Towards the Compression of First-Order Resolution Proofs by Lowering Unit Clauses

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Our Goal

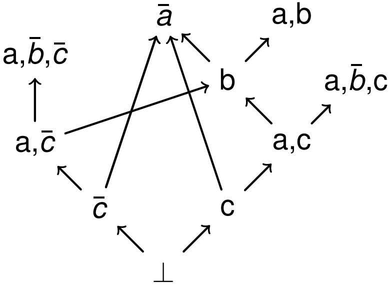
Lifting propositional proof compression algorithms to first-order logic.

This work: LowerUnits

Proof Compression Motivation

- The best, most efficient provers, do not generate the best, least redundant proofs.
- Many compression algorithms for propositional proofs; few for first-order proofs.

A Propositional Proof



LowerUnits

Definition (Unit)

A unit clause is a subproof with a conclusion clause (final clause) having exactly 1 literal

Theorem

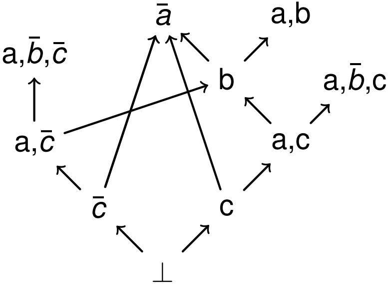
A unit clause can always be lowered

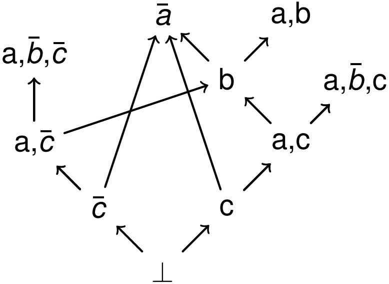
Compression is achieved by delaying resolution with unit clause subproofs.

Two Traversals

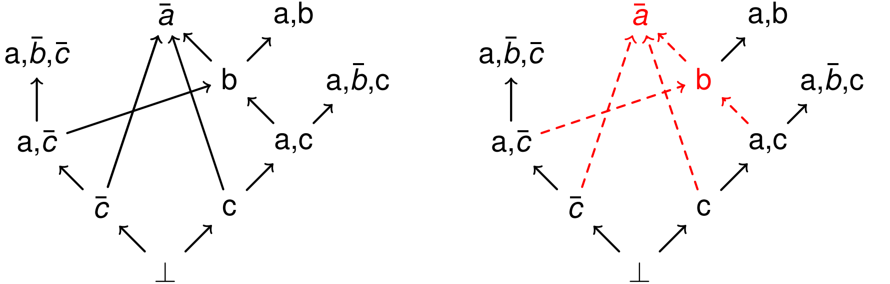
- $\uparrow$ Collect units with more than one resolvent
- $\downarrow$ Delete units and reintroduce them at the bottom of the proof

Propositional Example

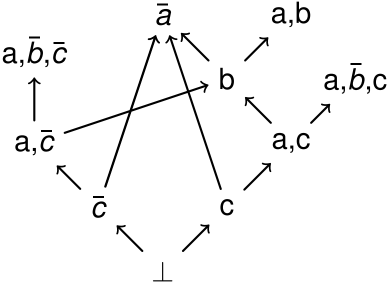


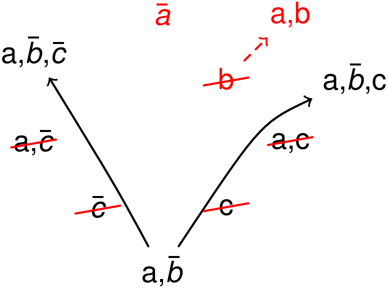


Propositional Example



Propositional Example





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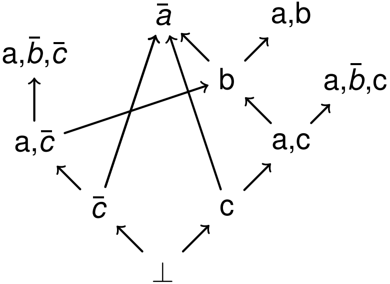

