Eckstein-Keane-Wolpin models

An invitation for transdiciplinary collaboration

OpenSourceEconomics*

Abstract

We present background material for a particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We describe the economic setup, mathematical formulation, and calibration procedures for so-called Eckstein-Keane-Wolpin (EKW) models. We analyze an example application using our group's research codes `respy` and `estimagic`. We draw on research outside economics to improve the computational implementation and explore possible extensions.

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Contents

1. Introduction

Structural models Structural economic models clearly specify an individual's objective and the institutional and informational constraints of their economic environment under which they operate. They are calibrated to reproduce data on observed individual individual decision and experiences. Based on the results, researchers can quantify the importance of competing economic mechanisms in determining economic outcomes and forecast the effects of alternative policies before their implementation (Wolpin, 2013).

EKW models We restrict us to the class of Eckstein-Keane-Wolpin (EKW) models (Keane & Wolpin, 1997; Blundell et al., 2016; Adda et al., 2017). Labor economists often apply these models for the analysis of human capital investment decisions. Human capital is the knowledge, skills, competencies, and attributes embodied in individuals that facilitate the creation of personal, social, and economic well-being (Becker, 1964). Differences in human capital attainment are a major determinant of inequality in a variety of life outcomes such as labor market success and health across and within countries (OECD, 2001).

In Bhuller et al. (2018), for example, we apply an EKW model to analyze the mechanisms determining schooling decisions in Norway. We calibrate the model using Norwegian population panel data with nearly career-long earnings histories. We find an important role for the option value of schooling, which measures the value of the information generated by each additional year of schooling. After validating their model using an increase in mandatory schooling, we then use the model to study the underlying mechanisms that generate the increase of average years of schooling and forecast the effects of several policy alternatives.

Stucture We present background material for this particular class structural economic models to facilitate transdiciplinary collaboration in their future development. We first describe the economic setup, mathematical formulation, and calibration procedures. We then analyze an example application using our group's research codes `respy` (OSE, 2020) and `estimagic` (Gabler, 2019). Finally, we review research outside economics to address computational challenges and explore possible conceptual extensions.

Notation Throughout, we only offer a limited number of seminal references and textbooks to invite further study. We will introduce acronyms and symbols as needed, but a full list of both is provided in the Appendix. Our notation draws form the related work by Puterman (1994), Aguirregabiria & Mira (2010), and Arcidiacono & Ellickson (2011).

2. Setup

We now present the basic setup of the EKW models. We first present the economic framework, outline its mathematical formulation, and briefly describe the calibration process.

2.1. Economic framework

Basic setup EKW models describe sequential decision-making under uncertainty (Machina & Viscusi, 2014; Gilboa, 2009). At time $t = 1, \dots, T$ each individual observes the state of the economic environment $s_t \in S$ and chooses an action a_t from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate reward $r(s_t, a_t)$ and the economy evolves to a new state s_{t+1}. The transition from s_t to s_{t+1} affected by the action but remains uncertain. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate reward. Instead, they take the future consequences of their current action into account.

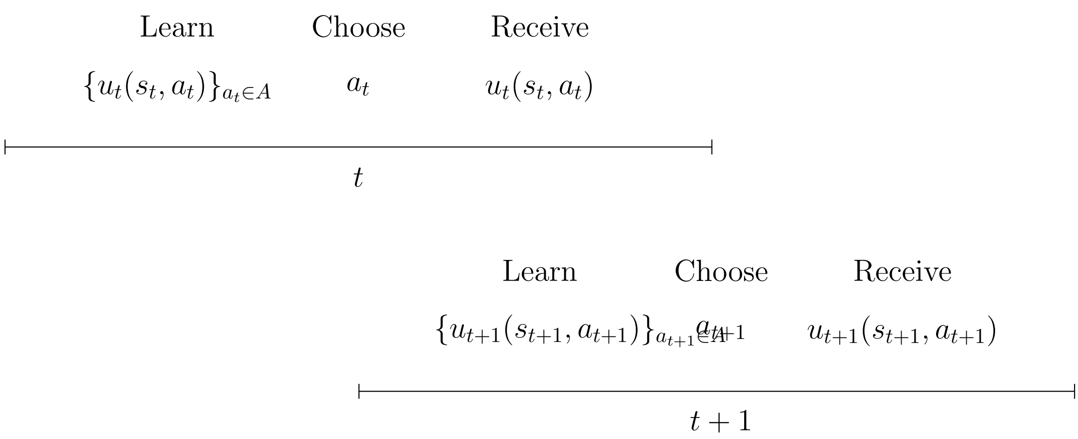
Decision rule A policy $\pi \equiv (a_1^\pi(s), \dots, a_T^\pi(s))$ provides the individual with a prescription for choosing an action in any possible future state, where the decision rule $a_t^\pi(s)$ specifies the action at a particular time t for any possible state s under π. It is a sequence of decision rules and its implementation generates a sequence of rewards. As the evolution of states over time is uncertain, individuals use a model about their economic environment to inform their subjective beliefs about the future. For a given model, individuals thus face risk as each induces a unique objective transition probability distribution $p(s_t, a_t)$ for the evolution of state s_t to s_{t+1} that depends on the action a_t. Individuals have rational expectations (Muth, 1961; Lucas, 1972) so their subjective beliefs correspond to the objective transition probabilities.

