Space and Congruence Compression of Proofs

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Master Thesis Defense

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Space and Congruence Compression of Proofs

Space and Congruence Compression of **Proofs**

Resolution

Resolution Rule

$$\frac{C \lor x \quad D \lor \neg x}{C \lor D}$$

Resolution

Resolution Rule

$$\begin{tikzcd}
C, x & & D, \neg x \\
& C, D \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee \neg x_3) \wedge (\neg x_1)$$

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$\neg x_1$

Example

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee \neg x_3) \wedge (\neg x_1)$$

$$\begin{tikzcd} x_1, x_2, x_3 & & \neg x_1 \\ & x_2, x_3 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Example

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$$\begin{tikzcd}
x_1, x_2, x_3 & & \neg x_1 & & x_1, \neg x_2 \\
& x_2, x_3 \arrow[ul] \arrow[ur] & & \neg x_2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example

$$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2) \land (x_1 \lor \neg x_3) \land (\neg x_1)$$

$$\begin{tikzcd}
x_1, x_2, x_3 & & \neg x_1 & & x_1, \neg x_2 & & x_1, \neg x_3 \\
& x_2, x_3 \arrow[ul] \arrow[ur] & & \neg x_2 \arrow[ul] \arrow[ur] & & \neg x_3 \arrow[ullll] \arrow[ur] &
\end{tikzcd}$$

Example

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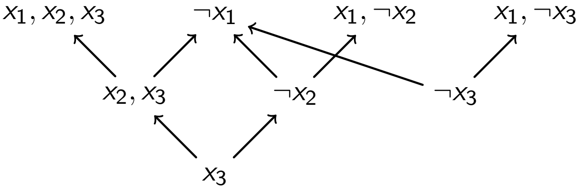

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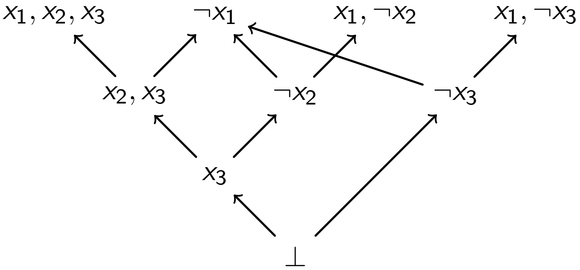

Example

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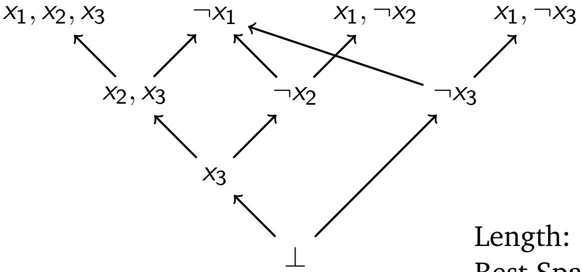

Length: 9
Best Space: 4

Example

$$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2) \land (x_1 \lor \neg x_3) \land (\neg x_1)$$

$$\begin{tikzcd}
x_1, x_2, x_3 & & x_1, \neg x_2 & & \\
& x_1, x_3 \arrow[ul] \arrow[ur] & & x_1, \neg x_3 & \\
& & x_1 \arrow[ul] \arrow[ur] & & \neg x_1 \\
& & & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Length: 7
Best Space: 3

Space and Congruence **Compression** of Proofs

Proof Compression

- Smaller unsat cores, interpolants
- Easier proof processing
- Smaller proofs libraries
- Easier trusted interaction of deductive systems
- Proof generalization
- Proof carrying code

Synthesizing Multiple Boolean Functions using Interpolation on a Single Proof

- Georg Hofferek, et al, 2013, TU Graz

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Method

1. Formulate problem in SMT theory of uninterpreted functions
2. Obtain proof of unsatisfiability from SMT solver
3. Transform proof (local first, colorable)
4. Extract a single n-interpolant from the proof
5. Extract multiple interpolants from the single interpolant

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Proof Compression using Skeptik

- Input proof: 1,870,407 nodes
- Output proof: 868,760 nodes (53,6% compression)

Space and **Congruence** Compression of Proofs

Example

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

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Proof

Equality is transitive, therefore from $f(a) = a$, $a = b$ and $b = f(b)$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

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Equality is transitive, therefore from $f(a) = a$, $a = b$ and $b = f(b)$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

A different Proof

$f(.)$ is a function, therefore from $a = b$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

A proof

$$
\begin{tikzcd}
f(a) \neq a, a \neq b, b \neq f(b), f(a) = f(b) & & & f(a) = a & \\
& a \neq b, b \neq f(b), f(a) = f(b) \arrow[ul] \arrow[urr] & & a = b & \\
& & b \neq f(b), f(a) = f(b) \arrow[ul] \arrow[ur] & & b = f(b) \\
& f(a) \neq f(b) & & f(a) = f(b) \arrow[ul] \arrow[ur] & \\
& & \perp \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

A different proof

$$\begin{tikzcd}
 & a \neq b, f(a) = f(b) & & a = b \\
f(a) \neq f(b) & & f(a) = f(b) \arrow[ul] \arrow[ur] & \\
 & \perp \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Congruence Closure

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Ground Terms

- Constants $a, b, c, \ldots$
- Compound Terms $f(t_1, \ldots, t_n)$

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- Constants $a, b, c, \ldots$
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Congruence Relation R

- Reflexive: $\forall t (t, t) \in R$
- Symmetric: $(s, t) \in R \Rightarrow (t, s) \in R$
- Transitive: $(t_1, t_2) \in R \ldots (t_{m-1}, t_m) \in R \Rightarrow (t_1, t_m) \in R$
- Compatible: $\forall i (t_i, s_i) \in R \Rightarrow (f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)) \in R$

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- Compatible: $\forall i (t_i, s_i) \in R \Rightarrow (f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)) \in R$

Congruence Closure E^* of set of equations E

- Smallest Congruence Relation containing E
- Computable in $O(n \log(n))$
- E is explanation for $(s, t) \in E^*$

Example revisited

Knowledge

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Example revisited

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1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ f(a) = a, a = b , b = f(b) \}$

Example revisited

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ \quad a = b \,, b = f(b) \}$

Example revisited

Knowledge

1. $f(a) = a$
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Explanation for $f(a) = f(b)$

$\{ \qquad a = b \qquad \}$

Example revisited

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ \quad a = b \quad \}$

Short explanation $\leadsto$ short (sub)proof

Short Explanation Decision Problem

Given a set of input equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

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Given a set of input equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

NP-complete

Short Explanation Decision Problem

Given a set of input equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

NP-complete

Reduction of SAT to the short explanation decision problem

Space and Congruence Compression of Proofs

Is it possible to remove parts of the proof from memory during proof processing?

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Which parts?

Space Measure Example

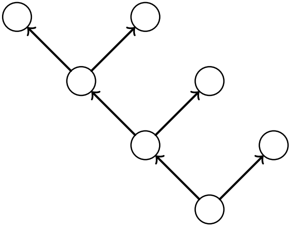
$$
\begin{tikzcd}
x_1, x_2, x_3 & & x_1, \neg x_2 & & \\
& x_1, x_3 \arrow[ul] \arrow[ur] & & x_1, \neg x_3 & \\
& & x_1 \arrow[ul] \arrow[ur] & & \neg x_1 \\
& & & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Space Measure Example

