Eckstein-Keane-Wolpin models

An invitation for transdiciplinary collaboration

OpenSourceEconomics*

Abstract

We present background material for a particular class of structural microeconometric models to facilitate transdiciplinary collaboration in their future development. We describe the economic framework, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models. We specify, simulate, and calibrate an example using our group's research codes `respy` and `estimagic`. We summarize our efforts to draw on research outside economics to ameliorate the computational challenges of working with these models and explore several conceptual extensions.

*Corresponding author: Philipp Eisenhauer, peisenha@uni-bonn.de. We thank Janoś Gabler, Annica Gehlen, James Heckman, Lars Hansen, Boryana Ilieva, Fedor Ishakov, Lena Janys, Tobias Raabe, John Rust, Michael Keane, and John Kennan for their valuable input.

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1 Introduction

Structural microeconometric models clearly specify an individual's objective, the economic environment, and their institutional and informational constraints under which they operate. They are calibrated to reproduce data on observed individual individual decision and experiences. Based on the results, researchers can quantify the importance of competing economic mechanisms in determining economic outcomes and forecast the effects of policy proposals (Wolpin, 2013).

We restrict us to the class of Eckstein-Keane-Wolpin (EKW) models (Adda et al., 2017; Blundell et al., 2016; Keane & Wolpin, 1997). Labor economists often apply these models for the analysis of human capital investment decisions. Human capital is the knowledge, skills, competencies, and attributes embodied in individuals that facilitate the creation of personal, social, and economic well-being (Becker, 1964). Differences in human capital attainment are a major determinant of inequality in a variety of life outcomes such as labor market success and health across and within countries (OECD, 2001).

In Bhuller et al. (2018), for example, we apply an EKW model to analyze the mechanisms determining schooling decisions in Norway. We calibrate the model using Norwegian population panel data with nearly career-long earnings histories After validating our model using an increase in mandatory schooling, we use it to study the underlying economic mechanisms that generate the resulting increase in average years of schooling and forecast the effect of several policy alternatives.

We present background material for this particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We first describe the economic framework, mathematical formulation, and calibration procedure. We then specify, simulate, and calibrate an example using our group's research codes `respy` and `estimagic`. We summarize our efforts to draw on research outside economics to ameliorate the computational challenges of working with these models and explore several conceptual extensions.

Throughout, we only offer a limited number of seminal references and textbooks that invite further study. We introduce acronyms and symbols as needed and our notation draws form the related work by Aguirregabiria & Mira (2010), Arcidiacono & Ellickson (2011), and Puterman (1994).

2 Setup

We now present the basic setup of the EKW models. We first describe the economic framework, then turn to its mathematical formulation, and finally outline the calibration procedure.

2.1 Economic framework

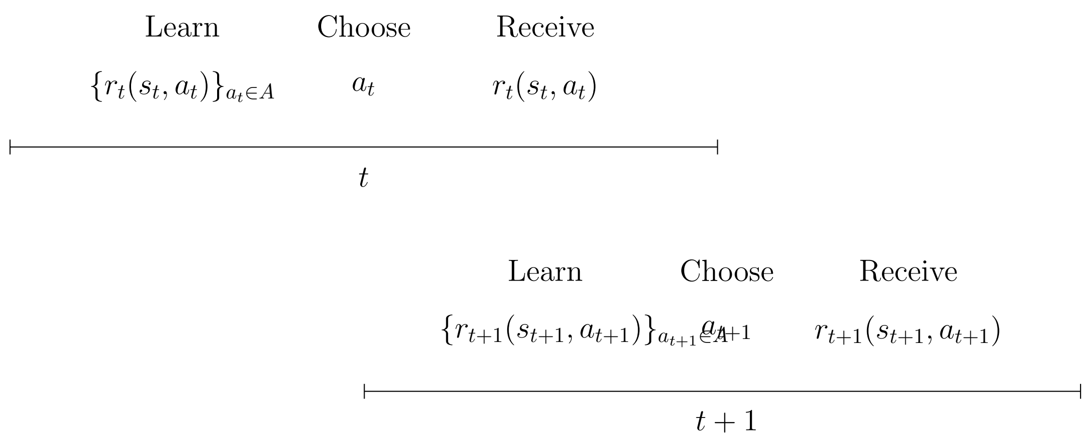
EKW models describe sequential decision-making under uncertainty (Gilboa, 2009; Machina & Viscusi, 2014). At time $t = 1, \ldots, T$ each individual observes the state of the economic environment $s_t \in S$ and chooses an action a_t from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate reward $r_t(s_t, a_t)$ and the economy evolves to a new state s_{t+1}. The transition from s_t to s_{t+1} is affected by the action but remains uncertain. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate reward. Instead, they take the future consequences of their current action into account.

