

# Worst-case analyses for first-order optimization methods

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## Foreword & Acknowledgements

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## 1 Introduction

This document provides a series of exercises for getting familiar with “performance estimation problems” and the use of semidefinite programming for analyzing the worst-case behaviors of first-order optimization methods. An informal introduction can be found in this [blog post](#).

Notation and necessary background material is provided in Section 4.

Note that worst-case analysis is a comfortable tool... but it might not always be representative of reality (but when it is... it is nice because comfortable!)

## 2 Getting familial with performance estimation problems

**Exercise 1 (Gradient method)** *For this exercise, consider the problem of “black-box” minimization of a smooth strongly convex function:*

$$f_\star \triangleq \min_{x \in \mathbb{R}^d} f(x), \quad (1)$$

where  $f$  is  $L$ -smooth and  $\mu$ -strongly convex (see Definition 2), and where  $x_\star \triangleq \operatorname{argmin}_x f(x)$  and  $f_\star \triangleq f(x_\star)$  its optimal value. For minimizing (1) we use gradient descent with a pre-determined sequence of step sizes  $\{\gamma_k\}_k$ ; that is, we iterate  $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$ . The goal of this exercise is to compute  $\tau(\mu, L, \gamma_k)$ , a.k.a. a convergence rate, the smallest value such that the inequality

$$\|x_{k+1} - x_\star\|^2 \leq \tau(\mu, L, \gamma_k) \|x_k - x_\star\|^2$$

is valid for any  $d \in \mathbb{N}$ , for any  $L$ -smooth  $\mu$ -strongly convex function  $f$  (notation  $f \in \mathcal{F}_{\mu, L}$ ) and for all  $x_k, x_{k+1} \in \mathbb{R}^d$  such that  $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$ , and  $x_\star = \operatorname{argmin}_x f(x)$ .

1. Show that

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \sup_{\substack{d, f \\ x_k, x_{k+1}, x_\star}} \frac{\|x_{k+1} - x_\star\|^2}{\|x_k - x_\star\|^2} \\ \text{s.t. } f \in \mathcal{F}_{\mu, L} \\ x_{k+1} = x_k - \gamma_k \nabla f(x_k) \\ \nabla f(x_\star) = 0, \end{aligned}$$

where  $f$ ,  $x_k$ ,  $x_{k+1}$ ,  $x_\star$ , and  $d$  are the variables and  $\mu$ ,  $L$ ,  $\gamma$  are parameters.

Note that we will (sometimes abusively) use  $\max$  instead of  $\sup$  in the sequel as the optimum is usually attained for such problems (for this exercise, this is actually easy to show as the optimization problem is over a compact set).

2. Show that

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \max_{\substack{x_k, x_{k+1}, x_\star \\ g_k, g_\star \\ f_k, f_\star}} \frac{\|x_{k+1} - x_\star\|^2}{\|x_k - x_\star\|^2} \\ \text{s.t. } \exists f \in \mathcal{F}_{\mu, L} \text{ such that } \begin{cases} f_i = f(x_i) & i = k, \star \\ g_i = f'(x_i) & i = k, \star \end{cases} \\ x_{k+1} = x_k - \gamma_k g_k \\ g_\star = 0. \end{aligned}$$

3. Using Theorem 2, show that

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \max_{\substack{x_k, x_{k+1}, x_\star \\ g_k, g_\star \\ f_k, f_\star}} & \frac{\|x_{k+1} - x_\star\|^2}{\|x_k - x_\star\|^2} \\ \text{s.t. } & f_\star \geq f_k + \langle g_k, x_\star - x_k \rangle + \frac{1}{2L} \|g_\star - g_k\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_\star - x_k - \frac{1}{L}(g_\star - g_k)\|^2 \\ & f_k \geq f_\star + \langle g_\star, x_k - x_\star \rangle + \frac{1}{2L} \|g_k - g_\star\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_k - x_\star - \frac{1}{L}(g_k - g_\star)\|^2 \\ & x_{k+1} = x_k - \gamma_k g_k \\ & g_\star = 0. \end{aligned}$$

4. Show that

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \max_{\substack{x_k, x_{k+1}, x_\star \\ g_k, g_\star \\ f_k, f_\star}} & \|x_{k+1} - x_\star\|^2 \\ \text{s.t. } & f_\star \geq f_k + \langle g_k, x_\star - x_k \rangle + \frac{1}{2L} \|g_\star - g_k\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_\star - x_k - \frac{1}{L}(g_\star - g_k)\|^2 \\ & f_k \geq f_\star + \langle g_\star, x_k - x_\star \rangle + \frac{1}{2L} \|g_k - g_\star\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_k - x_\star - \frac{1}{L}(g_k - g_\star)\|^2 \\ & \|x_k - x_\star\|^2 = 1 \\ & x_{k+1} = x_k - \gamma_k g_k \\ & g_\star = 0. \end{aligned}$$

