Trigonometry

$(\cos\theta, \sin\theta)$ is the coordinate on the unit circle that makes angle θ with the positive x-axis.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

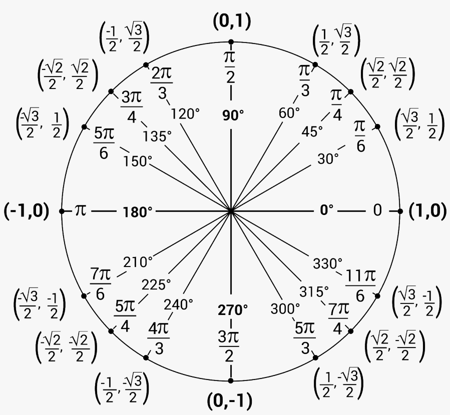
$$\text{Pythagorean identities} \begin{cases} \sin^2\theta + \cos^2\theta = 1 \\ \tan^2\theta + 1 = \sec^2\theta \\ 1 + \cot^2\theta = \csc^2\theta \end{cases}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$



Limits

Law	Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.
Sum	$\lim_{x\to a}(f(x)+g(x)) = L + M$
Scalar	$\lim_{x\to a} cf(x) = cL$
Product	$\lim_{x\to a}(f(x)\cdot g(x)) = L \cdot M$
Quotient	$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ for $M \neq 0$
Power	$\lim_{x\to a}(f(x))^n = L^n$
Root	$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ for all L if n is odd, and for $L \geq 0$ if n is even and $f(x) \geq 0$.

Squeeze Theorem:
Let f, g, and h be functions with $g(x) \leq f(x) \leq h(x)$ for all x and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} f(x) = L$.

Indeterminate Forms:
$\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, $\infty - \infty$, 1^∞, $0\cdot\infty$, ∞^0

$\varepsilon - \delta$ definition:
L is the limit of f as x approaches a if for all $\varepsilon > 0$, there is some $\delta > 0$, such that $|x - a| < \delta \implies |f(x) - L| < \varepsilon$.

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \qquad \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e \qquad \lim_{x\to\infty}\frac{ax^n + \dots}{bx^m + \dots} = \begin{cases} 0 & m > n \\ \infty & n > m \\ a/b & n = m \end{cases}$$

Continuity

Definition: f is *continuous* at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

- The following functions are **continuous on their domains**: polynomials, rational functions, trig and inverse trig functions, exponential functions, logarithms.
- The sum, product, and composition of continuous functions is continuous.

Composite Function Theorem:
If $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x)) = f(L)$.

Intermediate Value Theorem:
Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

Finding Derivatives

Limit definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}$$

Tangent line to $f(x)$ at $x = a$:

$$L(x) = f(a) + f'(a)(x - a)$$

L'Hôpital's Rule:
If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or ∞, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Scalar Rule	$[af]' = af'$
Sum Rule	$[f+g]' = f' + g'$
Product Rule	$[fg]' = f'g + fg'$
Quotient Rule	$\left[\frac{f}{g}\right]' = \frac{f'g - fg'}{g^2}$
Chain Rule	$[f(g(x))]' = f'(g(x))g'(x)$
Inverse Rule	$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$

Logarithmic Differentiation:
To find the derivative of $y = f(x)^{g(x)}$, take ln() of both sides, bring $g(x)$ down using the log rule ($\ln(a^b) = b\ln(a)$):

$$\ln(y) = \ln(f(x)^{g(x)}) = g(x)\ln(f(x))$$

Then implicitly differentiate and solve for y':

$$y' = f(x)^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right).$$

Power Rule	$[x^a]' = ax^{a-1}$	
Trig Rules	$[\sin(x)]' = \cos(x)$	$[\cos(x)]' = -\sin(x)$
(PSST!)	$[\tan(x)]' = \sec^2(x)$	$[\cot(x)]' = -\csc^2(x)$
	$[\sec(x)]' = \sec(x)\tan(x)$	$[\csc(x)]' = -\csc(x)\cot(x)$
Inverse Trig Rules	$[\arcsin(x)]' = \frac{1}{\sqrt{1-x^2}}$	$[\arccos(x)]' = \frac{-1}{\sqrt{1-x^2}}$
	$[\arctan(x)]' = \frac{1}{1+x^2}$	$[\text{arccot}(x)]' = \frac{-1}{1+x^2}$
	$[\text{arcsec}(x)]' = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$[\text{arccsc}(x)]' = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
Exponent Rule	$[a^x]' = \ln(a)a^x$	
Logarithm Rule	$[\log_a(x)]' = \frac{1}{x\ln(a)}$	

Integration

Definitions

- The *definite integral* of f on (a,b) is written $\int_a^b f(x)\,dx$ and is defined to be the *signed* area between the graph of f and the x-axis (if such a quantity exists).
- The *indefinite integral* (or *anti-derivative*) of f on is written $\int f(x)\,dx$ or $\int f$ is the family of functions whose derivative is f.

