Eckstein-Keane-Wolpin models

OpenSourceEconomics*

Abstract

We present background material for a particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models. We provide an example application using our group's research code `respy`. We draw on research outside economics to identify model components ripe for improvement and explore possible extensions.

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Keywords: life cycle model, human capital, risk, Markov decision process

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Contents

1. Introduction

Focus of handout We present background material for a particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for the so-called Eckstein-Keane-Wolpin (EKW) models (Aguirregabiria & Mira, 2010). We provide an example application using our group's research code `respy` (respy, 2018). We draw on research outside economics to identify model components ripe for improvement and explore possible extensions.

Structural models Structural economic models clearly specify an individual's objective and the constraints of their economic environment under which they operate. They are used to quantify the importance of competing economic mechanisms in determining economic outcomes and evaluate the effects of alternative policies before their implementation (Wolpin, 2013).

EKW models We restrict us to the class of Eckstein-Keane-Wolpin (EKW) models. Labor economists often apply these models for the analysis of human capital investment decisions. Differences in human capital attainment are a major determinant of inequality in a variety of life outcomes such as labor market success and educational attainment across and within countries.

Bhuller et al. (2018), for example, build a model to analyze the mechanisms determining schooling decisions in Norway. They find an important role for the option value of schooling, which measures the value of the information generated by each additional year of schooling. After validating their model using an increase in mandatory schooling, they then use the model to study the underlying mechanisms that generate the increase of average years of schooling and evaluate the effects of several policy alternatives.

Notation Throughout, we only provide a very limited number of references and only point to related textbooks instead. We will introduce acronyms and symbols as needed, but a full list of both is provided in the Appendix. The notation draws form the related work by Puterman (1994), Aguirregabiria & Mira (2010), and Arcidiacono & Ellickson (2011).

Stucture This handout is structured as follows. We first present the basic setup, then offer an example implementation using the `respy` package, and finally we outline possible improvements and extensions.

2. Setup

We now present the basic setup of the EKW models. We first present the economic framework, outline its mathematical formulation, and briefly describes the calibration process.

2.1. Economic framework

Basic setup EKW models describe sequential decision-making under risk (Machina & Viscusi, 2014; Gilboa, 2009). At time $t = 1, \ldots, T$ each individual observes the state of the economic environment $s_t \in S$ and chooses an action a_t from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate reward $r(s_t, a_t)$ and the economy evolves to a new state s_{t+1}. The transition from s_t to s_{t+1} is affected by the action. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate reward. Instead, they take the future consequences of their current action into account.

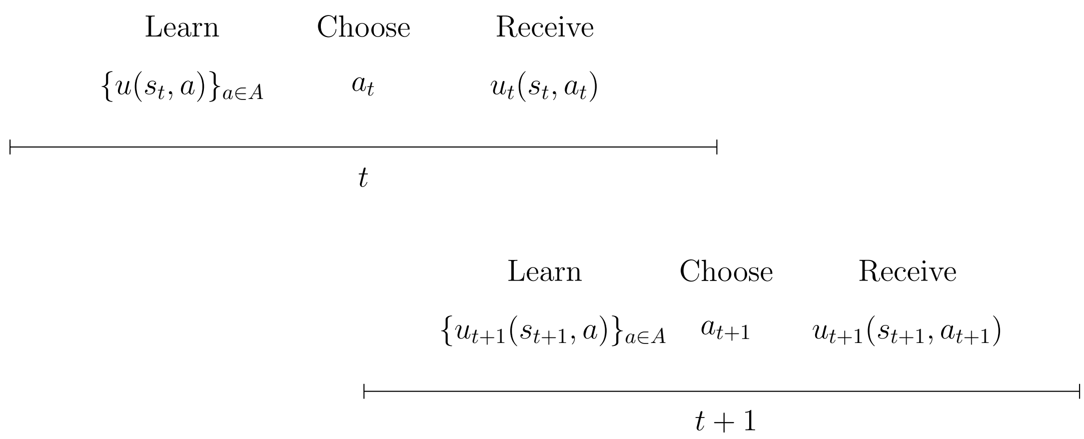
Decision rule A policy $\pi \equiv (a_1^\pi(s), \ldots, a_T^\pi(s))$ provides the individual with a prescription for choosing an action in any possible future state, where $a_t^\pi(s)$ specifies the action at a particular time t for any possible state under π. It is a sequence of decision rules and its implementation generates a sequence of rewards. The evolution of states over time is at least partly unknown as future rewards depend on, for example, shocks to preferences. Individuals use models about their economic environment to inform their beliefs about the future. For a given model, individuals thus face risk as each induces a unique objective transition probability distribution $p_t(s_t, a_t)$ for the evolution of state s_t to s_{t+1} that depends on the action a_t.

