Distributed Sparse Optimization

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Optimization meets big data: Background

- Tech. advances in data gathering

$\Longrightarrow$ rapid proliferation of massive data in many areas:

  - Internet (Google, Baidu, Facebook)
  - business (Walmart, SuNing, JingDong)
  - signal processing
  - climate, cosmology
  - medicine, medical imaging

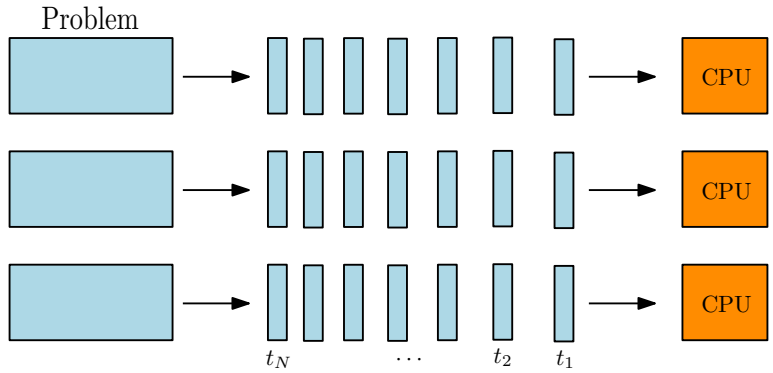
- To make sense of the data, new computational methods must be introduced

- Structured solutions (e.g., sparse vector, low-rank matrix, ...) have grown enormously important in many areas: statistics (LASSO, sparse logistic regression), machine learning (sparse SVM and PCA), image processing (total variation), seismic imaging, compressive sensing, natural language processing, bioinformatics, and many more...

Massive data $\Rightarrow$ parallel & distributed computing

- Many modern applications of sparse optimization has big data
- Many problems cannot be solved on a single workstation

$\Rightarrow$ Moving to parallel/distributed/cloud computing is a viable option



- This talk introduces parallel sparse optimization algorithms running on clusters or cloud computers

The basic idea

To explore two useful structures in the model and data:

1. **Separability**
   - (block)-separable: $f(\mathbf{x}) = \sum_{s=1}^{S} f_s(\mathbf{x}_s)$
   - partial (block)-separable: $f(\mathbf{x}) = \sum_{s=1}^{S} f_s(\mathbf{x})$
   - common, e.g., ℓ_1, $\ell_{1,2}$, Huber, elastic net, square/logistic/hinge loss

   $\Rightarrow$ enables computation *divide and conquer*, well known and widely used
2. **Data is "nearly orthogonal" and solution is sparse**

   $\Rightarrow$ enables only updating a few coordinates each time

   $\Rightarrow$ enables these coordinates are greedily selected
   (seemingly awkward for parallel computing, *but not really*)

   $\Rightarrow$ intermediate points are sparse, cheaper updates, fewer total iterations

Related prior art

- **ADMM** (alternating direction method of multipliers)
  - dates back to the 50', recently becomes popular
  - based on operator-splitting and variable-splitting
  - solves many convex signal processing problems
  - its distributed versions[1] *scale poorly* for big data (#.itr $\sim$ #.splitting)
- **Parallel coordinate descent**[2]
  - update multiple (blocks of) coordinates each time
  - (typically) select coordinates cyclically or at random
  - not tailored for nearly-orthogonal data and sparse solutions

This talk: parallelizes existing 1st-order algorithms and introduce an algorithm

Solve LASSO with 170GB data in 1.7 minutes to 5-digit accuracy under $1 cost

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[1] Boyd, Parikh, Chu, Peleato, and Eckstein [2011], Mota, Xavier, Aguiar, and Puschel [2012]

[2] Bradley, Kyrola, Bickson, Guestrin, and Guestrin [2011], Scherrer, Halappanavar, Tewari, and Haglin [2012a], Scherrer, Tewari, Halappanavar, and Haglin [2012b]

Model

Many sparse optimization model has the form

$$\min_{\mathbf{x}} \ \lambda \mathcal{R}(\mathbf{x}) + \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$$

- $\mathcal{R}(\mathbf{x})$ encourages $\mathbf{x}^*$ to have structure, often non-smooth but simple
- $\mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$ penalizes data fidelity loss, often differentiable

Consider

- very large $\mathbf{A}$
- one, or both, of $\mathcal{R}$ and $\mathcal{L}$ is (partially) separable