5. Define  $G$  and  $F$

$$G \triangleq \begin{bmatrix} \|x_k - x_\star\|^2 & \langle g_k, x_k - x_\star \rangle \\ \langle g_k, x_k - x_\star \rangle & \|g_k\|^2 \end{bmatrix}, \quad F \triangleq f_k - f_\star,$$

(note that  $G = [x_k - x_\star \quad g_k]^\top [x_k - x_\star \quad g_k] \succcurlyeq 0$ ). Show that  $\tau(\mu, L, \gamma_k)$  can be computed using the following  $2 \times 2$  semidefinite program (SDP):

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \max_{G, F} & G_{1,1} + \gamma_k^2 G_{2,2} - 2\gamma_k G_{1,2} \\ \text{s.t. } & F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0 \\ & -F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0 \\ & G_{1,1} = 1 \\ & G \succcurlyeq 0, \end{aligned} \tag{2}$$

6. Define  $h_k \triangleq \gamma_k L$  and  $\kappa = L/\mu$ . Show that  $\tau(\mu, L, \gamma_k) = \tau(1/\kappa, 1, h_k)$  (in other words: we can study the case  $L = 1$  only and deduce the dependence of  $\tau$  on  $L$  afterwards).

7. Complete the [PEPit code](#) (alternative in Matlab: [PESTO code](#)) for computing  $\tau(\mu, L, \gamma_k)$  and compute its value for a few numerical values of  $\mu$  and  $\gamma_k$ .

8. Using Lagrangian duality with the following primal-dual pairing ( $\tau, \lambda_1, \lambda_2$  are dual variables):

$$\begin{aligned} F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} &\leq 0 & : \lambda_1 \\ -F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} &\leq 0 & : \lambda_2 \\ G_{1,1} &= 1 & : \tau \end{aligned}$$

one can show that

$$\begin{aligned} \tau(\mu, L, \gamma_k) = \min_{\tau, \lambda_1, \lambda_2 \geq 0} & \tau \\ \text{s.t. } & S = \begin{bmatrix} \tau - 1 + \frac{\lambda_1 L \mu}{L-\mu} & \gamma_k - \frac{\lambda_1(\mu+L)}{2(L-\mu)} \\ \gamma_k - \frac{\lambda_1(\mu+L)}{2(L-\mu)} & \frac{\lambda_1}{L-\mu} - \gamma_k^2 \end{bmatrix} \succcurlyeq 0 \\ & 0 = \lambda_1 - \lambda_2. \end{aligned} \tag{3}$$

---

Note that equility holds due to strong duality (for going further: obtain this dual formulation and prove strong duality using a Slater condition).

Show that any feasible point  $(\tau, \lambda_1, \lambda_2)$  to (3) corresponds to an upper bound on  $\tau(\mu, L, \gamma_k)$  (i.e.,  $\tau(\mu, L, \gamma_k) \leq \tau$ ).

9. Is there a simple closed-form expression for  $\tau(\mu, L, \gamma_k)$ ? (hint: can we solve (3) in closed-form?) Does it match the numerical values obtained using the previous codes for computing  $\tau(\mu, L, \gamma_k)$  numerically?

10. How can we adapt the SDP formulation (2) for computing the smallest possible  $\tau$  such that the inequality

$$\|\nabla f(x_{k+1})\|^2 \leq \tau \|\nabla f(x_k)\|^2$$

is valid for any  $d \in \mathbb{N}$ , for any  $L$ -smooth  $\mu$ -strongly convex function  $f$  (notation  $f \in \mathcal{F}_{\mu,L}$ ) and for all  $x_k, x_{k+1} \in \mathbb{R}^d$  such that  $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$ ? Modify your previous code for computing such a bound. Can you guess a closed-form expression for it?

For going further: prove that feasible points to the (dual) problem

...

correspond to upper bounds on  $\tau$ . Are there simple closed-form solution for this problem?

11. How can we adapt the SDP formulation (2) for computing the smallest possible  $\tau$  such that the inequality

$$f(x_{k+1}) - f(x_\star) \leq \tau(f(x_k) - f(x_\star))$$

is valid for any  $d \in \mathbb{N}$ , for any  $L$ -smooth  $\mu$ -strongly convex function  $f$  (notation  $f \in \mathcal{F}_{\mu,L}$ ) and for all  $x_k, x_{k+1} \in \mathbb{R}^d$  such that  $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$ ? Modify your previous code for computing such a bound. Can you guess a closed-form expression for it?

For going further: prove that feasible points to the (dual) problem

...

correspond to upper bounds on  $\tau$ . Is there a simple closed-form solution for this problem?

12. Can we use this formalism for computing worst-case guarantees for a few iterations simultaneously? That is, to compute  $\tau(\mu, L, \{\gamma_k\}_{k=0,\dots,N-1})$  the smallest value such that the inequality

$$\|x_N - x_\star\|^2 \leq \tau(\mu, L, \{\gamma_k\}_{k=0,\dots,N-1}) \|x_0 - x_\star\|^2$$

is valid for any  $d \in \mathbb{N}$ , for any  $L$ -smooth  $\mu$ -strongly convex function  $f$  (notation  $f \in \mathcal{F}_{\mu,L}$ ) and for all  $x_0, x_1, \dots, x_N \in \mathbb{R}^d$  such that  $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$  ( $k = 0, \dots, N-1$ ), and  $x_\star = \operatorname{argmin}_x f(x)$ .

