Cαlculus

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Contents

A Note to Students

This text is meant to serve as a condensed resource for a student taking their first or second semester of calculus. I am writing with the assumption that the reader is familiar with some trigonometry/pre-calculus, but has perhaps forgotten some of it. While statements I write are always true, they will often not be entirely rigorous or stated with full generality. Use the big margins to make your own notes.

I'll leave prompts and questions in boxes like this.

My goal is to communicate what I believe are the key ideas in an introductory calculus class that will help the average student succeed.

Here are my suggestions for doing well in calculus (that should be cyclically repeated!):

1. Spend time fully understanding key concepts.

2. Identify and learn about common types of problems and exercises.

3. Find some really good examples.

4. Practice a lot!

When you begin learning calculus, it is OK not to understand everything right away. It takes time for some concepts to sink in. Doing a lot of practice will help you start developing intuition about how to tackle new problems, even if you don't fully comprehend everything you are doing. Eventually, your problem-solving skills and your abstract understanding will both be strong, but they need to grow together and build off each other. It is good to sit and actually think about something for a while without doing any writing. Keep at it and eventually you will be able to conjure pictures and animations in your head that relate ideas and succinctly encapsulate the idea of a problem or theorem.

P.S.

You might be thinking: "What the &@#$ is calculus?!".

Well, let me save you a trip to Wikipedia. At its core, calculus is the study of change. Its name comes from the Latin word for "pebbles" because the main idea of calculus is that to study things that are changing in non-linear ways, we can break them up into small pieces (like pebbles) that can be thought of as linear, and then put them back together.

P.P.S

Ok... I lied. The pebbles actually refer to the beads on an abacus, but a professor of mine once told me the other version, and I like that better!

Part I
Pre-Calculus

1 Trigonometry

The word "trigonometry" comes from the ancient Greek words for "triangle" and "measure", and it has found its way into just about every possible branch of mathematics and science. You would be putting yourself at a disadvantage not to make sure you are comfortable with at least the basic ideas of trigonometry before continuing to calculus.

You should think about trigonometry in two ways:

1. pragmatically (it is a useful tool for solving problems, and there are some things you should memorize)

2. abstractly (it is really just a different way to talk about similar right triangles)

1.1 Basic Definitions

The **unit circle** is the circle with (unit) radius 1, centered at the origin. When we draw a radius of the unit circle, it forms an angle θ with the positive x axis.

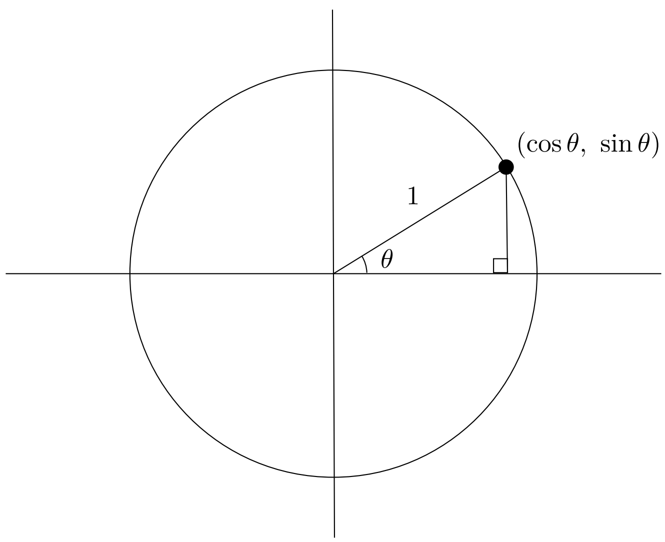


Figure 1: $\sin\theta$ and $\cos\theta$ are coordinates

This line segment always has length 1, and connects the origin to some point on the circle. We define $\cos\theta$ and $\sin\theta$ to be the x and y coordinates of this point. In other words, $\cos\theta$ is defined to be the (signed) length of the horizontal leg of the right triangle and similarly, $\sin\theta$ is defined to be the (signed) length of the vertical leg.

We measure angles in either degrees or radians. The angle θ can be any real number, but there are 2π radians (or 360°) in one full revolution. Imagine the angle θ changing, making the radius sweep around the circle like a SONAR on a submarine. The hypotenuse of the triangle stays constantly at 1, but the side lengths fluctuate between 1 and 0.

> Draw a picture: where on the unit circle is $\cos\theta$ positive and where is it negative? How about $\sin\theta$?

The two functions sin and cos are the basic building blocks, but there are four other commonly used trig functions. They are defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

These functions also have interesting visual representations on the unit circle. To explore these functions, check out this graph on Desmos.

1.2 Values on the Unit Circle

There are several angles θ for which you should be able to compute $\sin\theta$ and $\cos\theta$ (then computing the other four trig functions is easy). Luckily, there is a lot of symmetry involved: it is only necessary to memorize a few numbers and then you can easily fill in the rest.