Maximum number of nodes in memory: 0

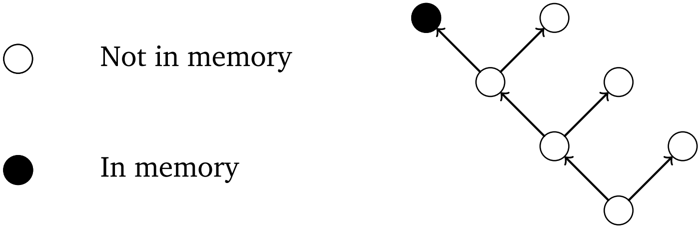
Not in memory

In memory



Space Measure Example

Maximum number of nodes in memory: 1

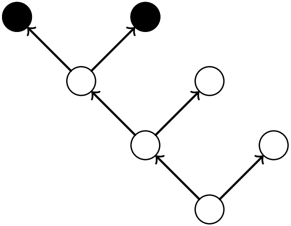


Space Measure Example

Maximum number of nodes in memory: 2

Not in memory

In memory

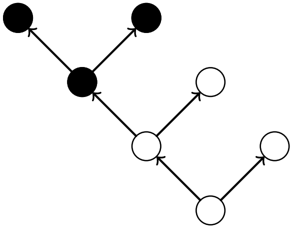


Space Measure Example

Maximum number of nodes in memory: 3

Not in memory

In memory

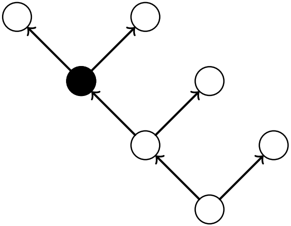


Space Measure Example

Maximum number of nodes in memory: 3

Not in memory

In memory

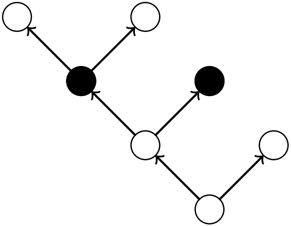


Space Measure Example

Maximum number of nodes in memory: 3

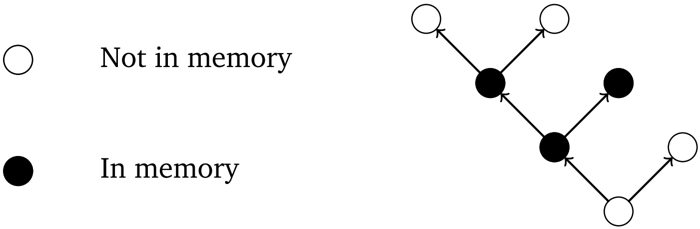
Not in memory

In memory



Space Measure Example

Maximum number of nodes in memory: 3

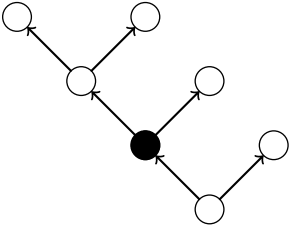


Space Measure Example

Maximum number of nodes in memory: 3

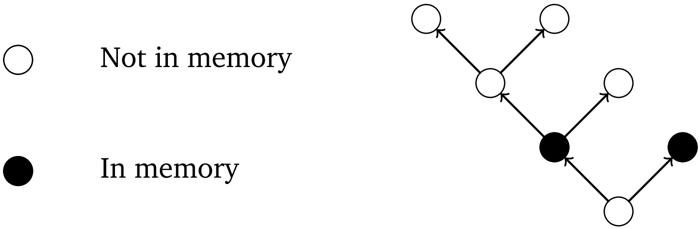
Not in memory

In memory



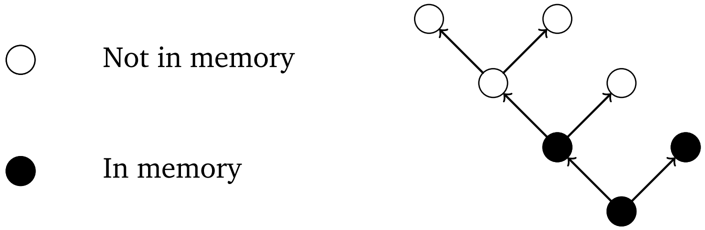
Space Measure Example

Maximum number of nodes in memory: 3



Space Measure Example

Maximum number of nodes in memory: 3

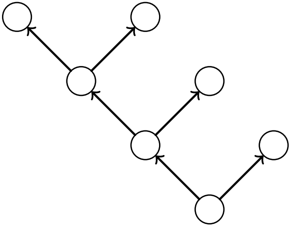


Space Measure Example

Maximum number of nodes in memory: 0

Not in memory

In memory

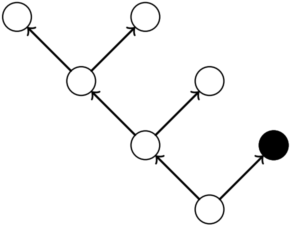


Space Measure Example

Maximum number of nodes in memory: 1

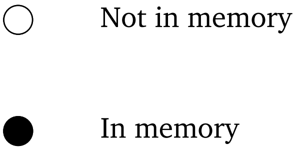
Not in memory

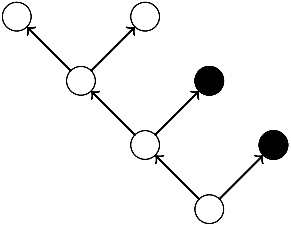
In memory



Space Measure Example

Maximum number of nodes in memory: 2



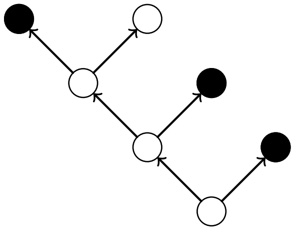


Space Measure Example

Maximum number of nodes in memory: 3

○ Not in memory

● In memory

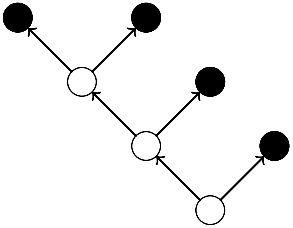


Space Measure Example

Maximum number of nodes in memory: 4

Not in memory

In memory

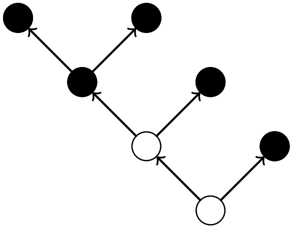


Space Measure Example

Maximum number of nodes in memory: 5

Not in memory

In memory

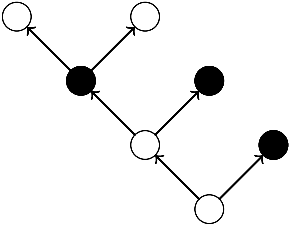


Space Measure Example

Maximum number of nodes in memory: 5

○ Not in memory

● In memory

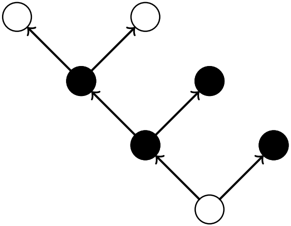


Space Measure Example

Maximum number of nodes in memory: 5

Not in memory

In memory

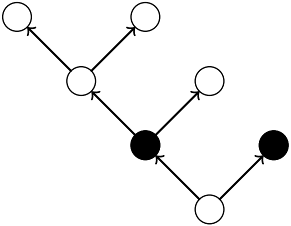


Space Measure Example

Maximum number of nodes in memory: 5

 Not in memory

 In memory



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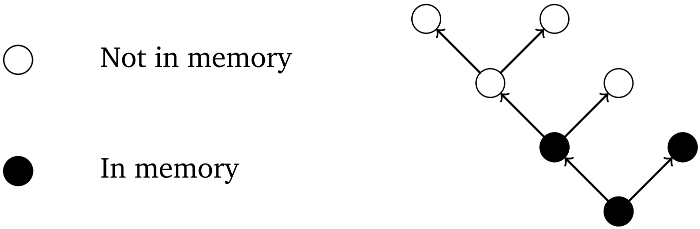
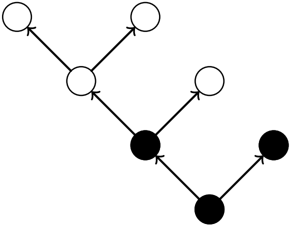


Space Measure Example

Maximum number of nodes in memory: 5

○ Not in memory

● In memory



Good traversal orders are essential!

Space Measure

Space measure of a proof and a traversal order

Maximal amount of nodes that have to be kept in memory at once while processing the proof following the traversal order

Construct Traversal Orders

Construct Optimal Order

- NP-complete
- Optimal strategy in some pebbling game

Construct Good Order

- Greedy Algorithms
- Heuristic choices
- Top-Down
- Bottom-Up

Experiments, Unsung Heroes & Conclusion

Experimental Results

Congruence Compression

- 3965 proofs from problems in QF_UF logic
- 2% average effective compression in proof length
- 28% compression in explanation length

Space Compression

- 7555 SAT and SMT proofs
- Bottom-Up outperforms Top-Down
- Average space measure is 44 times smaller than proof length

Unsung Heroes

- Explanation producing congruence closure algorithm
  - Using immutable data structures
  - Modified version of Dijkstra's shortest path algorithm
- Proof producing algorithm
- Resolution calculus extended with equality
- SAT translation of optimal traversal order
- Correctness & soundness proofs
- Implementation of all presented methods

Conclusion

- Proofs can be compressed in length and space
- Finding the shortest explanation is NP-complete
- Proof production is tricky
- Construct traversal orders Bottom-Up

Thank you for your attention!

NP-completeness proof sketch

From a propositional logic formula Φ obtain ...

- a set of equations E_Φ
- a target equation $s_\Phi = t_\Phi$
- $k_\Phi \in \mathbb{N}$

such that ...

