

# Eckstein-Keane-Wolpin models

OpenSourceEconomics\*

## Abstract

We present background material for a particular class structural economic models to facilitate transdisciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models. We provide an example application using our group's research code **respy**. We review research outside economics to identify model components ripe for improvement and explore possible model extensions.

**JEL Codes:** J24, D81, C44

**Keywords:** life cycle model, human capital, risk, Markov decision process

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# 1. Introduction

**Focus of handout** We present background material for a particular class structural economic models to facilitate transdisciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for the so-called Eckstein-Keane-Wolpin (EKW) models ([Aguirregabiria & Mira, 2010](#)). We provide an example application using our group’s research code `respy` ([OSE, 2020](#)). We draw on research outside economics to identify model components ripe for improvement and explore possible model extensions.

**Structural models** Structural economic models clearly specify an individual’s objective and the constraints of their economic environment under which they operate. They are calibrated to reproduce data on observed individual individual behaviors. This allows to quantify the importance of competing economic mechanisms in determining economic outcomes and forecast the effects of alternative policies before their implementation ([Wolpin, 2013](#)).

**EKW models** We restrict us to the class of Eckstein-Keane-Wolpin (EKW) models. Labor economists often apply these models for the analysis of human capital investment decisions. Human capital is the knowledge, skills, competencies, and attributes embodied in individuals that facilitate the creation of personal, social, and economic well-being ([OECD, 2001](#)). Differences in human capital attainment are a major determinant of inequality in a variety of life outcomes such as labor market success and health across and within countries.

In [Bhuller et al. \(2018\)](#), for example, we apply an EKW model to analyze the mechanisms determining schooling decisions in Norway. They calibrate the model using Norwegian population panel data with nearly career-long earnings histories. They find an important role for the option value of schooling, which measures the value of the information generated by each additional year of schooling. After validating their model using an increase in mandatory schooling, they then use the model to study the underlying mechanisms that generate the increase of average years of schooling and forecast the effects of several policy alternatives.

**Structure** This handout is structured as follows. We first present the basic setup, then offer an example implementation using the `respy` package, and finally we outline possible improvements and extensions.

**Notation** Throughout, we only provide a very limited number of seminal references and mostly point to related textbooks instead. We will introduce acronyms and symbols as needed, but a full list of both is provided in the Appendix. The notation draws from the related work by [Puterman \(1994\)](#), [Aguirregabiria & Mira \(2010\)](#), and [Arcidiacono & Ellickson \(2011\)](#).

## 2. Setup

We now present the basic setup of the EKW models. We first present the economic framework, outline its mathematical formulation, and briefly describe the calibration process.

### 2.1. Economic framework

**Basic setup** EKW models describe sequential decision-making under risk (Machina & Viscusi, 2014; Gilboa, 2009). At time  $t = 1, \dots, T$  each individual observes the state of the economic environment  $s_t \in S$  and chooses an action  $a_t$  from the set of admissible actions  $\mathcal{A}$ . The decision has two consequences: an individual receives an immediate reward  $r(s_t, a_t)$  and the economy evolves to a new state  $s_{t+1}$ . The transition from  $s_t$  to  $s_{t+1}$  is affected by the action. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate reward. Instead, they take the future consequences of their current action into account.

