Cαlculus

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Contents

A Note to Students

This text is meant to serve as a condensed resource for a student taking their first or second semester of calculus. I am writing with the assumption that the reader is familiar with some trigonometry/pre-calculus, but has perhaps forgotten some of it. While statements I write are always true, they will often not be entirely rigorous or stated with full generality. Use the big margins to make your own notes.

I'll leave prompts and questions in boxes like this.

My goal is to communicate what I believe are the key ideas in an introductory calculus class that will help the average student succeed. Here are my suggestions for doing well in calculus (that should be cyclically repeated!):

1. Spend time fully understanding key concepts.
2. Identify and learn about common types of problems and exercises.
3. Find some really good examples.
4. Practice a lot!

When you begin learning calculus, it is OK not to understand everything right away. It takes time for some concepts to sink in. Doing a lot of practice will help you start developing intuition about how to tackle new problems, even if you don't fully comprehend everything you are doing. Eventually, your problem-solving skills and your abstract understanding will both be strong, but they need to grow together and build off each other. It is good to sit and actually think about something for a while without doing any writing. Keep at it and eventually you will be able to conjure pictures and animations in your head that relate ideas and succinctly encapsulate the idea of a problem or theorem.

P.S.

You might be thinking: "What the &%$# is calculus?!".

Well, let me save you a trip to Wikipedia. At its core, calculus is the study of change. Its name comes from the Latin word for "pebbles" because the main idea of calculus is to study things that are changing in non-linear ways by breaking them up into small pieces (like pebbles) that can be thought of as linear, and then putting them back together.

P.P.S

Ok... I lied. The pebbles actually refer to the beads on an abacus, but a professor of mine once told me the other version, and I like that better!

Part I
Pre-Calculus

1 Trigonometry

The word "trigonometry" comes from the ancient Greek words for "triangle" and "measure", and it has found its way into just about every possible branch of mathematics and science. You would be putting yourself at a disadvantage not to make sure you are comfortable with at least the basic ideas of trigonometry before continuing to calculus.

You should think about trigonometry in two ways:

1. pragmatically (it is a useful tool for solving problems, and there are some things you should memorize)

2. abstractly (it is really just a way to talk about similar triangles)

1.1 Basic Definitions

The **unit circle** is the circle with (unit) radius 1, centered at the origin, $(0, 0)$. When we draw a radius of the unit circle, it forms an angle θ with the positive side of the horizontal axis.

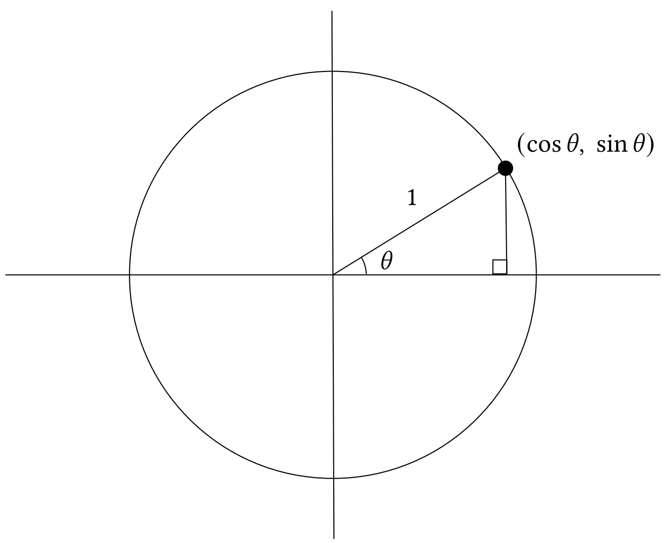


Figure 1: $\sin\theta$ and $\cos\theta$ are coordinates

This line segment always has length 1, and connects the origin to some point on the circle. We define $\cos\theta$ and $\sin\theta$ to be the x and y coordinates of this point. In other words, $\cos\theta$ is defined to be the (signed) length of the horizontal leg of the right triangle and similarly, $\sin\theta$ is defined to be the (signed) length of the vertical leg.

We measure angles in either degrees or radians. The angle θ can be any real number, but there are 2π radians (or 360°) in one full revolution. Imagine the angle θ changing, making the radius sweep around the circle like a SONAR on a submarine. The hypotenuse of the triangle stays constantly at 1, but the side lengths fluctuate between 1 and 0.

> Draw a picture: where on the unit circle is $\cos\theta$ positive and where is it negative? How about $\sin\theta$?

The two functions sin and cos are the basic building blocks, but there are four other commonly used trig functions. They are defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

These functions also have visual representations on the unit circle.

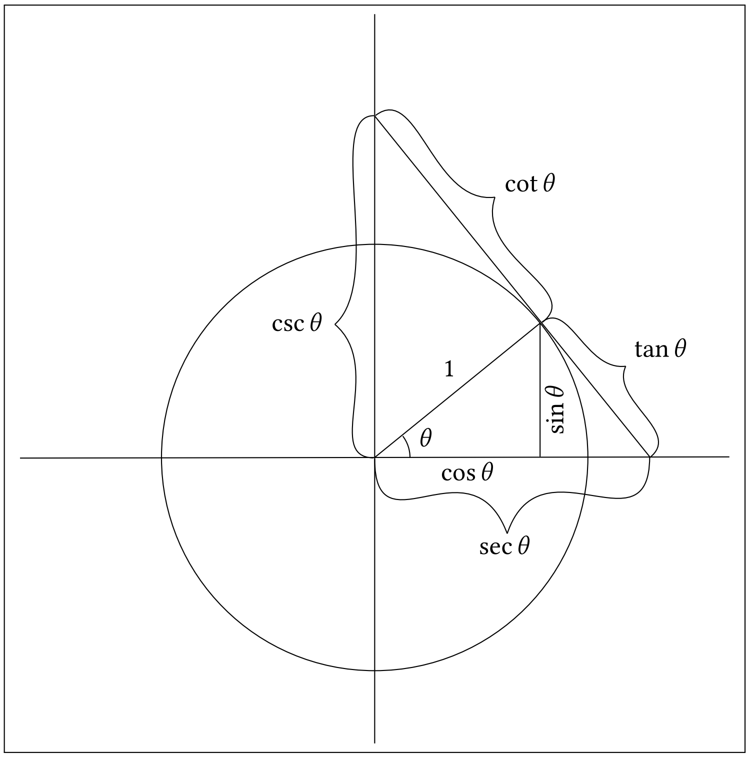


Figure 2: The six common trig functions on the unit circle

To explore these functions, check out this graph on Desmos.

1.2 Values on the Unit Circle

There are several angles θ for which you should be able to compute $\sin\theta$ and $\cos\theta$ (then computing the other four trig functions is easy). Luckily, there is a lot of symmetry involved: it is only necessary to memorize a few numbers and then you can easily fill in the rest.

Below is a diagram of all of the angles whose sin and cos you should be familiar with.

It might seem intimidating, but just focus on the first (upper-right) quadrant and the first five angles there:

θ	$\cos\theta$	$\sin\theta$
0	$\sqrt{4}/2$	$\sqrt{0}/2$
$\pi/6$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/2$	$\sqrt{0}/2$	$\sqrt{4}/2$

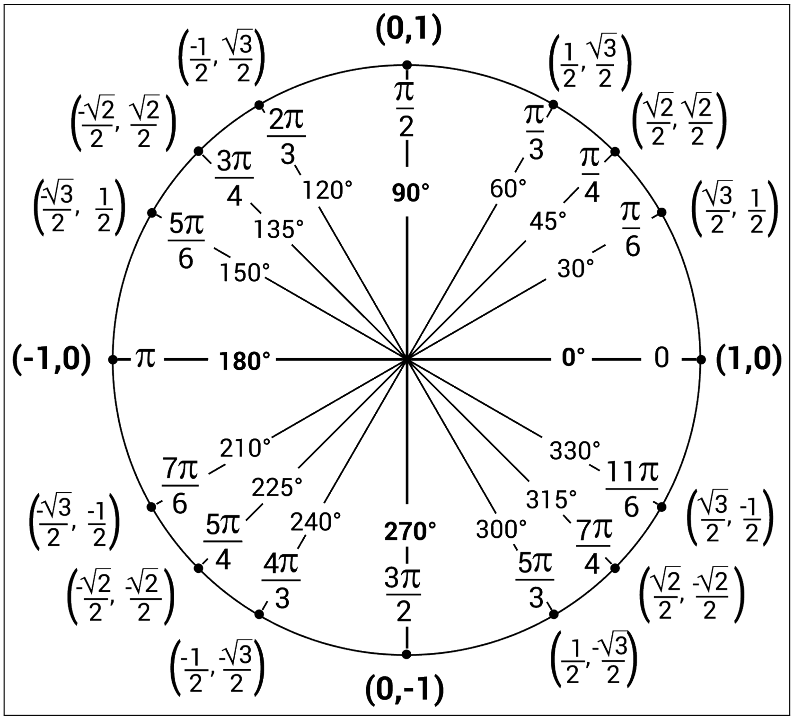


Figure 3: The Unit Circle

In this table, I wrote the numbers in an "un-simplified" way so you can see the pattern: their numerators change by 1 under a square-root, and as one goes up, the other goes down.

You can also see that the values for other angles are the same number, but with a different sign (according to the quadrant they are in), and that the unit circle is symmetric, reflecting over the x and y axis.

1.3 The Pythagorean and Other Identities

Remember the Pythagorean Theorem, $a^2 + b^2 = c^2$? Well, since $\sin\theta$ and $\cos\theta$ are the lengths of a right triangle with hypotenuse 1, we have the following **Pythagorean identity**:

$$\cos^2\theta + \sin^2\theta = 1.$$

Dividing this equation by $\sin^2\theta$ or $\cos^2\theta$, we get

$$\cot^2\theta + 1 = \csc^2\theta \qquad \text{and} \qquad 1 + \tan^2\theta = \sec^2\theta.$$

There is some nice symmetry between $\sin\theta$ and $\cos\theta$. The leg opposite the angle is θ is $\sin\theta$. Since the internal angles of a triangle sum to π, the other acute angle is $\frac{\pi}{2} - \theta$. Then, the leg opposite $\frac{\pi}{2} - \theta$ has length $\sin\left(\frac{\pi}{2} - \theta\right)$, but is also $\cos\theta$. Therefore,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta).$$

Similarly,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta).$$

Looking at the picture of the unit circle, we can also see that

$$\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \cos(-\theta) = \cos(\theta).$$

Notation check! Which of the following are the same?

$$\begin{array}{cc} \sin\theta^2 & \sin^2\theta \\ \sin(\theta^2) & \sin(\theta)^2 \\ (\sin(\theta))^2 & (\sin\theta)^2 \end{array}$$

Label the angle $\frac{\pi}{2} - \theta$ on Figure 1.1.

There is one more kind of identity that will be useful for us:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

These are called the **angle sum formulas**, and they are long but if you read them out loud, they make a little song! Try it while clapping on each word:

"SINE-COSINE-COSINE-SINE. COSINE-COSINE-SINE-SINE."

No seriously, sing the song. Now do it again.

Once you remember the order of the functions, you just have to remember that the As and Bs alternate and that the sign in the second equation flips ($\pm$ becomes $\mp$).

These simplify if A and B are the same angle θ to what are called the **double angle formulas**. In this case,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

and

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1.\end{aligned}$$

Check that these are correct. Hint: use the Pythagorean identity.

1.4 Applications

So, how do we use trigonometry to help solve problems? The main way that trigonometry is useful for you as a calculus student is two-fold. One is as a source of nice function examples to play around with (once we start taking limits and derivatives). The other is as a means to fill in missing information (this is the "SOH-CAH-TOA" you might remember).

Let's say you have a right triangle with hypotenuse 8 and one a 30° angle.

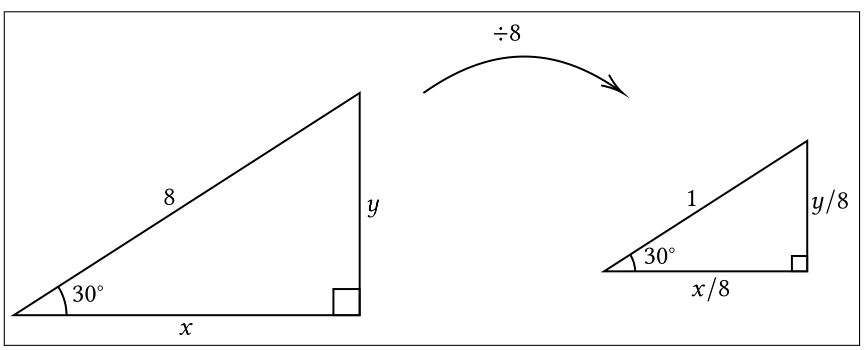


Figure 4: Similar Triangles

If you want to know what the other side lengths are, this is how you can use trigonometry to do it! First, scale the triangle down by the side length that you know (in this case 8). This doesn't change the angles, so we now have a right triangle with hypotenuse 1. By definition of sin and cos,

$$\sin 30^\circ = \frac{y}{8} \qquad \text{and} \qquad \cos 30^\circ = \frac{x}{8},$$

so we can solve for x and y:

$$y = 8\sin 30^\circ = 4 \qquad \text{and} \qquad x = 8\cos 30^\circ = 4\sqrt{3}.$$

The mnemonic device "SOH-CAH-TOA" reminds us that Sin of 30° is the Opposite leg (y) over the Hypotenuse (8), and Cos of 30° is the Adjacent leg (x) over the Hypotenuse (8).