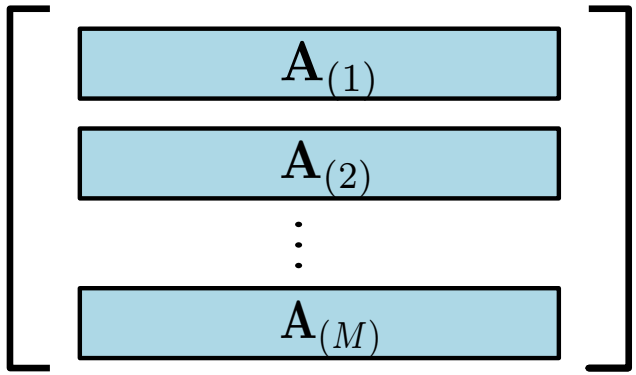
First-order approaches:

- prox-linear: $\mathbf{x}^{k+1} \leftarrow \min \lambda \mathcal{R}(\mathbf{x}) + \langle \mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}^k, \mathbf{b}), \mathbf{x} \rangle + \frac{1}{2\delta_k} \|\mathbf{x} - \mathbf{x}^k\|^2$
- accelerated prox-linear
- dual / Bregman / linearized Bregman / ...
- .....

Common bottleneck: $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$

Row partition of $\mathbf{A}$:

- let $\mathbf{A}_{(i)}$ be the ith row block of $\mathbf{A}$



- compute $\mathbf{A}\mathbf{x}$: *broadcast* $\mathbf{x}$, if needed, and *parallel* $\mathbf{A}_{(i)}\mathbf{x}$
- compute $\mathbf{A}^T\mathbf{y}$: *parallel* $\mathbf{A}_{(i)}^T\mathbf{y}_{(i)}$ and *reduce* $\mathbf{A}^T\mathbf{y} = \sum_{i=1}^M \mathbf{A}_{(i)}^T\mathbf{y}_{(i)}$

$\Rightarrow$ compute $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b}) = \sum_{i=1}^M \mathbf{A}_{(i)}^T \nabla \mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}, \mathbf{b}_i)$:

(1) *broadcast* $\mathbf{x}$, if needed;

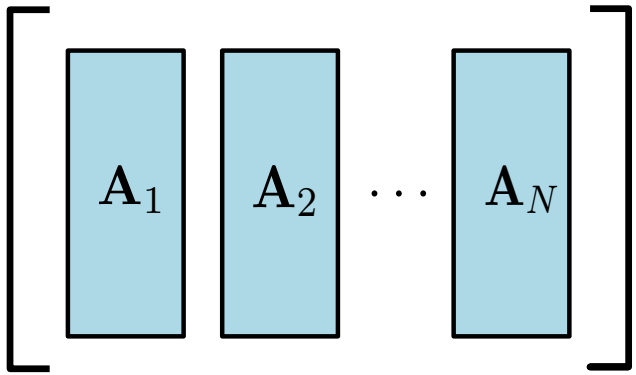
(2) *parallel* $\mathbf{A}_{(i)}^T \nabla \mathcal{L}_i(\mathbf{A}_{(i)}\mathbf{x}, \mathbf{b}_i)$;

(3) *reduce* $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$.

Bottleneck: $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$

Column partition of $\mathbf{A}$:

- let $\mathbf{A}_j$ be the jth column block of $\mathbf{A}$



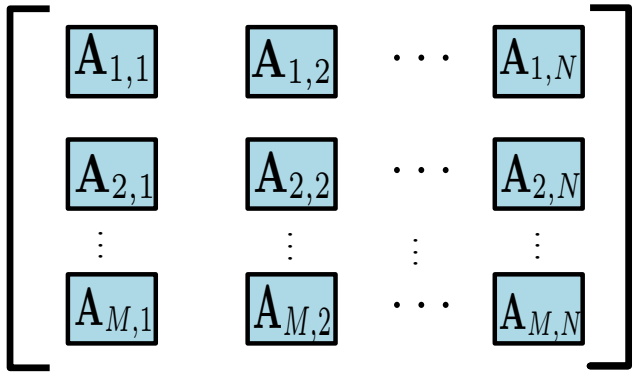
- compute $\mathbf{A}\mathbf{x}$: *parallel* $\mathbf{A}_j\mathbf{x}_j$ and *reduce* $\sum_{j=1}^{N} \mathbf{A}_j\mathbf{x}_j$
- compute $\mathbf{A}^T\mathbf{y}$: *broadcast* $\mathbf{y}$, if needed, and *parallel* $\mathbf{A}_j^T\mathbf{y}$