Timing of events Figure 1 depicts the timing of events in the model for two generic time periods. At the beginning of period t an individual fully learns about the immediate reward of each alternative, chooses one of them, and receives its immediate reward. Then the state evolves from s_t to s_{t+1} and the process is repeated in $t + 1$.

Decision theory Individuals make their decisions facing uncertainty and they maximize the expected total discounted rewards. An exponential discount factor $0 < \delta < 1$ parameterizes their time preference and captures a taste for immediate over future rewards.

Equation (1) provides the formal representation of the individual's objective. Given an initial state s_1, individuals seek to implement the policy π from the set of all possible policies Π that maximizes the expected total discounted rewards over all T decision periods given the

Figure 1: Timing of events



information $\mathcal{I}_1$ available in the first period.

$$\max_{\pi \in \Pi} \mathrm{E}^{\pi}_{s_1} \left[\sum_{t=1}^{T} \delta^{t-1} r(s_t, a^{\pi}_t(s_t)) \,\middle|\, \mathcal{I}_1 \right] \tag{1}$$

The superscript of the expectation emphasizes that each policy π induces a different unique probability distribution over the sequences of rewards. Note that in slight abuse of notation s_{t+1} is a random variable given the information available at $\mathcal{I}_t$ and $a^{\pi}_t(s_t)$ denotes the actual action that an individual chooses in time t if they encounter s_t and follow policy π.

2.2. Mathematical formulation

EKW models are set up as a standard Markov decision processes (MDP) (Puterman, 1994; White, 1993). When making sequential decisions under risk, the task is to determine the optimal policy π^* with the largest expected total discounted rewards $v_1^{\pi^*}$ as formalized in equation (1). In principle, this requires to evaluate the performance of all policies based on all possible sequences of rewards and to weight each by the probability that they occur. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems.

Optimal decisions in a MDP are a deterministic function of the current state s only, i.e. an optimal decision rule is always deterministic and Markovian. We restrict our notation to this special case right from the beginning.

The value function $v^{\pi}_t(s_t)$ captures the expected total discounted rewards under π from period

t onwards when in s_t:

$$v_t^\pi(s_t) \equiv \mathrm{E}_{s_t}^\pi \left[\sum_{j=0}^{T-t} \delta^j \, r_{t+j}(s_{t+j}, a_{t+j}^\pi(s_{t+j})) \,\middle|\, \mathcal{I}_t \right]$$

Then $v_1^\pi(s_t)$ can be determined for any policy by recursively evaluating equation (2):

$$v_t^\pi(s_t) = r(s_t, a_t^\pi(s_t)) + \delta \, \mathrm{E}_{s_t}^\pi \left[v_{t+1}^\pi(s_{t+1}) \,\middle|\, \mathcal{I}_t \right]. \tag{2}$$

Equation (2) expresses the total value $v_t^\pi(s_t)$ of adopting policy π going forward as the sum of its immediate rewards and all expected discounted future rewards.

The principle of optimality allows to construct the optimal policy π^* by solving the optimality equations (3) for all s and t recursively:

$$v_t^{\pi^*}(s_t) = \max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi^*} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}. \tag{3}$$

The optimal value function $v_t^{\pi^*}$ is the expected discounted rewards in t over the remaining time horizon assuming the optimal policy is implemented going forward.

The optimal decision is simply to choose the alternative with the highest total value:

$$a_t^{\pi^*}(s_t) \equiv \arg\max_{a_t \in A} \left\{ r_t(s_t, a_t) + \delta \, \mathrm{E}_{s_t}^{\pi^*} \left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] \right\}$$

Solution approach Algorithm 1 allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate rewards in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively following equation (3) as the calculation of their expected future rewards is straightforward given the relevant transition probabilities.