What does the "TOA" part say? What situation would it help you figure something out about a triangle?

The identities that we learned about also have many applications, but mostly are in simplifying calculuations. You should keep them in mind so you recognize when they might be helpful in simplifying an expression. Here is a cool example:

$$\begin{aligned}
\sin(2\theta)\left(\frac{\tan\theta + \cot\theta}{2}\right) &= 2\sin\theta\cos\theta\left(\frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{2}\right)\\
&= \sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)\\
&= \sin^2\theta + \cos^2\theta\\
&= 1.
\end{aligned}$$

Trig identities can also be a computationally useful tool: if you need to know the Sine (or Cosine, etc.) of an angle you are not familiar with, write the angle as a sum or difference of angles you know and use the angle sum formulas. For example,

$$\begin{aligned}
\cos(15^\circ) &= \cos(45^\circ - 30^\circ)\\
&= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ)\\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\\
&= \frac{\sqrt{6} + \sqrt{2}}{4}.
\end{aligned}$$

Can you compute $\tan(75^\circ)$?

1.5 There's Always More Trigonometry...

Trigonometry dates back to the 3rd century BCE and needless to say it has changed a bit over the years. We've only covered the basics and focused on the trigonometry involving right triangles. There is always more trigonometry to learn: more trigonometric functions and MANY more relations among them! I won't pester you with many more right now, but they might come up in the future, so just be aware.

The two very common laws of trigonometry that involve non-right triangles are the Law of Sines and the Law of Cosines. I will state them here (for culture).

Figure 1.5 shows a triangle with internal angles A, B, and C, and respective opposite side lengths a, b, and c. The triangle is **circumscribed** in a circle, meaning its vertices lie on the circle's boundary. It turns out that the circle has radius

$$R = \frac{abc}{\sqrt{(a+b+c)(a-b+c)(a+b-c)(-a+b+c)}}.$$

Theorem 1.1 (Law of Sines)**.** If a triangle has sides and angles as above, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = \frac{abc}{2\Delta},$$

where R is the radius of the circumscribed circle and Δ is the area of the triangle.

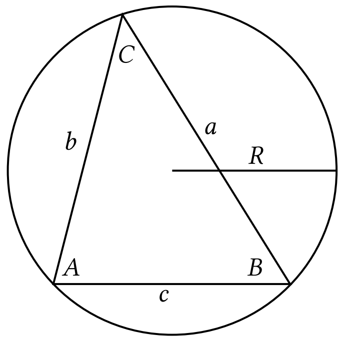


Figure 5: Circumscribed Triangle

Theorem 1.2 (Law of Cosines)**.** If a triangle has sides and angles as above, then

$$a^2 + b^2 = c^2 + 2ab\cos C.$$

Almost looks familiar, right?

When you use these formulas, keep in mind that solving for an angle can be tricky: if $\sin\theta = 1/2$, then θ could be 30°, 150°, 390°, etc..

Here is one last (very pretty) theorem:

Theorem 1.3 (Heron's Formula)**.** A triangle with side lengths a, b, and c has area

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}$.

Prove this formula.

2 Functions on the Real Line

2.1 Real Numbers

The set of real numbers ($\mathbb{R}$) is an uncountably infinite set, meaning I can not even begin to exhaustively list all of its members. However, here are some of my favorites: 3, 2/5, 4.11, 713, $\sqrt{37}$, π. Real numbers include both the rationals and irrationals.

Do you know any numbers that aren't real?

The real numbers can be thought of as a continuum that has no beginning and no end: real numbers can get infinitely small and infinitely large.

More concretely, imagine you are standing on a road that goes on forever, both in front of you and behind you. If you drop a traffic cone where you are standing, and call it "the origin", then any distance you walk on this road away from the cone is a real number. If you walk forward, it is positive, and if you walk backwards, it is negative. You can keep walking forever and reach any number you would like, but you will never reach an "end of the road" because it doesn't exist.

2.2 Subsets of the Real Line

A **subset** is a subcollection. There are a few ways we notate subsets of $\mathbb{R}$, so this section is mostly dedicated to notation. The first two examples of a subset are kind of stupid: the **empty set** $\emptyset$ (the set with no elements), and all of $\mathbb{R}$ are both subsets of $\mathbb{R}$.

You can define a **literal** subset of $\mathbb{R}$ by listing the elements it contains. For example, $\{1, 2, 3\}$ is the set containing 1, 2, and 3.

Intervals are—in my opinion—the most important kind of subset of $\mathbb{R}$. The following table shows the 8 kinds of intervals and the two ways we write them ($a \leq b$ are real numbers).

The set of real numbers that are...	**Interval Notation**	**Set-Builder Notation**
greater than a and less than b	(a, b)	$\{x \in \mathbb{R} \;:\; a < x < b\}$
greater than or equal to a and less than b	$[a, b)$	$\{x \in \mathbb{R} \;:\; a \leq x < b\}$
greater than a and less than or equal to b	$(a, b]$	$\{x \in \mathbb{R} \;:\; a < x \leq b\}$
greater than or equal to a and less than or equal to b	$[a, b]$	$\{x \in \mathbb{R} \;:\; a \leq x \leq b\}$
greater than a	(a, ∞)	$\{x \in \mathbb{R} \;:\; a < x\}$
greater than or equal to a	$[a, \infty)$	$\{x \in \mathbb{R} \;:\; a \leq x\}$
less than b	$(-\infty, b)$	$\{x \in \mathbb{R} \;:\; x < b\}$
less than or equal to b	$(-\infty, b]$	$\{x \in \mathbb{R} \;:\; x \leq b\}$

Interval notation is easy to write and read, once you get a hang of what the different parentheses mean. The regular parentheses () are called "open" and mean that the particular endpoint of the interval is not included. The [] brackets are called "closed" and indicate that the endpoint *is* included.

Why aren't $(a, \infty]$, $(a, \infty]$, $[-\infty, b)$, and $[-\infty, b]$ included in the table?

The set-builder notation is read as follows:

$$\{\underbrace{x}_{x}\ \underbrace{\in}_{\text{in}}\ \underbrace{\mathbb{R}}_{\text{the real numbers}}\ \underbrace{:}_{\text{such that}}\ \underbrace{a < x < b}_{x \text{ is strictly between } a \text{ and } b}\}.$$

All of the subsets of $\mathbb{R}$ that you'll want to use in this course can be written using combinations of intervals. There are two ways to do this:

- The **union** of two sets is the set that contains the elements in *either* set and is denoted with the "cup" symbol, $\cup$.
- The **intersection** of two sets is the set that contains the elements in *both* sets and is denoted with the "cap" symbol, $\cap$.

Here are some examples:

A	B	$A \cup B$	$A \cap B$
$\{1, 2, 3\}$	$\{3, 4, 5\}$	$\{1, 2, 3, 4, 5\}$	$\{3\}$
$\{0, 2, 3\}$	$(1, 3)$	$\{0\} \cup (1, 3]$	$\{2\}$
$(3, 4]$	$[4, 5)$	$(3, 5)$	$\{4\}$
$(-\infty, 5)$	$[5, \infty)$	$(-\infty, \infty) = \mathbb{R}$	$\emptyset$
$(-3, 0)$	$(2, 6)$	$(-3, 0) \cup (2, 6)$	$\emptyset$
$(1, 5)$	$(3, 7)$	$(1, 7)$	$(3, 5)$
$(1, 8)$	$(3, 4)$	$(1, 8)$	$(3, 4)$

Draw a picture of each of these examples. Does *Venn diagram* ring any bells?

2.3 Defining Functions

We will start with quite a few new (and abstract) definitions, but we will make things concrete very soon.

If A and B are two sets, a **function** f from A to B is an assignment of each element of A to a unique element of B. In other words, for each input a in A, f outputs exactly one b in B, which we denote as $f(a)$. The set A is called the **domain** of f, the set B is called the **codomain** of f, and we write

$$f : A \to B \qquad \text{or} \qquad A \xrightarrow{f} B.$$

Is f defined by

$$f(x) = \begin{cases} x + 1 & x > 0 \\ x^2 & x < 1 \end{cases}$$

a function?

Please remember that a function has three pieces of information: the domain, the codomain, and the rule of assignment. If one of these pieces is changed, the resulting function is different.

The **range** of f is the set of all outputs of f and is denoted $f(A)$. The range is always a subset of the codomain, but they are not always equal sets. The relationship between the domain, codomain, and the range are an important one:

- A function is called **surjective** (or **onto**) if its range and codomain are the same (i.e. $f(A) = B$). In other words, a function is surjective if every element of the codomain is an output of f.
- A function is called **injective** (or **one-to-one**) if each element of the range is the output of exactly one element of the domain. In other words, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- If a function is both surjective and injective, then it is called **bijective**.

What does the word "sur" mean in French?

The figure below is a cartoon of three functions. From left to right, (1) a surjection that is not injective, (2) a bijection, and (3) an injection that is not surjective.

Write down the domain, codomain, and range of each of these functions.

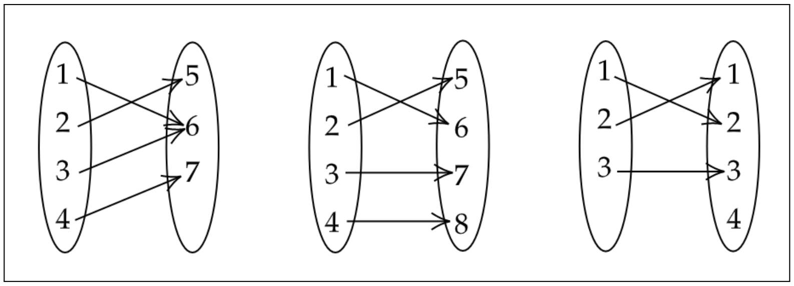


Figure 6: A Surjection, a Bijection, and an Injection

We will only use **real functions**, which are functions whose domain and codomain are subsets of $\mathbb{R}$. These functions take real numbers as inputs and give real numbers as outputs, so they can be graphed on a Cartesian grid, with the inputs on the horizontal axis and outputs on the vertical axis.

To check that a curve drawn on a plot is a function, you can use the **vertical line test**: any vertical line must intersect a graph of a real function at most once (i.e. if a vertical line intersects a curve more than once, then the curve cannot be the graph of a real function).

Draw a non-function that fails the vertical line test.

Some people may get sloppy and say things like

"the function $f(x) = x^2$"

but what they really mean is

"the real function f, defined by $f(x) = x^2$".

Distinguishing between a function and the formula defining it will eventually make things easier to understand. However, from now on, I might say "function" and mean "real-function" (use context clues).

Think of real functions that are (1) injective but not surjective, (2) surjective but not injective, (3) neither injective nor surjective, (4) bijective.

2.4 More Domains and Ranges

All real functions can have $\mathbb{R}$ as their codomain, but not all real functions have $\mathbb{R}$ as their domain and range. The following table lists domains and ranges for some common functions.

Function type		Domain	Range
Polynomial	$a_n x^n + \cdots + a_1 x + a_0$	$\mathbb{R}$	If the highest power is odd, $\mathbb{R}$. If the highest power is even and positive, then $[b, \infty)$, and $(-\infty, b)$ otherwise (for some b).
Exponential	a^x	$\mathbb{R}$	$(0, \infty)$
Logarithm	$\log_a(x)$	$(0, \infty)$	$\mathbb{R}$
Rational	$\frac{p(x)}{q(x)}$ (p, q are polynomials)	$\{x \in \mathbb{R} \,:\, q(x) \neq 0\}$	It depends

Let's do some concrete examples:

- The function f given by $f(x) = \frac{1}{x}$ does not have 0 in its domain, since we can't divide by 0. Also, f does not have 0 in its range, since there is no real number x for which $\frac{1}{x} = 0$. Therefore the domain and range of f are both $(-\infty, 0) \cup (0, \infty)$.

- Similarly, tan does not have $2n\pi + \pi/2$ in its domain for any integer n, since $\cos(2n\pi + \pi/2) = 0$ (we can write the domain of tan as $\{x \in \mathbb{R} : x \neq 2n\pi + \pi/2$ for any integer $n\}$). The range of tan is all of $\mathbb{R}$.
- The function f given by $f(x) = x^2$ has domain $\mathbb{R}$ and range $[0, \infty)$.

What is the domain and range of f given by $f(x) = n^n$ where n is a positive integer ($n = 1, 2, 3, \ldots$)?

2.5 Inverses

A function $f : A \longrightarrow B$ is **invertible** if there is some function $g : B \longrightarrow A$ such that two conditions hold:

(i) for all a in A, $g(f(a)) = a$;

(ii) for all b in B, $f(g(b)) = b$.

In this case, we call g the **inverse** of f and write $f^{-1} = g$. In other words, a function is **invertible** if it is reversible as a mapping: if f maps x to $y := f(x)$, then its inverse f^{-1} must map y to x.

Look abck at Figure 2.3. Which of these functions are invertible? On any function that is invertible, draw the arrows for the inverse function.