Below is a diagram of all of the angles whose sin and cos you should be familiar with.

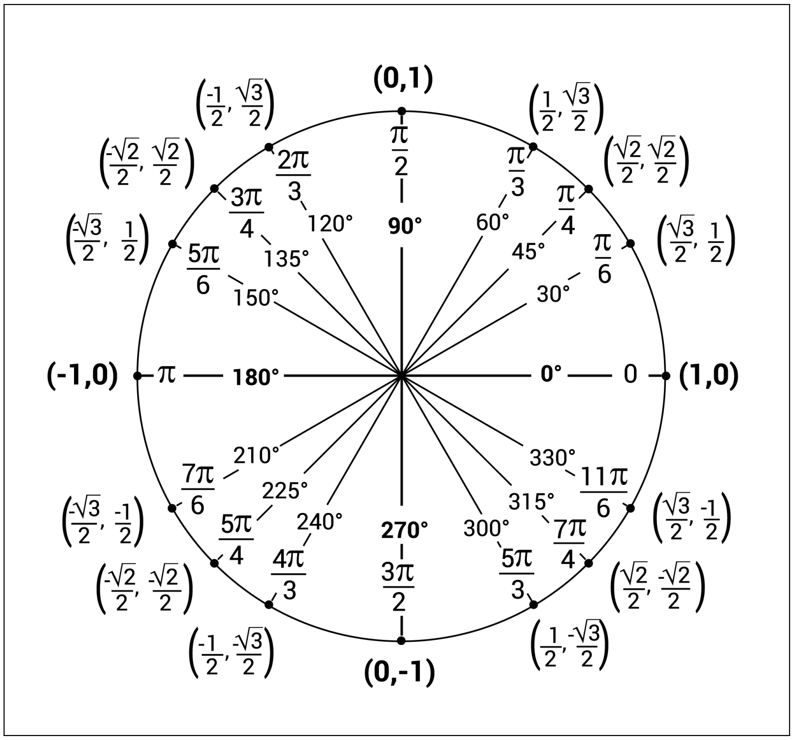


Figure 2: The Unit Circle

It might seem intimidating, but just focus on the first (upper-right) quadrant and the first five angles there:

θ	$\cos\theta$	$\sin\theta$
0	$\sqrt{4}/2$	$\sqrt{0}/2$
$\pi/6$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/2$	$\sqrt{0}/2$	$\sqrt{4}/2$

In this table, I wrote the numbers in an "un-simplified" way so you can see the pattern: their numerators change by 1 under a square-root, and as one goes up, the other goes down.

You can also see that the values for other angles are the same number, but with a different sign (according to the quadrant they are in), and that the unit circle is symmetric, reflecting over the x and y axis.

1.3 The Pythagorean and Other Identities

Remember the Pythagorean Theorem, $a^2 + b^2 = c^2$? Well, since $\sin\theta$ and $\cos\theta$ are the lengths of a right triangle with hypotenuse 1, we have the following **Pythagorean identity**:

$$\cos^2\theta + \sin^2\theta = 1.$$

Dividing this equation by $\sin^2\theta$ or $\cos^2\theta$, we get

$$\cot^2\theta + 1 = \csc^2\theta \qquad \text{and} \qquad 1 + \tan^2\theta = \sec^2\theta.$$

Which of the following are the same?

$$\begin{array}{cc} \sin\theta^2 & \sin^2\theta \\ \sin(\theta^2) & \sin(\theta)^2 \\ (\sin(\theta))^2 & (\sin\theta)^2 \end{array}$$

There is some nice symmetry between $\sin\theta$ and $\cos\theta$. The leg opposite the angle is θ is $\sin\theta$. Since the internal angles of a triangle sum to π, the other acute angle is $\frac{\pi}{2} - \theta$. Then, the leg opposite $\frac{\pi}{2} - \theta$ has length $\sin\left(\frac{\pi}{2} - \theta\right)$, but is also $\cos\theta$. Therefore,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta).$$

Label the angle $\frac{\pi}{2} - \theta$ on Figure 1.1.

Similarly,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta).$$

Looking at the picture of the unit circle, we can also see that

$$\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \cos(-\theta) = \cos(\theta).$$

There is one more kind of identity that will be useful for us:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

These are called the **angle sum formulas**, and they are long but if you read them out loud, they make a little song! Try it while clapping on each word:

"SINE-COSINE-COSINE-SINE. COSINE-COSINE-SINE-SINE."

No seriously, sing the song. Now do it again.

Once you remember the order of the functions, you just have to remember that the As and Bs alternate and that the sign in the second equation flips ($\pm$ becomes $\mp$).

These simplify if A and B are the same angle θ to what are called the **double angle formulas**. In this case,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

and

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1. \end{aligned}$$

Check that these are correct. Hint: use the Pythagorean identity.