Timing of events Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of time t an individual fully learns about the immediate reward of each alternative, chooses one of them, and receives its immediate reward. Then the state evolves from s_t to s_{t+1} and the process is repeated in $t + 1$.

Decision theory Individuals make their decisions facing risk and have rational expectations as their model about the future also turns out to be true. In this case, there is a consensus that rational choices are expressed by the maximization of their expected total discounted rewards. A constant discount factor ensures dynamic consistency of preferences as the individual's future actions agree with the planned-for contingencies. Beliefs are updated according to Bayes's rule.

Equation (1) provides the formal representation of the individual's objective. Given an initial state s_1, individuals seek to implement the policy π from the set of all possible policies Π that maximizes the expected total discounted rewards over all T decision periods given the

Figure 1: Timing of events



information $\mathcal{I}_1$ available.

$$\max_{\pi \in \Pi} \mathrm{E}^{\pi}_{s_1} \left[\sum_{t=1}^{T} \delta^{t-1} r(s_t, a^{\pi}_t(s_t)) \,\middle|\, \mathcal{I}_1 \right] \tag{1}$$

The exponential discount factor $0 < \delta < 1$ captures a preference for immediate over future rewards. The superscript of the expectation emphasizes that each policy π induces a different unique probability distribution over the sequences of rewards.

2.2. Mathematical formulation

EKW models are set up as a standard Markov decision processes (MDP) (Puterman, 1994; White, 1993). When making sequential decisions, the task is to determine the optimal policy π^* with the largest expected total discounted rewards $v^{\pi^*}_1$ as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of rewards and the probability that each occurs. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems. Note that in slight abuse of notation s_{t+1} is a random variable given the information available at $\mathcal{I}_t$.

Let $v^{\pi}_t(s)$ denote the expected total discounted rewards under π from period t onwards:

$$v^{\pi}_t(s_t) \equiv \mathrm{E}^{\pi}_{s_t} \left[\sum_{j=0}^{T-t} \delta^{j}\, r(s_{t+j}, a^{\pi}_{t+j}(s_{t+j})) \,\middle|\, \mathcal{I}_t \right]$$

Then $v^{\pi}_1(s_t)$ can be determined for any policy by recursively evaluating equation (2):

$$v^{\pi}_t(s) = r(s_t, a^{\pi}_t(s_t)) + \delta\, \mathrm{E}^{\pi}_{s_t} \left[v^{\pi}_{t+1}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right]. \tag{2}$$

Equation (2) expresses the rewards $v_t^\pi(s_t)$ of adopting policy π going forward as the sum of its immediate rewards and all expected discounted future rewards.

The principle of optimality allows to construct the optimal policy π^* by solving the optimality equations for all s and t in equation (3) recursively:

$$v_t^{\pi^*}(s_t) = \max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}. \tag{3}$$

The value function $v_t^{\pi^*}$ is the expected discounted rewards in t over the remaining time horizon assuming the optimal policy is implemented going forward.

The optimal decision is simply the alternative with the highest value:

$$a_t^{\pi^*}(s_t) \equiv \arg\max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \right\}$$

Solution approach Algorithm (1) allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate rewards in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively as the calculation of their expected future rewards is straightforward given the relevant transition probabilities.

Algorithm 1 Backward induction procedure

---

for $t = T, \ldots, 1$ **do**

  if t == T **then**

    $v_T^{\pi^*}(s_T) = \max_{a \in A} \left\{ r(s_T, a) \right\} \qquad \forall \, s_T \in S$

  else

    Compute $v_t^{\pi^*}(s_t)$ for each $s_t \in S$ by

      $v_t^{\pi^*}(s_t) = \max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \right\}$

    and set

      $a_t^{\pi^*}(s_t) = \arg\max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \right\}.$

  end if

end for

2.3. Calibration procedure

EKW models are calibrated to data to obtain information on preference parameters and transition probabilities (Davidson & MacKinnon, 2003; Gourieroux & Monfort, 1996). Given this information, the quantitative importance of competing economic mechanisms can be assessed and the effects of public policies predicted. This requires the parameterization of all elements of the model which we collect in θ.