Algorithm 1 Backward induction procedure

for $t = T, \dots, 1$ **do**

  if t == T **then**

  $$v_T^{\pi^*}(s_T) = \max_{a_T \in A} \Big\{ r_T(s_T, a_T) \Big\} \qquad \forall\, s_T \in S$$

  else

  Compute $v_t^{\pi^*}(s_t)$ for each $s_t \in S$ by

  $$v_t^{\pi^*}(s_t) = \max_{a_t \in A} \Big\{ r_t(s_t, a_t) + \delta\, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle|\, \mathcal{I}_t \right] \Big\}$$

  and set

  $$a_t^{\pi^*}(s_t) = \arg\max_{a_t \in A} \Big\{ r_t(s_t, a_t) + \delta\, \mathrm{E}_{s_t}^{\pi} \left[v_{t+1}^{\pi^*}(s_{t+1}) \middle|\, \mathcal{I}_t \right] \Big\}.$$

  end if

end for

2.3. Calibration procedure

EKW models are calibrated to data on observed individual decisions and experiences to obtain information on reward functions, preference parameters, and transition probabilities (Davidson & MacKinnon, 2003; Gourieroux & Monfort, 1996). Given this information, the quantitative importance of competing economic mechanisms can be assessed and the effects of alternative public policies forecasted. This requires the parameterization of all elements of the model which we collect in θ.

Data The econometrician has access to observations for $i = 1, \dots, N$ individuals in each time period t. For every observation (i, t) in the data, the researcher observes action a_{it} and a subset x_{it} of the state s_{it}. Therefore, from an researcher's point of view, we need to distinguish between two types of state variables $s_{it} = (x_{it}, \epsilon_{it})$. Variables x_{it} are observed by the econometrician and the individual i at time t, while ϵ_{it} are only observed by the individual. In addition, some realizations of the rewards $r_{it} = r(x_{it}, \epsilon_{it}, a_{it})$ are observed as well. In summary, the data $\mathcal{D}$ contains:

$$\mathcal{D} = \{a_{it}, x_{it}, r_{it} : i = 1, 2, \dots, N; t = 1, \dots, T_i\},$$

where T_i is the number of observations over which we observe individual i.

Procedures Different calibration procedures exist that address particularities of the available data. We briefly outline likelihood-based and simulation-based calibration Whatever the calibration criterion, in order to evaluate it for every candidate parameterization of the model θ it is necessary to construct the optimal policy π^*. Therefore at each trial value of θ, we need to solved the model using the backward induction procedure outlined in Algorithm 1.

Likelihood-based The individual chooses the alternative with the highest total value $a_t^{\pi^*}(s_t)$ which is determined by the complete state s_t. However, researchers only observe the subset x_t. Given parametric assumptions about the distribution of ϵ, we can determine the probability $p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)$ of individual i at time t choosing a_{it} and receiving r_{it} given x_{it}.

The likelihood function $\mathcal{L}(\theta \mid \mathcal{D})$ captures the probability of the observed data as a function of θ and the goal of likelihood-based estimation is to find the value of the model parameters θ that maximizes it

$$\hat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max} \underbrace{\prod_{i=1}^{N} \prod_{t=1}^{T_i} p_{it}(a_{it}, r_{it} \mid x_{it}, \theta)}_{\mathcal{L}(\theta|\mathcal{D})}$$

Simulation-based

- This issue is under active investigation in our group and this section will be fleshed out further by Annica Gehlen as our work progresses.

3. Example

We now provide an illustration of an EKW model. We start with an outline and discussion of a model of occupational choice by Keane & Wolpin (1994). We then discuss its analysis our group's research code `respy` (OSE, 2020).

3.1. Keane & Wolpin (1994)

Setup Individuals live for a total of T periods and choose each period t to either work in one of two occupations ($a_t = 1, 2$), attend school ($a_t = 3$), or stay at home ($a_t = 4$). Immediate rewards are determined as follows:

$$r(s_t, a_t) = \begin{cases} w_{1t} = \exp\{\alpha_{10} + \alpha_{11} g_t + \alpha_{12} x_{1t} + \alpha_{13} x_{1t}^2 + \alpha_{14} x_{2t} + \alpha_{15} x_{2t}^2 + \epsilon_{1t}\} & \text{if } a_t = 1 \\ w_{2t} = \exp\{\alpha_{20} + \alpha_{21} g_t + \alpha_{22} x_{1t} + \alpha_{23} x_{1t}^2 + \alpha_{24} x_{2t} + \alpha_{25} x_{2t}^2 + \epsilon_{2t}\} & \text{if } a_t = 2 \\ \beta_0 - \beta_1 \mathbb{I}[\, g_t \geq 12\,] - \beta_2 \mathbb{I}[\, a_{t-1} \neq 3\,] + \epsilon_{3t} & \text{if } a_t = 3 \\ \gamma_0 + \epsilon_{4t} & \text{if } a_t = 4. \end{cases}$$

g_t is the number of periods of schooling obtained by the beginning of period t, x_{1t} and x_{2t} are the number of periods that the individual worked in the two occupations respectively. The reward for each labor market alternative corresponds to its wage (w_{1t}, w_{2t}) and α_1 and α_2 are thus parameters associated with the wage functions. They capture the returns to schooling and occupation-specific human capital. Turning to the rewards from school attendance, β_0 is the consumption reward of schooling, β_1 is the post-secondary cost of schooling, and β_2 is an adjustment cost associated with returning to school. The mean reward of the home alternative is denoted γ_0. The ϵ_{at}'s are alternative-specific shocks to occupational productivity, the consumption reward of schooling, and the reward of home time.