1.4 Applications

So, how do we use trigonometry to help solve problems? The main way that trigonometry is useful for you as a calculus student is two-fold. One is as a source of nice function examples to play around with (once we start taking limits and derivatives). The other is as a means to fill in missing information (this is the "SOH-CAH-TOA" you might remember).

Let's say you have a right triangle with hypotenuse 8 and one a 30° angle.

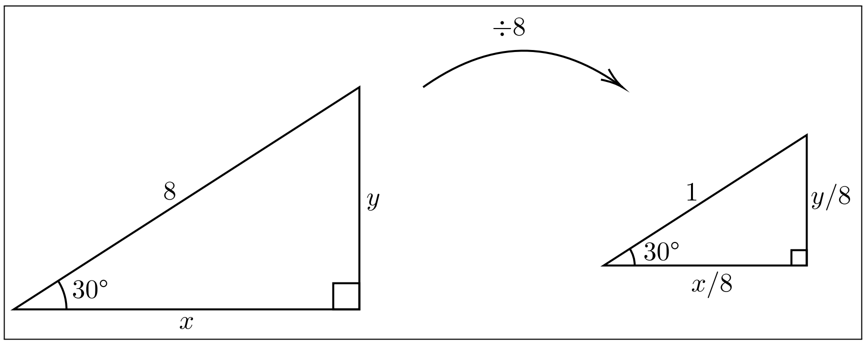


Figure 3: Similar Triangles

If you want to know what the other side lengths are, this is how you can use trigonometry to do it! First, scale the triangle down by the side length that you know (in this case 8). This doesn't change the angles, so we now have a right triangle with hypotenuse 1. By definition of sin and cos,

$$\sin 30^\circ = \frac{y}{8} \qquad \text{and} \qquad \cos 30^\circ = \frac{x}{8},$$

so we can solve for x and y:

$$y = 8\sin 30^\circ = 4 \qquad \text{and} \qquad x = 8\cos 30^\circ = 4\sqrt{3}.$$

What does the "TOA" part say? What situation would it help you figure something out about a triangle?

The mnemonic device "SOH-CAH-TOA" reminds us that Sin of 30° is the Opposite leg (y) over the Hypotenuse (8), and Cos of 30° is the Adjacent leg (x) over the Hypotenuse (8).

The identities that we learned about also have many applications, but mostly are in simplifying calculuations. You should keep them in mind so you recognize when they might be helpful in simplifying an expression. Here is a cool example:

$$\begin{aligned}
\sin(2\theta)\left(\frac{\tan\theta+\cot\theta}{2}\right) &= 2\sin\theta\cos\theta\left(\frac{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}{2}\right)\\
&= \sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)\\
&= \sin^2\theta+\cos^2\theta\\
&= 1.
\end{aligned}$$

Trig identities can also be a computationally useful tool: if you need to know the Sine (or Cosine, etc.) of an angle you are not familiar with, write the angle as a sum or difference of angles you know and use the angle sum formulas. For example,

$$\begin{aligned}\cos(15°) &= \cos(45° - 30°)\\ &= \cos(45°)\cos(30°) + \sin(45°)\sin(30°)\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

Can you compute $\tan(75°)$?

1.5 There's Always More Trigonometry...

Trigonometry dates back to the 3rd century BCE and needless to say it has changed a bit over the years. We've only covered the basics and focused on the trigonometry involving right triangles. There is always more trigonometry to learn: more trigonometric functions and MANY more relations among them! I won't pester you with many more right now, but they might come up in the future, so just be aware.

The two very common laws of trigonometry that involve non-right triangles are the Law of Sines and the Law of Cosines. I will state them here (for culture).

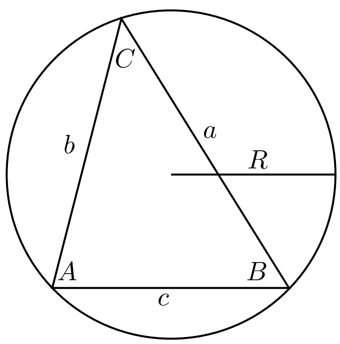


Figure 4: Circumscribed Triangle

Figure 1.5 shows a triangle with internal angles A, B, and C, and respective opposite side lengths a, b, and c. The triangle is **circumscribed** in a circle, meaning its vertices lie on the circle's boundary. It turns out that the circle has radius

$$R = \frac{abc}{\sqrt{(a+b+c)(a-b+c)(a+b-c)(-a+b+c)}}.$$

Theorem 1.1 (Law of Sines)**.** If a triangle has sides and angles as above, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = \frac{abc}{2\Delta},$$

where R is the radius of the circumscribed circle and Δ is the area of the triangle.

