Partial Regularization of First-Order Resolution Proofs

J. Gorzny[1] B. Woltzenlogel Paleo[2,3]

[1]University of Waterloo

[2]Australian National University

[3]Vienna University of Technology

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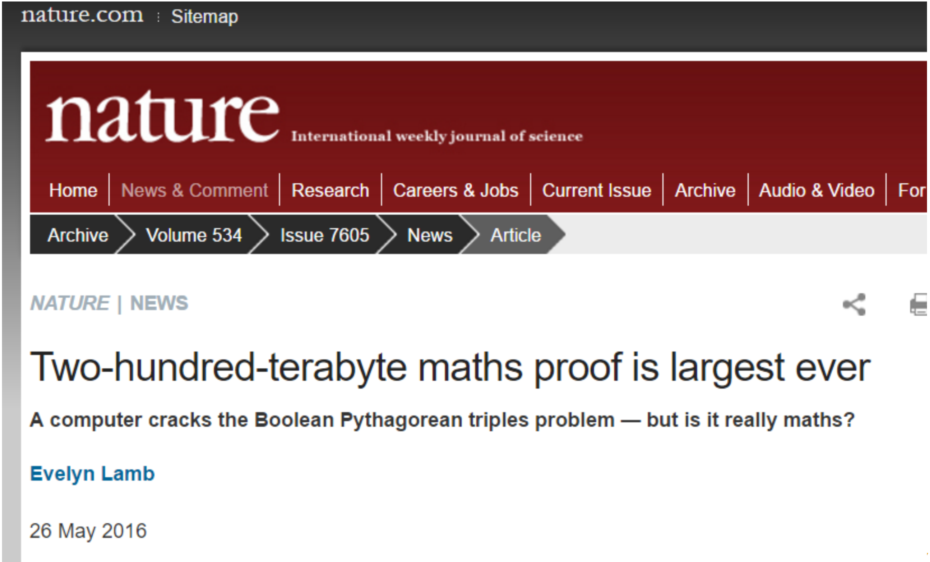
The Quest for Simple Proofs

"The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs."

–David Hilbert [Thi03]

The ‘Real World’



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Two-hundred-terabyte maths proof is largest ever

A computer cracks the Boolean Pythagorean triples problem — but is it really maths?

Evelyn Lamb

26 May 2016

(See [HKM16])

First-Order Proof Compression Motivation

- The best, most efficient provers, do not generate the best, least redundant proofs.
- Many compression algorithms for propositional proofs; few for first-order proofs.
- Finding a minimal proof is NP-hard, so use heuristics to find smaller proofs (see [FMP11])

Our Goal

Lifting propositional proof compression algorithms to first-order logic.

This work: `LowerUnits` [FMP11] and `RecyclePivotWithIntersection` [FMP11, BIFH$^+$08]

(Propositional) Proofs

Definition (Proof)

A directed acyclic graph $\langle V, E, \Gamma \rangle$, where

- V is a set of nodes
- E is a set of edges labeled by literals
- Γ (the proof clause) is inductively constructible using *axiom* and *resolution* nodes

Definition (Axiom)

A proof with a single node (so $E = \emptyset$)

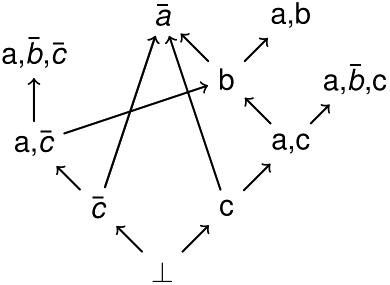
(Propositional) Resolution

Definition (Resolution)

Given two proofs ψ_L and ψ_R with conclusions Γ_L and Γ_R with some literal l such that $\bar{l} \in \Gamma_L$ and $l \in \Gamma_R$, the resolution proof ψ of ψ_L and ψ_R on l, denoted $\psi = \psi_L \psi_R$ is such that:

- ψ's nodes are the union of the nodes of ψ_L and ψ_R, and a new root node
- there is an edge from $\rho(\psi)$ to $\rho(\psi_L)$ labeled with $\bar{l}$
- there is an edge from $\rho(\psi)$ to $\rho(\psi_R)$ labeled with l
- ψ's conclusion is $(\Gamma_L \setminus \{\bar{l}\}) \cup (\Gamma_R \setminus \{l\})$

A Propositional Proof



Deletion

Deletion of an edge

- The resolvent is replaced by the other premise
- Some subsequent resolutions may have to be deleted too

Deletion of a subproof ψ

- Deletion of every edge coming to $\rho(\psi)$
- The operation is commutative and associative

First-Order Proofs

Definition (First-Order Proof)

A directed acyclic graph $\langle V, E, \Gamma \rangle$, where

- V is a set of nodes
- E is a set of edges labeled by literals **and substitutions**
- Γ (the proof clause) is inductively constructible using *axiom*, ***(first-order) resolution***, **and *contraction*** nodes

Axioms are unchanged

Substitutions and Unifiers

Definition (Substitution)

A mapping $\{X_1 \setminus t_1, X_2 \setminus t_2, \ldots\}$ from variables $X_1, X_2, \ldots$ to terms $t_1, t_2, \ldots$

Definition (Unifier)

A substitution that makes two terms equal when applied to them.