A policy $\pi \equiv (a_1^\pi(s), \ldots, a_T^\pi(s))$ provides the individual with a prescription for choosing an action in any possible future state. It is a sequence of decision rules that $a_t^\pi(s)$ specify the action at a particular time t for any possible state s under π. The implementation of a policy generates a sequence of rewards. As the evolution of states over time is uncertain, individuals use a model about their economic environment to inform their subjective beliefs about the future. Individuals face risk as the model induces an objective transition probability distribution $p_t(s_t, a_t)$ for the evolution of state s_t to s_{t+1} that depends on the action a_t. Individuals have rational expectations (Lucas, 1972; Muth, 1961) so their subjective beliefs correspond to the objective transition probabilities.

Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of period t an individual fully learns about the immediate reward of each alternative, chooses one of them, and receives its immediate reward. Then the state evolves from s_t to s_{t+1} and the process is repeated in $t + 1$. Individuals make their decisions facing uncertainty and they maximize the expected total discounted rewards. An exponential discount factor $0 < \delta < 1$ parameterizes their time preference and captures a taste for immediate over future rewards.

Equation (1) provides the formal representation of the individual's objective. Given an initial state s_1, individuals seek to implement the policy π from the set of all possible policies Π that maximizes the expected total discounted rewards over all T decision periods given the

Figure 1: Timing of events



information $\mathcal{I}_1$ available in the first period.

$$\max_{\pi \in \Pi} \mathrm{E}_{s_1}^{\pi} \left[\sum_{t=1}^{T} \delta^{t-1} r_t(s_t, a_t^{\pi}(s_t)) \,\middle|\, \mathcal{I}_1 \right] \tag{1}$$

The superscript of the expectation emphasizes that each policy π induces a different unique probability distribution over the sequences of rewards. Note that in slight abuse of notation s_{t+1} is a random variable given the information available at $\mathcal{I}_t$ and $a_t^{\pi}(s_t)$ denotes the actual action that an individual chooses in time t if they encounter s_t and follow policy π.

2.2 Mathematical formulation

EKW models are set up as a standard Markov decision processes (MDP) (Puterman, 1994; White, 1993). When making sequential decisions under uncertainty, the task is to determine the optimal policy π^* with the largest expected total discounted rewards $v_1^{\pi^*}(s_1)$ as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of rewards and to weight each by the probability that they occur. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems.

Optimal decisions in a MDP are a deterministic function of the current state s only, i.e. an optimal decision rule is always deterministic and Markovian. We restrict our notation to this special case right from the beginning.

The value function $v_t^{\pi}(s_t)$ captures the expected total discounted rewards under π from period

t onwards when in s_t:

$$v_t^\pi(s_t) \equiv \mathrm{E}_{s_t}^\pi \left[\sum_{j=0}^{T-t} \delta^j \, r_{t+j}(s_{t+j}, a_{t+j}^\pi(s_{t+j})) \,\middle|\, \mathcal{I}_t \right]$$

Then $v_1^\pi(s_1)$ can be determined for any policy by recursively evaluating equation (2):

$$v_t^\pi(s_t) = r_t(s_t, a_t^\pi(s_t)) + \delta \, \mathrm{E}_{s_t}^\pi \left[v_{t+1}^\pi(s_{t+1}) \,\middle|\, \mathcal{I}_t \right]. \tag{2}$$

Equation (2) expresses the total value $v_t^\pi(s_t)$ of adopting policy π going forward as the sum of its immediate rewards and all expected discounted future rewards.