$\Rightarrow$ compute $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{A}\mathbf{x}, \mathbf{b})$ ..... (left to the reader)

Bottleneck: $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{Ax}, \mathbf{b})$

Two-way block partition of $\mathbf{A}$:

- let $\mathbf{A}_{i,j}$ be the (i,j)th block of $\mathbf{A}$



- compute $\mathbf{Ax}$ and $\mathbf{A}^T\mathbf{y}$: *mixed use of broadcast, parallel, and reduce*

$\Rightarrow$ compute $\mathbf{A}^T \nabla \mathcal{L}(\mathbf{Ax}, \mathbf{b})$ ..... (left to the reader)

Load balancing:

- blocks can have different sizes
- assign larger blocks to faster nodes for better load balance

Example: parallel/distributed ISTA for LASSO

LASSO

$$\min \ f(\mathbf{x}) = \lambda \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$

ISTA algorithm

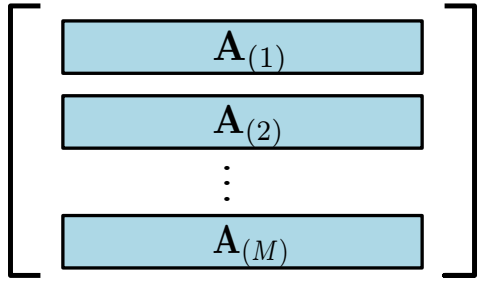
$$\mathbf{x}^{k+1} = \text{shrink}\left(\mathbf{x}^k - \delta_k \mathbf{A}^T\mathbf{A}\mathbf{x}^k + \delta_k \mathbf{A}^T\mathbf{b},\ \lambda\delta_k\right)$$

Serial code

1: initialize $\mathbf{x} = \mathbf{0}$ and δ;
2: pre-compute $\mathbf{A}^T\mathbf{b}$
3: **while** not converged **do**
4: $\quad \mathbf{g} = \mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{A}^T\mathbf{b}$
5: $\quad \mathbf{x} = \text{shrink}\,(\mathbf{x} - \delta\mathbf{g},\ \lambda\delta)$;
6: **end while**

Example: parallel/distributed ISTA for LASSO

distribute blocks of rows to M nodes

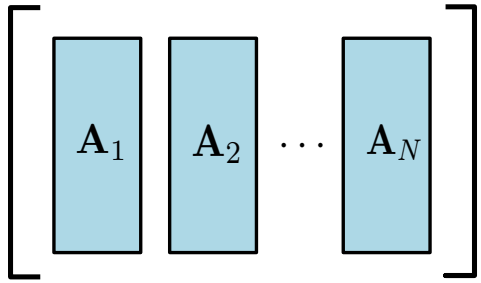


1: initialize $\mathbf{x} = \mathbf{0}$ and δ;
2: i = find_my_processor_id
3: processor i loads $\mathbf{A}_{(i)}$
4: pre-compute $\delta \mathbf{A}^T \mathbf{b}$
5: **while** not converged **do**
6: processor i computes $\mathbf{c}_i = \mathbf{A}_{(i)}^T \mathbf{A}_{(i)} \mathbf{x}$
7: $\mathbf{c}$ = allreduce($\mathbf{c}_i$, SUM)
8: $\mathbf{x} = \text{shrink}\left(\mathbf{x}^k - \delta \mathbf{c} + \delta \mathbf{A}^T \mathbf{b}, \lambda \delta\right)$;
9: **end while**

- assume $\mathbf{A} \in \mathbb{R}^{m \times n}$
- one allreduce per iteration
- speedup
  $\approx \frac{1}{\rho/M + (1-\rho) + O(\log(M))}$
- ρ is close to 1
- requires synchronization

