MULTI-FLUID PLASMA MODEL

1. Dimensional Equations

1.1. Maxwell's equations.

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} =& -\nabla \times \boldsymbol{E},\\ \frac{\partial \boldsymbol{D}}{\partial t} =& \nabla \times \boldsymbol{H} - \boldsymbol{J},\\ \nabla \cdot \boldsymbol{D} =& \rho_c,\\ \nabla \cdot \boldsymbol{B} =& 0,\end{aligned}$$

where we have permittivity ε_0 and permeability μ_0 in vacuum and the relative permittivity ε_r and permeability μ_r which characterise the electromagnetic properties of the medium. We also have the electric flux density $\boldsymbol{D}$, electric field intensity $\boldsymbol{E}$, magnetic flux density $\boldsymbol{B}$ and magnetic field intensity $\boldsymbol{H}$, which are related according to

$$\boldsymbol{D} = \varepsilon_0 \varepsilon_r \boldsymbol{E}, \qquad \boldsymbol{B} = \mu_0 \mu_r \boldsymbol{H}.$$

The free current density $\boldsymbol{J}$ and free charge density $\rho_c$are related according to

$$\frac{\partial \rho_c}{\partial t} = \nabla \cdot \boldsymbol{J}.$$

We can see that for $\varepsilon_r = \mu_r = 1$ we retrieve the classic Maxwells equations for a vacuum with,

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} =& -\nabla \times \boldsymbol{E},\\ \frac{\partial \boldsymbol{E}}{\partial t} =& c_0 \nabla \times \boldsymbol{B} - \frac{\boldsymbol{J}}{\varepsilon_0},\\ \nabla \cdot \boldsymbol{E} =& \frac{\rho_c}{\varepsilon_0},\end{aligned}$$

where $c_0 = {}^1/_{\sqrt{\varepsilon_0 \mu_0}}$ is the speed of light in a vacuum. Note that we are assuming the relative permittivity and permeability are diagonal tensors according to $\varepsilon_r \delta_{ij}$.

In the Lorenz gauge, Maxwell's equations can be written according to

$$\begin{aligned}\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} =& -\frac{\rho_c}{\epsilon_0},\\ \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} =& -\mu_0 \boldsymbol{J},\end{aligned}$$

where ϕ is the electric scalar potential and $\boldsymbol{A}$ is the magnetic vector potential. The electric and magnetic fields can be recovered according to,

$$\begin{aligned}\boldsymbol{E} =& -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t},\\ \boldsymbol{B} =& \nabla \times \boldsymbol{A}.\end{aligned}$$

If we assume an effectively infinite speed of light we recover the equations of electro- and mangneto-statics, according to

$$\begin{aligned}\nabla^2 \phi =& -\frac{\rho_c}{\epsilon_0},\\ \nabla^2 \boldsymbol{A} =& -\mu_0 \boldsymbol{J},\end{aligned}$$

where the electric and magnetic fields become invariant with time, $\boldsymbol{E} = -\nabla\phi$. Note that for $\boldsymbol{D}$ we have $\nabla^2\phi = -\epsilon_r \rho_c$.

1.2. **Hyperbolic/parabolic Maxwell's equations.** Maxwell's equations are given in mixed hyperbolic/parabolic form [1, 2] by

$$\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} =& - \nabla \times \boldsymbol{E} - \nabla \psi, \\
\frac{\partial \boldsymbol{D}}{\partial t} =& \nabla \times \boldsymbol{H} - \boldsymbol{J} - \nabla \phi, \\
\frac{\partial \phi}{\partial t} =& - c_h^2 \left(\nabla \cdot \boldsymbol{D} - \rho_c \right) - \frac{c_h^2}{c_p^2} \phi, \\
\frac{\partial \psi}{\partial t} =& - c_h^2 \nabla \cdot \mathbf{B} - \frac{c_h^2}{c_p^2} \psi,
\end{aligned}$$

where the divergence constraints are imposed by potentials ψ and ϕ with positive parameters c_h and c_p. Note that ψ has units of $\boldsymbol{E}$ and ϕ has units of $\boldsymbol{H}$ while $\frac{c_h^2}{c_p^2}$ has units of inverse time.

1.3. **Projection method.** Alternatively, we may use the projection method for coercing the electric and magnetic fields to comply with the divergence conditions. We solve,

$$\nabla \cdot \boldsymbol{X} - S = \nabla^2 \phi,$$

where $\boldsymbol{X}$ is some vector field and we have a scalar potential ϕ such that we can then perform the projection,

$$\boldsymbol{X}^* = \boldsymbol{X} - \nabla \phi,$$

where $\nabla \cdot \boldsymbol{X}^* - S = 0$. The source S is zero for the magnetic field, where the constraint is $\nabla \cdot \boldsymbol{B} = 0$, while for the $\boldsymbol{D}$ field we have $\nabla \cdot \boldsymbol{D} = \epsilon_r \rho_c$ and thus $S = \epsilon_r \rho_c$.

1.4. **Fluid equations.**

$$\begin{aligned}
\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot \left(\rho_\alpha \boldsymbol{u}_\alpha \right) =& S_\alpha, \\
\frac{\partial \rho_\alpha \boldsymbol{u}_\alpha}{\partial t} + \nabla \cdot \left(\rho_\alpha \boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha + \mathbb{P}_\alpha \right) =& n_\alpha q_\alpha \left(\mathbf{E} + \boldsymbol{u}_\alpha \times \mathbf{B} \right) + \sum_{\beta \neq \alpha} \boldsymbol{R}_{\alpha\beta}, \\
\frac{\partial \epsilon_\alpha}{\partial t} + \nabla \cdot \left(\epsilon_\alpha \boldsymbol{u}_\alpha + \boldsymbol{u}_\alpha \cdot \mathbb{P}_\alpha + \boldsymbol{h}_\alpha \right) =& \boldsymbol{u}_\alpha \cdot \left(n_\alpha q_\alpha \mathbf{E} + \sum_{\beta \neq \alpha} \boldsymbol{R}_{\alpha\beta} \right) + \sum_{\beta \neq \alpha} Q_{\alpha\beta}, \\
\frac{\partial p_{\alpha\|}}{\partial t} + \nabla \cdot \left(p_\alpha^{\|} \boldsymbol{u}_\alpha \right) =& - 2 p_\alpha^{\|} \boldsymbol{b} \cdot \left(\boldsymbol{b} \cdot \nabla \right) \boldsymbol{u}_\alpha + \frac{p_\alpha - p_\alpha^{\|}}{\tau_\alpha},
\end{aligned}$$

where

$$\begin{aligned}
\rho_\alpha =& n_\alpha m_\alpha, \\
\mathbb{P}_\alpha =& p_\alpha^{\perp} \mathbf{I} + \left(p_\alpha^{\|} - p_\alpha^{\perp} \right) \boldsymbol{b} \otimes \boldsymbol{b} \\
p_\alpha =& \frac{2 p_\alpha^{\perp} + p_\alpha^{\|}}{3} = n_\alpha k_B T_\alpha, \\
\epsilon_\alpha =& \frac{p_\alpha}{\gamma - 1} + \frac{\rho_\alpha \boldsymbol{u}_\alpha \cdot \boldsymbol{u}_\alpha}{2}, \\
\boldsymbol{b} =& \frac{\mathbf{B}}{|\mathbf{B}|}.
\end{aligned}$$

Note that $\mathbb{P}$ is the pressure tensor composed of the pressure perpendicular $p^{\perp}$ and parallel $p^{\|}$ to the magnetic field vector $\boldsymbol{b}$ [3]. The heat flux is given by $\boldsymbol{h}$ Relaxation of $p^{\|}$ towards the average pressure occurs at a rate τ which is chosen according to the flow regime. Closures for anisotropic terms are given in Appendix A.

1.4.1. *Collisional effects.* Collisions between species are handled according to [4, 5],

$$
\begin{aligned}
\boldsymbol{R}_{\alpha\beta} &= m_\alpha n_\alpha \nu_{\alpha\beta}\left(\boldsymbol{u}_\beta - \boldsymbol{u}_\alpha\right) = -\boldsymbol{R}_{\beta\alpha},\\
Q_{\alpha\beta} &= Q^{\text{fric}}_{\alpha\beta} + Q^{\text{eq}}_{\alpha\beta},\\
Q^{\text{fric}}_{\alpha\beta} &= m_{\alpha\beta} n_\alpha \nu_{\alpha\beta}\left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta\right)\cdot\left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta\right),\\
Q^{\text{eq}}_{\alpha\beta} &= 3 m_\alpha n_\alpha \frac{\nu_{\alpha\beta}}{m_\alpha + m_\beta}\left(k_B T_\beta - k_B T_\alpha\right) = -Q^{\text{eq}}_{\beta\alpha},
\end{aligned}
$$

where,

$$
\begin{aligned}
m_{\alpha\beta} =& \frac{m_\alpha m_\beta}{m_\alpha + m_\beta},\\
\nu_{\alpha\beta} =& \begin{cases} \frac{\sqrt{2} n_\beta Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\epsilon_0^2 m_{\alpha\beta} m_\alpha}\left[\left(\frac{2}{9\pi}\right)^{1/3}\left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta\right)\cdot\left(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta\right) + \frac{k_B T_\alpha}{m_\alpha} + \frac{k_B T_\beta}{m_\beta}\right]^{-3/2}, & Z_\alpha > 0 \wedge Z_\beta > 0,\\ \frac{4}{3}\frac{n_\beta m_\beta \sigma_{\alpha\beta}}{m_\alpha + m_\beta}\sqrt{\frac{8 k_B T_\alpha}{\pi m_\alpha} + \frac{8 k_B T_\beta}{\pi m_\beta}}, & Z_\alpha = 0 \vee Z_\beta = 0, \end{cases}
\end{aligned}
$$

where Z is the charge state, e is the unit charge of an electron, $\ln\Lambda_{\alpha\beta} \approx 10$ is the Coulomb logarithm, and $\sigma_{\alpha\beta}$ is the collision cross section.

