# Worst-case analyses for first-order optimization methods

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# 1 Introduction

This document provides a series of exercises for getting familiar with "performance estimation problems" and the use of semidefinite programming for analyzing the worst-case behaviors of first-order optimization methods. An informal introduction can be found in this blog post.

Notation and necessary background material is provided in Section 3.

Note that worst-case analysis is a confortable tool... but it might not always be representative of reality (but when it is... it is nice because confortable!)

# 2 Exercises

**Exercise 1 (Linear convergence of the gradient method)** *For this exercise, consider the problem of "black-box" minimization of a smooth strongly convex function:*

$$f_\star \triangleq \min_{x\in\mathbb{R}^d} f(x), \tag{1}$$

*where $f$ is $L$-smooth and $\mu$-strongly convex (see Definition 2), and where $x_\star \triangleq \operatorname{argmin}_x f(x)$ and $f_\star \triangleq f(x_\star)$ its optimal value. For minimizing (1) we use gradient descent with a pre-determined sequence of step sizes $\{\gamma_k\}_k$; that is, we iterate $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$. The goal of this exercise is to compute $\tau(\mu, L, \gamma_k)$, a.k.a. a convergence rate, the smallest value such that the inequality*

$$\|x_{k+1} - x_\star\|^2 \leqslant \tau(\mu, L, \gamma_k)\|x_k - x_\star\|^2$$

*is valid for any $d \in \mathbb{N}$, for any $L$-smooth $\mu$-strongly convex function $f$ (notation $f \in \mathcal{F}_{\mu,L}$) and for all $x_k, x_{k+1} \in \mathbb{R}^d$ such that $x_{k+1} = x_k - \frac{\gamma_k}{L}\nabla f(x_k)$, and $x_\star = \operatorname{argmin}_x f(x)$.*

1. *Show that*

$$\begin{aligned}\tau(\mu, L, \gamma_k) = \sup_{\substack{d,f\\ x_k, x_{k+1}, x_\star}} \;& \frac{\|x_{k+1} - x_\star\|^2}{\|x_k - x_\star\|^2}\\ \text{s.t. } & f \in \mathcal{F}_{\mu,L}\\ & x_{k+1} = x_k - \gamma_k \nabla f(x_k)\\ & \nabla f(x_\star) = 0,\end{aligned}$$

   *where $f$, $x_k$, $x_{k+1}$, $x_\star$, and $d$ are the variables and $\mu$, $L$, $\gamma$ are parameters.*

   *Note that we will (sometimes abusively) use* max *instead of* sup *in the sequel as the optimum is usually attained for such problems (for this exercise, this is actually easy to show as the optimization problem is over a compact set).*

2. *Show that*

$$\begin{aligned}\tau(\mu, L, \gamma_k) = \max_{\substack{x_k, x_{k+1}, x_\star\\ g_k, g_\star\\ f_k, f_\star}} \;& \frac{\|x_{k+1} - x_\star\|^2}{\|x_k - x_\star\|^2}\\ \text{s.t. } & \exists f \in \mathcal{F}_{\mu,L} \text{ such that } \begin{cases} f_i = f(x_i) & i = k, \star\\ g_i = f'(x_i) & i = k, \star\end{cases}\\ & x_{k+1} = x_k - \gamma g_k\\ & g_\star = 0.\end{aligned}$$

3. *Using Theorem 2, show that*

$$\tau(\mu, L, \gamma_k) = \max_{\substack{x_k, x_{k+1}, x_\star \\ g_k, g_\star \\ f_k, f_\star}} \quad \frac{\|x_{k+1} - x_\star\|^2}{\|x_k - x_\star\|^2}$$

$$\begin{aligned} \text{s.t. } & f_\star \geqslant f_k + \langle g_k, x_\star - x_k\rangle + \tfrac{1}{2L}\|g_\star - g_k\|^2 + \tfrac{\mu}{2(1-\mu/L)}\left\|x_\star - x_k - \tfrac{1}{L}(g_\star - g_k)\right\|^2 \\ & f_k \geqslant f_\star + \langle g_\star, x_k - x_\star\rangle + \tfrac{1}{2L}\|g_k - g_\star\|^2 + \tfrac{\mu}{2(1-\mu/L)}\left\|x_k - x_\star - \tfrac{1}{L}(g_k - g_\star)\right\|^2 \\ & x_{k+1} = x_k - \gamma g_k \\ & g_\star = 0. \end{aligned}$$