Φ is satisfiable if and only if there is an explanation $E' \subseteq E_\Phi$ of $s_\Phi = t_\Phi$ with $|E'| \leq k_\Phi$

NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

$\bot_1$ —— $\hat{x}_1$ —— $\top_1$ $\quad$ $\bot_2$ —— $\hat{x}_2$ —— $\top_2$ $\quad$ $\bot_3$ —— $\hat{x}_3$ —— $\top_3$

NP-completeness proof sketch example

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Translation to equations

$$\bot_1 \;—\; \hat{x}_1 \;—\; \top_1 \qquad \bot_2 \;—\; \hat{x}_2 \;—\; \top_2 \qquad \bot_3 \;—\; \hat{x}_3 \;—\; \top_3$$

$$\hat{c}_1 \;—\; t_1(\hat{x}_1), \quad \hat{c}_1 \;—\; t_1(\hat{x}_2), \quad \hat{c}_1 \;—\; f_1(\hat{x}_3)$$

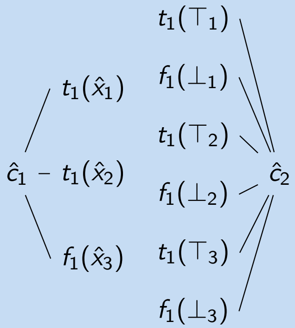
NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

$$\bot_1 \;—\; \hat{x}_1 \;—\; \top_1 \qquad \bot_2 \;—\; \hat{x}_2 \;—\; \top_2 \qquad \bot_3 \;—\; \hat{x}_3 \;—\; \top_3$$



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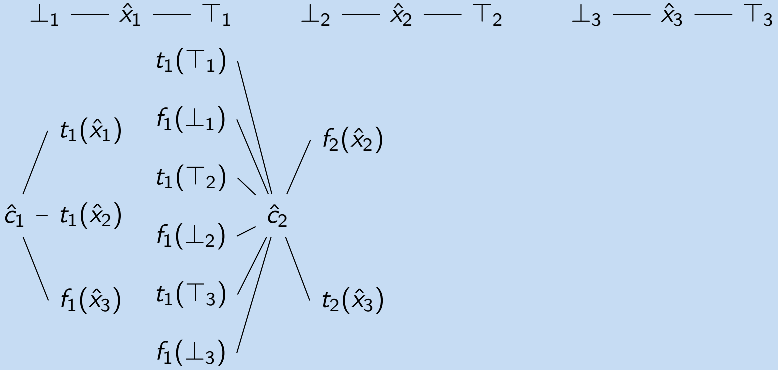

NP-completeness proof sketch example

Formula

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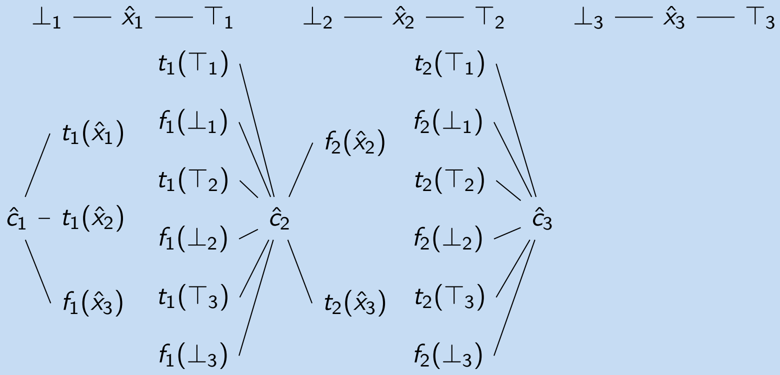
Translation to equations

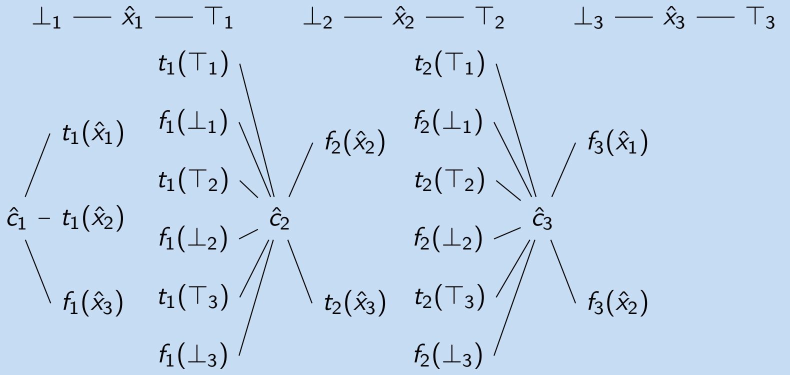


NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations

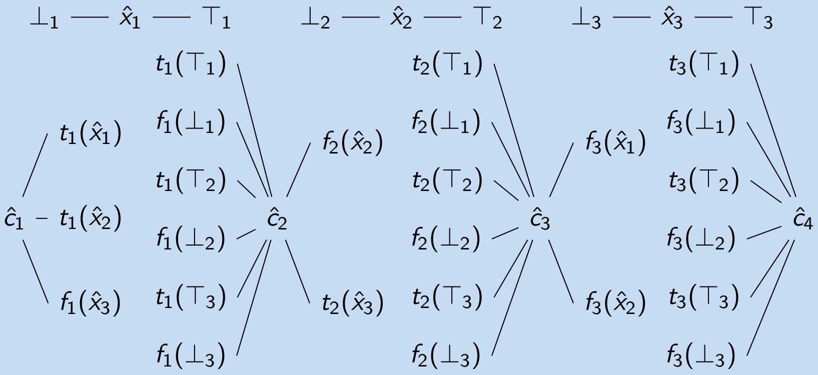


NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations



NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Small subset corresponding to satisfying assignment

$$\hat{x}_1 \longrightarrow \top_1 \qquad \bot_2 \longrightarrow \hat{x}_2 \qquad \hat{x}_3 \longrightarrow \top_3$$

$$\hat{c}_1 - t_1(\hat{x}_1) \quad t_1(\top_1) - \hat{c}_2 - f_2(\hat{x}_2) \quad f_2(\bot_2) - \hat{c}_3 - f_3(\hat{x}_2) \quad f_3(\bot_2) - \hat{c}_4$$

Short explanation, long proof

$$t_a \leftarrow f(f(a,b), f(a,a))$$

$$t_b \leftarrow f(f(b,a), f(b,b))$$

$$f(a,b) \neq f(b,a), f(a,a) \neq f(b,b), t_a = t_b$$

$$a \neq b, f(a,b) = f(b,a)$$

$$a \neq b, f(a,a) \neq f(b,b), t_a = t_b$$

$$a \neq b, f(a,a) = f(b,b)$$

$$a \neq b, t_a = t_b$$

Compatibility proof

$$t_a \neq a, a \neq b, b \neq t_b, t_a = t_b$$

Transitivity proof

Congruence Experimental Results

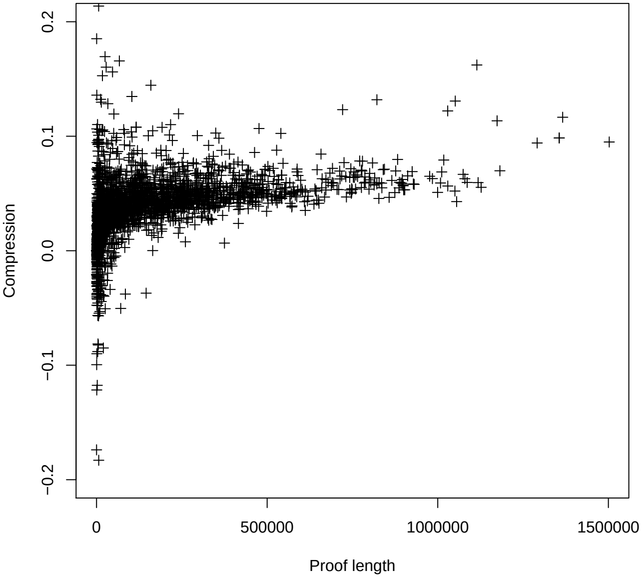
- 3965 proofs of problems of the SMT-LIB benchmark in the QF_UF logic

Method	**Compression**	**Min**	**Max**	**Speed**
EqGraph	**5.350** %	-18.302 %	**81.347** %	0.343
Proof Forest	5.196 %	-43.985 %	77.202 %	0.611
DAGify	3.368 %	**0.0** %	14.433 %	**1.655**

Table: Compression Results

Congruence Graph	**Compressed**	**Compression**
Equation Graph	12.42 %	28.34 %
Proof Forest	11.459 %	28.69 %

Table: Explanation Size Results


Compression
0.2
0.1
0.0
−0.1
−0.2
0
500000
1000000
1500000
Proof length

Space Compression Results

- VeriT: Problems from SMT-lib
- TraceCheck: Problems from SATLIB, computed with PicoSAT