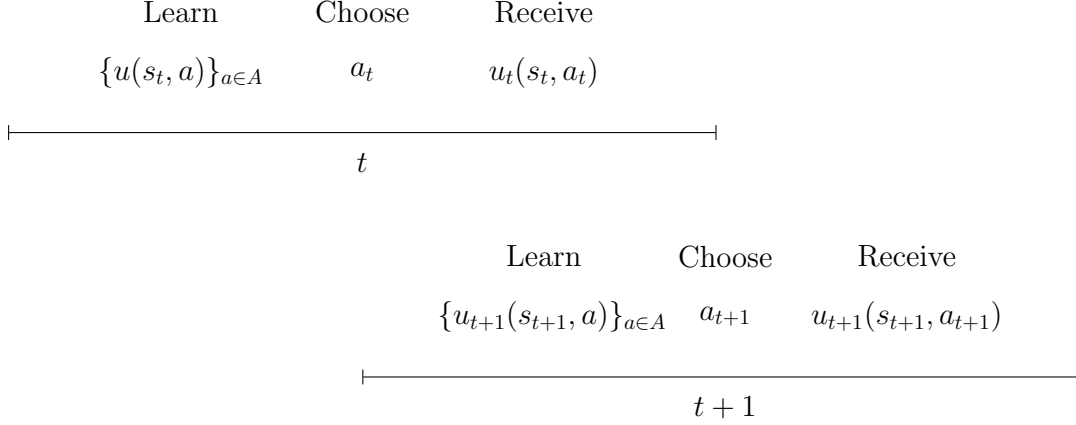
**Decision rule** A policy  $\pi \equiv (a_1^\pi(s), \dots, a_T^\pi(s))$  provides the individual with a prescription for choosing an action in any possible future state, where  $a_t^\pi(s)$  specifies the action at a particular time  $t$  for any possible state  $s$  under  $\pi$ . It is a sequence of decision rules and its implementation generates a sequence of rewards. The evolution of states over time is at least partly unknown and individuals use models about their economic environment to inform their subjective beliefs about the future. For a given model, individuals thus face risk as each induces a unique objective transition probability distribution  $p_t(s_t, a_t)$  for the evolution of state  $s_t$  to  $s_{t+1}$  that depends on the action  $a_t$ . Individuals have rational expectations (Muth, 1961; Lucas, 1972) so their subjective beliefs correspond to the objective transition probabilities.

**Timing of events** Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of time  $t$  an individual fully learns about the immediate reward of each alternative, chooses one of them, and receives its immediate reward. Then the state evolves from  $s_t$  to  $s_{t+1}$  and the process is repeated in  $t + 1$ .

**Decision theory** Individuals make their decisions facing risk and in this case, there is a consensus that rational choices are expressed by the maximization of their expected total discounted rewards. An exponential discount factor  $0 < \delta < 1$  captures a preference for immediate over future rewards and ensures dynamic consistency of preferences as the individual's future actions agree with the planned-for contingencies. Beliefs are updated according to Bayes's rule.

Equation (1) provides the formal representation of the individual's objective. Given an initial state  $s_1$ , individuals seek to implement the policy  $\pi$  from the set of all possible policies  $\Pi$  that maximizes the expected total discounted rewards over all  $T$  decision periods given the

**Figure 1:** Timing of events



information  $\mathcal{I}_1$  available.

$$\max_{\pi \in \Pi} \mathbb{E}_{s_1}^{\pi} \left[ \sum_{t=1}^T \delta^{t-1} r(s_t, a_t^{\pi}(s_t)) \mid \mathcal{I}_1 \right] \quad (1)$$

The superscript of the expectation emphasizes that each policy  $\pi$  induces a different unique probability distribution over the sequences of rewards.

## 2.2. Mathematical formulation

EKW models are set up as a standard Markov decision processes (MDP) (Puterman, 1994; White, 1993). When making sequential decisions under risk, the task is to determine the optimal policy  $\pi^*$  with the largest expected total discounted rewards  $v_1^{\pi^*}$  as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of rewards and to weight each by the probability that they occur. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems.

Note that in slight abuse of notation  $s_{t+1}$  is a random variable given the information available at  $\mathcal{I}_t$  and  $a_t^{\pi}(s_t)$  denotes the actual action that an individual chooses in time  $t$  if they encounter  $s_t$  and follow policy  $\pi$ . Optimal decisions in a MDP are a deterministic function of the current state  $s$  only, i.e. an optimal decision rule is always deterministic and Markovian. We restrict our notation to this special case right from the beginning.

Let the value function  $v_t^{\pi}(s)$  capture the expected total discounted rewards under  $\pi$  from period

$t$  onwards:

$$v_t^\pi(s_t) \equiv \mathbb{E}_{s_t}^\pi \left[ \sum_{j=0}^{T-t} \delta^j r(s_{t+j}, a_{t+j}^\pi(s_{t+j})) \mid \mathcal{I}_t \right]$$

Then  $v_1^\pi(s_t)$  can be determined for any policy by recursively evaluating equation (2):

$$v_t^\pi(s_t) = r(s_t, a_t^\pi(s_t)) + \delta \mathbb{E}_{s_t}^\pi [v_{t+1}^\pi(s_{t+1}) \mid \mathcal{I}_t]. \quad (2)$$

Equation (2) expresses the rewards  $v_t^\pi(s_t)$  of adopting policy  $\pi$  going forward as the sum of its immediate rewards and all expected discounted future rewards.