Given the structure of the reward functions and the lack of serial correlation, the state at time t is $s_t = \{g_t, x_{1t}, x_{2t}, a_{t-1}, \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$. While an individual's stock of human capital is observable $\{g_t, x_{1t}, x_{2t}, a_{t-1}\}$ to the individual and the researcher, the $\{\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$ are only observable by the individual. The observable components evolve deterministically according to the following rules:

$$\begin{aligned} x_{1,t+1} &= x_{1t} + \mathbb{I}[\, a_t = 1\,] \\ x_{2,t+1} &= x_{2t} + \mathbb{I}[\, a_t = 2\,] \\ g_{t+1} &= g_t \;\; + \mathbb{I}[\, a_t = 3\,]. \end{aligned}$$

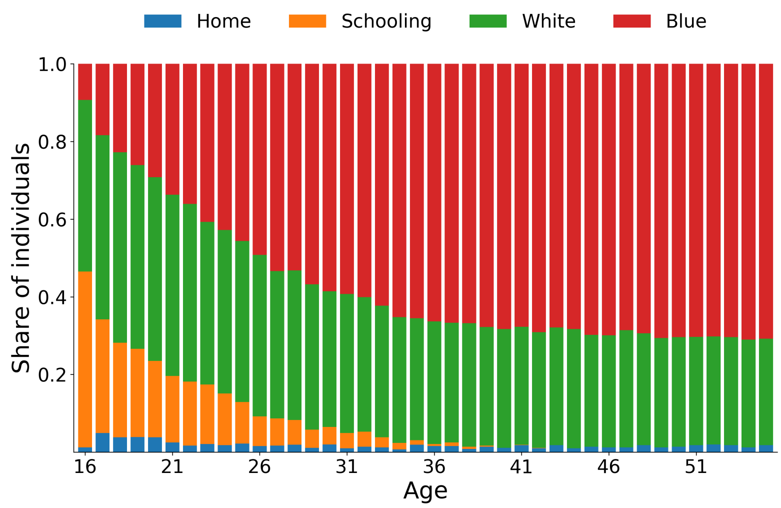
However, the unobservable components evolve randomly. They are jointly normally distributed $[\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}]^T \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ with mean zero and covariance matrix $\mathbf{\Sigma}$.

Parameterization We follow Keane & Wolpin (1994) and study this model under the following parameterization. When entering the model, individuals are identical and have no labor market experience ($x_{11} = x_{21} = 0$) but ten years of schooling ($g_1 = 10$). The basic idea is that individuals are about age 16 when entering the model and then follow it for $T = 40$ years until retirement. Different choices over the life cycle are then simply the cumulative effects of different shocks.

Schooling increases wages by only 4% in the first occupation compared to 8% in the second. We will thus refer to the former as blue-collar and the latter as white-collar going forward. Starting wages are considerably lower in the white-collar sector, but wages increase more rapidly with occupation-specific experience compared to blue-collar wages. Own-work experience is highly valuable in both occupations. However, while white-collar wages increase with blue-collar experience as well, the opposite is not true. There is a consumption value of schooling of $5,000, but the total cost of pursuing post-secondary education is considerable and amounts to $5,000. Once leaving school, individuals incur a nearly prohibitive cost of $15,000 for re-enrolling. Individuals are forward-looking with a discount factor of 0.95. The random shocks are uncorrelated across alternatives. Further details about the parameterization are available in our Appendix.

Descriptives We simulate the life cycle histories of 1,000 individuals. Figure 2 shows the share of individuals choosing each of the four alternatives by period. Initially, roughly 52% of individuals enroll in school, but this share declines rapidly, and only 19% attain any post-secondary education. Right away, about 35% of individuals are working in the blue-collar occupation. Blue-collar employment initially increases even further to peak at 67% as individuals are leaving school and entering the labor market. White-collar employment steadily rises over the life cycle but never reaches more than 35%. About 5% of individuals stay at home each period.

