Calculus 1 Notes and Thoughts
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Contents

1 Limits

Limits are very important in Calculus, since we need very small quantities to effectively study change. Yeah, that's pretty vague, but hopefully it'll clear up in a few pages.

We study limits for many reasons. One of them is to study functions at points where they are not defined, but are "close" to being defined. For example, the function f given by $f(x) = \frac{x(x+1)}{x}$ is not defined at 0, but looks like the line $x + 1$ everywhere else.

1.1 Intuitive Limits

The important idea of a **limit** is to examine what the output of a function does as the input moves close to a specific value. Since the functions that we will be studying are from $\mathbb{R}$ to $\mathbb{R}$, and there are two ways that you can approach a number on the real line (from the left and right) we will talk about left- and right-sided limits.

Before getting into the actual definition, we'll start intuitively. If a function is "continuous"[1] and is defined at a, then as x moves closer to a, $f(x)$ moves closer to $f(a)$, so the limit of f as x approaches a is $f(a)$.

Let's get a little more general: suppose a function f is defined on open intervals on either side of a.

<table>
<tr>
<td>If f is defined on an interval to the left of a and the values of $f(x)$ approach L as x approaches a from the left, we say that the "left-sided limit of f as x approaches a is L" and we write $$\lim_{x \to a^-} f(x) = L.$$</td>
<td>If f is defined on an interval to the right of a and the values of $f(x)$ approach L as x approaches a from the right, we say that the "right-sided limit of f as x approaches a is L" and we write $$\lim_{x \to a^+} f(x) = L.$$</td>
</tr>
<tr>
<td colspan="2">If the right and left sided limits match, then we say that the "limit of f as x approaches a is L" and write $$\lim_{x \to a} f(x) = L.$$</td>
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Note: f need not be defined at a to evaluate the limit of f as x approaches a.

One can compute $f(x)$ for values of x that get closer and closer to either side of a. If the values approach L, then you have good reason to believe that L is the (right- and/or left-sided) limit.

x	$f(x)$
$a - 0.1$	$f(a - 0.1)$
$a - 0.01$	$f(a - 0.01)$
$a - 0.001$	$f(a - 0.001)$
$a - 0.0001$	$f(a - 0.0001)$
$a + 0.0001$	$f(a + 0.0001)$
$a + 0.001$	$f(a + 0.001)$
$a + 0.01$	$f(a + 0.01)$
$a + 0.1$	$f(a + 0.1)$

Such calculations, however, cannot prove that a function limits to a specific value.

[1] I put this word in quotes because I haven't defined it yet, but you should have an intuitive idea of what this means: being able to draw it without lifting your pencil. Just wait for a few pages.

1.2 Infinite Limits

We can extend the idea of limits outside of the real numbers to include positive and negative infinity in place of both a and L.

- If $f(x)$ grows (in the $\pm$ direction) without bound as x approaches a from the left, then we write

$$\lim_{x \to a^-} f(x) = \pm\infty.$$

  Similarly for x approaching a from the right.

- We denote the value (if such a value exists) that $f(x)$ approaches as x approaches $\pm\infty$ as

$$\lim_{x \to \pm\infty} f(x).$$

1.3 The $\epsilon - \delta$ Definition

You may be dissatisfied with the intuitive approach to limits, so we'll make the definition more rigorous. The main idea that needs to be captured by a formal definition is *arbitrary precision.* That is, we need a way to say formally that "$f(x)$ approaches L as x approaches a."

By controlling the input x, we must be able to make the distance $|f(x) - L|$ between the output $f(x)$ and L to be as small as we want ("arbitrarily small"). In other words, if $\epsilon > 0$ is any small positive number, we must be able to ensure (by controlling x) that $|f(x) - L| < \epsilon$. To control x, we can make the distance $|x - a|$ between x and a smaller than some positive number $\delta > 0$ (that may depend on ϵ).

Now we're ready for the real "ϵ–δ" definition of a limit: we say that L **is the limit of f as x approaches** a if

$$\text{for all } \epsilon > 0, \text{ there exists some } \delta > 0, \text{ such that } |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Read that last line a few times, because statements with multiple quantifiers can be tricky! It may be helpful to think about the definition of a limit as a game/conversation between two people, Alex and Blake. Alex is trying to claim that the limit of f as $x \to a$ is L, and Blake is doing their best job to contest it. Here is how their conversation might go:

A: I think the limit of $f(x) = 3x + 1$ as x approaches 2 is 7.