The principle of optimality allows to construct the optimal policy  $\pi^*$  by solving the optimality equations for all  $s$  and  $t$  in equation (3) recursively:

$$v_t^{\pi^*}(s_t) = \max_{a \in A} \left\{ r(s_t, a) + \delta \mathbb{E}_{s_t}^{\pi^*} [v_{t+1}^{\pi^*}(s_{t+1}) \mid \mathcal{I}_t] \right\}. \quad (3)$$

The optimal value function  $v_t^{\pi^*}$  is the expected discounted rewards in  $t$  over the remaining time horizon assuming the optimal policy is implemented going forward.

The optimal decision is simply the alternative with the highest value:

$$a_t^{\pi^*}(s_t) \equiv \arg \max_{a \in A} \left\{ r(s_t, a) + \delta \mathbb{E}_{s_t}^{\pi^*} [v_{t+1}^{\pi^*}(s_{t+1}) \mid \mathcal{I}_t] \right\}$$

**Solution approach** Algorithm 1 allows to solve the MDP by a simple backward induction procedure. In the final period  $T$ , there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate rewards in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively following equation (3) as the calculation of their expected future rewards is straightforward given the relevant transition probabilities.

---

**Algorithm 1** Backward induction procedure

---

```
for  $t = T, \dots, 1$  do
  if  $t == T$  then
     $v_T^{\pi^*}(s_T) = \max_{a \in A} \left\{ r(s_T, a) \right\} \quad \forall s_T \in S$ 
  else
    Compute  $v_t^{\pi^*}(s_t)$  for each  $s_t \in S$  by
      
$$v_t^{\pi^*}(s_t) = \max_{a \in A} \left\{ r(s_t, a) + \delta \mathbb{E}_{s_t}^{\pi} [v_{t+1}^{\pi^*}(s_{t+1}) | \mathcal{I}_t] \right\}$$

    and set
      
$$a_t^{\pi^*}(s_t) = \arg \max_{a \in A} \left\{ r(s_t, a) + \delta \mathbb{E}_{s_t}^{\pi} [v_{t+1}^{\pi^*}(s_{t+1}) | \mathcal{I}_t] \right\}.$$

  end if
end for
```

---

### 2.3. Calibration procedure

EKW models are calibrated to data to obtain information on reward functions, preference parameters, and transition probabilities (Davidson & MacKinnon, 2003; Gourieroux & Monfort, 1996). Given this information, the quantitative importance of competing economic mechanisms can be assessed and the effects of alternative public policies predicted. This requires the parameterization of all elements of the model which we collect in  $\theta$ .

**Data** The econometrician has access to observations for  $i = 1, \dots, N$  individuals in each time period  $t$ . For every observation  $(i, t)$  in this panel data set, the researcher observes action  $a_{it}$  and a subset  $x_{it}$  of the state  $s_{it}$ . Therefore, from an econometricians point of view, we need to distinguish between two types of state variables  $s_{it} = (x_{it}, \epsilon_{it})$ . Variables  $x_{it}$  that are observed by the econometrician and the individual  $i$  at time  $t$  and those that are only observed by the individual  $\epsilon_{it}$ . In addition, also some realizations of the rewards  $r_{it} = r(x_{it}, \epsilon_{it}, a_{it})$  are often observed. In summary, the data set  $\mathcal{D}$  is

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, 2, \dots, N; t = 1, \dots, T_i\},$$

where  $T_i$  is the number of observations over which we observe individual  $i$ .

**Procedures** Different calibration procedures exist that address particularities of the available data set. We briefly outline likelihood-based and simulation-based calibration. Whatever the calibration criterion, in order to evaluate it for every candidate parameterization of the model  $\theta$  it is necessary to construct the optimal policy  $\pi^*$ . Therefore at each trial value of  $\theta$  the whole model needs to be solved by the backward induction procedure outlined in Algorithm 1.