Example: parallel ISTA for LASSO

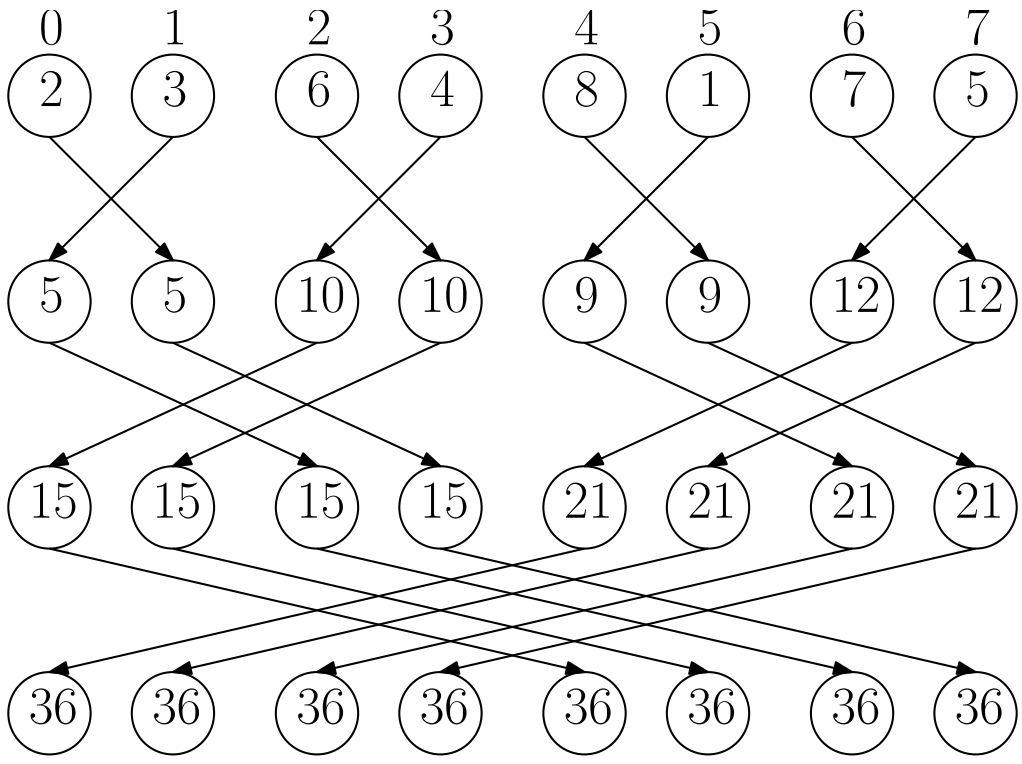
distribute blocks of columns to N nodes



1: initialize $\mathbf{x} = \mathbf{0}$ and δ;
2: i = find_my_processor_id
3: processor i loads $\mathbf{A}_i$
4: pre-compute $\delta \mathbf{A}_i^T \mathbf{b}$
5: **while** not converged **do**
6: processor i computes $\mathbf{y}_i = \mathbf{A}_i \mathbf{x}_i$
7: $\mathbf{y}$ = allreduce($\mathbf{y}_i$, SUM)
8: processor i computes $\mathbf{z}_i = \mathbf{A}_i^T \mathbf{y}$
9: $\mathbf{x}_i^{k+1} = \text{shrink}\left(\mathbf{x}_i^k - \delta \mathbf{z}_i + \delta \mathbf{A}_i^T \mathbf{b}, \lambda\delta\right)$;
10: **end while**

- one allreduce per iteration
- speedup $\approx \frac{1}{\rho/N + (1-\rho) + O(\frac{m}{n}\log(N))}$
- ρ is close to 1
- requires synchronization
- if $m \ll n$, this approach is faster

Example of allreduce SUM



$\log(N)$ layers, N parallel communications per layer

Example: parallel FISTA[3]

Algorithm P-FISTA

1: node j keeps $\mathbf{A}_j$, $\mathbf{b}$, initializes $\mathbf{x}_j^0 = \mathbf{x}_j^1 = 0$;
2: **for** $k = 1, 2, \ldots, K$ **do**
3: $\quad \bar{\mathbf{x}}_j \leftarrow \mathbf{x}_j^k + \frac{k-2}{k+1}\left(\mathbf{x}_j^k - \mathbf{x}_j^{k-1}\right)$;
4: $\quad \mathbf{w} \leftarrow \sum_{j=1}^{N} \mathbf{A}_j \bar{\mathbf{x}}_j$ by Allreduce;
5: $\quad \mathbf{y} \leftarrow \nabla \mathcal{L}(\mathbf{w}; \mathbf{b})$;
6: $\quad \mathbf{g}_j \leftarrow \mathbf{A}_j^T \mathbf{y}$;
7: $\quad \mathbf{x}_j^{k+1} \leftarrow \mathbf{prox}_{\lambda\|\cdot\|_1}(\bar{\mathbf{x}}_j - \delta_k \mathbf{g}_j)$;
8: **end for**

Also: parallel algorithms for sparse logistic regression, sparse SVM, ...