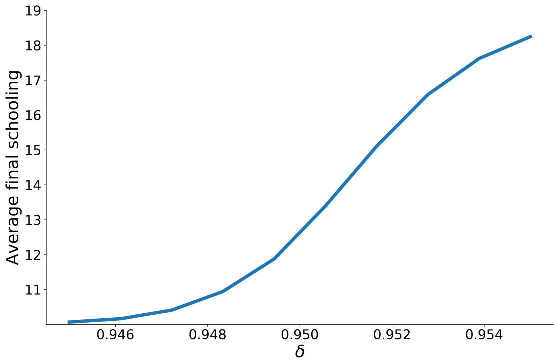
Figure 2: Choices over the life cycle



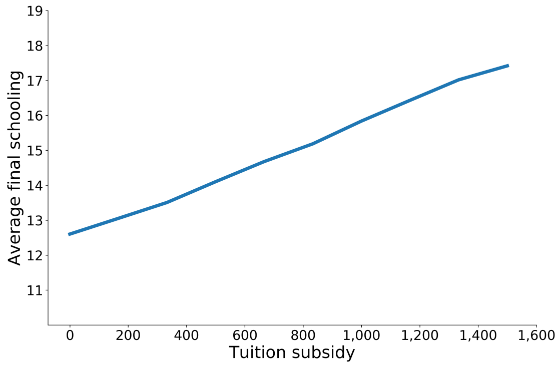
Overall, the level of average final schooling is slightly above a high school degree with 12.6 years. Individuals incur the immediate costs of their schooling investments in the form of tuition and foregone earnings right at the beginning of their life cycle. Doing so maximizes their ability to reap the reward of increased rewards over the remaining time periods.

Economic mechanisms and policy evaluation Figure 3 illustrates the ability of structural economic models to quantify the impact of economic mechanisms and to forecast the effect of public policies. On the left, we vary the discount factor δ between 0.945 and 0.955 while we reduce β_1 by the size of a tuition subsidy of up to $\$1,500$ on the right. In both cases, we are interested in the effect on average final schooling.

Figure 3: Economic mechanism and policy forecast



(a) Time preference



(b) Tuition subsidy

Increases in the discount factor and the tuition subsidy both result in increased levels of average final schooling. However, they do so very different reasons. While individuals put more emphasis on the future benefits of their schooling investment in the former, they react to a reduction in its immediate cost in the latter.

3.2. `respy`

We can flexibly specify, simulate, and calibrate EKW models using our group's research codes `respy` (OSE, 2020) and `estimagic` (Gabler, 2019). We briefly showcase a typical workflow for such an analysis. Instructions on how to use the packages, obtain the source code, replicate several seminal papers, and other implementation details are available in their respective online documentation at https://respy.readthedocs.io and https://estimagic.readthedocs.io.

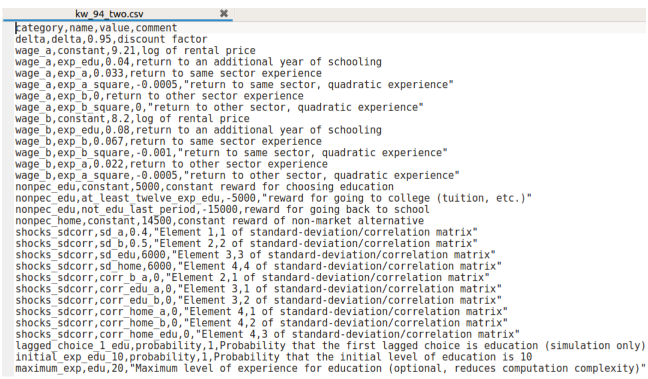
Workflow illustration Figure 4 illustrates the typical workflow with `respy`. Initially, the user provides the observed data, the parameterization of the model, and other options. All together define the structure of the model, and we can construct the functionality for the evaluation of the likelihood function and the simulation of the data. We can use our group's optimization toolbox `estimagic` of the model and then study the properties of data simulated based on the results.

Figure 4: Typical workflow

```python
from estimagic.optimization.optimize import maximize
import respy as rp

# obtain model input
df, params, options = get_model_input()

# process model
crit_func = rp.get_crit_func(params, options, df)
simulate = rp.get_simulate_func(params, options)

# perform likelihood-based calibration and simulate dataset
results, params_rslt = maximize(crit_func, params, "nlopt_bobyqa")
df_rslt = simulate(params_rslt)

# conduct analysis
...
```

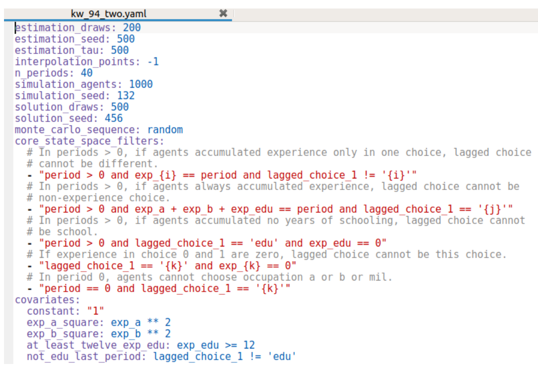
Figure 5 shows the model specification files for Keane & Wolpin (1994). The file on the left deals with the parameterization of the model. It specifies the coefficient values for the reward functions and the distribution of the unobservable state variables. On the right, we provide details on the construction of the observed state variables and tuning parameters for numerous numerical methods.