**Likelihood-based** The individual chooses the alternative with the highest total value  $a_t^{\pi^*}(s_t)$  which is determined by the complete state and rewards are also determined by  $s_t$ . However, the econometrician only observes the subset  $x_t$ . Given parametric assumptions about the distribution of  $\epsilon$ , we can determine the probability  $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$  of individual  $i$  at time  $t$  choosing  $d_{it}$  and receiving  $r_{it}$  given  $x_{it}$ .

The goal of likelihood-based estimation is to find the value of the model parameters  $\theta$  that maximize the likelihood function  $\mathcal{L}(\theta \mid \mathcal{D})$  which captures the probability of the observed data as a function of  $\theta$ .

$$\hat{\theta} \equiv \arg \max_{\theta \in \Theta} \underbrace{\prod_{i=1}^N \prod_{t=1}^{T_i} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)}_{\mathcal{L}(\theta \mid \mathcal{D})}$$

### Simulation-based

- This issue is under active investigation in our group and this section will be fleshed out further by Annica Gehlen as our work progresses.



### 3. Example

We now provide one concrete illustration of an EKW model. We start with an outline and discussion of the seminal example of a model of occupational choice by [Keane & Wolpin \(1994\)](#) and then discuss our group's research code `respy` ([OSE, 2020](#)).

#### 3.1. Keane & Wolpin (1994)

**Setup** Individuals live for a total of  $T$  periods and choose each period  $t$  to either work in one of two occupations ( $a_t = 1, 2$ ), attend school ( $a_t = 3$ ), or stay at home ( $a_t = 4$ ). The immediate reward from each alternative is the following:

$$r(s_t, a_t) = \begin{cases} w_{1t} = \exp\{\alpha_{10} + \alpha_{11}g_t + \alpha_{12}x_{1t} + \alpha_{13}x_{1t}^2 + \alpha_{14}x_{2t} + \alpha_{15}x_{2t}^2 + \epsilon_{1t}\} & \text{if } a_t = 1 \\ w_{2t} = \exp\{\alpha_{20} + \alpha_{21}g_t + \alpha_{22}x_{1t} + \alpha_{23}x_{1t}^2 + \alpha_{24}x_{2t} + \alpha_{25}x_{2t}^2 + \epsilon_{2t}\} & \text{if } a_t = 2 \\ \beta_0 - \beta_1\mathbb{I}[g_t \geq 12] - \beta_2\mathbb{I}[a_{t-1} \neq 3] + \epsilon_{3t} & \text{if } a_t = 3 \\ \gamma_0 + \epsilon_{4t} & \text{if } a_t = 4. \end{cases}$$

$g_t$  is the number of periods of schooling obtained by the beginning of period  $t$ ,  $x_{1t}$  and  $x_{2t}$  are the number of periods that the individual worked in the two occupations respectively. The reward for each labor market alternative corresponds to its wage ( $w_{1t}, w_{2t}$ ) and  $\alpha_1$  and  $\alpha_2$  are thus parameters associated with the wage functions. They capture the returns to schooling and occupation-specific human capital.  $\beta_0$  is the consumption reward of schooling,  $\beta_1$  is the post-secondary cost of schooling, and  $\beta_2$  is an adjustment cost associated with returning to school. The mean reward of the home alternative is denoted  $\gamma_0$ . The  $\epsilon_{at}$ 's are alternative-specific shocks to occupational productivity, the consumption reward of schooling, and the reward of home time. They are serially uncorrelated.

Given the structure of the reward functions and the lack of serial correlation, the state at time  $t$  is  $s_t = \{g_t, x_{1t}, x_{2t}, a_{t-1}, \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$ . While an individual's stock of human capital is observable  $\{g_t, x_{1t}, x_{2t}, a_{t-1}\}$  to the individual and the econometrician, the  $\{\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$  are only observable by the individual. The observable components evolve deterministically according to the following rules:

$$\begin{aligned} x_{1,t+1} &= x_{1t} + \mathbb{I}[a_t = 1] \\ x_{2,t+1} &= x_{2t} + \mathbb{I}[a_t = 2] \\ g_{t+1} &= g_t + \mathbb{I}[a_t = 3]. \end{aligned}$$