Data The econometrician has access to panel data for N individuals. For every observation (i,t) in the panel data set, the researcher observes action a_{it} and a subvector x_{it} of the state vector. Therefore, from an econometricians point of view, we need to distinguish between two types of state variables $s_{it} = (x_{it}, \epsilon_{it})$. Variables x_{it} that are observed by the econometrician and the individual i at time t and those that are only observed by the individual ϵ_{it}. In addition, also some realizations of the rewards $r_{it} = r(x_{it}, \epsilon_{it}, a_{it})$.

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, 2, \ldots, N; t = 1, \ldots, T_i\},$$

where T_i is the number of observations over which we observe individual i.

Procedures We briefly outline maximum likelihood estimation and the method of simulated moments. Whatever the estimation criterion, in order to evaluate it for a particular value of θ it is necessary to construct the optimal policy π^*. Therefore at each trial value of θ the whole model needs to solved by the backward induction algorithm.

Likelihood-based The individual chooses the alternative with the highest total value $a_t^{\pi^*}(s_t)$ which is determined by the complete state and rewards are also determined by s. However, the econometrician only observes the subset x. Given parametric assumptions about the distribution of ϵ, we can determine the probability $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$ of individual i at time t choosing d_{it} and receiving r_{it} given x_{it}.

$$\mathcal{L}(\theta \mid \mathcal{D}) \equiv \prod_{i=1}^{N} \prod_{t=1}^{T_i} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$$

The goal of likelihood-based estimation is to find the value of the model parameters θ that maximize the likelihood function:

$$\hat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max}\, \mathcal{L}(\theta \mid \mathcal{D})$$

Simulation-based

- This issue is under active investigation in our group and this section will be fleshed out further as our work progresses.

3. Example

Our research group is actively developing the the `respy` package which allows for the flexible specification, simulation, and estimation of the typical EKW models. Details are available in its online documentation at `https://respy.readthedocs.io`. We now provide the seminal implementation of Keane & Wolpin (1994) as an example.

Setup Individuals live for a total of T periods and make a decision about their human capital investment each period. They choose to either work in one of two occupations (a = 1, 2), attend school (a = 3), or stay at home (a = 4). The immediate utility from each alternative is the following:

$$u_t(s,a) = \begin{cases} w_{1t} = \exp\{\alpha_{10} + \alpha_{11} g_t + \alpha_{12} x_{1t} + \alpha_{13} x_{1t}^2 + \alpha_{14} x_{2t} + \alpha_{15} x_{2t}^2 + \epsilon_{1t}\} & \text{if } a = 1 \\ w_{2t} = \exp\{\alpha_{20} + \alpha_{21} g_t + \alpha_{22} x_{1t} + \alpha_{23} x_{1t}^2 + \alpha_{24} x_{2t} + \alpha_{25} x_{2t}^2 + \epsilon_{2t}\} & \text{if } a = 2 \\ \beta_0 - \beta_1 \mathbb{I}[\, g_t \geq 12\,] - \beta_2 (1 - \mathbb{I}[\, a_{t-1} = 3\,]) + \epsilon_{3t} & \text{if } a = 3 \\ \gamma_0 + \epsilon_{4t} & \text{if } a = 4. \end{cases}$$

g_t is the number of periods of schooling obtained by the beginning of period t, x_{1t} and x_{2t} are the number of periods that the individual worked in the two occupations respectively. The utility for each labor market alternative corresponds to its wage (w_{1t}, w_{2t}) and α_1 and α_2 are thus parameters associated with the wage functions. They capture the returns to schooling and occupation-specific human capital. β_0 is the consumption utility of schooling, β_1 is the post-secondary cost of schooling, and β_2 is an adjustment cost associated with returning to school. The mean utility of the home alternative is denoted γ_0. The ϵ_{at}'s are alternative-specific shocks to occupational productivity, the consumption utility of schooling, and the utility of home time. They are serially uncorrelated.