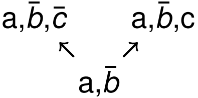

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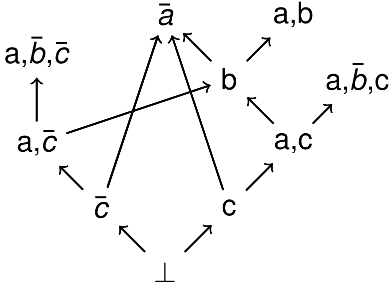

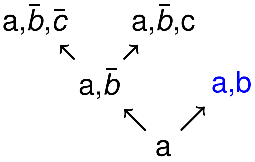

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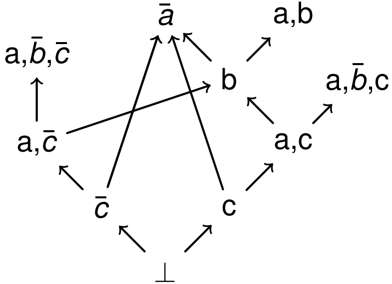

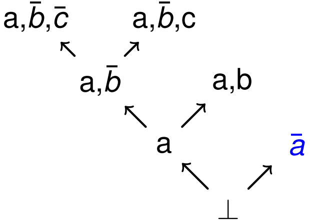

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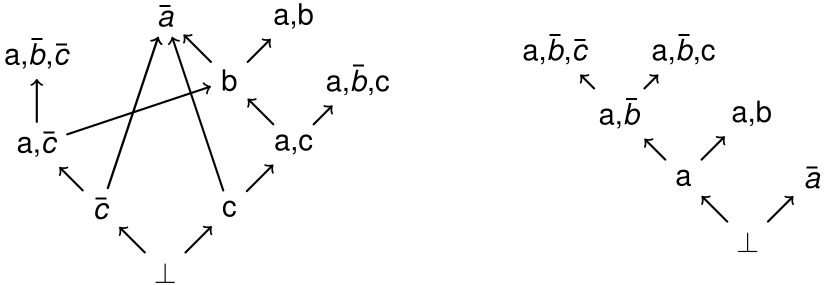

First-Order Change: Helpful Contractions

$$
\begin{tikzcd}
\eta_2\colon \vdash p(Y) & & \eta_1\colon p(W) \vdash q(Z) & \\
& \eta_3\colon \vdash q(Z) \arrow[ul] \arrow[ur] & & \eta_4\colon p(X), q(Z) \vdash \\
& & \eta_5\colon p(X) \vdash \arrow[ul] \arrow[ur] & \\
& \bot \arrow[ur] \arrow[uuul, bend left] & &
\end{tikzcd}
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First-Order Change: Helpful Contractions

$$
\begin{tikzcd}
{\color{red}\eta_2\colon \vdash p(Y)} & & \eta_1\colon p(W) \vdash q(Z) & \\
& \eta_3\colon \vdash q(Z) \arrow[ul] \arrow[ur] & & \eta_4\colon p(X), q(Z) \vdash \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
\eta'_1\colon p(W) \vdash q(Z) & & \eta'_4\colon p(X), q(Z) \vdash \\
& \eta'_5 : p(X), p(Y) \vdash \arrow[ul] \arrow[ur] &
\end{tikzcd}
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& \bot \arrow[ur] \arrow[uuul] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\eta_1'\colon p(W) \vdash q(Z) & & \eta_4'\colon p(X), q(Z) \vdash \\
& \eta_5'\colon p(X), p(Y) \vdash \arrow[ul] \arrow[ur] & \\
\lfloor \eta_5' \rfloor\colon p(U) \vdash \arrow[ur] & &
\end{tikzcd}
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& \eta'_5 : p(X), p(Y) \vdash \arrow[ul] \arrow[ur] & \\
\lfloor \eta'_5 \rfloor\colon p(U) \vdash \arrow[ur] & & \eta'_2\colon \vdash p(Y) \\
& \bot \arrow[ul] \arrow[ur] &
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$$

First-Order Challenge: Pre-Deletion Check

$$
\begin{tikzcd}
\eta_5 \colon q(Y) \vdash p(a) & & \eta_4 \colon p(X) \vdash & & \eta_3 \colon \vdash p(b), q(Y) \\
& \eta_1 \colon q(Y) \vdash \arrow[ul] \arrow[ur] & & \eta_2 \colon \vdash q(Y) \arrow[ul] \arrow[ur] & \\
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& & \cancel{\lfloor \eta \rfloor} \arrow[u] & &
\end{tikzcd}$$

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& & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Definition (Pre-Deletion Property)

η unit, $l \in \eta$, such that l is resolved with literals $l_1, \ldots, l_n$ in a proof ψ. η satisfies the *pre-deletion unifiability* property in ψ if $l_1, \ldots, l_n$ and $\bar{l}$ are unifiable.