The principle of optimality allows to construct the optimal policy π^* by solving the optimality equations (3) for all s and t recursively:

$$v_t^{\pi^*}(s_t) = \max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi^*} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}. \tag{3}$$

The optimal value function $v_t^{\pi^*}$ is the expected discounted rewards in t over the remaining time horizon assuming the optimal policy is implemented going forward. The optimal decision is simply to choose the alternative with the highest total value:

$$a_t^{\pi^*}(s_t) = \arg\max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi^*} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}.$$

Algorithm 1 allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate rewards in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively following equation (3) as the calculation of their expected future rewards is straightforward given the relevant transition probabilities.

Algorithm 1 Backward induction procedure

for $t = T, \ldots, 1$ **do**

  if t == T **then**

    $v_T^{\pi^*}(s_T) = \max_{a_T \in A} \Big\{ r_T(s_T, a_T) \Big\} \qquad \forall\, s_T \in S$

  else

    Compute $v_t^{\pi^*}(s_t)$ for each $s_t \in S$ by

$$v_t^{\pi^*}(s_t) = \max_{a_t \in A} \Big\{ r_t(s_t, a_t) + \delta\, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \Big\}$$

    and set

$$a_t^{\pi^*}(s_t) = \arg\max_{a_t \in A} \Big\{ r_t(s_t, a_t) + \delta\, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \Big\}.$$

  end if

end for

2.3 Calibration procedure

EKW models are calibrated to data on observed individual decisions and experiences to back out information on reward functions, preference parameters, and transition probabilities (Davidson & MacKinnon, 2003; Gourieroux & Monfort, 1996). Given this information, the quantitative importance of competing economic mechanisms can be assessed and the effects of alternative public policies forecasted. This requires the parameterization of all elements of the model which we collect in θ.

The econometrician has access to observations for $i = 1, \ldots, N$ individuals in each time period t. For every observation (i, t) in the data, the researcher observes action a_{it} and a subset x_{it} of the state s_{it}. Therefore, from an researcher's point of view, we need to distinguish between two types of state variables $s_{it} = (x_{it}, \epsilon_{it})$. Variables x_{it} are observed by the econometrician and the individual i at time t, while ϵ_{it} are only observed by the individual. In addition, some realizations of the rewards $r_{it} = r_t(x_{it}, \epsilon_{it}, a_{it})$ are recorded as well. In summary, the data $\mathcal{D}$ contains:

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, 2, \ldots, N; t = 1, \ldots, T_i\},$$

where T_i is the number of observations for which we observe individual i.

Different calibration procedures exist that address particularities of the available data. We briefly outline likelihood-based and simulation-based calibration Whatever the calibration criterion, in order to evaluate it for every candidate parameterization of the model $\hat{\theta}$, it is necessary to solve the model anew and construct the optimal policy π^*.

Likelihood-based The individual chooses the alternative with the highest total value $a_t^{\pi^*}(s_t)$ which is determined by the complete state s_t. However, researchers only observe the subset x_t. Given parametric assumptions about the distribution of ϵ, we can determine the probability $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$ of individual i at time t choosing a_{it} and receiving r_{it} given x_{it}.

The likelihood function $\mathcal{L}(\theta \mid \mathcal{D})$ captures the probability of the observed data as a function of θ and the goal of likelihood-based estimation is to find the value of the model parameters θ that maximizes it

$$\hat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max} \underbrace{\prod_{i=1}^{N} \prod_{t=1}^{T_i} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)}_{\mathcal{L}(\theta|\mathcal{D})}$$

Simulation-based

- This issue is under active investigation in our group and this section will be fleshed out further by Annica Gehlen as our work progresses.

3 Example

We now provide an illustration of an EKW model. We start with an outline and discussion of a model of occupational choice by Keane & Wolpin (1994). We then discuss its specification, simulation, and calibration using our group's research codes `respy` and `estimagic`.