First-Order (Unifying) Resolution

Definition (First-Order Resolution)

Given two proofs ψ_L and ψ_R with conclusions Γ_L and Γ_R with some literal l such that $l_L \in \Gamma_L$ and $l_R \in \Gamma_R$, and σ_L and σ_R are substitutions usch that $l_L\sigma_L = \overline{l_R}\sigma_R$, and the variables in $(\Gamma_L \setminus l_L)\sigma_L$ and $(\Gamma_R \setminus l_R)\sigma_R$ are disjoint, then the resolution proof ψ of ψ_L and ψ_R on l, denoted $\psi = \psi_L\psi_R$ is such that:

- ψ's nodes are the union of the nodes of ψ_L and ψ_R, and a new root node
- there is an edge from $\rho(\psi)$ to $\rho(\psi_L)$ labeled with l_L and σ_L
- there is an edge from $\rho(\psi)$ to $\rho(\psi_R)$ labeled with l_R and σ_R
- ψ's conclusion is $(\Gamma_L \setminus l_L)\sigma_L \cup (\Gamma_R \setminus l_R)\sigma_R$

Unifying Resolution Example

$$\begin{tikzcd} \eta_1\colon p(a) \vdash & & \eta_2\colon q(Y,X) \vdash p(Y) \\ & \psi\colon q(a,X) \vdash \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\sigma = \{Y \rightarrow a\}$$

Refutation when $\psi = \bot$

Contraction

Definition (Contraction)

If ψ' is a proof and σ is a unifier of $\{l_1, \ldots, l_n\} \subset \Gamma'$, then a contraction ψ is a proof where

- ψ's nodes are the union of the nodes of ψ' and a new node v
- There is an edge from $\rho(\psi')$ to v labeled with $\{l_1, \ldots, l_n\}$ and σ
- The conclusion is $(\Gamma' \setminus \{l_1, \ldots, l_n\})\sigma \cup \{l\}$, where $l = l_k\sigma$ for $k \in \{1, \ldots, n\}$

Contraction Example

$$\eta_1\colon p(X, Y), p(a, Z), p(a, f(b)) \vdash q(Z)$$

$$\psi\colon p(a, f(b)) \vdash q(f(b))$$

$$\sigma = \{X \rightarrow a, Y \rightarrow f(b), Z \rightarrow f(b)\}$$

Contraction Example

$$\eta_1\colon p(X,Y), p(X,Z), p(U,V) \vdash q(Z)$$

$$\uparrow$$

$$\psi\colon p(X,Z) \vdash q(Z)$$

$$\sigma = \{Y \rightarrow Z, U \rightarrow Z, V \rightarrow Z\}$$

Contraction Example

$$\eta_1\colon p(X, Y), p(a, Z), p(a, f(b)) \vdash q(Z)$$

$$\uparrow$$

$$\psi\colon p(X, Y), p(a, f(b)) \vdash q(f(b))$$

$$\sigma = \{Z \rightarrow f(b)\}$$

Lowering Units

Definition (Unit)

A unit clause is a subproof with a conclusion clause (final clause) having exactly 1 literal

Theorem ([FMP11])

A unit clause can always be lowered

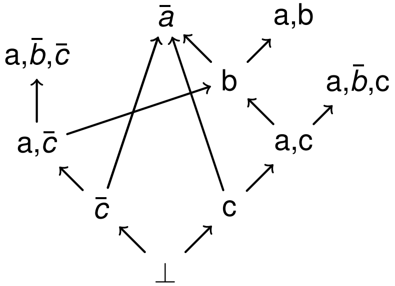
Compression is achieved by delaying resolution with unit clause subproofs.

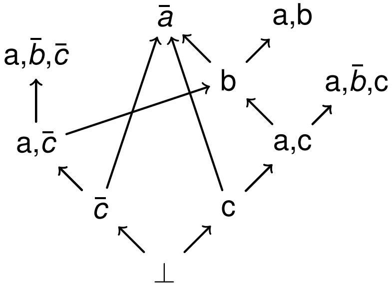
LowerUnits

Two Traversals of the proof

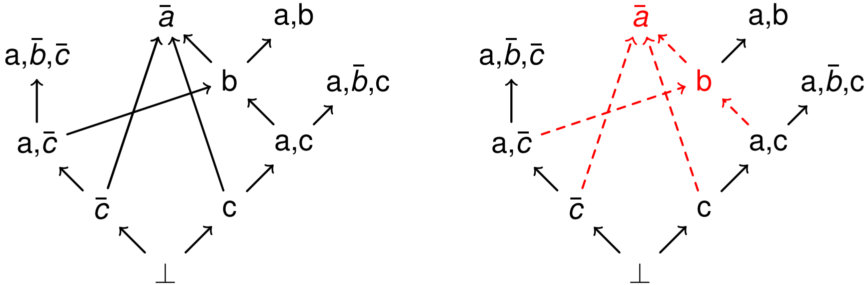
- $\uparrow$ Collect units with more than one resolvent
- $\downarrow$ Delete units and reintroduce them at the bottom of the proof