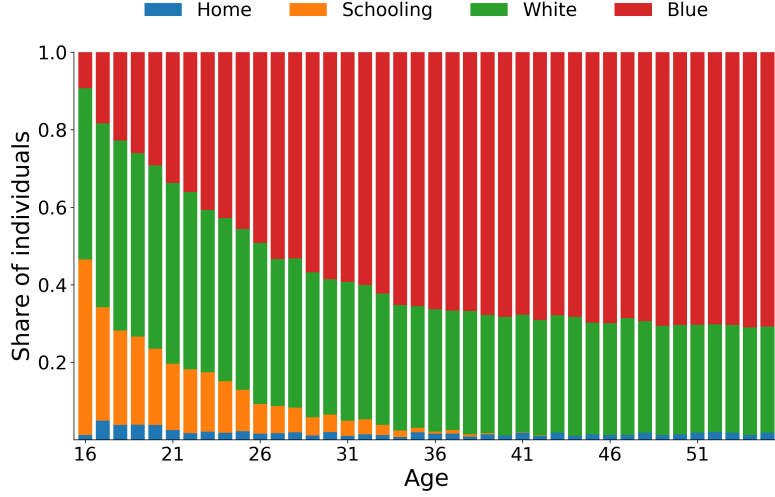
However, the unobservable components evolve randomly. They are jointly normally distributed  $[\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}]^T \sim \mathcal{N}(\mathbf{0}, \Sigma)$  with mean zero and covariance matrix  $\Sigma$ .

When entering the model, individuals have no labor market experience ( $x_{11} = x_{21} = 0$ ) but ten years of schooling ( $g_1 = 10$ ). The idea is that individuals are about age 16 when entering the model in the first period and start identically, different choices over the life cycle are then simply the cumulative effects of different shocks.

**Parameterization** We follow [Keane & Wolpin \(1994\)](#) and study this model under the following parameterization. Schooling increases wages by only 4% in the first occupation compared to 8% in the second. We will thus refer to the former as blue-collar and the latter as white-collar going forward. Starting wages are considerably lower in the white-collar sector, but wages increase more rapidly with occupation-specific experience compared to blue-collar wages. Own-work experience is highly valuable in both occupations. However, while white-collar wages increase with blue-collar experience as well, the opposite is not true. There is a consumption value of schooling of \$5,000, but the total cost of pursuing post-secondary education is considerable and amounts to \$5,000. Once leaving school, individuals incur a nearly prohibitive cost of \$15,000 for re-enrolling. Individuals are forward-looking with a discount factor of 0.95. The random shocks are uncorrelated across alternatives. Further details about the parameterization are available in our Appendix.

**Descriptives** We simulate the life cycle histories of 1,000 individuals for 40 periods. [Figure 2](#) shows the share of individuals choosing each of the four alternatives by period. Initially, roughly 52% of individuals enroll in school, but this share declines rapidly and only 19% attain any post-secondary education. Right away, about 35% of individuals are working in the blue-collar occupation. Blue-collar employment initially increases even further to peak at 67% as individuals are leaving school and entering the labor market. White-collar employment steadily rises over the life cycle but never reaches more than 35%. About 5% of individuals stay at home each period.

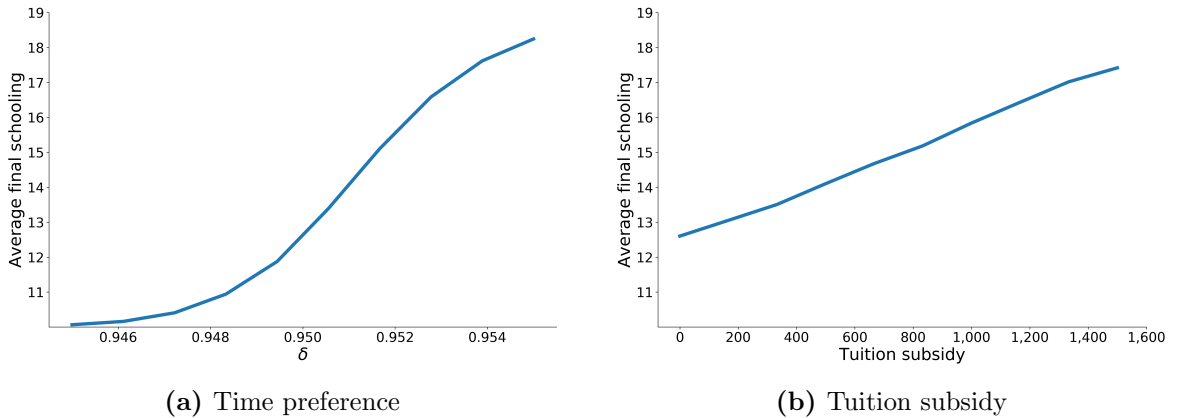
**Figure 2: Choices over the life cycle**