1.5. **System of equations.**

$$
\frac{\partial}{\partial t}\begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \\ \phi \\ \psi \end{pmatrix} + \nabla\cdot\begin{pmatrix} \phi & -\frac{B_z}{\mu_0\mu_z} & \frac{B_y}{\mu_0\mu_y} \\ \frac{B_z}{\mu_0\mu_z} & \phi & -\frac{B_x}{\mu_0\mu_x} \\ -\frac{B_y}{\mu_0\mu_y} & \frac{B_x}{\mu_0\mu_x} & \phi \\ \psi & \frac{D_z}{\varepsilon_0\varepsilon_z} & -\frac{D_y}{\varepsilon_0\varepsilon_y} \\ -\frac{D_z}{\varepsilon_0\varepsilon_z} & \psi & \frac{D_x}{\varepsilon_0\varepsilon_x} \\ \frac{D_y}{\varepsilon_0\varepsilon_y} & -\frac{D_x}{\varepsilon_0\varepsilon_x} & \psi \\ c_h^2 D_x & c_h^2 D_y & c_h^2 D_z \\ c_h^2 B_x & c_h^2 B_y & c_h^2 B_z \end{pmatrix} = \begin{pmatrix} -j_x \\ -j_y \\ -j_z \\ 0 \\ 0 \\ 0 \\ c_h^2 \rho_c - \frac{c_h^2}{c_p^2}\phi \\ -\frac{c_h^2}{c_p^2}\psi \end{pmatrix}
$$

$$
\frac{\partial}{\partial t}\begin{pmatrix} \rho_\alpha \\ \rho_\alpha u_\alpha \\ \rho_\alpha v_\alpha \\ \rho_\alpha w_\alpha \\ \epsilon_\alpha \\ p_{\alpha\|} \end{pmatrix} + \nabla\cdot\left(\begin{matrix} \rho_\alpha u_\alpha & \rho_\alpha v_\alpha \\ \rho_\alpha u_\alpha^2 + b_x^2\left(p^{\|} - p^{\perp}\right) + p^{\perp} & \rho_\alpha u_\alpha v_\alpha + b_x b_y\left(p^{\|} - p^{\perp}\right) \\ \rho_\alpha u_\alpha v_\alpha + b_x b_y\left(p^{\|} - p^{\perp}\right) & \rho_\alpha v_\alpha^2 + b_y^2\left(p^{\|} - p^{\perp}\right) + p^{\perp} \\ \rho_\alpha u_\alpha w_\alpha + b_x b_z\left(p^{\|} - p^{\perp}\right) & \rho_\alpha v_\alpha w_\alpha + b_y b_z\left(p^{\|} - p^{\perp}\right) \\ u_\alpha\left(\epsilon_\alpha + p^{\perp}\right) + b_x\left(p^{\|} - p^{\perp}\right)\left(u b_x + v b_y + w b_z\right) & v_\alpha\left(\epsilon_\alpha + p^{\perp}\right) + b_y\left(p^{\|} - p^{\perp}\right)\left(u b_x + v b_y + \right. \\ p_\alpha^{\|} u_\alpha & p_\alpha^{\|} v_\alpha \end{matrix}\right.
$$

$$
= \begin{pmatrix} S_\alpha \\ r_\alpha \rho_\alpha\left(E_x + v_\alpha B_z - w_\alpha B_y\right) + \sum_{\beta\neq\alpha} R^x_{\alpha\beta} \\ r_\alpha \rho_\alpha\left(E_y + w_\alpha B_x - u_\alpha B_z\right) + \sum_{\beta\neq\alpha} R^y_{\alpha\beta} \\ r_\alpha \rho_\alpha\left(E_z + u_\alpha B_y - v_\alpha B_x\right) + \sum_{\beta\neq\alpha} R^z_{\alpha\beta} \\ \boldsymbol{u}_\alpha\cdot\left(n_\alpha q_\alpha \mathbf{E} + \sum_{\beta\neq\alpha}\boldsymbol{R}_{\alpha\beta}\right) + \sum_{\beta\neq\alpha} Q_{\alpha\beta} \\ -2 p_\alpha^{\|}\boldsymbol{b}\cdot\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}_\alpha + \frac{p_\alpha - p_\alpha^{\|}}{\tau_\alpha} \end{pmatrix}
$$

1.6. **Plasma parameters.** The Larmor radius, d_L, and Debye length, d_D, are characterising values of the plasma. The Larmor radius is given by,

$$
d_{L,\alpha} = \frac{u_{\alpha\perp} m_\alpha}{\left|q_\alpha\right| B},
$$

where $u_{\alpha\perp} = \left|\mathbf{u}_\alpha - \frac{\mathbf{u}_\alpha\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}\right|$ is the velocity perpendicular to the magnetic field of strength $B = |\mathbf{B}|$. The Debye length is calculated according to

$$
\begin{aligned}
\omega_{p,\alpha} &\equiv \sqrt{\frac{n_\alpha q_\alpha^2}{\epsilon_0 m_\alpha}}, \\
v_{th,\alpha} &= \sqrt{\frac{k_B T_\alpha}{m_\alpha}}, \\
d_{D,\alpha} &= \sqrt{\frac{\epsilon_0 k_B T_\alpha}{n_\alpha q_\alpha^2}} = \frac{v_{th,\alpha}}{\omega_{p,\alpha}} = \sqrt{\frac{v_{th,\alpha}^2 \epsilon_0 m_\alpha}{n_\alpha q_\alpha^2}},
\end{aligned}
$$

where ω_p is the plasma frequency and v_{th} is the one dimensional thermal velocity.
The interaction between electromagnetic fields and matter may be described by the parameter β,

$$
\beta_\alpha = \frac{2 p_\alpha \mu_0}{B^2}.
$$

Using this interaction parameter we may re-define the Larmor radius in terms of β and plasma skin depth, d_S, as follows,

$$
d_D = \frac{d_S}{c}, \qquad d_L = \sqrt{\frac{\beta}{2}} d_S,
$$

where

$$
\begin{aligned}
d_{S,\alpha} &\equiv \frac{1}{q_\alpha} \sqrt{\frac{m_\alpha}{\mu_0 n_\alpha}}, \\
\beta_\alpha &\equiv \frac{2 \mu_0 n_\alpha m_\alpha u_\alpha^2}{B_\alpha^2}.
\end{aligned}
$$

1.6.1. *Permittivity.* The frequency dependent relative permittivity for a cold plasma, also known as the dielectric tensor, is given for arbitrary field direction $\boldsymbol{B}$ according to,

$$
\begin{aligned}
\boldsymbol{\epsilon}(\omega) &= \mathbf{1} + \sum_s \left[-\mathbf{1} \frac{\omega_{p,s}^2}{\omega^2} + \frac{\omega_{p,s}^2 (\alpha \cdot \boldsymbol{\Omega}_s)}{\omega (\omega^2 - \Omega_s^2)} - \frac{\omega_{p,s}^2 (\alpha \cdot \boldsymbol{\Omega}_s)^2}{\omega^2 (\omega^2 - \Omega_s^2)} \right] \\
\Omega_s &= \frac{q_s \boldsymbol{B}}{m_\alpha}, \\
\alpha &= (\alpha_x, \alpha_y, \alpha_z)
\end{aligned}
$$

$$
\alpha_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \alpha_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad \alpha_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

For a field aligned according to $\boldsymbol{B} = B_0 \mathbf{e}_z$ (background magnetic field is assumed to align with local z axis) we have,

$$
\varepsilon_r = K_{ij} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \begin{cases} S = 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2 - \Omega_\alpha^2}, \\ D = \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega} \frac{\Omega_\alpha}{\omega^2 - \Omega_\alpha^2} \\ P = 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2}. \end{cases},
$$

In the low freqency limit we have $\omega \ll \Omega_\alpha$ leading to

$$
\begin{aligned}
S =& 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2 - \Omega_\alpha^2} & D =& \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega} \frac{\Omega_\alpha}{\omega^2 - \Omega_\alpha^2} & P =& 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \\
S =& 1 + \sum_\alpha \frac{\rho_\alpha \mu_0 c_0^2}{B_0^2}, & D \approx& -\sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega} \frac{\Omega_\alpha}{\Omega_\alpha^2} = -\sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega \Omega_\alpha} \\
S \approx& 1 + \frac{c_0^2}{v_A^2}, & D \approx& -\sum_\alpha \frac{1}{\omega} \frac{n_\alpha q_\alpha^2}{\epsilon_0 m_\alpha} \frac{m_\alpha}{q_s B_0} \\
& & D \approx& -\sum_\alpha \frac{1}{\omega} \frac{n_\alpha q_\alpha}{\epsilon_0 B_0} \\
& & D \approx& -\sum_\alpha \frac{u_\alpha d_{L,\alpha}}{\omega d_{D,\alpha}}
\end{aligned}
$$

where $v_A = \frac{B}{\sqrt{\mu_0 \rho}}$ is the Alfven speed. For the high frequency limit $\omega \gg \Omega_\alpha$ we have

$$
\begin{aligned}
S =& 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2 - \Omega_\alpha^2} & D =& \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega} \frac{\Omega_\alpha}{\omega^2 - \Omega_\alpha^2} & P =& 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \\
S \approx& 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} & D \approx& \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \frac{\Omega_\alpha}{\omega} \\
& & D \approx& 0
\end{aligned}
$$