Fundamental Theorem of Calculus If $F' = f$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Scalar Rule	$\int af = a\int f.$
Sum Rule	$\int f + \int g = \int f + \int g$
Int. by Parts	$\int f'g = fg - \int fg'$
u-substitution	$\int f'(g(x))g'(x)\,dx = f(g(x))$
Power Rule	$\int x^a\,dx = \begin{cases} \frac{1}{a+1}x^{a+1} + C & a \neq -1 \\ \ln\lvert x\rvert + C & a = -1 \end{cases}$
Trig Rules	$\int \sin(x)\,dx = -\cos(x) + C$
	$\int \cos(x)\,dx = \sin(x) + C$
Exp. Rules	$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C$

Partial Fractions

Factor	Term in decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Trig Substitutions

Integrand	Substitution	Result
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a\tan\theta$

Other Antiderivatives

$\int \ln(x)\,dx = x\ln(x) - x + C$

$\int \tan(x)\,dx = \ln\lvert\cos(x)\rvert + C$

$\int \sec(x)\,dx = \ln\lvert\sec(x)+\tan(x)\rvert + C$

$\int \csc(x)\,dx = \ln\lvert\csc(x)-\cot(x)\rvert + C$

$\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$

Riemann Sums

$R_n = \sum_{k=1}^n f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$

$L_n = \sum_{k=1}^n f\left(a + (k-1)\frac{b-a}{n}\right)\frac{b-a}{n}$

$T_n = \sum_{i=k}^n \frac{f\left(a+(k-1)\frac{b-a}{n}\right)+f\left(a+k\frac{b-a}{n}\right)}{2}\frac{b-a}{n}$

$$\lim_{n\to\infty} R_n, L_n, T_n = \int_a^b f(x)\,dx$$

Sums of Powers

- $\sum_{k=1}^n 1 = n$
- $\sum_{k=1}^n k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$

Power Series

Near $x = a$,

$$f(x) \approx T_k(x) = \sum_{n=0}^{k}\frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and on some interval centered at a,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Function	Taylor Series	Interval of Convergence
e^x	$\sum_{n=0}^{\infty}\frac{1}{n!}x^n$	$\mathbb{R}$
$\sin(x)$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$\mathbb{R}$
$\cos(x)$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$	$\mathbb{R}$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty}x^n$	$(-1,1)$
$\ln(1-x)$	$\sum_{n=1}^{\infty}\frac{-1}{n}x^n$	$[-1,1)$
$(x+1)^k$	$\sum_{n=0}^{\infty}\binom{k}{n}x^n$	$\mathbb{R}$

Test for Convergence and Divergence

- **Divergence Test:** $\lim_{n\to\infty} a_n \neq 0 \implies \sum a_n$ diverges.
- **Integral Test:** If $f(x) > 0$ is CTS and decreasing on $[k,\infty)$, and $f(n) = a_n$,

$$\int_k^\infty f(x)\,dx \text{ converges} \iff \sum_{n=k}^{\infty} a_n \text{ converges.}$$

- **The p-series Test:** If $k > 0$, $\sum_{n=k}^{\infty}\frac{1}{n^p}$ converges $\iff p > 1$.
- **Comparison Test:** If $0 \leq a_n \leq b_n$ for all n, then

$$\sum b_n \text{ converges} \implies \sum a_n \text{ converges.}$$

- **Limit Comparison Test:** If $a_n \geq 0$ and $b_n > 0$ for all n and $\lim_{n\to\infty} a_n/b_n$ is positive and finite, then

$$\sum a_n \text{ converges} \iff \sum b_n \text{ converges.}$$

- **Alternating Series Test:** If $b_n \geq 0$ decreases to 0, $\sum b_n(-1)^n$ converges.
- **Absolute Convergence Test:**

$$\sum \lvert a_n\rvert \text{ converges} \implies \sum a_n \text{ converges.}$$

- **Ratio Test:** Define $L = \lim_{n\to\infty}\lvert a_{n+1}/a_n\rvert$.
  If $L < 1$, $\sum a_n$ converges. If $L > 1$, $\sum a_n$ diverges.
- **Root Test:** Define $L = \lim_{n\to\infty}\lvert a_n\rvert^{1/n}$.
  If $L < 1$, $\sum a_n$ converges. If $L > 1$, $\sum a_n$ diverges.

Bounds

If $f(x) > 0$ is CTS and decreasing on $[1, \infty)$ with $f(n) = a_n$, then for any N,

$$\left(\sum_{n=1}^{N} a_n + \int_{N+1}^{\infty} f(x)\, dx\right) \le \sum_{n=1}^{\infty} a_n \le \left(\sum_{n=1}^{N} a_n + \int_{N}^{\infty} f(x)\, dx\right).$$

Suppose $S = \sum_{n=1}^{\infty} b_n(-1)^n$ where $b_n \ge 0$, and b_n decreases to 0. Then for any N,

$$|S - S_N| \le b_{N+1}$$

where $S_N = \sum_{n=1}^{N} b_n(-1)^n$.

Taylor's Inequality: If for all $x \in [a-d, a+d]$,

$$\left|f^{(k+1)}(x)\right| \le M,$$

then for all $x \in [a-d, a+d]$,

$$|f(x) - T_k(x)| \le \frac{M}{(k+1)!}|x-a|^{k+1}.$$

Solving ODEs

To solve the separable ODE $\dfrac{dy}{dx} = f(x)g(y)$, separate the x and y variables and integrate:

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$$

To solve a 2nd order ODE of the form $ay'' + by' + c = 0$, find the roots λ of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

Case	Roots	Solutions
$b^2 - 4ac \ne 0$	λ_1, λ_2	$y = pe^{\lambda_1 x} + qe^{\lambda_2 x}$
$b^2 - 4ac = 0$	λ	$y = pe^{\lambda x} + qxe^{\lambda x}$

Common ODEs

Name	ODE	Solution
Exponential	$y' = ky$	$y = e^{kx+C}$
Logistic	$y' = ky\left(1 - \frac{y}{M}\right)$	$y = \frac{M}{1 \pm Ce^{-kx}}$
Harmonic Oscillator	$-ky = my''$	$y = A \cdot \cos\left(\sqrt{\frac{k}{m}}x + C\right)$