Eckstein-Keane-Wolpin models

An invitation for transdisciplinary collaboration

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Abstract

We present background material on a class of structural microeconometric models to facilitate transdisciplinary collaboration in their future development. We describe the economic framework, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models. We specify, simulate, and calibrate an example using our group's research codes `respy` and `estimagic`. We summarize our efforts drawing on research outside economics to address their computational challenges and improve their reliability and interpretability.

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Contents

1 Introduction

Economists use structural microeconometric models to study individual decision-making. These models specify the objective of individuals, their economic environment, and the institutional and informational constraints under which they operate. Calibration of the model to observed data on individual decisions and experiences allows quantifying the importance of competing economic mechanisms in determining economic outcomes and forecasting the effects of policy proposals (Wolpin, 2013).

We restrict our exposition to the class of Eckstein-Keane-Wolpin (EKW) models (Adda et al., 2017; Blundell et al., 2016; Keane & Wolpin, 1997). Labor economists use them to study human capital investment decisions. Human capital comprises the knowledge, skills, competencies, and attributes embodied in individuals that facilitate the creation of personal, social, and economic well-being (Becker, 1964). Differences in human capital attainment lead to inequality in a variety of life outcomes such as labor market success and health across and within countries (OECD, 2001).

In Bhuller et al. (2018), for example, we apply an EKW model to analyze the mechanisms determining schooling decisions in Norway. We calibrate the model using Norwegian population panel data with nearly career-long earnings histories. After validating our model using a mandatory schooling reform, we gain insights into the underlying economic mechanisms that generate the effects of the policy and can forecast the impacts of several policy alternatives.

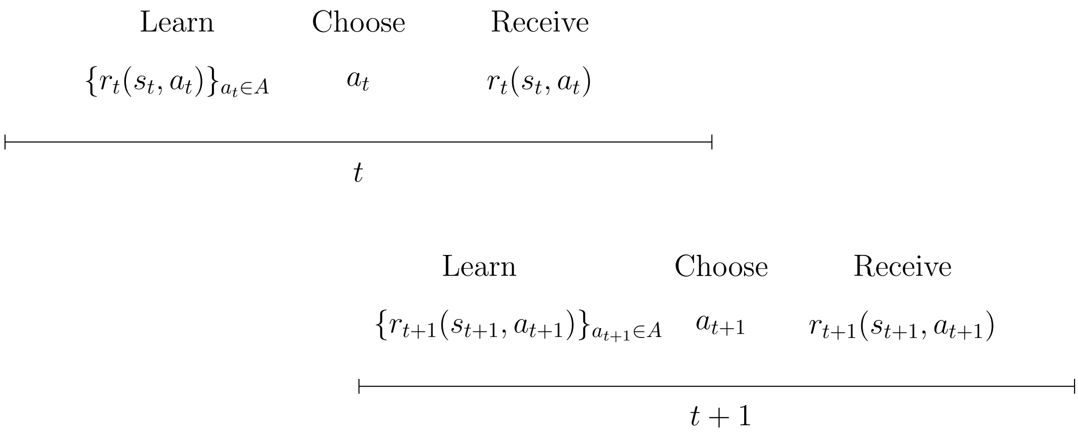
We offer this handout to facilitate transdisciplinary collaboration in the future development of EKW models. We first describe their economic framework, mathematical formulation, and calibration procedure. We then specify, simulate, and calibrate an example using our group's research codes `respy` (OSE, 2020) and `estimagic` (Gabler, 2019). Finally, we summarize our efforts drawing on research outside economics to address their computational challenges and improve their reliability and interpretability.

Throughout, we only offer a limited number of seminal references and textbooks that invite further study. We introduce acronyms and symbols as needed, and our notation draws on the reviews by Aguirregabiria & Mira (2010), Arcidiacono & Ellickson (2011), and Puterman (1994).

2 Setup

We now present the basic setup of the EKW models. We first describe the economic framework, then turn to its mathematical formulation, and finally outline the calibration procedure.

