Eckstein-Keane-Wolpin models*

OpenSourceEconomics

Abstract

This notes provides some background material to understand the class of Eckstein-Keane-Wolpin (EKW) models. EKW models are commonly used in the context of labor economics. These can be solved, simulated, and estimated using the `respy` package.

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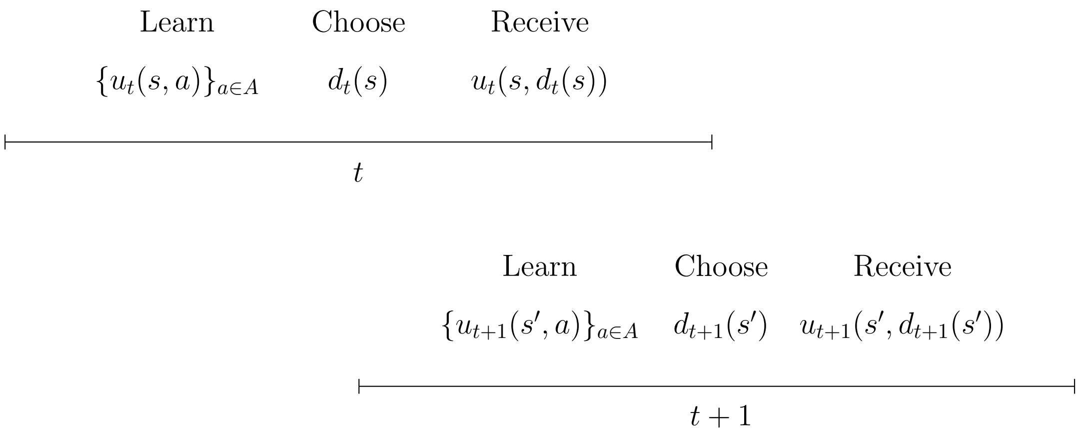
1. Introduction

This handout provides an overview regarding our efforts in the implementation ...
We now present the economic, mathematical, and computational model for the class of Eckstein-Keane-Wolpin (EKW) model. We will start with a discussion of the basic economic environment and then turn to the corresponding mathematical model of standard Markov decision process (MPD). We continue with a brief description of the estimation procedure before concluding this section with a discussion of some computational challenges when working with these models. Throughout, we will introduce acronyms and symbols as needed, but a full list of both is provided in Appendix A. The notation draws form the related work by Puterman (1994), Aguirregabiria & Mira (2010), and Arcidiacono & Ellickson (2011).

Computational challenges The implementation and analysis of this class of models entails several computational challenges. Among them integration of a high-dimensional non-differentiable function, large-scale global optimization of a noisy and non-smooth criterion function, function approximation, and parallelization strategies. We briefly outline each of them.

Stucture The remainder of this handout is structured as follows. We first present the economic model with a discussion of the basic economic environment. In Section 3 we present the corresponding mathematical model of a standard finite-horizon discrete Markov decision process (MPD) and outline its solution approach. When then outline the estimation step in Section 4. With this overview at hand, we discuss selected computational challenges in Section 5. This note concludes with an example model in Section 6.

Figure 1: Timing of events



2. Economic motivation

Basic setup At time $t = 1, \ldots, T$ each individual observes the state of the economic environment $s \in S$ and chooses an action a from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate utility $u_t(s, a)$ and the economy evolves to a new state s'. The transition from s to s' is affected by the action. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate utility. Instead, they take the future consequences of their current action into account.

Timing of events Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of time t an individual fully learns about the immediate utility of each alternative, chooses one of them, and receives its immediate utility. Then the state evolves from s to s' and the process is repeated in $t + 1$.

Decision rule A decision rule d_t specifies the action at a particular time t for any possible state. A policy $\pi \equiv (d_1, \ldots, d_T)$ provides the individual with a prescription for choosing an action in any possible future state. It is a sequence of decision rules and its implementation generates a sequence of utilities. The evolution of states over time is at least partly unknown as future utilities depend on, for example, shocks to preferences. Let X_t denote the random variable for the state at time t. Individuals use models about their economic environment to inform their beliefs about the future. For a given model, individuals thus face risk as each induces a unique objective transition probability distribution $p_t(s, a)$ for the evolution of state s to s' that depends on the action a.

Decision theory Individuals make their decisions facing risk and have rational expectations (Muth, 1961; Lucas, 1972) as their model about the future also turns out to be true. In this case, there is a consensus that rational choices are expressed by expected utility preferences (Bernoulli, 1738; von Neumann & Morgenstern, 1944, 1947).