3.1 Keane & Wolpin (1994)

Individuals live for a total of T periods and choose each period t to either work in one of two occupations ($a_t = 1, 2$), attend school ($a_t = 3$), or stay at home ($a_t = 4$). Immediate rewards are determined as follows:

$$r(s_t, a_t) = \begin{cases} w_{1t} = \exp\{\alpha_{10} + \alpha_{11} g_t + \alpha_{12} e_{1t} + \alpha_{13} e_{1t}^2 + \alpha_{14} e_{2t} + \alpha_{15} e_{2t}^2 + \epsilon_{1t}\} & \text{if } a_t = 1 \\ w_{2t} = \exp\{\alpha_{20} + \alpha_{21} g_t + \alpha_{22} e_{1t} + \alpha_{23} e_{1t}^2 + \alpha_{24} e_{2t} + \alpha_{25} e_{2t}^2 + \epsilon_{2t}\} & \text{if } a_t = 2 \\ \beta_0 - \beta_1 \mathbb{I}[\, g_t \geq 12\,] - \beta_2 \mathbb{I}[\, a_{t-1} \neq 3\,] + \epsilon_{3t} & \text{if } a_t = 3 \\ \gamma_0 + \epsilon_{4t} & \text{if } a_t = 4. \end{cases}$$

g_t is the number of periods of schooling obtained by the beginning of period t, e_{1t} and e_{2t} are the number of periods that the individual worked in the two occupations respectively. The reward for each labor market alternative corresponds to its wage (w_{1t}, w_{2t}) and α_1 and α_2 are thus parameters associated with the wage functions. They capture the returns to schooling and occupation-specific human capital. Turning to the rewards from school attendance, β_0 is the consumption reward of schooling, β_1 is the post-secondary cost of schooling, and β_2 is an adjustment cost associated with returning to school. The mean reward of the home alternative is denoted γ_0. The ϵ_{at}'s are alternative-specific shocks to occupational productivity, the consumption value of schooling and home time.

Given the structure of the reward functions and imposing a that the shocks are not correlated across choices or time, the state at time t is $s_t = \{g_t, e_{1t}, e_{2t}, a_{t-1}, \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$. An individual's stock of human capital is observable $\{g_t, e_{1t}, e_{2t}, a_{t-1}\}$ to the individual and the researcher. It evolves deterministically according to the following rules:

$$\begin{aligned}
e_{1,t+1} &= e_{1t} + \mathbb{I}\,[\, a_t = 1\,] \\
e_{2,t+1} &= e_{2t} + \mathbb{I}\,[\, a_t = 2\,] \\
g_{t+1} &= g_t \;\; + \mathbb{I}\,[\, a_t = 3\,].
\end{aligned}$$

The shocks $\{\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$ are only observable by the individual. They evolve randomly and follow a joint normal distribution $[\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}]^T \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ with mean zero and covariance matrix $\boldsymbol{\Sigma}$.

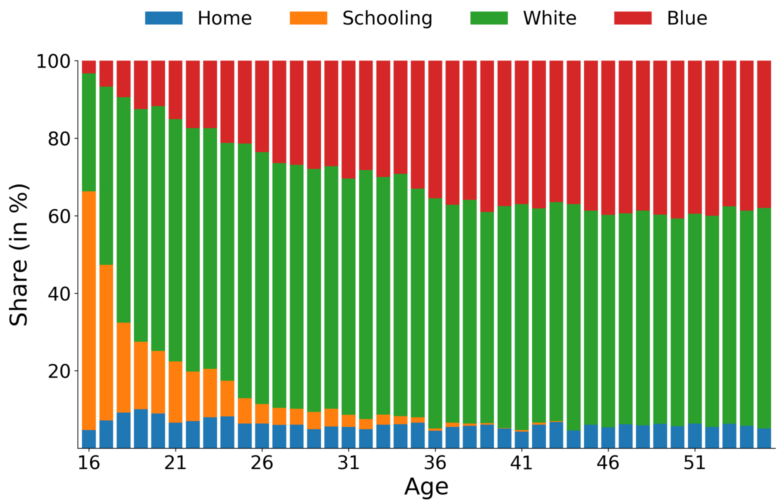
Keane & Wolpin (1994) analyze three different parameterizations of the model and we reproduce the second one for this example. Initially, individuals are identical and have no labor market experience ($e_{11} = e_{21} = 0$) but ten years of schooling ($g_1 = 10$). The basic idea is that individuals are about age 16 when entering the model and then follow it for $T = 40$ years until retirement. Different choices over the life cycle are then simply the cumulative effects of different shocks.