First-Order Challenge: Post-Deletion Check

$$\begin{tikzcd}
 & \eta_1\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash & & \eta_2\colon \vdash p(U, V) \\
\eta_4\colon \vdash r(W) & & \eta_3\colon r(V), p(U, q(V, b)) \vdash \arrow[ul] \arrow[ur] & \\
 & \eta_5\colon p(U, q(W, b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur, bend right] &
\end{tikzcd}$$

First-Order Challenge: Post-Deletion Check

η_1: $r(Y), p(X, q(Y, b)), p(X, Y) \vdash$

η_2: $\vdash p(U, V)$

η_4: $\vdash r(W)$

η_3: $r(V), p(U, q(V, b)) \vdash$

η_5: $p(U, q(W, b)) \vdash$

$\perp$

First-Order Challenge: Post-Deletion Check

$$\eta_1\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash \qquad \qquad \color{red}{\eta_2\colon \vdash p(U, V)}$$

$$\eta_4\colon \vdash r(W) \qquad \qquad \eta_3\colon r(V), p(U, q(V, b)) \vdash$$

$$\eta_5\colon p(U, q(W, b)) \vdash$$

$$\bot$$

$$\eta_4'\colon \vdash r(W) \qquad \qquad \eta_1'\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash$$

$$\eta_5'\colon p(X, q(W, b)), p(X, W) \vdash$$

First-Order Challenge: Post-Deletion Check

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 & \eta_5\colon p(U, q(W, b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\eta'_4\colon \vdash r(W) & & \eta'_1\colon r(Y), p(X, q(Y, b)), p(X, Y) \vdash \\
 & \eta'_5\colon p(X, q(W, b)), p(X, W) \vdash \arrow[ul] \arrow[ur] & \\
 & \lfloor \cancel{\eta'_5} \rfloor \arrow[u] &
\end{tikzcd}$$

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 & \eta_1\colon r(Y), p(X, q(Y,b)), p(X,Y) \vdash & & \color{red}{\eta_2\colon \vdash p(U,V)} \\
\eta_4\colon \vdash r(W) & & \eta_3\colon r(V), p(U, q(V,b)) \vdash \arrow[ul] \arrow[ur] & \\
 & \eta_5\colon p(U, q(W,b)) \vdash \arrow[ul] \arrow[ur] & & \\
 & & \bot \arrow[ul] \arrow[uuur, bend right] &
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Definition (Post-Deletion Property)

η unit, $l \in \eta$, such that l is resolved with literals $l_1, \ldots, l_n$ in a proof ψ. η satisfies the *post-deletion unifiability* property in ψ if $l_1^{\dagger\downarrow}, \ldots, l_n^{\dagger\downarrow}$ and $\overline{l^\dagger}$ are unifiable, where $l^\dagger$ is the literal in $\psi' = \psi \setminus \{\eta\}$ corresponding to l in ψ, and $l^{\dagger\downarrow}$ is the descendant of $l^\dagger$ in the roof of ψ'.

First-Order Lower Units Challenges

- Deletion changes literals
- Unit collection depends on whether contraction is possible after propagation down the proof

Deletion of units require knowledge of proof after deletion, and deletion depends on what will be lowered.

