Lagrange Equations of the Second Kind

The basis of this multibody dynamics formulation is the Lagrange Equations of the second kind:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_{iq}}\right)-\frac{\partial T}{\partial q_{iq}}-Q_{iq}-\sum_{i\bar{G}=1}^{n\bar{G}}\frac{\partial \bar{G}_{i\bar{G}}}{\partial \dot{q}_{iq}}\bar{\lambda}_{i\bar{G}}-\sum_{iG=1}^{n_G}\frac{\partial G_{iG}}{\partial q_{iq}}\lambda_{iG}=0 \qquad iq=1,\cdots,nq$$

$$p_{i_q}=\frac{\partial T}{\partial \dot{q}_{i_q}}$$

$$\frac{dp_{i_q}}{dt}-\frac{\partial T}{\partial q_{i_q}}-Q_{i_q}-\sum_{i_G=1}^{n_G}\frac{\partial G_{i_G}}{\partial q_{i_q}}\lambda_{i_G}-\sum_{i_{\bar{G}}=1}^{n_{\bar{G}}}\bar{G}_{i_G}\bar{\lambda}_{i_G}=0 \qquad i_q=1,\cdots,n_q$$

In vector form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial \mathbf{q}}-\mathbf{Q}-\frac{\partial \mathbf{G}^T}{\partial \mathbf{q}}\boldsymbol{\lambda}-\bar{\mathbf{G}}^T\bar{\boldsymbol{\lambda}}=\mathbf{0}$$

$$\mathbf{p}=\frac{\partial T}{\partial \dot{\mathbf{q}}}$$

$$\dot{\mathbf{p}}-\frac{\partial T}{\partial \mathbf{q}}-\mathbf{Q}-\frac{\partial \mathbf{G}^T}{\partial \mathbf{q}}\boldsymbol{\lambda}-\bar{\mathbf{G}}^T\bar{\boldsymbol{\lambda}}=\mathbf{0}$$

The terms in the equations will be defined and derived below.

Inertial Frame

$OXYZ$ form the Cartesian Frame in inertial frame.

$\vec{\mathbf{I}}\,\vec{\mathbf{J}}\,\vec{\mathbf{K}}$ are base unit vectors for. $OXYZ$

Assembly Frame

$o_a x_a y_a z_a$ form the Cartesian Frame in assembly frame.

$\vec{\mathbf{i}}_a\,\vec{\mathbf{j}}_a\,\vec{\mathbf{k}}_a$ are base unit vectors for. $o_a x_a y_a z_a$

Position from Inertial Frame, $\mathbf{r}_{OaO}(t)$

$\mathbf{r}_{OaO}(t)$ = displacement from Inertial Frame to assembly frame.

Assembly motion can be prescribed so that computed variables are kept small.

For example a satellite in orbit can have the assembly frame prescribed as a huge ellipse and computed values are just the deviations.

Orientation from Inertial Frame, $[\mathbf{A}(t)]_{Oa}$

$$\begin{Bmatrix}\vec{\mathbf{I}}\\ \vec{\mathbf{J}}\\ \vec{\mathbf{K}}\end{Bmatrix}=[\mathbf{A}(t)]_{Oa}\begin{Bmatrix}\vec{\mathbf{i}}_a\\ \vec{\mathbf{j}}_a\\ \vec{\mathbf{k}}_a\end{Bmatrix}$$

$\mathbf{v}_{OaO}(t)$ = velocity from Inertial Frame to assembly frame in Inertial Frame components.

$[\dot{\mathbf{A}}(t)]_{Oa}$ = time derivative of $[\mathbf{A}(t)]_{Oa}$

$\boldsymbol{\omega}_{OaO}(t)$ = angular velocity from Inertial Frame to assembly frame in Inertial Frame components.

$\boldsymbol{\omega}_{Oaa}(t)$ = angular velocity from Inertial Frame to assembly frame in assembly frame components.

$\mathbf{a}_{OaO}(t)$ = acceleration from Inertial Frame to assembly frame in Inertial Frame components.

$\left[\ddot{\mathbf{A}}(t)\right]_{Oa}$ = second time derivative of $\left[\mathbf{A}(t)\right]_{Oa}$

$\boldsymbol{\alpha}_{OaO}(t)$ = angular acceleration from Inertial Frame to assembly frame in Inertial Frame components.

$\boldsymbol{\alpha}_{Oaa}(t)$ = angular acceleration from Inertial Frame to assembly frame in assembly frame components.

Principal Part Frame

$o_p x_p y_p z_p$ is a Cartesian Frame fixed to a part. It is at the center of mass (cm) and its axes are along the principal axis of the rigid body . $J_x < J_y < J_z$

$\vec{\mathbf{i}}_p \vec{\mathbf{j}}_p \vec{\mathbf{k}}_p$ are base unit vectors for. $o_p x_p y_p z_p$

Coordinate System, $\mathbf{q}$

$\mathbf{r}_{apa}(\mathbf{q}_X) = \mathbf{q}_X$ = displacement from assembly frame to part frame.

$$\begin{Bmatrix} \vec{\mathbf{i}}_a \\ \vec{\mathbf{j}}_a \\ \vec{\mathbf{k}}_a \end{Bmatrix} = \left[\mathbf{A}(\mathbf{q}_E)\right]_{ap} \begin{Bmatrix} \vec{\mathbf{i}}_p \\ \vec{\mathbf{j}}_p \\ \vec{\mathbf{k}}_p \end{Bmatrix}$$

Cartesian Coordinates, $\mathbf{q}_X$

$$\mathbf{q}_X = \begin{Bmatrix} q_{X1} \\ q_{X2} \\ q_{X3} \end{Bmatrix} = \begin{Bmatrix} x_{apa} \\ y_{apa} \\ z_{apa} \end{Bmatrix} = \mathbf{r}_{apa}$$

Cartesian Coordinate Velocity, $\mathbf{v}_X$

$$\mathbf{v}_X = \begin{Bmatrix} v_{X1} \\ v_{X2} \\ v_{X3} \end{Bmatrix} = \dot{\mathbf{q}}_X = \begin{Bmatrix} v_{apa1} \\ v_{apa2} \\ v_{apa3} \end{Bmatrix} = \mathbf{v}_{apa}$$

Cartesian Coordinate Acceleration, $\mathbf{a}_X$

$\mathbf{a}_X = \dot{\mathbf{v}}_X = \mathbf{a}_{apa}$

Orientation Coordinates, $\mathbf{q}_E$

For any physical vector

$\vec{\mathbf{V}} = V_{xa}\vec{\mathbf{i}}_a + V_{ya}\vec{\mathbf{j}}_a + V_{za}\vec{\mathbf{k}}_a = V_{xp}\vec{\mathbf{i}}_p + V_{yp}\vec{\mathbf{j}}_p + V_{zp}\vec{\mathbf{k}}_p$

$$\begin{Bmatrix} \vec{\mathbf{i}}_a \\ \vec{\mathbf{j}}_a \\ \vec{\mathbf{k}}_a \end{Bmatrix} = \left[\mathbf{A}(\mathbf{q}_E)\right]_{ap} \begin{Bmatrix} \vec{\mathbf{i}}_p \\ \vec{\mathbf{j}}_p \\ \vec{\mathbf{k}}_p \end{Bmatrix}$$

$$\begin{Bmatrix} V_{xa} \\ V_{ya} \\ V_{za} \end{Bmatrix} = \left[\mathbf{A}(\mathbf{q}_E)\right]_{ap} \begin{Bmatrix} V_{xp} \\ V_{yp} \\ V_{zp} \end{Bmatrix}$$

Euler Parameters, $\mathbf{q}_E$

$E_{1,}E_{2,}E_{3,}E_4$ = Euler parameters of the orientation of the part principal frame wrt assembly frame.

$\mathbf{n}$ = unit vector along the axis of rotation of $o_p x_p y_p z_p$ from $o_a x_a y_a z_a$.