Schooling increases wages by only 4% in the first occupation compared to 8% in the second. We will thus refer to the former as blue-collar and the latter as white-collar going forward. Starting wages are considerably lower in the white-collar sector, but wages increase more rapidly with occupation-specific experience compared to blue-collar wages. Own-work experience is highly valuable in both occupations. However, while white-collar wages increase with blue-collar experience as well, the opposite is not true. There is a consumption value of schooling of \$5,000, but the total cost of pursuing post-secondary education is considerable and amounts to \$5,000. Once leaving school, individuals incur a nearly prohibitive cost of \$15,000 for re-enrolling. Individuals are forward-looking with a discount factor of 0.95.

We simulate the life cycle histories of 1,000 individuals. Figure 2 shows the share of individuals choosing each of the four alternatives by period. Initially, roughly 60% of individuals enroll in school, but this share declines rapidly, and only 19% attain any post-secondary education.

Right away, about 35% of individuals are working in the white-collar occupation. White-collar employment initially increases even further to peak at 67% as individuals are leaving school and entering the labor market. Blue-collar employment steadily rises over the life cycle but never reaches more than 40%. About 5% of individuals stay at home each period.

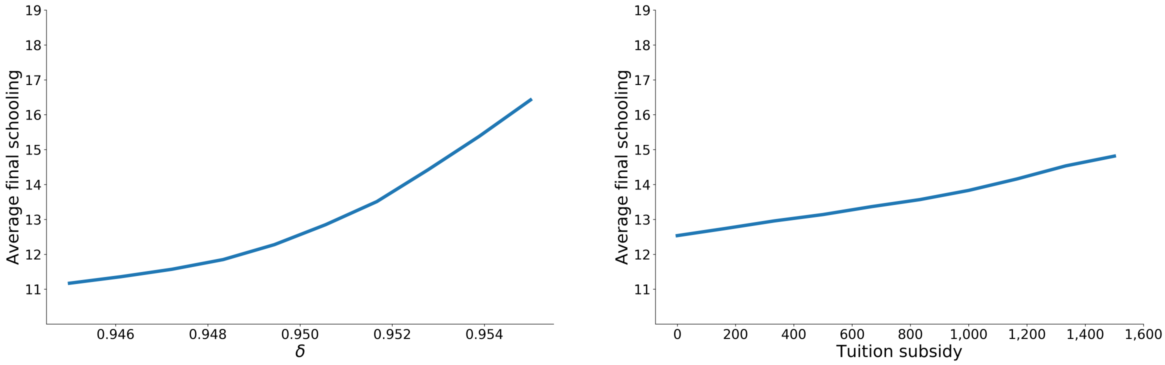
Figure 2: Choices over the life cycle



Overall, the level of average final schooling is slightly above a high school degree with 12.6 years. Individuals incur the immediate costs of their schooling investments in the form of tuition and foregone earnings right at the beginning of their life cycle. Doing so maximizes their ability to reap the reward of increased wages over the remaining time periods.

Figure 3 illustrates the ability of structural economic models to quantify the impact of economic mechanisms and to forecast the effect of public policies. On the left, we vary time preferences parameterized by δ between 0.945 and 0.955 while we reduce β_1 by the size of a tuition subsidy of up to $\$1,500$ on the right. In both cases, we are interested in the effect on average final schooling.