Propositional Example

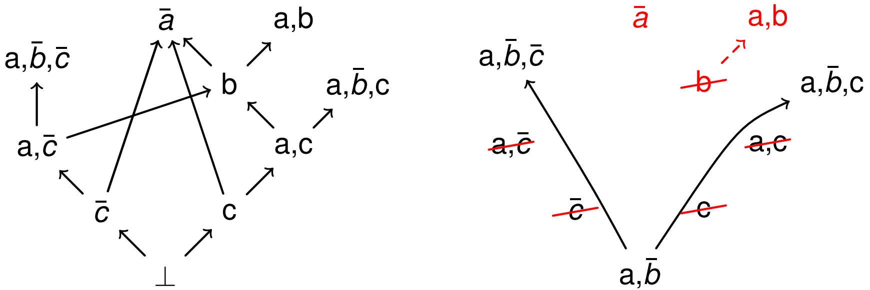




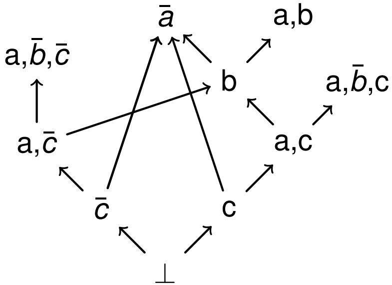
Propositional Example

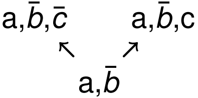


Propositional Example

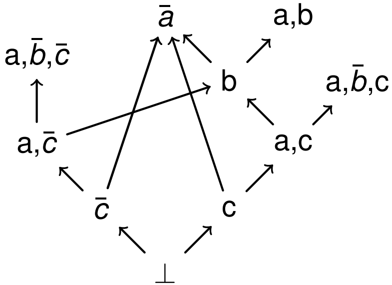


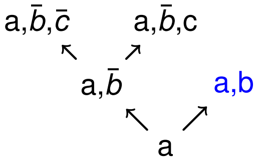
Propositional Example



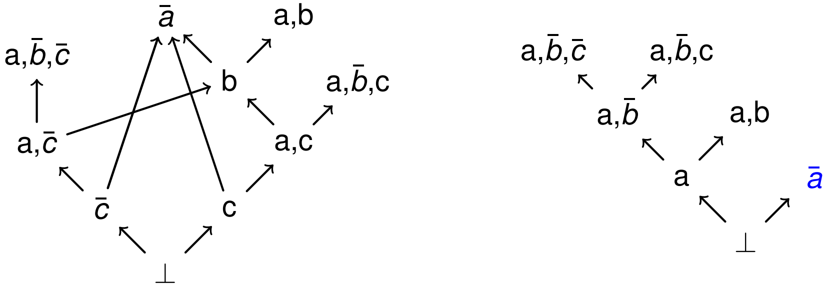


Propositional Example

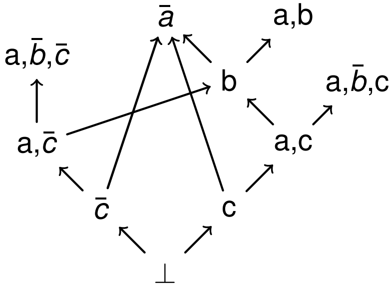


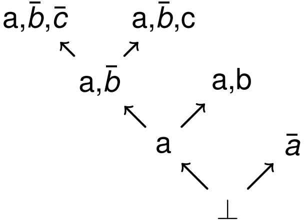


Propositional Example

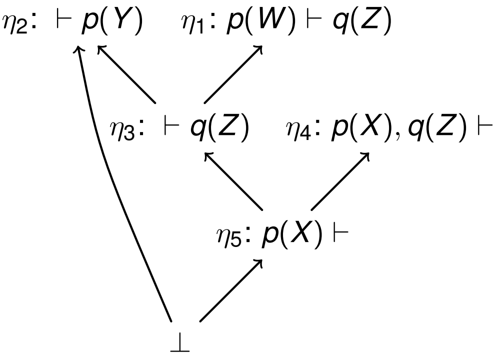


Propositional Example





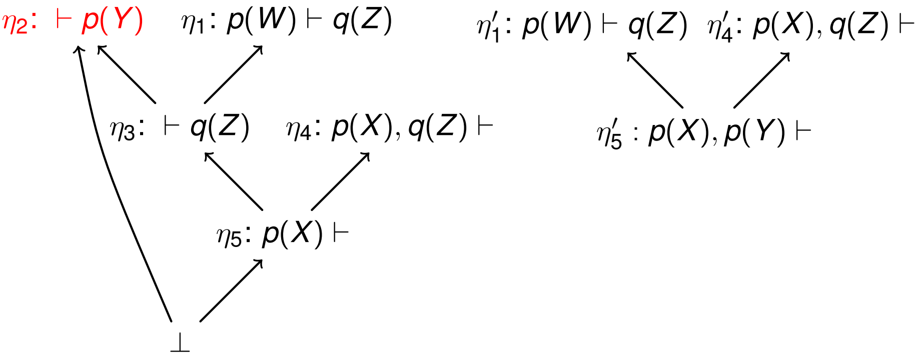
First-Order Change: Helpful Contractions



First-Order Change: Helpful Contractions

$$\begin{tikzcd}
\color{red}{\eta_2\colon \vdash p(Y)} & & \eta_1\colon p(W) \vdash q(Z) & \\
& \eta_3\colon \vdash q(Z) \arrow[ul] \arrow[ur] & & \eta_4\colon p(X), q(Z) \vdash \\
& & \eta_5\colon p(X) \vdash \arrow[ul] \arrow[ur] & \\
& \bot \arrow[uuul] \arrow[ur] & &
\end{tikzcd}$$

First-Order Change: Helpful Contractions

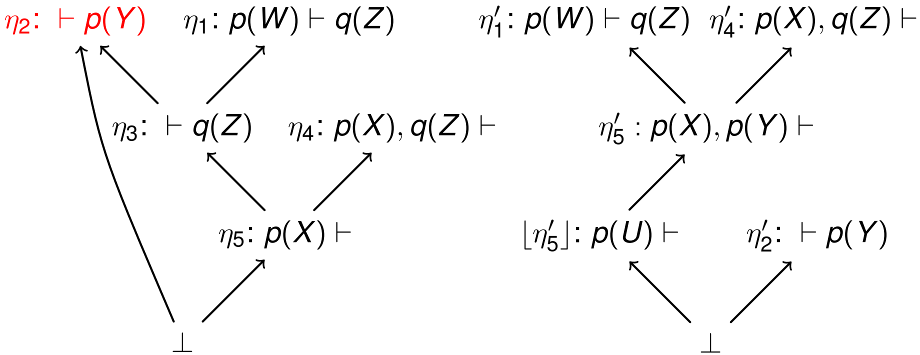


First-Order Change: Helpful Contractions

$$
\begin{tikzcd}
\eta_2\colon \vdash p(Y) & & \eta_1\colon p(W) \vdash q(Z) & \\
& \eta_3\colon \vdash q(Z) \arrow[ul] \arrow[ur] & & \eta_4\colon p(X), q(Z) \vdash \\
& & \eta_5\colon p(X) \vdash \arrow[ul] \arrow[ur] & \\
& \bot \arrow[uuul] \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\eta_1'\colon p(W) \vdash q(Z) & & \eta_4'\colon p(X), q(Z) \vdash \\
& \eta_5'\colon p(X), p(Y) \vdash \arrow[ul] \arrow[ur] & \\
\lfloor \eta_5' \rfloor\colon p(U) \vdash \arrow[ur] & &
\end{tikzcd}
$$

First-Order Change: Helpful Contractions



First-Order Challenge: Pre-Deletion Check

$$
\begin{tikzcd}
\eta_5\colon q(Y) \vdash p(a) & & \eta_4\colon p(X) \vdash & & \eta_3\colon \vdash p(b), q(Y) \\
& \eta_1\colon q(Y) \vdash \arrow[ul] \arrow[ur] & & \eta_2\colon \vdash q(Y) \arrow[ul] \arrow[ur] & \\
& & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

