Limits and Derivatives Practice Problems

A. Are the following true or false? If true, explain why. If false, give a counter-example.

1. If $\lim_{x\to a} f(x)$ does not exist, then f is undefined at the point $x = a$.
2. If f and g are continuous on their domains which contain a, then $\lim_{x\to a} f(x) + g(x) = f(a) + g(a)$.
3. If a function is continuous at a, then $f'(a)$ exists.

B. Evaluate the following limits (or say that the limit DNE):

1. $\lim_{x\to 3} \frac{x^2 - 9}{x + 3}$
2. $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$
3. $\lim_{x\to \pi/2} \frac{\cot(x)}{\cos(x)}$
4. $\lim_{x\to 0} \frac{(\cos^2(x) - 1)(x + 3)}{x}$

C. For each function f, find a value of c so that f is continuous on $\mathbb{R}$:

1. $f(x) = \begin{cases} 2x & x \leq c \\ x^2 + 1 & x > c. \end{cases}$
2. $f(x) = \begin{cases} 2x + c & x < 2 \\ x^2 + cx + 1 & x \geq 2. \end{cases}$

D. For each f, find f' using the limit definition of the derivative.

1. $f(x) = \sqrt{x + 4}$
2. $f(x) = 2x^2 + 3x$

E. For each f, find f' and f'' using any method you want.

1. $f(x) = 3x^3 - \frac{4}{x^2}$
2. $f(x) = x\sin(x)$
3. $f(x) = \frac{x^2+3}{x-4}$
4. $f(x) = x^2\cos(x) + x\tan(x)$
5. $f(x) = \sin(x)\cos(x)e^x$

F. When are the following functions increasing/decreasing? When are they concave up/down?

1. $f(x) = 2x^2 + 3x$
2. $f(x) = x^3 + 17x^2 + 2$

Answers (in no particular order)

- $f'(x) = e^x(\cos^2(x)+\sin(x)\cos(x)-\sin^2(x))$, $f''(x) = f'(x)+e^x\left(-4\cos(x)\sin(x)+\cos^2(x)-\sin^2(x)\right)$
- 0
- $f'(x) = 4x+3$
- 6
- increasing: $x > \frac{-3}{4}$, decreasing: $x < \frac{-3}{4}$, concave up: all of $\mathbb{R}$, concave down: nowhere
- $f'(x) = \frac{1}{2}(x+4)^{-1/2}$
- False (e.g. $f(x) = \frac{x^2}{x}$ is not defined at 0, but $\lim_{x\to 0} f(x) = 0$)
- $f'(x) = \sin(x) + x\cos(x)$, $f''(x) = \cos(x) + \cos(x) - x\sin(x)$
- 1
- True (Since f and g are continuous, so is $f+g$. Then by the def. of continuity, $\lim_{x\to a} f(x)+g(x) = f(a)+g(a)$)
- $f'(x) = 9x^2 + 8x^{-3}$, $f''(x) = 18x - 24x^{-4}$
- 0
- -1
- False ($f(x) = |x|$ is continuous at 0, but $f'(0)$ DNE)
- $f'(x) = \frac{(2x)(x-4)-(x^2+3)}{(x-4)^2}$, $f''(x) = \frac{(2x-8)(x^2-8x+16)-(x^2-8x-3)(2x-8)}{(x^2-8x+16)^2}$
- 1
- increasing: $x > 0$ or $x < \frac{-3}{4}$, decreasing: $\frac{-3}{4} < x < 0$, concave up: $x > \frac{-17}{3}$, concave down: $x < \frac{-17}{3}$
- $f'(x) = 2x\cos(x) - x^2\sin(x) + \tan(x) + x\sec^2(x)$, $f''(x) = \left(2-x^2\right)\cos(x) - 4x\sin(x) + \sec^2(x)\left(2+2x\tan(x)\right)$