Overall, the level of average final schooling is slightly above a high school degree with 12.6 years. Individuals incur the immediate costs of their schooling investments in the form of tuition and foregone earnings right at the beginning of their life cycle. Doing so allows them to get the most out of their increased wages in the future.

**Economic mechanisms and policy evaluation** Figure 3 illustrates the ability of structural economic models to quantify the impact of economic mechanisms and to forecast the effect of public policies. For both cases, we are interested in the effect on average final schooling. On the left, we vary the discount factor  $\delta$  between 0.945 and 0.955 while we reduce  $\beta_1$  by the size of a tuition subsidy of up to \$1,500.

**Figure 3: Economic mechanism and policy forecast**



Increases in the discount factor and the tuition subsidy both result in increased levels of average final schooling. However, they do so very different reasons. While individuals put more emphasis on the future benefits of their schooling investment in the former, they react to a

reduction in its immediate cost in the latter.

### 3.2. respy

Our research group is actively developing the **respy** package, which allows for the flexible specification, simulation, and estimation of the typical EKW models. Instructions on how to use the package, obtain the source code, replicate several seminal papers, and details on the numerical methods are all available in its online documentation at <https://respy.readthedocs.io>.

**Workflow illustration** Figure 4 illustrates the typical workflow with the **respy** package.<sup>1</sup> The model is specified in the parameters and options specification.

**Figure 4:** Workflow example

---

```
import respy as rp

params, options, _ = rp.get_example_model("kw_94_one")
simulate = rp.get_simulate_func(params, options)
df = simulate(params)
```

---

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<sup>1</sup>The complete set of replication codes for the material included in this handout is available at <http://bit.ly/ekw-handout-replication>.

## 4. Improvements

The implementation and analysis of this class of models entails several computational challenges. Among them integration of a high-dimensional non-differentiable function, large-scale global optimization of a noisy and non-smooth criterion function, function approximation, and parallelization strategies. We now describe some of our own efforts to improve the current practice within economics. Given the advanced state of our projects, we now actively seek the input from domain experts for further improvement and subsequent joint publication.

### 4.1. Numerical integration

In [Gabler, Eisenhauer, & Suchy \(2020\)](#) we draw on the extensive literature in applied math on numerical integration ([Davis & Rabinowitz, 2007](#)).

To clarify the structure of the integral determining the future value of a state, it is useful to consider the optimality equation in a generic time period  $t$ . This allows to focus on action-specific rewards instead of future values. Let  $v_t^\pi(s_t, a_t)$  denote the action-specific value function of choosing action  $a$  in state  $s$  while continuing with the optimal policy going forward. but sticking to the optimal policy  $\pi^*$  going forward.

$$v_t^\pi(s_t, a_t) = u(s_t, a_t) + \delta \mathbb{E}_{s_t} [v_{t+1}^{\pi^*}(s_{t+1}) \mid \mathcal{I}_t] \quad (4)$$

$$= u(s_t, a_t) + \delta \int_S v_{t+1}^{\pi^*}(s_{t+1}) \, dp_t(a_t, s_t) \quad (5)$$

$$= u(s_t, a_t) + \delta \underbrace{\int_S \max_{a \in A} \left\{ v_{t+1}^\pi(s_{t+1}, a_{t+1}) \right\} dp_t(a_t, s_t)}_{\mathcal{I}(a_{t+1})}. \quad (6)$$

The evaluation of such an integral is required millions of times during the backward induction procedure. The current practice is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities ([Judd & Skrainka, 2011](#)).