You might wonder: *when is a function invertible?* It turns out that we already understand exactly what we want:

Theorem 2.1. A function is invertible exactly when it is bijective.

Lets think through this: looking back at Figure 2.3, the first function is not invertible because it is not injective. Both 1 and 3 map to 6, but condition (i) says that we must have $f^{-1}(f(1)) = 1$ and $f^{-1}(f(3)) = 3$. Since $f(1)$ and $f(3)$ are both equal to 6, we must have that $1 = f^{-1}(6) = 3$, crazy talk!

The third function in Figure 2.3 is not invertible because it is not surjective. This is a problem: since the element 4 in the codomain is not mapped to by eny element of the domain, no matter what we might chose $f^{-1}(4)$ to be, $f(f^{-1}(4)) \neq 4$, which violates condition (ii).

The condition that invertible functions must be injective is sometimes called the **horizontal line test** for real functions: the graph of an invertible real function cannot intersect any horizontal line more than once (i.e. if a horizontal line intersects the graph of a function more than once, then the function cannot be invertible).

Draw the graph of a non-invertible function failing the horizontal line test.

The general strategy for finding a functions inverse is to switch the place of x and y in the equation, and then solve for y. The result, if it exists, will give you an inverse for the original function (on a possibly smaller domain). Graphically, the inverse of a function is a reflection of the function across the line $y = x$. If a function is not bijective, we can "fix" it by making its domain or codomain smaller. For any function there are several ways to do this, but there are some agreed upon conventions that I'll outline below.

- The function f defined by $f(x) = x^2$ has domain $\mathbb{R}$, but is not invertible: the line $y = 4$ intersects the graph in two places, $(-2, 4)$ and $(2, 4)$. However, we can can change both its domain and codomain to be $[0, \infty)$. To find its inverse, solve the equation $x = y^2$ for y. We get $y = \pm\sqrt{x}$, but since we are changing the domain of f to only the non-negative real numbers, we forget about the negative square root. Then the inverse of f is given by $f^{-1} = \sqrt{x}$, and for any non-negative x and y ($x, y \in [0, \infty)$),

$$f^{-1}(f(y)) = f(x^2) = \sqrt{(x)^2}) = x$$

and

$$f(f^{-1}(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y.$$

- The function sin has domain $\mathbb{R}$ and range $[-1, 1]$ but is **periodic** (meaning its values cyclically repeat) so cannot be invertible without an adjustment to its domain. The smaller domain we choose is $[-\pi/2, \pi/2]$ and the modified codomain is its range, $[-1, 1]$. The inverse of sin is called arcsin or $\sin^{-1}$ and has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

- The function $f(x) = e^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. If we shrink the codomain $\mathbb{R}$ to be equal to the range, then its inverse is given by $f^{-1}(x) = \ln(x)$ and domain $(0, \infty)$ and range $\mathbb{R}$.

Look up the modified domains and codomains for the other inverse trig functions.

Part II
Limits and Continuity

Limits are important in calculus since we need very small quantities to effectively study change. Yeah, that's pretty vague, but hopefully it'll clear up in a few pages.

We study limits for many reasons. One of them is to study functions at points where they are only "close" to being defined. Another is to make quantities very (arbitrarily) small, without making them 0.

Here is an example of a practical use of limits. The following two functions' graphs are exactly the same, except at $x = 1$.

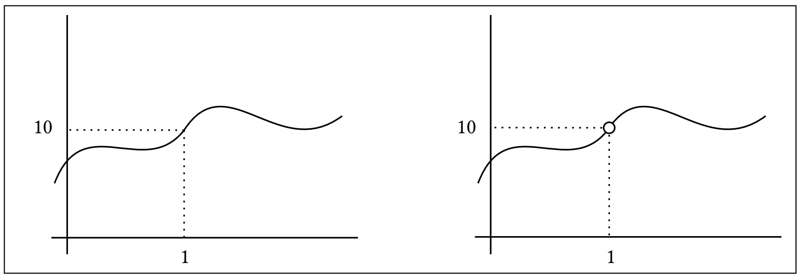


Figure 7: Two functions with the same limits

The first function is defined at $x = 1$, while the second function is not. However, the values of the second function are near 10 when x is near 1. There is still something useful we can say about the second function at 1: "the *limit* at 1 is 10."

3 Definitions

3.1 The Limit

The intuitive idea of a **limit** is to examine what the output of a function does as the input moves close to a specific value. We say that the limit of f at a is L if $f(x)$ moves closer to L as as x moves closer to a. Note that f need not be defined at a to find the limit of f as x approaches a (in fact, that's the whole point!).

Since the functions we care about take real numbers as inputs, there are two ways you can approach a number on the real line (from the left and from the right). We will talk about left- and right-sided limits.

If f is defined on an interval to the *left* of a (with a as an endpoint) and the values of $f(x)$ approach L as x approaches a from the *left*, we say that the "**left-sided limit of f as x approaches** a is L" and we write

$$\lim_{x \to a^-} f(x) = L.$$

If f is defined on an interval to the *right* of a (with a as an endpoint) and the values of $f(x)$ approach L as x approaches a from the *right*, we say that the **right-sided limit of f as x approaches** a is L" and we write

$$\lim_{x \to a^+} f(x) = L.$$

If the right and left sided limits match, then we say that the **limit of f as x approaches a is** L and write

$$\lim_{x \to a} f(x) = L.$$

If the left and right sided limits, do not match, then we say that the limit **does not exist** (DNE).

One can compute $f(x)$ for values of x that get closer and closer to either side of a. If the values approach L, then you have good reason to believe that L is the (right- and/or left-sided) limit.

x	$f(x)$
$a - 0.1$	$f(a - 0.1)$
$a - 0.01$	$f(a - 0.01)$
$a - 0.001$	$f(a - 0.001)$
$a - 0.0001$	$f(a - 0.0001)$
$a + 0.0001$	$f(a + 0.0001)$
$a + 0.001$	$f(a + 0.001)$
$a + 0.01$	$f(a + 0.01)$
$a + 0.1$	$f(a + 0.1)$

Such calculations, however, cannot prove that a function limits to a specific value.

3.2 Infinite Limits

We can extend the idea of limits outside of the real numbers to include positive and negative infinity in place of both a and L.

- If $f(x)$ grows without bound as x approaches a from the left, then we write

$$\lim_{x\to a^-} f(x) = \infty.$$

  (Similarly for x approaching a from the right, and also if $f(x)$ becomes increasingly negative without bound).

- We denote the value (if such a value exists) that $f(x)$ approaches as x approaches ∞ as

$$\lim_{x\to\infty} f(x).$$

  (Similarly if x approaches $-\infty$).

3.3 Optional: The $\varepsilon - \delta$ Definition

You may be dissatisfied with the intuitive approach to limits, so we can make the definition more rigorous. The main idea that needs to be captured by a formal definition is *arbitrary precision*. That is, we need a way to say formally that "$f(x)$ approaches L as x approaches a."

By controlling the input x, we must be able to make the distance $|f(x) - L|$ between the output $f(x)$ and L to be as small as we want ("arbitrarily small"). In other words, if $\varepsilon > 0$ is any small positive number, we must be able to ensure (by controlling x) that $|f(x) - L| < \varepsilon$. To control x, we can make the distance $|x - a|$ between x and a smaller than some positive number $\delta > 0$ (that may depend on ε).

Now we're ready for the real "$\varepsilon - \delta$" definition of a limit: we say that L **is the limit of f as x approaches a** if

$$\text{for all } \varepsilon > 0,\ \text{there is some } \delta > 0,\ \text{such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Read that definition a few times, because statements with multiple quantifiers can be tricky!

Figure 8 is the classic picture you should try to commit to memory, or at very least, thoroughly understand. Here is how you should think about it:

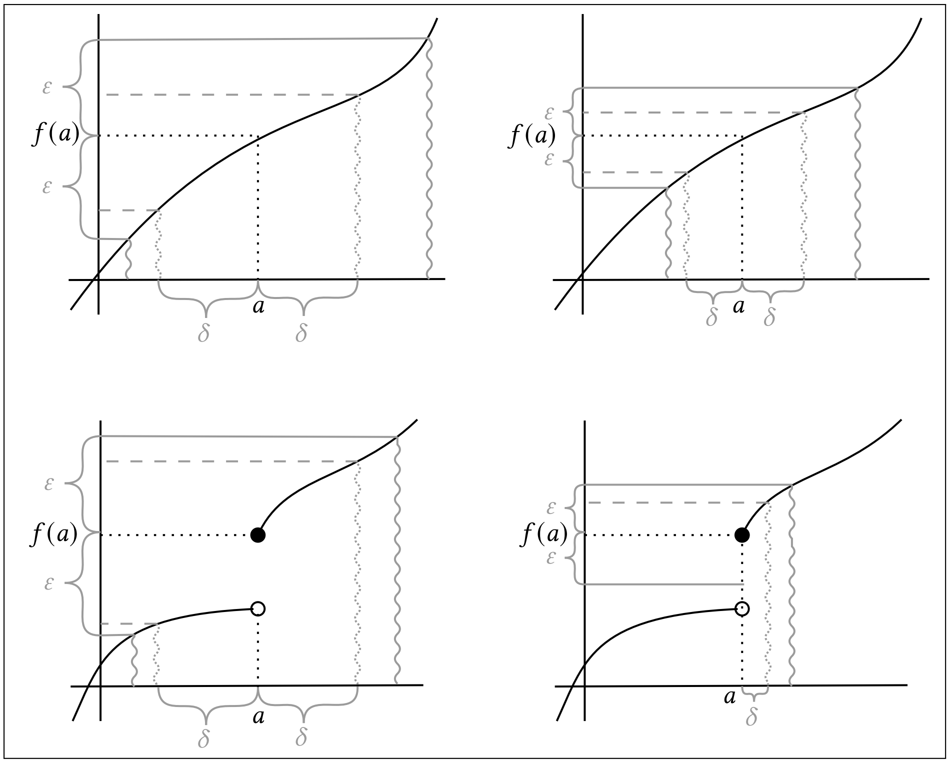


Figure 8: A cartoon of the $\varepsilon - \delta$ definition of a limit

1. Let ε define an interval around $f(a)$ on the y-axis.
2. Then trace the interval right until it hits the graph of the function (solid gray line), and wherever the solid gray line meets the function, draw the solid squiggly line down to meet the x-axis.
3. Choose δ small enough so that when the reverse action is done (trace the δ interval up with the dotted squiggly line to the function and then left with the dashed line to the y-axis), the resulting interval is contained within the ε interval.

While the function graphed in the top two plots has a limit at a, the function that is graphed in the bottom two plots does not. For this function the first value of ε (graph on the left) is not small enough to detect the jump discontinuity, but on the second (graph on the right), the ε given is so small that no value of δ can be big enough to ignore jump.

It may be helpful to think about the definition of a limit as a game/conversation between two people, Alex and Blake. Alex is trying to claim that the limit of f as $x \to a$ is L, and Blake is doing their best job to contest it. Here is how their conversation might go:

A: I think the limit of $f(x) = 3x + 1$ as x approaches 2 is 7.

B: Well if you think so, can you ensure that $|f(x) - 7| < \frac{1}{10}$?

A: Yes! If we take x such that $|x - 2| < 1/30$, then

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < 3(1/30) = 1/10.$$

In this example, $\varepsilon = 1/10$ and $\delta = 1/30$. However, if we want to really prove that the limit is 7, we let $\epsilon > 0$ be arbitrary, and take $\delta = \varepsilon/3$. Then for any x such that $|x - 2| < \delta$,

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2| < 3(\varepsilon/3) = \varepsilon.$$

Prove that the limit of $x^2 + 1$ at 1 is 2.
Hint: let $\delta = \min(\varepsilon/3, 1/10)$.

For simple examples, it is pretty easy to work backwards and figure out what δ should be, but for more complicated functions f, it can be more difficult. Here is a Desmos example that shows dynamically how δ can depend on ε.

3.4 Continuity

Intuitively, a function is continuous if you can draw its graph without lifting up your pencil. We can use limits to write a more precise definition: a function f is **continuous at** a if three conditions are satisfied:

(a) f is defined at a (i.e. $f(a)$ makes sense)

(b) $\lim_{x \to a^+} f(x) = f(a)$

(c) $\lim_{x \to a^-} f(x) = f(a)$

In other words, a function is continuous at a point if its value and limit match at that point.

If U is a subset of $\mathbb{R}$ and f is continuous at every point in U, then we say that f is **continuous on** U.

In the definition above, if (b) and/or (c) is not true (or one of the sided limits doesn't exist), then f is **discontinuous at** a. Here are three types of discontinuities that you may encounter: if f is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists and is a real number, (but $f(a) \neq \lim_{x \to a} f(x)$).
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and are real numbers, but are different.
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Draw a picture of each kind of discontinuity.

Note that if a is not in the domain of f, then f is neither continuous or discontinuous at a.

4 Properties of Limits

4.1 Limit Laws

There are several *limit laws* that will be useful for computing limits. They can all be proven using the $\varepsilon - \delta$ definition of the limit, but you can just believe them.

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, let c be a constant, and let n be a positive integer.