θ = right hand rule rotation about $\mathbf{n}$.

$$\mathbf{q}_E=\begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{Bmatrix}=\begin{Bmatrix} \vec{\mathbf{n}}\cdot\vec{\mathbf{i}}_a \sin\frac{\theta}{2} \\ \vec{\mathbf{n}}\cdot\vec{\mathbf{j}}_a \sin\frac{\theta}{2} \\ \vec{\mathbf{n}}\cdot\vec{\mathbf{k}}_a \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{Bmatrix}$$

Euler Parameter Constraint, G_E

$$G_E=E_1^2+E_2^2+E_3^2+E_4^2-1=0$$

Euler Parameter Matrices, $[\mathbf{B}],[\mathbf{C}]$

$$[\mathbf{B}]=\begin{bmatrix} E_4 & -E_3 & E_2 & -E_1 \\ E_3 & E_4 & -E_1 & -E_2 \\ -E_2 & E_1 & E_4 & -E_3 \end{bmatrix}$$

$$[\mathbf{C}]=\begin{bmatrix} E_4 & E_3 & -E_2 & -E_1 \\ -E_3 & E_4 & E_1 & -E_2 \\ E_2 & -E_1 & E_4 & -E_3 \end{bmatrix}$$

$$[\mathbf{B}][\mathbf{B}^T]=[\mathbf{I}]$$

$$[\mathbf{C}][\mathbf{C}^T]=[\mathbf{I}]$$

Angular Velocity Coordinate $\mathbf{v}_\omega$

$$\mathbf{v}_\omega=\begin{Bmatrix} v_{\omega 1} \\ v_{\omega 2} \\ v_{\omega 3} \end{Bmatrix}=\boldsymbol{\omega}_{apa}=\begin{Bmatrix} \omega_{apa1} \\ \omega_{apa2} \\ \omega_{apa3} \end{Bmatrix}=2[\mathbf{B}]_{ap}\dot{\mathbf{q}}_E$$

$$\dot{\mathbf{q}}_E=\frac{1}{2}[\mathbf{B}]_{ap}^T\boldsymbol{\omega}_{apa}=\frac{1}{2}[\mathbf{B}]_{ap}^T\mathbf{v}_\omega$$

Angular Acceleration Coordinate $\mathbf{a}_\alpha$

$$\mathbf{a}_\alpha=\begin{Bmatrix} a_{\alpha 1} \\ a_{\alpha 2} \\ a_{\alpha 3} \end{Bmatrix}=\boldsymbol{\alpha}_{apa}=\begin{Bmatrix} \alpha_{apa1} \\ \alpha_{apa2} \\ \alpha_{apa3} \end{Bmatrix}=\dot{\boldsymbol{\omega}}_{apa}=\dot{\mathbf{v}}_\omega=2\left([\dot{\mathbf{B}}]_{ap}\dot{\mathbf{q}}_E+[\mathbf{B}]_{ap}\ddot{\mathbf{q}}_E\right)$$

State Vectors and Relations, $\mathbf{q}, \mathbf{v}, \mathbf{a}$

$$\mathbf{q}=\begin{Bmatrix} q_{1X} \\ q_{1E} \\ \vdots \\ q_{nX} \\ q_{nE} \end{Bmatrix}=\begin{Bmatrix} r_{ap1a} \\ q_{1E} \\ \vdots \\ r_{apna} \\ q_{nE} \end{Bmatrix} \qquad \mathbf{v}=\begin{Bmatrix} v_{1X} \\ v_{1\omega} \\ \vdots \\ v_{nX} \\ v_{n\omega} \end{Bmatrix}=\begin{Bmatrix} v_{ap1a} \\ \omega_{ap1a} \\ \vdots \\ v_{apna} \\ \omega_{apna} \end{Bmatrix} \qquad \mathbf{a}=\begin{Bmatrix} a_{1X} \\ a_{1\alpha} \\ \vdots \\ a_{nX} \\ a_{n\alpha} \end{Bmatrix}=\begin{Bmatrix} a_{ap1a} \\ \alpha_{ap1a} \\ \vdots \\ a_{apna} \\ \alpha_{apna} \end{Bmatrix}$$

$$\dot{\mathbf{q}}=\begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}[\mathbf{B}]_{ap1}^{T} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \frac{1}{2}[\mathbf{B}]_{apn}^{T} \end{bmatrix}\mathbf{v}$$

$$\mathbf{v}=\begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2[\mathbf{B}]_{ap1} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 2[\mathbf{B}]_{apn} \end{bmatrix}\dot{\mathbf{q}}$$

$$\dot{\mathbf{v}}=\mathbf{a}$$

Define Kinematic Matrix, $[\mathbf{V}]$

$$[\mathbf{V}]=\begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}[\mathbf{B}]_{ap1}^{T} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \frac{1}{2}[\mathbf{B}]_{apn}^{T} \end{bmatrix}$$

Position from Assembly Frame, $\mathbf{r}_{apa}(\mathbf{q}_X)$

$$\mathbf{r}_{apa}=\mathbf{r}_{apa}(\mathbf{q}_X)$$

Velocity from Assembly Frame, $\mathbf{v}_{apa}(\mathbf{v}_X)$

$$\mathbf{v}_{apa}=\frac{d\,\mathbf{r}_{apa}}{d\,t}=\frac{\partial\,\mathbf{r}_{apa}(\mathbf{q}_X)}{\partial\,\mathbf{q}_X}\dot{\mathbf{q}}_X=\dot{\mathbf{q}}_X=\mathbf{v}_X$$

Acceleration from Assembly Frame, $\mathbf{a}_{apa}(\mathbf{a}_X)$

$$\mathbf{a}_{apa}=\frac{d\,\mathbf{v}_{apa}}{d\,t}=\frac{\partial\,\mathbf{v}_{apa}(\mathbf{v}_X)}{\partial\,\mathbf{v}_X}\dot{\mathbf{v}}_X=\dot{\mathbf{v}}_X=\mathbf{a}_X$$

Orientation from Assembly Frame, $\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}$

$$\begin{Bmatrix}\vec{\mathbf{i}}_a\\ \vec{\mathbf{j}}_a\\ \vec{\mathbf{k}}_a\end{Bmatrix}=\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}\begin{Bmatrix}\vec{\mathbf{i}}_p\\ \vec{\mathbf{j}}_p\\ \vec{\mathbf{k}}_p\end{Bmatrix}$$

$$[\mathbf{A}]_{ap}=[\mathbf{B}]_{ap}[\mathbf{C}]_{ap}^T=\begin{bmatrix}E_4 & -E_3 & E_2 & -E_1\\ E_3 & E_4 & -E_1 & -E_2\\ -E_2 & E_1 & E_4 & -E_3\end{bmatrix}_{ap}\begin{bmatrix}E_4 & -E_3 & E_2\\ E_3 & E_4 & -E_1\\ -E_2 & E_1 & E_4\\ -E_1 & -E_2 & -E_3\end{bmatrix}_{ap}$$

$$[\mathbf{A}]_{ap}=\begin{bmatrix}E_1^2-E_2^2-E_3^2+E_4^2 & 2(E_1E_2-E_3E_4) & 2(E_1E_3+E_2E_4)\\ 2(E_1E_2+E_3E_4) & -E_1^2+E_2^2-E_3^2+E_4^2 & 2(E_2E_3-E_1E_4)\\ 2(E_1E_3-E_2E_4) & 2(E_2E_3+E_1E_4) & -E_1^2-E_2^2+E_3^2+E_4^2\end{bmatrix}_{ap}$$