If we assume transmission at speed c and have wavelengths on the order of the Debye length, then

$$
\begin{aligned}
\omega^2 &= \frac{c^2}{\lambda^2} \\
\omega_\alpha^2 &= \frac{c^2 n_\alpha q_\alpha^2}{\epsilon_0 k_B T_\alpha} \\
\omega_\alpha^2 &= \frac{c^2 n_\alpha q_\alpha^2}{\epsilon_0 k_B T_\alpha}
\end{aligned}
$$

which gives

$$
\begin{aligned}
\frac{\omega_{p,\alpha}^2}{\omega^2} =& \frac{n_\alpha q_\alpha^2}{\epsilon_0 m_\alpha} \frac{\epsilon_0 k_B T_\alpha}{c^2 n_\alpha q_\alpha^2} \\
=& \frac{k_B T_\alpha}{m_\alpha c^2} \\
=& \frac{u_{T,\alpha}^2}{c^2}
\end{aligned}
$$

So, for $\omega \approx {}^{c}/_{d_D} \gg \Omega_\alpha$ we have

$$
\begin{aligned}
S \approx P \approx& 1 - \frac{u_{T,e}^2}{c^2} \\
\approx& 1 - \frac{\hat{T}}{\hat{m}\hat{c}^2} \\
\approx& 1 - \frac{\hat{p}}{\hat{\rho}\hat{c}^2}
\end{aligned}
$$

2. Reference parameters

To define our simulation we define a number of reference parameters including length, x_0, and temperature, T_0, this gives a speed, $u_0 = \sqrt{\frac{k_B T_0}{m_0}}$, and mass, m_0. The reference time then follows as $t_0 = \frac{x_0}{u_0}$. Our reference pressure is in terms of number density, n_0, mass and speed, such that $p_0 = n_0 m_0 u_0^2$. Note that we can define a reference density as $\rho_0 = n_0 m_0$ A reference magnetic field strength is also defined as B_0

and electric field as $E_0 = cB_0$. The charge of an electron is defined as $q_0 = e$. We may also use a reference Larmor radius and Debye length according to

$$
\begin{aligned}
d_{D,0} =& \sqrt{\frac{\epsilon_0 k_B T_0}{n_0 q_0^2}} = \sqrt{\frac{u_0^2 \epsilon_0 m_0}{n_0 q_0^2}} = \sqrt{\frac{u_0^2 m_0}{c^2 \mu_0 n_0 q_0^2}}, \\
d_{L,0} =& \frac{u_0 m_0}{q_0 B_0},
\end{aligned}
$$

which gives

$$
\begin{aligned}
u_0 &= \sqrt{\frac{k_B T_0}{m_0}} \\
n_0 &= \frac{u_0^2 \epsilon_0 m_0}{d_{D,0}^2 q_0^2}
\end{aligned}
$$

Likewise for d_S and β we have,

$$
d_{S,0} \equiv \frac{1}{q_0}\sqrt{\frac{m_0}{\mu_0 n_0}} = \frac{u_0 \hat{c}}{q_0}\sqrt{\frac{\epsilon_0 m_0}{n_0}} = \hat{c} d_{D,0}, \qquad \beta_0 \equiv \frac{2\mu_0 n_0 m_0 u_0^2}{B_0^2} = \frac{2}{\hat{c}^2}\frac{n_0 m_0}{\epsilon_0 B_0^2} = 2\frac{d_{L,0}^2}{d_{S,0}^2}.
$$

The reference parameters are given in Table 1

Table 1. Reference parameters

Name	Definition	Name	Definition
Length	x_0	Larmor radius	$d_{L,0} = \frac{u_0 m_0}{q_0 B_0}$
Speed	$u_0 = \sqrt{\frac{k_B T_0}{m_0}}$	Debye length	$d_{D,0} = \sqrt{\frac{u_0^2 m_0}{c^2 \mu_0 n_0 q_0^2}}$
Time	$t_0 = \frac{x_0}{u_0}$	Number density	$n_0 = \frac{u_0^2 \epsilon_0 m_0}{d_{D,0}^2 q_0^2}$
Charge	$q_0 = e$		$D_0 = \varepsilon_0 E_0,$
Mass	m_0	Fields	$H_0 = c_0 D_0$
Temperature	T_0		$B_0 = \mu_0 H_0$
Pressure	$p_0 = n_0 m_0 u_0^2$		$E_0 = c_0 B_0$

3. Non-dimensionalization

Using the reference parameters listed above we may obtain our non-dimensional parameters (ignoring species subscript),

$$
\begin{aligned}
&\hat{n} = \frac{n}{n_0}, && \hat{m} = \frac{m}{m_0}, && \hat{\rho} = \frac{\rho}{\rho_0}, && \hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{u_0}, && \hat{p} = \frac{p}{p_0}, && \hat{\epsilon} = \frac{\epsilon}{p_0}, && \hat{T} = \frac{T}{T_0}, \\
&\hat{\boldsymbol{B}} = \frac{\boldsymbol{B}}{B_0}, && \hat{\boldsymbol{D}} = \frac{\boldsymbol{D}}{D_0}, && \hat{\boldsymbol{E}} = \frac{\boldsymbol{E}}{E_0}, && \hat{\boldsymbol{H}} = \frac{\boldsymbol{H}}{H_0}, && \hat{\psi} = \frac{\psi}{E_0}, && \hat{\phi} = \frac{\phi}{H_0}, && \hat{c}_h = \frac{c_h}{u_0} \\
&\hat{q} = \frac{q}{q_0}, && \hat{t} = \frac{t}{t_0}, && \hat{c} = \frac{c}{u_0}, && \hat{d}_D = \frac{d_D}{x_0}, && \hat{d}_L = \frac{d_L}{x_0}, && \hat{d}_S = \frac{d_S}{x_0}, && \hat{c}_p = \frac{c_p}{\sqrt{x_0 u_0}}.
\end{aligned}
$$

3.1. **Interaction parameter.** The interaction parameter must be scaled to accomodate the non-dimensionalizat

$$
\beta = \frac{\beta_0 \hat{p}}{\hat{B}^2}.
$$

3.2. **Plasma parameters.** If we desire to calculate non-dimensional plasma parameters from non-dimensional quantities we can do so via the following equations

$$\begin{aligned}\hat{d}_{L,\alpha} =&\hat{d}_{L,0}\frac{\hat{u}_{\alpha\perp}\hat{m}_\alpha}{|\hat{q}_\alpha|\left|\hat{\mathbf{B}}\right|},\\ \hat{d}^2_{D,\alpha} =&\hat{d}^2_{D,0}\frac{\hat{T}}{\hat{n}\hat{q}^2}.\end{aligned}$$

3.3. **Defining a simulation.** In order to define a simulation we must provide the spatial definition of our primitive (or conserved) variables that we would like to evolve; $\hat{\rho}_\alpha$, $\hat{\mathbf{u}}_\alpha$, $\hat{p}_\alpha$, $\hat{\mathbf{E}}$, $\hat{\mathbf{B}}$ along with parameters that define our (non-dimensional) plasma regime; $\hat{c}$, $\hat{m}_\alpha$, $\hat{q}_\alpha$, $\hat{n}_0$ and $\hat{d}_{L,0}$, $\hat{d}_{D,0}$ or β_0, $\hat{d}_{S,0}$. These quantities allow for the complete calculation of all the equations listed below. Note that $\hat{\Gamma}$ values are initialized to zero. To relate our non-dimensional quantities back to physical quantities, we must provide values for some of our reference parameters. In particular we must provide,

$$x_0,\, c,\, q_0,\, m_0,\, n_0,$$

which then allows us to calculate the remaining reference parameters according to

$$\begin{aligned}u_0 =&\frac{c}{\hat{c}}, & t_0 =&\frac{x_0}{u_0}, & T_0 =&\frac{u_0^2 m_0}{k_B},\\ \rho_0 =&m_0 n_0, & p_0 =&\rho_0 u_0^2, & E_0 =&cB_0.\end{aligned}$$

3.4. **Fluid equations.**

$$\begin{aligned}\frac{\partial\hat{\rho}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\rho}_\alpha\hat{\mathbf{u}}_\alpha\right) =&\hat{S}_\alpha,\\ \frac{\partial\hat{\rho}_\alpha\hat{\mathbf{u}}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\rho}_\alpha\hat{\mathbf{u}}_\alpha\hat{\mathbf{u}}_\alpha+\hat{p}_\alpha\mathbb{I}+\hat{\Pi}_\alpha\right) =&\frac{\hat{\rho}_\alpha\hat{r}_\alpha}{\hat{d}_{L,0}}\left(\hat{c}\hat{\mathbf{E}}+\hat{\mathbf{u}}_\alpha\times\hat{\mathbf{B}}\right)+\sum_\beta\hat{\boldsymbol{R}}_\alpha^{\alpha\beta},\\ \frac{\partial\hat{\epsilon}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\epsilon}_\alpha\hat{\boldsymbol{u}}_\alpha+\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbb{P}}_\alpha+\hat{\boldsymbol{h}}_\alpha\right) =&\hat{\boldsymbol{u}}_\alpha\cdot\left(\frac{\hat{\rho}_\alpha\hat{r}_\alpha\hat{c}}{\hat{d}_{L,0}}\hat{\mathbf{E}}+\sum_{\beta\neq\alpha}\hat{\boldsymbol{R}}_{\alpha\beta}\right)+\sum_{\beta\neq\alpha}\hat{Q}_{\alpha\beta},\\ \frac{\partial\hat{p}_\alpha^{\|}}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{p}_\alpha^{\|}\hat{\boldsymbol{u}}_\alpha\right) =&-2\hat{p}_\alpha^{\|}\boldsymbol{b}\cdot\left(\boldsymbol{b}\cdot\hat{\nabla}\right)\hat{\boldsymbol{u}}_\alpha+\frac{\hat{p}_\alpha-\hat{p}_\alpha^{\|}}{\hat{\tau}_\alpha}\end{aligned}$$