Figure 5: Model specification


kw_94_two.csv

```
category,name,value,comment
delta,delta,0.95,discount factor
wage_a,constant,9.21,log of rental price
wage_a,exp_edu,0.04,return to an additional year of schooling
wage_a,exp_a,0.033,return to same sector experience
wage_a,exp_a_square,-0.0005,"return to same sector, quadratic experience"
wage_a,exp_b,0,return to other sector experience
wage_a,exp_b_square,0,"return to other sector, quadratic experience"
wage_b,constant,8.2,log of rental price
wage_b,exp_edu,0.08,return to an additional year of schooling
wage_b,exp_b,0.067,return to same sector experience
wage_b,exp_b_square,-0.001,"return to same sector, quadratic experience"
wage_b,exp_a,0.022,return to other sector experience
wage_b,exp_a_square,-0.0005,"return to other sector, quadratic experience"
nonpec_edu,constant,5000,constant reward for choosing education
nonpec_edu,at_least_twelve_exp_edu,-5000,"reward for going to college (tuition, etc.)"
nonpec_edu,not_edu_last_period,-15000,reward for going back to school
nonpec_home,constant,14500,constant reward of non-market alternative
shocks_sdcorr,sd_a,0.4,"Element 1,1 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_b,0.5,"Element 2,2 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_edu,6000,"Element 3,3 of standard-deviation/correlation matrix"
shocks_sdcorr,sd_home,6000,"Element 4,4 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_b_a,0,"Element 2,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_edu_a,0,"Element 3,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_edu_b,0,"Element 3,2 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_a,0,"Element 4,1 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_b,0,"Element 4,2 of standard-deviation/correlation matrix"
shocks_sdcorr,corr_home_edu,0,"Element 4,3 of standard-deviation/correlation matrix"
lagged_choice_1_edu,probability,1,Probability that the first lagged choice is education (simulation only)
initial_exp_edu_10,probability,1,Probability that the initial level of education is 10
maximum_exp,edu,20,"Maximum level of experience for education (optional, reduces computation complexity)"
```

(a) Parameterization


kw_94_two.yaml

```yaml
estimation_draws: 200
estimation_seed: 500
estimation_tau: 500
interpolation_points: -1
n_periods: 40
simulation_agents: 1000
simulation_seed: 132
solution_draws: 500
solution_seed: 456
monte_carlo_sequence: random
core_state_space_filters:
  # In periods > 0, if agents accumulated experience only in one choice, lagged choice
  # cannot be different.
  - "period > 0 and exp_{i} == period and lagged_choice_1 != '{i}'"
  # In periods > 0, if agents always accumulated experience, lagged choice cannot be
  # non-experience choice.
  - "period > 0 and exp_a + exp_b + exp_edu == period and lagged_choice_1 == '{j}'"
  # In periods > 0, if agents accumulated no years of schooling, lagged choice cannot
  # be school.
  - "period > 0 and lagged_choice_1 == 'edu' and exp_edu == 0"
  # If experience in choice 0 and 1 are zero, lagged choice cannot be this choice.
  - "lagged_choice_1 == '{k}' and exp_{k} == 0"
  # In period 0, agents cannot choose occupation a or b or mil.
  - "period == 0 and lagged_choice_1 == '{k}'"
covariates:
  constant: "1"
  exp_a_square: exp_a ** 2
  exp_b_square: exp_b ** 2
  at_least_twelve_exp_edu: exp_edu >= 12
  not_edu_last_period: lagged_choice_1 != 'edu'
```

(b) Options

4. Improvements

The implementation of EKW models poses several computational challenges. Among them numerical integration, global optimization, function approximation, and efficient parallelization. We now describe some of our efforts to align `respy` and `estimagic` with the state-of-the-art in computational methods. We have concluded our own preparatory work and actively seek input from domain experts for further improvements and subsequent joint publication.

4.1. Numerical integration

In Gabler, Eisenhauer, & Suchy (2020) we draw on the extensive literature in applied math on numerical integration (Davis & Rabinowitz, 2007; Gerstner & Griebel, 1998).