Given the structure of the utility functions and the lack of serial correlation, the state at time t is:

$$s_t = \{g_t, x_{1t}, x_{2t}, a_{t-1}, \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}.$$

The observable components of s_t evolve according to the following rules:

$$\begin{aligned} x_{1,t+1} &= x_{1t} + \mathbb{I}[\, a_t = 1\,] \\ x_{2,t+1} &= x_{2t} + \mathbb{I}[\, a_t = 2\,] \\ g_{t+1} &= g_t \;\; + \mathbb{I}[\, a_t = 3\,]. \end{aligned}$$

The transitions of all observable components of s_t are deterministic. However, there is uncertainty about the realization of its unobservable components. All unobservable components are jointly normally distributed with mean zero and covariance matrix $\mathbf{\Sigma}$.

$$[\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}]^T \sim \mathcal{N}_0(\mathbf{0}, \mathbf{\Sigma})$$

When entering the model, individuals have no labor market experience ($x_{11} = x_{21} = 0$) but ten years of schooling ($g_1 = 10$). The idea is that individuals are about age 16 when entering the model in the first period and start out identically, different choices over the life cycle are then simply the cumulative effects of different shocks.

Parameterization Keane & Wolpin (1994) outline a life cycle model of human capital investment under risk, so we rely on their parameterization as a baseline but instill individuals with an additional fear of model misspecification. The returns to schooling differ considerably between the two occupations. Schooling increases wages by only 4% in the first occupation compared to 8% in the second. We will thus refer to the former as blue-collar and the latter as white-collar going forward. Starting wages are considerably lower in the white-collar sector but wages increase more rapidly with occupation-specific experience compared to blue-collar wages. Own-work experience is highly valuable in both occupations. However, while white-collar wages increase with blue-collar experience as well, the opposite is not true. There is a consumption value of schooling of \$5,000 but the total cost of pursuing post-secondary education is considerable and amounts to \$5,000. Once leaving school, individuals incur a nearly prohibitive cost of \$15,000 for re-enrolling. Individuals are forward-looking but the future is discounted by 5%. The random shocks are not correlated across alternatives. Further details about the parameterization are available in Appendix 3.

Interface Figure 2 illustrates the typical workflow with the `respy` package. The model is specified in the parameters and options specification.

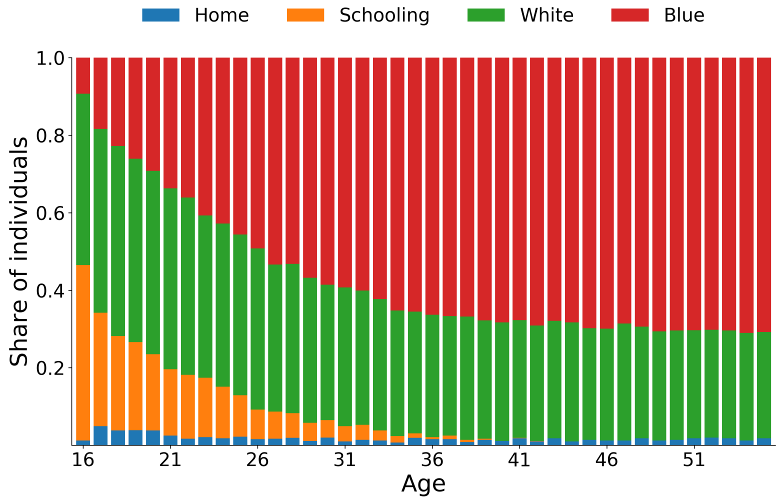
Figure 2: Workflow example

```
import respy as rp

params, options, _ = rp.get_example_model("kw_94_one")
simulate = rp.get_simulate_func(params, options)
df = simulate(params)
```

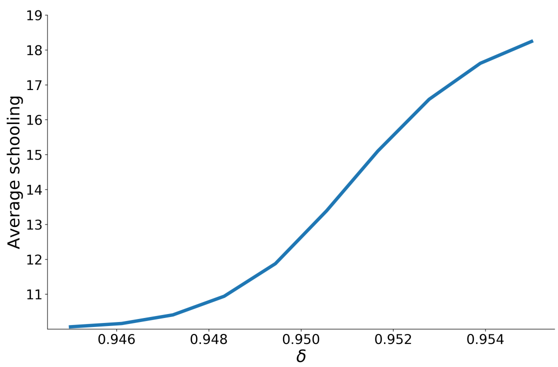
Descriptives We simulate the life cycle histories of 1,000 individuals for 40 periods. Figure 3 shows the share of individuals choosing each of the four alternatives by period. Initially, roughly 52% of individuals are enrolled in school but this share declines rapidly and only 19% attain any post-secondary education. Right away, about 35% of individuals are working in the blue-collar occupation. Blue-collar employment initially increases even further to peak at 67% as individuals are leaving school and entering the labor market. White-collar employment rises constantly over the life cycle but never reaches more than 35%. About 5% of individuals stay at home each period.