where

$$
\begin{aligned}
\hat{p}_\alpha =& \frac{2\hat{p}_\alpha^{\perp} + \hat{p}_\alpha^{\parallel}}{3} = \hat{n}_\alpha \hat{T}_\alpha, \\
\mathbb{P}_\alpha =& \begin{cases} \hat{p}_\alpha^{\perp}\mathbf{I} + \left(\hat{p}_\alpha^{\parallel} - \hat{p}_\alpha^{\perp}\right) \boldsymbol{b}\otimes\boldsymbol{b}, & \left|\hat{\mathbf{B}}\right| > 0 \wedge \hat{q}_\alpha \neq 0 \\ \hat{p}_\alpha \mathbb{I} + \hat{\Pi}_\alpha, & \text{otherwise,} \end{cases}, \\
p_\alpha^{\perp}\mathbf{I} + \left(p_\alpha^{\parallel} - p_\alpha^{\perp}\right) \boldsymbol{b}\otimes\boldsymbol{b} =& \begin{bmatrix} b_x^2\left(p^\parallel - p^\perp\right) + p^\perp & b_x b_y \left(p^\parallel - p^\perp\right) & b_x b_z \left(p^\parallel - p^\perp\right) \\ b_x b_y \left(p^\parallel - p^\perp\right) & b_y^2\left(p^\parallel - p^\perp\right) + p^\perp & b_y b_z \left(p^\parallel - p^\perp\right) \\ b_x b_z \left(p^\parallel - p^\perp\right) & b_y b_z \left(p^\parallel - p^\perp\right) & b_z^2\left(p^\parallel - p^\perp\right) + p^\perp \end{bmatrix}, \\
\boldsymbol{b} =& \frac{\hat{\mathbf{B}}}{\left|\hat{\mathbf{B}}\right|}, \\
\hat{\epsilon}_\alpha =& \frac{\hat{p}_\alpha}{\gamma - 1} + \frac{\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \cdot \hat{\mathbf{u}}_\alpha}{2}, \\
\hat{\boldsymbol{R}}_{\alpha\beta} =& \hat{m}_\alpha \hat{n}_\alpha \hat{\nu}_{\alpha\beta} \left(\hat{\boldsymbol{u}}_\beta - \hat{\boldsymbol{u}}_\alpha\right), \\
\hat{Q}_{\alpha\beta} =& \hat{Q}_{\alpha\beta}^{\text{fric}} + \hat{Q}_{\alpha\beta}^{\text{eq}}, \\
\hat{Q}_{\alpha\beta}^{\text{fric}} =& \hat{m}_{\alpha\beta} \hat{n}_\alpha \hat{\nu}_{\alpha\beta} \left(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta\right) \cdot \left(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta\right), \\
Q_{\alpha\beta}^{\text{eq}} =& 3\hat{m}_\alpha \hat{n}_\alpha \frac{\hat{\nu}_{\alpha\beta}}{\hat{m}_\alpha + \hat{m}_\beta} \left(\hat{T}_\beta - \hat{T}_\alpha\right), \\
\hat{\nu}_{\alpha\beta} =& \begin{cases} \frac{\hat{n}_0}{\hat{d}_{D,0}^4} \frac{\sqrt{2}\hat{n}_\beta \hat{q}_\alpha^2 \hat{q}_\beta^2 \ln\mathit{\Lambda}_{\alpha\beta}}{12\pi^{3/2}\hat{m}_{\alpha\beta}\hat{m}_\alpha} \left[\left(\frac{2}{9\pi}\right)^{1/3} \left(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta\right) \cdot \left(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta\right) + \frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta}\right]^{-3/2}, & \hat{q}_\alpha > 0 \wedge \hat{q}_\beta > 0, \\ \frac{4}{3} \frac{\hat{n}_0 \hat{n}_\beta \hat{m}_\beta \hat{\sigma}_{\alpha\beta}}{\hat{m}_\alpha + \hat{m}_\beta} \sqrt{\frac{8}{\pi}\left(\frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta}\right)}, & \hat{q}_\alpha = 0 \vee \hat{q}_\beta = 0. \end{cases}
\end{aligned}
$$

3.5. Maxwell's equations.

$$
\begin{aligned}
\frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} - \hat{c}_0 \hat{\nabla} \times \hat{\boldsymbol{H}} =& - \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2 \hat{c}_0} \hat{\boldsymbol{J}} - \hat{c}_0 \hat{\nabla} \hat{\phi}, \\
\frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \hat{c}_0 \hat{\nabla} \times \hat{\mathbf{E}} =& - \hat{c}_0 \hat{\nabla} \hat{\psi}, \\
\frac{\partial \hat{\phi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} \frac{\hat{c}_h^2}{\hat{c}_0^2} \hat{\rho}_c - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\frac{\partial \hat{\psi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\mathbf{B}} =& - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\psi}
\end{aligned}
$$

$$
\frac{\partial}{\partial t} \begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \\ \phi \\ \psi \end{pmatrix} + c_0 \nabla \cdot \begin{pmatrix} \phi & -\frac{B_z}{\mu_z} & \frac{B_y}{\mu_y} \\ \frac{B_z}{\mu_z} & \phi & -\frac{B_x}{\mu_x} \\ -\frac{B_y}{\mu_y} & \frac{B_x}{\mu_x} & \phi \\ \psi & \frac{D_z}{\varepsilon_z} & -\frac{D_y}{\varepsilon_y} \\ -\frac{D_z}{\varepsilon_z} & \psi & \frac{D_x}{\varepsilon_x} \\ \frac{D_y}{\varepsilon_y} & -\frac{D_x}{\varepsilon_x} & \psi \\ \frac{c_h^2}{c_0^2} D_x & \frac{c_h^2}{c_0^2} D_y & \frac{c_h^2}{c_0^2} D_z \\ \frac{c_h^2}{c_0^2} B_x & \frac{c_h^2}{c_0^2} B_y & \frac{c_h^2}{c_0^2} B_z \end{pmatrix} = \begin{pmatrix} -\frac{d_{L,0}}{d_{D,0}^2 c_0} j_x \\ -\frac{d_{L,0}}{d_{D,0}^2 c_0} j_y \\ -\frac{d_{L,0}}{d_{D,0}^2 c_0} j_z \\ 0 \\ 0 \\ 0 \\ \frac{d_{L,0} c_h^2}{d_{D,0}^2 c_0^2} \rho_c - \frac{c_h^2}{c_p^2} \phi \\ -\frac{c_h^2}{c_p^2} \psi \end{pmatrix}
$$

The potential form (for the static solution of the fields from the charge and current densities) are given by

$$\hat{\nabla}^2\left(\epsilon_r\hat{\phi}\right) = -\frac{\hat{d}_{L,0}}{\hat{c}\hat{d}_{D,0}^2}\sum\hat{\rho}_\alpha\frac{\hat{q}_\alpha}{\hat{m}_\alpha}, \qquad \hat{\nabla}^2\hat{\boldsymbol{A}} = -\frac{\hat{d}_{L,0}\mu_r}{\hat{c}^2\hat{d}_{D,0}^2}\sum\frac{\hat{\rho}_\alpha\hat{\boldsymbol{u}}_\alpha\hat{q}_\alpha}{\hat{m}_\alpha},$$

$$\hat{\boldsymbol{D}} = -\hat{\nabla}\left(\epsilon_r\hat{\phi}\right), \qquad \hat{\boldsymbol{B}} = \hat{\nabla}\times\hat{\boldsymbol{A}}.$$

Note that full working may be found in Appendix B.

We will henceforth refrain from using the hat $\hat{()}$ to denote a dimensionless parameter as all will be assumed thus, unless specified otherwise.