Figure 1: Timing of events



2.1 Economic framework

EKW models describe sequential decision-making under uncertainty (Gilboa, 2009; Machina & Viscusi, 2014). At time $t = 1, \ldots, T$ each individual observes the state of the economic environment $s_t \in S$ and chooses an action a_t from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate reward $r_t(s_t, a_t)$ and the economy evolves to a new state s_{t+1}. The transition from s_t to s_{t+1} is affected by the action but remains uncertain. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate reward. Instead, they take the future consequences of their current action into account.

A policy $\pi \equiv (a_1^\pi(s_1), \ldots, a_T^\pi(s_T))$ provides the individual with instructions for choosing an action in any possible future state. It is a sequence of decision rules $a_t^\pi(s_t)$ that specify the action at a particular time t for any possible state s_t under π. The implementation of a policy generates a sequence of rewards that depends on the objective transition probability distribution $p_t(s_t, a_t)$ for the evolution of state s_t to s_{t+1} induced by the model. Individuals have rational expectations (Lucas, 1972; Muth, 1961) so their subjective beliefs about the future agree with the objective transition probabilities of the model.

Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of period t an individual fully learns about the immediate reward of each alternative, chooses one of them, and receives its immediate reward. Then the state evolves from s_t to s_{t+1} and the process is repeated in $t+1$. Individuals face uncertainty and they seek to maximize the expected total discounted rewards. An exponential discount factor $0 < \delta < 1$ parameterizes their time preference and captures a taste for immediate over future rewards.

Equation (1) provides the formal representation of the individual's objective. Given an initial state s_1, individuals implement the policy π from the set of all possible policies Π that maximizes the expected total discounted rewards over all T decision periods given the information $\mathcal{I}_1$ available in the first period.

$$\max_{\pi\in\Pi} \mathrm{E}_{s_1}^{\pi}\left[\sum_{t=1}^{T}\delta^{t-1}r_t(s_t, a_t^{\pi}(s_t))\,\middle|\,\mathcal{I}_1\right] \tag{1}$$

The superscript of the expectation emphasizes that each policy π induces a different probability distribution over the sequences of rewards.

2.2 Mathematical formulation

EKW models are set up as a standard Markov decision process (MDP) (Puterman, 1994; White, 1993). When making sequential decisions under uncertainty, the task is to determine the optimal policy π^* with the largest expected total discounted rewards $v_1^{\pi^*}(s_1)$ as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of rewards each weighted by the probability with which they occur. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems.[1]

The value function $v_t^{\pi}(s_t)$ captures the expected total discounted rewards under policy π from period t onwards for an individual experiencing state s_t:

$$v_t^{\pi}(s_t) \equiv \mathrm{E}_{s_t}^{\pi}\left[\sum_{j=0}^{T-t}\delta^{j}\, r_{t+j}(s_{t+j}, a_{t+j}^{\pi}(s_{t+j}))\,\middle|\,\mathcal{I}_t\right].$$

Then we can determine $v_1^{\pi}(s_1)$ for any policy by recursively evaluating equation (2):

$$v_t^{\pi}(s_t) = r_t(s_t, a_t^{\pi}(s_t)) + \delta\,\mathrm{E}_{s_t}^{\pi}\left[v_{t+1}^{\pi}(s_{t+1})\,\middle|\,\mathcal{I}_t\right]. \tag{2}$$

Equation (2) expresses the total value $v_t^{\pi}(s_t)$ of adopting policy π going forward as the sum of its immediate reward and all expected discounted future rewards.

The principle of optimality (Bellman, 1954) allows to construct π^* by solving the optimality

[1]Optimal decisions in a MDP are a deterministic function of the current state s only, i.e. an optimal decision rule is always deterministic and Markovian. We restrict our notation to this special case right from the beginning.

equations (3) for all s and t recursively:

$$v_t^{\pi^*}(s_t) = \max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi^*} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}. \tag{3}$$

The optimal value function $v_t^{\pi^*}$ is the sum of the expected discounted rewards in t over the remaining time horizon assuming the optimal policy is implemented going forward. The optimal action is choosing the alternative with the highest total value:

$$a_t^{\pi^*}(s_t) = \arg\max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi^*} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}.$$

Algorithm 1 allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and the optimal action is choosing the alternative with the highest immediate reward in each state. With the decision rule for the final period at hand, the other optimal decisions can be determined recursively following equation (3) as the calculation of their expected future rewards is straightforward given the relevant transition probabilities.