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& \eta_1\colon q(Y) \vdash \arrow[ul] \arrow[ur] & & \eta_2\colon \vdash q(Y) \arrow[ul] \arrow[ur] & \\
& & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

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$$
\begin{tikzcd}[column sep=small]
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& \eta_1\colon q(Y) \vdash \arrow[ul] \arrow[ur] & & \eta_2\colon \vdash q(Y) \arrow[ul] \arrow[ur] & \\
& & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\eta'_5\colon q(Y) \vdash p(a) & & \eta'_3\colon \vdash p(b), q(Y) \\
& \eta\colon \vdash p(a), p(b) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

First-Order Challenge: Pre-Deletion Check

$$
\begin{tikzcd}
\eta_5\colon q(Y) \vdash p(a) & & \color{red}{\eta_4\colon p(X) \vdash} & & \eta_3\colon \vdash p(b), q(Y) \\
& \eta_1\colon q(Y) \vdash \arrow[ul] \arrow[ur] & & \eta_2\colon \vdash q(Y) \arrow[ul] \arrow[ur] & \\
& & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\eta_5'\colon q(Y) \vdash p(a) & & \eta_3'\colon \vdash p(b), q(Y) \\
& \eta\colon \vdash p(a), p(b) \arrow[ul] \arrow[ur] & \\
& \cancel{\lfloor \eta \rfloor} \arrow[u] &
\end{tikzcd}
$$

First-Order Challenge: Pre-Deletion Check

$$
\begin{tikzcd}
\eta_5\colon q(Y) \vdash p(a) & & \color{red}{\eta_4\colon p(X) \vdash} & & \eta_3\colon \vdash p(b), q(Y) \\
& \eta_1\colon q(Y) \vdash \arrow[ul] \arrow[ur] & & \eta_2\colon \vdash q(Y) \arrow[ul] \arrow[ur] & \\
& & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Definition (Pre-Deletion Property)

η unit, $l \in \eta$, such that l is resolved with literals $l_1, \ldots, l_n$ in a proof ψ. η satisfies the *pre-deletion unifiability* property in ψ if $l_1, \ldots, l_n$ and $\bar{l}$ are unifiable.

First-Order Challenge: Post-Deletion Check

$$
\begin{tikzcd}
\eta_1\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash & & & \eta_2\colon \vdash p(U, V) \\
\eta_4\colon \vdash r(W) & & \eta_3\colon r(V), p(U, q(V, b)) \vdash \arrow[ull] \arrow[ur] & \\
& \eta_5\colon p(U, q(W, b)) \vdash \arrow[ul] \arrow[ur] & & \\
& & \bot \arrow[ul] \arrow[uuur, bend right] &
\end{tikzcd}
$$

First-Order Challenge: Post-Deletion Check

$$\begin{tikzcd}
 & \eta_1 \colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash & & \eta_2 \colon \vdash p(U, V) \\
\eta_4 \colon \vdash r(W) & & \eta_3 \colon r(V), p(U, q(V, b)) \vdash \arrow[ul] \arrow[ur] & \\
 & \eta_5 \colon p(U, q(W, b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur, bend right] &
\end{tikzcd}$$

First-Order Challenge: Post-Deletion Check

$$
\begin{tikzcd}[column sep=small]
 & \eta_1\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash & & \textcolor{red}{\eta_2\colon \vdash p(U, V)} \\
\eta_4\colon \vdash r(W) & & \eta_3\colon r(V), p(U, q(V, b)) \vdash \arrow[ul] \arrow[ur] & \\
 & \eta_5\colon p(U, q(W, b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur, bend right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\eta_4'\colon \vdash r(W) & & \eta_1'\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash \\
 & \eta_5'\colon p(X, q(W, b)), p(X, W) \vdash \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

First-Order Challenge: Post-Deletion Check

$$
\begin{tikzcd}
 & \eta_1 \colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash & & \eta_2 \colon \vdash p(U, V) \\
\eta_4 \colon \vdash r(W) & & \eta_3 \colon r(V), p(U, q(V, b)) \vdash \arrow[ul] \arrow[ur] & \\
 & \eta_5 \colon p(U, q(W, b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur, bend right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\eta'_4 \colon \vdash r(W) & & \eta'_1 \colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash \\
 & \eta'_5 \colon p(X, q(W, b)), p(X, W) \vdash \arrow[ul] \arrow[ur] & \\
 & \lfloor \cancel{\eta'_5} \rfloor \arrow[u] &
\end{tikzcd}
$$

First-Order Challenge: Post-Deletion Check

$$
\begin{tikzcd}
 & \eta_1\colon r(Y), p(X, q(Y,b)), p(X,Y) \vdash & & \eta_2\colon \vdash p(U,V) \\
\eta_4\colon \vdash r(W) & & \eta_3\colon r(V), p(U, q(V,b)) \vdash \arrow[ul] \arrow[ur] & \\
 & \eta_5\colon p(U, q(W,b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur, bend right] &
\end{tikzcd}
$$

Definition (Post-Deletion Property)

η unit, $l \in \eta$, such that l is resolved with literals $l_1, \ldots, l_n$ in a proof ψ. η satisfies the *post-deletion unifiability* property in ψ if $l_1^{\dagger\downarrow}, \ldots, l_n^{\dagger\downarrow}$ and $\overline{l^{\dagger}}$ are unifiable, where $l^{\dagger}$ is the literal in $\psi' = \psi \setminus \{\eta\}$ corresponding to l in ψ, and $l^{\dagger\downarrow}$ is the descendant of $l^{\dagger}$ in the roof of ψ'.

First-Order Lower Units Challenges

- Deletion changes literals
- Unit collection depends on whether contraction is possible after propagation down the proof

Deletion of units require knowledge of proof after deletion, and deletion depends on what will be lowered.