Orientation Velocity from Assembly Frame, $\left[\dot{\mathbf{A}}(\mathbf{q}_E,\mathbf{v}_\omega)\right]_{ap}$

$$[\dot{\mathbf{A}}]_{ap}=\frac{d[\mathbf{A}]_{ap}}{dt}=[\dot{\mathbf{B}}]_{ap}[\mathbf{C}]_{ap}^T+[\mathbf{B}]_{ap}[\dot{\mathbf{C}}]_{ap}^T=2[\mathbf{B}]_{ap}[\dot{\mathbf{C}}]_{ap}^T=2[\mathbf{B}]_{ap}\frac{\partial[\dot{\mathbf{C}}]_{ap}^T}{\partial\dot{\mathbf{q}}_E}\dot{\mathbf{q}}_E=2[\mathbf{B}]_{ap}\frac{\partial[\mathbf{C}]_{ap}^T}{\partial\mathbf{q}_E}\dot{\mathbf{q}}_E$$

$$=2[\mathbf{B}]_{ap}\frac{\partial[\mathbf{C}]_{ap}^T}{\partial\mathbf{q}_E}\frac{1}{2}[\mathbf{B}]_{ap}^T\mathbf{v}_\omega=[\mathbf{B}]_{ap}\frac{\partial[\mathbf{C}]_{ap}^T}{\partial\mathbf{q}_E}[\mathbf{B}]_{ap}^T\mathbf{v}_\omega$$

$$[\dot{\mathbf{A}}]_{ap}=[\mathbf{B}]_{ap}\frac{\partial[\mathbf{C}]_{ap}^T}{\partial\mathbf{q}_E}[\mathbf{B}]_{ap}^T\mathbf{v}_\omega$$

Orientation Velocity and Angular Velocity Relationship

From Haug pp 330. Define skew-symmetric matrix

$$[\tilde{\mathbf{v}}]_\omega=[\tilde{\boldsymbol{\omega}}]_{apa}=\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}_{apa}$$

$$[\tilde{\boldsymbol{\omega}}]_{apa}=[\dot{\mathbf{A}}]_{ap}[\mathbf{A}^T]_{ap}$$

$$[\dot{\mathbf{A}}]_{ap}=[\tilde{\boldsymbol{\omega}}]_{apa}[\mathbf{A}]_{ap}=[\tilde{\mathbf{v}}]_\omega[\mathbf{A}]_{ap}$$

Orientation Acceleration from Assembly Frame, $\left[\ddot{\mathbf{A}}(\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)\right]_{ap}$

$$[\ddot{\mathbf{A}}]_{ap}=\frac{d[\dot{\mathbf{A}}]_{ap}}{dt}=[\dot{\tilde{\boldsymbol{\omega}}}]_{apa}[\mathbf{A}]_{ap}+[\tilde{\boldsymbol{\omega}}]_{apa}[\dot{\mathbf{A}}]_{ap}$$

$$=[\dot{\tilde{\boldsymbol{\omega}}}]_{apa}[\mathbf{A}]_{ap}+[\tilde{\boldsymbol{\omega}}]_{apa}[\tilde{\boldsymbol{\omega}}]_{apa}[\mathbf{A}]_{ap}$$

$$=\left([\dot{\tilde{\boldsymbol{\omega}}}]_{apa}+[\tilde{\boldsymbol{\omega}}]_{apa}[\tilde{\boldsymbol{\omega}}]_{apa}\right)[\mathbf{A}]_{ap}$$

$$[\ddot{\mathbf{A}}]_{ap}=\frac{d[\dot{\mathbf{A}}]_{ap}}{dt}=[\tilde{\mathbf{a}}]_\alpha[\mathbf{A}]_{ap}+[\tilde{\mathbf{v}}]_\omega[\dot{\mathbf{A}}]_{ap}$$

$$=[\tilde{\mathbf{a}}]_\alpha[\mathbf{A}]_{ap}+[\tilde{\mathbf{v}}]_\omega[\tilde{\mathbf{v}}]_\omega[\mathbf{A}]_{ap}$$

$$=\left([\tilde{\mathbf{a}}]_\alpha+[\tilde{\mathbf{v}}]_\omega[\tilde{\mathbf{v}}]_\omega\right)[\mathbf{A}]_{ap}$$

Angular Velocity from Assembly Frame, $\vec{\boldsymbol{\omega}}_{ap}$

$$\vec{\boldsymbol{\omega}}_{ap}=\omega_{apax}\vec{\mathbf{i}}_a+\omega_{apay}\vec{\mathbf{j}}_a+\omega_{apaz}\vec{\mathbf{k}}_a=\omega_{appx}\vec{\mathbf{i}}_p+\omega_{appy}\vec{\mathbf{j}}_p+\omega_{appz}\vec{\mathbf{k}}_a$$

Angular Velocity from Assembly Frame, $\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)$

$$\boldsymbol{\omega}_{apa}=\mathbf{v}_\omega$$

Angular Velocity from Assembly Frame, $\boldsymbol{\omega}_{app}(\mathbf{q}_E,\mathbf{v}_\omega)$

$$\boldsymbol{\omega}_{app}=[\mathbf{A}]_{ap}^T\boldsymbol{\omega}_{apa}=[\mathbf{A}]_{ap}^T\mathbf{v}_\omega$$

Angular Acceleration from Assembly Frame, $\vec{\boldsymbol{\alpha}}_{ap}$

$$\vec{\boldsymbol{\alpha}}_{ap}=\alpha_{apax}\vec{\mathbf{i}}_a+\alpha_{apay}\vec{\mathbf{j}}_a+\alpha_{apaz}\vec{\mathbf{k}}_a=\alpha_{appx}\vec{\mathbf{i}}_p+\alpha_{appy}\vec{\mathbf{j}}_p+\alpha_{appz}\vec{\mathbf{k}}_a$$

Angular Acceleration from Assembly Frame, $\boldsymbol{\alpha}_{apa}(\mathbf{a}_\alpha)$

$$\boldsymbol{\alpha}_{apa}=\frac{d\,\boldsymbol{\omega}_{apa}}{dt}=\frac{d\,\mathbf{v}_\omega}{dt}=\dot{\mathbf{v}}_\omega=\mathbf{a}_\alpha$$

Angular Acceleration from Assembly Frame, $\boldsymbol{\alpha}_{app}(\mathbf{q}_E,\mathbf{a}_\alpha)$

$$\boldsymbol{\alpha}_{app}=[\mathbf{A}]_{ap}^T\boldsymbol{\alpha}_{apa}=[\mathbf{A}]_{ap}^T\mathbf{a}_\alpha$$

Angular Acceleration from Assembly Frame, $\boldsymbol{\alpha}_{app}(\mathbf{q}_E,\mathbf{a}_\alpha)$

$$\begin{aligned}\boldsymbol{\alpha}_{app}&=\frac{d\,\boldsymbol{\omega}_{app}}{dt}=[\dot{\mathbf{A}}]_{ap}^T\mathbf{v}_\omega+[\mathbf{A}]_{ap}^T\mathbf{a}_\alpha\\&=[\mathbf{A}]_{ap}^T[\tilde{\mathbf{v}}]_\omega^T\mathbf{v}_\omega+[\mathbf{A}]_{ap}^T\mathbf{a}_\alpha\\&=[\mathbf{A}]_{ap}^T\mathbf{a}_\alpha\end{aligned}$$

Absolute Position, $\mathbf{r}_{OpO}(t,\mathbf{q}_X)$

$$\mathbf{r}_{OpO}=\mathbf{r}_{OaO}(t)+[\mathbf{A}(t)]_{Oa}\mathbf{r}_{apa}(\mathbf{q}_X)$$

Absolute Velocity, $\mathbf{v}_{OpO}(t,\mathbf{q}_X,\mathbf{v}_X)$

$$\mathbf{v}_{OpO}=\mathbf{v}_{OaO}(t)+[\dot{\mathbf{A}}(t)]_{Oa}\mathbf{r}_{apa}(\mathbf{q}_X)+[\mathbf{A}(t)]_{Oa}\mathbf{v}_{apa}(\mathbf{v}_X)$$

Absolute Acceleration, $\mathbf{a}_{OpO}(t,\mathbf{q}_X,\mathbf{v}_X,\dot{\mathbf{v}}_X)$

$$\mathbf{a}_{OpO}=\mathbf{a}_{OaO}(t)+[\ddot{\mathbf{A}}(t)]_{Oa}\mathbf{r}_{apa}(\mathbf{q}_X)+2[\dot{\mathbf{A}}(t)]_{Oa}\mathbf{v}_{apa}(\mathbf{v}_X)+[\mathbf{A}(t)]_{Oa}\mathbf{a}_{apa}(\dot{\mathbf{v}}_X)$$

