# Robust investments under risk and ambiguity

## A robust decision rule for Harold Zurcher

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# Revisiting Rust (1987)

# A short summary

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- Monthly data on bus mileage and engine replacements for 104 buses over 10 years, is provided by the modeled agent (Harold Zurcher).

Model:

- Single agent Markov decision problem over infinite horizon.
- Each month $t$ the agent decides to maintain or replace the bus engine.
- Mileage since last replacement $x_t$ is discretisized in states of 5000 miles.
- If the engine is replaced, $x_t$ is set to 0.

# A short summary

In Rust (1987) and Rust (1988) it is shown, that there exists an optimal stationary decision rule, which can be deviated from the Bellman equation, (Bellman (1954)):

$$V_\theta(x_t) = \max_{i_t \in \{0,1\}} \left[ u(x_t, i_t, \theta_1) + \epsilon_t(i_t) + \beta EV_\theta(x_t, i_t) \right]$$

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with

$$u(x_t, i_t, \theta_1) = \begin{cases} -c(x_t, \theta_1) & \textbf{if} \quad i_t = 0 \\ -[RC + c(0, \theta_1)] & \textbf{if} \quad i_t = 1 \end{cases}$$

# Model parameters

The decision process is shaped by the following parameters:

| | |
|---|---|
| $\beta$ | Discount factor. |
| $\theta_1$ | Parameters determining the costs of each alternative. |
| $\theta_2$ | Probability distribution parameters of unobserved costs $\epsilon_t$. |
| $\theta_3$ | Probability distribution parameters of the driven mileage $x_t$. |

## A short summary

- Rust (1988) imposes the conditional independence (CI):

$$p(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t, i_t, \theta_2, \theta_3) = q(\epsilon_{t+1} | x_{t+1}, \theta_2) p(x_{t+1} | x_t, i_t, \theta_3)$$

which yields a computational traceable solution for the estimation of the expected value of future periods.

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First:

$$l^1(x_1, ..., x_T, i_1, ..., i_T | x_0, i_0, \theta) = \prod_{t=1}^{T} p(x_t | x_{t-1}, i_{t-1}, \theta_3)$$

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Second:

$$l^2(x_1, ..., x_T, i_1, ..., i_T | \theta) = \prod_{t=1}^{T} P(i_t | x_t, \theta)$$

# Questions addressed in our research

- How sensible is the performance of the decision rule to the estimation quality of the transition probabilities?
- Can the agent account for uncertainty in his estimates?
- Does accounting for uncertainty lead to a better decision strategy?

# Assessing the quality of estimation

# Motivation

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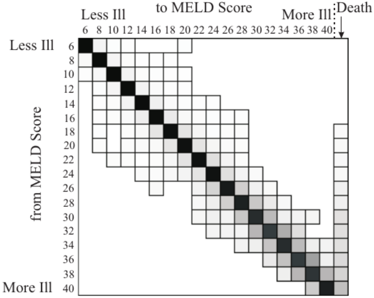


Transition matrix

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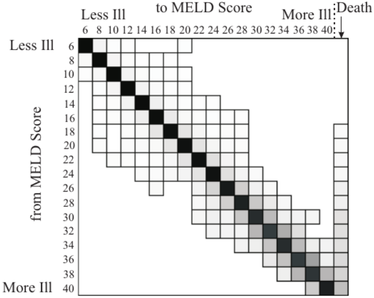
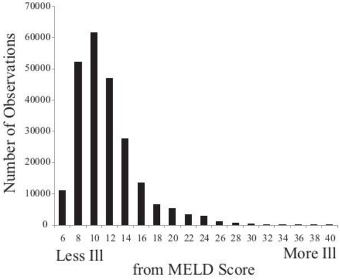


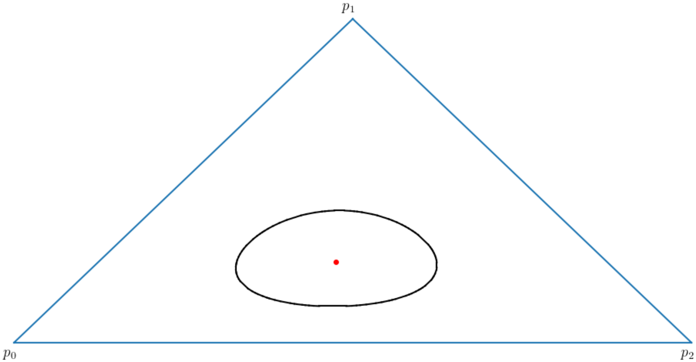

Transition matrix




NumObs per state

# Uncertainty sets

- The more precise the estimate, the smaller the uncertainty and therefore the smaller the set.