4. Update procedure

4.1. **Unsplit Transverse Method.** We use the unsplit-transverse-method as implemented in [6]. This is composed of the following steps

(1) Calculate the reconstructed face values for all quantities
(2) Calculate updated face values according to (c_i is the local characteristic speed in the i-th direction)

$$Q^{n+1/2}_{i\pm1/2} = Q^n_{i\pm1/2} - \frac{c_i}{2}\frac{\Delta t}{\Delta x}\left[Q^n_{i+1/2} - Q^n_{i-1/2} \mp 3\left(1 - c_i\frac{2}{3}\frac{\Delta t}{\Delta x}\right)\left(Q^n_{i-1/2} + Q^n_{i+1/2}\right)\right]$$

(3) Calculate fluxes
(4) Correct face values by using the transverse flux information according to
   (a) 2D

$$Q^{n+1/2,*}_{i\pm1/2,j} = Q^{n+1/2}_{i\pm1/2,j} + \frac{1}{2}\frac{\Delta t}{\Delta y}\left(F^{n+1/2}_{y,i,j-1/2} - F^{n+1/2}_{y,i,j+1/2}\right)$$
$$Q^{n+1/2,*}_{i,j\pm1/2} = Q^{n+1/2}_{i,j\pm1/2} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{x,i-1/2,j} - F^{n+1/2}_{x,i+1/2,j}\right)$$

   (b) 3D

$$\begin{aligned}
Q^{n+1/2,*}_{i\pm1/2,j,k} =& Q^{n+1/2}_{i\pm1/2,j,k} + \frac{1}{2}\frac{\Delta t}{\Delta y}\left(F^{n+1/2}_{y,i,j-1/2,k} - F^{n+1/2}_{y,i,j+1/2,k}\right) \\
&+ \frac{1}{2}\frac{\Delta t}{\Delta z}\left(F^{n+1/2}_{z,i,j,k-1/2} - F^{n+1/2}_{z,i,j,k+1/2}\right) \\
Q^{n+1/2,*}_{i,j\pm1/2,k} =& Q^{n+1/2}_{i,j\pm1/2,k} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{x,i-1/2,j,k} - F^{n+1/2}_{x,i+1/2,j,k}\right) \\
&+ \frac{1}{2}\frac{\Delta t}{\Delta z}\left(F^{n+1/2}_{z,i,j,k-1/2} - F^{n+1/2}_{z,i,j,k+1/2}\right) \\
Q^{n+1/2,*}_{i,j,k\pm1/2} =& Q^{n+1/2}_{i,j,k\pm1/2} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{x,i-1/2,j,k} - F^{n+1/2}_{x,i+1/2,j,k}\right) \\
&+ \frac{1}{2}\frac{\Delta t}{\Delta y}\left(F^{n+1/2}_{y,i,j-1/2,k} - F^{n+1/2}_{y,i,j+1/2,k}\right)
\end{aligned}$$

(5) Recalculate the fluxes
(6) Update cell centred value.

Note that characteristic speed used to update the face values to $n + 1/2$ should really be calculated on a per component basis according to [7]. The current approach is correct for Maxwell's equations, due to the single wave speed, but for systems with more complicated wave structure a more accurate estimate is desirable.

4.2. **ODE solver.** We split the update equations such that fluxes and source terms are solved separately according to,

$$\begin{aligned}
\hat{Q} =& Q^n + \Delta t F\left(Q^n\right), \\
Q^{n+1} =& \hat{Q} + \Delta t S\left(\hat{Q}\right),
\end{aligned}$$

where $F(Q)$ is the update due to fluxes and $S(Q)$ is that due to source terms. The fluxes are calculated explicitly, as discussed in section 5, while the source terms are solved via one of a selection of ODE solvers.

4.2.1. *Euler ODE solvers.* The simplest approach is to use the forwards Euler form such that,

$$Q^{n+1} = \hat{Q} + \Delta t S\left(\hat{Q}\right).$$

Alternatively we may use the backwards Euler form such that,

$$S\left(\hat{Q}\right) = \left(\boldsymbol{I} - \Delta t \frac{\partial S}{\partial \hat{Q}}\right) S\left(Q^{n+1}\right),$$

where $\frac{\partial S}{\partial \hat{Q}}$is the Jacobian of the source term update equation. This forms a linear system that may be solved for $S\left(Q^{n+1}\right)$ and which may then be applied according to,

$$Q^{n+1} = \hat{Q} + \Delta t S\left(Q^{n+1}\right).$$

4.2.2. *Stiff ODE solvers.* For problems where the source terms themselves are quite stiff we may also use stiff ODE solvers. Possibilities are LSODA[8]and CVODE[9].

4.2.3. *Jacobian evaluation.* The Jacobian of the source terms may be calculated by explicit definition (if provided), numerical differentiation (finite differences), or with autodiff [10], a C++ library for automatic differentiation.

5. Fluxes

5.1. **Reconstruction.** Reconstruction is performed on primitive values along each dimension using a selection of TVD, central, or polynomial based functions.

5.2. **Fluxes.** The hydrodynamic fluxes are calculated according to a range of approximate Riemann solvers, the HLLE and HLLC solvers are given as follows [11].

5.2.1. *HLLE.* The HLLE solver is given as follows

$$\begin{aligned}
S_L &= \min\left(u_L - a_L,\, u_R - a_R\right), \\
S_R &= \max\left(u_L + a_L,\, u_R + a_R\right), \\
F^{\text{HLLE}} &= \begin{cases} F_L, & S_L > 0, \\ \frac{S_R F_L - S_L F_R + S_L S_R (U_R - U_L)}{S_R - S_L}, & S_L \leq 0 \leq S_R, \\ F_R, & S_R < 0, \end{cases}
\end{aligned}$$

where u is the local speed, a is the local sound speed, F is the flux vector, and U the state vector for the left L and right R sides of the interface.

5.2.2. *HLLC.* The HLLC solver is given according to

$$F^{\text{HLLC}} = \begin{cases} F_L, & S_L > 0, \\ F_L^*, & S_L \leq 0 \leq S_M, \\ F_R^*, & S_M \leq 0 \leq S_M, \\ F_R, & S_R < 0, \end{cases}$$

where

$$S_M = \frac{\rho_R u_R \left(S_R - u_R\right) - \rho_L u_L \left(S_L - u_L\right) + p_L - p_R}{\rho_R \left(S_R - u_R\right) - \rho_L \left(S_L - u_L\right)}.$$

5.2.3. *Electromagnetic Fluxes.* The characteristic speeds of Maxwells equations are

$$u^{\mathrm{EM}} = \left(0, \pm c, \pm c\Gamma_B, \pm c\Gamma_E\right),$$

thus the HLL flux is simplified to give,

$$S_L = -S_R = \min\left(0, u^{\mathrm{EM}}\right),$$
$$F^{\mathrm{HLL,EM}} = \frac{F_L + F_R + S_R\left(U_L - U_R\right)}{2}$$

where

$$U = \begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \\ \phi \\ \psi \end{pmatrix}, \qquad F_x = c_0 \begin{pmatrix} \phi \\ \frac{B_z}{\mu_z} \\ -\frac{B_y}{\mu_y} \\ \psi \\ -\frac{D_z}{\varepsilon_z} \\ \frac{D_y}{\varepsilon_y} \\ \frac{c_h^2}{c_0^2} D_x \\ \frac{c_h^2}{c_0^2} B_x \end{pmatrix}.$$

Alternatively the exact flux for the system of equations may be derived using the Rankine-Hugoniot relations [12] according to

$$\hat{F}_{D_y} = \frac{\hat{c}_0}{c_R\mu_R + c_L\mu_L}\left(s\left(c_R\hat{B}_z^R + c_L\hat{B}_z^L\right) + \left(\frac{\hat{D}_y^L}{\varepsilon_L} - \frac{\hat{D}_y^R}{\varepsilon_R}\right)\right), \quad \hat{F}_{D_z} = \frac{\hat{c}_0}{c_R\mu_R + c_L\mu_L}\left(-\left(c_R\hat{B}_y^R + c_L\hat{B}_y^L\right) + \left(\frac{\hat{D}_z^L}{\varepsilon_L} - \right.\right.$$
$$\hat{F}_{B_z} = \frac{\hat{c}_0}{c_R\varepsilon_R + c_L\varepsilon_L}\left(s\left(c_R\hat{D}_y^R + c_L\hat{D}_y^L\right) + \left(\frac{\hat{B}_z^L}{\mu_L} - \frac{\hat{B}_z^R}{\mu_R}\right)\right), \quad \hat{F}_{B_y} = \frac{\hat{c}_0}{c_R\varepsilon_R + c_L\varepsilon_L}\left(-\left(c_R\hat{D}_z^R + c_L\hat{D}_z^L\right) + \left(\frac{\hat{B}_y^L}{\mu_L} - \right.\right.$$
$$\hat{F}_{D_x} = \frac{\hat{c}_0}{2}\left(\hat{\phi}^L + \hat{\phi}^R\right) - \frac{\hat{c}_h}{2}\left(D_x^R - D_x^L\right), \quad \hat{F}_{B_x} = \frac{\hat{c}_0}{2}\left(\hat{\psi}^L + \hat{\psi}^R\right) - \frac{\hat{c}_h}{2}\left(B_x^R - B_x^L\right),$$
$$\hat{F}_{\phi} = \frac{\hat{c}_h^2}{2\hat{c}_0}\left(\hat{D}_x^L + \hat{D}_x^R\right) - \frac{\hat{c}_h}{2}\left(\hat{\phi}^R - \hat{\phi}^L\right), \quad \hat{F}_{\psi} = \frac{\hat{c}_h^2}{2\hat{c}_0}\left(\hat{B}_x^L + \hat{B}_x^R\right) - \frac{\hat{c}_h}{2}\left(\hat{\psi}^R - \hat{\psi}^L\right),$$

where $c_r = \frac{1}{\sqrt{\varepsilon_r\mu_r}}$, also note that ε and μ can be axis dependent and will then align with the field axis i.e. $\frac{D_y}{\epsilon_y}$. See Appendix E for further details.

5.2.4. *Blending.* While the HLLC solver is preferable, due to its ability to capture contact discontinuities exactly, it can also suffer from numerical instabilities such as the carbuncle effect. The HLLE scheme is not plagued by this problem and thus a switch is defined that blends the fluxes of the two schemes and thus support stable operation of the numerical method. The blending function is defined according to

$$F = \left(1 - \phi\right) F^{\mathrm{HLLC}} + \phi F^{\mathrm{HLLE}},$$

where

$$\phi = \frac{1 + \tanh\left(5\delta\left(\varphi - {}^{3}/_{4\delta}\right)\right)}{2}$$

and

$$\varphi = \left|\frac{p_R - p_L}{p_R + p_L}\right|.$$

This gives a detector that ramps up to 1 when a shock is detected at $\varphi \gtrsim {}^{1}/_{\delta}$, and zero otherwise.