Absolute Orientation, $[\mathbf{A}(t,\mathbf{q}_E)]_{Op}$

$$[\mathbf{A}]_{Op}=[\mathbf{A}(t)]_{Oa}[\mathbf{A}(\mathbf{q}_E)]_{ap}$$

Absolute Orientation Velocity, $[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)]_{Op}$

$$[\dot{\mathbf{A}}]_{Op}=[\dot{\mathbf{A}}(t)]_{Oa}[\mathbf{A}(\mathbf{q}_E)]_{ap}+[\mathbf{A}(t)]_{Oa}[\dot{\mathbf{A}}(\mathbf{q}_E,\mathbf{v}_\omega)]_{ap}$$

Absolute Orientation Acceleration, $\left[\ddot{\mathbf{A}}\left(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha\right)\right]_{Op}$

$$[\ddot{\mathbf{A}}]_{Op}=[\ddot{\mathbf{A}}(t)]_{Oa}[\mathbf{A}(\mathbf{q}_E)]_{ap}+2[\dot{\mathbf{A}}(t)]_{Oa}[\dot{\mathbf{A}}(\mathbf{q}_E,\mathbf{v}_\omega)]_{ap}+[\mathbf{A}(t)]_{Oa}[\ddot{\mathbf{A}}(\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)]_{ap}$$

Absolute Angular Velocity, $\boldsymbol{\omega}_{OpO}(t,\mathbf{v}_\omega)$

$$\boldsymbol{\omega}_{OpO}=\boldsymbol{\omega}_{OaO}(t)+[\mathbf{A}(t)]_{Oa}\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)$$

Absolute Angular Velocity, $\boldsymbol{\omega}_{Opp}(t,\mathbf{q}_E,\mathbf{v}_\omega)$

$$\boldsymbol{\omega}_{Opp}=[\mathbf{A}(\mathbf{q}_E)]_{ap}^T[\mathbf{A}(t)]_{Oa}^T\{\boldsymbol{\omega}_{OaO}(t)+[\mathbf{A}(t)]_{Oa}\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)\}$$

$$\boldsymbol{\omega}_{Opp}=[\mathbf{A}(\mathbf{q}_E)]_{ap}^T\{[\mathbf{A}(t)]_{Oa}^T\boldsymbol{\omega}_{OaO}(t)+\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)\}$$

Absolute Angular Velocity alternative, $\boldsymbol{\omega}_{Opp}(t,\mathbf{q}_E,\mathbf{v}_\omega)$

$$\boldsymbol{\omega}_{Opp}=[\mathbf{A}(t,\mathbf{q}_E)]_{Op}^T\boldsymbol{\omega}_{OpO}(t,\mathbf{v}_\omega)$$

Absolute Angular Acceleration, $\boldsymbol{\alpha}_{OpO}\left(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha\right)$

$$\boldsymbol{\alpha}_{OpO}=\boldsymbol{\alpha}_{OaO}(t)+[\dot{\mathbf{A}}(t)]_{Oa}\boldsymbol{\omega}_{apa}(\mathbf{q}_E,\mathbf{v}_\omega)+[\mathbf{A}(t)]_{Oa}\boldsymbol{\alpha}_{apa}(\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)$$

Absolute Angular Acceleration, $\boldsymbol{\alpha}_{Opp}\left(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha\right)$

$$\boldsymbol{\omega}_{Opp}=[\mathbf{A}(\mathbf{q}_E)]_{ap}^T\{\boldsymbol{\omega}_{Oaa}(t)+\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)\}$$

$$\boldsymbol{\alpha}_{Opp}=[\dot{\mathbf{A}}(\mathbf{q}_E,\mathbf{v}_\omega)]_{ap}^T\boldsymbol{\omega}_{Oaa}(t)+[\mathbf{A}(\mathbf{q}_E)]_{ap}^T\boldsymbol{\alpha}_{Oaa}(t)+\boldsymbol{\alpha}_{app}(\mathbf{q}_E,\mathbf{a}_\alpha)$$

Marker Frame

$o_m x_m y_m z_m$ is a Cartesian Frame fixed to a part. It can be located anywhere on a part.
$\vec{\mathbf{i}}_m\vec{\mathbf{j}}_m\vec{\mathbf{k}}_m$ are base unit vectors for $o_m x_m y_m z_m$.

Position from Part Frame, $\mathbf{r}_{pmp}()$

$$\mathbf{r}_{pmp}=\mathbf{r}_{pmp}()$$

Velocity from Part Frame, $\mathbf{v}_{pmp}()$

$$\mathbf{v}_{pmp}=\mathbf{0}$$

Orientation from Part Frame, $[\mathbf{A}()]_{pm}$

$$\begin{Bmatrix}\vec{\mathbf{i}}_p\\\vec{\mathbf{j}}_p\\\vec{\mathbf{k}}_p\end{Bmatrix}=[\mathbf{A}()]_{pm}\begin{Bmatrix}\vec{\mathbf{i}}_m\\\vec{\mathbf{j}}_m\\\vec{\mathbf{k}}_m\end{Bmatrix}$$

Orientation Velocity from Part Frame, $[\dot{\mathbf{A}}()]_{pm}$

$$[\dot{\mathbf{A}}]_{pm}=\mathbf{0}$$

Orientation Acceleration from Part Frame, $[\ddot{\mathbf{A}}()]_{pm}$

$$[\ddot{\mathbf{A}}]_{pm}=\mathbf{0}$$

Absolute Position, $\mathbf{r}_{OmO}(t,\mathbf{q})$

$$\mathbf{r}_{OmO}=\mathbf{r}_{OpO}(t,\mathbf{q}_X)+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Op}\mathbf{r}_{pmp}()$$

Absolute Velocity, $\mathbf{v}_{OmO}(t,\mathbf{q},\mathbf{v})$

$$\mathbf{v}_{OmO}=\mathbf{v}_{OpO}(t,\mathbf{q}_X,\mathbf{v}_X)+\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Op}\mathbf{r}_{pmp}()$$

Absolute Acceleration, $\mathbf{a}_{OmO}(t,\mathbf{q},\mathbf{v},\mathbf{a})$

$$\mathbf{a}_{OmO}=\mathbf{a}_{OpO}(t,\mathbf{q}_X,\mathbf{v}_X,\mathbf{a}_X)+\left[\ddot{\mathbf{A}}\right]_{Op}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)\mathbf{r}_{pmp}()$$

Absolute Orientation, $\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}$

$$\begin{Bmatrix}\vec{\mathbf{I}}\\ \vec{\mathbf{J}}\\ \vec{\mathbf{K}}\end{Bmatrix}=\left[\mathbf{A}(t)\right]_{Oa}\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}\left[\mathbf{A}()\right]_{pm}\begin{Bmatrix}\vec{\mathbf{i}}_m\\ \vec{\mathbf{j}}_m\\ \vec{\mathbf{k}}_m\end{Bmatrix}$$

$$\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}=\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Op}\left[\mathbf{A}()\right]_{pm}$$

Absolute Orientation Velocity, $\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Om}$

$$\left[\dot{\mathbf{A}}\right]_{Om}=\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Op}\left[\mathbf{A}()\right]_{pm}$$

Absolute Orientation Acceleration, $\left[\ddot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)\right]_{Om}$

$$\left[\ddot{\mathbf{A}}\right]_{Om}=\left[\ddot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)\right]_{Op}\left[\mathbf{A}()\right]_{pm}$$

Absolute Angular Velocity, $\boldsymbol{\omega}_{OmO}(t,\mathbf{q}_E,\mathbf{v}_\omega)$

$$\boldsymbol{\omega}_{OmO}=\boldsymbol{\omega}_{OpO}(t,\mathbf{q}_E,\mathbf{v}_\omega)$$

Absolute Angular Acceleration, $\boldsymbol{\alpha}_{OmO}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)$

$$\boldsymbol{\alpha}_{OmO}=\boldsymbol{\alpha}_{OpO}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)$$

End Frame

$o_e x_e y_e z_e$ is a fixed or moving Cartesian Frame referenced to a marker. If moving, it may lie on a curve, surface or be prescribed, all wrt to the marker. The curve or surface is parameterized by $\mathbf{s}$ which is one or two dimensional. These definitions are stored in the marker so that many end frames can use the same curve or surface. The end frame stores the $\mathbf{s}$ parameter.