13. What happens if  $\mu = 0$ ? Can you isolate the problem on a simple counter example? You can, for example, use this [PEPit code](#) (alternative in Matlab: [PESTO code](#)). Can you imagine a solution for avoiding such pathological behaviors in the analyses? What about studying guarantees of type

$$f(x_N) - f_\star \leq \tau(\mu, L, \{\gamma_k\}_{k=0,\dots,N-1}) \|x_0 - x_\star\|^2$$

instead? Modify your code for studying such worst-case bounds, and try it numerically for the choice  $\gamma_k = 1/L$ ,  $L = 1$  and  $\mu = 0$ . Can you guess the depece on  $N$ ?

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### 3 Further exercises

**Exercise 2 (Sublinear convergence of gradient descent and acceleration)** *Show that the smallest  $\tau$  such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *numerical trials (code: XXX). Ex: modify the code to compute worst gradient norm, and worst best gradient norm among iterates*

**Exercise 3 (Subgradient method)** *Show that the smallest  $\tau$  such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *numerical trials (code: XXX). Ex: modify the code to compute worst gradient norm, and worst best gradient norm among iterates*

**Exercise 4 (Acceleration and Lyapunov analyses)** *Show that the smallest  $\tau$  such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *numerical trials*

**Exercise 5 (Fixed-point iterations)** *Show that the smallest  $\tau$  such that the inequality*

...

1. *two methods: Halpern and Kras...*
2. *show that it can be formulated as ...*
3. *show that the previous problem can be framed using a discrete version...*
4. *show that it is equivalent to the SDP XXXX*
5. *using duality show that ... (dual SDP)*
6. *numerical trials*

**Exercise 6 (Stochastic gradient descent)** *Show that the smallest  $\tau$  such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *using duality show that ... (dual SDP)*

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5. numerical trials

**Exercise 7 (Proximal point method)** Show that the smallest  $\tau$  such that the inequality

...

1. show that it can be formulated as ...
2. show that the previous problem can be framed using a discrete version...
3. show that it is equivalent to the SDP XXXX
4. numerical trials

**Exercise 8 (Proximal gradient method)** Show that the smallest  $\tau$  such that the inequality

...

1. show that it can be formulated as ...
2. show that the previous problem can be framed using a discrete version...
3. show that it is equivalent to the SDP XXXX
4. numerical trials

**Exercise 9 (Douglas-Rachford splitting)** Show that the smallest  $\tau$  such that the inequality

...

1. show that it can be formulated as ...
2. show that the previous problem can be framed using a discrete version...
3. show that it is equivalent to the SDP XXXX
4. numerical trials

**Exercise 10 (Frank-Wolfe)** Show that the smallest  $\tau$  such that the inequality

...

1. show that it can be formulated as ...
2. show that the previous problem can be framed using a discrete version...
3. show that it is equivalent to the SDP XXXX
4. numerical trials

**Exercise 11 (Alternate projections & Dykstra)** Show that the smallest  $\tau$  such that the inequality

...

1. show that it can be formulated as ...
2. show that the previous problem can be framed using a discrete version...
3. show that it is equivalent to the SDP XXXX
4. numerical trials

---

**Exercise 12 (Mirror descent)** *Show that the smallest  $\tau$  such that the inequality*

...

1. *show that it can be formulated as ...*
2. *show that the previous problem can be framed using a discrete version...*
3. *show that it is equivalent to the SDP XXXX*
4. *numerical trials*

**Exercise 13 (Primal-dual methods)** *Monotone inclusion (modelling KKT system) and VI*

1. *DRS*
2. *PRS*
3. *PPM*
4. *EG*
5. *OG*
6. *...*

**Exercise 14 (Adaptive methods)** 1. *ELS*

2. *GFOM*
3. *Polyak steps*

## 4 Background material and useful facts

### 4.1 Standard definitions

smoothness, strong convexity...

**Definition 1** *cpp*

**Definition 2** *sm str cvx (notation...)*

**Definition 3** *lip cvx*

**Definition 4** *lip cvx*

### 4.2 Interpolation/extension theorems

This section gathers useful elements allowing to answer certain questions ...

**Theorem 1** *interpolation 1... (ccp)*

**Theorem 2** *interpolation 2... (smooth str convex)*

**Theorem 3** *interpolation 3... (smooth nonconvex)*

**Theorem 4** *nonsmooth*

**Theorem 5** *indicator*

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### 4.3 Other useful inequalities

### 4.4 SDP duality

++primal and dual SDP formulations

**Theorem 6** *slater*

### 4.5 Going further - readings