Theorem 1.2 (Law of Cosines)**.** If a triangle has sides and angles as above, then

$$a^2 + b^2 = c^2 + 2ab\cos C.$$

Almost looks familiar, doesn't it?

When you use these formulas, keep in mind that solving for an angle can be tricky: if $\sin\theta = 1/2$, then θ could be 30°, 150°, 390°, etc..

2 Functions on the Real Line

2.1 Real Numbers

The set of real numbers ($\mathbb{R}$) is an uncountably infinite set, meaning I can not even begin to exhaustively list all of its members. However, here are some of my favorites: 3, 2/5, 4.11, 713, $\sqrt{37}$, π. Real numbers include both the rationals and irrationals.

Do you know any numbers that aren't real?

The real numbers can be thought of as a continuum that has no beginning and no end: real numbers can get infinitely small and infinitely large.

More concretely, imagine you are standing on a road that goes on forever, both in front of you and behind you. If you drop a traffic cone where you are standing, and call it "the origin", then any distance you walk on this road away from the cone is a real number. If you walk forward, it is positive, and if you walk backwards, it is negative. You can keep walking forever and reach any number you would like, but you will never reach an "end of the road" because it doesn't exist.

2.2 Subsets of the Real Line

A **subset** is a subcollection. There are a few ways we notate subsets of $\mathbb{R}$, so this section is mostly dedicated to notation. The first two examples of a subset are kind of stupid: the **empty set** $\emptyset$ (the set with no elements), and all of $\mathbb{R}$ are both subsets of $\mathbb{R}$.

You can define a **literal** subset of $\mathbb{R}$ by listing the elements it contains. For example, $\{1, 2, 3\}$ is the set containing 1, 2, and 3.

Intervals are—in my opinion—the most important kind of subset of $\mathbb{R}$. The following table shows the 8 kinds of intervals and the two ways we write them ($a \leq b$ are real numbers).

The set of real numbers that are...	**Interval Notation**	**Set-Builder Notation**
greater than a and less than b	(a, b)	$\{x \in \mathbb{R} \ : \ a < x < b\}$
greater than or equal to a and less than b	$[a, b)$	$\{x \in \mathbb{R} \ : \ a \leq x < b\}$
greater than a and less than or equal to b	$(a, b]$	$\{x \in \mathbb{R} \ : \ a < x \leq b\}$
greater than or equal to a and less than or equal to b	$[a, b]$	$\{x \in \mathbb{R} \ : \ a \leq x \leq b\}$
greater than a	(a, ∞)	$\{x \in \mathbb{R} \ : \ a < x\}$
greater than or equal to a	$[a, \infty)$	$\{x \in \mathbb{R} \ : \ a \leq x\}$
less than b	$(-\infty, b)$	$\{x \in \mathbb{R} \ : \ x < b\}$
less than or equal to b	$(-\infty, b]$	$\{x \in \mathbb{R} \ : \ x \leq b\}$

Interval notation is easy to write and read, once you get a hang of what the different parentheses mean. The regular parentheses () are called "open" and mean that the particular endpoint of the interval is not included. The [] brackets are called "closed" and indicate that the endpoint *is* included.

Why aren't $(a, \infty]$, $(a, \infty]$, $[-\infty, b)$, and $[-\infty, b]$ included in the table?

The set-builder notation is read as follows:

$$\{\underbrace{x}_{x}\underbrace{\in}_{\text{in}}\underbrace{\mathbb{R}}_{\substack{\text{the real}\\\text{numbers}}}\quad\underbrace{:}_{\text{such that}}\quad\underbrace{a < x < b}_{\substack{x\text{ is strictly}\\\text{between }a\text{ and }b}}\}.$$

All of the subsets of $\mathbb{R}$ that you'll want to use in this course can be written using intervals. There are two ways to do this:

- The **union** of two sets is the set that contains all of the elements in either set and is denoted with the cup symbol, $\cup$. Here are some examples:
    1. $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$
    2. $(3,4] \cup [4,5) = (3,5)$
    3. $(-\infty,5) \cup [5,\infty) = (-\infty,\infty) = \mathbb{R}$
    4. $(-3,0) \cup (2,6)$
    5. $(1,5) \cup (3,7) = (1,7)$
    6. $(1,8) \cup (3,4) = (1,8)$
- The **intersection** of two sets is the set that contains the elements in both sets and is denoted with the cap symbol, $\cap$. Here are some examples:
    1. $\{1,2,3\} \cap \{3,4,5\} = \{3\}$
    2. $(3,4] \cap [4,5) = \{4\}$
    3. $(-\infty,5) \cap [5,\infty) = \emptyset$
    4. $(-3,0) \cap (2,6) = \emptyset$
    5. $(1,5) \cap (3,7) = (3,5)$
    6. $(1,8) \cap (3,4) = (3,4)$

Draw a picture of each of these examples. Does *Venn diagram* ring any bells?