Constant Law	$\lim\limits_{x\to a} b = b$
Identity Law	$\lim\limits_{x\to a} x = a$
Sum law	$\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = L + M$
Difference law	$\lim\limits_{x\to a}(f(x)-g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = L - M$
Constant Multiple law	$\lim\limits_{x\to a} cf(x) = c \cdot \lim\limits_{x\to a} f(x) = cL$
Product law	$\lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = L \cdot M$
Quotient law	$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$ for $M \neq 0$
Power law	$\lim\limits_{x\to a}(f(x))^n = \left(\lim\limits_{x\to a} f(x)\right)^n = L^n$
Root law	$\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd,
	and for $L \geq 0$ if n is even and $f(x) \geq 0$.

Write out what each limit law means in plain English.

The first two laws should be obvious. The last two laws are special cases of the following theorem.

Theorem 4.1 (Composite Function Theorem)**.** If $f(x)$ is continuous at L and $\lim\limits_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

4.2 Using Continuity

You should believe that the following types of functions are continuous at every point in their domains:

- polynomials
- rational functions
- trig and inverse trig functions
- exponential functions
- logarithms

An **algebraic combination of continuous functions** (ACCF) is a sum, product, exponentiation, or composition of continuous functions.

Theorem 4.2. Any ACCF is continuous on its domain.

To finish, another very useful theorem:

Theorem 4.3 (The Intermediate Value Theorem)**.** Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

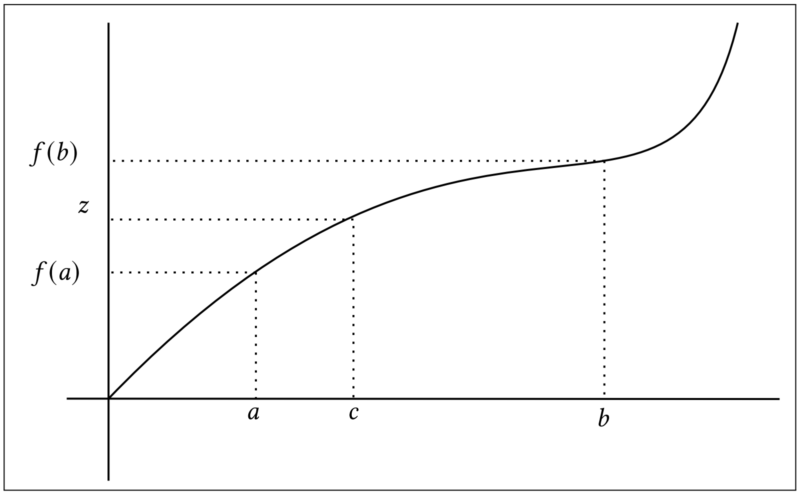


Figure 9: A Cartoon of the Intermediate Value Theorem

Phrased properly, the Intermediate Value Theorem is very intuitive. If you imagine the x-axis is time and the y-axis is position, the theorem says if you start at location $f(a)$ and end at location $f(b)$, then for any location z between $f(a)$ and $f(b)$, there must have been some time that you were at z (as long as you can't teleport!).

5 Evaluating Limits

Now that you understand the definitions and some properties of limits, we will focus on how to evaluate a limit (algebraically) when you come across one:

$$\lim_{x \to a} f(x).$$

There are two main types of functions that you might be asked to evaluate limits for. The first is an algebraic combination of continuous functions (ACCF). This means any sum, product, composition, etc. of simple functions like trig, exponential, polynomial, etc. The second type is a piece-wise function composed of type 1 functions.

The process of evaluating both types are the same, except for at boundary points of the piece-wise components. At these points, you must compare right- and left-sided limits. For every other point (and also any point for an ACCF) use the flow chart in figure 10. The following sections will explain the three possible outcomes.

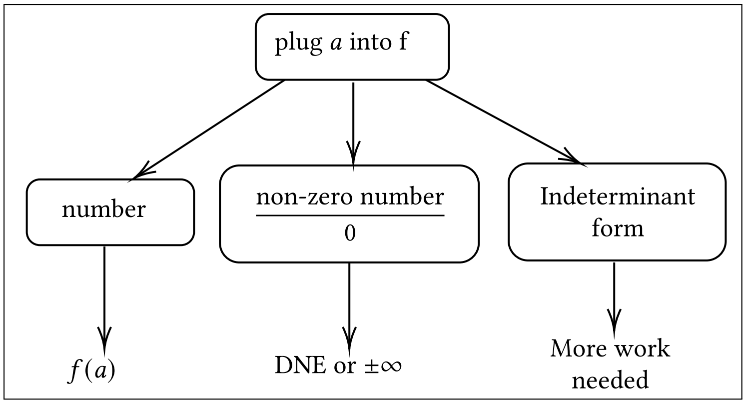


Figure 10: Flowchart for evaluating limits

5.1 A Number

If f is an ACCF and $f(a)$ is a number, then a is in the domain of f. Since any ACCF is continuous on its domain,

$$\lim_{x \to a} f(x) = f(a)$$

Re-read this until you understand it!

by the definition of continuity.

5.2 A (non-zero) Number over Zero

If f is an ACCF and attempting to evaluate $f(a)$ results in a non-zero number over 0, then f has a vertical asymptote at $x = a$. Near $x = a$, the values of $f(x)$ get very close to either positive or negative infinity. The right and left sided limits are either positive or negative infinity, so there are only four cases, as seen in figure 11.

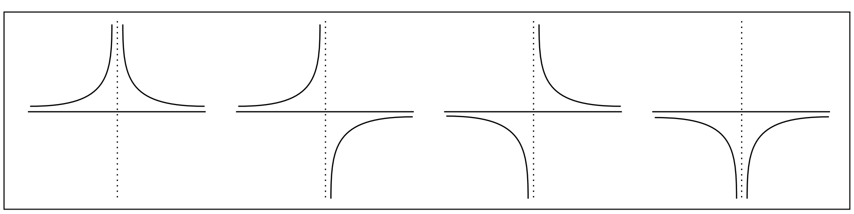


Figure 11: The four cases of vertical asymptotes

Label each function in figure 11 with its respective limit.

Therefore, the only possibilities for the value of these limits are ∞, $-\infty$, and DNE. To figure it out which one it is, evaluate both of the two sided limits. For each, plug in a value that is slightly to the right (or the left) of a. If the result is positive, the sided limit is positive infinity, and if the result is negative, negative infinity.

5.3 Indeterminant Form

There are seven indeterminant forms:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0^0 \qquad \infty-\infty \qquad 1^\infty \qquad 0\cdot\infty \qquad \infty^0$$

They are called **indeterminant** because in this form, we cannot be sure of its value (or even if its value makes sense). Take 0^0 for example: zero raised to any power should be 0, but anything raised to the power of 0 should be 1... so which is it?

Write out similar question for the other 6 indeterminant forms.

When we get an indeterminant form after trying to evaluate $f(a)$, we need to do more work. There are several things you can try:

- Simplify. If the function is rational and has common factor in its numerator and denominator, by "canceling" the term, the resulting function is not the same: the original function has a hole at the point where the factored term is 0. For example, $f(x) = \frac{x(x-1)}{(x-1)}$ has a hole at $x = 1$, but everywhere else is the line $y = x$. However, this cancellation does not change the value of the limit.

- Multiply the numerator and denominator of a rational function by the conjugate of the denominator or the numerator (the **conjugate** of a binomial $a + b$ is $a - b$).

- Use a special-case limit:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$$

- In the case of $x \to \pm\infty$ for a rational function, divide the numerator and denominator by the highest degree term of the denominator.

- Use the Squeeze Theorem:

  Theorem 5.1 (Squeeze Theorem)**.** Let f, g, and h be functions with $g(x) \leq f(x) \leq h(x)$ for all x in some interval around a (except possibly at a). If

  $$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

  then

  $$\lim_{x\to a} f(x) = L$$

  as well.

  The most common application of squeeze theorem is squeezing

  $$-1 \leq \sin\theta \leq 1.$$

Later, we will learn a trick called L'Hôpital's Rule to more easily compute limits with indeterminant forms. But first we need to learn about derivatives!

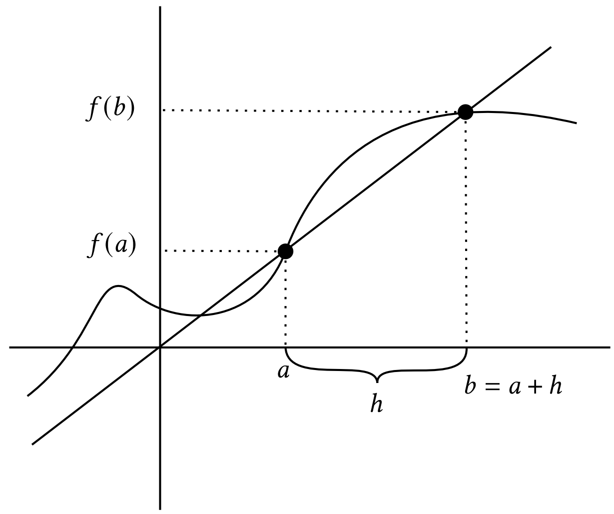
Part III
Derivatives and Applications

6 The Derivative

6.1 The Limit Definition

Recall that a **secant line** is a line between two points on the graph of a function. If f is a function and $(a, f(a))$ and $(b, f(b))$ are *different* points on the function, then the secant line that they determine has equation

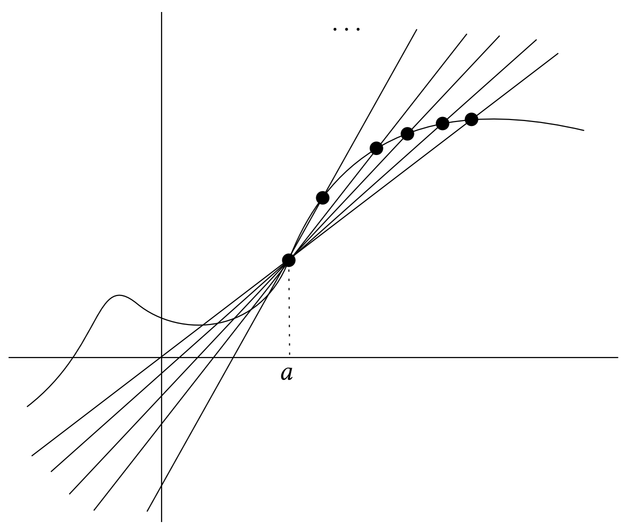
$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$



We can assume $b > a$ and write $b = a + h$ for some $h > 0$. Then the equation for the secant line through $(a, f(a))$ and $(b, f(b)) = (a + h, f(a + h))$ is

$$y = f(a) + \frac{f(a + h) - f(a)}{h}(x - a).$$

Now we're going to use *limits* to move one point $(b, f(b))$ closer to the other $(a, f(a))$, and see what happens to the secant line. Since the secant line passes through the point $(a, f(a))$, we only have to track what happens to the slope. As b moves to a, the value of h goes to 0, so the slope of the secant line approaches



$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

We'll denote this quantity $f'(a)$, and call this the **derivative of f at a** (if this limit exists). The line through the point $(a, f(a))$ with slope $f'(a)$ is called the **tangent line of f at a** and has equation

$$y = f(a) + f'(a)(x - a).$$

You'll notice that the tangent line captures some *local information* about f at a. In other words, it is a good approximation of f at a; that is, f and its tangent line are "similar" at a. Intuitively, you should think about the tangent line at a to be the line that "hugs f the best at a."

Note that this definition of the derivative only makes sense if f is defined on an open interval containing a. We can view the derivative as a function, and write $[f(x)]'$ or $f'(x)$. The derivative of $f'(x)$ is called the **second derivative** and is denoted $f''(x)$. In general, the nth derivative $f^{(n)}(x)$ is defined to be the derivative of $f^{(n-1)}(x)$.

6.2 The Two Part "Program" for Finding Derivatives

So now we have a goal: find derivatives of functions. The problem is that the limit definition is difficult to use (that pesky $h \to 0$ in the denominator). So instead, we will do the following two steps (let a be a real number and f and g be functions):

1. rules to break functions up and reassemble:

The Scalar Multiple Rule	$[af]' = af'$.
The Sum Rule	$[f+g]' = f' + g'$
The Product Rule	$[fg]' = f'g + fg'$
The Quotient Rule	$\left[\frac{f}{g}\right]' = \frac{f'g-fg'}{g^2}$
The Chain Rule	$[f(g(x))]' = f'(g(x))g'(x)$
The Inverse Rule	$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$

2. find derivatives of "basic" functions:

Constant Rule	$[a]' = 0$
Power Rule	$[x^a]' = ax^{a-1}$
Trig Rules	$[\sin(x)]' = \cos(x)$
	$[\cos(x)]' = -\sin(x)$
Exponential Rules	$[a^x]' = \ln(a)a^x$
Logarithmic Rules	$[\log_a(x)]' = \frac{1}{x\ln(a)}$

Most of the time, you will be able to find the derivatives using a combination of these rules. For instance, the derivatives of the other 4 trig functions can be found using the quotient rule and the derivatives of sin and cos:

$$[\tan(x)]' = \sec^2(x) \qquad [\sec(x)]' = \sec(x)\tan(x)$$

$$[\cot(x)]' = -\csc^2(x) \qquad [\csc(x)]' = -\csc(x)\cot(x)$$

Try to find a pattern with these four derivatives.