[3] Beck and Teboulle [2009]

Block-coordinate descent

Definition: update one block of variables each time, keeping others fixed

(a better name would be *block coordinate update*)

Advantage: each update is simple

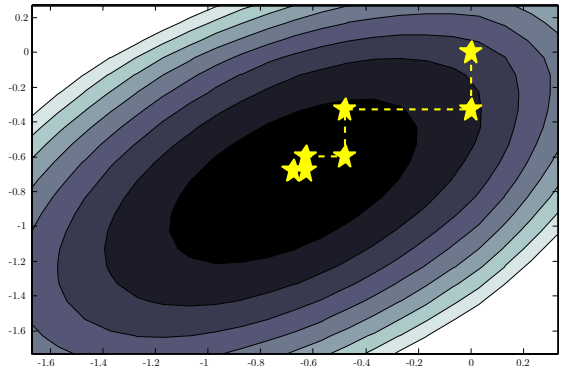
Disadvantages:

- more iterations (with exceptions)
- may stuck at non-stationary points if problem is non-convex and/or non-smooth (with exceptions)

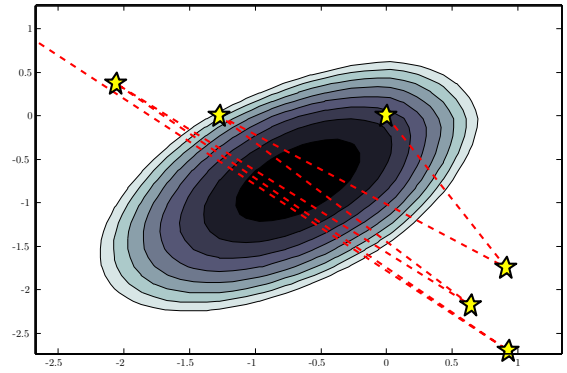
Selection rules: cycle (Gauss-Seidel), parallel (Jacobi), random, greedy

How many coordinates can be updated non-cooperatively?

- **Example 1**: (figures show 2D projection of a 3D problem)

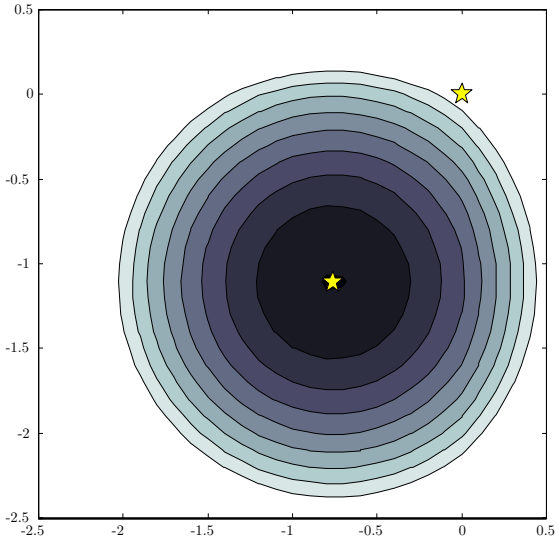


1 coordinate update



3 coordinate update

- **Example 2**:



3 coordinate update

How many coordinates can be updated non-cooperatively?

P (#. non-cooperative coordinate updated) depends on **block spectral radius**

$$\rho_P = \max_{\mathbf{M} \in \mathcal{M}} \rho(\mathbf{M}),$$

where $\mathcal{M}$ is the set of all $P \times P$ submatrices that we can obtain from $\mathbf{A}^T\mathbf{A}$ corresponding to selecting exactly one column from each of the P blocks

Theorem

Assume each column of $\mathbf{A}$ *has unit 2-norm. If* $\rho_P < 2$*, then any* P *parallel (prox-linear) updates per iteration gives*

$$\mathcal{F}(\mathbf{x} + \mathbf{d}) - \mathcal{F}(\mathbf{x}) \leq \frac{\rho_P - 2}{2}\beta\|\mathbf{d}\|_2^2$$

moreover,

$$\mathcal{F}(\mathbf{x}^k) - \mathcal{F}(\mathbf{x}^*) \;\leq\; \frac{2C^2\left(2L + \beta\sqrt{\frac{N}{P}}\right)^2}{(2-\rho_P)\beta} \cdot \frac{1}{k}.$$

(see our paper for parameter definitions and proof; partial credits to Scherrer, Tewari, Halappanavar, and Haglin [2012b])

Greedy coordinate selection

Greed is good for sparse optimization!