- $O(n^2)$ solution to have full knowledge
- Difficult bookkeeping required for implementation

Greedy First-Order Lower Units - A Quicker Alternative

- Ignore post-deletion satisfaction
- Focus on pre-deletion satisfaction
- Greedy contraction

Faster run-time (linear; one traversal)
Easier to implement

- Doesn't always compress (returns original proof sometimes)

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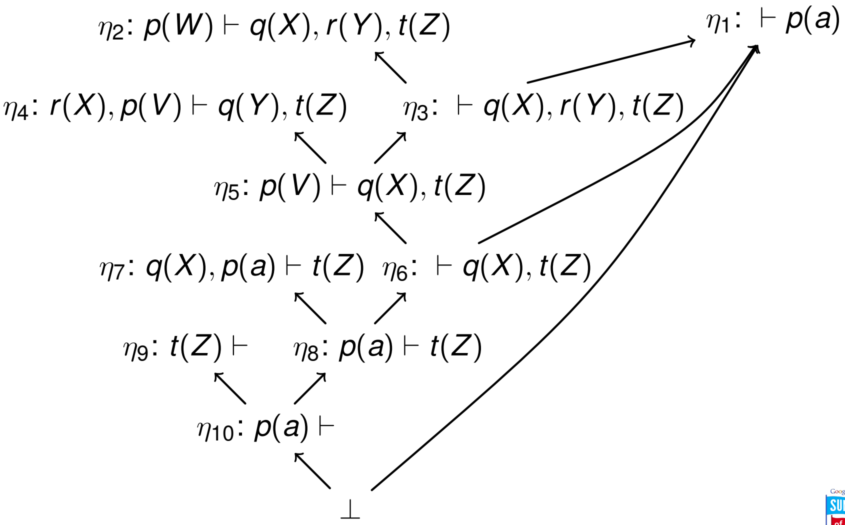
First-Order Example



First-Order Example

η_2: $p(W) \vdash q(X), r(Y), t(Z)$ $\quad$ η_1: $\vdash p(a)$

η_4: $r(X), p(V) \vdash q(Y), t(Z)$ $\quad$ η_3: $\vdash q(X), r(Y), t(Z)$

η_5: $p(V) \vdash q(X), t(Z)$

η_7: $q(X), p(a) \vdash t(Z)$ $\quad$ η_6: $\vdash q(X), t(Z)$

η_9: $t(Z) \vdash$ $\quad$ η_8: $p(a) \vdash t(Z)$

η_{10}: $p(a) \vdash$

$\perp$

$$\sigma = \{W \rightarrow a, V \rightarrow a\}$$

First-Order Example

$\eta_1\colon\ \vdash p(a)$

$\eta_2\colon p(W) \vdash q(X), r(Y), t(Z)$

$\{W \rightarrow a\}$

$\eta_4\colon r(X), p(V) \vdash q(Y), t(Z)$

$\eta_3\colon\ \vdash q(X), r(Y), t(Z)$

$\eta_5\colon p(V) \vdash q(X), t(Z)$

$\eta_7\colon q(X), p(a) \vdash t(Z)$

$\eta_6\colon\ \vdash q(X), t(Z)$

$\eta_9\colon t(Z) \vdash$

$\eta_8\colon p(a) \vdash t(Z)$

$\eta_{10}\colon p(a) \vdash$

$\perp$

First-Order Example

$\eta_1\colon \vdash p(a)$

$\eta_4\colon r(X), p(V) \vdash q(Y), t(Z)$ $\quad$ $\eta_3'\colon p(a) \vdash q(X), r(Y), t(Z)$

$\eta_5'\colon p(V), p(a) \vdash q(X), t(Z)$

$\eta_7\colon q(X), p(a) \vdash t(Z)$ $\quad$ $\eta_6\colon \vdash q(X), t(Z)$

$\eta_9\colon t(Z) \vdash$ $\quad$ $\eta_8\colon p(a) \vdash t(Z)$

$\eta_{10}\colon p(a) \vdash$

$\perp$

First-Order Example

$$\eta_1 \colon \vdash p(a)$$

$$\eta_4 \colon r(X), p(V) \vdash q(Y), t(Z) \qquad \eta_3 \colon p(a) \vdash q(X), r(Y), t(Z)$$

$$\eta_5' \colon p(V), p(a) \vdash q(X), t(Z)$$

$$\{V \rightarrow a\}$$

$$\eta_7 \colon q(X), p(a) \vdash t(Z) \qquad \eta_6 \colon \vdash q(X), t(Z)$$

$$\eta_9 \colon t(Z) \vdash \qquad \eta_8 \colon p(a) \vdash t(Z)$$

$$\eta_{10} \colon p(a) \vdash$$

$$\bot$$

First-Order Example

$\eta_1\colon \vdash p(a)$

$\eta_4\colon r(X), p(V) \vdash q(Y), t(Z)$ $\quad$ $\eta_3\colon p(a) \vdash q(X), r(Y), t(Z)$

$\eta_6'\colon p(a), p(a) \vdash q(X), t(Z)$

$\eta_7\colon q(X), p(a) \vdash t(Z)$ $\quad$ $\lfloor \eta_6' \rfloor\colon p(a) \vdash q(X), t(Z)$

$\eta_9\colon t(Z) \vdash$ $\quad$ $\eta_8\colon p(a) \vdash t(Z)$

$\eta_{10}\colon p(a) \vdash$

$\perp$

First-Order Example

$$
\begin{tikzcd}
 & & & & \color{red}{\eta_1\colon\ \vdash p(a)} \\
 & & \eta_4\colon r(X), p(V) \vdash q(Y), t(Z) & \eta_3\colon p(a) \vdash q(X), r(Y), t(Z) & \\
 & & & \eta_6'\colon p(a), p(a) \vdash q(X), t(Z) \arrow[ul] \arrow[u] & \\
 & \eta_7\colon q(X), p(a) \vdash t(Z) & & \lfloor \eta_6' \rfloor\colon p(a) \vdash q(X), t(Z) \arrow[u] & \\
\eta_9\colon t(Z) \vdash & & \eta_8\colon p(a) \vdash t(Z) \arrow[ul] \arrow[ur] & & \\
 & \eta_{10}\colon p(a) \vdash \arrow[ul] \arrow[ur] \arrow[dr, blue, "\color{blue}{\emptyset}"'] & & & \\
 & & \bot \arrow[uuuuuurr, red, bend right] & &
\end{tikzcd}
$$