Algorithm 1 Backward induction procedure

for $t = T, \ldots, 1$ **do**

  if t == T **then**

    $v_T^{\pi^*}(s_T) = \max_{a_T \in A} \left\{ r_T(s_T, a_T) \right\} \qquad \forall \, s_T \in S$

  else

    Compute $v_t^{\pi^*}(s_t)$ for each $s_t \in S$ by

      $v_t^{\pi^*}(s_t) = \max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}$

    and set

      $a_t^{\pi^*}(s_t) = \arg\max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}.$

  end if

end for

2.3 Calibration procedure

EKW models are calibrated to data on observed individual decisions and experiences under the hypothesis that the individual's behavior is generated from the solution to the model. The goal is to back out information on reward functions, preference parameters, and transition probabilities. This requires the full parameterization θ of the model.

Economists have access to information for $i = 1, \ldots, N$ individuals in each time period t. For every observation (i,t) in the data, we observe action a_{it}, reward r_{it}, and a subset x_{it} of the state s_{it}. Therefore, from an economist's point of view, we need to distinguish between two types of state variables $s_{it} = (x_{it}, \epsilon_{it})$. At time t, the economist and individual both observe x_{it} while ϵ_{it} is only observed by the individual. In summary, the data $\mathcal{D}$ has the following structure:

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, \ldots, N; t = 1, \ldots, T_i\},$$

where T_i is the number of observations for which we observe individual i.

Numerous calibration procedures for different settings exist (Davidson & MacKinnon, 2003; Gourieroux & Monfort, 1996). We briefly outline likelihood-based and simulation-based calibration. Independent of the calibration criterion, it is necessary to solve for the optimal policy π^* at each candidate parameterization of the model.

Likelihood-based calibration seeks to find the parameterization $\hat{\theta}$ that maximizes the likelihood function $\mathcal{L}(\theta \mid \mathcal{D})$, i.e. the probability of observing the given data as a function of θ. As we only observe a subset x_t of the state, we can determine the probability $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$ of individual i at time t in x_{it} choosing a_{it} and receiving r_{it} given parametric assumptions about the distribution of ϵ_{it}. The objective function takes the following form:

$$\hat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max} \underbrace{\prod_{i=1}^{N} \prod_{t=1}^{T_i} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)}_{\mathcal{L}(\theta \mid \mathcal{D})}.$$

In simulation-based calibration, our goal is to find the parameterization $\hat{\theta}$ that yields a simulated data set from the model that closest resembles the observed data. More precisely, the goal is often to minimize the weighted squared distance between a set of moments M_D computed on the observed data and the same set of moments computed on the simulated data $M_S(\theta)$. The objective function takes the following form:

$$\hat{\theta} \equiv \underset{\theta \in \Theta}{\arg\min} (M_D - M_S(\theta))' W (M_D - M_S(\theta)).$$

3 Example

We now analyze a canonical EKW model. We start with an outline and discussion of the occupational choice model presented in Keane & Wolpin (1994). The model is deliberately simple, but already contains the core components required for serious research applications. We will parameterize it so that we can simulate a data set that mirrors the basic patterns often documented in actual data on observed individual decisions. We conclude by sketching its specification, simulation, and calibration using our group's research codes `respy` and `estimagic`.

3.1 Keane & Wolpin (1994)

We study individuals over their working life for $T = 40$ years from age 16 to 55. Individuals can choose to either work in one of two occupations ($a_t = 1, 2$), attend school ($a_t = 3$), or stay at home ($a_t = 4$). Immediate rewards are determined as follows:

