Theorem of ________

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Thm

For a m-dimensional hypercube and a n-dimensional hypercube s.t. $m, n \in \mathbb{Z}$, $m \geqslant n \geqslant 0$, there are(is) exactly $2^{m-n} C_m^n$ of n-dimensional hypercube in a m-dimensional hypercube.

Notation: $W_m^n = 2^{m-n} C_m^n$

Remark: 1) $W_m^m = 2^{m-m} C_m^m = 1$

2) $W_m^0 = 2^{m-0} C_m^0 = 2^m$

Def (1)

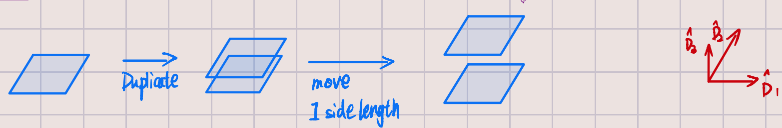
Dimension Ascending

When forming a hypercube (HHC) that is one dimension higher than another hypercube (LHC), a. the LHC duplicate itself and moves 1 "side length" towards the new dimension b. connecting the elements of the relatively lower dimension with those of the relatively higher dimension.

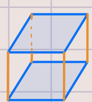
Ex(1)

Example of Dimension Ascending: form a cube (3-Dimensional Hyper cube) from a square (2-Dimensional Hyper cube).

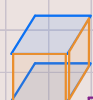
a.



b.



connect points with lines



connect lines with surfaces

connecting the elements of the relatively lower dimension with those of the relatively higher dimension.

Def(2)

N-Dimensional Hypercube

A point is a 0-Dimensional Hypercube, after n times of Dimension Ascending, it forms a N-Dimensional Hypercube

Notation

W_m^n : the number of n-D Hypercube in m-D Hypercube $\quad m,n \in \mathbb{Z},\ m \geq n \geq 0$

Pf. ① from the mode of Dimension Ascending we know that $W_m^n = \underset{\text{Duplication}}{2} W_{m-1}^n + \underbrace{W_{m-1}^{n-1}}_{\text{connection}} \ (\mathrm{I})$

Def(2) Ex(1)

② $C_m^n = C_{m-1}^n + C_{m-1}^{n-1} \ (\mathrm{II}),\quad m,n \in \mathbb{Z},\quad m \geq n \geq 0$

pf. ②:

Way[1] Using Febonacci Sequence

1   1

2   1 1

3   1 2 1

4   1 3 3 1

5 1 4 6 4 1

. . . .

$C_{m-1}^{n-1} \ C_{m-1}^n$

m ... C_m^n ...

Way[2]

$$C_m^n \overset{?}{=} C_{m-1}^n + C_{m-1}^{n-1}$$

$$RHS = \frac{(m-1)\cdot(m-1-1)\cdots(m-1-n+2)\overset{m-n}{(m-1-n+1)}}{1\cdot 2\cdots n} + \frac{(m-1)(m-1-1)\cdots\overset{m-n+1}{(m-1-(n-1)+1)}}{1\cdot 2\cdots(n-1)}$$

$$RHS = \frac{\boxed{(m-1)\cdot(m-2)\cdots(m-n+1)}(m-n)}{\boxed{1\cdot 2\cdots n}} + \frac{\boxed{(m-1)(m-2)\cdots(m-n+1)}\cdot n}{\boxed{1\cdot 2\cdots(n-1)}\cdot n}$$

$$RHS = \frac{\boxed{(m-1)(m-2)\cdots(m-n+1)}(m-n+n)}{\boxed{1\cdot 2\cdot\cdots\cdot n}}$$

$$RHS = \frac{(m-1)(m-2)\cdots(m-n+1)m}{1\cdot 2\cdot\ldots\ldots n}$$

$$= LHS = \frac{m\cdot(m-1)\cdots(m-n+1)}{1\cdot 2\cdots\cdot n}$$

$\because 2^{m-n} \neq 0$

$\therefore (\mathrm{II}) \times 2^{m-n}$:

$$2^{m-n} C_m^n = 2^{m-n} C_{m-1}^n + 2^{m-n} C_{m-1}^{n-1}$$

$$2^{m-n} C_m^n = 2\cdot 2^{m-n-1} C_{m-1}^n + 2^{m-n-1+1} C_{m-1}^{n-1}$$

$$\boxed{2^{m-n} C_m^n} = 2\cdot\boxed{2^{(m-1)-(n)} C_{m-1}^n} + \boxed{2^{(m-1)-(n-1)} C_{m-1}^{n-1}}$$

Confirming to the form of (I), which is $\boxed{W_m^n} = 2\boxed{W_{m-1}^n} + \boxed{W_{m-1}^{n-1}}$

So we can say that $W_m^n = 2^{m-n} C_m^n$

Q.E.D.