Last Time:

- Friction
- ILC

Today:

- Stochastic Optimal Control
- Linear-Quadratic Gaussian (LQG)

Stochastic Control:

- So far we have assumed we know the system's state exactly.
- What happens when all we have are noisy measurements of quantities related to the state?

$$y = g(x)$$

<u>Deterministic</u>		<u>Stochastic</u>
X	$\longrightarrow$	$p(x \mid y)$

PDF of the state conditioned on the measurements

* Stochastic Optimal Control Problem:

$$\min_{u} \; E[J(x,u)]$$

- In principle, can solve with DP
- Very hard to do in general

* LQG

- Special case we can solve

  Linear dynamics

  Quadratic Cost

  Gaussian Noise

- Dynamics

$$x_{n+1} = Ax_n + Bu_n + \underbrace{w_n}_{\text{"process noise"}}$$

$$y_n = Cx_n + \underbrace{v_n}_{\text{"Measurement noise"}}$$

$$w_n \sim N(0, W) \qquad v_n \sim N(0, V)$$

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* Multivariate Gaussians

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(P)}} \exp\left(-\frac{1}{2}(x-\mu)^T P^{-1}(x-\mu)\right)$$

mean: $\mu = E[x] \in \mathbb{R}^n$

covariance: $P = E[(x-\mu)(x-\mu)^T] \in S_+^n$

$$E[f(x)] = \int_{\text{All space}} f(x)\,p(x)\,dx$$

"Uncorrelated" $\Rightarrow$ $E[(x-\hat{x})(y-\hat{y})^T] = 0$

- Cost Function

$$J = E\left[x_N^T Q_N x_N + \sum_{n=1}^{N-1} x_n^T Q x_n + u_n^T R u_n\right]$$

- D.P. Recursion:

$$V_N(x) = E[x_N^T Q_N x_N] = E[x_N^T P_N x_N]$$

$$V_{N-1}(x) = \min_u E\left[x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + (Ax_{N-1} + B_{N-1} + w_{N-1})^T Q \cdots \right.$$

$$\left. \cdots P_N (Ax_{N-1} + Bu_{N-1} + w_{N-1})\right]$$

$$= \min_u E\Big[\underbrace{x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + (Ax_{N-1} + Bu_{N-1})^T P_N (Ax_{N-1} + Bu_{N-1})}_{\text{standard LQR Terms}}\Big]$$

$$+ E\Big[\underbrace{\cancelto{0}{(Ax_{N-1} + Bu_{N-1})^T P_N w_{N-1}} + \cancelto{0}{w_{N-1}^T P_N (Ax_{N-1} + Bu_{N-1})} + w_{N-1}^T P_N w_{N-1}}_{\text{Noise Terms}}\Big]$$

constant

w_{N-1} and x_{N-1} are uncorrelated!

$\Rightarrow$ Noise terms have no effect on the controller!

(You just get a higher cost)

* "Certainty-Equivalence Principle"

  - The optimal LQG controller is just an LQR controller with x replaced by $E[x]$

* "Separation Principle"

  - For LQG, we can design an optimal controller and an optimal estimator seperately and the hook them together. The resulting feedback policy will be optimal.

* Neither of these holds in general (only for LQG) but they are very frequently used in practice to design sub-optimal policies.

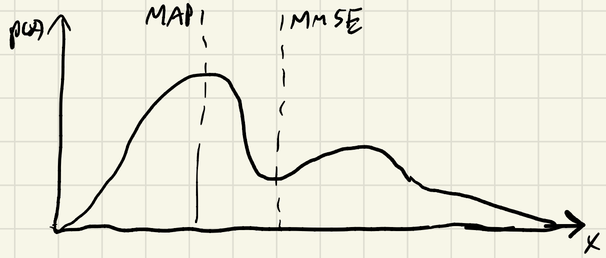
Optimal Estimation

- What should we try to optimize?

- Minimum mean-squared error (MMSE) estimator:

$$\underset{x}{\operatorname{argmin}} \; E[(x-\hat{x})^T(x-\hat{x})]$$ "least squares" "minimum variance"

$$= E[\operatorname{tr}((x-\hat{x})^T(x-\hat{x}))] = E[\operatorname{tr}((x-\hat{x})(x-\hat{x})^T)] = \operatorname{tr}(P)$$



- Maximum a-posteriori (MAP) estimator

$$\underset{x}{\operatorname{argmax}} \; p(x|y)$$

- For a Gaussian, these are the same (the mean)

Kalman Filter:

- Recursive linear MMSE estimator

- Assume we have an estimate of the state that includes all measurement information up to time t_n:

$$\hat{x}_{n|k} = E[x_n \mid y_{1:k}]$$

- Assume we also know the error covariance:

$$P_{k|k} = E[(x_n - \hat{x}_{n|n})(x_n - \hat{x}_{n|n})^T]$$

- We want to update $\hat{x}$ and P to include a new measurement at t_{n+1}

- The KF can be broken into 2 steps:

Prediction Step:

$$\hat{x}_{n+1|n} = E[Ax_n + Bu_n + w_n \mid y_{1:n}]$$
$$= A\hat{x}_{k|n} + Bu_n$$

$$P_{n+1|n} = E\left[(x_{n+1} - \hat{x}_{n+1|n})(x_{n+1} - \hat{x}_{n+1|n})^T\right]$$

$$= E\left[(Ax_n + \cancelto{0}{Bu_n} + w_n - A\hat{x}_{n|n} - \cancelto{0}{Bu_n})(\cdots)^T\right]$$

$$= A\,\underbrace{E\left[(x_n - \hat{x}_{n|n})(x_n - \hat{x}_{n|n})^T\right]}_{P_{n|n}} A^T + \underbrace{E\left[w_n w_n^T\right]}_{W}$$

$$= A P_{n|n} A^T + W$$

Measurement Update:

- Define "innovation"

$$z_{n+1} = y_{n+1} - C\hat{x}_{n+1|n} = C x_{n+1} + V_{n+1} - C\hat{x}_{n+1|n}$$

- Innovation Covariance

$$S_{n+1} = E\left[z_{n+1} z_{n+1}^T\right] = E\left[(C x_{n+1} + V_{n+1} + C\hat{x}_{n+1|n})(\cdots)^T\right]$$

* V_{n+1} and X_{n+1} are uncorrelated

$$\Rightarrow S_{n+1} = C\,\underbrace{E\left[(x_{n+1} - \hat{x}_{n+1|n})(\cdots)^T\right]}_{P_{n+1|n}} C^T + \underbrace{E\left[V_{n+1} V_{n+1}^T\right]}_{V}$$

$$= C P_{n+1|n} C^T + V$$

- Innovation is what we feed back into the estimator

- State Update :

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + \underbrace{L_{n+1}}_{\text{"Kalman Gain"}} z_{n+1}$$

- Covariance Update :

$$P_{n+1|n+1} = E[(x_{n+1} - \hat{x}_{n+1|n+1})(\cdots)^T]$$

$$= E[(x_{n+1} - \hat{x}_{n+1|n} - L_{n+1}(\underbrace{Cx_{n+1} + v_{n+1}} - C\hat{x}_{n+1|n}))(\cdots)^T]$$

* v_{n+1} and x_{n+1} are uncorrelated ...

$$= \underbrace{(I - L_{n+1}C)P_{n+1|n}(I - L_{n+1}C)^T + L_{n+1}VL_{n+1}^T}_{\text{"Joseph Form"}}$$

- Kalman Gain

$$\text{MMSE} \Rightarrow \text{Minimize } E[(x_n - \hat{x}_{n|n})^T(x_n - \hat{x}_{n|n})]$$

$$= tr(P_{n|n})$$

$\Rightarrow$ set $\dfrac{\partial tr(P_{n|n})}{\partial L_{n+1}} = 0$ and solve for L_{n+1}

$$\Rightarrow -2P_{n+1|n}C^T + 2L_{n+1}CP_{n+1|n}C^T + 2L_{n+1}V = 0$$

$$\Rightarrow -P_{n+1|n}C^T + L_{n+1}S_{n+1} = 0$$

$$\Rightarrow \boxed{L_{n+1} = P_{n+1|n}C^TS_{n+1}}$$

* KF Algorithm Summary:

1) Start with $\hat{x}_{0|0}$, $P_{0|0}$, W, V

2) Predict:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k \qquad P_{k+1|k} = AP_{k|k}A^T + W$$

3) Calculate Innovation + Covariance:

$$z_{k+1} = y_{k+1} - C\hat{x}_{k+1|k} \qquad S_{k+1} = CP_{k+1|k}C^T + V$$

4) Calculate Kalman Gain:

$$L_{k+1} = P_{k+1|k}C^T S_{k+1}^{-1}$$

5) Update:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + L_{k+1}z_{k+1}$$

$$P_{k+1|k+1} = (I - L_{k+1}C)P_{k+1|k}(I - L_{k+1}C)^T + L_{k+1}VL_{k+1}^T$$

6) Goto 2

* How do we apply this to nonlinear systems?

  - Extended KF (EKF): Linearize about $\hat{x}$ and proceed as in standard KF

  - Many other generalizations