Let's consider an atemporal version of the typical integral from [Keane & Wolpin \(1997\)](#). In their model, individuals can choose among five alternatives. Each of the alternative-specific rewards is in part determined by a random continuous state variable that follows a normal distribution which happens to be unobserved. The transition of all observable state variables is deterministic. This results in a five-dimensional integrals the dimensionality is determined by the random state variables. The integral takes the following form:

$$\int_{\epsilon} \max_{a \in A} \left\{ v_{t+1}^\pi(x_{t+1}, \epsilon, a) \right\} \phi_{\mu, \Sigma}(\epsilon) \, d\epsilon.$$

where  $\epsilon = (\epsilon_1, \dots, \epsilon_5) \sim \mathcal{N}(\mu, \Sigma)$  follows a multivariate normal distribution with mean  $\mu \in \mathbb{R}^5$ , covariance matrix  $\Sigma \in \mathbb{R}^{5 \times 5}$ , and probability density function  $\phi_{\mu, \Sigma}$ . A key features of this integral is the lack of general separability between the random and deterministic state variables. This makes the closed-form solutions impossible even under suitable distributional assumptions (McFadden, 1978; Rust, 1987).

## 4.2. Global optimization

In Gabler, Eisenhauer, & Roehl (2020) we draw on the specialized literature on global optimization to improve the reliability of calibrations (Locatelli & Schoen, 2013).

Depending on the calibration procedure special challenges arise. Likelihood-based estimation requires smoothing of the choice probabilities, while simulation-based calibration requires the application of noisy function optimization. The further development of our groups optimization toolbox `estimagic` (Gabler, 2019) is part of this project.

## 4.3. Miscellaneous

Other areas of improvement that require our future attention is the approximate solution of the model using function approximation techniques and the issue of using massive parallelism for the analysis of the models.

## 5. Extensions

We briefly discuss selected extensions to the baseline model that are our active areas of research. Again, given the advanced state of efforts here projects, we now actively seek the input from domain experts for further improvement and subsequent joint publication.

### 5.1. Robust decision-making

The uncertainties involved in human capital investments are ubiquitous ([Becker, 1964](#)). Individuals usually make investments early in life when they are still uncertain about their own abilities and tastes. In addition, returns also depend on demographic, economic, and technological trends that only start to unfold years from now. However, the treatment of uncertainty in life cycle models of human capital investment is very narrow. A model provides individuals with a formalized view about their economic environment and implies unique probabilities for all possible future events. Individuals have no fear of model misspecification.

In [Eisenhauer & Suchy \(2020\)](#), we address this shortcoming by formulating, implementing, and exploring a life cycle model of robust human capital investment where individuals face risk within a model and ambiguity about the model ([Arrow, 1951](#)). Ambiguity arises as individuals simply do not know the true model and consider a whole set of models as reasonable descriptions of their economic environment. Individuals fear model misspecification and thus seek robust decisions, i.e. decisions that perform well over the whole range of models.

We incorporate methods from robust optimization ([Ben-Tal et al., 2009](#); [Wiesemann et al., 2014](#); [Rahimian & Mehrotra, 2019](#)) and robust Markov decision processes ([Iyengar, 2005](#); [Nilim & El Ghaoui, 2005](#)) that allow us to construct decision rules that explicitly take potential model misspecification into account.

### 5.2. Uncertainty quantification

We investigate this issue in [Gabler, Eisenhauer, Janys, & Mensinger \(2020\)](#) and this section will be fleshed out further as our work progresses.

### 5.3. Model validation

In [Bhuller et al. \(2018\)](#), we calibrate an EKW model on Norwegian population panel data with nearly career-long earnings histories. Due to the richness of the data, we can validate the model using a mandatory schooling reform. Our data includes substantial geographic variation in compulsory schooling across Norway between 1960 and 1975 as mandatory schooling increased from 7 to 9 years at different points in time across municipalities. We split our data in an

estimation and validation sample. We only use pre-reform data in our estimation, predict the effect of increasing mandatory schooling by 2 years, and compare our forecast with the post-reform outcome. This allows us to assess the ability of our model to extrapolate outside the support of our estimation data. Such a validation of computational models is a prerequisite for their use in other disciplines ([Adams, 2012](#); [Oberkampff & Roy, 2010](#)). It is extremely rare in economics as drastic regime shifts are seldom available in observational data and costly to implement in large-scale experiments.