6. EB

To combat the small cell problem with embedded boundaries we employ a form of cell merging. For each cut cell with volume-fraction less than some threshold value we generate an exclusive super-cell S_j by recursively absorbing the cell s_i that is attached to the first largest face of the current cell, such that $s_i \in S_j$, until the total volume fraction exceeds the threshold (typically 0.5). We then calculate the mean value of the super cell for each conserved quantity according to,

$$\rho_j = \frac{\sum_{i \in S_j} \phi_i \left(u_i + \Delta u_i\right)}{\sum_{i \in S_j} \phi_i}.$$

We then adjust the Δu_i values (calculated from the divergence operation) such that $u_i + \Delta u_i = \rho_j$. Essentially, we assume a constant value for our conserved quantity at time t^{n+1} across the entire super-cell and adjust the individual cell updates to achieve this. This approach allows for cell merging to be applied as a post-processing step following the divergence calculation.

7. Isotopes

In order to approximate isotopes within the solver we make each fluid a mixture where the mixture fraction is tracked by a passive scalar. Within a cell we have the total density, ρ, and the mass fraction α. We also know the mass, m_i, charge, q_i, and ratio of specific heats, γ_i, of each component. We desire to know the overall average quantities m, q, and γ.

Given

$$\begin{aligned} \rho &= mn, \\ \rho_0 &= m_0 n_0 = \left(1-\alpha\right)\rho, \\ \rho_1 &= m_1 n_1 = \alpha\rho, \end{aligned}$$

then

$$n_0 = \frac{\left(1-\alpha\right)\rho}{m_0}, \qquad n_1 = \frac{\alpha\rho}{m_1}.$$

Given that $n = \sum_i n_i$ we have

$$m = \frac{\rho}{n} = \frac{\rho}{\frac{(1-\alpha)\rho}{m_0} + \frac{\alpha\rho}{m_1}} = \frac{m_0 m_1}{m_0\alpha + m_1\left(1-\alpha\right)}.$$

The total charge density, $nq = \sum_i n_i q_i$, allows us to get the charge of the cell according to

$$q = \frac{\sum_i n_i q_i}{n} = \frac{\sum_i n_i q_i}{\sum_i n_i} = \frac{\alpha m_0 q_1 + \left(1-\alpha\right) m_1 q_0}{m_0\alpha + m_1\left(1-\alpha\right)}.$$

To find γ we have

$$\gamma = \frac{c_p}{c_v}, \qquad R = c_p - c_v = \frac{k_B}{m},$$

$$c_p = \frac{k_B\gamma}{m\left(\gamma-1\right)}, \qquad c_v = \frac{k_B}{m\left(\gamma-1\right)}.$$

The specific heats of a mixture are given by $c = \sum_i w_i c_i$ where w_i is the component mass fraction. We thus have (including some mathematica simplification)

$$
\begin{aligned}
\gamma &= \frac{\sum_i w_i c_{p,i}}{\sum_i w_i c_{v,i}}, \\
&= \frac{(1-\alpha)\, c_{p,0} + \alpha c_{p,1}}{(1-\alpha)\, c_{v,0} + \alpha c_{v,1}}, \\
&= \frac{(1-\alpha)\, \frac{k_B \gamma_0}{m_0(\gamma_0-1)} + \alpha \frac{k_B \gamma_1}{m_1(\gamma_1-1)}}{(1-\alpha)\, \frac{k_B}{m_0(\gamma_0-1)} + \alpha \frac{k_B}{m_1(\gamma_1-1)}}, \\
&= \frac{\frac{\gamma_0(1-\alpha)}{m_0(\gamma_0-1)} + \frac{\gamma_1 \alpha}{m_1(\gamma_1-1)}}{\frac{1-\alpha}{m_0(\gamma_0-1)} + \frac{\alpha}{m_1(\gamma_1-1)}}, \\
\gamma &= \frac{m_0 \alpha\, (\gamma_0 - 1)\, \gamma_1 + m_1 \gamma_0\, (\alpha - 1 + (1-\alpha)\, \gamma_1)}{m_0 \alpha\, (\gamma_0 - 1) + m_1\, (\alpha - 1 + (1-\alpha)\, \gamma_1)}.
\end{aligned}
$$

References

[1] C.-D. Munz, P. Ommes, and R. Schneider. A three-dimensional finite-volume solver for the maxwell equations with divergence cleaning on unstructured meshes. *Computer Physics Communications*, 130(1-2):83 – 117, 2000.

[2] Su Yan and Jian-Ming Jin. A continuity-preserving and divergence-cleaning algorithm based on purely and damped hyperbolic maxwell equations in inhomogeneous media. *Journal of Computational Physics*, 334:392 – 418, 2017.

[3] Zhenguang Huang, Gabor Toth, Bart van der Holst, Yuxi Chen, and Tamas Gombosi. A six-moment multi-fluid plasma model. *Journal of Computational Physics*, 387:134 – 153, 2019.

[4] D. Ghosh, T.D. Chapman, R.L. Berger, A. Dimits, and J.W. Banks. A multispecies, multifluid model for laser-induced counterstreaming plasma simulations. *Computers & Fluids*, 186:38–57, 2019.

[5] W. Fundamenski and O.E. Garcia. Comparison of coulomb collision rates in the plasma physics and magnetically confined fusion literature. Technical report, EUROfusion Consortium, 2007.

[6] Jose A. Moreno, Eduardo Oliva, and Pedro Velarde. EMcLAW: An unsplit Godunov method for Maxwell's equations including polarization, metals, divergence control and AMR. *Computer Physics Communications*, page 107268, 2020.

[7] Phillip Colella. Multidimensional upwind methods for hyperbolic conservation laws. *Journal of Computational Physics*, 87(1):171 – 200, 1990.

[8] A. C. Hindmarsh and L. R. Petzold. Lsoda, ordinary differential equation solver for stiff or non-stiff system, 2005.

[9] Alan C. Hindmarsh, Peter N. Brown, Keith E. Grant, Steven L. Lee, Radu Serban, Dan E. Shumaker, and Carol S. Woodward. Sundials: Suite of nonlinear and differential/algebraic equation solvers. *ACM Trans. Math. Softw.*, 31(3):363–396, September 2005.

[10] Allan M. M. Leal et al. autodiff, a modern, fast and expressive C++ library for automatic differentiation. `https://autodiff.github.io`, 2018.

[11] E. F. Toro, M. Spruce, and W. Speares. Restoration of the contact surface in the HLL-riemann solver. *Shock Waves*, 4(1):25–34, 1994.

[12] Alfonso Barbas and Pedro Velarde. Development of a godunov method for maxwell's equations with adaptive mesh refinement. *Journal of Computational Physics*, 300:186 – 201, 2015.

[13] John A. Krommes. Projection-operator methods for classical transport in magnetized plasmas. part 1. linear response, the braginskii equations and fluctuating hydrodynamics. *Journal of Plasma Physics*, 84(4):925840401, 2018.

[14] S. I. Braginskii. Transport processes in plasma. *Reviews of Plasma Physics*, 1:205, 1965.

Appendix A. Higher order terms closure

A.1. Neutral fluids. We need expressions for Π and $\boldsymbol{h}$ in order to close the system of equations. Assuming an isotropic viscosity for unmagnetized flow we may apply the following for the viscous stress tensor,

$$
\Pi = -\mu \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) - \lambda \mathbf{I} \nabla \cdot \boldsymbol{u},
$$

$$
\Pi = - \begin{pmatrix}
2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & 2\mu \frac{\partial v}{\partial y} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) & 2\mu \frac{\partial w}{\partial z} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\end{pmatrix},
$$

where μ is the viscosity and $\lambda = -\frac{2}{3}\mu$ is the bulk viscosity according to Stokes hypothesis. We may use a constant viscosity or any of the temperature dependent relationships such as a power law model or Sutherlands equation.

The heat flux is given by,

$$\boldsymbol{h} = -\kappa \nabla T,$$
$$\kappa = c_p \frac{\mu}{\mathrm{Pr}}.$$

A.2. **Magnetised fluids (NOT IMPLEMENTED).** For a magnetised fluid we have [13, 14]

$$
\begin{aligned}
\Pi &= -nm\boldsymbol{m} \colon \boldsymbol{S} \\
\Pi &= -nm \sum_{p=0}^{4} n\mu_p \boldsymbol{V}_p \boldsymbol{:} \boldsymbol{S} \\
\boldsymbol{V}_0 \colon \boldsymbol{S} &= \frac{3}{2}\left(\boldsymbol{B} - \frac{1}{3}\boldsymbol{I}\right)\left(\boldsymbol{B} - \frac{1}{3}\boldsymbol{I}\right) \boldsymbol{:} \boldsymbol{W} \\
\boldsymbol{V}_1 \colon \boldsymbol{S} &= \boldsymbol{\delta}^{\perp} \cdot \boldsymbol{W}.\boldsymbol{\delta}^{\perp} + \frac{1}{2}\left(\hat{\boldsymbol{b}} \cdot \boldsymbol{W}.\hat{\boldsymbol{b}}\right)\boldsymbol{\delta}^{\perp} \\
\boldsymbol{V}_2 \colon \boldsymbol{S} &= \boldsymbol{\delta}^{\perp} \cdot \boldsymbol{W}.\boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{W}.\boldsymbol{\delta}^{\perp} \\
\boldsymbol{V}_3 \colon \boldsymbol{S} &= \frac{1}{2}\left(\boldsymbol{\delta}^{\perp} \cdot \boldsymbol{W}.\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \boldsymbol{W}.\boldsymbol{\delta}^{\perp}\right) \\
\boldsymbol{V}_4 \colon \boldsymbol{S} &= \boldsymbol{B} \cdot \boldsymbol{W}.\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \boldsymbol{W}.\boldsymbol{B} \\
\boldsymbol{W} &= 2\left(\boldsymbol{S} - \frac{1}{3}\nabla \cdot \boldsymbol{u}\boldsymbol{I}\right) \\
\boldsymbol{S} &= \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right] \\
\boldsymbol{B} &= \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} = b_i b_j \\
\boldsymbol{\delta}^{\perp} &= \boldsymbol{I} - \boldsymbol{B} = \delta^{\perp}_{ij} \\
\boldsymbol{\beta} &= -\epsilon_{ijk} \cdot b_j \\
\hat{\boldsymbol{b}} &= {}^{\boldsymbol{B}}/_{|\boldsymbol{B}|}
\end{aligned}
$$