First-Order Example

$$\eta_1 \colon \vdash p(a)$$

$$\eta_4 \colon r(X), p(V) \vdash q(Y), t(Z) \qquad \eta_3 \colon p(a) \vdash q(X), r(Y), t(Z)$$

$$\eta_6' \colon p(a), p(a) \vdash q(X), t(Z)$$

$$\eta_7 \colon q(X), p(a) \vdash t(Z) \qquad \lfloor \eta_6' \rfloor \colon p(a) \vdash q(X), t(Z)$$

$$\eta_9 \colon t(Z) \vdash \qquad \eta_8 \colon p(a) \vdash t(Z)$$

$$\eta_{10} \colon p(a) \vdash$$

First-Order Example

$$
\begin{tikzcd}[column sep=small]
 & & & \eta_1\colon\ \vdash p(a) \\
\eta_4\colon r(X), p(V) \vdash q(Y), t(Z) & & \eta_3\colon p(a) \vdash q(X), r(Y), t(Z) & \\
 & & \eta_6'\colon p(a), p(a) \vdash q(X), t(Z) \arrow[ull] \arrow[u] & \\
 & \eta_7\colon q(X), p(a) \vdash t(Z) & \lfloor \eta_6' \rfloor\colon p(a) \vdash q(X), t(Z) \arrow[u] & \\
\eta_9\colon t(Z) \vdash & \eta_8\colon p(a) \vdash t(Z) \arrow[u] \arrow[ur] & & \\
 & \eta_{10}\colon p(a) \vdash \arrow[ul] \arrow[u] & & \\
 & & \bot \arrow[ul] \arrow[uuuuuur, bend right] &
\end{tikzcd}
$$

Recycling Pivots

Removes *irregularities*: inferences η where the pivot occurs as a pivot of another inference below η on the path to the root

- Store a set of *safe* $\mathcal{S}(\eta)$ literals for each node η
- If there are multiple paths, take *intersection* of safe literals
- Bottom-up: compute safe literals; mark deletions
- Top-down: regularize

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- Bottom-up: compute safe literals; mark deletions
- Top-down: regularize

Regularization Can Be Bad

Resolution without irregularities is still complete. But:

Theorem ([Tse70])

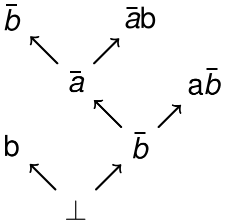
There are unsatisfiable formulas whose shortest regular resolution refutations are exponentially longer than their shortest unrestricted resolution refutations.

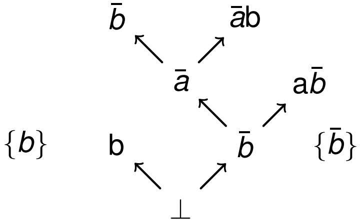
Propositional Example

$$
\begin{tikzcd}
\bar{b} & & \bar{a}b & \\
& \bar{a} \arrow[ul] \arrow[ur] & & a\bar{b} \\
b & & \bar{b} \arrow[ul] \arrow[ur] & \\
& \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bar{b} & & \bar{a}b & \\
& \bar{a} \arrow[ul] \arrow[ur] & & a\bar{b} \\
b & & \bar{b} \arrow[ul] \arrow[ur] & \\
& \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

$\uparrow$ safe literal collection

Propositional Example





$\uparrow$ safe literal collection

Propositional Example

$$
\begin{tikzcd}
\bar b & & \bar a b & \\
 & \bar a \arrow[ul] \arrow[ur] & & a\bar b \\
b & & \bar b \arrow[ul] \arrow[ur] & \\
 & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \bar b & & \bar a b & & \\
\{\bar a \bar b\} & & \bar a \arrow[ul] \arrow[ur] & & a\bar b & \{a\bar b\} \\
\{b\} & b & & \bar b \arrow[ul] \arrow[ur] & \{\bar b\} & \\
 & & \bot \arrow[ul] \arrow[ur] & & &
\end{tikzcd}
$$

$\uparrow$ safe literal collection

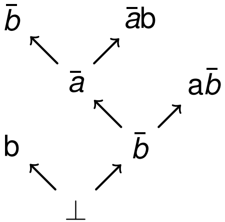
Propositional Example

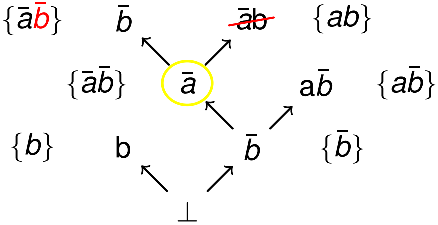
$$
\begin{tikzcd}
\bar{b} & & \bar{a}b & \\
& \bar{a} \arrow[ul] \arrow[ur] & & a\bar{b} \\
b & & \bar{b} \arrow[ul] \arrow[ur] & \\
& \perp \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\{\bar{a}\bar{b}\} & \bar{b} & & \bar{a}b & \{ab\} & \\
& \{\bar{a}\bar{b}\} & \bar{a} \arrow[ul] \arrow[ur] & & a\bar{b} & \{a\bar{b}\} \\
\{b\} & b & & \bar{b} \arrow[ul] \arrow[ur] & \{\bar{b}\} & \\
& & \perp \arrow[ul] \arrow[ur] & & &
\end{tikzcd}
$$

$\uparrow$ safe literal collection

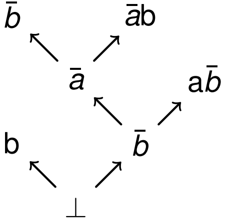
Propositional Example

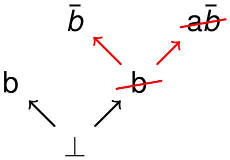




↑ safe literal collection

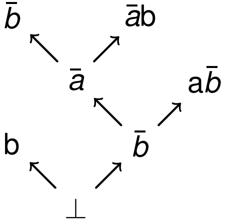
Propositional Example

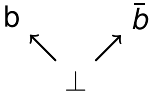




$\downarrow$ regularization

Propositional Example





$\downarrow$ regularization

Pre-Regularization Checks I

$\eta_1\colon \vdash p(W, X)$

$\{\vdash q(c), \color{red}{p(a, X)}\}$

$\eta_2\colon p(W, X) \vdash q(c)$

$\{p(W, X) \vdash q(c), p(a, X)\}$

$\eta_3\colon \vdash q(c)$

$\{\vdash q(c), p(a, X)\}$

$\eta_4\colon q(c) \vdash p(a, X)$

$\{q(c) \vdash p(a, X)\}$

$\eta_6\colon p(Y, b) \vdash$

$\{p(Y, b) \vdash\}$

$\eta_5\colon \vdash p(a, X)$

$\{\vdash p(a, X)\}$

$\perp$

$$\sigma = \{W \to a\} \implies \sigma\eta_1 \in \mathcal{S}(\eta_1)$$

Pre-Regularization Checks I

$$
\begin{tikzcd}
\eta_6\colon p(Y,b) \vdash & & \eta_1\colon \vdash p(W,X) \\
& \perp \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$\sigma = \{W \to Y, X \to b\}$$