Figure 3: Economic mechanism and policy forecast



(a) Time preference

(b) Tuition subsidy

Increases in the discount factor and the tuition subsidy both result in increased levels of average final schooling. However, they do so very different reasons. While individuals put more emphasis on the future benefits of their schooling investment in the former, they react to a reduction in its immediate cost in the latter.

3.2 `respy` and `estimagic`

Our research code `respy` allows to flexibly specify and simulate EKW models, while the can use our groups optimization toolbox `estimagic` for their calibration. We briefly showcase a typical workflow for such an analysis. Instructions on how to use the packages, obtain the source code, assess the testing harness, replicate several seminal papers, and other implementation details are available in their respective online documentations at `https://respy.readthedocs.io` and `https://estimagic.readthedocs.io`.

Figure 4 illustrates a typical workflow. Initially, the user provides the observed data, the parameterization of the model, and other options. All together define the structure of the model, and we can construct the functionality for the evaluation of the likelihood function and the simulation of the data. We can use `estimagic` for the calibration of the the model and then study the properties of data simulated based on the results.

Figure 4: Typical workflow

```python
from estimagic.optimization.optimize import maximize
import respy as rp

# obtain model input
df, params, options = get_model_input()

# process model
crit_func = rp.get_crit_func(params, options, df)
simulate = rp.get_simulate_func(params, options)

# perform likelihood-based calibration and simulate dataset
results, params_rslt = maximize(crit_func, params, "nlopt_bobyqa")
df_rslt = simulate(params_rslt)

# conduct analysis
...
```

Figure 5 shows the model specification files for Keane & Wolpin (1994). The file on the left deals with the parameterization of the model. We set the coefficient values for the reward functions and the distribution of the unobservable state variables. On the right, we provide details on the construction of the observed state variables and tuning parameters for numerous numerical methods.

Figure 5: Model specification

kw_94_two.csv

```
category,name,value,comment
delta,delta,0.95,discount factor
wage_a,constant,9.21,log of rental price
wage_a,exp_edu,0.04,return to an additional year of schooling
wage_a,exp_a,0.033,return to same sector experience
wage_a,exp_a_square,-0.0005,"return to same sector, quadratic experience"
wage_a,exp_b,0,return to other sector experience
wage_a,exp_b_square,0,"return to other sector, quadratic experience"
wage_b,constant,8.2,log of rental price
wage_b,exp_edu,0.08,return to an additional year of schooling
wage_b,exp_b,0.067,return to same sector experience
wage_b,exp_b_square,-0.001,"return to same sector, quadratic experience"
wage_b,exp_a,0.022,return to other sector experience
wage_b,exp_a_square,-0.0005,"return to other sector, quadratic experience"
nonpec_edu,constant,5000,constant reward for choosing education
nonpec_edu,at_least_twelve_exp_edu,-5000,"reward for going to college (tuition, etc.)"
nonpec_edu,not_edu_last_period,-15000,reward for going back to school
nonpec_home,constant,14500,constant reward of non-market alternative
shocks_sdcorr,sd_a,0.4,"Element 1,1 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_b,0.5,"Element 2,2 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_edu,6000,"Element 3,3 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_home,6000,"Element 4,4 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_b_a,0,"Element 2,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_edu_a,0,"Element 3,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_edu_b,0,"Element 3,2 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_a,0,"Element 4,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_b,0,"Element 4,2 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_edu,0,"Element 4,3 of standard-deviation/correlation matrix"
lagged_choice_1_edu,probability,1,Probability that the first lagged choice is education (simulation only)
initial_exp_edu_10,probability,1,Probability that the initial level of education is 10
maximum_exp,edu,20,"Maximum level of experience for education (optional, reduces computation complexity)"
```

