Eckstein-Keane-Wolpin models

OpenSourceEconomics*

Abstract

We present background material for a particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models. We provide an example application using our group's research code `respy`. We draw on research outside economics to identify model components ripe for improvement and explore possible extensions.

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1 Introduction

Focus of handout We present background material for a particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models (Aguirregabiria & Mira, 2010). We provide an example application using our group's research code `respy` (respy, 2018). We draw on research outside economics to identify model components ripe for improvement and explore possible extensions.

Structural models Structural economic models clearly specify an individual's objective and the constraints of their economic environment under which they operate. They are used to quantify the importance of competing economic mechanisms in determining economic outcomes and predict the impact alternative policies before their implementation (Wolpin, 2013).

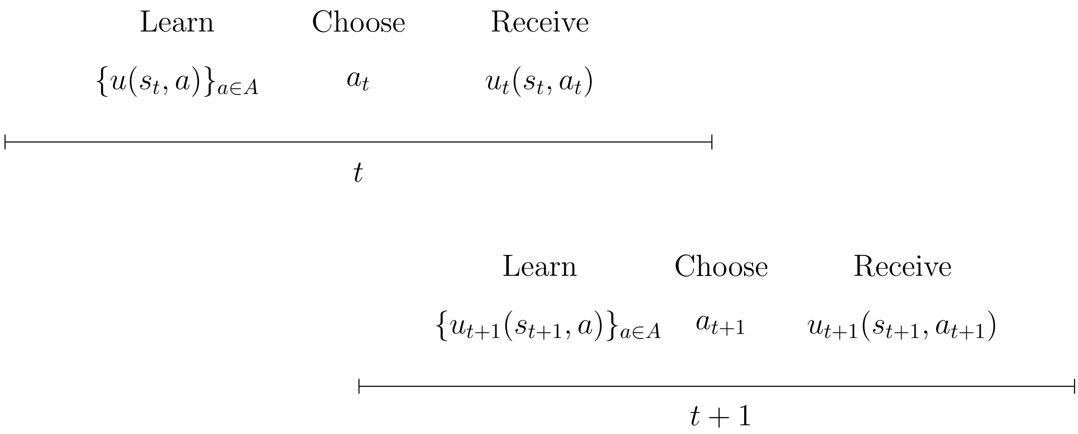
EKW models Our discussion is restricted to the class of Eckstein-Keane-Wolpin (EKW) models. These type of models is often used in labor economics for the analysis of human capital investment decisions. Differences in human capital attainment are a major determinant of inequality in a variety of life outcomes such as labor market success and educational attainment across and within countries.

Bhuller et al. (2018), for example, build a model to analyze the mechanisms determining schooling decisions in Norway. They validate their model using an increase in mandatory schooling reform and then use the model to study the underlying mechanisms the that generate the increase in schooling level and then evaluate policy alternatives.

Notation Throughout, we focus on textbook references. We will introduce acronyms and symbols as needed, but a full list of both is provided in Appendix **??**. The notation draws form the related work by Puterman (1994), Aguirregabiria & Mira (2010), and Arcidiacono & Ellickson (2011).

Stucture This handout is structured as follows. We first present the basic setup and introduce the economic framework, its mathematical formalization, and the calibration procedure. We then present one example. Finally, we outline possible improvements and extensions.

Figure 1: Timing of events



2 Setup

2.1 Economic framework

Basic setup EKW models describe sequential decision-making under risk (Machina & Viscusi, 2014; Gilboa, 2009). At time $t = 1, \ldots, T$ each individual observes the state of the economic environment $s_t \in S$ and chooses an action a_t from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate reward $r(s_t, a_t)$ and the economy evolves to a new state s_{t+1}. The transition from s_t to s_{t+1} is affected by the action. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate reward. Instead, they take the future consequences of their current action into account.

Decision rule A policy $\pi \equiv (a_1^\pi(s), \ldots, a_T^\pi(s))$ provides the individual with a prescription for choosing an action in any possible future state, where $a_t^\pi(s)$ specifies the action at a particular time t for any possible state under π. It is a sequence of decision rules and its implementation generates a sequence of rewards. The evolution of states over time is at least partly unknown as future rewards depend on, for example, shocks to preferences. Individuals use models about their economic environment to inform their beliefs about the future. For a given model, individuals thus face risk as each induces a unique objective transition probability distribution $p_t(s_t, a_t)$ for the evolution of state s_t to s_{t+1} that depends on the action a_t.

Timing of events Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of time t an individual fully learns about the immediate reward of each alternative, chooses one of them, and receives its immediate reward. Then the state evolves from s_t to s_{t+1} and the process is repeated in $t + 1$.

Decision theory Individuals make their decisions facing risk and have rational expectations as their model about the future also turns out to be true. In this case, there is a consensus that rational choices are expressed by the maximization of their expected total discounted rewards. A constant discount factor ensures dynamic consistency of preferences as the individual's future actions agree with the planned-for contingencies. Beliefs are updated according to Bayes's rule.