Figure 3: Choice patterns

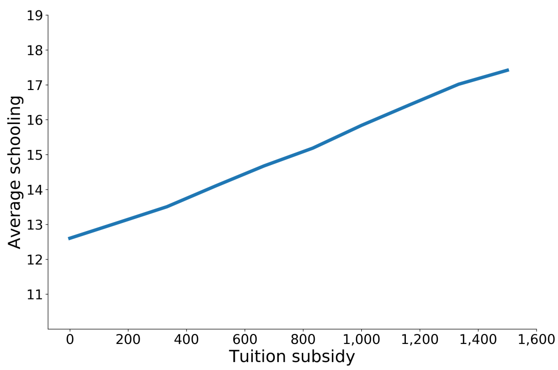


Economic mechanisms and policy evaluation Figure 4 illustrates the ability of structural economic models to quantify the impact of economic mechanisms on average schooling and predict the effect of a tuition subsidy. We evaluate the effect of a tuition subsidy of up to \$1.500 where we simulate a sample of 1,000 individuals but decrease $\hat{\beta}_1$ according to the tuition subsidy.

Figure 4: Economic mechanisms and policy evaluation



(a) Time preferences



(b) Tuition subsidy

4. Improvements

The implementation and analysis of this class of models entails several computational challenges. Among them integration of a high-dimensional non-differentiable function, large-scale global optimization of a noisy and non-smooth criterion function, function approximation, and parallelization strategies. We briefly outline each of them.

4.1. Numerical integration

We want to draw on the extensive literature in applied math on numerical integration (Davis & Rabinowitz, 2007). To clarify the structure of the integral determining the future value of a state, it is useful to consider the optimality equation in a generic time period t. This allows to focus on action-specific rewards instead of future values. Let $v_t^\pi(s_t, a_t)$ denote the action-specific value function of choosing action a in state s while continuing with the optimal policy going forward. but sticking to the optimal policy π^* going forward.

$$
\begin{aligned}
v_t^\pi(s_t, a_t) &= u(s_t, a_t) + \delta\, \mathrm{E}_{s_t}\left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] && (4)\\
&= u(s_t, a_t) + \delta \int_S v_{t+1}^{\pi^*}(s_{t+1})\; \mathrm{d}p_t(a_t, s_t) && (5)\\
&= u(s_t, a_t) + \delta \underbrace{\int_S \max_{a \in A} \left\{ v_{t+1}^{\pi}(s_{t+1}, a_{t+1}) \right\} \mathrm{d}p_t(a_t, s_t)}_{\mathcal{I}(a_{t+1})}. && (6)
\end{aligned}
$$

The evaluation of such an integral is required millions of times during the backward induction procedure. The current practice is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities (Judd & Skrainka, 2011).

Let's consider an atemporal version of the typical integral from Keane & Wolpin (1997). In their model, individuals can choose among five alternatives. Each of the alternative-specific rewards is in part determined by a random continuous state variable that follows a normal distribution which happens to be unobserved. The transition of all observable state variables is deterministic. This results in a five-dimensional integrals the dimensionality is determined by the random state variables. The integral takes the following form:

$$
\int_\epsilon \max_{a \in A} \left\{ v_{t+1}^{\pi}(x_{t+1}, \epsilon, a) \right\} \phi_{\mu,\Sigma}(\epsilon)\, \mathrm{d}\epsilon.
$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_5) \sim \mathcal{N}(\mu, \Sigma)$ follows a multivariate normal distribution with mean $\mu \in \mathbb{R}^5$, covariance matrix $\Sigma \in \mathbb{R}^{5\times 5}$, and probability density function $\phi_{\mu,\Sigma}$. A key features of this integral is the lack of general separability between the random and deterministic state variables. This makes the closed-form solutions impossible even under suitable distributional assumptions

(McFadden, 1978; Rust, 1987).

4.2. Global optimization

We want to draw on the specialized literature on global optimization to improve the reliability of calibrations (Locatelli & Schoen, 2013). Depending on the calibration procedure special challenges arise. Likelihood-based estimation requires smoothing of the choice probabilities, while simulation-based calibration requires the application of noisy function optimization.

- This issue is under active investigation in our group and this section will be fleshed out further as our work progresses.

4.3. Miscellaneous

Other areas of improvement that require future attention is the approximate solution of the model using function approximation techniques and the issue of using massive parallelism in the model.