$$r_t(s_t, a_t) = \begin{cases} w_{1t} = \exp\{\alpha_{10} + \alpha_{11} g_t + \alpha_{12} e_{1t} + \alpha_{13} e_{1t}^2 + \alpha_{14} e_{2t} + \alpha_{15} e_{2t}^2 + \epsilon_{1t}\} & \text{if } a_t = 1 \\ w_{2t} = \exp\{\alpha_{20} + \alpha_{21} g_t + \alpha_{22} e_{1t} + \alpha_{23} e_{1t}^2 + \alpha_{24} e_{2t} + \alpha_{25} e_{2t}^2 + \epsilon_{2t}\} & \text{if } a_t = 2 \\ \beta_0 - \beta_1 \mathbb{I}[\, g_t \geq 12 \,] - \beta_2 \mathbb{I}[\, a_{t-1} \neq 3 \,] + \epsilon_{3t} & \text{if } a_t = 3 \\ \gamma_0 + \epsilon_{4t} & \text{if } a_t = 4. \end{cases}$$

g_t is the number of periods of schooling obtained by the beginning of period t, e_{1t} and e_{2t} are the number of periods that the individual worked in the two occupations respectively. The reward for each labor market alternative corresponds to its wage (w_{1t}, w_{2t}) and α_1 and α_2 are thus parameters associated with the wage functions. They capture the returns to schooling and occupation-specific human capital. Turning to the rewards from school attendance, β_0 is the consumption reward of schooling, β_1 is the post-secondary cost of schooling, and β_2 is an adjustment cost associated with returning to school after dropping out. The mean reward of the home alternative is denoted γ_0. The ϵ_{at}'s are alternative-specific shocks to occupational productivity, the consumption value of schooling, and home time.

Given the structure of the reward functions and imposing that the shocks are not correlated across time, the state at time t is $s_t = \{g_t, e_{1t}, e_{2t}, a_{t-1}, \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$. The current stock of human capital is observable $\{g_t, e_{1t}, e_{2t}, a_{t-1}\}$ to the individual and the economist. It evolves deterministically according to the following rules:

$$
\begin{aligned}
e_{1,t+1} &= e_{1t} + \mathbb{I}\,[\,a_t = 1\,] \\
e_{2,t+1} &= e_{2t} + \mathbb{I}\,[\,a_t = 2\,] \\
g_{t+1} &= g_t \;\; + \mathbb{I}\,[\,a_t = 3\,].
\end{aligned}
$$

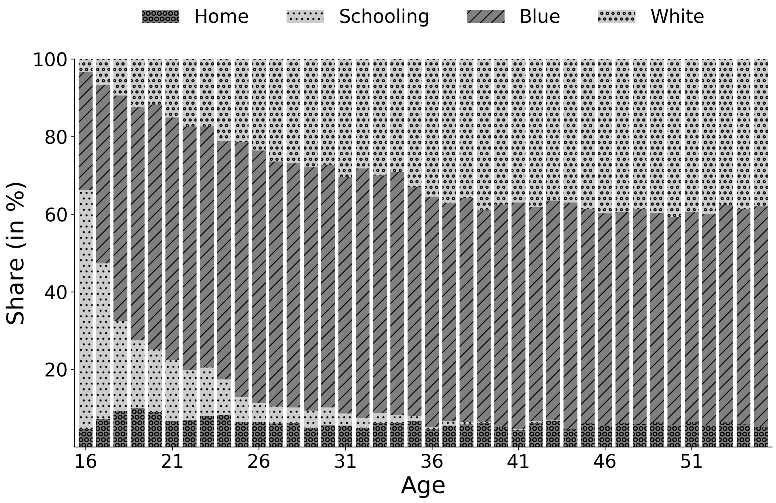
The shocks $\{\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$ are only observable by the individual and evolve stochastically according to a joint normal distribution.

Keane & Wolpin (1994) analyze three different parameterizations of the model, and we reproduce the second one for this example. Initially, individuals are identical and have no labor market experience ($e_{11} = e_{21} = 0$) but ten years of schooling ($g_1 = 10$). Different choices over the life cycle are then simply the cumulative effects of different shocks.