2.3 Defining Functions

The functions that we'll use are all **real functions**, so we can think about a function as a map that takes real numbers as inputs and also outputs real numbers. A real function must map each input to *exactly one* output. Since a real function can be graphed on a Cartesian grid, with inputs on the horizontal axis and outputs on the vertical axis, this condition can be restated as the **vertical line test**: every vertical line must intersect a graph of a real function at most once.

Draw a non-function that fails the vertical line test.

Some people may get sloppy and say things like

$$\text{"the function } f(x) = x^2\text{"}$$

but what they really mean is

$$\text{"the real function } f\text{, defined by } f(x) = x^2\text{".}$$

Is f defined by

$$f(x) = \begin{cases} x+1 & x > 0 \\ x^2 & x < 1 \end{cases}$$

a function?

From now on, when I say "function" I mean "real-function". However, distinguishing between a function and the formula defining it will eventually make things easier to understand.

2.4 Domain and Range

The **domain** of a function is the set of possible inputs. The **range** of a function is the set of outputs. Not all real functions have $\mathbb{R}$ as their domain and range. The following table domains and ranges for some common functions.

Function type		Domain	Range
Polynomial	$a_n x^n + \cdots + a_1 x + a_0$	$\mathbb{R}$	If the highest power is odd, $\mathbb{R}$. If the highest power is even and positive, then $[b, \infty)$, and $(-\infty, b)$ otherwise (for some b).
Exponential	a^x	$\mathbb{R}$	$(0, \infty)$
Logarithm	$\log_a(x)$	$(0, \infty)$	$\mathbb{R}$
Rational	$\frac{p(x)}{q(x)}$ (p, q are polynomials)	$\{x \in \mathbb{R} \ : \ q(x) \neq 0\}$	It depends

Let's do some concrete examples:

- The function f given by $f(x) = \frac{1}{x}$ does not have 0 in its domain, since we can't divide by 0. Also, f does not have 0 in its range, since there is no real number x for which $\frac{1}{x} = 0$. Therefore the domain and range are both $(-\infty, 0) \cup (0, \infty)$.
- Similarly, tan does not have $2n\pi + \pi/2$ in its domain for any integer n, since $\cos(2n\pi + \pi/2) = 0$ (we can write the domain of tan as $\{x \in \mathbb{R} \ : \ x \neq 2n\pi + \pi/2 \text{ for some integer } n\}$). The range of tan is all of $\mathbb{R}$.
- The function f given by $f(x) = x^2$ has domain $\mathbb{R}$ and range $[0, \infty)$.

2.5 Inverses

Most simply, a function has an **inverse** (and is called **invertible**) if it is reversible as a mapping. In other words, if f maps x to $y := f(x)$, then its inverse f^{-1} must map y to x. The immediate problem we encounter is if f maps—say—both 2 and -2 to 4, then where should f^{-1} map 4? It can't map 4 to both 2 and -2, so f must not be invertible.

Can you think of such a function f?

Formally, the **inverse** of f is the function f^{-1} such that

$$f(f^{-1}(y)) = y \quad \text{and} \quad f^{-1}(f(x)) = x$$

for all x in the domain of f and y in the range of f.

So the question is: when is a function invertible? In order to find an answer, we'll have to make a few more definitions:

- A function is called **injective** (or **one-to-one**) if every distinct input gets mapped to a distinct output. In other words, f is injective if $f(a) = f(b)$ only happens when $a = b$.
- A function is called **surjective** (or **onto**) if every possible output gets mapped to. In other words, f is surjective if for every b in $\mathbb{R}$, there is some a in $\mathbb{R}$ such that $f(a) = b$.
- If a function is both injective and surjective, it is called **bijective**.

Think of functions that are (1) injective but not surjective, (2) surjective but not injective, (3) neither injective nor surjective, (4) bijective.

Following our example from above, it seems like f needs to be injective in order to be invertible. This is sometimes the **horizontal line test**: the graph of an invertible function cannot intersect any horizontal line more than once.

Draw the graph of a non-invertible function failing the horizontal line test.

Invertible functions must also be surjective: otherwise there is some possible output of f that does not have somewhere to be sent by f^{-1}.

It turns out that a function is invertible exactly when it is bijective. If a function is not bijective, we can "fix" it by making its domain smaller.

The general strategy for finding a functions inverse is to switch the place of x and y in the equation, and then solve for y. The result, if it exists, will give you an inverse for the original function (on a possibly smaller domain). Graphically, the inverse of a function is a reflection of the function across the line $y = x$.