Equation (1) provides the formal representation of the individual's objective. Given an initial state s_1, individuals seek to implement the policy π from the set of all possible policies Π that maximizes the expected total discounted rewards over all T decision periods given the information $\mathcal{I}_1$ available.

$$\max_{\pi \in \Pi} \mathrm{E}_{s_1}^{\pi} \left[\sum_{t=1}^{T} \delta^{t-1} r(s_t, a_t^{\pi}(s_t)) \,\middle|\, \mathcal{I}_1 \right] \tag{1}$$

The exponential discount factor $0 < \delta < 1$ captures a preference for immediate over future rewards. The superscript of the expectation emphasizes that each policy π induces a different unique probability distribution over the sequences of rewards.

2.2 Mathematical formulation

EKW models are set up as a standard Markov decision processes (MDP) (Puterman, 1994; White, 1993). When making sequential decisions, the task is to determine the optimal policy π^* with the largest expected total discounted rewards $v_1^{\pi^*}$ as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of rewards and the probability that each occurs. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems. Note that in slight abuse of notation s_{t+1} is a random variable given the information available at $\mathcal{I}_t$.

Let $v_t^{\pi}(s)$ denote the expected total discounted rewards under π from period t onwards:

$$v_t^{\pi}(s_t) \equiv \mathrm{E}_{s_t}^{\pi} \left[\sum_{j=0}^{T-t} \delta^{j}\, r(s_{t+j}, a_{t+j}^{\pi}(s_{t+j})) \,\middle|\, \mathcal{I}_t \right]$$

Then $v_1^{\pi}(s_t)$ can be determined for any policy by recursively evaluating equation (2):

$$v_t^{\pi}(s) = r(s_t, a_t^{\pi}(s_t)) + \delta\, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right]. \tag{2}$$

Equation (2) expresses the rewards $v_t^{\pi}(s_t)$ of adopting policy π going forward as the sum of its immediate rewards and all expected discounted future rewards.

The principle of optimality allows to construct the optimal policy π^* by solving the optimality equations for all s and t in equation (3) recursively:

$$v_t^{\pi^*}(s_t) = \max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}. \tag{3}$$

The value function $v_t^{\pi^*}$ is the expected discounted rewards in t over the remaining time horizon assuming the optimal policy is implemented going forward.

The optimal decision is simply the alternative with the highest value:

$$a_t^{\pi^*}(s_t) \equiv \arg\max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \right\}$$

Solution approach Algorithm (1) allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate rewards in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively as the calculation of their expected future rewards is straightforward given the relevant transition probabilities.

Algorithm 1 Backward induction procedure

for $t = T, \ldots, 1$ **do**

  if t == T **then**

    $v_T^{\pi^*}(s_T) = \max_{a \in A} \left\{ r(s_T, a) \right\} \qquad \forall \, s_T \in S$

  else

    Compute $v_t^{\pi^*}(s_t)$ for each $s_t \in S$ by

      $v_t^{\pi^*}(s_t) = \max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \right\}$

    and set

      $a_t^{\pi^*}(s_t) = \arg\max_{a \in A} \left\{ r(s_t, a) + \delta \, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle| \, \mathcal{I}_t \right] \right\}.$

  end if

end for

2.3 Calibration procedure

EKW models are calibrated to data to obtain information on preference parameters and transition probabilities (Davidson & MacKinnon, 2003; Gourieroux & Monfort, 1996). Given this information, the quantitative importance of competing economic mechanisms can be assessed and the effects of public policies predicted. This requires the parameterization of all elements of the model which we collect in θ.

Data The econometrician has access to panel data for N individuals. For every observation (i, t) in the panel data set, the researcher observes action a_{it} and a subvector x_{it} of the state vector. Therefore, from an econometricians point of view, we need to distinguish between two types of state variables $s_{it} = (x_{it}, \epsilon_{it})$. Variables x_{it} that are observed by the econometrician and the individual i at time t and those that are only observed by the individual ϵ_{it}. In addition, also some realizations of the rewards $r_{it} = r(x_{it}, \epsilon_{it}, a_{it})$.

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, 2, \ldots, N; t = 1, \ldots, T_i\},$$

where T_i is the number of observations over which we observe individual i.

Procedures We briefly outline maximum likelihood estimation and the method of simulated moments. Whatever the estimation criterion, in order to evaluate it for a particular value of θ it is necessary to construct the optimal policy π^*. Therefore at each trial value of θ the whole model needs to solved by the backward induction algorithm.

Likelihood-based The individual chooses the alternative with the highest total value $a_t^*(s_t)$ which is determined by the complete state and rewards are also determined by s. However, the econometrician only observes the subset x. Given parametric assumptions about the distribution of ϵ, we can determine the probability $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$ of individual i at time t choosing d_{it} and receiving r_{it} given x_{it}.

$$\mathcal{L}(\theta \mid \mathcal{D}) = \prod_{i=1}^{N} \prod_{t=1}^{T_i} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$$

The goal of likelihood-based estimation is to find the value of the model parameters θ that maximize the likelihood function:

$$\hat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max}\, \mathcal{L}(\theta \mid \mathcal{D})$$

Simulation-based ... to be written

3 Example

We now provide either Keane & Wolpin (1994) or Keane & Wolpin (1997) here. Given the purpose of this note as a high level overview, it is probably fine to just restrict to (Keane & Wolpin, 1994).