- $O(n^2)$ solution to have full knowledge
- Difficult bookkeeping required for implementation

Greedy First-Order Lower Units - A Quicker Alternative

- Ignore post-deletion satisfaction
- Focus on pre-deletion satisfaction
- Greedy contraction

Faster run-time (linear; one traversal)
Easier to implement

- Doesn't always compress (returns original proof sometimes)

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First-Order Example

$\eta_2\colon p(W) \vdash q(X), r(Y), t(Z)$ $\qquad$ $\eta_1\colon \vdash p(a)$

$\eta_4\colon r(X), p(V) \vdash q(Y), t(Z)$ $\qquad$ $\eta_3\colon \vdash q(X), r(Y), t(Z)$

$\eta_5\colon p(V) \vdash q(X), t(Z)$

$\eta_7\colon q(X), p(a) \vdash t(Z)$ $\qquad$ $\eta_6\colon \vdash q(X), t(Z)$

$\eta_9\colon t(Z) \vdash$ $\qquad$ $\eta_8\colon p(a) \vdash t(Z)$

$\eta_{10}\colon p(a) \vdash$

$\perp$

First-Order Example

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 & & \eta_2\colon p(W) \vdash q(X), r(Y), t(Z) & & \eta_1\colon \vdash p(a) \\
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 & & \eta_5\colon p(V) \vdash q(X), t(Z) \arrow[ul] \arrow[ur] & & \\
 & \eta_7\colon q(X), p(a) \vdash t(Z) & & \eta_6\colon \vdash q(X), t(Z) \arrow[ul] \arrow[uuur] & \\
 \eta_9\colon t(Z) \vdash & & \eta_8\colon p(a) \vdash t(Z) \arrow[ul] \arrow[ur] & & \\
 & \eta_{10}\colon p(a) \vdash \arrow[ul] \arrow[ur] & & & \\
 & & \bot \arrow[ul] \arrow[uuuuuurr] & &
\end{tikzcd}
$$

$$\sigma = \{W \to a, V \to a\}$$

First-Order Example

$\eta_2\colon p(W) \vdash q(X), r(Y), t(Z)$ $\qquad$ $\color{red}{\eta_1\colon \vdash p(a)}$

$\color{blue}{\{W \to a\}}$

$\eta_4\colon r(X), p(V) \vdash q(Y), t(Z)$ $\qquad$ $\eta_3\colon \vdash q(X), r(Y), t(Z)$

$\eta_5\colon p(V) \vdash q(X), t(Z)$

$\eta_7\colon q(X), p(a) \vdash t(Z)$ $\qquad$ $\eta_6\colon \vdash q(X), t(Z)$

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\begin{tikzcd}[column sep=small]
 & & & & & & \eta_1\colon\ \vdash p(a) \\
 & \eta_4\colon r(X), p(V) \vdash q(Y), t(Z) & & \eta_3'\colon p(a) \vdash q(X), r(Y), t(Z) & & & \\
 & & \eta_5'\colon p(V), p(a) \vdash q(X), t(Z) \arrow[ul] \arrow[ur] & & & & \\
 & \eta_7\colon q(X), p(a) \vdash t(Z) & & \eta_6\colon\ \vdash q(X), t(Z) \arrow[ul] \arrow[uuurrr, bend right] & & & \\
 \eta_9\colon t(Z) \vdash & & \eta_8\colon p(a) \vdash t(Z) \arrow[ul] \arrow[ur] & & & & \\
 & \eta_{10}\colon p(a) \vdash \arrow[ul] \arrow[ur] & & & & & \\
 & & \bot \arrow[ul] \arrow[uuuuuurrrr, bend right] & & & &
\end{tikzcd}
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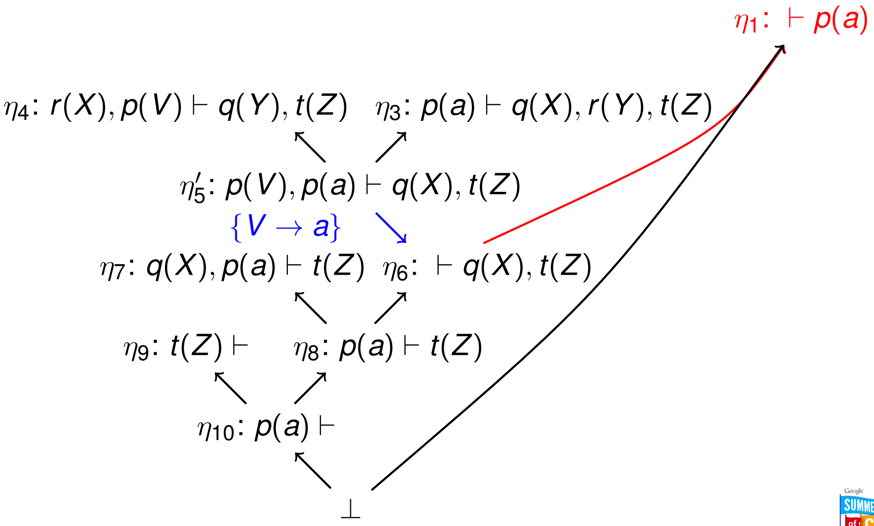