where (cgs units, quasi-neutral, ${}^{m_e}/_{m_i} \ll 1$)

$$
\begin{aligned}
\mu_0^i &= 0.96 n_i T_i \tau_i, & \mu_0^e &= 0.733 n_e T_e \tau_e, \\
\mu_2^i &= \frac{n_i T_i \tau_i}{\Delta}\left(\frac{6}{5}x^2 + 2.23\right) & \mu_2^e &= \frac{n_e T_e \tau_e}{\Delta}\left(2.05x^2 + 8.5\right) \\
\mu_4^i &= \frac{n_i T_i \tau_i x}{\Delta}\left(x^2 + 2.38\right) & \mu_4^e &= -\frac{n_e T_e \tau_e x}{\Delta}\left(x^2 + 7.91\right) \\
\mu_1^i &= \mu_2^i\,(2x) & \mu_1^e &= \mu_2^e\,(2x) \\
\mu_3^i &= \mu_4^i\,(2x) & \mu_3^e &= \mu_4^e\,(2x) \\
x &= \omega_i \tau_i & x &= \omega_e \tau_e \\
\Delta &= x^4 + 4.03x^2 + 2.33 & \Delta &= x^4 + 13.8x^2 + 11.6 \\
\omega_i &= \frac{ZeB}{m_i c} & \omega_e &= \frac{eB}{m_e c} \\
\tau_i &= \frac{3\sqrt{m_i} T_i^{3/2}}{4\sqrt{\pi}\lambda e^4 Z^4 n_i} & \tau_e &= \frac{3\sqrt{m_e} T_e^{3/2}}{4\sqrt{2\pi}\lambda e^4 Z^2 n_i} \\
\lambda &= \begin{cases} 23.4 - 1.15\log n + 3.45\log T_e, & T_e < 50eV, \\ 25.3 - 1.15\log n + 2.3\log T_e, & T_e > 50eV \end{cases}
\end{aligned}
$$

TODO: magnetised heat flux terms

Appendix B. Non-dimensionalizations

B.1. Interaction parameter.

$$
\begin{aligned}
\beta =& \frac{p}{{B^2}/{2\mu_0}} = \frac{2\mu_0 p}{B^2} \\
=& \left[\frac{\mu_0 m_0 n_0 u_0^2}{B_0^2}\right]\frac{2\hat{p}}{\hat{B}^2} \\
=& \left[\frac{\mu_0 m_0 n_0 u_0^2 \beta_0}{2\mu_0 n_0 m_0 u_0^2}\right]\frac{2\hat{p}}{\hat{B}^2} \\
=& \left[\frac{\beta_0}{2}\right]\frac{2\hat{p}}{\hat{B}^2} \\
\beta =& \frac{\beta_0 \hat{p}}{\hat{B}^2}
\end{aligned}
$$

B.2. Alfven speed.

$$
v_{A,x} = \sqrt{\frac{B_x^2}{\mu_0 \rho} + \frac{p_\perp - p_\parallel}{\rho} b_x^2}
$$

where $\rho = \sum_\alpha \rho_\alpha$, $p_\perp = \sum_\alpha p_\alpha^\perp$, $p_\parallel = \sum_\alpha p_\alpha^\parallel$ and $\boldsymbol{b} = \boldsymbol{B}/\left|\boldsymbol{B}\right|$.

$$
\beta_0 \equiv \frac{2\mu_0 n_0 m_0 u_0^2}{B_0^2}.
$$

$$
\begin{aligned}
v_A &= \sqrt{\frac{B^2}{\mu_0 \rho} + \frac{p_\perp - p_\parallel}{\rho} b^2} \\
\left[\frac{u_0^2}{1}\right]\hat{v}_A^2 &= \left[\frac{B_0^2}{\mu_0 \rho_0}\right]\frac{\hat{B}^2}{\hat{\rho}} + \left[\frac{p_0}{\rho_0}\right]\frac{\hat{p}_\perp - \hat{p}_\parallel}{\hat{\rho}} b^2 \\
\hat{v}_A^2 &= \left[\frac{B_0^2}{\mu_0 \rho_0 u_0^2}\right]\frac{\hat{B}^2}{\hat{\rho}} + \left[\frac{p_0}{\rho_0 u_0^2}\right]\frac{\hat{p}_\perp - \hat{p}_\parallel}{\hat{\rho}} b^2 \\
\hat{v}_A^2 &= \frac{2}{\beta_0}\frac{\hat{B}^2}{\hat{\rho}} + \frac{\hat{p}_\perp - \hat{p}_\parallel}{\hat{\rho}} b^2 \\
\hat{v}_A^2 &= \frac{2\hat{B}^2}{\beta_0 \hat{\rho}} + \frac{b^2}{\hat{\rho}}\left(\hat{p}_\perp - \hat{p}_\parallel\right)
\end{aligned}
$$

B.3. Plasma length scales.

B.3.1. *Reference.*

$$
\begin{aligned}
d_{D,0} =& \sqrt{\frac{\epsilon_0 k_B T_0}{n_0 q_0^2}} = \sqrt{\frac{u_0^2 \epsilon_0 m_0}{n_0 q_0^2}} = \sqrt{\frac{u_0^2 m_0}{c^2 \mu_0 n_0 q_0^2}}, \\
d_{L,0} =& \frac{u_0 m_0}{q_0 B_0},
\end{aligned}
$$

$$
\hat{d}_{D,0} = \frac{d_{D,0}}{x_0}, \qquad \hat{d}_{L,0} = \frac{d_{L,0}}{x_0}
$$

$$
\begin{aligned}
\frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} &= \frac{u_0 m_0}{x_0 q_0 B_0}\frac{x_0^2 n_0 q_0^2}{u_0^2 \epsilon_0 m_0} \\
\frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} &= \frac{x_0 n_0 q_0}{u_0 \epsilon_0 B_0}
\end{aligned}
$$

$$
d_{S,0} \equiv \frac{1}{q_0}\sqrt{\frac{m_0}{\mu_0 n_0}} = \frac{u_0 \hat{c}}{q_0}\sqrt{\frac{\epsilon_0 m_0}{n_0}} = \hat{c} d_{D,0}, \qquad \beta_0 \equiv \frac{2\mu_0 n_0 m_0 u_0^2}{B_0^2} = \frac{2}{\hat{c}^2}\frac{n_0 m_0}{\epsilon_0 B_0^2} = 2\frac{d_{L,0}^2}{d_{S,0}^2}.
$$

$$
\begin{aligned}
\sqrt{\frac{\beta_0}{2}}\frac{1}{\hat{d}_{S,0}} &= \sqrt{\frac{\mu_0 n_0 m_0 u_0^2}{B_0^2}}\frac{x_0 q_0}{1}\sqrt{\frac{\mu_0 n_0}{m_0}} \\
&= \sqrt{\frac{\mu_0^2 n_0^2 u_0^2 x_0^2 q_0^2}{B_0^2}} \\
&= \frac{\mu_0 n_0 u_0 x_0 q_0}{B_0} \\
&= \frac{n_0 u_0 x_0 q_0}{c^2 \epsilon_0 B_0} \\
\sqrt{\frac{\beta_0}{2}}\frac{1}{\hat{d}_{S,0}} &= \frac{1}{\hat{c}^2}\frac{n_0 x_0 q_0}{u_0 \epsilon_0 B_0} \\
\sqrt{\frac{\beta_0}{2}}\frac{\hat{c}^2}{\hat{d}_{S,0}} &= \frac{n_0 x_0 q_0}{u_0 \epsilon_0 B_0}
\end{aligned}
$$

$$
\begin{aligned}
\beta_0 &= 2\frac{d_{L,0}^2}{d_{S,0}^2}. \\
d_{L,0}^2 &= d_{S,0}^2 \frac{\beta_0}{2} \\
d_{L,0} &= d_{S,0}\sqrt{\frac{\beta_0}{2}}
\end{aligned}
$$

B.3.2. *Larmor.*

$$
\begin{aligned}
d_{L,\alpha} =& \frac{u_{\alpha\perp} m_\alpha}{|q_\alpha| B}, \\
[x_0]\, \hat{d}_{L,\alpha} =& \left[\frac{u_0 m_0}{q_0 B_0}\right] \frac{\hat{u}_{\alpha\perp}\hat{m}_\alpha}{|\hat{q}_\alpha| \hat{B}}, \\
\hat{d}_{L,\alpha} =& \left[\frac{u_0 m_0}{q_0 B_0 x_0}\right] \frac{\hat{u}_{\alpha\perp}\hat{m}_\alpha}{|\hat{q}_\alpha| \hat{B}}, \\
\hat{d}_{L,\alpha} =& \hat{d}_{L,0}\frac{\hat{u}_{\alpha\perp}\hat{m}_\alpha}{|\hat{q}_\alpha| \hat{B}}, \\
\hat{d}_{L,\alpha} =& \hat{c}\hat{d}_{D,0}\frac{\hat{u}_{\alpha\perp}\hat{m}_\alpha}{|\hat{q}_\alpha|}\sqrt{\frac{\beta}{2\hat{p}}},
\end{aligned}
$$