Pre-Regularization Checks II

$\eta_1 \colon \vdash p(W, c)$ $\qquad$ $\eta_2 \colon p(W, X) \vdash q(c)$

$\{\vdash q(c), p(a, X)\}$ $\qquad$ $\{p(W, X) \vdash q(c), p(a, X)\}$

$\eta_3 \colon \vdash q(c)$ $\qquad$ $\eta_4 \colon q(c) \vdash p(a, X)$

$\{\vdash q(c), p(a, X)\}$ $\qquad$ $\{q(c) \vdash p(a, X)\}$

$\eta_6 \colon p(Y, b) \vdash$ $\qquad$ $\eta_5 \colon \vdash p(a, X)$

$\{p(Y, b) \vdash\}$ $\qquad \perp \qquad$ $\{\vdash p(a, X)\}$

$$\sigma = \{W \to a, X \to c\} \implies \sigma\eta_1 \in \mathcal{S}(\eta_1)$$

but...

Pre-Regularization Checks II

$$\begin{tikzcd} \eta_6\colon p(Y,b) \vdash & & \eta_1\colon \vdash p(c,a) \\ & \cancel{\bot} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

no σ!

Pre-Regularization Unifiability

Definition

Let η be a node with pivot ℓ' unifiable with safe literal ℓ which is resolved against literals $\ell_1, \ldots, \ell_n$ in a proof ψ. η is said to satisfy the *pre-regularization unifiability property* in ψ if ℓ_1,...,ℓ_n, and $\bar{\ell}'$ are unifiable.

Post-Regularization Checks

$$
\begin{tikzcd}
\eta_1\colon p(U,V) \vdash q(f(a,V),U) & & \eta_2\colon q(f(a,X),Y), q(T,X) \vdash q(f(a,Z),Y) \\
\eta_4\colon \vdash q(R,S) & & \eta_3\colon p(U,V), q(T,V) \vdash q(f(a,Z),U) \arrow[ull] \arrow[u] \\
\eta_6\colon \vdash p(c,d) & & \eta_5\colon p(U,V) \vdash q(f(a,Z),U) \arrow[ull] \arrow[u] \\
\eta_8\colon q(f(a,e),c) \vdash & & \eta_7\colon \vdash q(f(a,Z),c) \arrow[ull] \arrow[u] \\
& \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$\mathcal{S}(\eta_3) = \{q(T,V), p(c,d) \vdash q(f(a,e),c)\}$$

Post-Regularization Checks

$$\begin{tikzcd}
\eta_1 \colon p(U,V) \vdash q(f(a,V),U) & \cancel{\eta_2 \colon q(f(a,X),Y), q(T,X) \vdash q(f(a,Z),Y)} \\
\eta_4 \colon \vdash q(R,S) & \eta_3 \colon p(U,V), q(T,V) \vdash q(f(a,Z),U) \arrow[ul] \arrow[u] \\
\eta_6 \colon \vdash p(c,d) & \eta_5 \colon p(U,V) \vdash q(f(a,Z),U) \arrow[ul] \arrow[u] \\
\eta_8 \colon q(f(a,e),c) \vdash & \eta_7 \colon \vdash q(f(a,Z),c) \arrow[ul] \arrow[u] \\
\bot \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Post-Regularization Checks

$$
\begin{tikzcd}
\eta_4 \colon \vdash q(R, S) & & \eta_1 \colon p(U, V) \vdash q(f(a, V), U) \\
\eta_6 \colon \vdash p(c, d) & & \eta_5 \colon p(U, V) \vdash q(f(a, Z), U) \arrow[ull] \arrow[u] \\
\eta_8 \colon q(f(a, e), c) \vdash & & \eta_7 \colon \vdash q(f(a, Z), c) \arrow[ull] \arrow[u] \\
 & \perp \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Post-Regularization Checks

$$
\begin{tikzcd}
\cancel{\eta_4\colon \vdash q(R,S)} & & \eta_1\colon p(U,V) \vdash q(f(a,V),U) \\
\eta_6\colon \vdash p(c,d) & & \eta_5\colon p(U,V) \vdash q(f(a,Z),U) \arrow[ull] \arrow[u] \\
\eta_8\colon q(f(a,e),c) \vdash & & \eta_7\colon \vdash q(f(a,Z),c) \arrow[ull] \arrow[u] \\
 & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Post-Regularization Checks

$$
\begin{tikzcd}
\eta_6\colon \vdash p(c,d) & & {\color{blue}\eta_1\colon p(U,V) \vdash q(f(a,V),U)} \\
\eta_8\colon q(f(a,e),c) \vdash & & \eta_7\colon \vdash q(f(a,Z),c) \arrow[ull] \arrow[u] \\
& \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Post-Regularization Checks

$$
\begin{tikzcd}
\eta_6\colon \vdash p(c,d) & & \eta_1\colon p(U,V) \vdash q(f(a,V),U) \\
\eta_8\colon q(f(a,e),c) \vdash & & \eta_7\colon \vdash q(f(a,\color{blue}{d}),c) \arrow[ull] \arrow[u] \\
& \cancel{\bot} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Regularization Unifiability

Definition

Let η be a node with safe literals $\mathcal{S}(\eta) = \phi$ that is marked for regularization with parents η_1 and η_2, where η_2 is marked as a `deletedNode` in a proof ψ. η is said to satisfy the *regularization unifiability property* in ψ if there exists a substitution σ such that $\eta_1\sigma \subseteq \phi$.