Here are some examples:

- The function f defined by $f(x) = x^2$ has domain $\mathbb{R}$, but is not invertible: the line $y = 4$ intersects the graph in two places, $(-2, 4)$ and $(2, 4)$. However, we can can change the domain of f to be $[0, \infty)$. To find its inverse, solve the equation $x = y^2$ for y. We get $y = \pm\sqrt{x}$, but since we are changing the domain of f to only the non-negative real numbers, we forget about the negative square root. Then the inverse of f is given by $f^{-1} = \sqrt{x}$, and for any non-negative x and y $(x, y \in [0, \infty))$,

  $$f^{-1}(f(y)) = f(x^2) = \sqrt{(x)^2)} = x$$

  and

  $$f(f^{-1}(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y.$$

- The function sin has domain $\mathbb{R}$ and range $[-1, 1]$ but is **periodic** (meaning its values cyclicly repeat) so cannot be invertible without an adjustment to its domain. The

Part II
Limits and Continuity

3 Limits

Limits are important in calculus since we need very small quantities to effectively study change. Yeah, that's pretty vague, but hopefully it'll clear up in a few pages.

We study limits for many reasons. One of them is to study functions at points where they are only "close" to being defined. For example, the function f given by $f(x) = \frac{x(x+1)}{x}$ is not defined at 0, but looks like the line $x+1$ everywhere else.

3.1 Intuitive Limits

The intuitive idea of a **limit** is to examine what the output of a function does as the input moves close to a specific value. Since the functions we care about take real numbers as inputs, there are two ways you can approach a number on the real line (from the left and from the right). We will talk about left- and right-sided limits.

Before getting into the actual definition, we'll start intuitively. If a function is "continuous"[1] and is defined at a, then as x moves closer to a, $f(x)$ moves closer to $f(a)$, so the limit of f as x approaches a is $f(a)$.

Let's get a little more general: suppose a function f is defined on open intervals on either side of a.

<table>
<tr><td>If f is defined on an interval to the left of a and the values of $f(x)$ approach L as x approaches a from the left, we say that the "left-sided limit of f as x approaches a is L" and we write $$\lim_{x\to a^-} f(x) = L.$$</td><td>If f is defined on an interval to the right of a and the values of $f(x)$ approach L as x approaches a from the right, we say that the "right-sided limit of f as x approaches a is L" and we write $$\lim_{x\to a^+} f(x) = L.$$</td></tr>
<tr><td colspan="2">If the right and left sided limits match, then we say that the "limit of f as x approaches a is L" and write $$\lim_{x\to a} f(x) = L.$$</td></tr>
</table>

Note: f need not be defined at a to find the limit of f as x approaches a.

One can compute $f(x)$ for values of x that get closer and closer to either side of a. If the values approach L, then you have good reason to believe that L is the (right- and/or left-sided) limit.

[1] I put this word in quotes because I haven't defined it yet, but you should have an intuitive idea of what this means: being able to draw it without lifting your pencil. Just wait for a few pages.

x	$f(x)$
$a-0.1$	$f(a-0.1)$
$a-0.01$	$f(a-0.01)$
$a-0.001$	$f(a-0.001)$
$a-0.0001$	$f(a-0.0001)$
$a+0.0001$	$f(a+0.0001)$
$a+0.001$	$f(a+0.001)$
$a+0.01$	$f(a+0.01)$
$a+0.1$	$f(a+0.1)$

Such calculations, however, cannot prove that a function limits to a specific value.

3.2 Infinite Limits

We can extend the idea of limits outside of the real numbers to include positive and negative infinity in place of both a and L.

- If $f(x)$ grows (in the $\pm$ direction) without bound as x approaches a from the left, then we write

  $$\lim_{x\to a^-} f(x) = \pm\infty.$$

  Similarly for x approaching a from the right.

- We denote the value (if such a value exists) that $f(x)$ approaches as x approaches $\pm\infty$ as

  $$\lim_{x\to\pm\infty} f(x).$$

3.3 Optional: The $\epsilon - \delta$ Definition

You may be dissatisfied with the intuitive approach to limits, so we can make the definition more rigorous. The main idea that needs to be captured by a formal definition is *arbitrary precision*. That is, we need a way to say formally that "$f(x)$ approaches L as x approaches a."

By controlling the input x, we must be able to make the distance $|f(x)-L|$ between the output $f(x)$ and L to be as small as we want ("arbitrarily small"). In other words, if $\epsilon > 0$ is any small positive number, we must be able to ensure (by controlling x) that $|f(x) - L| < \epsilon$. To control x, we can make the distance $|x - a|$ between x and a smaller than some positive number $\delta > 0$ (that may depend on ϵ).

Now we're ready for the real "$\epsilon - \delta$" definition of a limit: we say that L **is the limit of** f **as** x **approaches** a if

$$\text{for all } \epsilon > 0, \text{ there is some } \delta > 0, \text{ such that } |x-a| < \delta \implies |f(x)-L| < \epsilon.$$

Read that last line a few times, because statements with multiple quantifiers can be tricky! It may be helpful to think about the definition of a limit as a game/conversation between two people, Alex and Blake. Alex is trying to claim that the limit of f as $x \to a$ is L, and Blake is doing their best job to contest it. Here is how their conversation might go:

A: I think the limit of $f(x) = 3x + 1$ as x approaches 2 is 7.

