". In: *Physics Education* 53.5, p. 055007. DOI: 10.1088/1361-6552/aac487

[8] T. W. Baumgarte and S. L. Shapiro (2021). *Numerical Relativity: Starting from Scratch*. Cambridge University Press. DOI: 10.1017/9781108933445

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Simulation - Integration Method

1. Symplectic for energy conservation
2. 4^{th} order ($4\mathcal{O}$) for big time step
3. Similar to leapfrog integration
4. Could also use
   1. $4\mathcal{O}$ symplectic Runge-Kutta[9]
   2. $2\mathcal{O}$ symplectic (Verlet)
   3. $1\mathcal{O}$ symplectic (Euler-Cromer)
5. Literature only for conservative forces; naively include v_i^{n-1} in acceleration

$4\mathcal{O}$ symplectic Yoshida integrator[10]

$$
\begin{aligned}
x_i^0 &= x_i & x_i^n &= x_i^{n-1} + c_n v_i^{n-1} \Delta t & x_{i+1} &= x_i^4 \\
v_i^0 &= v_i & v_i^n &= v_i^{n-1} + d_n \underbrace{a\left(x_i^n, v_i^{n-1}\right)}_{\text{“leapfrog”}} \Delta t & v_{i+1} &= v_i^4
\end{aligned}
$$

[9] S. Blanes and P. Moan (2002). “Practical symplectic partitioned Runge–Kutta and Runge–Kutta–Nyström methods”. In: *Journal of Computational and Applied Mathematics* 142.2, pp. 313–330. ISSN: 0377-0427. DOI: https://doi.org/10.1016/S0377-0427(01)00492-7

[10] H. Yoshida (1990). “Construction of higher order symplectic integrators”. In: *Physics Letters A* 150.5, pp. 262–268. ISSN: 0375-9601. DOI: https://doi.org/10.1016/0375-9601(90)90092-3

Simulation - Error Analysis

1. $4\mathcal{O}$ Yoshida similar to $2\mathcal{O}$ Wisdom-Holman (WH)[11]
2. Previous analysis of similar system[12]
3. Error remains small for many orbits
4. Error grows linearly after critical time

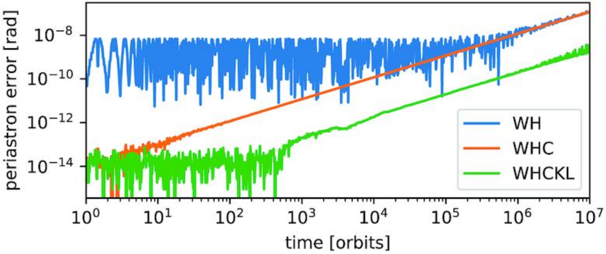

Figure 5. Periastron error in a long-term simulation with one planet and general relativistic corrections using the WH, WHC, and WHCKL integrators.

Mon Not R Astron Soc, Volume 490, Issue 4, December 2019, Pages 5122–5133, https://doi.org/10.1093/mnras/stz2942

[11] J. Wisdom and M. Holman (Oct. 1991). "Symplectic maps for the N-body problem.". In: *aj* 102, pp. 1528–1538. DOI: 10.1086/115978

[12] H. Rein, G. Brown, and D. Tamayo (Oct. 2019). "On the accuracy of symplectic integrators for secularly evolving planetary systems". In: *Monthly Notices of the Royal Astronomical Society* 490.4, pp. 5122–5133. ISSN: 0035-8711. DOI: 10.1093/mnras/stz2942. eprint: https://academic.oup.com/mnras/article-pdf/490/4/5122/30725851/stz2942.pdf

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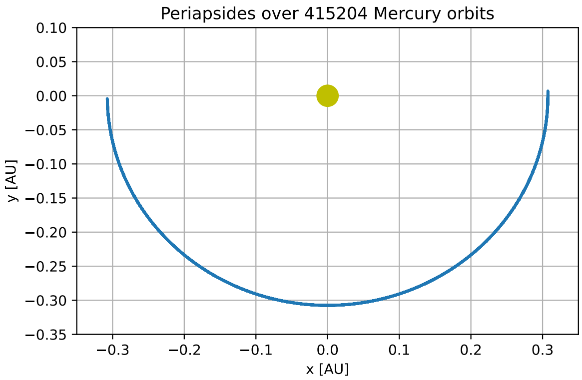
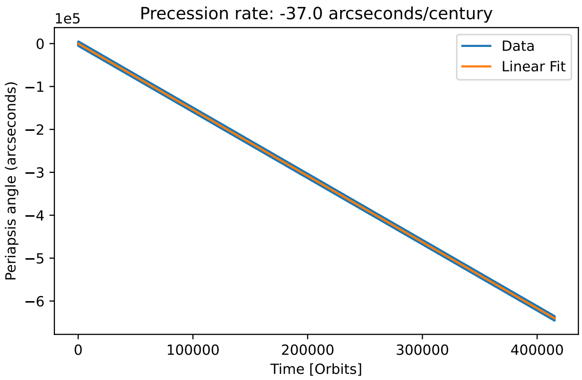
Results - Perihelion Trajectory