One year of additional schooling increases wages by only 4% in the first occupation compared to 8% in the second. We will thus refer to the former as blue-collar and the latter as white-collar going forward. Starting wages are considerably lower in the white-collar sector, but wages increase more rapidly with occupation-specific experience compared to blue-collar wages. Own-work experience is highly valuable in both occupations. However, while white-collar wages increase with blue-collar experience as well, the opposite is not true. There is a consumption value of schooling of $5,000, but the total cost of pursuing post-secondary education is considerable and amounts to $5,000. Once leaving school, individuals incur a nearly prohibitive cost of $15,000 for re-enrolling. Individuals are forward-looking with a discount factor δ of 0.95.

We simulate the life cycle histories of 1,000 individuals. Figure 2 shows the share of individuals choosing each of the four alternatives by period. Initially, roughly 60% of individuals enroll in school, but this share declines rapidly, and only 19% attain any post-secondary education. Right away, about 30% of individuals are working in the blue-collar occupation. Blue-collar employment initially increases even further to peak at 65% as individuals are leaving school and entering the labor market. White-collar employment steadily rises over the life cycle but never reaches more than 40%. About 5% of individuals stay at home each period.

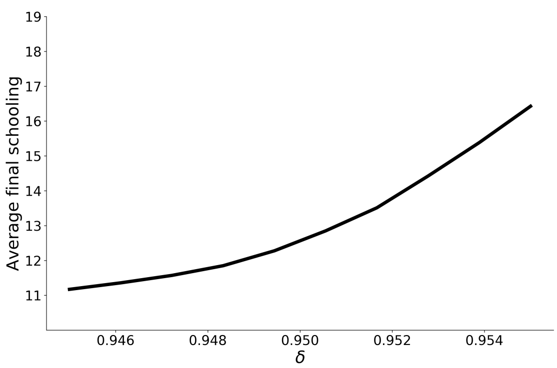
Figure 2: Choices over the life cycle



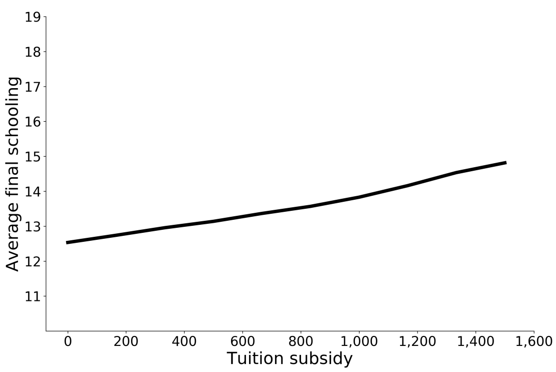
Overall, average final schooling is slightly above a high school degree with 12.5 years. Individuals choose to incur the immediate costs of their schooling investments in the form of tuition and foregone wages right at the start. Doing so maximizes their ability to reap the reward of increased wages over the remaining time periods.

Figure 3 illustrates the ability of the model to quantify the impact of economic mechanisms and to forecast the effect of public policies. On the left, we vary the discount factor capturing time preferences between 0.945 and 0.955 while we reduce β_1 by the size of a tuition subsidy of up to $\$1,500$ on the right. In both cases, we are interested in the changes to average final schooling.

Figure 3: Economic mechanism and policy forecast



(a) Time preference



(b) Tuition subsidy

Increases in the discount factor and the tuition subsidy both result in higher average final schooling. However, they do so for very different reasons. While individuals put more emphasis on the future benefits of their schooling investment in the former, they react to a reduction of its immediate cost in the latter.

3.2 respy and estimagic

Our research group is actively developing two research codes that allow analyzing EKW models. `respy` allows for their flexible specification and simulation, and `estimagic` provides the infrastructure for their calibration. We briefly showcase the typical workflow of using both packages in our research.

Figure 4 illustrates a typical workflow. Initially, the user provides the observed data, the parameterization of the model, and other options to `respy`. All together define the structure of the model, and we can construct the functionality for the simulation of data and the evaluation of the criterion function. `estimagic` allows calibrating the model to the observed data. The results from the calibration steps are used to, for example, analyze the economic mechanisms underlying the observed behaviors.