**A. Appendix**

## A.1. Acronyms and Symbols

**Table 1:** List of Acronyms

| Acronym | Meaning                 |
|---------|-------------------------|
| MDP     | Markov decision process |
| EKW     | Eckstein-Keane-Wolpin   |

**Table 2:** List of Symbols

| Symbol              | Meaning   |
|---------------------|---|
| $\mathbb{R}$        | set of real numbers   |
| $\mathbb{I}[A]$     | indicator function that takes value one if event $A$ is true  |
| Economic Model      |   |
| $t$                 | decision period   |
| $T$                 | number of decision periods  |
| $a \in \mathcal{A}$ | set of admissible actions   |
| $s \in \mathcal{S}$ | set of possible states with generic state $s$   |
| $s_t$               | realization of state $s$ in period $t$  |
| $a_t^\pi(s)$        | decision rule that specifies an action for all states $s$ in period $t$ following $\pi$                 |
| $a_t^\pi(s_t)$      | actual decision in period $t$ when in state $s_t$ following policy $\pi$                                |
| $a_t$               | actual decision at time $t$ by individual   |
| $a_{it}$            | actual decision observed by individual $i$ at time $t$  |
| $p_t(s, a)$         | conditional probability distributions for $s_{t+1}$ when choosing action $a$ in state $s$ in period $t$ |
| $r(s, a)$           | rewards when choosing action $a$ in state $s$   |
| $\delta$            | discount factor   |

**Table 2:** List of Symbols

| Symbol               | Meaning  |
|----------------------|--|
| $v_t^\pi$            | expected total discounted rewards of adopting policy $\pi$ from period $t$ going forward |
| $\pi \in \Pi$        | set of all policies  |
| Estimation procedure |  |
| Computational Model  |  |
| $\epsilon_{at}$      | random shock to rewards of alternative $a$ in period $t$                                 |
| $x_{jt}$             | number of periods worked in occupation $j$ by the beginning of period $t$                |
| $g_t$                | number of periods enrolled in school by the beginning of period $t$                      |
| $\mathcal{N}_0$      | true multivariate normal distribution for random shocks                                  |
| $\Sigma$             | covariance matrix of random shocks   |
| $v$                  | admissible realization of means for future labor market shocks                           |
| $\alpha_j$           | parameters for rewards function when working in occupation $j$                           |
| $\beta$              | parameters for rewards function when enrolling in school                                 |
| $\gamma$             | parameter for rewards function when staying at home                                      |

## A.2. Parameterization

Table 3 presents the full parameterization for the simulation of the baseline sample. [Keane & Wolpin \(1994\)](#) analyze three different parameterizations of the model. Our example is based on their second parameterization as the cost of post-secondary education is set to zero in their first parameterization.

**Table 3:** Parameterization

| Parameter     | Value            |
|---------------|------------------|
| $\eta$        | 0.01000          |
| $\delta$      | 0.95000          |
| $\alpha_{10}$ | 9.21000          |
| $\alpha_{11}$ | 0.04000          |
| $\alpha_{12}$ | 0.03300          |
| $\alpha_{13}$ | -0.00050         |
| $\alpha_{14}$ | 0.00000          |
| $\alpha_{15}$ | 0.00000          |
| $\alpha_{20}$ | 8.20000          |
| $\alpha_{21}$ | 0.08000          |
| $\alpha_{22}$ | 0.02200          |
| $\alpha_{23}$ | -0.00050         |
| $\alpha_{24}$ | 0.06700          |
| $\alpha_{25}$ | -0.00100         |
| $\beta_0$     | 5,000.00000      |
| $\beta_1$     | 5,000.00000      |
| $\beta_2$     | 15,000.00000     |
| $\gamma_0$    | 14,500.00000     |
| $\sigma_{11}$ | 0.16000          |
| $\sigma_{12}$ | 0.00000          |
| $\sigma_{13}$ | 0.00000          |
| $\sigma_{14}$ | 0.00000          |
| $\sigma_2$    | 0.25000          |
| $\sigma_{23}$ | 0.00000          |
| $\sigma_{24}$ | 0.00000          |
| $\sigma_{33}$ | 36,000,000.00000 |
| $\sigma_{34}$ | 0.00000          |
| $\sigma_{44}$ | 36,000,000.00000 |

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