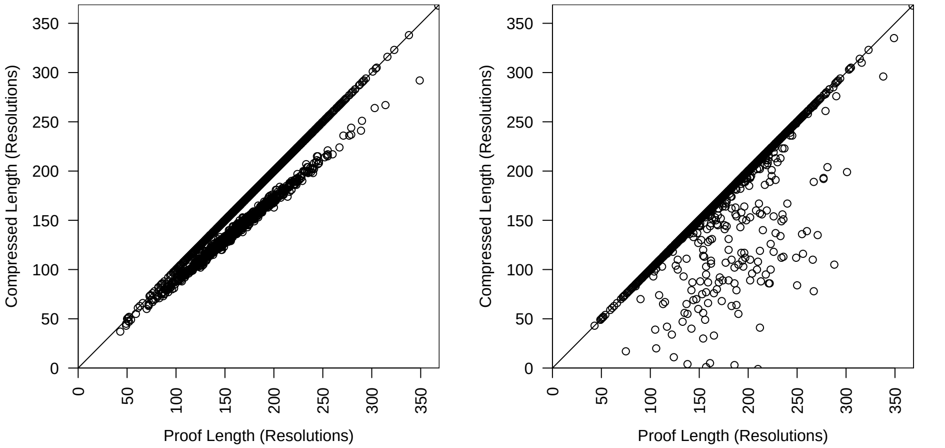
First-Order RPI

- Traverse bottom up, collect safe literals (apply unifiers to pivots), check pre-regularization property
- Traverse top-down, check regularization property

Experiment Setup

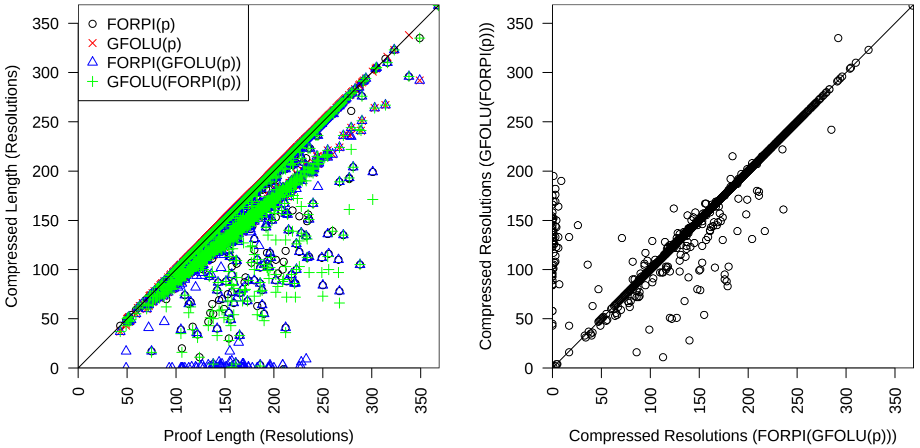
- Greedy First-Order Lower Units, Recycle Pivots With Intersection implemented as part of Skeptik (in Scala)
- $>$ 2400 randomly generated resolution proofs
- minutes to generate, seconds to compress

Results



Compressed Length (Resolutions)
Proof Length (Resolutions)
0
50
100
150
200
250
300
350
Compressed Length (Resolutions)
Proof Length (Resolutions)
0
50
100
150
200
250
300
350

Results



FORPI(p)
GFOLU(p)
FORPI(GFOLU(p))
GFOLU(FORPI(p))
Compressed Length (Resolutions)
Proof Length (Resolutions)
0
50
100
150
200
250
300
350
Compressed Resolutions (GFOLU(FORPI(p)))
Compressed Resolutions (FORPI(GFOLU(p)))

Results I

Percent of proofs compressed:

- LU(p): 36%
- RPI(p): 9%
- RPI(LU(p)): 43%
- LU(RPI(p)): 42%

Results II

Successful cumulative compression ratio:

- LU(p): 0.95
- RPI(p): 0.72
- RPI(LU(p)): 0.85
- LU(RPI(p)): 0.89

Conclusion

- Two simple, quick algorithms lifted from propositional to first-order logic for proof compression
  - LowerUnits compresses more often
  - RPI compresses more
- Future work:
  - Explore other proof compression algorithms?
  - Explore ways of dealing with the post-deletion property quickly

Thank you for your attention.
Any questions?

- Source code: `https://github.com/jgorzny/Skeptik`
- Data: `https://cs.uwaterloo.ca/~jgorzny/data/`

References I

- Omer Bar-Ilan, Oded Fuhrmann, Shlomo Hoory, Ohad Shacham, and Ofer Strichman, *Linear-time reductions of resolution proofs*, Haifa Verification Conference, Springer, 2008, pp. 114–128.
- Pascal Fontaine, Stephan Merz, and Bruno Woltzenlogel Paleo, *Compression of propositional resolution proofs via partial regularization*, International Conference on Automated Deduction, Springer, 2011, pp. 237–251.
- Marijn J. H. Heule, Oliver Kullmann, and Victor W. Marek, *Solving and verifying the boolean pythagorean triples problem via cube-and-conquer*, CoRR **abs/1605.00723** (2016).
- Rüdger Thiele, *Hilbert's twenty-fourth problem*, The American mathematical monthly **110** (2003), no. 1, 1–24.
- Gregory Tseitin, *On the complexity of proofs in propositional logics*, Seminars in Mathematics, vol. 8, 1970, pp. 466–483.

To-do

$$
\dfrac{\dfrac{\dfrac{\dfrac{\eta_8\colon p(X), q(X), r(X) \vdash \qquad \eta_7\colon \vdash p(Y)}{\eta_6\colon q(Y), r(Y) \vdash} \qquad \eta_5\colon p(Z) \vdash q(Z)}{\eta_4\colon p(Z), r(Z) \vdash} \qquad \eta_3\colon \vdash r(W)}{\eta_2\colon p(W) \vdash} \qquad \eta_1\colon \vdash p(U)}{\psi\colon \bot}
$$