Figure 4: Typical workflow

```
from estimagic.optimization.optimize import maximize
import respy as rp

# obtain model input
params, options, df = rp.get_example_model("kw_94_two")

# process model specification
crit_func = rp.get_crit_func(params, options, df)
simulate = rp.get_simulate_func(params, options)

# perform calibration
results, params_rslt = maximize(crit_func, params, "nlopt_bobyqa")

# conduct analysis
df_rslt = simulate(params_rslt)
```

Figure 5 shows the model specification files for Keane & Wolpin (1994). The file on the left sets the parameter values for the reward functions and the distribution of the unobservable state variables. On the right, we provide details on the construction of the observed state variables and numerous tuning parameters for the numerical solution of the model.

Figure 5: Model specification

kw_94_two.csv

```
category,name,value,comment
delta,delta,0.95,discount factor
wage_a,constant,9.21,log of rental price
wage_a,exp_edu,0.04,return to an additional year of schooling
wage_a,exp_a,0.033,return to same sector experience
wage_a,exp_a_square,-0.0005,"return to same sector, quadratic experience"
wage_a,exp_b,0,return to other sector experience
wage_a,exp_b_square,0,"return to other sector, quadratic experience"
wage_b,constant,8.2,log of rental price
wage_b,exp_edu,0.08,return to an additional year of schooling
wage_b,exp_b,0.067,return to same sector experience
wage_b,exp_b_square,-0.001,"return to same sector, quadratic experience"
wage_b,exp_a,0.022,return to other sector experience
wage_b,exp_a_square,-0.0005,"return to other sector, quadratic experience"
nonpec_edu,constant,5000,constant reward for choosing education
nonpec_edu,at_least_twelve_exp_edu,-5000,"reward for going to college (tuition, etc.)"
nonpec_edu,not_edu_last_period,-15000,reward for going back to school
nonpec_home,constant,14500,constant reward of non-market alternative
shocks_sdcorr,sd_a,0.4,"Element 1,1 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_b,0.5,"Element 2,2 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_edu,6000,"Element 3,3 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_home,6000,"Element 4,4 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_b_a,0,"Element 2,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_edu_a,0,"Element 3,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_edu_b,0,"Element 3,2 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_a,0,"Element 4,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_b,0,"Element 4,2 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_edu,0,"Element 4,3 of standard-deviation/correlation matrix"
lagged_choice_1_edu,probability,1,Probability that the first lagged choice is education
initial_exp_edu_10,probability,1,Probability that the initial level of education is 10
```

(a) Parameterization

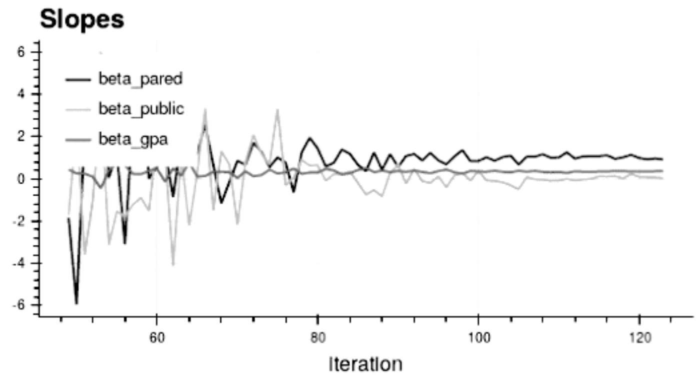
kw_94_two.yaml

```
estimation_draws: 200
estimation_seed: 500
estimation_tau: 500
interpolation_points: -1
n_periods: 40
simulation_agents: 1000
simulation_seed: 132
solution_draws: 500
solution_seed: 456
monte_carlo_sequence: random
core_state_space_filters:
  # In periods > 0, if agents accumulated experience only in one choice, lagged choice
  # cannot be different.
  - "period > 0 and exp_{i} == period and lagged_choice_1 != '{i}'"
  # In periods > 0, if agents always accumulated experience, lagged choice cannot be
  # non-experience choice.
  - "period > 0 and exp_a + exp_b + exp_edu == period and lagged_choice_1 == '{j}'"
  # In periods > 0, if agents accumulated no years of schooling, lagged choice cannot
  # be school.
  - "period > 0 and lagged_choice_1 == 'edu' and exp_edu == 0"
  # If experience in choice 0 and 1 are zero, lagged choice cannot be this choice.
  - "lagged_choice_1 == '{k}' and exp_{k} == 0"
  # In period 0, agents cannot choose occupation a or b or mil.
  - "period == 0 and lagged_choice_1 == '{k}'"
covariates:
  constant: "1"
  exp_a_square: exp_a ** 2
  exp_b_square: exp_b ** 2
  at_least_twelve_exp_edu: exp_edu >= 12
  not_edu_last_period: lagged_choice_1 != 'edu'
```