To clarify the structure of the integral determining the future value of a state, it is useful to consider the optimality equation in a generic time period t. Let $v_t^\pi(s_t, a_t)$ denote the action-specific value function of choosing action a in state s while continuing with the optimal policy going forward.

$$
\begin{aligned}
v_t^\pi(s_t, a_t) &= u(s_t, a_t) + \delta \,\mathrm{E}_{s_t}\left[v_{t+1}^{\pi^*}(s_{t+1}) \,\middle|\, \mathcal{I}_t \right] && (4)\\
&= u(s_t, a_t) + \delta \int_S v_{t+1}^{\pi^*}(s_{t+1}) \; \mathrm{d}p_t(a_t, s_t) && (5)\\
&= u(s_t, a_t) + \delta \underbrace{\int_S \max_{a \in A} \left\{ v_{t+1}^{\pi}(s_{t+1}, a_{t+1}) \right\} \mathrm{d}p_t(a_t, s_t)}_{\mathcal{I}(a_{t+1})}. && (6)
\end{aligned}
$$

We need to solve the integral $\mathcal{I}(a_{t+1})$ millions of times during the backward induction procedure. The current practice is to implement a random Monte Carlo integration which introduces considerable numerical error and computational instabilities (Judd & Skrainka, 2011).

Let's consider an atemporal version of the typical integral from Keane & Wolpin (1994). As outlined earlier, individuals can choose among four alternatives. Each of the alternative-specific rewards is in part determined by a random continuous state variable that follows a normal distribution. The transition of all other state variables is deterministic. This results in a four-dimensional integral of the following form:

$$
\int_\epsilon \max_{a \in A} \left\{ v_{t+1}^{\pi}(x_{t+1}, \epsilon, a) \right\} \phi_{\mu,\Sigma}(\epsilon) \, \mathrm{d}\epsilon.
$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_4) \sim \mathcal{N}(\mu, \Sigma)$ follows a multivariate normal distribution with mean $\mu \in \mathbb{R}^4$, covariance matrix $\Sigma \in \mathbb{R}^{4\times4}$, and probability density function $\phi_{\mu,\Sigma}$.

4.2. Global optimization

In Gabler, Eisenhauer, & Röhl (2020), we draw on the specialized literature on global optimization to assess and improve the reliability of the model calibration (Nocedal & Wright, 2006; Locatelli & Schoen, 2013).

We conduct a benchmarking exercise using Keane & Wolpin (1994, 1997) as a well-known and empirically-grounded test case. Depending on the calibration procedure, particular challenges arise. For example, likelihood-based estimation requires smoothing of the choice probabilities, while simulation-based calibration involves noisy function optimization.

5. Extensions

We are actively pursuing several extensions to the basic setup and standard analysis of EKW models. For example, we draw on the methodological literature on robust-decision making and uncertainty quantification to account for the uncertainties in our analysis. Also, we are working with the German Institute for Economic Research and Statistics Norway to improve the available data for the calibration of the model models. Again, we have concluded our own preparatory work and actively seek input from domain experts for further improvements and subsequent joint publication.

5.1. Robust decision-making

The uncertainties involved in human capital investments are ubiquitous (Becker, 1964). Individuals usually make investments early in life when they are still uncertain about their abilities and tastes. Their returns also depend on demographic, economic, and technological trends that only start to unfold years from now. However, the treatment of uncertainty in life cycle models of human capital investment is very narrow. A model provides individuals with a formalized view of their economic environment and implies unique probabilities for all possible future events. Individuals have no fear of model misspecification.

In Eisenhauer & Suchy (2020), we address this shortcoming by formulating, implementing, and exploring a life cycle model of robust human capital investment where individuals face risk within a model and ambiguity about the model (Arrow, 1951). Ambiguity arises as individuals do not know the true model and consider a whole set of models as reasonable descriptions of their economic environment. Individuals fear model misspecification and thus seek robust decisions, i.e., decisions that perform well over the whole range of models.

We incorporate methods from robust optimization (Ben-Tal et al., 2009; Wiesemann et al., 2014; Rahimian & Mehrotra, 2019) and robust Markov decision processes (Iyengar, 2005; Nilim & El Ghaoui, 2005) that allow us to construct decision rules that explicitly take potential model misspecification into account.

5.2. Uncertainty quantification

There are numerous sources of uncertainties in the policy predictions inferred from a calibrated EKW model. The model is subject to misspecification, its numerical implementation introduces approximation error, the data is subject to measurement error, and the estimated parameters remain partly uncertain. However, economists display incredible certitude as they disregard all uncertainty (Manski, 2013).