Name	Number of proofs	Maximum length	Average length
TraceCheck_1	2239	90756	5423
TraceCheck_2	215	1768249	268863
veriT_1	4187	2241042	103162
veriT_2	914	120075	5391

Table: Proof Benchmark Sets

Space Compression Results

Algorithm Heuristic	**Relative Performance** (%)	**Speed** (nodes/ms)
Bottom-Up		
Children	17.52	**88.6**
LastChild	**26.31**	84.5
Distance(1)	9.46	21.2
Distance(3)	-0.40	0.5
Top-Down		
Children	-27.47	0.3
LastChild	-31.98	1.9
Distance(1)	-70.14	0.6
Distance(3)	**-74.33**	**0.1**

Table: Experimental Results

Construct Order Top-Down

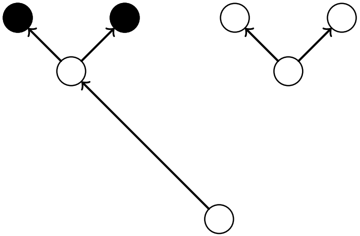
Maximum number of nodes in memory: 0

Construct Order Top-Down

Maximum number of nodes in memory: 1

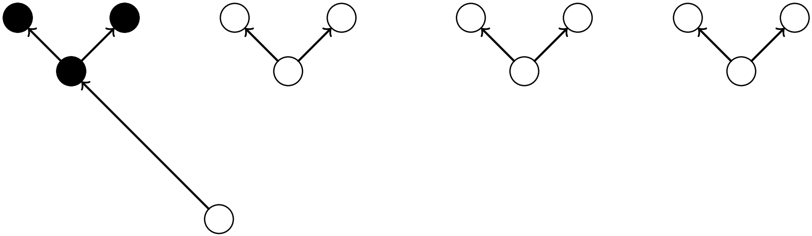
Construct Order Top-Down

Maximum number of nodes in memory: 2



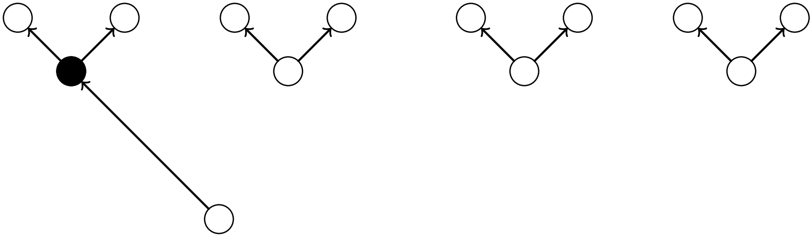
Construct Order Top-Down

Maximum number of nodes in memory: 3



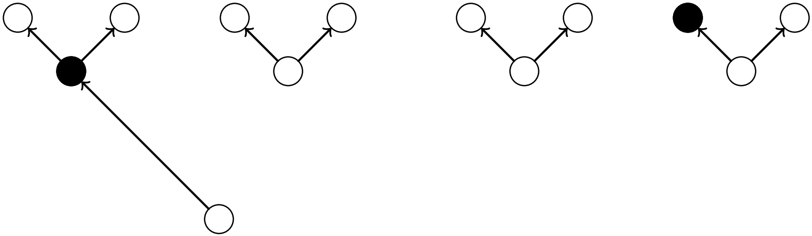
Construct Order Top-Down

Maximum number of nodes in memory: 3



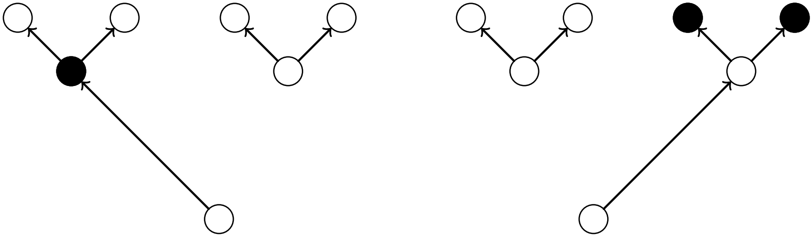
Construct Order Top-Down

Maximum number of nodes in memory: 3



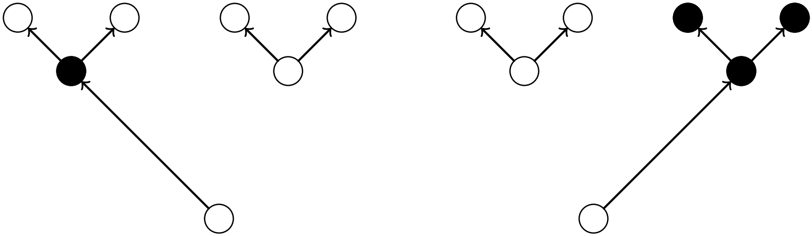
Construct Order Top-Down

Maximum number of nodes in memory: 3



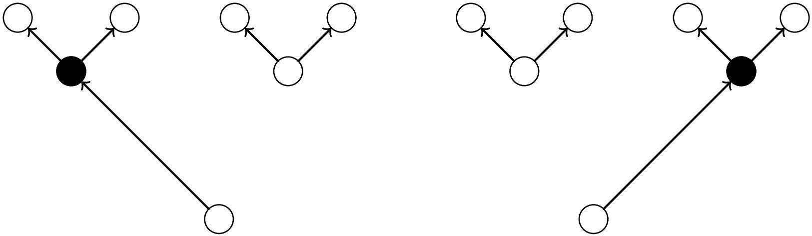
Construct Order Top-Down

Maximum number of nodes in memory: 4



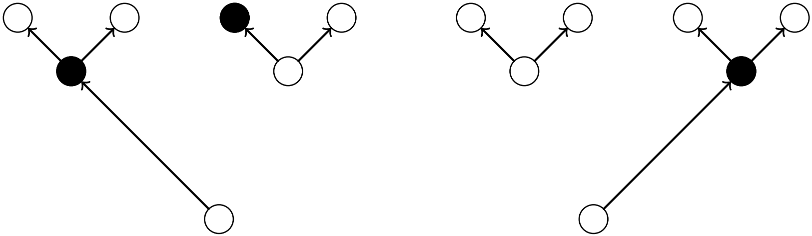
Construct Order Top-Down

Maximum number of nodes in memory: 4



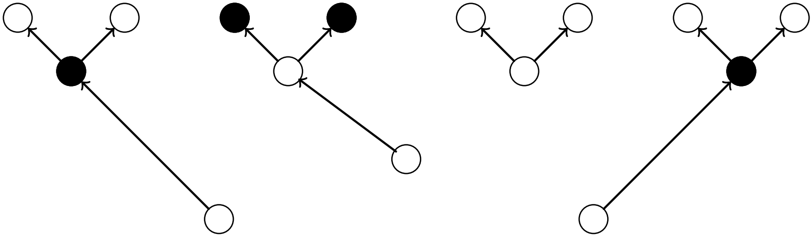
Construct Order Top-Down

Maximum number of nodes in memory: 4