(b) Options

Figure 6 depicts the dashboard provided by `estimagic` to monitor the progress and parameter values of the calibration in real-time. This allows us to detect problems during calibration right away and facilitates the debugging process.

Figure 6: Dashboard



We adopt a modern software engineering workflow in the development of both packages and tutorials, source code, testing harness, as well as implementation details are available in their respective online documentations at https://respy.readthedocs.io and https://estimagic.readthedocs.io.

4 Improvements

The implementation of EKW models poses several computational challenges. Among them are numerical integration, global optimization, function approximation, and efficient parallelization. We now describe some of our efforts to align `respy` and `estimagic` with the state-of-the-art in computational methods. We have concluded our preparatory work and actively seek input from domain experts for further improvements and joint publication.

4.1 Numerical integration

The solution of EKW models requires the evaluation of millions of integrals to determine the future value of each action in each state. In Gabler, Eisenhauer, & Suchy (2020), we draw on the extensive literature on numerical integration (Davis & Rabinowitz, 2007; Gerstner & Griebel, 1998) to improve the precision and reliability of their solution. The current practice in economics is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities (Judd & Skrainka, 2011).

We consider the optimality equation in a generic time period t to clarify the structure of the integral. Let $v_t^{\pi}(s_t, a_t)$ denote the action-specific value function of choosing action a_t in state s_t while continuing with the optimal policy going forward.

$$
\begin{aligned}
v_t^{\pi}(s_t, a_t) &= r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}\left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \\
&= r_t(s_t, a_t) + \delta \int_S v_{t+1}^{\pi^*}(s_{t+1}) \; \mathrm{d}p_t(a_t, s_t) \\
&= r_t(s_t, a_t) + \delta \int_S \max_{a_{t+1} \in A} \left\{ v_{t+1}^{\pi^*}(s_{t+1}, a_{t+1}) \right\} \mathrm{d}p_t(a_t, s_t).
\end{aligned}
$$

Let's consider an atemporal version of the typical integral from Keane & Wolpin (1994) as an example. As outlined earlier, individuals can choose among four alternatives. Each of the alternative-specific rewards is, in part, determined by a stochastic continuous state variable ϵ. The transition of all other state variables x is deterministic. This results in a four-dimensional integral of the following form:

$$
\int_{\epsilon} \max_{a \in A} \left\{ v^{\pi^*}(x, \epsilon, a) \right\} \phi_{\mu,\Sigma}(\epsilon) \, \mathrm{d}\epsilon \quad \forall \, x \in X,
$$

where ϵ follows a multivariate normal distribution with mean μ, covariance matrix Σ, and probability density function $\phi_{\mu,\Sigma}$.

4.2 Global optimization

The calibration of EKW models is challenging due to a large number of parameters and multiplicity of local minima. In Gabler, Eisenhauer, & Röhl (2020), we draw on the literature on global optimization to assess and improve the reliability of the calibrations (Locatelli & Schoen, 2013; Nocedal & Wright, 2006).

We conduct a benchmarking exercise using Keane & Wolpin (1994, 1997) as a well-known and empirically-grounded test case. Depending on the calibration procedure, particular challenges arise. For example, while likelihood-based calibration requires smoothing of the choice probabilities, simulation-based calibration involves the optimization of a noisy function. We provide guidelines for the selection of the appropriate algorithm in each setting and showcase diagnostics to assess the reliability of the calibration results.

5 Extensions

We are actively pursuing several extensions to the standard analysis of EKW models. For example, we draw on the methodological literature on robust-decision making and uncertainty quantification to account for the uncertainties within and outside the model (Hansen, 2015). We also work with the German Institute for Economic Research and Statistics Norway to improve the available data for the calibration of the models. Again, we have concluded our preparatory work and actively seek input from domain experts for further improvements and joint publication.