6.3 Implicit Differentiation

We'll refer to the set of points that satisfy an equation of two variables as a **curve in the plane**. Some examples you might have seen before are circles and ellipses. The circle with center (h,k) and radius r is the set of points (x,y) in the plane that satisfy the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

Look up the similar equation for an ellipse and write it down here.

In general, curves are not functions (they might not pass the vertical line test), but they can still have tangent lines, so we should be able to describe their slope at a point. We do this by "zooming in" on a point of the curve, and forgetting about the other parts of the curve. Now (locally), it looks like a function, so we can treat the y variable as a function of x. This process is called "implicit" differentiation, and consists of two steps:

1. differentiate both sides of the equation (keeping in mind that since y is a function of x, the derivative of $f(y)$ is $f'(y) \cdot y'$ by the chain rule).

2. solve for y'.

Other notation is sometimes used: if $y = f(x)$, $\frac{dy}{dx} = \frac{d}{dx}[f(x)]$ is the derivative of f, and $\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}[f(x)]$ is the nth derivative of f.

When you solve for y', it is possible that the expression will consist of both y and x variables. That is because the formula for y' is not a function of just x, it is a function of both x and y!

Let's do an example:

- The curve of points satisfying the equation $y^3 = x^2 - xy - 1$ is not a function, since it does not pass the vertical line test: both $(-1, -1)$ and $(-1, 1)$ are solutions. However, we can still find the derivative at each point on this curve! Start by taking the (x) derivative of both sides.

  On the left, the x derivative of y^3 is $3y^2y'$, by the power rule (and the chain rule). The derivative of x^2 is $2x$, the derivative of xy is $y+xy'$ by the product rule (and chain rule), and the derivative of 1 is 0, of course. Comparing the derivatives of each side of the equation, we get

  $$3y^2y' = 2x - (y + xy'),$$

  so solving for y', we have

  $$y' = \frac{2x - y}{3y^2 + x}.$$

  At $(-1, 1)$, $y' = \frac{-3}{2}$, and at $(-1, -1)$, $y' = \frac{-1}{2}$. The tangent line to the curve at the point $(-1, 1)$ is $y = \frac{-3}{2}(x + 1) + 1$

6.4 Parametric Curves

A **parametric curve** is a curve defined by two coordinate functions $(x(t), y(t))$. You should think of (x, y) as the position of some particle and the particle's coordinates $x(t)$ and $y(t)$ are functions of time.

To find the slope (AKA derivative) of a parametric curve at time t, simply compute

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

An example:

- The derivative of the curve $(x, y) = (\sin(t), t^2)$ is

  $$\frac{y'(t)}{x'(t)} = \frac{2t}{\cos(t)}.$$

  The equation of the tangent line at $t = \pi$ is $y = \pi^2 + \frac{2\pi}{-1}(x - 0)$.

6.5 Logarithmic Differentiation

Sometimes you will be faced with a function whose formula includes an expression involving x raised to a power that also involves x. Finding the derivative of such a function has a specific procedure:

Strategy: Logarithmic Differentiation

Suppose $y = f(x)^{g(x)}$ and we want to find y'. First apply the ln function to both sides of the equation and use the log property $\ln(a^b) = b\ln(a)$:

$$\ln(y) = \ln(f(x)^{g(x)}) = g(x)\ln(f(x)).$$

Then implicit differentiation yields

$$\frac{1}{y}y' = g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)},$$

and solving for y' and substituting $y = f(x)^{g(x)}$ we get

$$y' = f(x)^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right).$$

Here are some examples:

- $[x^x]' = x^x(\ln(x) + 1) = \ln(x^{x^x}) + x^x$. (Here $f(x) = g(x) = x$).
- $[\sin(x)^x]' = \sin(x)^x(\ln(\sin(x)) + x\cot(x))$. (Here $f(x) = \sin(x)$ and $g(x) = x$).

Work these examples out.

I don't recommend that you memorize the formula for $[f(x)^{g(x)}]'$. Instead, remember the procedure and re-create when you need it.

An alternate method for logarithmic differentiation is to transform $f(x)^{g(x)}$ into a form more easily differentiable by first taking the logarithm and then exponentiating:

$$f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\ln(f(x))}.$$

Take the derivative of $e^{g(x)\ln(f(x))}$ and see that it matches the expression for y' in the strategy box.

6.6 Using The Inverse Rule

When we want to find the derivative of an inverse of a function f, the inverse rule tells us that

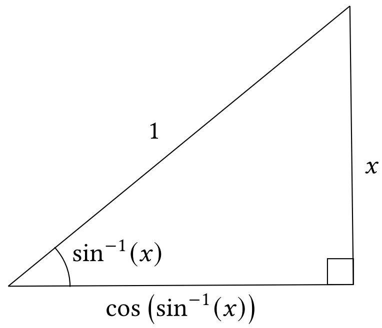
$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))},$$

but how is this helpful? It is actually very easy to see why this rule is true:

$$\begin{aligned}
& f(f^{-1}(x)) = x && \text{(definition of inverse)}\\
\implies & f'(f^{-1}(x))[f^{-1}(x)]' = 1 && \text{(chain rule)}\\
\implies & [f^{-1}(x)]' = \frac{1}{f'(y)} && \text{(solve for } [f^{-1}(x)]'\text{)}\\
\implies & [f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))} && \text{(substitute)}
\end{aligned}$$

Let's use this rule to find the derivative of inverse trig functions:

- The rule says that $[\sin^{-1}(x)]' = \frac{1}{\cos(\sin^{-1}(x))}$, but we should be able to simplify this expression using the unit circle.



Argue this using the Pythagorean identitiy ($\sin^2\theta + \cos^2\theta = 1$) instead.

if a (non-right) angle of a right triangle (with hypotenuse 1) is $\sin^{-1}(x)$, the leg opposite $\sin^{-1}(x)$ has length x. Then $\cos(\sin^{-1}(x))$ is the length of the other leg of the triangle, so $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ by the Pythagorean theorem. Therefore,

$$[\sin^{-1}(x)]' = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Figure these equations out by yourself using the same method as for $\sin^{-1}$. Hint: refer to Figure 2

- The derivatives of the other inverse trig functions are

$$[\cos^{-1}(x)]' = \frac{-1}{\sqrt{1-x^2}}$$
$$[\csc^{-1}(x)]' = \frac{-1}{|x|\sqrt{x^2-1}}$$
$$[\sec^{-1}(x)]' = \frac{1}{|x|\sqrt{x^2-1}}$$
$$[\tan^{-1}(x)]' = \frac{1}{1+x^2}$$
$$[\cot^{-1}(x)]' = \frac{-1}{1+x^2}$$

7 Applications of Derivatives

7.1 Linear Approximation

This first application of derivatives is nothing too new! We first described the tangent line of a function f as the line that "hugs f the best at a" and now we are going to take of advantage of that fact.

The main idea is that if we have a function f and we know $f(a)$ and $f'(a)$ for some a, then we can use the tangent line of f at a to estimate values of a function near a. That is, if $a \approx b$, then

$$f(b) \approx f(a) + f'(a)(b - a).$$

When is this approximation an overestimate? Underestimate?

The right side of this approximation is exactly the tangent line of f at a, evaluated at b.

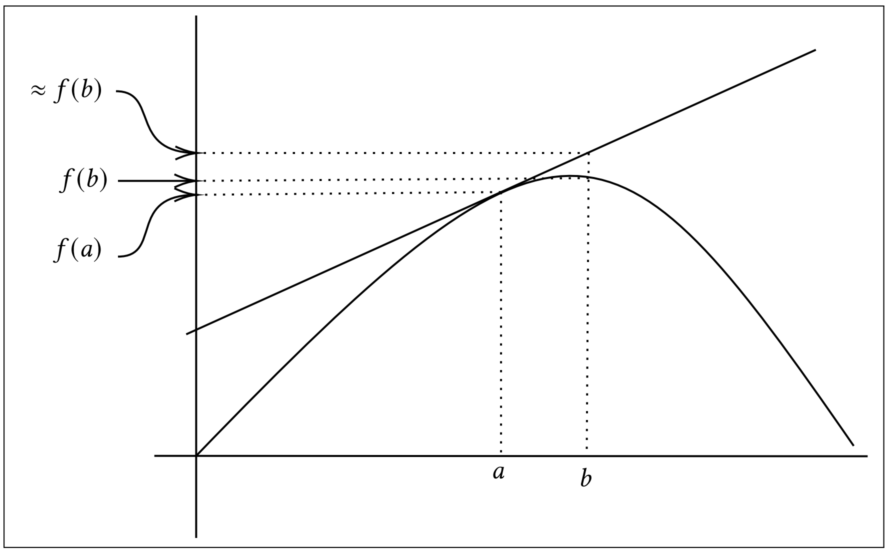


Figure 12: Linear Approximation

Let's pretend we don't have a calculator, and try to estimate $\sqrt{4.1}$. In this case $f(x) = \sqrt{x}$, and we should set $a = 4$, since $4.1 \approx 4$ and we can figure out $f(4)$ and $f'(4)$. By the power rule, $f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(4) = \frac{1}{2}(4^{-1/2}) = \frac{1}{4}$, and of course $f(4) = \sqrt{4} = 2$. Using the formula above,

$$\sqrt{4.1} = f(4.1) \approx f(4) + f'(4)(4.1 - 4) = 2 + \frac{1}{4}(0.1) = 2.025.$$

If we check a calculator for a better approximation, we find that

$$\sqrt{4.1} \approx 2.02484567313,$$

Estimate $\sqrt{3.9}$ and compare to the actual value.

so we weren't that far off!

7.2 L'Hôpital's Rule

L'Hôpital's Rule is a way to easily tackle some of those limit problems that were in an indeterminate form, namely $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Theorem 7.1. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or ∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

as long as all of these limits are defined.

What this means for you: if you are trying to take the limit of a rational function and after plugging in you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then take the derivative of the numerator and the denominator, and try to evaluate the limit again.

It is possible that one use of L'Hôptial's rule won't be enough: you may have to use it multiple times. Here are some examples:

- $\lim_{x\to 0} \frac{\sin(x)}{x} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$
- $\lim_{x\to\infty} \frac{3x^2+1}{4x^2+3x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{6x}{8x+3} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$

Sometimes you will encounter an indeterminant form that is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and in these cases, we can transform it into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hôptial's rule. Here are some illustiative examples:

- $\lim_{x\to 1} \frac{1}{x-1} - \frac{1}{\ln(x)} = \lim_{x\to 1} \frac{\ln(x) - x + 1}{(x-1)\ln(x)} \overset{\text{LH}}{=} \lim_{x\to 1} \frac{1/x - 1}{\ln(x) + 1 - \frac{1}{x}}$
  $\overset{\text{LH}}{=} \lim_{x\to 1} \frac{-1/x^2}{1/x + 1/x^2} = -\frac{1}{2}.$
- $\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{\text{LH}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$
- $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = e^{\lim_{x\to 0^+} \ln(x^x)} = e^{\lim_{x\to 0^+} x\ln(x)} = e^0 = 1.$

Label each of these examples with original and trasnsformed type of indeterminant forms they demonstrate.

7.3 Kinematics

Kinematics is the study of motion. In this section, we'll concern ourselves with only one basic situation: an object moving in a straight path (only forwards and backwards).

Suppose the position of an object at time t is given by $x(t)$. The object's **velocity** at time t is given by

$$v(t) = x'(t)$$

since velocity is the rate at which position is changing. Similarly, the object's **acceleration** at time t is given by

$$a(t) = v'(t) = x''(t),$$

since acceleration is the rate at which velocity is changing.

How do we interpret these quantities? One way is to think about **speed**, which is the absolute value of velocity. In other words, if you are moving backward at 2 m/s, then your velocity is −2 m/s and you speed is 2 m/s, but if you are moving forward at the speed 2 m/s, then your speed and velocity are both 2 m/s.

What is an example of a position function whose velocity is positive?

7.4 Related Rates

As we saw in kinematics, the time-derivative of a function tells us the rate at which the function is changing. So, if we have two related quantities A and B that are changing over time, we can find the rate at which A is changing at time T if we know the rate at which B is changing at time T.

Note: sometimes the following different notations are used for the time derivative of A at time T:

$$A'(T) = \left.\frac{dA}{dt}\right|_{t=T} = \dot{A}(T)$$

Here is the general problem-solving strategy:

Strategy: Related Rates

1. Draw a picture.
2. Label the relevant quantities (including A and B).
3. Find an equation that relates A and B (and nothing else).
4. Implicitly differentiate the equation with respect to time t. (Remember that A and B are functions of t, so use the chain rule!)
5. Solve for $A'(t)$ in terms of $A(t)$, $B(t)$, and $B'(t)$
6. Plug in $t = T$.

7.5 Qualities of Graphs

In this section we will understand how derivatives tell us qualitative information about of graphs of functions. To start,

$$f \text{ is } \begin{cases} \text{increasing at } x & \text{if } f'(x) > 0 \\ \text{decreasing at } x & \text{if } f'(x) < 0 \end{cases}$$

and

$$f \text{ is } \begin{cases} \text{concave up at } x & \text{if } f''(x) > 0 \\ \text{concave down at } x & \text{if } f''(x) < 0. \end{cases}$$

You already know what **increasing** and **decreasing** mean. The graph of a function that is **concave up at** x looks like a bowl near x, and the graph of a function that is **concave down at** x looks like a hill near x. Points on a graph where concavity changes (from up to down, or down to up) are called **inflection points**.