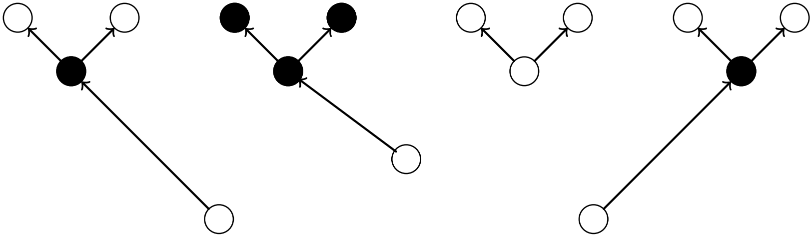
Construct Order Top-Down

Maximum number of nodes in memory: 4



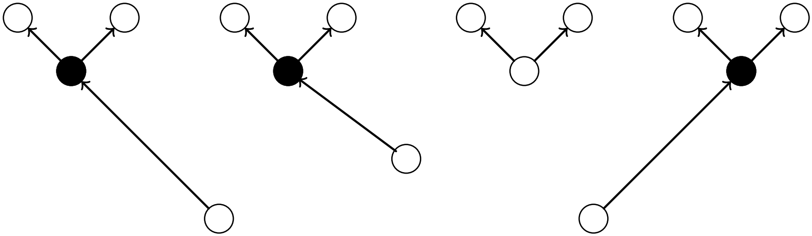
Construct Order Top-Down

Maximum number of nodes in memory: 5



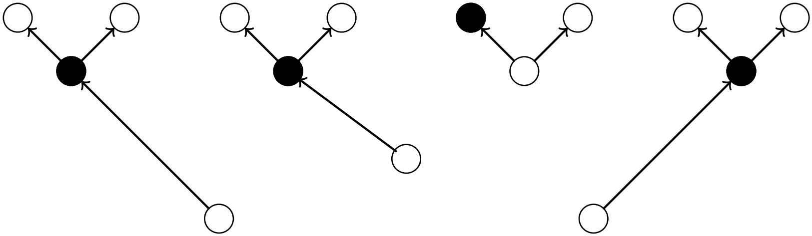
Construct Order Top-Down

Maximum number of nodes in memory: 5



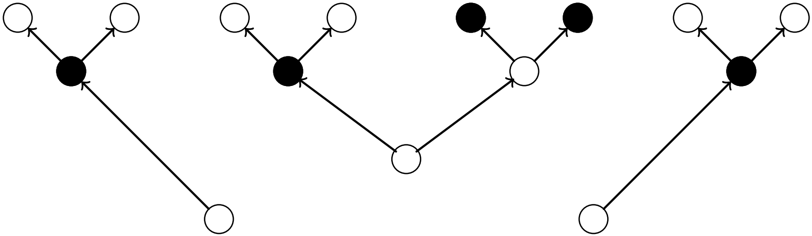
Construct Order Top-Down

Maximum number of nodes in memory: 5



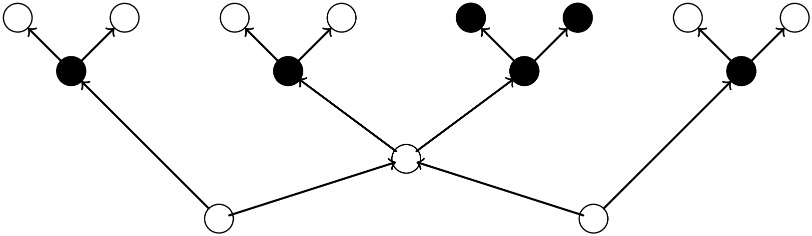
Construct Order Top-Down

Maximum number of nodes in memory: 5



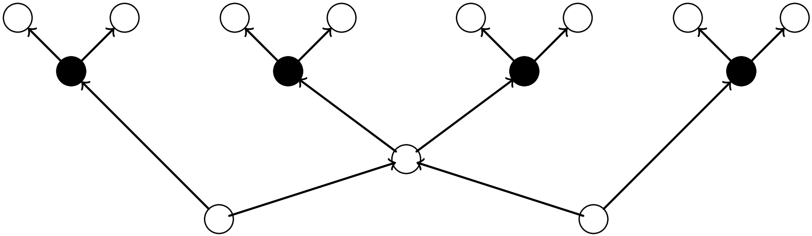
Construct Order Top-Down

Maximum number of nodes in memory: 6



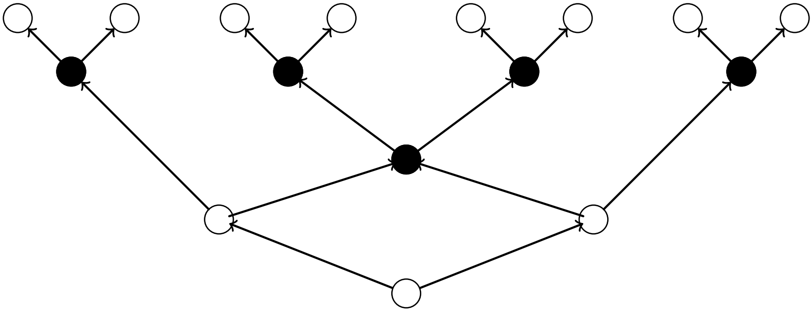
Construct Order Top-Down

Maximum number of nodes in memory: 6



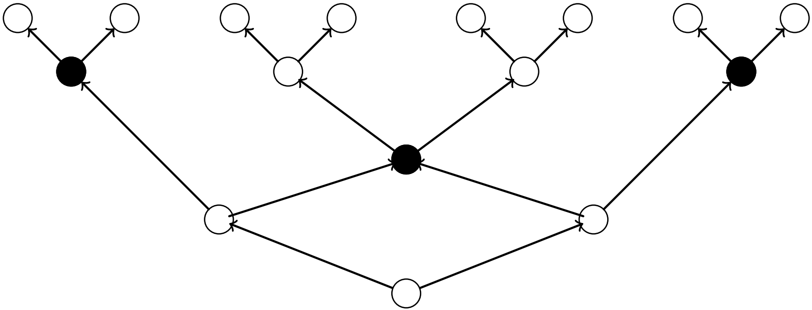
Construct Order Top-Down

Maximum number of nodes in memory: 6



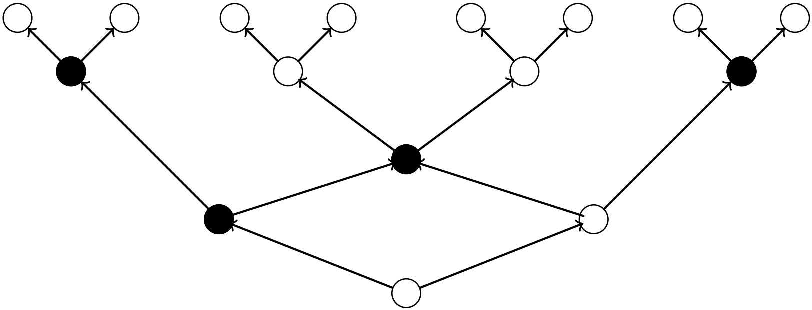
Construct Order Top-Down

Maximum number of nodes in memory: 6



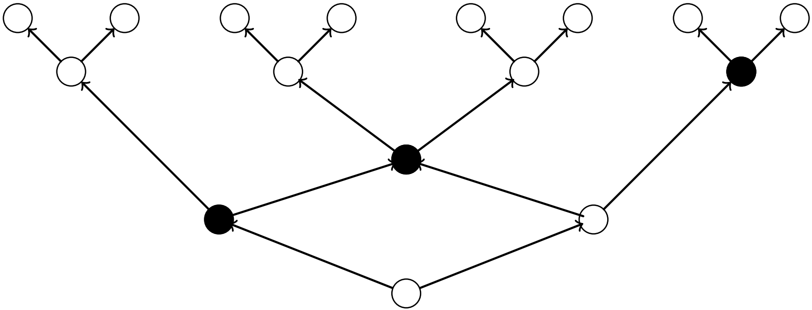
Construct Order Top-Down

Maximum number of nodes in memory: 6



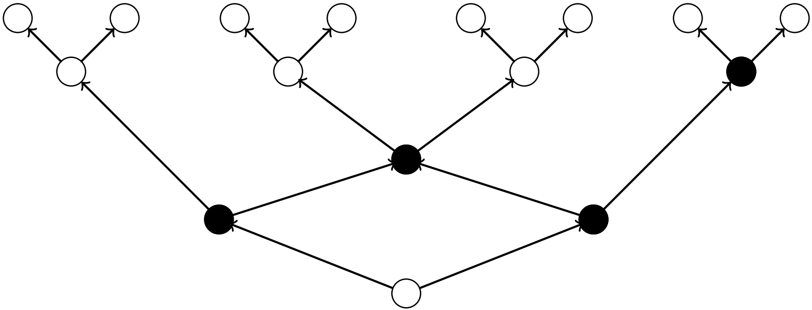
Construct Order Top-Down

Maximum number of nodes in memory: 6



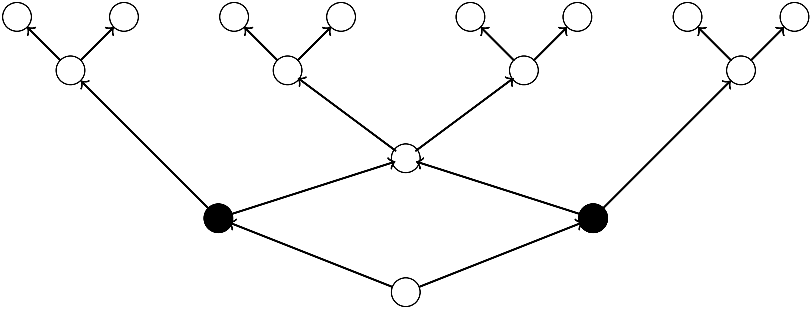
Construct Order Top-Down

Maximum number of nodes in memory: 6



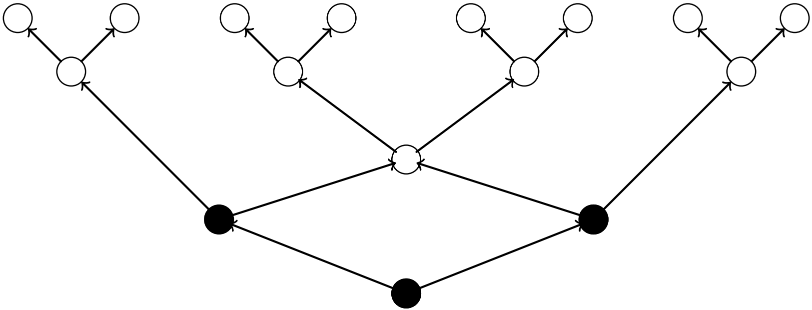
Construct Order Top-Down

Maximum number of nodes in memory: 6



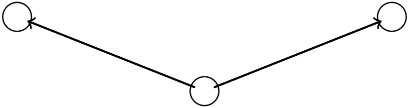
Construct Order Top-Down

Maximum number of nodes in memory: 6



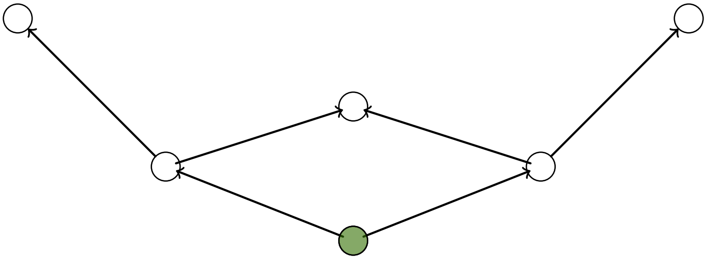
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