B: Well if you think so, then can you ensure that $|f(x) - 7| < \frac{1}{10}$?

A: Yes! If we take x such that $|x - 2| < 1/30$, then

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < 3(1/30) = 1/10.$$

In this example, $\epsilon = 1/10$ and $\delta = 1/30$. However, if we want to really prove that the limit is 7, we let $\epsilon > 0$ be arbitrary, and take $\delta = \epsilon/3$. Then for any x such that $|x - 2| < \delta$,

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < 3(\epsilon/3) = \epsilon.$$

For simple examples, it is pretty easy to work backwards and figure out what δ should be, but for more complicated functions f, it can be more difficult. Here is a Desmos example that shows dynamically how δ can depend on ϵ.

1.4 Limit Laws

So far we have used the $\epsilon - \delta$ definition to prove that a limit of a function is a certain value, we estimated limits using computation, and we might be able to eye-ball a limit of a "continuous" function, but we still need a more sophisticated way to figure out what the limit of a function is at a certain point. There are several *limit laws* that make computing limits easier. We also have two basic facts that should be obvious: for any $a, b \in \mathbb{R}$,

$$\lim_{x \to a} b = b \quad \text{and} \quad \lim_{x \to a} x = a.$$

Together with the following laws, you'll be able to evaluate the limits of many functions. Suppose $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, and let c be a constant. Then the following hold.

Sum law	$\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$
Difference law	$\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = L - M$
Constant multiple law	$\lim\limits_{x \to a} cf(x) = c \cdot \lim\limits_{x \to a} f(x) = cL$
Product law	$\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = L \cdot M$
Quotient law	$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{L}{M}$ for $M \neq 0$
Power law	$\lim\limits_{x \to a} (f(x))^n = \left(\lim\limits_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.
Root law	$\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

1.5 Limit "Tricks" and Key Examples

There are some other tricks that will help you evaluate limits. Try these things if you're not sure what else to do; they might be helpful if you end up in a situation where you are trying to divide $0/0$.

- Simplify. Perhaps the function is rational, and has common factor in its numerator and denominator. By "canceling" the term, the resulting function is not the same: the original function has a hole at the point where the factored term is 0. For example, $f(x) = \frac{x(x-1)}{(x-1)}$ has a hole at $x = 1$, but everywhere else is the line $y = x$.

- Multiply by the conjugate of the denominator or the numerator of a rational function (the conjugate of a binomial $a + b$ is $a - b$).

- $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1.$

- $\lim\limits_{x \to 0} \dfrac{\cos(x) - 1}{x} = 0.$

- $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e.$

2 Continuity

2.1 Definitions

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at** a if three conditions are satisfied:

(a) f is defined at a (i.e. $f(a)$ makes sense)

(b) $\lim\limits_{x\to a^+} f(x) = f(a)$

(c) $\lim\limits_{x\to a^-} f(x) = f(a)$

If (a), (b), or (c) are not true, then f is **discontinuous at** a. There are three types of discontinuities: if f is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists and is a real number.
2. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist and are real numbers, but are different.
3. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

2.2 Using Continuity

The following functions are continuous at every point in their domains:

- polynomials
- rational functions
- trig and inverse trig functions
- exponential functions
- logarithms

Therefore, if you want to evaluate a limit of any of these functions f at a point a in its domain, the limit is equal to $f(a)$ since f is continuous.

Here's another limit law, now that you know about continuous functions:

Theorem (Composite Function Theorem)**.** If $f(x)$ is continuous at L and $\lim\limits_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

To finish off the section, a very useful theorem:

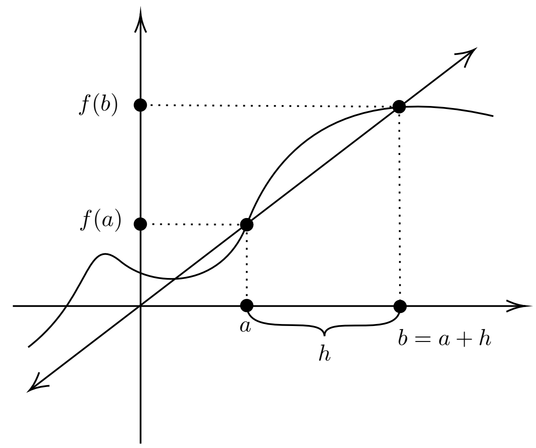
Theorem (The Intermediate Value Theorem)**.** Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