5. Extensions

We briefly discuss selected extensions to the baseline model that are our active areas of research:

5.1. Robust decision-making

Individuals face ubiquitous uncertainties when faced with important decisions. Policy makers vote for climate change mitigation efforts facing uncertainty about future costs and benefits (Barnett et al., 2019), while doctors decide on the timing of an organ transplant in light of uncertainty about future patient health (Kaufman et al., 2017). Economic models formalize the objectives, trade-offs, and uncertainties for such decisions. In these models, the treatment of uncertainty is often limited to risk as the model induces a unique probability distribution over sequences of possible futures. There is no role for ambiguity about the true model (Knight, 1921; Arrow, 1951) and thus no fear of model misspecification. However, limits to knowledge lead to considerable ambiguity about how the future unfolds (Hayek, 1975; Hansen, 2015).

This creates the need for robust decision rules that work well over a whole range of different models, instead of a decision rule that is optimal for one particular model. Optimal decision rules are designed without any fear of misspecification, using a single model to inform decisions. They thus perform very well if that model turns out to be true. However, their performance is very sensitive and deteriorates rapidly in light of model misspecification. Robust decision rules explicitly account for such a possibility and their performance is less affected.

Methods from distributionally robust optimization (Ben-Tal et al., 2009; Wiesemann et al., 2014; Rahimian & Mehrotra, 2019) and robust Markov decision processes (Iyengar, 2005; Nilim & El Ghaoui, 2005; Wiesemann et al., 2014) allow to construct decision rules that explicitly take potential model misspecification into account.

5.2. Uncertainty quantification

- This issue is under active investigation in our group and this section will be fleshed out further as our work progresses.

5.3. Model validation

- This issue is under active investigation in our group and this section will be fleshed out further as our work progresses.

A. Appendix

A.1. Acronyms and Symbols

Table 1: List of Acronyms

Acronym	**Meaning**
MDP	Markov decision process
EKW	Eckstein-Keane-Wolpin

Table 2: List of Symbols

Symbol	Meaning
$\mathbb{R}$	set of real numbers
$\mathbb{I}[A]$	indicator function that takes value one if event A is true
	Economic Model
t	decision period
T	number of decision periods
$a \in \mathcal{A}$	set of admissible actions
$s \in \mathcal{S}$	set of possible states with generic state s
s_t	realization of state s in period t
$a_t^\pi(s)$	decision rule that specifies an action for all states s in period t following π
$a_t^\pi(s_t)$	actual decision in period t when in state s_t following policy π
a_t	actual decision at time t by individual
a_{it}	actual decision observed by individual i at time t
$p_t(s,a)$	conditional probability distributions for s_{t+1} when choosing action a in state s in period t
$r(s,a)$	rewards when choosing action a in state s
δ	discount factor

Table 2: List of Symbols

Symbol	**Meaning**
v_t^π	expected total discounted rewards of adopting policy π from period t going forward
$\pi \in \Pi$	set of all policies
	Estimation procedure
	Computational Model
ϵ_{at}	random shock to rewards of alternative a in period t
x_{jt}	number of periods worked in occupation j by the beginning of period t
g_t	number of periods enrolled in school by the beginning of period t
$\mathcal{N}_0$	true multivariate normal distribution for random shocks
Σ	covariance matrix of random shocks
υ	admissible realization of means for future labor market shocks
α_j	parameters for rewards function when working in occupation j
β	parameters for rewards function when enrolling in school
γ	parameter for rewards function when staying at home

A.2. Parameterization

Table 3 presents the full parameterization for the simulation of the baseline sample. Keane & Wolpin (1994) analyze three different parameterizations of the model. Our example is based on their second parameterization as the cost of post-secondary education is set to zero in their first parameterization.

Table 3: Parameterization

Parameter	Value
η	0.01000
δ	0.95000
α_{10}	9.21000
α_{11}	0.04000
α_{12}	0.03300
α_{13}	-0.00050
α_{14}	0.00000
α_{15}	0.00000
α_{20}	8.20000
α_{21}	0.08000
α_{22}	0.02200
α_{23}	-0.00050
α_{24}	0.06700
α_{25}	-0.00100
β_0	5,000.00000
β_1	5,000.00000
β_2	15,000.00000
γ_0	14,500.00000
σ_{11}	0.16000
σ_{12}	0.00000
σ_{13}	0.00000
σ_{14}	0.00000
σ_2	0.25000
σ_{23}	0.00000
σ_{24}	0.00000
σ_{33}	36,000,000.00000
σ_{34}	0.00000
σ_{44}	36,000,000.00000

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