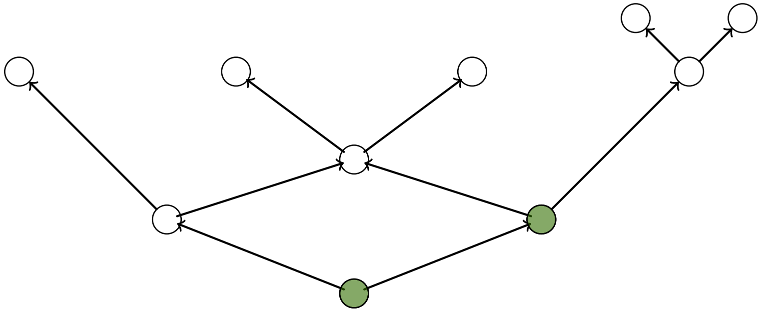
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



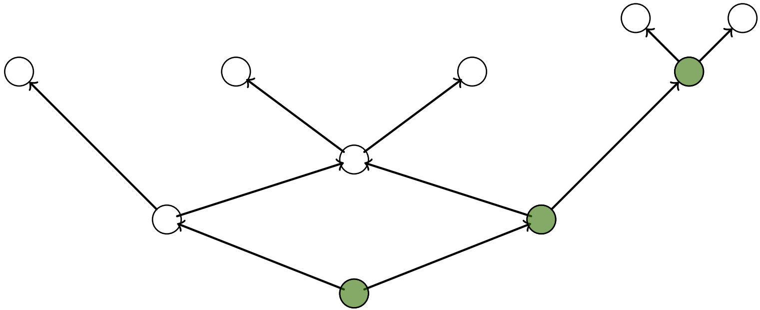
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



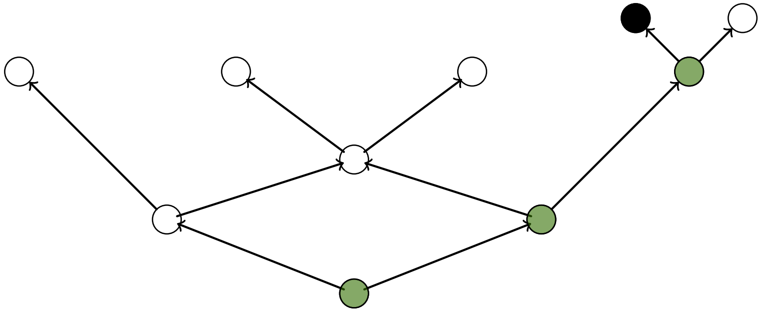
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



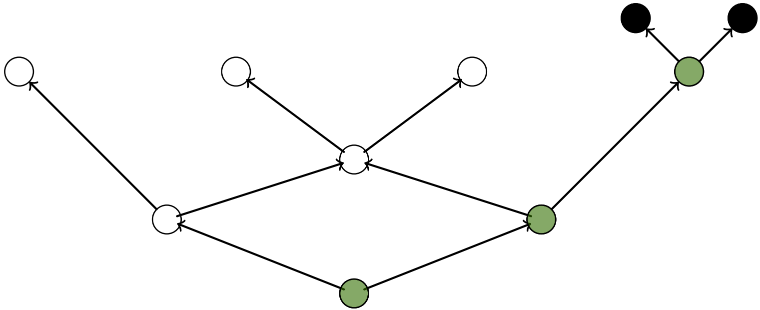
Construct Order Bottom-Up

Maximum number of nodes in memory: 1



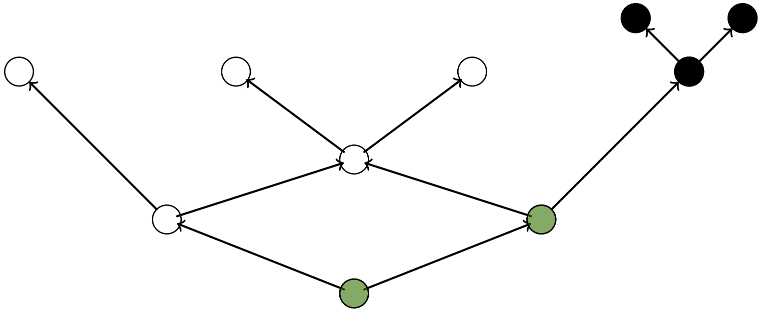
Construct Order Bottom-Up

Maximum number of nodes in memory: 2



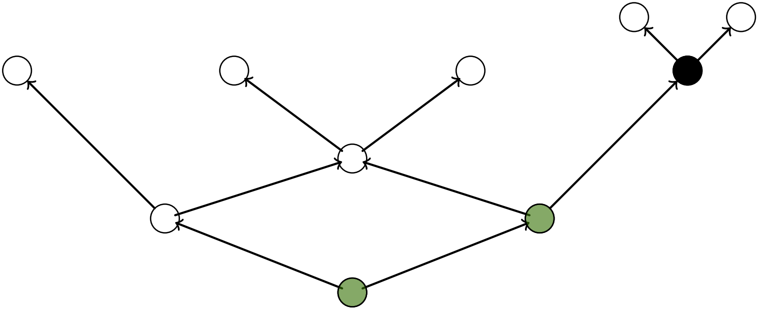
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



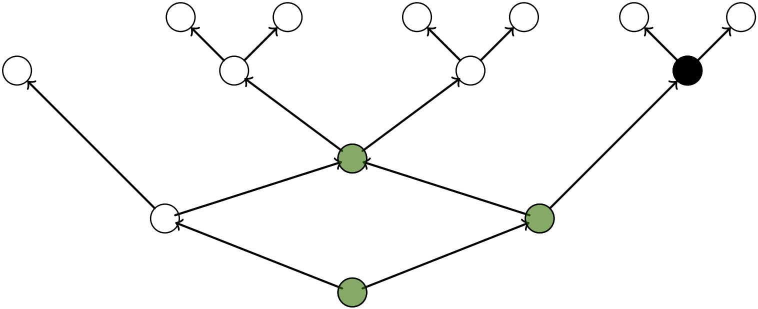
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



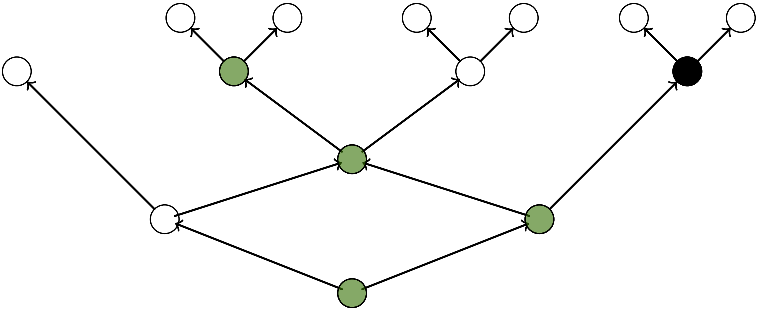
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



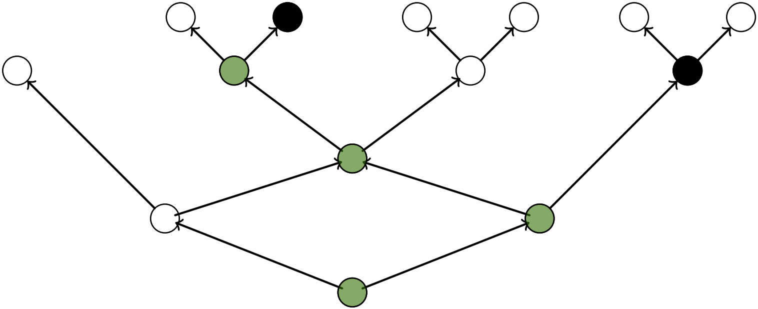
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