First-Order Example

$\eta_1\colon\ \vdash p(a)$

$\eta_4\colon r(X), p(V) \vdash q(Y), t(Z) \qquad \eta_3\colon p(a) \vdash q(X), r(Y), t(Z)$

$\eta_6'\colon p(a), p(a) \vdash q(X), t(Z)$

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$\eta_{10}\colon p(a) \vdash$

$\bot$

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$\emptyset$

$\perp$

First-Order Example

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$\eta'_6\colon p(a), p(a) \vdash q(X), t(Z)$

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\eta_9\colon t(Z) \vdash & \eta_8\colon p(a) \vdash t(Z) \arrow[ul] \arrow[ur] & & \\
 & \eta_{10}\colon p(a) \vdash \arrow[ul] \arrow[u] & & \\
 & & \bot \arrow[ul] \arrow[uuuuuur, bend right] &
\end{tikzcd}
$$

Experiment Setup

- Simple First-Order Lower units implemented as part of Skeptik (in Scala)
- 308 real first-order proofs generated by SPASS from problems from TPTP Problem Library
  - 2280 initial problems (1032 known unsatisfiable)
  - SPASS asked to use only resolution and contraction rules
  - 300s timeout
- proofs *generated* on cluster at the University of Victoria
- proofs *compressed* on *this laptop*

Time to generate proofs: $\approx$ 40 minutes
Time to compress proofs: $\approx$ 5 seconds

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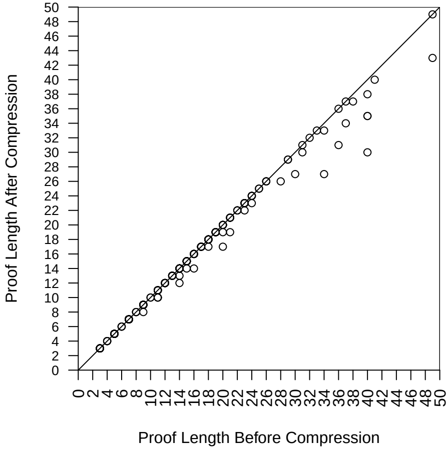
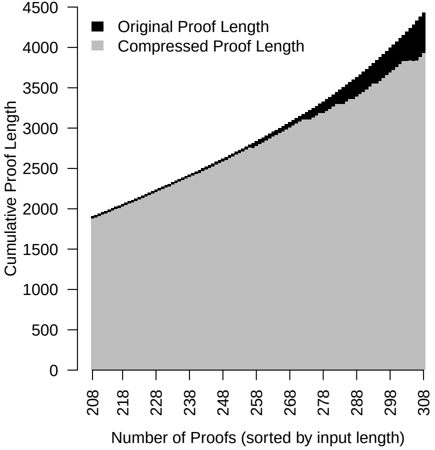
Results



Proof Length After Compression
Proof Length Before Compression



Original Proof Length
Compressed Proof Length
Cumulative Proof Length
Number of Proofs (sorted by input length)

Results

Higher compression in longer proofs:
13/18 proofs with length $\geq$ 30 nodes successfully compressed.

Total compression ratio 11.3%: 4429 vs. 3929 nodes.
18.4% for 100 longest proofs.

Only 14/308 proofs failed to satisfy the post-deletion unifiability property

Conclusion

- Simple First-Order Lower Units is a quick algorithm for first-order proof compression
- Future work:
  - Explore other proof compression algorithms, e.g. Recycle Pivots with Intersection
  - Explore ways of dealing with the post-deletion property quickly

Thank you for your attention.
Any questions?

- Source code: `https://github.com/jgorzny/Skeptik`
- Data: `http://www.math.uvic.ca/~jgorzny/data/`