In Gabler, Eisenhauer, Janys, & Mensinger (2020), we draw on a rich literature in other disciplines where a proper accounting of the uncertainty in forecasts from complex computational models is mandatory (Smith, 2014; Saltelli et al., 2004, 2008). However, uncertainty quantification for EKW models poses several unique challenges. They usually have a large number of uncertain and correlated parameters and the quantity of interest is a complex nonlinear function of the parameter and time-consuming to compute. Using methods from machine learning, we set up a surrogate model that is fast to evaluate. We analyze the seminal model of (Keane & Wolpin, 1994, 1997) to showcase our approach in a well-known and empirically-motivated setting and characterize the uncertainty in their key findings.

5.3. Model validation

In Bhuller et al. (2018), we calibrate an EKW model on Norwegian population panel data with nearly career-long earnings histories. Due to the richness of the data, we can validate the model using a mandatory schooling reform. Our data includes substantial geographic variation in compulsory schooling across Norway between 1960 and 1975 as mandatory schooling increased from 7 to 9 years at different points in time across municipalities. We split our data into an estimation and validation sample. We only use pre-reform data in our estimation, forecast the effect of increasing mandatory schooling by two years, and compare our forecast with the post-reform outcome. Doing so allows us to assess the ability of our model to extrapolate outside the support of our estimation data.

Such validation of computational models is a prerequisite for their use in other disciplines (Adams, 2012; Oberkampf & Roy, 2010). It is extremely rare in economics as drastic regime shifts are seldom available in observational data and costly to implement in large-scale experiments.

5.4. Nonstandard expectations

Eisenhauer et al. (2020), we relax the assumption that individual expectations are perfectly in line with the calibrated EKW model. Instead, we analyze and quantify the effect of biased expectations about wage growth in part-time employment on life cycle wage profiles. We design specific survey questions for the German Socio-Economic Panel (Haisken-DeNew & Frick, 2005) and elicit the expected wage trajectories for full-time and part-time employment directly. Thus, we can incorporate the belief elicitation directly in the design of a life cycle model of female labor supply.

A. Appendix

A.1. Acronyms and Symbols

Table 1: List of Acronyms

Acronym	**Meaning**
MDP	Markov decision process
EKW	Eckstein-Keane-Wolpin

Table 2: List of Symbols

Symbol	Meaning
$\mathbb{R}$	set of real numbers
$\mathbb{I}[A]$	indicator function that takes value one if event A is true
	Economic Model
t	decision period
T	number of decision periods
$a \in \mathcal{A}$	set of admissible actions
$s \in \mathcal{S}$	set of possible states with generic state s
s_t	realization of state s in period t
$a_t^\pi(s)$	decision rule that specifies an action for all states s in period t following π
$a_t^\pi(s_t)$	actual decision in period t when in state s_t following policy π
a_t	actual decision at time t by individual
a_{it}	actual decision observed by individual i at time t
$p_t(s, a)$	conditional probability distributions for s_{t+1} when choosing action a in state s in period t
$r(s, a)$	rewards when choosing action a in state s
δ	discount factor

Table 2: List of Symbols

Symbol	Meaning
v_t^{π}	expected total discounted rewards of adopting policy π from period t going forward
$\pi \in \Pi$	set of all policies
Estimation procedure	
Computational Model	
ϵ_{at}	random shock to rewards of alternative a in period t
x_{jt}	number of periods worked in occupation j by the beginning of period t
g_t	number of periods enrolled in school by the beginning of period t
$\mathcal{N}_0$	true multivariate normal distribution for random shocks
Σ	covariance matrix of random shocks
υ	admissible realization of means for future labor market shocks
α_j	parameters for rewards function when working in occupation j
β	parameters for rewards function when enrolling in school
γ	parameter for rewards function when staying at home

A.2. Parameterization

Table 3 presents the full parameterization for the simulation of the baseline sample. Keane & Wolpin (1994) analyze three different parameterizations of the model. Our example is based on their second parameterization as the cost of post-secondary education is set to zero in their first parameterization.

Table 3: Parameterization

Parameter	Value
η	0.01000
δ	0.95000
α_{10}	9.21000
α_{11}	0.04000
α_{12}	0.03300
α_{13}	-0.00050
α_{14}	0.00000
α_{15}	0.00000
α_{20}	8.20000
α_{21}	0.08000
α_{22}	0.02200
α_{23}	-0.00050
α_{24}	0.06700
α_{25}	-0.00100
β_0	5,000.00000
β_1	5,000.00000
β_2	15,000.00000
γ_0	14,500.00000
σ_{11}	0.16000
σ_{12}	0.00000
σ_{13}	0.00000
σ_{14}	0.00000
σ_2	0.25000
σ_{23}	0.00000
σ_{24}	0.00000
σ_{33}	36,000,000.00000
σ_{34}	0.00000
σ_{44}	36,000,000.00000

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