# Distance of transition probabilities

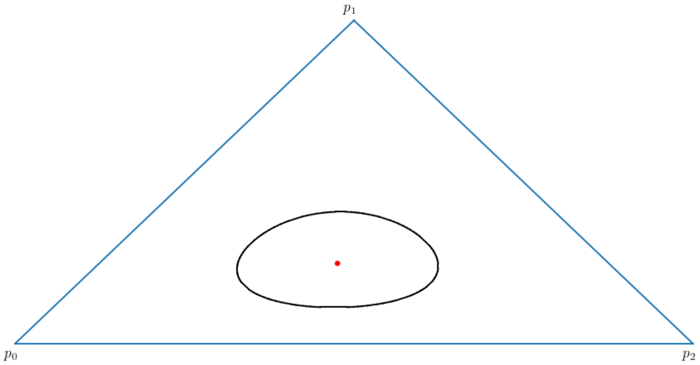
The distance between any point in the set and the estimate (red) is calculated by the Kullback Leibler divergence (Kullback and Leibler (1951)):

$$D(p||\hat{p}) = \sum_{i=0}^{2} \hat{p}(i) log \frac{p(i)}{\hat{p}(i)}$$

where $\hat{p}$ is the estimate and $p$ any point in the simplex.

# Uncertainty sets

$D(p||\hat{p}) = 0.06$



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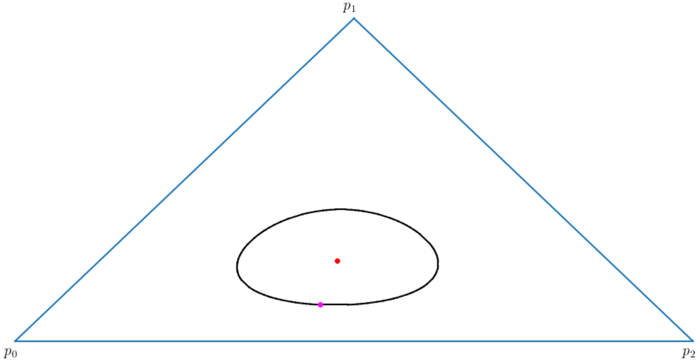

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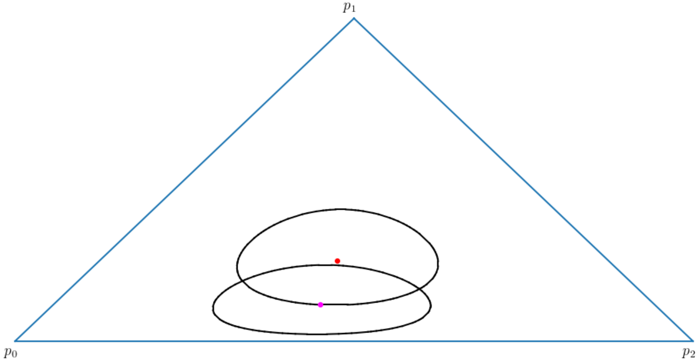

# Size of uncertainty set

Following the ideas of Ben-Tal, den Hertog, De Waegenaere, Melenberg, and Rennen (2013):

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| $\omega$ | Probability, that $\tilde{p}_s \in P_s$. |

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Therefore

$$\rho = F^{-1}_{|S|-1}(\omega)/2N_s$$

# Robust decision making

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- Robust decision making is a game nature vs. agent.
- First the agent chooses his action to maximize his present value.
- Then nature chooses the transition probabilities to minimize the agent's value.
- The agent and nature have common information and therefore the agent chooses in the first step the alternative with the highest (max) minimal (min) value.

