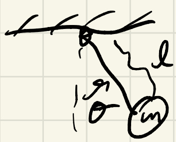
Continuous-Time Dynamics

- Most general/generic for smooth systems:

$$\underbrace{\dot{x}}_{\text{"dynamics"}} = f(\underbrace{x}_{x \in \mathbb{R}^n \text{ "state"}}, \overbrace{u}^{u \in \mathbb{R}^m \text{ "input"}})$$

- for a mechanical system $x = \begin{bmatrix} q \\ v \end{bmatrix}$ — q: "configuration" (not always a vector); v: velocity

- Example (Pendulum):



$$\boxed{ml^2\ddot{\theta} + mgl\sin(\theta) = \underbrace{\tau}_{u}}$$

$$q = \theta, \quad v = \dot{\theta}$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) + \frac{1}{ml^2}u \end{bmatrix}}_{f(x,u)}$$

$x \in S^1 \times \mathbb{R}$ (cylinder)

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* Control-Affine Systems

$$\dot{x} = \underbrace{f_0(x)}_{\text{"drift"}} + G(x)u$$

- most problems can be put in this form

- Pendulum: $f_0(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}$

* Manipulator Dynamics:

$$\underbrace{M(q)}_{\text{"Mass matrix"}}\dot{v} + \underbrace{C(q,v)}_{\substack{\text{"Dynamics Bias"} \\ \text{(Coriolis + Gravity)}}} = \underbrace{B(q)}_{\text{"Input Jacobian"}}u$$

$$\underbrace{\dot{q} = G(q)v}_{\text{kinematics}} \qquad \dot{x} = f(x,u) = \begin{bmatrix} G(q)v \\ M(q)^{-1}(B(q)u - C) \end{bmatrix}$$

- Pendulum:

$$M(q) = ml^2, \quad C(q,v) = gl\sin(\theta), \quad B = I, \quad G = I$$

- All mechanical systems can be written in this form

- This is just a way of re-writing the Euler-Lagrange equation for:

$$L = \frac{1}{2}v^T M(q) v - V(q)$$

(Bonus points for showing this)

* Linear Systems

$$\dot{x} = A(t)x + B(t)u$$

- Called "time invariant" $A(t) = A$, $B(t) = B$
- Called "time varying" otherwise
- Super important in control design
- We often approximate nonlinear systems with linear ones:

$$\dot{x} = f(x,u) \Rightarrow A \approx \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}$$

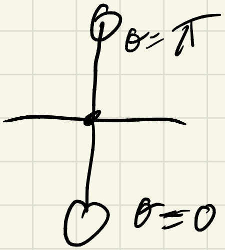
Equilibria:

- A point where system will remain "at rest"

$$\Rightarrow \dot{x} = f(x,u) = 0$$

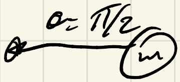
- Algebraically, roots of the dynamics
- Pendulum

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} \dot{\theta} = 0 \\ \theta = 0, \pi \end{array}$$



* First Control Problem:

- Can we move the equilibrium point?



$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\pi/2) + \frac{1}{m\ell^2}u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

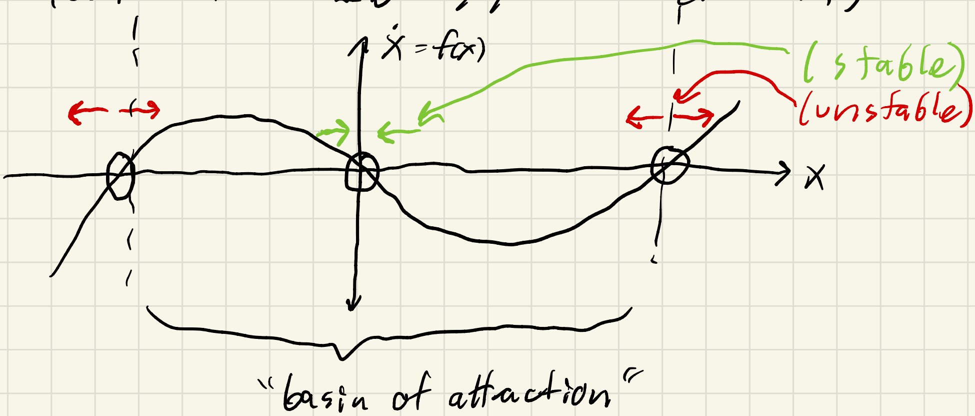
$$\frac{1}{m\ell^2}u = -\frac{g}{\ell}\underbrace{\sin(\pi/2)}_{=1} \Rightarrow u = mg\ell$$

- In general, we get a root-finding problem in u:

$$f(x^*, u) = 0$$

Stability of Equilibria:

- When will a system stay "near" an equilibrium point under perturbations.
- Let's look at a 1D system ($x \in \mathbb{R}$)



$$\frac{\partial f}{\partial x} < 0 \Rightarrow \text{stable}, \quad \frac{\partial f}{\partial x} > 0 \Rightarrow \text{unstable}$$

* To generalize to higher dimensions:

- $\frac{\partial f}{\partial x}$ is a Jacobian matrix

- Take an eigendecomposition $\Rightarrow$ decouple into n 1D systems

$$\mathrm{Re}\left[\mathrm{eig}\left(\frac{\partial f}{\partial x}\right)\right] < 0 \Rightarrow \text{stable}$$

$$\text{otherwise} \Rightarrow \text{unstable}$$

- Pendulum:

$$f(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(\theta) & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} \Rightarrow \mathrm{eig}\left(\left.\frac{\partial f}{\partial x}\right|_{\theta=\pi}\right) = \pm\sqrt{\frac{g}{\ell}}$$

$$\Rightarrow \text{unstable}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=0} = \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix} \Rightarrow \mathrm{eig}\left(\frac{\partial f}{\partial x}\right) = 0 \pm i\sqrt{\frac{g}{\ell}}$$

- Pure imaginary case is called "marginally stable"

$\Rightarrow$ undamped oscillations

- Adding damping (e.g. $u = -K_d\dot{\theta}$) results in strictly negative eigenvalues