4. *Show that*

$$\tau(\mu, L, \gamma_k) = \max_{\substack{x_k, x_{k+1}, x_\star \\ g_k, g_\star \\ f_k, f_\star}} \quad \|x_{k+1} - x_\star\|^2$$

$$\begin{aligned} \text{s.t. } & f_\star \geqslant f_k + \langle g_k, x_\star - x_k\rangle + \tfrac{1}{2L}\|g_\star - g_k\|^2 + \tfrac{\mu}{2(1-\mu/L)}\left\|x_\star - x_k - \tfrac{1}{L}(g_\star - g_k)\right\|^2 \\ & f_k \geqslant f_\star + \langle g_\star, x_k - x_\star\rangle + \tfrac{1}{2L}\|g_k - g_\star\|^2 + \tfrac{\mu}{2(1-\mu/L)}\left\|x_k - x_\star - \tfrac{1}{L}(g_k - g_\star)\right\|^2 \\ & \|x_k - x_\star\|^2 = 1 \\ & x_{k+1} = x_k - \gamma g_k \\ & g_\star = 0. \end{aligned}$$

5. *Define $G$ and $F$*

$$G \triangleq \begin{bmatrix} \|x_k - x_\star\|^2 & \langle g_k, x_k - x_\star\rangle \\ \langle g_k, x_k - x_\star\rangle & \|g_k\|^2 \end{bmatrix}, \quad F \triangleq f_k - f_\star,$$

*(note that $G = [x_k - x_\star \quad g_k]^\top [x_k - x_\star \quad g_k] \succcurlyeq 0$. Show that previous problem can be reformulated as a $2 \times 2$ SDP*

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \max_{G, F} \quad & G_{1,1} + \gamma_k^2 G_{2,2} - 2\gamma_k G_{1,2} \\ \text{s.t.} \quad & F + \tfrac{L\mu}{2(L-\mu)} G_{1,1} + \tfrac{1}{2(L-\mu)} G_{2,2} - \tfrac{L}{L-\mu} G_{1,2} \leqslant 0 \\ & -F + \tfrac{L\mu}{2(L-\mu)} G_{1,1} + \tfrac{1}{2(L-\mu)} G_{2,2} - \tfrac{\mu}{L-\mu} G_{1,2} \leqslant 0 \\ & G_{1,1} = 1 \\ & G \succcurlyeq 0, \end{aligned}$$

6. *Define $h_k \triangleq \gamma_k L$ and $\kappa = L/\mu$. Show that $\tau(\mu, L, \gamma_k) = \tau(1/\kappa, 1, h_k)$ (in other words: we can study the case $L = 1$ only and deduce the dependence of $\tau$ on $L$ afterwards).*

7. *Complete the PEPit code (alternative in Matlab: PESTO code) for computing $\tau(\mu, L, \gamma_k)$ and compute its value for a few numerical values of $\mu$ and $\gamma_k$.*

8. *Using Lagrangian duality with the following primal-dual pairing ($\tau, \lambda_1, \lambda_2$ are dual variables):*

$$\begin{aligned} & F + \tfrac{L\mu}{2(L-\mu)} G_{1,1} + \tfrac{1}{2(L-\mu)} G_{2,2} - \tfrac{L}{L-\mu} G_{1,2} \leqslant 0 && : \lambda_1 \\ & -F + \tfrac{L\mu}{2(L-\mu)} G_{1,1} + \tfrac{1}{2(L-\mu)} G_{2,2} - \tfrac{\mu}{L-\mu} G_{1,2} \leqslant 0 && : \lambda_2 \\ & G_{1,1} = 1 && : \tau \end{aligned}$$

*one can show that*

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \min_{\tau, \lambda_1, \lambda_2 \geqslant 0} \; & \tau \\ \text{s.t. } & S = \begin{bmatrix} \tau - 1 + \frac{\lambda_1 L\mu}{L-\mu} & \gamma_k - \frac{\lambda_1(\mu+L)}{2(L-\mu)} \\ \gamma_k - \frac{\lambda_1(\mu+L)}{2(L-\mu)} & \frac{\lambda_1}{L-\mu} - \gamma_k^2 \end{bmatrix} \succcurlyeq 0 \\ & 0 = \lambda_1 - \lambda_2. \end{aligned} \tag{2}$$

*Note that equility holds due to strong duality (for going further: prove strong duality using a Slater condition).*

*Show that any feasible point* $(\tau, \lambda_1, \lambda_2)$ *to* (2) *corresponds to an upper bound on* $\tau(\mu, L, \gamma_k)$ *(i.e.,* $\tau(\mu, L, \gamma_k) \leqslant \tau$.