# Theory

This leads to a robust Bellman equation, which in the framework of Rust (1987) is:

$$V_\theta(x_t) = \max_{i_t \in \{0,1\}} \left[ u(x_t, i_t, \theta_1) + \epsilon_t(i_t) + \beta \min_{\theta_3 \in P_s^{i_t}} EV_\theta(x_t, i_t) \right]$$

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compared to the standard Bellman equation:

$$V_\theta(x_t) = \max_{i_t \in \{0,1\}} \left[ u(x_t, i_t, \theta_1) + \epsilon_t(i_t) + \beta EV_\theta(x_t, i_t) \right]$$

# Economic consequence

A stylized example:

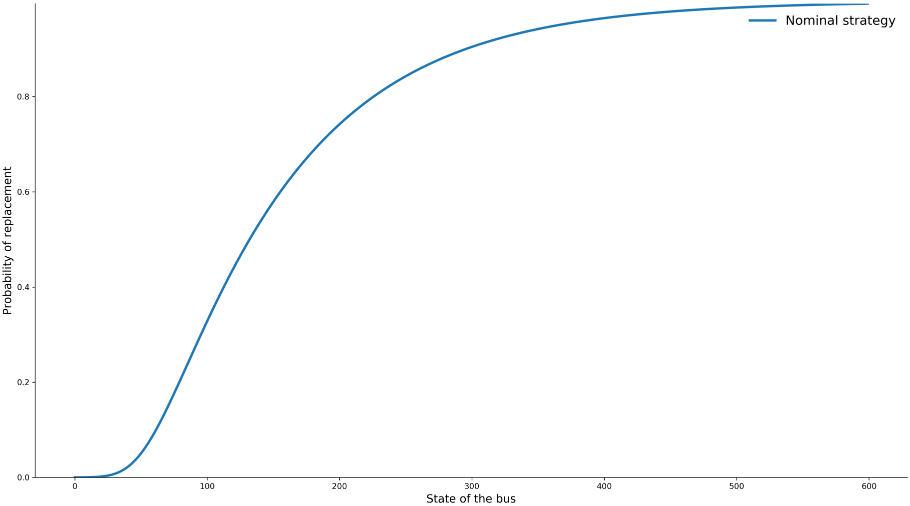
- $RC = 10$
- $c(x_t, \theta_1) = 0.01 * x_t$

# Set size vs. transition probabilities in %

$N_s = \frac{4292}{389}$

| $\omega$ | $\rho$ | $p_0$ | $p_1$ | $p_2$ | $p_3$ | ... | $p_9$ | $p_{10}$ | $p_{11}$ | $p_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2.9 | 8.1 | 27.9 | 24.6 | ... | 0.07 | 0.09 | 0.05 | 0.02 |
| 0.54 | 0.536 | 0.3 | 1.6 | 9.9 | 15.5 | ... | 1.2 | 2.7 | 2.3 | 1.9 |
| 0.99 | 0.953 | 0.1 | 0.8 | 5.8 | 10.7 | ... | 2.2 | 5.9 | 5.9 | 5.7 |

# Maintenance probabilities



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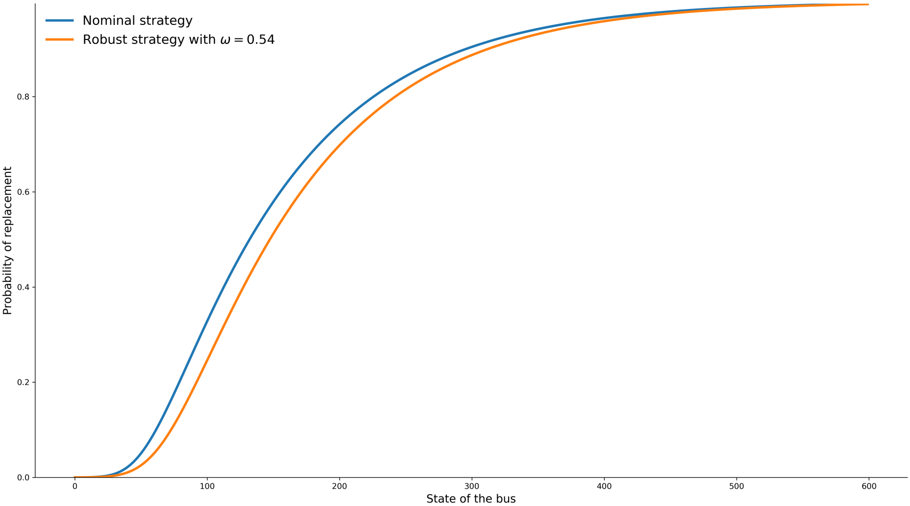