$\vec{\mathbf{i}}_e\vec{\mathbf{j}}_e\vec{\mathbf{k}}_e$ are base unit vectors for $o_e x_e y_e z_e$.

$\vec{\mathbf{r}}_{me}(t,\mathbf{s})$ = displacement from marker frame to end frame.

$$\begin{Bmatrix}\vec{\mathbf{i}}_m\\ \vec{\mathbf{j}}_m\\ \vec{\mathbf{k}}_m\end{Bmatrix}=\left[\mathbf{A}(t,\mathbf{s})\right]_{me}\begin{Bmatrix}\vec{\mathbf{i}}_e\\ \vec{\mathbf{j}}_e\\ \vec{\mathbf{k}}_e\end{Bmatrix}$$

Point

$\vec{\mathbf{r}}_{me}(t)$ = displacement from marker frame to end frame.

$$\begin{Bmatrix} \vec{\mathbf{i}}_m \\ \vec{\mathbf{j}}_m \\ \vec{\mathbf{k}}_m \end{Bmatrix} = [\mathbf{A}(t)]_{me} \begin{Bmatrix} \vec{\mathbf{i}}_e \\ \vec{\mathbf{j}}_e \\ \vec{\mathbf{k}}_e \end{Bmatrix} = \begin{bmatrix} \mathbf{i}_{em} & \mathbf{j}_{em} & \mathbf{k}_{em} \end{bmatrix} \begin{Bmatrix} \vec{\mathbf{i}}_e \\ \vec{\mathbf{j}}_e \\ \vec{\mathbf{k}}_e \end{Bmatrix}$$

Curve

If s_1 is a parameter, $\mathbf{r}_{mem}(t, s_1)$ is a time varying curve wrt a marker.

Curve tangent $\mathbf{t}_{1\mathrm{em}}(t, s_1)$

$$\mathbf{t}_{1\mathrm{em}} = \frac{\partial \mathbf{r}_{mem}(t, s_1)}{\partial s_1}$$

Curve tangent $\mathbf{t}_{1\mathrm{eO}}(t, \mathbf{q}_E, s_1)$

$$\mathbf{t}_{1\mathrm{eO}} = [\mathbf{A}(t, \mathbf{q}_E)]_{Om} \mathbf{t}_{1\mathrm{em}}(t, s_1)$$

Outward normal, $\mathbf{n}_{em}(t, s_1)$

$$\mathbf{n}_{em} = -\frac{\partial^2 \mathbf{r}_{mem}(t, s_1)}{\partial s_1 \partial s_1}$$

Binormal, $\mathbf{b}_{em}(t, s_1)$

$$\mathbf{b}_{em} = \mathbf{n}_{em}(t, s_1) \times \mathbf{t}_{1\mathrm{em}}(t, s_1)$$

Curve End Frame

$\vec{\mathbf{i}}_e = \dfrac{\vec{\mathbf{t}}_{1e}}{t_{1e}}$ is tangent to the curve in increasing s_1 .

$\vec{\mathbf{k}}_e = \dfrac{\vec{\mathbf{n}}_e}{n_e}$ is outward normal to the curve.

$\vec{\mathbf{j}}_e = \vec{\mathbf{k}}_e \times \vec{\mathbf{i}}_e = \dfrac{\vec{\mathbf{n}}_{2e}}{n_{2e}}$ is binormal to the curve.

Problem: $\vec{\mathbf{j}}_e$ and $\vec{\mathbf{k}}_e$ can be discontinuous and indeterminant.

A possible solution is to chose a fixed vector $\vec{\mathbf{t}}_{2\mathrm{e}}$ that is never parallel to the curve. Then

$\vec{\mathbf{i}}_e = \dfrac{\vec{\mathbf{t}}_{1e}}{t_{1e}}$ is tangent to the curve in increasing s_1 .

$$\vec{\mathbf{k}}_e = \frac{\vec{\mathbf{t}}_{1e}}{t_{1e}} \times \frac{\vec{\mathbf{t}}_{2e}}{t_{2e}}$$

$$\vec{\mathbf{j}}_e = \vec{\mathbf{k}}_e \times \vec{\mathbf{i}}_e$$

Surface

If s_1 and s_2 are parameters, then $\mathbf{r}_{mem}(t, s_1, s_2)$ is a time varying surface wrt a marker.

Surface with Constant Profile (Prism)

Profile is defined in the marker xy plane. Therefore the parametric equations are

$x = f_x(s_1) \quad y = f_y(s_1) \quad z = s_2$

It is possible to simplify to use s_1 only.

Plane Surface

The surface is the marker xy plane.

$x = s_1 \quad y = s_2 \quad z = 0$

It is possible to avoid using $\mathbf{s}$.

Surface tangent $\mathbf{t}_{1\mathrm{em}}(t, s_1, s_2)$

$$\mathbf{t}_{1\mathrm{em}} = \frac{\partial \mathbf{r}_{mem}(t, s_1, s_2)}{\partial s_1}$$

Similar to curve tangent.

Surface tangent $\mathbf{t}_{2\mathrm{em}}(t, s_1, s_2)$

$$\mathbf{t}_{2\mathrm{em}} = \frac{\partial \mathbf{r}_{mem}(t, s_1, s_2)}{\partial s_2}$$

Similar to curve tangent.

Surface normal $\mathbf{n}_{em}(t, s_1, s_2)$

$$\mathbf{n}_{em} = \mathbf{t}_{1\mathrm{em}} \times \mathbf{t}_{2\mathrm{em}} = \frac{\partial \mathbf{r}_{mem}}{\partial s_1} \times \frac{\partial \mathbf{r}_{mem}}{\partial s_2}$$

Surface normal $\mathbf{n}_{eO}(t, \mathbf{q}_E, s_1, s_2)$

$$\mathbf{n}_{eO} = [\mathbf{A}(t, \mathbf{q}_E)]_{Om} \mathbf{n}_{em}(t, s_1, s_2)$$

Surface End Frame

$\vec{\mathbf{i}}_e = \frac{\vec{\mathbf{t}}_{1e}}{t_{1e}}$ is tangent to the curve in increasing s_1 .

$$\vec{\mathbf{k}}_e = \frac{\vec{\mathbf{t}}_{1e}}{t_{1e}} \times \frac{\vec{\mathbf{t}}_{2e}}{t_{2e}}$$

$$\vec{\mathbf{j}}_e = \vec{\mathbf{k}}_e \times \vec{\mathbf{i}}_e$$

$$\begin{Bmatrix} \vec{\mathbf{i}}_m \\ \vec{\mathbf{j}}_m \\ \vec{\mathbf{k}}_m \end{Bmatrix} = [\mathbf{A}(t, \mathbf{s})]_{me} \begin{Bmatrix} \vec{\mathbf{i}}_e \\ \vec{\mathbf{j}}_e \\ \vec{\mathbf{k}}_e \end{Bmatrix} = [\mathbf{i}_{em} \quad \mathbf{j}_{em} \quad \mathbf{k}_{em}] \begin{Bmatrix} \vec{\mathbf{i}}_e \\ \vec{\mathbf{j}}_e \\ \vec{\mathbf{k}}_e \end{Bmatrix}$$