(a) Parameterization

kw_94_two.yaml

```yaml
estimation_draws: 200
estimation_seed: 500
estimation_tau: 500
interpolation_points: -1
n_periods: 40
simulation_agents: 1000
simulation_seed: 132
solution_draws: 500
solution_seed: 456
monte_carlo_sequence: random
core_state_space_filters:
  # In periods > 0, if agents accumulated experience only in one choice, lagged choice
  # cannot be different.
  - "period > 0 and exp_{i} == period and lagged_choice_1 != '{i}'"
  # In periods > 0, if agents always accumulated experience, lagged choice cannot be
  # non-experience choice.
  - "period > 0 and exp_a + exp_b + exp_edu == period and lagged_choice_1 == '{j}'"
  # In periods > 0, if agents accumulated no years of schooling, lagged choice cannot
  # be school.
  - "period > 0 and lagged_choice_1 == 'edu' and exp_edu == 0"
  # If experience in choice 0 and 1 are zero, lagged choice cannot be this choice.
  - "lagged_choice_1 == '{k}' and exp_{k} == 0"
  # In period 0, agents cannot choose occupation a or b or mil.
  - "period == 0 and lagged_choice_1 == '{k}'"
covariates:
  constant: "1"
  exp_a_square: exp_a ** 2
  exp_b_square: exp_b ** 2
  at_least_twelve_exp_edu: exp_edu >= 12
  not_edu_last_period: lagged_choice_1 != 'edu'
```

(b) Options

4 Improvements

The implementation of EKW models poses several computational challenges. Among them numerical integration, global optimization, function approximation, and efficient parallelization. We now describe some of our efforts to align `respy` and `estimagic` with the state-of-the-art in computational methods. We have concluded our own preparatory work and actively seek input from domain experts for further improvements and subsequent joint publication.

4.1 Numerical integration

In Gabler, Eisenhauer, & Suchy (2020) we draw on the extensive literature in applied math on numerical integration (Davis & Rabinowitz, 2007; Gerstner & Griebel, 1998).

To clarify the structure of the integral determining the future value of a state, it is useful to consider the optimality equation in a generic time period t. Let $v_t^\pi(s_t, a_t)$ denote the action-specific value function of choosing action a in state s while continuing with the optimal policy going forward.

$$\begin{aligned}
v_t^\pi(s_t, a_t) &= u(s_t, a_t) + \delta \, \mathrm{E}_{s_t}\left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \\
&= u(s_t, a_t) + \delta \int_S v_{t+1}^{\pi^*}(s_{t+1}) \, \mathrm{d}p_t(a_t, s_t) \\
&= u(s_t, a_t) + \delta \underbrace{\int_S \max_{a \in A} \left\{ v_{t+1}^\pi(s_{t+1}, a_{t+1}) \right\} \mathrm{d}p_t(a_t, s_t)}_{\mathcal{I}(a_{t+1})}.
\end{aligned}$$

We need to solve the integral $\mathcal{I}(a_{t+1})$ millions of times during the backward induction procedure. The current practice is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities (Judd & Skrainka, 2011).

Let's consider an atemporal version of the typical integral from Keane & Wolpin (1994). As outlined earlier, individuals can choose among four alternatives. Each of the alternative-specific rewards is in part determined by a random continuous state variable that follows a normal distribution. The transition of all other state variables is deterministic. This results in a four-dimensional integral of the following form:

$$\int_\epsilon \max_{a \in A} \left\{ v_{t+1}^\pi(x_{t+1}, \epsilon, a) \right\} \phi_{\mu,\Sigma}(\epsilon) \, \mathrm{d}\epsilon.$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_4) \sim \mathcal{N}(\mu, \Sigma)$ follows a multivariate normal distribution with mean $\mu \in \mathbb{R}^4$, covariance matrix $\Sigma \in \mathbb{R}^{4\times 4}$, and probability density function $\phi_{\mu,\Sigma}$.

4.2 Global optimization

In Gabler, Eisenhauer, & Röhl (2020), we draw on the specialized literature on global optimization to assess and improve the reliability of the model calibration (Locatelli & Schoen, 2013; Nocedal & Wright, 2006).

We conduct a benchmarking exercise using Keane & Wolpin (1994, 1997) as a well-known and empirically-grounded test case. Depending on the calibration procedure, particular challenges arise. For example, likelihood-based estimation requires smoothing of the choice probabilities, while simulation-based calibration involves noisy function optimization.