B: Well if you think so, can you ensure that $|f(x) - 7| < \frac{1}{10}$?

A: Yes! If we take x such that $|x-2| < 1/30$, then

$$|f(x)-7| = |3x+1-7| = |3x-6| = 3|x-2| < 3(1/30) = 1/10.$$

In this example, $\epsilon = 1/10$ and $\delta = 1/30$. However, if we want to really prove that the limit is 7, we let $\epsilon > 0$ be arbitrary, and take $\delta = \epsilon/3$. Then for any x such that $|x-2| < \delta$,

$$|f(x)-7| = |3x+1-7| = |3x-6| = 3|x-2| < 3(\epsilon/3) = \epsilon.$$

For simple examples, it is pretty easy to work backwards and figure out what δ should be, but for more complicated functions f, it can be more difficult. Here is a Desmos example that shows dynamically how δ can depend on ϵ.

3.4 Limit Laws

We still need a more sophisticated way to figure out what the limit of a function is at a certain point. There are several *limit laws* that make computing limits easier. We also have two basic facts that should be obvious: for any $a, b \in \mathbb{R}$,

$$\lim_{x\to a} b = b \quad \text{and} \quad \lim_{x\to a} x = a.$$

Together with the following laws, you'll be able to evaluate the limits of many functions. Suppose $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, let c be a constant, and let n be a positive integer.

Sum law	$\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = L + M$
Difference law	$\lim\limits_{x\to a}(f(x)-g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = L - M$
Constant Multiple law	$\lim\limits_{x\to a} cf(x) = c\cdot\lim\limits_{x\to a} f(x) = cL$
Product law	$\lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x) = L\cdot M$
Quotient law	$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$ for $M \neq 0$
Power law	$\lim\limits_{x\to a}(f(x))^n = \left(\lim\limits_{x\to a} f(x)\right)^n = L^n$
Root law	$\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd, and for $L \geq 0$ if n is even and $f(x) \geq 0$.

3.5 Limit "Tricks" and Key Examples

There are some other tricks that will help you evaluate limits. Try these things if you're not sure what else to do; they might be helpful if you end up in a situation where you are trying to divide 0/0.

- Simplify. Perhaps the function is rational, and has common factor in its numerator and denominator. By "canceling" the term, the resulting function is not the same: the original function has a hole at the point where the factored term is 0. For example, $f(x) = \frac{x(x-1)}{(x-1)}$ has a hole at $x = 1$, but everywhere else is the line $y = x$. However, this cancellation does not change the value of the limit.

- Multiply by the conjugate of the denominator or the numerator of a rational function (the conjugate of a binomial $a+b$ is $a-b$).
- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.
- $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$.
- $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

4 Continuity

4.1 Definitions

A function f is **continuous at** a if three conditions are satisfied:

1. f is defined at a (i.e. $f(a)$ makes sense)
2. $\lim_{x\to a^+} f(x) = f(a)$
3. $\lim_{x\to a^-} f(x) = f(a)$

If (a), (b), or (c) are not true, then f is **discontinuous at** a. There are three types of discontinuities: if f is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists and is a real number.
2. f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are real numbers, but are different.
3. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

4.2 Using Continuity

The following functions are continuous at every point in their domains:

- polynomials
- rational functions
- trig and inverse trig functions
- exponential functions
- logarithms

Therefore, if you want to evaluate a limit of any of these functions f at a point a in its domain, the limit is equal to $f(a)$ since f is continuous.

Here's another limit law, now that you know about continuous functions:

Theorem (Composite Function Theorem)**.** If $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

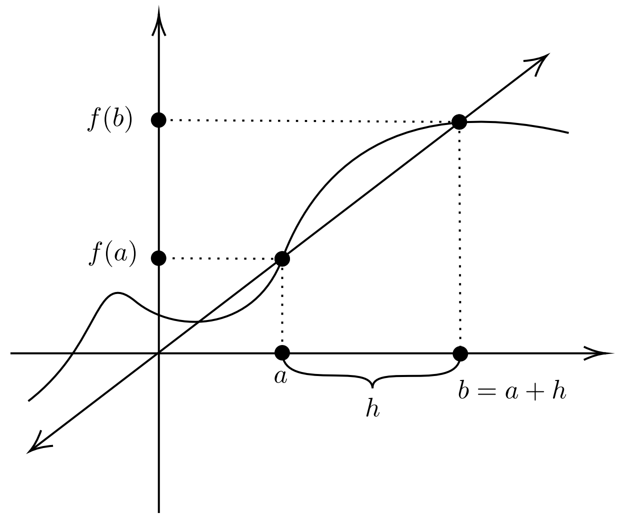
To finish off the section, a very useful theorem:

Theorem (The Intermediate Value Theorem)**.** Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

Part III
Derivatives and Applications

5 The Derivative

5.1 The Limit Definition

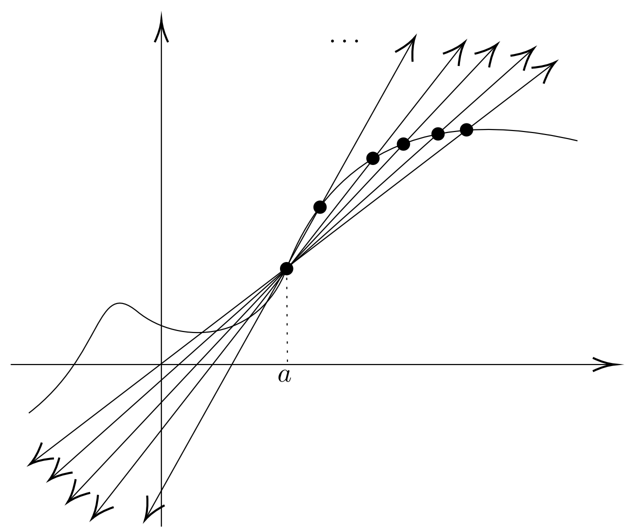


Recall that a **secant line** is a line between two points on the graph of a function. If f is a function and $(a, f(a))$ and $(b, f(b))$ are *different* points on the function, then the secant line that they determine has equation

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

We can assume $b > a$ and write $b = a + h$ for some $h > 0$. Then the equation for the secant line through $(a, f(a))$ and $(b, f(b)) = (a+h, f(a+h))$ is

$$y = f(a) + \frac{f(a + h) - f(a)}{h}(x - a).$$



Now we're going to use *limits* to move one point $(b, f(b))$ closer to the other $(a, f(a))$, and see what happens to the secant line. Since the secant line passes through the point $(a, f(a))$, we only have to track what happens to the slope. As b moves to a, the value of h goes to 0, so the slope of the secant line approaches

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

We'll denote this quantity $f'(a)$, and call this the **derivative of** f **at** a (if this limit exists). The line through the point $(a, f(a))$ with slope $f'(a)$ is called the **tangent line of** f **at** a and has equation

$$y = f(a) + f'(a)(x - a).$$

You'll notice that the tangent line captures some *local information* about f at a. In other words, it is a good approximation of f at a; that is, f and its tangent line are "similar" at a. Intuitively, you should think about the tangent line at a to be the line that "hugs f the best."

Note that this definition of the derivative only makes sense if f is defined on an open interval containing a. We can view the derivative as a function, and write $f'(x)$. The derivative of $f'(x)$ is called the **second derivative** and is denoted $f''(x)$. In general, the nth derivative $f^{(n)}(x)$ is defined to be the derivative of $f^{(n-1)}(x)$. Other notation is sometimes used: if $y = f(x)$, $\frac{dy}{dx} = \frac{d}{dx}[f(x)]$ is the derivative of f, and $\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}[f(x)]$ is the nth derivative of f.

5.2 The Two Part "Program" for Finding Derivatives

So now we have a goal: find derivatives of functions. The problem is that the limit definition is difficult to use (that pesky $h \to 0$ in the denominator). So instead, we will do the following two steps (let a be a real number and f and g be functions):

1. "break functions up into simpler ones"

The Scalar Multiple Rule	$(af)' = af'$.
The Sum Rule	$(f+g)' = f' + g'$
The Product Rule	$(fg)' = f'g + fg'$
The Quotient Rule	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

2. find derivatives of "simple" functions:

The Constant Rule	$\frac{d}{dx}[a] = 0$
The Power Rule	$\frac{d}{dx}[x^a] = ax^{a-1}$
Basic Trig Functions	$\frac{d}{dx}[\sin(x)] = \cos(x)$ and $\frac{d}{dx}[\cos(x)] = -\sin(x)$
Exponential Functions	$\frac{d}{dx}[a^x] = \ln(a)a^x$
Logarithmic Functions	$\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$

Most of the time, you will be able to find the derivatives using a combination of these rules. For instance, the derivatives of the other 4 trig functions can be found using the quotient rule and the derivatives of sin and cos:

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \qquad \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x) \qquad \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

5.3 The Chain Rule

In progress.

5.4 Comparing the Graphs of f, f', and f''

It is helpful to be able to look at a graph of a function and understand what it means for its first and second derivative. The table below summarizes:

f	f'	f''
increasing	positive	
constant	0	
decreasing	negative	
concave up	increasing	positive
linear	constant	0
concave down	decreasing	negative

5.5 Implicit Differentiation

In progress.

5.6 Related Rates

In progress.

5.7 Optimization

In progress.