4 Improvements

The implementation and analysis of this class of models entails several computational challenges. Among them integration of a high-dimensional non-differentiable function, large-scale global optimization of a noisy and non-smooth criterion function, function approximation, and parallelization strategies. We briefly outline each of them.

4.1 Numerical integration

We want to draw on the extensive literature in applied math on numerical integration (Davis & Rabinowitz, 2007). To clarify the structure of the integral determining the future value of a state, it is useful to consider the optimality equation in the second to last period. This allows to focus on action-specific rewards instead of future values. Let $v^{\pi}_{T-1}(s_t, a_t)$ denote the action-specific value function of choosing action a in state s while continuing with the optimal policy going forward. but sticking to the optimal policy π^* going forward.

$$
\begin{aligned}
v^{\pi}_t(s_t, a_t) &= u(s_t, a_t) + \delta \,\mathrm{E}_{s_t}\left[v^{\pi^*}_{t+1}(s_{t+1})\right] && (4)\\
&= u(s_t, a_t) + \delta \int_S v^{\pi^*}_{t+1}(s_{t+1}) \;\mathrm{d}p_t(a_t, s_t) && (5)\\
&= u(s_t, a_t) + \delta \underbrace{\int_S \max_{a_{t+1}\in A}\left\{v^{\pi}_{t+1}(s_{t+1}, a_{t+1})\right\} \mathrm{d}p_t(a_t, s_t)}_{\mathcal{I}(a_{t+1})}. && (6)
\end{aligned}
$$

The evaluation of such an integral is required millions of times during the backward induction procedure. The current practice is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities (Judd & Skrainka, 2011).

Let's consider an atemporal version of the typical integral from Keane & Wolpin (1997). In their model, individuals can choose among five alternatives. Each of the alternative-specific rewards is in part determined by a random continuous state variable that follows a normal distribution which happens to be unobserved. The transition of all observable state variables is deterministic. This results in a five-dimensional integral $\mathcal{I}(a')$ as the dimensionality is determined by the random state variables. The integral takes the following form:

$$
\mathcal{I}(a) = \int_{\epsilon} \max_{a\in A}\left\{v^{\pi}_{t+1}(x, \epsilon, a)\}\right\}\phi_{\mu,\Sigma}(\epsilon)\,\mathrm{d}\epsilon.
$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_5) \sim \mathcal{N}(\mu, \Sigma)$ follows a multivariate normal distribution with mean $\mu \in \mathbb{R}^5$, covariance matrix $\Sigma \in \mathbb{R}^{5\times 5}$, and probability density function $\phi_{\mu,\Sigma}(\epsilon)$. A key features of this integral is the lack of general separability between the random and deterministic state variables. This makes the closed-form solutions impossible even under suitable distributional assumptions

(McFadden, 1978; Rust, 1987).

4.2 Global optimization

We want to draw on the specialized literature ().

- **Likelihood-based estimation** This approach requires smoothing of the choice probabilities.

$$p_t(d_{it} \mid x_{it}, \theta) = \int \mathbb{I}\left[\delta(x_{it}, \epsilon_{it}, \theta) = a_{it}\right] g(\epsilon)\, \mathrm{d}\epsilon$$

- **Simulation-based estimation** This approach requires the optimization of a noisy function.

4.3 Miscellaneous

Function approximation, prallelization

5 Extensions

We briefly discuss selected extensions to the baseline model that are our active areas of research:

5.1 Uncertainty quantification

5.2 Robust decision-making

Individuals face ubiquitous uncertainties when faced with important decisions. Policy makers vote for climate change mitigation efforts facing uncertainty about future costs and benefits (Barnett et al., 2019), while doctors decide on the timing of an organ transplant in light of uncertainty about future patient health (Kaufman et al., 2017). Economic models formalize the objectives, trade-offs, and uncertainties for such decisions. In these models, the treatment of uncertainty is often limited to risk as the model induces a unique probability distribution over sequences of possible futures. There is no role for ambiguity about the true model (Knight, 1921; Arrow, 1951) and thus no fear of model misspecification. However, limits to knowledge lead to considerable ambiguity about how the future unfolds (Hayek, 1975; Hansen, 2015).

This creates the need for robust decision rules that work well over a whole range of different models, instead of a decision rule that is optimal for one particular model. Optimal decision rules are designed without any fear of misspecification, using a single model to inform decisions. They thus perform very well if that model turns out to be true. However, their performance is very sensitive and deteriorates rapidly in light of model misspecification. Robust decision rules explicitly account for such a possibility and their performance is less affected.

Methods from distributionally robust optimization (Ben-Tal et al., 2009; Wiesemann et al., 2014; Rahimian & Mehrotra, 2019) and robust Markov decision processes (Iyengar, 2005; Nilim & El Ghaoui, 2005; Wiesemann et al., 2014) allow to construct decision rules that explicitly take potential model misspecification into account.

5.3 Model validation

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