- Sun-Mercury system
- 10^5 Earth-years
- Significant perihelion precession



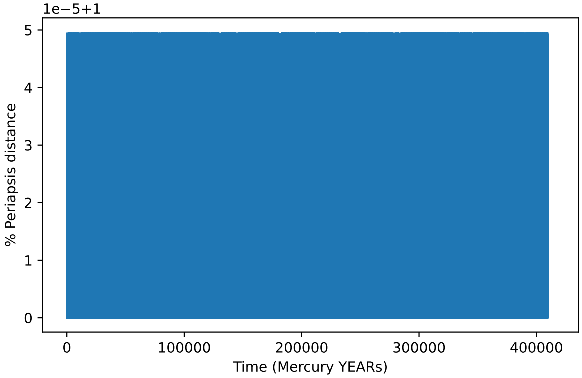
Results - Perihelion Precession

- Sun-Mercury system
- 10^5 Earth-years
- Significant perihelion precession
- Precession rate in-line with theory and observation



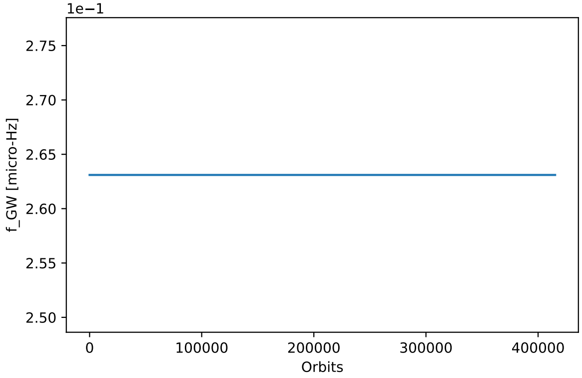
Results - Inspiral

- Sun-Mercury system
- 10^5 Earth-years
- Insignificant change on this time scale
- "The block"



Results - Gravitational Wave Frequency

- Sun-Mercury system
- 10^5 Earth-years
- Insignificant change on this time scale
- "The line"



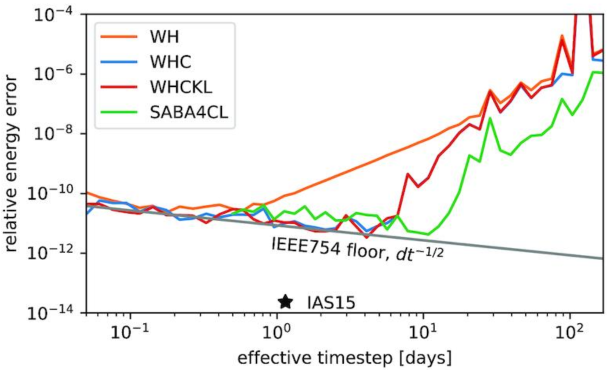
Conclusion

- Simulate because theory is hard at best
- Lot's of community interest on weak-field simulation
- Post-Newtonian approximations are valid for weak-field systems
- 4-th order symplectic integration seems a suitable approach
- The scale on which the dynamics of the solar system change is very long

Future Work

- Use a different integrator like WHCKL or SABA for even better energy conservation
- Use Parameterized Post-Newtonian or BSSN formalism[13]
- Use Einstein–Infeld–Hoffmann equations[14]
- Compare data to VSOP[15]
- Evolve until adiabatic limit

Figure 1. Relative energy error in 20 Myr integrations of the Solar System using different integrators.



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Thank you & Code source

Thank you for your attention.

The code for this project is available on:

- GitHub (QR code): `github.com/nondairyneutrino/mercury_chirp`
- My website: `nathanwchapman.com`