9. *Is there a simple closed form expression for* $\tau(\mu, L, \gamma_k)$*? (hint: can we solve* (2) *in closed form?) Does it match the numerical values obtained using the previous codes for computing* $\tau(\mu, L, \gamma_k)$ *numerically?*

10. *How can we adapt the (primal) SDP formulation for computing the smallest possible* $\tau$ *such that the inequality*
$$\|\nabla f(x_{k+1}\|^2 \leqslant \tau \|\nabla f(x_k)\|^2$$
*is valid for any* $d \in \mathbb{N}$*, for any* $L$*-smooth* $\mu$*-strongly convex function* $f$ *(notation* $f \in \mathcal{F}_{\mu,L}$*) and for all* $x_k, x_{k+1} \in \mathbb{R}^d$ *such that* $x_{k+1} = x_k - \frac{\gamma_k}{L}\nabla f(x_k)$*? Modify your previous code for computing such a bound. Can you guess a closed-form expression for it?*

    *For going further: prove that feasible points to the (dual) problem*
$$\ldots$$
    *correspond to upper bounds on* $\tau$*. Are there simple closed-form solution for this problem?*

11. *How can we adapt the (primal) SDP formulation for computing the smallest possible* $\tau$ *such that the inequality*
$$f(x_{k+1}) - f(x_\star) \leqslant \tau(f(x_k) - f(x_\star))$$
*is valid for any* $d \in \mathbb{N}$*, for any* $L$*-smooth* $\mu$*-strongly convex function* $f$ *(notation* $f \in \mathcal{F}_{\mu,L}$*) and for all* $x_k, x_{k+1} \in \mathbb{R}^d$ *such that* $x_{k+1} = x_k - \frac{\gamma_k}{L}\nabla f(x_k)$*? Modify your previous code for computing such a bound. Can you guess a closed-form expression for it?*

    *For going further: prove that feasible points to the (dual) problem*
$$\ldots$$
    *correspond to upper bounds on* $\tau$*. Is there a simple closed-form solution for this problem?*

**Exercise 2 (Sublinear convergence of gradient descent and acceleration)** *Show that the smallest* $\tau$ *such that the inequality*
$$\ldots$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *numerical trials (code: XXX). Ex: modify the code to compute worst gradient norm, and worst best gradient norm among iterates*

**Exercise 3 (Subgradient method)** *Show that the smallest* $\tau$ *such that the inequality*
$$\ldots$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *numerical trials (code: XXX). Ex: modify the code to compute worst gradient norm, and worst best gradient norm among iterates*

**Exercise 4 (Acceleration and Lyapunov analyses)** *Show that the smallest* $\tau$ *such that the inequality*
$$\ldots$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *numerical trials*

**Exercise 5 (Fixed-point iterations)** *Show that the smallest $\tau$ such that the inequality*

$$...$$

1. *two methods: Halpern and Kras...*
2. *show that it can be formulated as ...*
3. *show that the previous problem can be framed using a discrete version...*
4. *show that it is equivalent to the SDP XXXX*
5. *using duality show that ... (dual SDP)*
6. *numerical trials*

**Exercise 6 (Stochastic gradient descent)** *Show that the smallest $\tau$ such that the inequality*

$$...$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *using duality show that ... (dual SDP)*
5. *numerical trials*

**Exercise 7 (Proximal point method)** *Show that the smallest $\tau$ such that the inequality*

$$...$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

**Exercise 8 (Proximal gradient method)** *Show that the smallest $\tau$ such that the inequality*

$$...$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

**Exercise 9 (Douglas-Rachford splitting)** *Show that the smallest $\tau$ such that the inequality*

$$...$$

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

**Exercise 10 (Frank-Wolfe)** *Show that the smallest $\tau$ such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

**Exercise 11 (Alternate projections & Dykstra)** *Show that the smallest $\tau$ such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

**Exercise 12 (Mirror descent)** *Show that the smallest $\tau$ such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

# 3 Background material and useful facts

## 3.1 Standard definitions

smoothness, strong convexity...

**Definition 1** *cpp*

**Definition 2** *sm str cvx (notation...)*

**Definition 3** *lip cvx*

## 3.2 Interpolation/extension theorems

This section gathers useful elements allowing to answer certain questions ...

**Theorem 1** *interpolation 1... (ccp)*

**Theorem 2** *interpolation 2... (smooth str convex)*

**Theorem 3** *interpolation 3... (smooth nonconvex)*

**Theorem 4** *nonsmooth*

**Theorem 5** *indicator*