- select P coordinates based on some merit (e.g., most descent, largest step)
- widely used in *matching pursuit* (build solution by coordinate)
- applied to sparse optimization with $P = 1$ by Li and Osher [2009]
- we extend it to $P \geq 2$ (or dynamic P) in parallel
- if solution is sparse, then exceptionally effective!
  - ▶ coordinates in $\mathrm{supp}(\mathbf{x}^*)$ are selected and updated
  - ▶ most zero variables stay zero *the whole time*
  - ▶ since $\mathbf{x}^k - \mathbf{x}^*$ is sparse, objective is effectively strongly convex
  - ▶ "problem dimension" is reduced $\Rightarrow$ #.iterations is reduced
  - ▶ *caution*: selection requires "sorting", which can be expensive

GRock: greedy coordinate-block descent

Consider:

$$\min \lambda\|\mathbf{x}\|_1 + f(\mathbf{A}\mathbf{x} - \mathbf{b})$$

- decompose $\mathbf{A}\mathbf{x} = \sum_j \mathbf{A}_i\mathbf{x}_j$; block $\mathbf{A}_j$ and $\mathbf{x}_j$ are kept on node j

Parallel GRock:

1. (*parallel*) compute coordinate-merit, for each coordinate i

$$d_i = \arg\min_d \ \lambda \cdot r(x_i + d) + g_i d + \frac{1}{2}d^2, \quad \text{where } g_i = \mathbf{A}_{(i)}^T \nabla \mathbf{f}(\mathbf{A}\mathbf{x} - \mathbf{b})$$

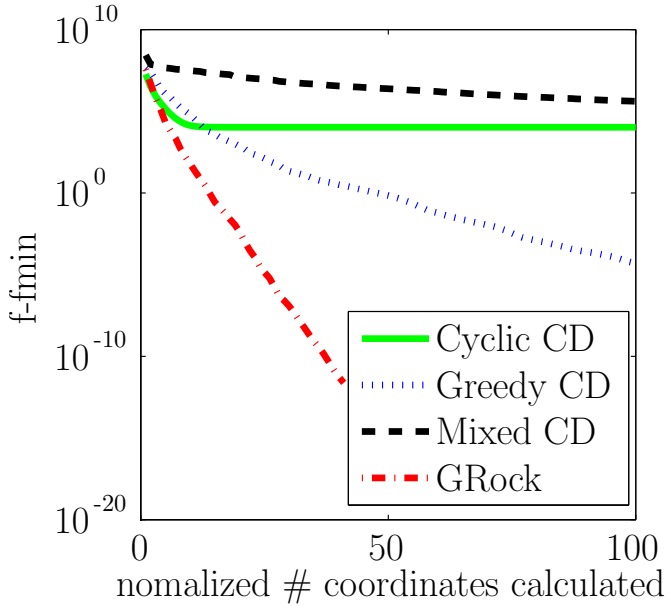
2. (*parallel*) compute block-merit, for each block j

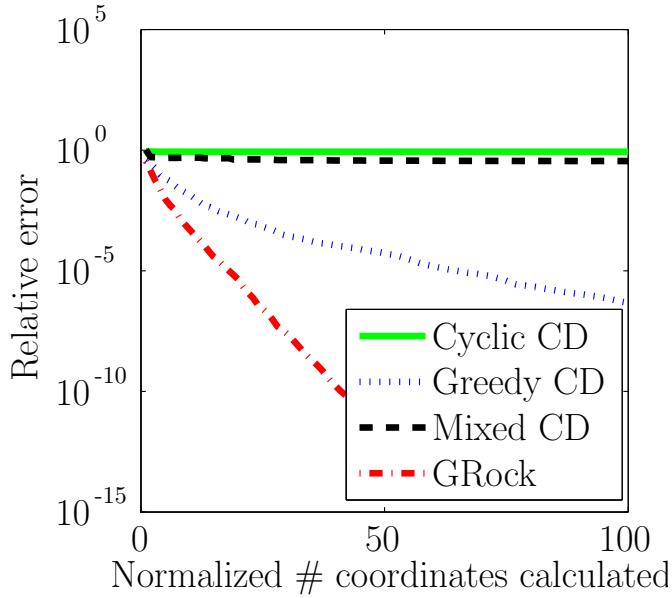
$$m_j = \max\{|d| : d \text{ is an element of } \mathbf{d}_j\}$$

   s_j denotes the index of the maximal coordinate in block j

3. (*allreduce*) $\mathcal{P} \leftarrow$ select P blocks with largest m_j, $2 \le P \le N$
4. (*parallel*) update $x_{s_j} \leftarrow x_{s_j} + d_{s_j}$ for all $j \in \mathcal{P}$
5. (*allreduce*) update $\mathbf{A}\mathbf{x}$