Position from Marker Frame, $\mathbf{r}_{mem}(t, \mathbf{s})$

$$\mathbf{r}_{mem} = \mathbf{r}_{mem}(t, \mathbf{s})$$

Velocity from Marker Frame, $\mathbf{v}_{mem}(t, \mathbf{s}, \dot{\mathbf{s}})$

$$\mathbf{v}_{mem} = \frac{d\,\mathbf{r}_{mem}}{d\,t} = \frac{\partial \mathbf{r}_{mem}(t, \mathbf{s})}{\partial \mathbf{s}} \dot{\mathbf{s}} + \frac{\partial \mathbf{r}_{mem}(t, \mathbf{s})}{\partial t}$$

Acceleration from Marker Frame, $\mathbf{a}_{mem}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$

$$\mathbf{a}_{mem}=\frac{d\,\mathbf{v}_{mem}}{d\,t}=\dot{\mathbf{s}}^{T}\frac{\partial^{2}\mathbf{r}_{mem}(t,\mathbf{s})}{\partial\mathbf{s}\partial\mathbf{s}}\dot{\mathbf{s}}+\frac{\partial\,\mathbf{r}_{mem}(t,\mathbf{s})}{\partial\mathbf{s}}\ddot{\mathbf{s}}+\frac{\partial^{2}\mathbf{r}_{mem}(t,\mathbf{s})}{\partial t\,\partial\mathbf{s}}\dot{\mathbf{s}}+\frac{\partial^{2}\mathbf{r}_{mem}(t,\mathbf{s})}{\partial t^{2}}$$

Orientation from Marker Frame, $[\mathbf{A}(t,\mathbf{s})]_{me}$

$$\begin{Bmatrix}\vec{\mathbf{i}}_{m}\\ \vec{\mathbf{j}}_{m}\\ \vec{\mathbf{k}}_{m}\end{Bmatrix}=[\mathbf{A}(t,\mathbf{s})]_{me}\begin{Bmatrix}\vec{\mathbf{i}}_{e}\\ \vec{\mathbf{j}}_{e}\\ \vec{\mathbf{k}}_{e}\end{Bmatrix}$$

Orientation Velocity from Marker Frame, $[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}})]_{me}$

$$[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}})]_{me}=\frac{d\,[\mathbf{A}(t,\mathbf{s})]_{me}}{d\,t}=\frac{\partial[\mathbf{A}(t,\mathbf{s})]_{me}}{\partial\,\mathbf{s}}\dot{\mathbf{s}}+\frac{\partial[\mathbf{A}(t,\mathbf{s})]_{me}}{\partial\,t}$$

Orientation Acceleration from Marker Frame, $[\ddot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})]_{me}$

$$[\ddot{\mathbf{A}}]_{me}=\frac{d\,[\dot{\mathbf{A}}]_{me}}{d\,t}=\dot{\mathbf{s}}^{T}\frac{\partial^{2}[\mathbf{A}(t,\mathbf{s})]_{me}}{\partial\mathbf{s}\partial\mathbf{s}}\dot{\mathbf{s}}+\frac{\partial[\mathbf{A}(t,\mathbf{s})]_{me}}{\partial\,\mathbf{s}}\ddot{\mathbf{s}}+\frac{\partial^{2}[\mathbf{A}(t,\mathbf{s})]_{me}}{\partial\,t\,\partial\,\mathbf{s}}\dot{\mathbf{s}}+\frac{\partial^{2}[\mathbf{A}(t,\mathbf{s})]_{me}}{\partial\,t^{2}}$$

Angular Velocity in Marker Frame, $\boldsymbol{\omega}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})$

$$\boldsymbol{\omega}_{mem}=\boldsymbol{\omega}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})$$

Angular Acceleration in Marker Frame, $\boldsymbol{\alpha}_{mem}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$

$$\boldsymbol{\alpha}_{mem}=\boldsymbol{\alpha}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})$$

Position from Part Frame, $\mathbf{r}_{pep}(t,\mathbf{s})$

$$\mathbf{r}_{pep}=\mathbf{r}_{pmp}()+[\mathbf{A}()]_{pm}\mathbf{r}_{mem}(t,\mathbf{s})$$

Velocity from Part Frame, $\mathbf{v}_{pep}(t,\mathbf{s},\dot{\mathbf{s}})$

$$\mathbf{v}_{pep}=[\mathbf{A}()]_{pm}\mathbf{v}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})$$

Acceleration from Part Frame, $\mathbf{a}_{pep}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$

$$\mathbf{a}_{pep}=[\mathbf{A}()]_{pm}\mathbf{a}_{mem}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$$

Orientation from Part Frame, $[\mathbf{A}(t,\mathbf{s})]_{pe}$

$$\begin{Bmatrix}\vec{\mathbf{i}}_{p}\\ \vec{\mathbf{j}}_{p}\\ \vec{\mathbf{k}}_{p}\end{Bmatrix}=[\mathbf{A}()]_{pm}[\mathbf{A}(t,\mathbf{s})]_{me}\begin{Bmatrix}\vec{\mathbf{i}}_{e}\\ \vec{\mathbf{j}}_{e}\\ \vec{\mathbf{k}}_{e}\end{Bmatrix}$$

$$[\mathbf{A}(t,\mathbf{s})]_{pe}=[\mathbf{A}()]_{pm}[\mathbf{A}(t,\mathbf{s})]_{me}$$

Orientation Velocity from Part Frame, $[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}})]_{pe}$

$$[\dot{\mathbf{A}}]_{pe}=[\mathbf{A}()]_{pm}[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}})]_{me}$$

Orientation Acceleration from Part Frame, $\left[\ddot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})\right]_{pe}$

$$\left[\ddot{\mathbf{A}}\right]_{pe}=\left[\mathbf{A}()\right]_{pm}\left[\ddot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})\right]_{me}$$

Angular Velocity in Part Frame, $\boldsymbol{\omega}_{pep}(t,\mathbf{s},\dot{\mathbf{s}})$

$$\boldsymbol{\omega}_{pep}=\left[\mathbf{A}\right]_{pm}\boldsymbol{\omega}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})$$

Angular Acceleration in Part Frame, $\boldsymbol{\alpha}_{pep}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$

$$\boldsymbol{\alpha}_{pep}=\left[\mathbf{A}\right]_{pm}\boldsymbol{\alpha}_{mem}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$$

Absolute Position, $\mathbf{r}_{OeO}(t,\mathbf{q},\mathbf{s})$

$$\mathbf{r}_{OeO}=\mathbf{r}_{OaO}(t)+\left[\mathbf{A}(t)\right]_{Oa}\left\{\mathbf{r}_{apa}(\mathbf{q}_X)+\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}\left\{\mathbf{r}_{pmp}()+\left[\mathbf{A}()\right]_{pm}\mathbf{r}_{mem}(t,\mathbf{s})\right\}\right\}$$
$$\mathbf{r}_{OeO}=\mathbf{r}_{OaO}(t)+\left[\mathbf{A}(t)\right]_{Oa}\mathbf{r}_{apa}(\mathbf{q}_X)+\left[\mathbf{A}(t)\right]_{Oa}\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}\mathbf{r}_{pmp}()+\left[\mathbf{A}(t)\right]_{Oa}\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}\left[\mathbf{A}()\right]_{pm}\mathbf{r}_{mem}(t,\mathbf{s})$$
$$\mathbf{r}_{OeO}=\mathbf{r}_{OpO}(t,\mathbf{q}_X)+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Op}\mathbf{r}_{pep}(t,\mathbf{s})$$
$$\mathbf{r}_{OeO}=\mathbf{r}_{OmO}(t,\mathbf{q})+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}\mathbf{r}_{mem}(t,\mathbf{s})$$

Note: The last form is best because the partFrame level need not be involved. There is no repetition of computation at the partFrame level. The price is to store $\mathbf{r}_{OmO}(t,\mathbf{q})$, $\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}$ and their partials in markerFrame. This suggests that Part controls MarkerFrame, MarkerFrame controls EndFrame, and Joint controls Constraints. System should control KinematicsIJ since KinematicsIJ is used by both Joints and Forces.