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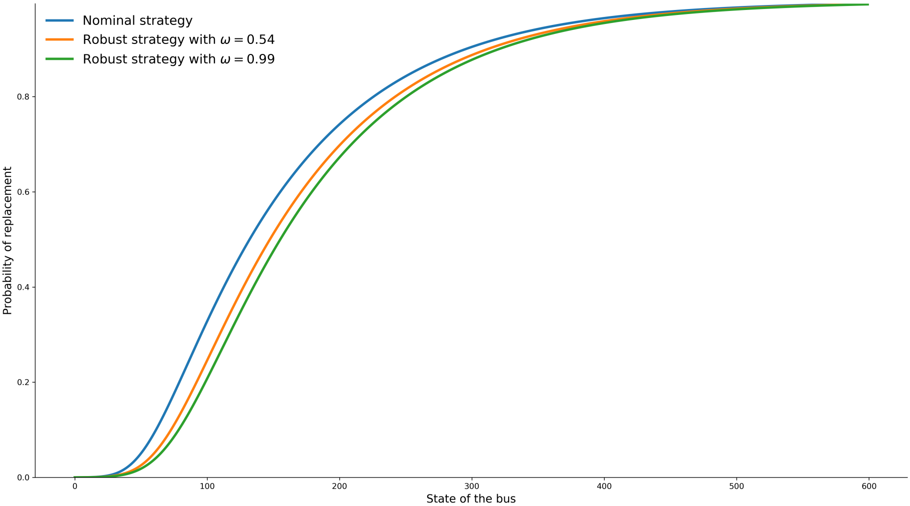
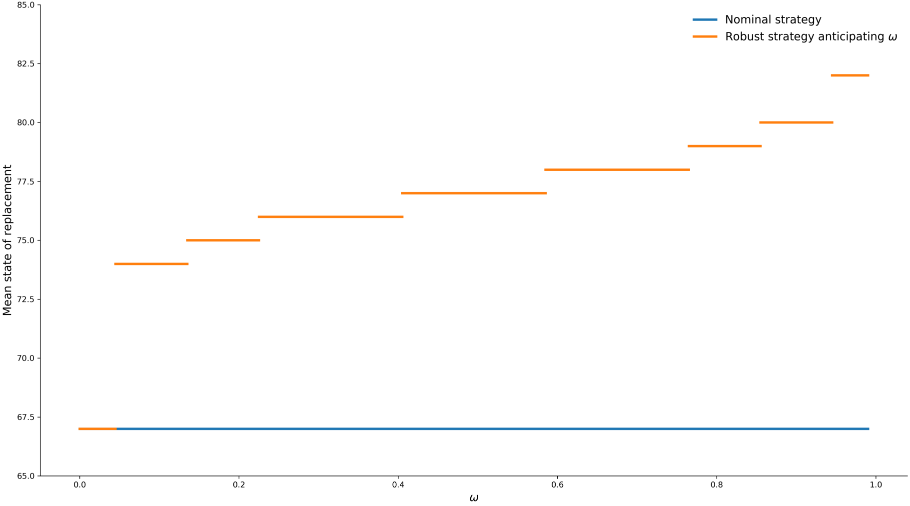
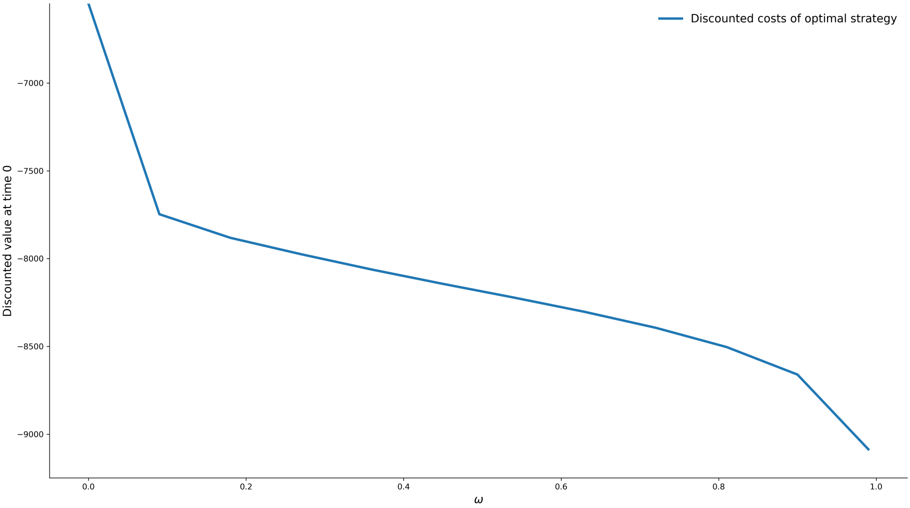