Draw a sketch of an inflection point.

Theorem 7.2 (The Mean Value Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$.

In plain English, the mean value theorem says that the average value of the slope on an interval is always attained on that interval.

7.6 Extremal Points

A function f has a

$$\textbf{local} \begin{cases} \textbf{maximum} \\ \textbf{minimum} \end{cases} \text{at } a \text{ if } \begin{cases} f(a) \geq f(x) \\ f(a) \leq f(x) \end{cases} \text{for all } x \text{ in some interval } I \ni a;$$

$$\textbf{global} \begin{cases} \textbf{maximum} \\ \textbf{minimum} \end{cases} \text{at } a \text{ if } \begin{cases} f(a) \geq f(x) \\ f(a) \leq f(x) \end{cases} \text{for all } x \text{ in the domain of } f.$$

Any such point is called an **extremal point** of f.

Global maximum *values* are unique, but points at which they occur might not be. For example, $\sin(x)$ has global maximum value of 1, even though this value occurs infinitely many times on the domain of $\sin(x)$. All global extremal points are local extremal points.

The point a is called a **critical point** of f if $f'(a) = 0$ or f is not differentiable at a ($f'(a)$ doesn't exist). The following theorem tells us how find extremal points using critical points.

Theorem 7.3. If a is an extremal point of f, a is a critical point of f.

Name and draw a function with a non-extremal critical point.

WARNING: The converse is not true! (i.e. not all critical points are extremal!)

Theorem 7.4 (First Derivative Test)**.** Suppose a is a critical point of f.

$$\text{If } f' \begin{cases} \text{changes from } (+) \text{ to } (-) \\ \text{changes from } (-) \text{ to } (+) \\ \text{doesn't change sign} \end{cases} \text{at } a\text{, then } a \text{ is} \begin{cases} \text{a local maximum of } f. \\ \text{a local minimum of } f. \\ \text{not a extremal point.} \end{cases}$$

Theorem 7.5 (Second Derivative Test)**.** Suppose $f'(a) = 0$ and f'' is continuous at a.

$$\text{If } \begin{cases} f''(a) > 0 \\ f''(a) < 0 \end{cases} \text{then } a \text{ is a} \begin{cases} \text{local minimum} \\ \text{local maximum} \end{cases} \text{of } f.$$

Strategy: Finding Extremal Points

1. Solve $f'(x) = 0$ for x.
2. Find other critical points (points of non-differentiability such as end-points of the domain, cusps, peaks, and points of discontinuity).
3. If you are looking for *local* extremal points, use the first or second derivative tests as necessary to classify points the critical points you found.
4. If you are looking for *global* extremal points, evaluate f at each critical points to find the largest and smallest.

7.7 Optimization

"Optimization"-style questions will ask you to optimize (maximize or minimize) a certain quantity A under specified constraints. The constraints will still allow one variable x of change. This means you will need to find the value of x for which A is maximal or minimal.

The strategy for optimization problems is very similar to the strategy for related rates, but uses the theory of extremal points from the previous section.

Strategy: Optimization

1. Draw a picture.
2. Label the relevant quantities.
3. Find an equation for the quantity that you want to optimize A in terms of the thing that you can change x (and note the domain of A).
4. Find the extremal points of A.

7.8 Newton's Method for Finding Roots

Imagine that you have an function f and you want to find the zeros (or roots) of f. That is, you want to find the values of x such that $f(x) = 0$. **Newton's Method** is a computational algorithm that can sometimes be used to approximate such a value. First, we'll describe the algorithm, and then we'll show how to do use your TI calculator to implement it.

Imagine that x_0 is an initial guess of a root of the function f. If we can take the derivative of f, then we know that the tangent line of f at x_0 is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

and so we find (setting $y = 0$ and solving for x) this tangent line intersects the x-axis at the point $(x_1, 0)$, where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We continue this, setting

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

and so on:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In many cases, this sequence of numbers should approach a zero of f. Here is the intuitive reason why: (1) if x_n is not a zero of f, then x_{n+1} moves in the direction of a zero, and (2) if x_n is a zero of f, then $x_{n+1} = x_n$.

The second point is obvious, but let's look into the first point. If $f(x_n)$ is not 0, then we have two cases:

- $x_n < x_{n+1}$ (move to the right). Algebraically, this occurs exactly when $f(x_n)$ and $f'(x_n)$ have opposite sign (when f is increasing below the x-axis, or decreasing above the x-axis).
- $x_{n+1} < x_n$ (move to the left). Similarly, this occurs exactly when $f(x_n)$ and $f'(x_n)$ have the same sign (when f is increasing above the x-axis, or decreasing below the x-axis).

Draw a picture showing the four cases.

Seems reasonable, right? In either case we try to move in the direction of a root.

Now, how do we use Newton's method? Get out your TI calculator, and type the function f into Y$_1$=, and f' into Y$_2$=. Then return to the main calculator screen and type

$$0 \rightarrow A$$

(or replace the 0 with a different initial guess x_0) and click ENTER. Then type

$$A - Y_1(A)/Y_2(A) \rightarrow A$$

and click ENTER. The variable A is now set as x_1, and this value should be displayed. Click ENTER again and A is now x_2. Continue clicking ENTER to find x_n for higher n (clicking 100 times will show you x_{100}).

Although Newton's method is generally an easy algorithm to implement and use, it does not always work.

Consider $f(x) = x^3$ and try to use Newtons method to detect the root 0 with the initial guess $x_0 = 1$.

Part IV
Integration and Applications

8 The Definite Integral

For reasons that will not be immediately obvious, integration is opposite differentiation on the 'calculus coin.'

8.1 The Definite Integral

The **definite integral of f on (a, b)** is written

$$\int_a^b f(x)\,dx$$

and is defined to be the *signed* area between the graph of f and the x-axis (if such a quantity exists).

When might this quantity not exist?

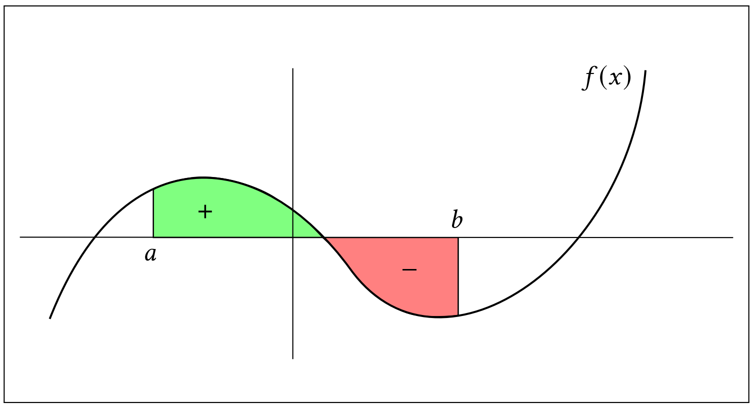


Figure 13: The Geometric Definition of the Definite Integral

Compute

$$\int_{-3}^{7} 2x + 4\,dx$$

using geometry.

The figure shows the meaning of the word "signed": areas that are above the x-axis are counted as positive, while those below are negative.

With no other tools available to you, the only way to compute definite integrals right now is to draw the graph of f and find the desired area using your knowledge of geometry. This can be easy if f is made up of lines and circular arcs, but if not, the way to compute a definite integral is not immediately obvious.

8.2 Reimann Sums

The first way that we will attempt to compute definite integrals is with a limit of a Riemann sum. We will start by approximating the area by a collection of vertical rectangles. Refer to the figure as an example as we construct a Riemann sum.

Divide the interval (a, b) into n equal sub-intervals, each with width $\Delta x = \frac{b-a}{n}$. Then for each sub-interval draw a rectangle with base length Δx on the x-axis and height determined by the value of f at the right endpoint of the sub-interval. The right endpoint of the kth sub-interval is $x_k = a + k\Delta x$, so the area of the kth rectangle is $f(x_k)\Delta x$. The area of all n rectangles together is

$$\sum_{k=1}^{n} f(x_k)\Delta x.$$

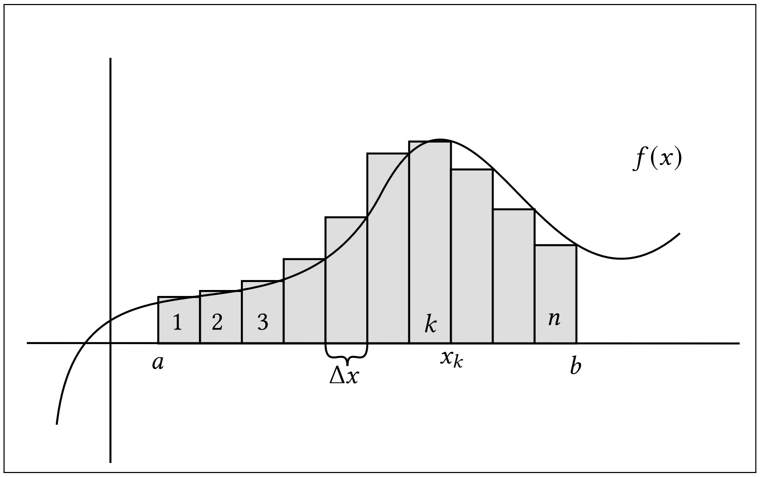


Figure 14: A Right Reimann Sum

Since we used the *right* endpoint of each sub-interval, we call this quantity the **nth Right Riemann Sum**:

$$R_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}.$$

We could also have used the *left* endpoint of each sub-interval to determine the height of each box, in which case we would get the **nth Left Riemann Sum**:

Draw a picture of L_n.

$$L_n = \sum_{k=1}^{n} f\left(a + (k-1)\frac{b-a}{n}\right)\frac{b-a}{n}.$$

If instead we use trapezoids above each sub-interval, we get the **nth Trapezoid Riemann Sum**:

Draw a picture of T_n.

$$T_n = \sum_{i=k}^{n} \frac{f\left(a + (k-1)\frac{b-a}{n}\right) + f\left(a + k\frac{b-a}{n}\right)}{2}\frac{b-a}{n} = \frac{R_n + L_n}{2},$$

where the kth trapezoid has width Δx and heights determined by the left and right endpoints of the kth sub-interval. (Remember that a trapezoid with width w and heights h_1 and h_2 has area $A = \frac{h_1+h_2}{2}w$.

When are R_n, L_n, and T_n overestimates? Underestimates?

In practice, it doesn't matter which you choose to compute because of the following theorem:

Theorem 8.1. With the notation as above,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} T_n = \int_a^b f(x)\,dx.$$

This theorem should be intuitive: as n gets larger, the width of each rectangle gets smaller and the approximation gets better. More abstractly, this tells us that integration and summation are intimately related: you can think of a definite integral as a sum of infinitely thin boxes with height $f(x)$ and width dx, one for "each" value of x in (a, b). This perspective will be very helpful later.

Note the similarity in notation between a definite integral and the Riemann sum

$$\sum_{k=1}^{n} f(x_k)\Delta x.$$

For large n, a Riemann sum can give a very good approximation, but in general it is not feasible to do by hand. However, if f is a polynomial it is not hard to compute R_n algebraically and then take a limit. In order to do so, we'll need the following summation identities:

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Here is an example:

- To compute $\int_0^1 x^2 + x\,dx$, first find R_n:

$$\begin{aligned}
R_n &= \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n} \\
&= \sum_{k=1}^{n} \left(\left(0 + k\frac{1-0}{n}\right)^2 + \left(0 + k\frac{1-0}{n}\right)\right)\frac{1-0}{n} && \text{(plug in)} \\
&= \sum_{k=1}^{n} \left(\frac{k^2}{n^2} + \frac{k}{n}\right)\frac{1}{n} && \text{(expand and simplify)} \\
&= \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n^2}\sum_{k=1}^{n} k && \text{(factor)} \\
&= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^2}\frac{n(n+1)}{2} && \text{(use sum identities)}
\end{aligned}$$

Then take the limit:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{2}{6} + \frac{1}{2} = \frac{5}{6}.$$

> Compute
> $$\int_0^1 x^3\,dx$$
> using the method of Riemann Sums.

8.3 The Fundamental Theorem of Calculus

Let's cut right to the chase:

> **Theorem 8.2** (The Fundamental Theorem of Calculus)**.**
> $$\int_a^b f(x)\,dx = F(b) - F(a),$$
> where $F' = f$.

This theorem is the connection between differentiation and integration! It is also how we will compute definite integrals from now on:

1. Find a function F for which $F' = f$.
2. Evaluate $F(b) - F(a)$.

The first point is called **anti-differentiation**, and is the harder step. Before we get to that, let's try to understand how this theorem agrees with our intuition about Riemann Sums. If we take the b-derivative of the equation in the theorem, we get

$$\frac{d}{db}\left[\int_a^b f(x)\,dx\right] = \frac{d}{db}\left[F(b) - F(a)\right] = f(b).$$

This means that the rate at which the area is changing is $f(b)$. This makes sense in terms of Riemann sums: if we increase b by a small amount Δb, the area will change by about $f(b)\Delta b$. Thus the derivative of the area is $f(b)\Delta b/\Delta b = f(b)$.