Absolute Velocity, $\mathbf{v}_{OeO}(t,\mathbf{q},\mathbf{s},\mathbf{v},\dot{\mathbf{s}})$

$$\mathbf{v}_{OeO}=\mathbf{v}_{OmO}(t,\mathbf{q},\mathbf{v})+\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Om}\mathbf{r}_{mem}(t,\mathbf{s})+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}\mathbf{v}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})$$

Absolute Acceleration, $\mathbf{a}_{OeO}(t,\mathbf{q},\mathbf{s},\mathbf{v},\dot{\mathbf{s}},\mathbf{a},\ddot{\mathbf{s}})$

$$\mathbf{a}_{OeO}=\mathbf{a}_{OmO}(t,\mathbf{q},\mathbf{v},\mathbf{a})+\left[\ddot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)\right]_{Om}\mathbf{r}_{mem}(t,\mathbf{s})$$
$$+2\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Om}\mathbf{v}_{mem}(t,\mathbf{s},\dot{\mathbf{s}})+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}\mathbf{a}_{mem}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})$$

Absolute Orientation, $\left[\mathbf{A}(t,\mathbf{q},\mathbf{s})\right]_{Oe}$

$$\begin{Bmatrix}\vec{\mathbf{i}}_O\\ \vec{\mathbf{j}}_O\\ \vec{\mathbf{k}}_O\end{Bmatrix}=\left[\mathbf{A}(t)\right]_{Oa}\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}\left[\mathbf{A}()\right]_{pm}\left[\mathbf{A}(t,\mathbf{s})\right]_{me}\begin{Bmatrix}\vec{\mathbf{i}}_e\\ \vec{\mathbf{j}}_e\\ \vec{\mathbf{k}}_e\end{Bmatrix}$$
$$\left[\mathbf{A}(t,\mathbf{q}_E,\mathbf{s})\right]_{Oe}=\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}\left[\mathbf{A}(t,\mathbf{s})\right]_{me}$$

Absolute Orientation Velocity, $\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{s},\mathbf{v}_\omega,\dot{\mathbf{s}})\right]_{Oe}$

$$\left[\dot{\mathbf{A}}\right]_{Oe}=\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Om}\left[\mathbf{A}(t,\mathbf{s})\right]_{me}+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}\left[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}})\right]_{me}$$

Absolute Orientation Acceleration, $\left[\ddot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{s},\mathbf{v}_\omega,\dot{\mathbf{s}},\mathbf{a}_\alpha,\ddot{\mathbf{s}})\right]_{Oe}$

$$\left[\ddot{\mathbf{A}}\right]_{Oe}=\left[\ddot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega,\mathbf{a}_\alpha)\right]_{Om}\left[\mathbf{A}(t,\mathbf{s})\right]_{me}$$
$$+2\left[\dot{\mathbf{A}}(t,\mathbf{q}_E,\mathbf{v}_\omega)\right]_{Om}\left[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}})\right]_{me}+\left[\mathbf{A}(t,\mathbf{q}_E)\right]_{Om}\left[\dot{\mathbf{A}}(t,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}})\right]_{me}$$

Absolute Angular Velocity, $\boldsymbol{\omega}_{OeO}\left(t, \mathbf{q}_E, \mathbf{s}, \mathbf{v}_\omega, \dot{\mathbf{s}}\right)$

$$\boldsymbol{\omega}_{OeO}=\boldsymbol{\omega}_{OmO}\left(t, \mathbf{q}_A, \mathbf{v}_\omega\right)+\left[\mathbf{A}\left(t, \mathbf{q}_A\right)\right]_{Om}\boldsymbol{\omega}_{mem}\left(t, \mathbf{s}, \dot{\mathbf{s}}\right)$$

Absolute Angular Acceleration, $\boldsymbol{\alpha}_{OeO}\left(t, \mathbf{q}_E, \mathbf{s}, \mathbf{v}_\omega, \dot{\mathbf{s}}, \mathbf{a}_\alpha, \ddot{\mathbf{s}}\right)$

$$\boldsymbol{\alpha}_{OeO}=\boldsymbol{\alpha}_{OmO}\left(t, \mathbf{q}_E, \mathbf{v}_\omega, \mathbf{a}_\alpha\right)+\left[\dot{\mathbf{A}}\left(t, \mathbf{q}_E, \mathbf{v}_\omega\right)\right]_{Om}\boldsymbol{\omega}_{mem}\left(t, \mathbf{s}, \dot{\mathbf{s}}\right)$$
$$+\left[\mathbf{A}\left(t, \mathbf{q}_E\right)\right]_{Om}\boldsymbol{\alpha}_{mem}\left(t, \mathbf{s}, \dot{\mathbf{s}}, \ddot{\mathbf{s}}\right)$$

Kinetic Energy, $T\left(t, \mathbf{q}, \mathbf{v},\right)$

$$\mathbf{v}_{OpO}=\mathbf{v}_{OaO}(t)+\left[\dot{\mathbf{A}}(t)\right]_{Oa}\mathbf{r}_{apa}(\mathbf{q}_X)+\left[\mathbf{A}(t)\right]_{Oa}\mathbf{v}_{apa}(\mathbf{v}_X)$$
$$\boldsymbol{\omega}_{OpO}=\boldsymbol{\omega}_{OaO}(t)+\left[\mathbf{A}(t)\right]_{Oa}\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)$$
$$\boldsymbol{\omega}_{Opp}=\left[\mathbf{A}(\mathbf{q}_E)\right]_{ap}^T\left\{\left[\mathbf{A}(t)\right]_{Oa}^T\boldsymbol{\omega}_{OaO}(t)+\boldsymbol{\omega}_{apa}(\mathbf{v}_\omega)\right\}$$
$$T=\frac{1}{2}\sum_{i=1}^{np}\left(m_i\mathbf{v}_{OipO}^T\mathbf{v}_{OipO}+\boldsymbol{\omega}_{Oipip}^T\left[\mathbf{J}_{pp}\right]_i\boldsymbol{\omega}_{Oipip}\right)$$

Generalized Inertia, $\dot{\mathbf{p}}$

$$\dot{\mathbf{p}}_I=\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\mathbf{q}}_I}\right)$$
$$\dot{\mathbf{p}}_{IX}=\frac{d}{dt}\left(\frac{\partial T}{\partial\mathbf{v}_{IX}}\right)$$
$$\dot{\mathbf{p}}_{IX}=\frac{d}{dt}\left(m_p\left[\mathbf{A}_{OIa}\right]^T\mathbf{v}_{OIpO}\right)=m_I\left(\left[\dot{\mathbf{A}}_{OIa}\right]^T\mathbf{v}_{OIpO}+\left[\mathbf{A}_{OIa}\right]^T\mathbf{a}_{OIpO}\right)$$
$$\dot{\mathbf{p}}_{I\omega}=\frac{d}{dt}\left(\frac{\partial T}{\partial\mathbf{v}_{I\omega}}\right)$$
$$\dot{\mathbf{p}}_{I\omega}=\frac{d}{dt}\left(\frac{\partial\boldsymbol{\omega}_{OIpIp}^T}{\partial\mathbf{v}_{I\omega}}\left[\mathbf{J}_{pp}\right]_I\boldsymbol{\omega}_{OIpIp}\right)$$
$$\dot{\mathbf{p}}_{I\omega}=\frac{d}{dt}\left(\left[\mathbf{A}\right]_{IaIp}\left[\mathbf{J}_{pp}\right]_I\boldsymbol{\omega}_{OIpIp}\right)$$
$$\dot{\mathbf{p}}_{I\omega}=\left[\dot{\mathbf{A}}\right]_{IaIp}\left[\mathbf{J}_{pp}\right]_I\boldsymbol{\omega}_{OIpIp}+\left[\mathbf{A}\right]_{IaIp}\left[\mathbf{J}_{pp}\right]_I\boldsymbol{\alpha}_{OIpIp}$$
$$\dot{\mathbf{p}}_{I\omega}=\left[\tilde{\boldsymbol{\omega}}\right]_{IaIpIa}\left[\mathbf{A}\right]_{IaIp}\left[\mathbf{J}_{pp}\right]_I\boldsymbol{\omega}_{OIpIp}+\left[\mathbf{A}\right]_{IaIp}\left[\mathbf{J}_{pp}\right]_I\boldsymbol{\alpha}_{OIpIp}$$