Formalization Individuals maximize their expected total discounted utility (Samuelson, 1937; Koopmans, 1960). A constant discount factor ensures dynamic consistency of preferences as the individual's future actions agree with the planned-for contingencies. Beliefs are updated according to Bayes's rule.[1]

Equation (1) provides the formal representation of the individual's objective. Given an initial state s, individuals seek to implement the optimal policy π^* from the set of all possible policies Π that maximizes the expected total discounted utility over all T decision periods $v_1^{\pi^*}(s)$ given the information available at the time $\mathcal{I}_1$.

$$v_1(s_t) = \max_{\pi \in \Pi} \mathrm{E}_{s_1}^{\pi} \left[\sum_{t=1}^{T} \delta^{t-1} u_t(s_t, a_t) \middle| \, \mathcal{I}_1 \right] \tag{1}$$

The exponential discount factor $0 < \delta < 1$ captures a preference for immediate over future utility. The superscript of the expectation emphasizes that each policy π induces a different unique probability distribution over the sequences of utilities.

[1]See Frederick et al. (2002) for a critical review of the literature on time discounting and time preference. Fang & Silverman (2009), Fang & Yang (2015), and Chan (2017) are examples of hyperbolic discounting and thus potentially time-inconsistent preferences in settings similar to the one discussed here.

3. Mathematical formulation

EKW models are set up as a standard MDP where there is a unique transition probability distribution $p_t(s, a)$ associated with each state and action.[2]

When making sequential decisions, the task is to determine the optimal policy π^* with the largest expected total discounted utility $v_1^{\pi^*}$ as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of utilities and the probability that each occurs. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems.

Let $v_t^{\pi}(s)$ denote the expected total discounted utility under π from period t onwards:

$$v_t^{\pi}(s) = \mathrm{E}_s^{\pi}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} u_\tau(X_\tau, d_\tau(X_\tau))\right].$$

Then $v_1^{\pi}(s)$ can be determined for any policy by recursively evaluating equation (2):

$$v_t^{\pi}(s) = u_t(s, d_t(s)) + \delta \mathrm{E}_s^{p}\left[v_{t+1}^{\pi}(X_{t+1})\right]. \tag{2}$$

Equation (2) expresses the utility $v_t^{\pi}(s)$ of adopting policy π going forward as the sum of its immediate utility and all expected discounted future utilities.

The principle of optimality (Bellman, 1957; Puterman, 1994) allows to construct the optimal policy π^* by solving the optimality equations for all s and t in equation (3) recursively:

$$\begin{aligned} v_t^{\pi^*}(s) &= \max_{a \in A}\left\{u_t(s, a) + \delta \mathrm{E}_s^{p}\left[v_{t+1}^{\pi^*}(X_{t+1})\right]\right\} && (3)\\ &= \max_{a \in A}\left\{u_t(s, a) + \delta \int_S v_{t+1}^{\pi^*}(u)\, p(u \mid a, s)\, \mathrm{d}u\right\}. && (4) \end{aligned}$$

The value function $v_t^{\pi^*}$ is the expected discounted utility in t over the remaining time horizon assuming the optimal policy is implemented going forward.

The optimal decision is simply the alternative with the highest value:

$$d_t^{\pi^*}(s) = \arg\max_{a \in A}\left\{u_t(s, a) + \delta \mathrm{E}_s^{p}\left[v_{t+1}^{\pi^*}(X_{t+1})\right]\right\}$$

[2] See Puterman (1994) and White (1993) for a textbook introduction to standard Markov decision processes and Rust (1994) for a review of its use in structural estimation.

Solution approach Algorithm (1) allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate utility in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively as the calculation of their expected future utility is straightforward given the relevant transition probability distribution.

Algorithm 1 Backward Induction Algorithm for MDP

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for $t = T, \ldots, 1$ **do**

  if t == T **then**

$$v_T^{\pi^*}(s) = \max_{a \in A} \Big\{ u_T(s, a) \Big\} \qquad \forall\, s \in S$$

  else

   Compute $v_t^{\pi^*}(s)$ for each $s \in S$ by

$$v_t^{\pi^*}(s) = \max_{a \in A} \Big\{ u_t(s, a) + \delta \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \Big\}$$

   and set

$$d_t^{\pi^*}(s) = \arg\max_{a \in A} \Big\{ u_t(s, a) + \delta \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \Big\}.$$

  end if

end for

Issues to address

- The variable X is the same as x denoting the observable components of the state space.
- The $p(u \mid \ldots)$ is misleading as u is also the utility

4. Estimation procedures

Available information The econometrician has access to panel data for N individuals. For every observation (i,t) in the panel data set, the researcher observes action a_{it} and a subvector x_{it} of the state vector. Therefore, from an econometricians point of view, it is useful to distinguish between two types of state variables $s = (x, \epsilon)$. Variables x that are observed by the econometrician and the individual and those that are only observed by the individual ϵ. In addition, also some realizations of the rewards $r_{it} = r(x_{it}, \epsilon_{it}, a_{it})$.

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, 2, \ldots, N; t = 1, \ldots, T_i\}\},$$

where T_i is the number of observations over which we observe individual i.

Parameterization The models need to be parameterized functional forms and the distribution of unobservables. EKW models are calibrated to obtain information on structural parameters of preferences and the transition probabilities that reproduce key economic patterns of interest in observed sources such as administrative data sets and the like. This allows to assess the quantitative importance of competing economic mechanisms and then predict the effects of public policies. We collect all parameters of the model in θ.

