Derivative Application Exercises

A. Estimate each of the following using linear approximation.

1. $\sqrt{9.1}$ 2. $\sqrt[5]{31.9}$ 3. $\sin(\pi + 0.1)$ 4. $\ln(e - 0.1)$ 5. $e^{0.1}$

B. Evaluate the following limits using L'Hôpital's Rule.

1. $\displaystyle\lim_{x\to 0} \frac{x^2 - 2x}{4x^2}$
2. $\displaystyle\lim_{x\to\infty} \frac{4x^3 - 2x + 3}{3x^3 + 8x + 100}$
3. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^{x^2}$
4. $\displaystyle\lim_{x\to\infty} \frac{2^x + x^2}{e^x - \pi}$
5. $\displaystyle\lim_{x\to\infty} \left(e^x + \sqrt{x}\right)^{\frac{1}{x}}$
6. $\displaystyle\lim_{x\to\infty} \left(\frac{1}{x^2} + 2^{-x}\right)^{\frac{1}{x}}$
7. $\displaystyle\lim_{x\to\infty} 2^x - x^2$
8. $\displaystyle\lim_{x\to 0^+} x \ln\left(\frac{x}{x+1}\right)$
9. $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} \ln\left(x^2\right)$

C. Kinematics

1. What is the difference between speed and velocity?
2. If the displacement from the origin of a particle after time t is given by $d(t) = 3\cos^2(t) + 4^t$, write the formula for the function of the particles velocity and acceleration.
3. A ball is thrown up into the air. Its height after t seconds is given by $h(t) = -2t^2 + 10t$. (a) how high is the ball after 2 seconds? (b) What is its average velocity during the first two seconds? (c) How fast is the bullet traveling after 2 seconds? (d) What is the acceleration of the bullet at 2 seconds?

D. Related Rates

1. For all positive values of c, the parabola $f(x) = x^2 + 2x - c$ has one positive root. Find the rate at which this root is changing with respect to the rate at which c is changing.
2. Two ships start at the same port depart at the same time. One travels North at a rate of 30 m/s and the other travels East at a rate of 40 m/s. How fast is the distance between the two ships changing after 10 minutes?
3. I have a cylindrical vat of water 3 ft in diameter, and a hose that fills it up at a rate of 4 ft^3/min. It also leaks $\frac{1}{2}$ ft^3 every minute. If the vat is initially empty, how fast is the height of the water changing after 10 min?
4. When I look at a tree in the distance, it looks small, but as I walk closer, the tree appears to be growing. I start 100 m away from the tree and hold my arm (0.5 m long) out with ruler in my hand and the tree reaches the 10 cm mark. I then start walking towards the tree at 1.5 m/s. How fast is the observed height of the tree changing after 10 seconds? How about when I am 50 m from the tree?
5. Sand is falling out of a pipe at a rate of 4 m^3/s, forming a conical pile. Due to the crystalline structure of the sand, the radius of the pile is always twice the height. How fast is the height of the sand pile changing when the height is 10 m?
6. A 10 ft ladder is leaning against a wall but the ground is slippery and it starts to slide. After time t, the bottom end of the ladder is $x(t) = 2t^2 + 5$ ft away from the wall. How fast is the height of the top of the ladder changing when the bottom of the ladder is 7 feet from the wall? How fast is the area under the ladder changing at the same time?
7. A hot air balloon is taking off 80 m away from me and is rising at 7 m/s. As it does, I adjust my head's angle to keep looking at it. How fast is this angle changing when the balloon is 70 m high?

E. For each of the following functions, find (a) the intervals on which they are increasing/decreasing, (b) the intervals on which they are concave up/down, (c) the coordinates of any local/global maxima or minima, (d) points of inflection. Then sketch the graph.

1. $f(x) = 2x^2 + 3x$
2. $f(x) = x^3 + 17x^2 + 2$
3. $f(x) = \sin(x) - \cos(x)$ on $[-2\pi, 2\pi]$
4. $f(x) = e^{\cos(x)}$

F. Optimization

1. A rectangle has its base on the x-axis and its top two vertices lie on the parabola given by $f(x) = 16 - x^2$. What is the maximum area of this rectangle?
2. A movie company is trying to decide how many explosions they should put in their movie. The producers will allow anywhere between 0 and 100. The cost for x explosions is $10 + \sqrt{x}$ thousand dollars. Market research suggests that x explosions will bring in $12 + 10\arctan(\sqrt{x})$ thousand dollars at the box office. How many explosions should the movie company use to maximize profits?
3. We want to construct a box with no lid. What is the most volumetric box I can obtain with only 10 ft^2 of material if I insist the base of the box is a square?
4. I want to make a cylindrical metal can that holds 1 Liter (1000 cm^3) of water. What are the proportions of the can that will minimize the amount of metal I need to use?
5. The determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the quantity $D = ad - bc$. What is the largest possible determinant if both rows and both columns sum to 1, and all the entries are non-negative?
6. If a clock's minute hand is twice as long as its hour hand, what time between 1 and 2 o'clock are the tips of the hands the furthest from each other? When are they closest?
7. A rectangular grass field is 100m long in the North-South direction and 50m in the other. There is a sidewalk along its western side. You are standing on the South-West corner of the park and want to go to the North-East corner. You can walk 1m/s on the grass and 3m/s on the sidewalk. How far should you walk on the sidewalk before cutting across the field to minimize your travel time?