5.1 Robust decision-making

The uncertainties involved in human capital investments are ubiquitous (Becker, 1964). Individuals usually make investments early in life when they are still uncertain about their abilities and tastes. Their returns also depend on demographic, economic, and technological trends that only start to unfold years from now. However, the treatment of uncertainty in EKW models of human capital investment is very narrow. A model provides individuals with a formalized view of their economic environment and implies unique probabilities for all possible future events. Individuals have no fear of model misspecification.

In Eisenhauer & Suchy (2020), we address this shortcoming by formulating, implementing, and exploring an EKW model of robust human capital investment where individuals face risk within a model and ambiguity about the model (Arrow, 1951). Ambiguity arises as individuals do not know the true model and consider a whole set of models as reasonable descriptions of their economic environment. Individuals fear model misspecification and thus seek robust decisions, i.e., decisions that perform well over the whole range of models.

We incorporate methods from robust optimization (Ben-Tal et al., 2009; Rahimian & Mehrotra, 2019; Wiesemann et al., 2014) and robust Markov decision processes (Iyengar, 2005; Nilim & El Ghaoui, 2005) that allow to construct decision rules that explicitly take potential model misspecification into account.

5.2 Uncertainty quantification

There are numerous sources of uncertainty in the policy forecasts produced by a calibrated EKW model. The model is subject to misspecification, its numerical implementation introduces approximation error, the data is subject to measurement error, and the calibrated parameters remain partly uncertain. However, economists display incredible certitude as they disregard all uncertainty (Manski, 2013) in their forecasts.

In Gabler, Eisenhauer, Janys, & Mensinger (2020), we draw on a rich literature in other disciplines where a proper accounting of the uncertainty in forecasts from complex computational models is mandatory (Saltelli et al., 2004, 2008; Smith, 2014). However, uncertainty quantification for EKW models poses several unique challenges. They usually have a large number of uncertain and correlated parameters, and the quantity of interest is time-consuming to compute and a complex function of the model parameters. Using machine learning methods, we set up an emulator that approximates the full model but is fast to evaluate. We revisit the analysis of (Keane & Wolpin, 1994, 1997) to showcase our approach in a well-known and empirically-motivated setting and characterize the uncertainty in their key findings.

We construct our approximating emulator using recent advances in surrogate modeling (Forrester et al., 2008) and machine learning (Hastie et al., 2008; Murphy, 2012).

5.3 Model validation

The validation of computational models is a prerequisite for their use in other disciplines (Adams, 2012; Oberkampf & Roy, 2010). However, it is extremely rare in economics as drastic regime shifts are seldom available in observational data and costly to implement in large-scale experiments.

In Bhuller et al. (2018), we calibrate an EKW model on Norwegian population panel data with nearly career-long earnings histories. Due to the richness of the data, we can validate the model using a mandatory schooling reform. Our data includes substantial geographic variation in compulsory schooling across Norway between 1960 and 1975 as mandatory schooling increased from seven to nine years at different points in time across municipalities. We split our data into a calibration and validation sample. We only use pre-reform data in our calibration, forecast the effect of increasing mandatory schooling by two years, and compare our forecast with the post-reform outcome. Doing so allows us to assess the ability of our model to extrapolate individual responses outside the support of our calibration data. We use the validated model to gain insights into the underlying economic mechanisms that generate the effects of the policy and forecast the effects of several policy alternatives.

5.4 Nonstandard expectations

When economists analyze individual decision-making through the lens of an EKW model, they impose rational expectations. The subjective beliefs about the future correspond to the objective transition probabilities induced by the model.

In Eisenhauer et al. (2020), we relax this assumption. We analyze and quantify the effect of biased expectations about wage growth in part-time employment on life cycle wage profiles of female workers. We design specific survey questions for the German Socio-Economic Panel (Goebel et al., 2019) and elicit the expected wage trajectories for full-time and part-time employment directly. Thus, we can incorporate the belief elicitation directly in our life cycle model.

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