> Draw a picture of b increasing by Δb. Then sit and really think about this until it makes sense.

9 The Indefinite Integral

Since the Fundamental Theorem of Calculus tells us that we have to understand anti-derivatives in order to compute definite integrals, this chapter will focus on this task.

9.1 Anti-differentiation

If $F' = f$, we call F an **anti-derivative** (or **indefinite integral**) of f and write

$$F(x) = \int f(x)\,dx \quad \text{or} \quad F = \int f.$$

Check out this XKCD Comic.

Unfortunately, such a function F does not always exist.

The first thing to understand is that unlike derivatives, anti-derivatives are not unique! This is because constant terms are always killed by differentiation. For example, the derivative of x^2 and x^2+1 are both $2x$, so both are worthy anti-derivatives for $2x$. To solve this problem, we consider *the* anti-derivative of f to be the collection of *all* functions whose derivative is f. These functions only differ by a constant so we write "$+C$" at the end of *an* anti-derivative to denote *the* anti-derivative. For example, $\int 2x\,dx = x^2 + C$.

Similarly to differentiation, we will build up a two-part collection of tools to tackle integration problems.

1. rules to break functions up and reassemble:

Scalar Multiplication	$\int af = a\int f.$
Sum	$\int f + \int g = \int f + \int g$
Integration by Parts	$\int f'g = fg - \int fg'$
u-**substitution**	$\int f'(g(x))g'(x)\,dx = f(g(x))$

2. anti-derivatives of "basic" functions:

Power Rule	$\int x^a\,dx = \begin{cases} \frac{1}{a+1}x^{a+1} + C & a \neq -1 \\ \ln\|x\| + C & a = -1 \end{cases}$
Trig Rules	$\int \sin(x)\,dx = -\cos(x) + C$
	$\int \cos(x)\,dx = \sin(x) + C$
Exponential Rules	$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C$

Which differentiation rules do Integration by Parts and u-substitution correspond to?

Each of the integration rules are true because of a corresponding rule for derivatives.

Strategy: u-substitution

1. When presented with $\int f'(g(x))g'(x)\,dx$, set $u = g(x)$.
2. Then $du = g'(x)\,dx$.
3. Substitute: $\int f'(g(x))g'(x)\,dx = \int f'(u)\,du$.

The main idea of u-substitution is "change the variable to make the integral easier."

Some examples:

- By setting $u = \sin(x)$, we get $du = \cos(x)\ dx$, so

$$\int \sin^2(x)\cos(x)\ dx = \int u^2\ du.$$

- By setting $u = x + 1$, $du = dx$, so

$$\int x\sqrt{x+1} = \int (u-1)\sqrt{u}\ du.$$

9.2 Partial Fraction Decomposition

When attempting to integrate a rational function, partial fraction decomposition is a common trick that comes in handy. It effectively breaks a rational function into a sum of rational functions with lower degree denominators.

Factor in denominator	Term in decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Do you see the pattern? What terms would you add for a quartic factor in the denominator?

While this table is not exhaustive, it covers the range of complexity required in a calculus class.

Strategy: Partial Fraction Decomposition

1. Factor the denominator of the original rational expression.
2. Set the original expression equal to the sum of appropriate terms (see table).
3. Write the sum as a single rational expression by finding a common denominator.
4. Use the two numerators to solve a system of equations for the unknown values, A, B, C, etc.

9.3 Trigonometric Substitution

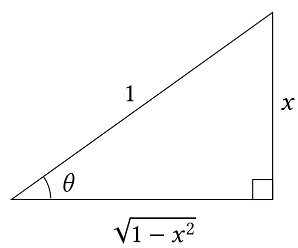
Another useful integration trick is Trig Substitution. The moral of this method is to replace a square-root containing a quadratic with an equivalent trigonometric expression that is easier to integrate. There are only three common examples of this. The table below summarizes them.

Integrand	Substitution	Result
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a\tan\theta$

- As an example, consider $\int \sqrt{1-x^2}\, dx$. Set $x = \sin\theta$, so $dx = \cos\theta\, d\theta$. Then we can do the change of variables:

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-(\sin\theta)^2}\cos\theta\, d\theta = \int \cos^2\theta\, d\theta.$$

This is explained geometrically by drawing the following triangle.



Draw the triangles for the other two examples.

10 More Definite Integrals

10.1 Improper Integrals

Improper integrals are integrals involving infinity. There are two types:

1. $\pm\infty$ is in a bound of the integral, or
2. the function being integrated has a vertical asymptote on the interval of integration.

To evaluate an improper integral, rewrite the integral as a limit of a proper integral, compute the integral as normal, and then take the limit. When evaluating integrals of the second type, make sure you use a sided limit from the inside of the bound of integration.

If the value of the improper integral is a finite quantity, we say the integral **converges**. If the value is an infinite quantity or the limit DNE, then we say the integral is **divergent**. Sometimes you might be asked the nature of the divergence (i.e. is the limit equal to $\pm\infty$ or non-existent).

Here's an example of each type:

- $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left[\frac{-1}{x}\right]_1^a = \lim_{a\to\infty}\left(1-\frac{1}{a}\right) = 1.$
- $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\frac{-1}{x}\right]_a^1 = \lim_{a\to 0^+}\left(\frac{1}{a}-1\right) = \infty.$

For what values of p does

$$\int_0^1 x^p\,dx$$

converge?

Many times, a vertical asymptote won't be one of the endpoints of the interval. When this happens, split the integral in two, so the VA now lies on the endpoints of the intervals. Here is an example:

- $\displaystyle\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx.$

When the convergence of an integral is all you care about, the following theorem can be helpful.

Theorem 10.1. If $f(x) \le g(x)$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

In particular, this means that if $\int_a^b f(x)\,dx$ diverges, then $\int_a^b g(x)\,dx$ diverges, and if $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.

To finish the section, here is an algebraic property of integrals that can come in handy:

Theorem 10.2. The following are always true if each integral exists and is finite.

1. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$
2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

Use 1 to prove 2.

10.2 Even and Odd Functions

A function f is called **even** if $f(-x) = f(x)$, and is called **odd** if $f(-x) = -f(x)$. Even functions are symmetric about the y-axis, and odd functions are rotationally symmetric around the origin. Therefore, if f is odd,

$$\int_{-a}^{a} f(x)\,dx = 0$$

since the areas bounded by f to the left and right of $x = 0$ differ only in sign. Similarly, if f is even,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

since the areas bounded by f to the left and right of $x = 0$ are the same. Here are some examples:

- $\int_{-1}^{1} \sin(x)\,dx = 0$ since $\sin(-x) = -\sin(x)$.
- $\int_{-1}^{1} \cos(x)\,dx = 2\int_{0}^{1} \cos(x)\,dx$ since $\cos(-x) = \cos(x)$.

11 Applications of Integrals

Many applications of integrals use the following heuristic idea: you have a value A that you want to compute; you break the problem up into infinitesimal pieces and compute the corresponding infinitesimal value "dA"; then sum up the small values by integrating

$$A = \int dA.$$

Understanding integrals as sums will make understanding these integral applications easier. The first example of this you will see is computing areas.

Try to find all such examples.

11.1 Average Values

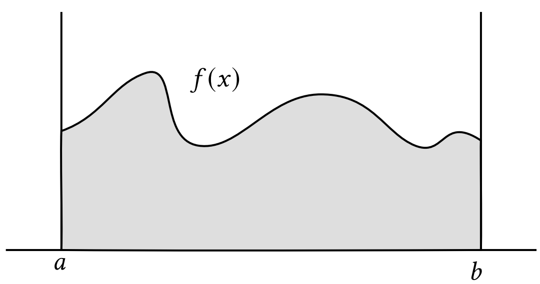
The average value of f on $[a, b]$ is

$$\frac{\int_a^b f(x)\,dx}{b-a}.$$

There are various intuitive explanations of this; here are two.

11.1.1 Water in a tank

Imagine water is sloshing around in a tank and the height of the water at a particular time is described by a function $f(x)$ (see the figure below).



The average value of $f(x)$ on $[a, b]$ is the height of the water once it settles. To find this height, find the area of the water $\left(\int_a^b f(x)\,dx\right)$ and divide by the width of the tank $(b - a)$.

11.1.2 Using the FTC and Average Slope

Let F be an antiderivative of f. Then the average value of $f(x)$ on $[a, b]$ is equal to the average value of F' on $[a, b]$. We know how to calculate this using the slope of a secant line:

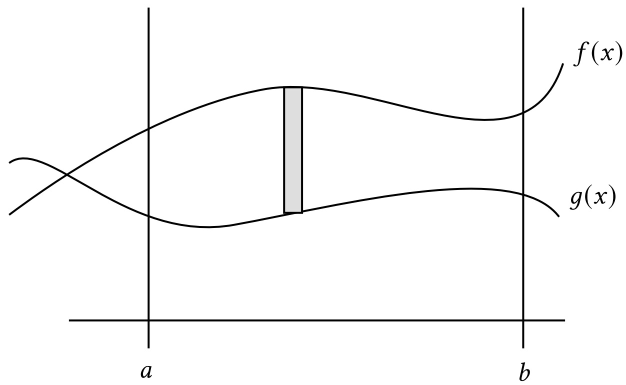
$$\frac{F(b) - F(a)}{b-a} \overset{FTC}{=} \frac{\int_a^b f(x)\,dx}{b-a}.$$

Use Reimann sums to get another explanation.

11.2 Areas

To compute the area A between $f(x)$ and $g(x)$ on $[a, b]$, imagine dividing up the area into infinitesimally small strips, each with height $f(x) - g(x)$ and (infinitesimal) width dx. The area of the strip is therefore $dA = (f(x) - g(x))\,dx$. One such strip is shown on the graph below (the width enlarged to be visible). To compute the desired area, simply integrate dA over $[a, b]$:

$$A = \int dA = \int_a^b (f(x) - g(x))\,dx.$$



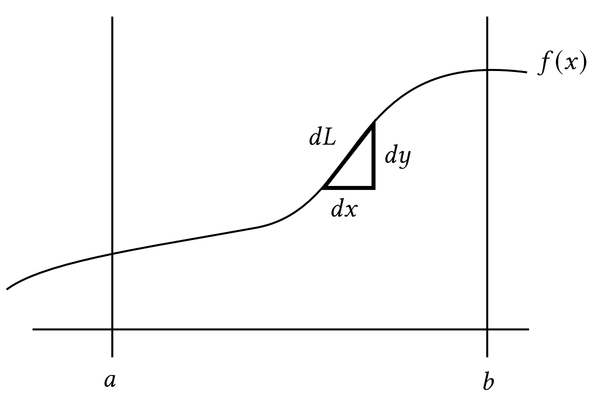
11.3 Arc Length

The **arc length** of $f(x)$ on $[a, b]$ is the length of string needed to trace the graph of f on the interval $[a, b]$. Similarly to computing areas, to compute the arc length L of a function f's graph, divide the length into infinitesimally short segments (that we assume to be linear). Then each segment's length can be computed using the pythagorean theorem (see picture below):

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx = \sqrt{1 + f'(x)^2}\,dx.$$

Then to compute the arc length L, sum dL over $[a, b]$ by integrating:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx.$$



11.4 Probability

Imagine we have an experiment, and every time it is performed the outcome is a real number. (Think throwing a dart at the plane, and the outcome is the x coordinate of the point it lands on).

To start, the we will define an **event** to be a subset of $\mathbb{R}$. Given such a subset $A \subset \mathbb{R}$, our goal is to describe the probability that the outcome X is in A. We denote this $\mathbb{P}(X \in A)$.

For the kinds of experiments we will consider, we will assume that it is never possible to get the same outcome twice. This means that the probability of any *particular* outcome is 0: for any $a \in \mathbb{R}$,

$$\mathbb{P}(X \in \{a\}) = \mathbb{P}(X = a) = 0.$$

This might seem paradoxical, because the dart must hit *some* point. However, it should make sense since there are infinitely many possible outcomes.

Think about this carefully.