3 The Derivative

3.1 The Limit Definition

Recall that a **secant line** is a line between two points on the graph of a function. If f is a function and $(a, f(a))$ and $(b, f(b))$ are *different* points on the function, then the secant line that they determine has equation

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

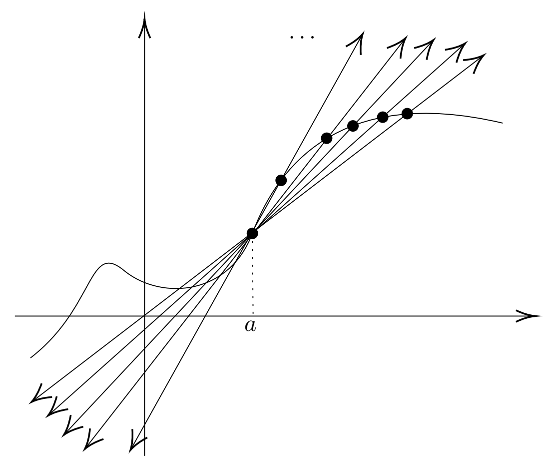


We can assume $b > a$ and write $b = a + h$ for some $h > 0$. Then the equation for the secant line through $(a, f(a))$ and $(b, f(b)) = (a + h, f(a + h))$ is

$$y = f(a) + \frac{f(a + h) - f(a)}{h}(x - a).$$

Now we're going to use *limits* to move one point $(b, f(b))$ closer to the other $(a, f(a))$, and see what happens to the secant line. Since the secant line passes through the point $(a, f(a))$, we only have to track what happens to the slope. As b moves to a, the value of h goes to 0, so the slope of the secant line approaches

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$



We'll denote this quantity $f'(a)$, and call this the **derivative of f at a** (if this limit exists). The line through the point $(a, f(a))$ with slope $f'(a)$ is called the **tangent line of f at a** and has equation

$$y = f(a) + f'(a)(x - a).$$

You'll notice that the tangent line captures some *local information* about f at a. In other words, it is a good approximation of f at a; that is, f and its tangent line are "similar" at a. Intuitively, you should think about the tangent line at a to be the line that "hugs f the best."

Note that this definition of the derivative only makes sense if f is defined on an open interval containing a. We can view the derivative as a function, and write $f'(x)$. The derivative of $f'(x)$ is called the **second derivative** and is denoted $f''(x)$. In general, the nth derivative $f^{(n)}(x)$ is defined to be the derivative of $f^{(n-1)}(x)$. Other notation is sometimes used: if $y = f(x)$, $\frac{dy}{dx} = \frac{d}{dx}[f(x)]$ is the derivative of f, and $\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}[f(x)]$ is the nth derivative of f.

3.2 The Two Part "Program" for Finding Derivatives

So now we have a goal: find derivatives of functions. The problem is that the limit definition is difficult to use (that pesky $h \to 0$ in the denominator). So instead, we will do the following two steps (let a be a real number):

1. "break functions up into simpler ones"

The Scalar Multiple Rule	$(au)' = au'$.
The Sum Rule	$(u+v)' = u' + v'$
The Product Rule	$(uv)' = u'v + uv'$
The Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

2. find derivatives of "simple" functions:

The Constant Rule	$\frac{d}{dx}[a] = 0$
The Power Rule	$\frac{d}{dx}[x^a] = ax^{a-1}$
Basic Trig Functions	$\frac{d}{dx}[\sin(x)] = \cos(x)$ and $\frac{d}{dx}[\cos(x)] = -\sin(x)$
Exponential Functions	$\frac{d}{dx}[a^x] = \ln(a)a^x$
Logarithmic Functions	$\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$

Most of the time, you will be able to find the derivatives using a combination of these rules. For instance, the derivatives of the other 4 trig functions can be found using the quotient rule and the derivatives of sin and cos:

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \qquad \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x) \qquad \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

3.3 Comparing the Graphs of f, f', and f''

f	f'	f''
increasing	positive	
constant	0	
decreasing	negative	
concave up	increasing	positive
linear	constant	0
concave down	decreasing	negative