# Threshold analysis



# Performance comparison



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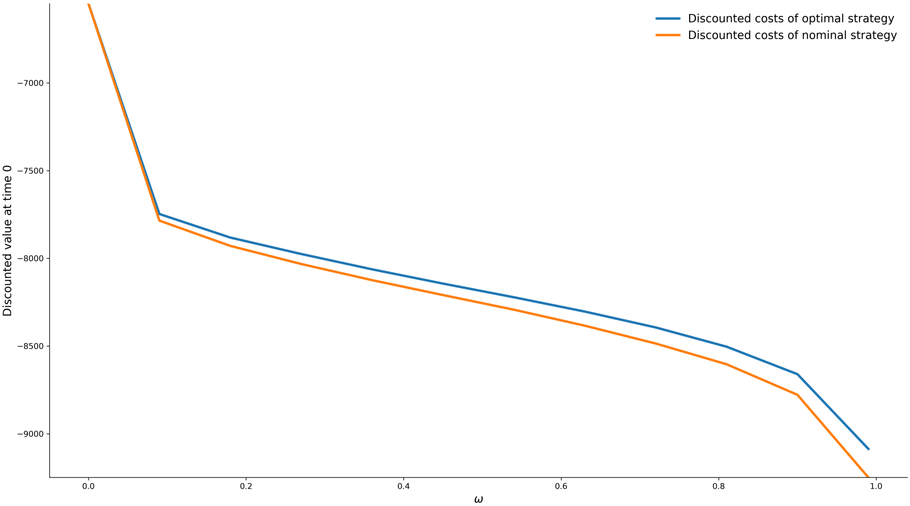

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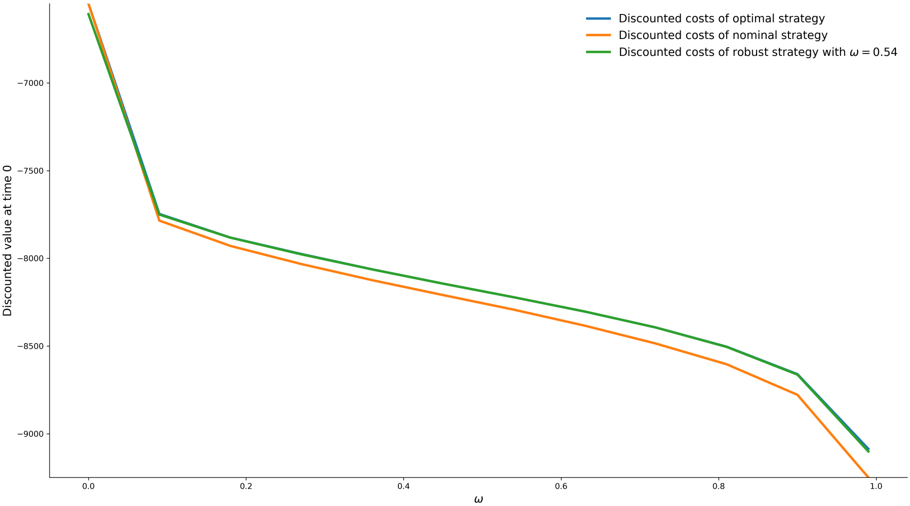

**Thank you for your attention**

# Our project online

Code, documentation, examples, and much more available online at

`https://github.com/OpenSourceEconomics`

**Visit us!**

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