To compute $\mathbb{P}(X \in A)$ in general, we need a way to describe an allocation of probability specific to the experiment that is happening. For this (of course) we use a a function! A bounded function $f : \mathbb{R} \to \mathbb{R}$ is called a **probability density function** (PDF) if it is non-negative and $\int_{\infty}^{\infty} f(x)\, dx = 1$. To compute a probability, we just have to integrate:

What does the condition

$$\int_{\infty}^{\infty} f(x)\, dx = 1$$

mean in terms of probability?

$$\mathbb{P}(X \in A) = \int_A f(x)\, dx.$$

This notation means that if A is the union of disjoint intervals, integrate over each interval and add the results. In particular, if $A = (a, b)$, then

$$\mathbb{P}(X \in A) = \int_a^b f(x)\, dx.$$

The anti-derivative function

$$F(x) = \mathbb{P}(X < x) = \int_{-\infty}^{x} f(t)\, dt$$

is called the **cumulative distribution function** (CDF). Here are some examples:

- The "uniform" PDF on $(0, 1)$ is defined as

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \qquad \text{and} \qquad F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

  is its CDF. The probability that an outcome of an experiment governed by this PDF is greater than $1/4$ is

$$\mathbb{P}\left(X \in (1/4, \infty)\right) = \int_{\frac{1}{4}}^{\infty} f(x)\, dx = \int_{\frac{1}{4}}^{1} 1\, dx + \int_{1}^{\infty} 0\, dx = \frac{3}{4}.$$

  The probability that an outcome is in $(0, 1/2) \cup (2/3, 1)$ is

$$\mathbb{P}\left(X \in (0, 1/2) \cup (2/3, 1)\right) = \int_0^{\frac{1}{2}} 1\, dx + \int_{\frac{2}{3}}^{1} 1\, dx = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

- The "exponential" PDF is defined as

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \ge 0 \end{cases} \qquad \text{and} \qquad F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \ge 0 \end{cases}$$

is its CDF. This PDF is often used to model the wait time between phone calls. The probability that an outcome of an experiment governed by this PDF is greater than 1/4 is

$$\mathbb{P}\left(X \in (1/4, \infty)\right) = \int_{\frac{1}{4}}^{\infty} e^{-x}\, dx = e^{-\frac{1}{4}}.$$

Draw a picture of these PDFs and CDFs.

11.5 Volumes and Surface Area

11.6 Moments and Centroids

11.7 Work

Part V
Sequences and Series

12 Sequences

A **sequence** of is an ordered and infinite list. A sequence of real numbers can be thought of as a function whose domain is the positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$ and codomain is $\mathbb{R}$. For example,

$$a = (1, 4, -7, \pi, 0, \dots)$$

is a sequence where $a_2 = a(2) = 4$. Sometimes a sequence is written (a_n).

Sequences can be defined by a formula in terms of the index n, or with a recursive formula, by specifying the first few terms, and then defining the subsequent terms using the previous. For example:

- The sequence a defined by $a_n = n^2$ is $(1, 4, 9, 16, 25, \dots)$.
- The sequence a defined by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ is the Fibonacci sequence $(1, 1, 2, 3, 5, 8, 13, 21, \dots)$.

The limit of a sequence a_n can be evaluated just as the limit of a function $f(n) = a_n$. It is written

$$\lim a_n = \lim_{n\to\infty} a_n.$$

A sequence (a_n) is **bounded** if there is some B for which $-B < a_n < B$ for all n.$Asequenceis$**monotonic**$ifitisalways$

13 Series

13.1 Introduction

Let (a_n) be a sequence, and define a new sequence (s_k) of "partial sums" by

$$s_k = \sum_{n=1}^{k} a_n = a_1 + \cdots + a_k.$$

We write

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \sum_{n=1}^{\infty} a_n = \sum a_n$$

if this limit exists, and call $\sum a_n$ a **series**. We are interested in when this quantity is finite:

- If $\lim\limits_{k\to\infty} s_k$ is finite, we say that $\sum a_n$ is **convergent** (or that $\sum a_n$ converges).
- When $\lim\limits_{k\to\infty} s_k$ is infinite, we say that $\sum a_n$ is **divergent** (or that $\sum a_n$ diverges).

There are two special cases when we can actually find the value of a series:

1. A series of the form $\sum\limits_{n=k}^{\infty} r^n$ is called a **geometric series** and is convergent if and only if $|r| < 1$. In that case,
$$\sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r}.$$
For example,
$$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{-1}{3}.$$

2. A **telescoping series** is one where all but finitely many terms cancel. To evaluate these series, write out several terms of the sum, and see which terms are cancelled. Many times partial fractions will come in handy. For example,
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1.$$

Cross out terms that cancel. What does $\frac{1}{4}$ cancel with?

13.2 Tests for Convergence and Divergence

The following theorems are all used to determine when a series is convergent. Note: Nothing in this section will tell you the *actual value* of a series.

Theorem 13.1 (Divergence Test)**.** If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ will diverge.

While reading forward, look for a counter-example for the converse.

Be careful: the converse of this theorem is not true! (If the limit of a_n is 0, that does not mean that $\sum a_n$ converges). Some examples:

- $\lim \frac{2n}{n+1} = 2 \neq 0$, so $\sum\limits_{n=1}^{\infty} \frac{2n}{n+1}$ diverges.
- $\lim n = \infty \neq 0$, so $\sum\limits_{n=1}^{\infty} n$ diverges.
- $\lim\, (-1)^n$ DNE, so $\sum\limits_{n=1}^{\infty} (-1)^n$ diverges.

Theorem 13.2 (Integral Test)**.** Suppose that $f(x)$ is a continuous, positive, and decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$. Then
$$\int_k^{\infty} f(x)\, dx \text{ is convergent} \iff \sum_{n=k}^{\infty} a_n \text{ is convergent.}$$

- $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges since $\int_2^{\infty} \frac{1}{x\ln(x)}\, dx$ diverges.
- $\sum\limits_{n=7}^{\infty} \frac{1}{n\ln(n)^2}$ converges since $\int_7^{\infty} \frac{1}{x\ln(x)^2}\, dx = \frac{1}{\ln(7)} < \infty$.

Theorem 13.3 (The p-series Test).
If $k > 0$, then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

- $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since $p = 3 > 1$.
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since $p = 1$.
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since $p = \frac{1}{2} < 1$.

Theorem 13.4 (Comparison Test). If $0 \le a_n \le b_n$ for all n, then

$$\sum b_n \text{ converges} \implies \sum a_n \text{ converges}$$

and (by contraposition)

$$\sum a_n \text{ diverges} \implies \sum b_n \text{ diverges.}$$

- $\sum_{n=1}^{\infty} \frac{\sin(n)^2}{n^3} \le \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.
- $\sum_{n=1}^{\infty} \frac{n^2+10}{n^3} \ge \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 13.5 (Limit Comparison Test). Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n \ge 0$ and $b_n > 0$ for all n. Define

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

If c is positive and finite, then either both series converge or both series diverge.

- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges since $\lim \frac{1/\sqrt{n}}{1/\sqrt{n+1}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.
- $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges since $\lim \frac{1/n^2}{1/(n+1)^2} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Theorem 13.6 (Alternating Series Test)**.** Suppose that we have a series $\sum a_n$ and either

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n+1} b_n$$

where $b_n \geq 0$ for all n. Then if,

1. $\lim_{n\to\infty} b_n = 0$ and
2. $\{b_n\}$ is a decreasing sequence

the series $\sum a_n$ is convergent.

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges since $\lim \frac{1}{n} = 0$ and $f(x) = \frac{1}{x}$ is a decreasing function on $[1, \infty)$ (since $f'(x) = \frac{-1}{x^2} < 0$ for all $x > 0$) .

Theorem 13.7 (Absolute Convergence Test)**.**

$$\sum |a_n| \text{ converges } \implies \sum a_n \text{ converges.}$$

If $\sum |a_n|$ converges, then $\sum a_n$ is called **absolutely convergent**. If not, $\sum a_n$ is called **conditionally convergent**.

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Theorem 13.8 (Ratio Test)**.** Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

- $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges since $L = \lim \left| \frac{1/(n+1)!}{1/n!} \right| = \lim \left| \frac{1}{n+1} \right| = 0 < 1$.
- $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges since $L = \lim \left| \frac{(n+1)!/2^{n+1}}{n!/2^n} \right| = \lim \left| \frac{n+1}{2} \right| = \infty > 1$.
- The ratio test is inconclusive for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ since $L = \lim \left| \frac{1/(n+1)^2}{1/n^2} \right| = 1$.

Theorem 13.9 (Root Test)**.** Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

13.3 Estimating Sums

Theorem 13.10. If f is a positive, continuous, and decreasing function on $[1, \infty)$ with $f(n) = a_n$, then for any N,

$$\left(\sum_{n=1}^{N} a_n + \int_{N+1}^{\infty} f(x)\, dx\right) \le \sum_{n=1}^{\infty} a_n \le \left(\sum_{n=1}^{N} a_n + \int_{N}^{\infty} f(x)\, dx\right).$$

In other words, if you estimate that the value of the entire sum $\sum_{n=1}^{\infty} a_n$ by $\sum_{n=1}^{N} a_n$, you would be off by a number between $\int_{N+1}^{\infty} f(x)\, dx$ and $\int_{N}^{\infty} f(x)\, dx$. In *other* other words,

$$\int_{N+1}^{\infty} f(x)\, dx \le \sum_{n=N+1}^{\infty} a_n \le \int_{N}^{\infty} f(x)\, dx.$$

Theorem 13.11. Suppose $S = \sum_{n=1}^{\infty} b_n(-1)^n$ where $b_n \ge 0$, and b_n decreases to 0. Then for any N,

$$|S - S_N| \le b_{N+1}$$

where $S_N = \sum_{n=1}^{N} b_n(-1)^n$.

In other words, the sum of the first N terms in an alternating series is different from the entire sum by at most the $N + 1$st term. This means that for any odd N,

$$\sum_{n=1}^{N} b_n(-1)^n \le \sum_{n=1}^{\infty} b_n(-1)^n \le \sum_{n=1}^{N+1} b_n(-1)^n.$$

14 Power Series

14.1 General Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where x is a variable. This means that the series itself is a function of x, so we can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

For some values of x, the power series will converge, and for others it will diverge. In other words, the domain of f is the set of $x \in \mathbb{R}$ for which $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges. The main topic in this section is to figure out the domain of f, which is called the **interval of convergence**.

The tool we use to find the interval of convergence is the ratio test: define

$$L(x) = \lim \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n} \right| = |x-c| \cdot \lim \left| \frac{a_{n+1}}{a_n} \right|.$$

We know that the power series converges at the values of x for which $L(x) < 1$ and diverges when $L(x) > 1$. That is, the domain of f contains every value of x for which

$$|x-c| < \lim \left| \frac{a_n}{a_{n+1}} \right|.$$

Write out this inequality as an interval. I.e.

$$x \in (\dots).$$

The value c is called the **center**, and $\lim \left| \frac{a_n}{a_{n+1}} \right|$ is called the **radius of convergence**. Since the ratio test doesn't say anything about when $L(x) = 1$ (the endpoints of the interval of convergence), we will need to test those cases separately. Here are some examples:

- $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(x-1)^n$

Use the ratio test and compute

$$L(x) = |x-1| \cdot \lim \left| \frac{(-1)^{n+1}}{n+1} \frac{n}{(-1)^n} \right| = |x-1|.$$

Then $L(x) < 1$ (the series converges) when $x \in (0,2)$. Now we test the endpoints $x = 0, 2$ directly:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(0-1)^n = \sum_{n=0}^{\infty} 1$$

diverges, and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(2-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

converges. Therefore $x = 2$ is included in the interval of convergence, but $x = 0$ is not. The interval of convergence is $(0,2]$, the radius of convergence is 1 and the center is $x = 1$.

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Compute

$$L(x) = |x| \cdot \lim \left| \frac{1}{(n+1)!} \frac{n!}{1} \right| = |x| \cdot \lim \left| \frac{1}{n+1} \right| = 0.$$

Then $L(x) < 1$ (the series converges) for all $x \in R$. Therefore, the interval of convergence is $(-\infty, \infty)$, the radius of convergence is ∞ and the center is $x = 0$.

- $f(x) = \sum_{n=0}^{\infty} n!x^n$. Compute

$$L(x) = |x| \cdot \lim \left| \frac{(n+1)!}{n!} \right| = |x| \cdot \lim |n+1| .$$

Then $L(0) = 0 < 1$ and $L(x) = \infty$ (the series diverges) for all $x \neq 0$. Therefore, the interval of convergence is $(0, 0) = \{0\}$, the radius of convergence is 0 and the center is $x = 0$.

14.2 Taylor Series

Previously, you learned that "near $x = a$",

$$f(x) \approx f(a) + f'(a)(x - a).$$

Notice that the left and right hand side agree in their 0th and 1st derivatives at $x = a$. This explains why linear approximation is a good one...but we can make it better: we can make the approximation agree with the $f(x)$ in its 2nd derivative at $x = a$ as well:

Check that the 0th, 1st, and 2nd derivatives of this 2nd degree polynomial approximation agrees with f at $x = a$.

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

Note that the 2 in the denominator is there for power rule to cancel with. In that same spirit, we can continue adding more terms to make the approximation agree with the function at higher derivatives:

$$f(x) \approx f(a)+f^{(1)}(a)(x-a)+\frac{f^{(2)}(a)}{2}(x-a)^2+\frac{f^{(3)}(a)}{2 \cdot 3}(x-a)^3+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Writing this in sumation notation,

$$f(x) \approx \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

This is called the **degree-k Taylor polynomial** for f at a. Note that $f^{(k)}(a)$ denotes the kth derivative of f evaluated at a. Of course we can take k to infinity, and by doing so we will get equality on a particular interval of convergence around $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

This is called the **Taylor series** for f at a. When $a = 0$, the series is called a **Maclaurin series**. The following table lists various common Taylor series.

Function	Taylor Series	Interval of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$\mathbb{R}$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\mathbb{R}$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\mathbb{R}$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\ln(1-x)$	$\sum_{n=1}^{\infty} \frac{-1}{n} x^n$	$(-1, 1)$
$(x+1)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$\mathbb{R}$

Verify the Taylor series in the table.