Procedures We briefly outline maximum likelihood estimation (Fisher, 1922) and the method of simulated moments (McFadden, 1989). Whatever the estimation criterion, in order to evaluate it for a particular value of θ it is necessary to know the optimal decision rules d^{π^*}. Therefore at each trial value of θ the dynamic programming model needs to be solved. More detailed information is available in the excellent textbooks by Davidson & MacKinnon (2003) and Gourieroux & Monfort (1996).

Likelihood-based estimation The individual chooses the alternative with the highest total value $d^*(s)$ which is determined by the complete state space and rewards are also determined by s. However, the econometrician only observes the subset x and thus can only determine the probability $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$ of individual i at time t choosing d_{it} and receiving r_{it} given x_{it}.

$$\mathcal{L}(\theta \mid \mathcal{D}) = \prod_{i=1}^{N} \prod_{t=1}^{T} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$$

The goal of likelihood-based estimation is to find the value of the model parameters θ that maximize the likelihood function:

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta \mid \mathcal{D})$$

Simulation-based estimation ...

The goal of simulation-based estimation

5. Computational challenges

The class of EKW models has several key characteristics that distinguish it from other classes of models but entail several computational complications. We will first discuss its key features and then describe some of the computational challenges.

Structure of integral To clarify the structure of the integral determining the future value of a state, it is useful to consider the optimality equation in the second to last period. This allows to focus on action-specific rewards instead of future values.

$$v_{T-1}^{\pi^*}(s) = \max_{a \in A} \left\{ u_{T-1}(s,a) + \delta \int_S v_T^{\pi^*}(u)\, p(u \mid a, s)\, \mathrm{d}u \right\} \tag{5}$$

$$= \max_{a \in A} \left\{ u_{T-1}(s,a) + \delta \int_S \max_{a \in A}\{u(u,a)\}\, p(u \mid a, s)\, \mathrm{d}u \right\}. \tag{6}$$

The evaluation of such an integral is required millions of times during the backward induction procedure. The current practice is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities (Judd & Skrainka, 2011).

Let's consider a typical integral from Keane & Wolpin (1997). There this integral has five dimensions and takes the following form.

$$\mathcal{I}(x) = \int_{\epsilon_1} \cdots \int_{\epsilon_5} \max \left\{ f_1(x,\epsilon_1), f_2(x,\epsilon_2), f_3(x,\epsilon_3), f_4(x) + \epsilon_4, f_5(x) + \epsilon_5 \right\} \phi_{\mu,\Sigma}(\epsilon)\, \mathrm{d}\epsilon.$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_5) \sim \mathcal{N}(\mu, \Sigma)$ follows a multivariate normal distribution with mean $\mu \in \mathbb{R}^5$, covariance matrix $\Sigma \in \mathbb{R}^{5\times 5}$, and probability density function $\phi_{\mu,\Sigma}(\epsilon)$.

Global optimization ...

- **Likelihood-based estimation** This approach requires smoothing of the choice probabilities.

$$p_t(d_{it} \mid x_{it}, \theta) = \int \mathbb{I}\left[\delta(x_{it}, \epsilon_{it}, \theta) = a_{it}\right] g(\epsilon)\, \mathrm{d}\epsilon$$

- **Simulation-based estimation** This approach requires the optimization of a noisy function.

Uncertainty quantification ...

Function approximation ...

Parallelization . . .

6. Example

We now provide either Keane & Wolpin (1994) or Keane & Wolpin (1997) here. Given the purpose of this note as a high level overview, it is probably fine to just restrict to (Keane & Wolpin, 1994).

A. Acronyms and Symbols

Table 1: List of Acronyms

Acronym	**Meaning**
MDP	standard Markov decision process

Table 2: List of Symbols

Symbol	**Meaning**
$\mathbb{R}$	set of real numbers
$\mathbb{I}[A]$	indicator function that takes value one if event A is true
	Economic Model
t	decision period
T	number of decision periods
$a \in \mathcal{A}$	set of admissible actions
$s \in \mathcal{S}$	set of possible states
Δ	set of all probabilities on states
$p_t(s, a)$	conditional probability distributions for s_{t+1} when choosing action a in state s in period t
d_t	decision rule that specifies an action for all states s in period t
X_t	random variable for the state at t

Table 2: List of Symbols

Symbol	Meaning
$u_t(s, a)$	utility when choosing action a in state s in period t
δ	discount factor
v_t^π	expected total discounted utility of adopting policy π from period t going forward
$\pi \in \Pi$	set of all policies
Mathematical Model	
Computational Model	
ϵ_{at}	random shock to utility of alternative a in period t
x_{jt}	number of periods worked in occupation j by the beginning of period t
g_t	number of periods enrolled in school by the beginning of period t
$\mathcal{N}_0$	true multivariate normal distribution for random shocks
Σ	covariance matrix of random shocks
υ	admissible realization of means for future labor market shocks
α_j	parameters for utility function when working in occupation j

Table 2: List of Symbols

Symbol	Meaning
β	parameters for utility function when enrolling in school
γ	parameter for utility function when staying at home

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