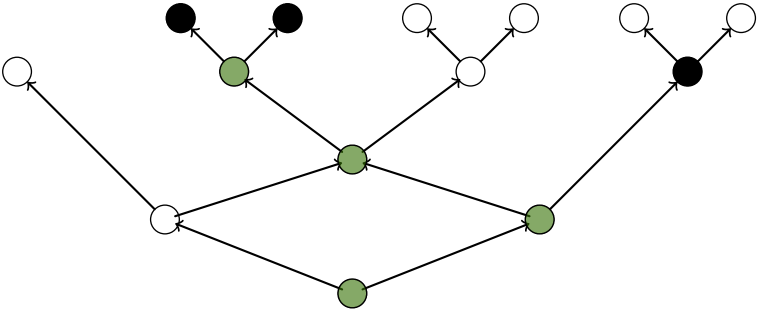
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



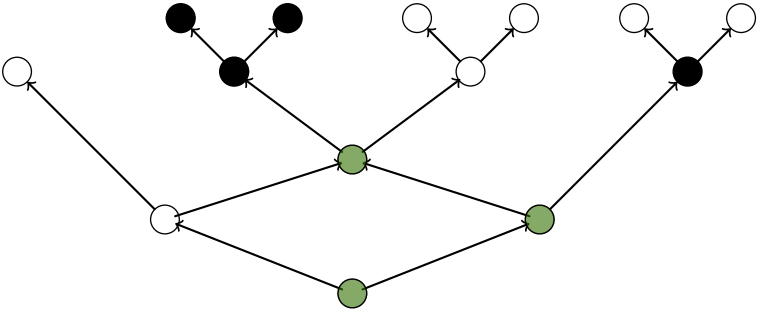
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



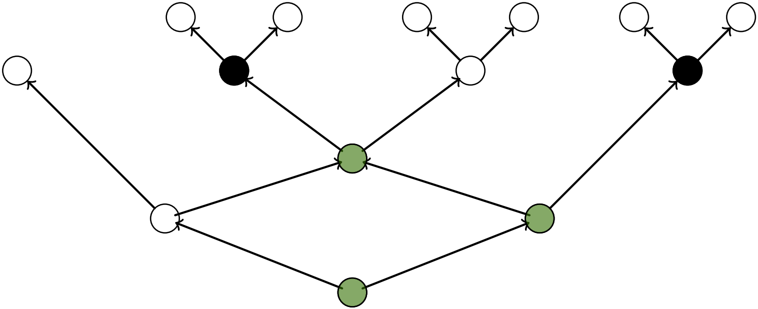
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



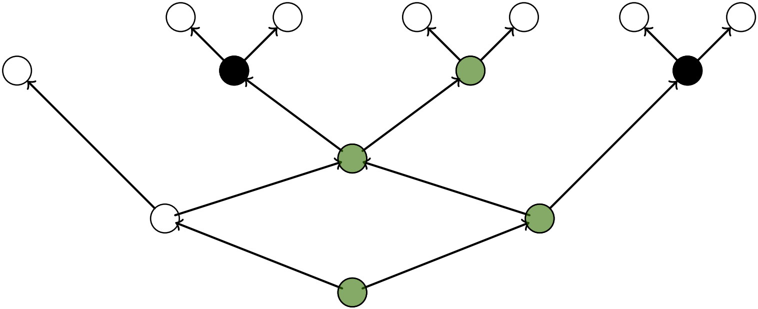
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



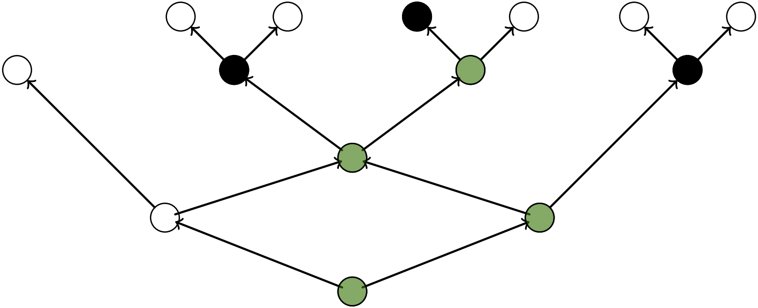
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



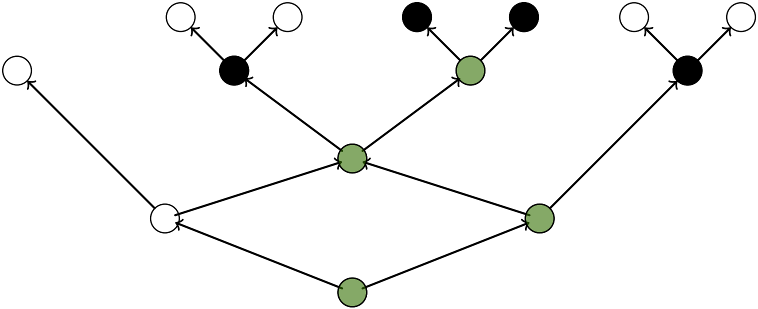
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



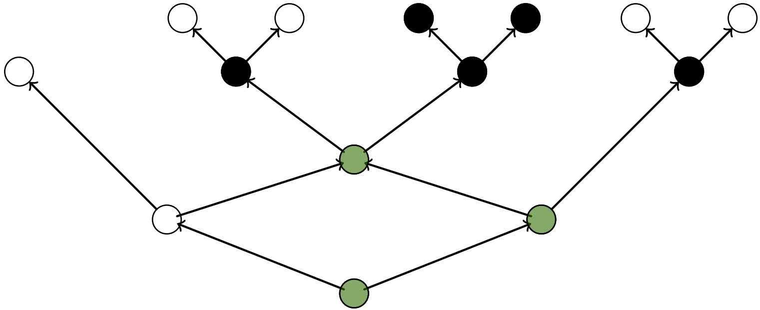
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



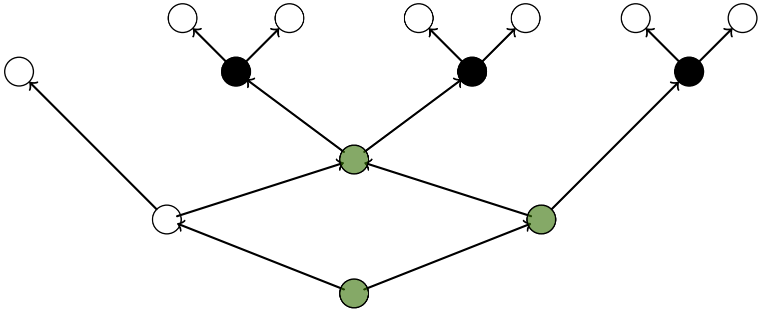
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



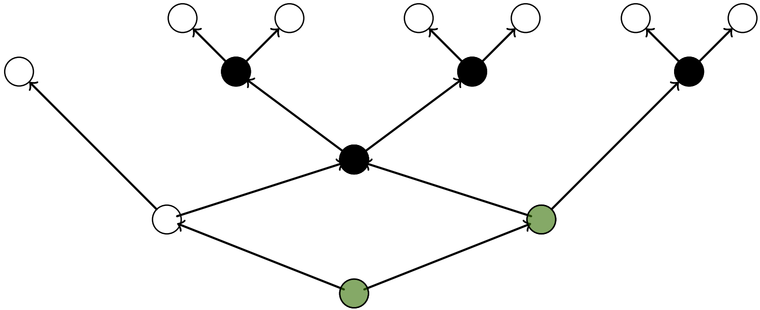
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



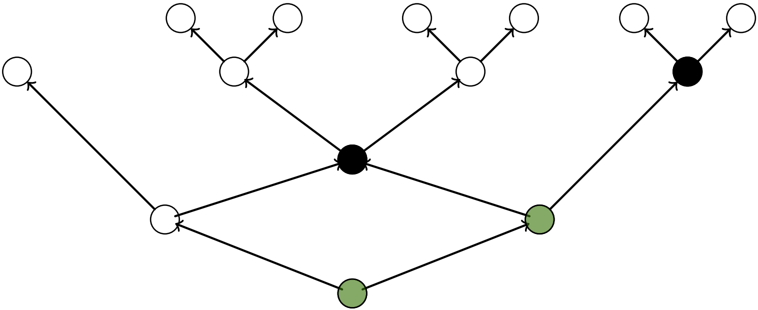
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