Virtual Work, δW

For an instant in time,

$$\delta W=\sum_{if}^{nf}\vec{\mathbf{F}}_{if}\cdot\delta\vec{\mathbf{r}}_{if}+\sum_{it}^{nt}\vec{\mathbf{T}}_{it}\cdot\delta\vec{\boldsymbol{\theta}}_{it}$$
$$\delta W=\left(\sum_{if}^{nf}\mathbf{F}_{Oif}^T\frac{\partial\mathbf{r}_{OifO}}{\partial\mathbf{q}}+\sum_{it}^{nt}\mathbf{T}_{Oit}^T\frac{\partial\boldsymbol{\theta}_{OitO}}{\partial\mathbf{q}}\right)\delta\mathbf{q}+\left(\sum_{if}^{nf}\mathbf{F}_{Oif}^T\frac{\partial\mathbf{r}_{OifO}}{\partial\mathbf{s}}+\sum_{it}^{nt}\mathbf{T}_{Oit}^T\frac{\partial\boldsymbol{\theta}_{OitO}}{\partial\mathbf{s}}\right)\delta\mathbf{s}$$
$$=\mathbf{Q}_q^T\delta\mathbf{q}+\mathbf{Q}_s^T\delta\mathbf{s}$$

Note that $\vec{\mathbf{r}}_{if}$ is the position vector to a material point on which $\vec{\mathbf{F}}_{if}$ acts at that instant. Do not include the part of $\delta\vec{\mathbf{r}}_{if}$ in change of $\vec{\mathbf{r}}_{if}$ due to changing material point.

Generalized Force, $\mathbf{Q}$

$$\mathbf{Q}_q = \sum_{if}^{nf} \frac{\partial \mathbf{r}_{OifO}^T}{\partial \mathbf{q}} \mathbf{F}_{Oif} + \sum_{it}^{nt} \frac{\partial \boldsymbol{\theta}_{OitO}^T}{\partial \mathbf{q}} \mathbf{T}_{Oit}$$

$$\mathbf{Q}_{IX} = \sum_{if}^{force\,on\,Ip} \frac{\partial \mathbf{r}_{OifO}^T}{\partial \mathbf{q}_{IX}} \mathbf{F}_{Oif}$$

$$\mathbf{Q}_{I\omega} = \sum_{if}^{forces\,on\,Ip} \frac{\partial \mathbf{r}_{OifO}^T}{\partial \mathbf{q}_{I\theta}} \mathbf{F}_{Oif} + \sum_{it}^{torques\,on\,Ip} \frac{\partial \boldsymbol{\theta}_{OitO}^T}{\partial \mathbf{q}_{I\theta}} \mathbf{T}_{Oit}$$

$$\mathbf{Q}_{I\omega} = \sum_{if}^{forces\,on\,Ip} \frac{\partial \mathbf{r}_{OifO}^T}{\partial \mathbf{q}_{IE}} \frac{\partial \mathbf{q}_{IE}}{\partial \mathbf{q}_{I\theta}} \mathbf{F}_{Oif} + \sum_{it}^{torques\,on\,Ip} \frac{\partial \boldsymbol{\theta}_{OitO}^T}{\partial \mathbf{q}_{I\theta}} \mathbf{T}_{Oit}$$

$$\mathbf{Q}_{I\omega} = \sum_{if}^{forces\,on\,Ip} \frac{\partial \mathbf{r}_{OifO}^T}{\partial \mathbf{q}_{IE}} \frac{\partial \dot{\mathbf{q}}_{IE}}{\partial \boldsymbol{\omega}_{IaIpIa}} \mathbf{F}_{Oif} + \sum_{it}^{torques\,on\,Ip} \frac{\partial \boldsymbol{\omega}_{OitO}^T}{\partial \boldsymbol{\omega}_{IaIpIa}} \mathbf{T}_{Oit}$$

$$\mathbf{Q}_{I\omega} = \sum_{if}^{forces\,on\,Ip} \frac{\partial \mathbf{r}_{OifO}^T}{\partial \mathbf{q}_{IE}} \frac{1}{2} [\mathbf{B}]_{ap}^T \mathbf{F}_{Oif} + \sum_{it}^{torques\,on\,Ip} [\mathbf{A}]_{Oa}^T \mathbf{T}_{Oit}$$

$\mathbf{Q}_s = \mathbf{0}$ since changing $\mathbf{s}$ changes the point of application of force which is not valid for virtual work.

Generalized Force due to Reaction, $\mathbf{Q}_{re}$

Let the reaction force be applied at a material point of part J which coincide instantaneously with the point of application of the action on part I. Hence,

$$\mathbf{r}_{OJeO} = \mathbf{r}_{OJpO}(t, \mathbf{q}_{XJ}) + [\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp} \mathbf{r}_{JpIeJp}(t, \mathbf{q}_J, \mathbf{s}_J, \mathbf{q}_I, \mathbf{s}_I) = \mathbf{r}_{OIeO}(t, \mathbf{q}_I, \mathbf{s}_I)$$

$$\mathbf{F}_{JeO} = -\mathbf{F}_{IeO}$$

$$\mathbf{T}_{JeO} = -\mathbf{T}_{IeO}$$

$$\mathbf{r}_{JpIeJp}(t, \mathbf{q}_J, \mathbf{s}_J, \mathbf{q}_I, \mathbf{s}_I) = [\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp}^T \{\mathbf{r}_{OIeO}(t, \mathbf{q}_I, \mathbf{s}_I) - \mathbf{r}_{OJpO}(t, \mathbf{q}_{XJ})\}$$

$$\frac{\partial \mathbf{r}_{JpIeJp}}{\partial \mathbf{q}_{XJ}} = -[\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp}^T \frac{\partial \mathbf{r}_{OJpO}(t, \mathbf{q}_{XJ})}{\partial \mathbf{q}_{XJ}}$$

$$\frac{\partial \mathbf{r}_{JpIeJp}}{\partial \mathbf{q}_{JE}} = \frac{\partial [\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp}^T}{\partial \mathbf{q}_{JE}} \{\mathbf{r}_{OIeO}(t, \mathbf{q}_I, \mathbf{s}_I) - \mathbf{r}_{OJpO}(t, \mathbf{q}_{XJ})\}$$

$\mathbf{r}_{JpIeJp}$ is assumed constant for the definition of $\mathbf{Q}_{re}$ since Je must be a material point.

However, the partials wrt $\mathbf{q}_I$ are need for Newton Raphson.

$$\frac{\partial \mathbf{r}_{JpIeJp}}{\partial \mathbf{q}_{IX}} = [\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp}^T \frac{\partial \mathbf{r}_{OIeO}(t, \mathbf{q}_I, \mathbf{s}_I)}{\partial \mathbf{q}_{IX}}$$

$$\frac{\partial \mathbf{r}_{JpIeJp}}{\partial \mathbf{q}_{IE}} = [\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp}^T \frac{\partial \mathbf{r}_{OIeO}(t, \mathbf{q}_I, \mathbf{s}_I)}{\partial \mathbf{q}_{IE}}$$

Therefore

$$\mathbf{Q}_{reJX} = \frac{\partial \mathbf{r}_{OJpO}^T(t, \mathbf{q}_{JX})}{\partial \mathbf{q}_{JX}} \mathbf{F}_{JeO}$$

$$\mathbf{Q}_{reJ\omega} = \mathbf{r}_{JpIeJp}^T(t, \mathbf{q}_J, \mathbf{s}_J, \mathbf{q}_I, \mathbf{s}_I) \frac{\partial [\mathbf{A}(t, \mathbf{q}_{JE})]_{OJp}^T}{\partial \mathbf{q}_{JE}} \frac{1}{2} [\mathbf{B}]_{JaJp}^T \mathbf{F}_{JeO} + [\mathbf{A}]_{OJa}^T \mathbf{T}_{JeO}$$