Compare different selection rules





- LASSO test with $\mathbf{A} \in \mathbb{R}^{512\times 1024}$, $N = 64$ column blocks
- GRock uses $P = 8$ updates each iteration
- greedy CD[4] uses $P = 1$
- mixed CD[5] selects $P = 8$ random blocks and best coordinate

[4] Li and Osher [2009]

[5] Scherrer, Tewari, Halappanavar, and Haglin [2012b]

Numerical comparison

Compared algorithms: flops+communication, per iteration, N parallel nodes

- parallel FISTA[6]: $O(\frac{mn}{N}) + O(n \log N)$
- parallel dual ADMM: $O(\frac{mn}{N} + 2m^2) + O(mN \log N)$, matrix factor cache
- GRock: $O(\frac{mn}{N} + Pm) + O(n \log N + N \log N)$

At each iteration

- parallel FISTA and GRock have comparable per-iteration cost
- parallel dual-ADMM needs more communication

[6] Beck and Teboulle [2009]

Test on cluster STIC @ Rice U.

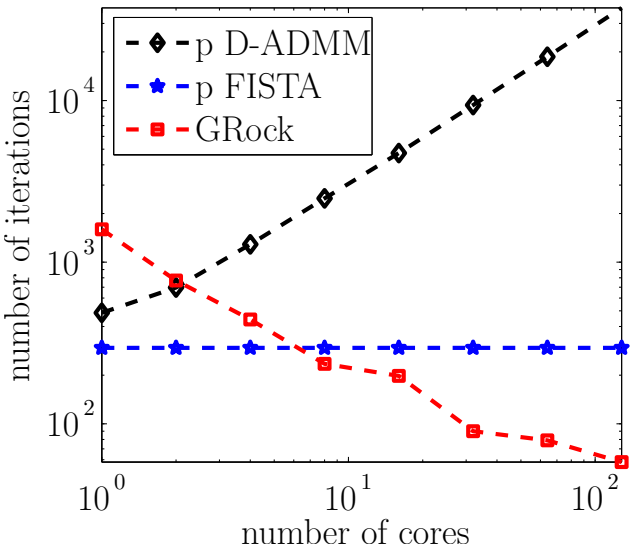
Hardware:

- 170 Appro Greenblade E5530 nodes each with two quad-core 2.4GHz Xeon (Nahalem) CPUs
- each node has 12GB of memory shared by all 8 cores
- # of processes used on each node = 8

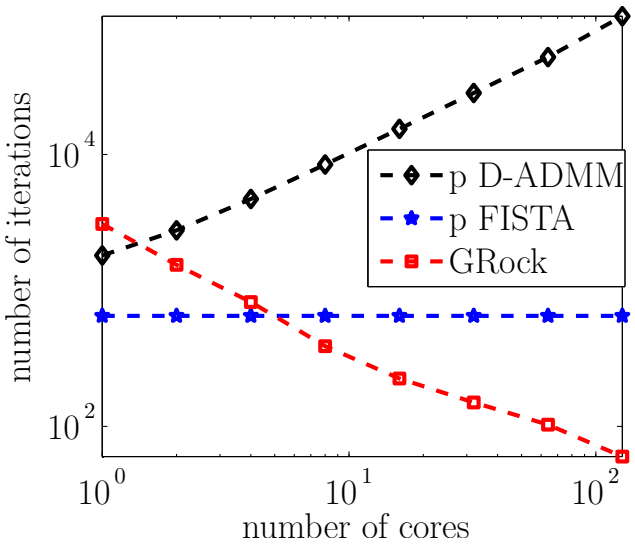
Test dataset:

	$\mathbf{A}$ type	$\mathbf{A}$ size	λ	sparsity of $\mathbf{x}^*$
dataset I	Gaussian	1024×2048	0.1	100
dataset II	Gaussian	2048×4096	0.01	200

iterations vs cores



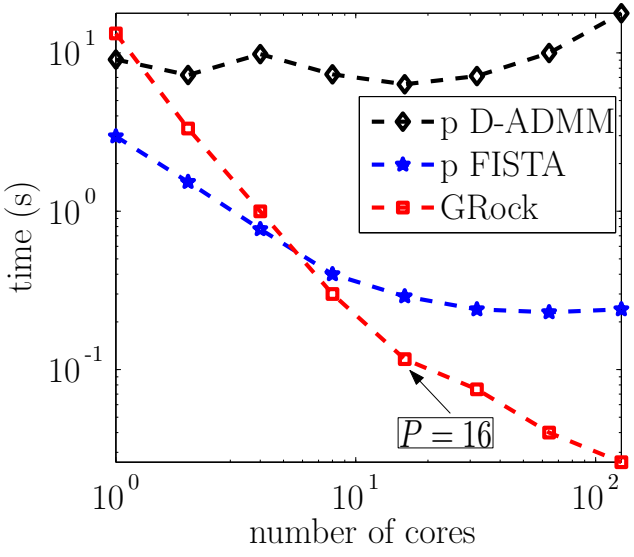
dataset I



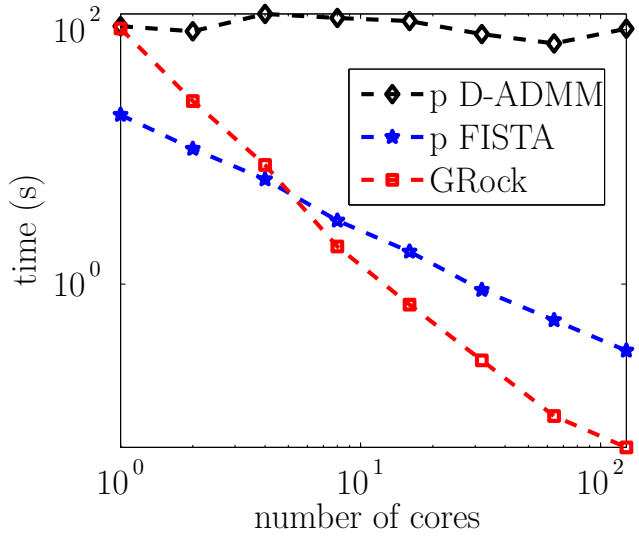
dataset II

Note: we set $P =$ number of cores

time vs cores



dataset I



dataset II

Note: we set $P =$ number of cores

Big data LASSO on Amazon EC2

Amazon EC2 is an elastic, pay-as-you-use cluster

Advantage: no hardware investment, everyone can have an account

Test dataset

- $\mathbf{A}$ dense matrix, 20 billion entries, and **170GB** size
- $\mathbf{x}$: 200K entries, 4K nonzeros, Gaussian values

Requested system from Amazon

- 20 "high-memory quadruple extra-large instances"
- each instance has 8 cores and 60GB memory

Code

- written in C using GSL (for matrix-vector multiplication) and MPI

parallel versions of dual-ADMM, FISTA, GRock

	p D-ADMM	p FISTA	GRock
estimate stepsize (min.)	n/a	1.6	n/a
matrix factorization (min.)	51	n/a	n/a
iteration time (min.)	105	40	1.7
number of iterations	2500	2500	104
communication time (min.)	30.7	9.5	0.5
stopping relative error	**1E-1**	**1E-3**	**1E-5**
total time (min.)	**156**	**41.6**	**1.7**
Amazon charge	**$85**	**$22.6**	**$0.93**

- ADMM performance is sensitive to penalty parameter β, which we picked as the best out of only a few trials (we cannot afford more)
- parallel dual-ADMM and FISTA were capped at 2500 iterations
- GRock used adaptive P and stopped at relative error 1E-5

Summary

- Big data applications requires parallel and distributed computing
- Separability structure enables parallel algorithms
- Greed is good (for sparse solutions) and greedy algorithms are parallelizable
- Algorithms are rather simple and must be simple to parallelize

Overheads of parallelism

- **computing overhead:** *start up time, synchronization wait time, data communication time, termination (data collection time)*
- **I/O overhead:** slow read and write of *non-local* data
- **algorithm overhead:** *extra variables, data duplication*
- **coding overhead:** language, library, operating system, debug, maintenance

Software codes can be found in the author's website.

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