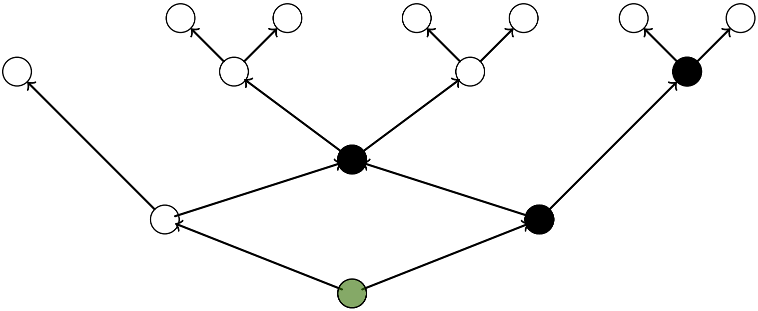
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



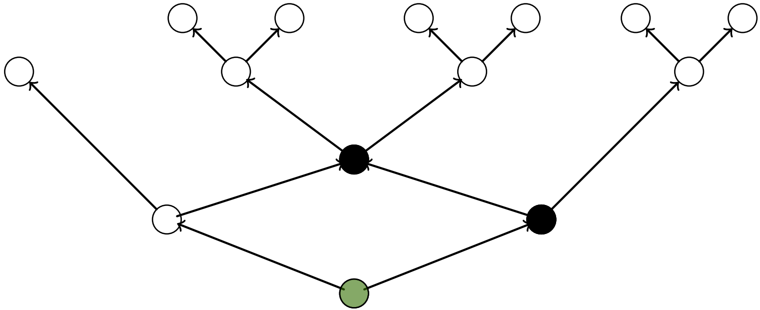
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



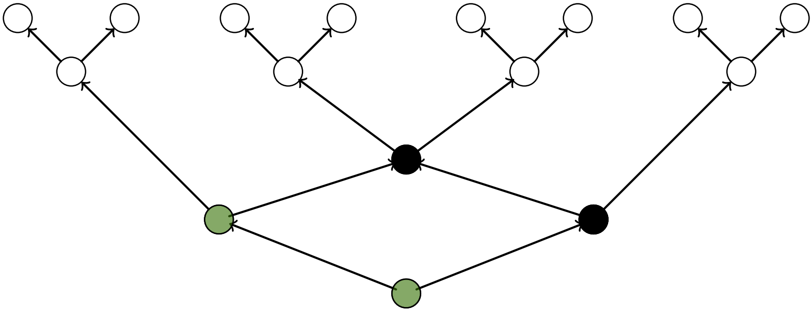
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



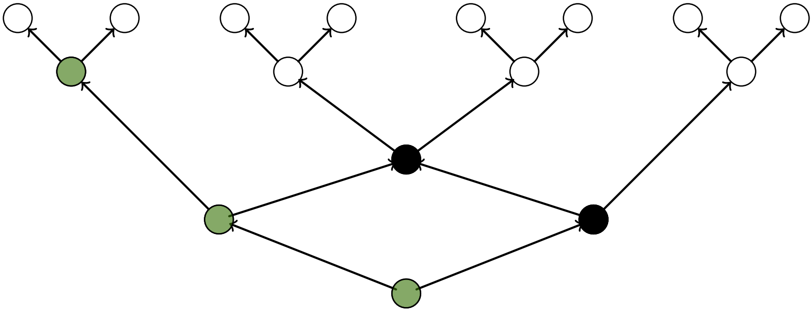
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



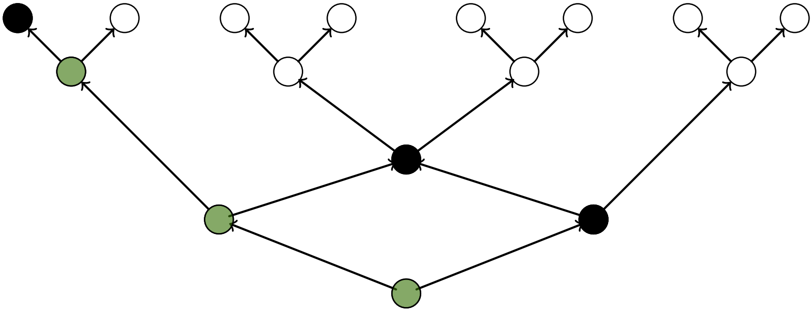
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



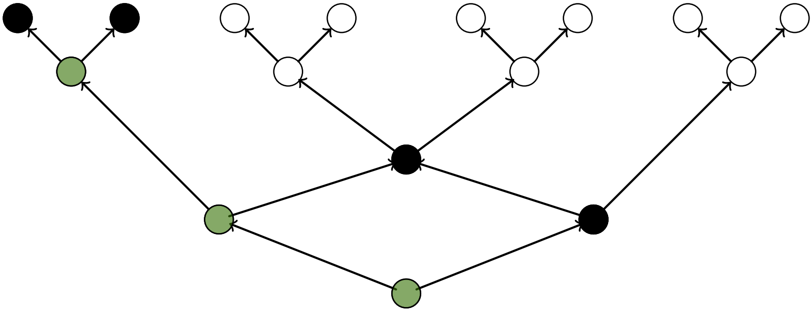
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



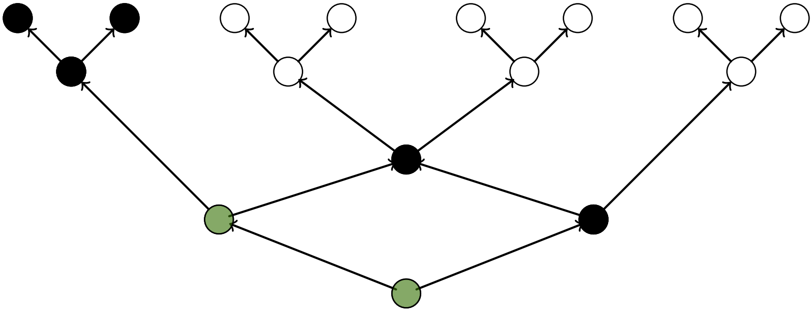
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



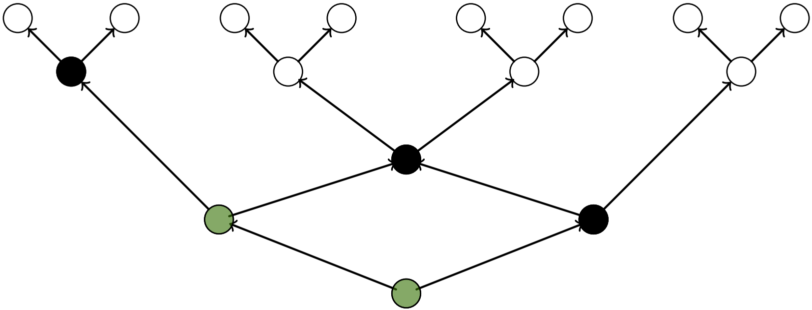
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



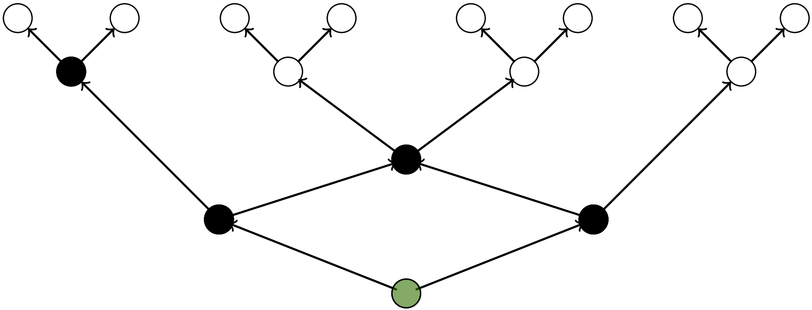
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



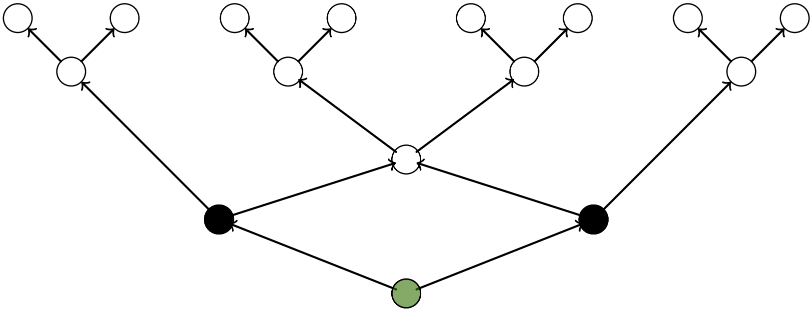
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



Construct Order Bottom-Up

Maximum number of nodes in memory: 5



Construct Order Bottom-Up

Maximum number of nodes in memory: 5