5 Extensions

We are actively pursuing several extensions to the basic setup and standard analysis of EKW models. For example, we draw on the methodological literature on robust-decision making and uncertainty quantification to account for the uncertainties in our analysis. Also, we are working with the German Institute for Economic Research and Statistics Norway to improve the available data for the calibration of the model models. Again, we have concluded our own preparatory work and actively seek input from domain experts for further improvements and subsequent joint publication.

5.1 Robust decision-making

The uncertainties involved in human capital investments are ubiquitous (Becker, 1964). Individuals usually make investments early in life when they are still uncertain about their abilities and tastes. Their returns also depend on demographic, economic, and technological trends that only start to unfold years from now. However, the treatment of uncertainty in life cycle models of human capital investment is very narrow. A model provides individuals with a formalized view of their economic environment and implies unique probabilities for all possible future events. Individuals have no fear of model misspecification.

In Eisenhauer & Suchy (2020), we address this shortcoming by formulating, implementing, and exploring a life cycle model of robust human capital investment where individuals face risk within a model and ambiguity about the model (Arrow, 1951). Ambiguity arises as individuals do not know the true model and consider a whole set of models as reasonable descriptions of their economic environment. Individuals fear model misspecification and thus seek robust decisions, i.e., decisions that perform well over the whole range of models.

We incorporate methods from robust optimization (Ben-Tal et al., 2009; Rahimian & Mehrotra, 2019; Wiesemann et al., 2014) and robust Markov decision processes (Iyengar, 2005; Nilim & El Ghaoui, 2005) that allow us to construct decision rules that explicitly take potential model misspecification into account.

5.2 Uncertainty quantification

There are numerous sources of uncertainties in the policy predictions inferred from a calibrated EKW model. The model is subject to misspecification, its numerical implementation introduces approximation error, the data is subject to measurement error, and the estimated parameters remain partly uncertain. However, economists display incredible certitude as they disregard all uncertainty (Manski, 2013).

In Gabler, Eisenhauer, Janys, & Mensinger (2020), we draw on a rich literature in other disciplines where a proper accounting of the uncertainty in forecasts from complex computational models is mandatory (Saltelli et al., 2004, 2008; Smith, 2014). However, uncertainty quantification for EKW models poses several unique challenges. They usually have a large number of uncertain and correlated parameters, and the quantity of interest is time consuming to compute and a complex function of the model parameters. Using methods from machine learning, we set up a surrogate model that is fast to evaluate. We analyze the seminal model of (Keane & Wolpin, 1994, 1997) to showcase our approach in a well-known and empirically-motivated setting and characterize the uncertainty in their key findings.

5.3 Model validation

In Bhuller et al. (2018), we calibrate an EKW model on Norwegian population panel data with nearly career-long earnings histories. Due to the richness of the data, we can validate the model using a mandatory schooling reform. Our data includes substantial geographic variation in compulsory schooling across Norway between 1960 and 1975 as mandatory schooling increased from 7 to 9 years at different points in time across municipalities. We split our data into an estimation and validation sample. We only use pre-reform data in our estimation, forecast the effect of increasing mandatory schooling by two years, and compare our forecast with the post-reform outcome. Doing so allows us to assess the ability of our model to extrapolate outside the support of our estimation data.

Such validation of computational models is a prerequisite for their use in other disciplines (Adams, 2012; Oberkampf & Roy, 2010). It is extremely rare in economics as drastic regime shifts are seldom available in observational data and costly to implement in large-scale experiments.

5.4 Nonstandard expectations

Eisenhauer et al. (2020), we relax the assumption that individual expectations are perfectly in line with the calibrated EKW model. Instead, we analyze and quantify the effect of biased expectations about wage growth in part-time employment on life cycle wage profiles. We design specific survey questions for the German Socio-Economic Panel (Haisken-DeNew & Frick, 2005) and elicit the expected wage trajectories for full-time and part-time employment directly. Thus, we can incorporate the belief elicitation directly in the design of a life cycle model of female labor supply.

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