B.3.3. *Debye.*

$$
\begin{aligned}
d_{D,\alpha} &= \sqrt{\frac{\epsilon_0 k_B T_\alpha}{n_\alpha q_\alpha^2}} \\
\left[x_0^2\right] \hat{d}_{D,\alpha}^2 &= \left[\frac{\epsilon_0 k_B T_0}{n_0 q_0^2}\right]\frac{\hat{T}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2} \\
\hat{d}_{D,\alpha}^2 &= \left[\frac{\epsilon_0 k_B T_0}{x_0^2 n_0 q_0^2}\right]\frac{\hat{T}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2} \\
\hat{d}_{D,\alpha}^2 &= \hat{d}_{D,0}^2\frac{\hat{T}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2} \\
\hat{d}_{D,\alpha} &= \hat{d}_{D,0}\sqrt{\frac{\hat{T}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2}} \\
\hat{d}_{D,\alpha} &= \frac{\hat{d}_{D,0}}{\hat{n}_\alpha \hat{q}_\alpha}\sqrt{\hat{p}_\alpha}
\end{aligned}
$$

B.3.4. *Skin depth.*

$$
\begin{aligned}
d_{S,\alpha} &= \frac{1}{q_\alpha}\sqrt{\frac{m_\alpha}{\mu_0 n_\alpha}}, \\
[x_0]\,\hat{d}_{S,\alpha} &= \left[\frac{1}{q_0}\right]\frac{1}{\hat{q}_\alpha}\sqrt{\left[\frac{m_0}{\mu_0 n_0}\right]\frac{\hat{m}_\alpha}{\hat{n}_\alpha}}, \\
\hat{d}^2_{S,\alpha} &= \left[\frac{m_0}{q_0^2 x_0^2 \mu_0 n_0}\right]\frac{\hat{m}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2}, \\
\hat{d}^2_{S,\alpha} &= \left[\frac{m_0 \epsilon_0 c_0^2}{q_0^2 x_0^2 n_0}\right]\frac{\hat{m}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2}, \\
\hat{d}^2_{S,\alpha} &= \left[\frac{m_0 \epsilon_0 u_0^2}{q_0^2 x_0^2 n_0}\right]\frac{\hat{c}^2\hat{m}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2}, \\
\hat{d}^2_{S,\alpha} &= \hat{d}^2_{D,0}\frac{\hat{c}^2\hat{m}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2}, \\
\hat{d}^2_{S,\alpha} &= \frac{\hat{d}^2_{S,0}\hat{m}_\alpha}{\hat{n}_\alpha \hat{q}_\alpha^2},
\end{aligned}
$$

B.4. **Frequencies.**

B.4.1. *Plasma.*

$$
\begin{aligned}
\omega_p &= \sqrt{\frac{nq^2}{\epsilon_0 m}} \\
\left[\frac{u_0^2}{x_0^2}\right]\hat{\omega}_p^2 &= \left[\frac{n_0 q_0^2}{\epsilon_0 m_0}\right]\frac{\hat{n}\hat{q}^2}{\hat{m}} \\
\hat{\omega}_p^2 &= \left[\frac{x_0^2 n_0 q_0^2}{u_0^2 \epsilon_0 m_0}\right]\frac{\hat{n}\hat{q}^2}{\hat{m}} \\
\hat{\omega}_p^2 &= \frac{\hat{n}\hat{q}^2}{\hat{d}^2_{D,0}\hat{m}} \\
\hat{\omega}_p^2 &= \frac{\hat{c}^2\hat{n}\hat{q}^2}{\hat{d}^2_{S,0}\hat{m}}
\end{aligned}
$$

B.4.2. *Cyclotron.*

$$
\begin{aligned}
\omega_c &= \frac{qB}{m} \\
\left[\frac{u_0}{x_0}\right]\hat{\omega}_c &= \left[\frac{q_0 B_0}{m_0}\right]\frac{\hat{q}\hat{B}}{\hat{m}} \\
\hat{\omega}_c &= \left[\frac{x_0 q_0 B_0}{u_0 m_0}\right]\frac{\hat{q}\hat{B}}{\hat{m}} \\
\hat{\omega}_c &= \frac{\hat{q}\hat{B}}{\hat{d}_{L,0}\hat{m}} \\
\hat{\omega}_c &= \frac{\hat{q}\hat{B}}{d_{S,0}\hat{m}}\sqrt{\frac{2}{\beta_0}}
\end{aligned}
$$

Appendix C. Collisional

C.1. **Viscosity.** $\mu \to \frac{kg}{ms}$, $c_1 \to \frac{kg}{msK^{1/2}}$, $C_2 \to K$, $\mathrm{Re}_0 = \frac{\rho_0 u_0 x_0}{\mu_0}$

TABLE 2. Properties of common gases at 1atm and 20°C

Gas	$\mu_0 \times 10^6$, $\frac{Ns}{m^2}$	power-law exponent, n
H_2	9.05	0.68
He	19.7	0.67
Ar	22.4	0.72

$$\mu = \frac{C_1 T^{3/2}}{T + C_2}$$
$$\left[\frac{\rho_0 x_0 u_0}{1}\right] \hat{\mu} = \left[\frac{\rho_0 x_0 u_0}{T_0^{1/2}} \frac{T_0^{3/2}}{T_0}\right] \frac{\hat{C}_1 \hat{T}^{3/2}}{\hat{T} + \hat{C}_2}$$
$$\hat{\mu} = \frac{\hat{C}_1 \hat{T}^{3/2}}{\hat{T} + \hat{C}_2}$$

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T + S}$$
$$\hat{\mu} = \hat{\mu}_0 \left(\frac{\hat{T}}{\hat{T}_0}\right)^{3/2} \frac{\hat{T}_0 + \hat{S}}{\hat{T} + \hat{S}}$$

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^n$$
$$\left[\frac{\rho_0 x_0 u_0}{1}\right] \hat{\mu} = \left[\frac{\rho_0 x_0 u_0}{1} \frac{T_0^n}{T_0^n}\right] \hat{\mu}_0 \left(\frac{\hat{T}}{\hat{T}_0}\right)^n$$
$$\hat{\mu} = \hat{\mu}_0 \left(\frac{\hat{T}}{\hat{T}_0}\right)^n$$

$$\hat{\mu}_0 = \frac{\mu_0}{\rho_0 x_0 u_0} = \frac{1}{\mathrm{Re}_0}$$

C.2. **Heat flux.**

$$\boldsymbol{h} = -\kappa \nabla T,$$
$$\kappa = c_p \frac{\mu}{\mathrm{Pr}},$$

$c_p \to \frac{m^2}{s^2 K}$, $\kappa \to \frac{kgm}{s^3 K}$, $\boldsymbol{h} \to \frac{kg}{s^3}$

$$\kappa = c_p \frac{\mu}{\mathrm{Pr}},$$
$$\left[\frac{m_0 u_0^3 x_0}{x_0^3 T_0}\right] \hat{\kappa} = \left[\frac{x_0^2 u_0^2}{x_0^2 T_0} \frac{m_0 u_0}{x_0^2}\right] \hat{c}_p \frac{\hat{\mu}}{\mathrm{Pr}}$$
$$\hat{\kappa} = \hat{c}_p \frac{\hat{\mu}}{\mathrm{Pr}}$$

C.3. **Collision frequency.**

C.3.1. *Charged.*

$$\nu_{\alpha\beta} = \frac{\sqrt{2} n_\beta Z_\alpha^2 Z_\beta^2 e^4 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\epsilon_0^2 m_{\alpha\beta} m_\alpha} \left[\left(\frac{2}{9\pi}\right)^{1/3} (\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta)\cdot(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta) + \frac{k_B T_\alpha}{m_\alpha} + \frac{k_B T_\beta}{m_\beta} \right]^{-3/2}$$

$$\left[\frac{u_0}{x_0}\right] \hat{\nu}_{\alpha\beta} = \left[\frac{n_0 q_0^4}{\epsilon_0^2 m_0^2}\right] \frac{\sqrt{2}\hat{n}_\beta \hat{q}_\alpha^2 \hat{q}_\beta^2 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\hat{m}_{\alpha\beta}\hat{m}_\alpha} \left[[u_0^2] \left(\left(\frac{2}{9\pi}\right)^{1/3} (\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta)\cdot(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta) + \frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta} \right) \right]^{-3/2}$$

$$\left[\frac{u_0}{x_0}\right] \hat{\nu}_{\alpha\beta} = \left[\frac{n_0 q_0^4}{\epsilon_0^2 m_0^2 u_0^3}\right] \frac{\sqrt{2}\hat{n}_\beta \hat{q}_\alpha^2 \hat{q}_\beta^2 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\hat{m}_{\alpha\beta}\hat{m}_\alpha} \left[\left(\frac{2}{9\pi}\right)^{1/3} (\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta)\cdot(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta) + \frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta} \right]^{-3/2}$$

$$\hat{\nu}_{\alpha\beta} = \left[\frac{n_0 q_0^4 x_0}{\epsilon_0^2 m_0^2 u_0^4}\right] \frac{\sqrt{2}\hat{n}_\beta \hat{q}_\alpha^2 \hat{q}_\beta^2 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\hat{m}_{\alpha\beta}\hat{m}_\alpha} \left[\left(\frac{2}{9\pi}\right)^{1/3} (\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta)\cdot(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta) + \frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta} \right]^{-3/2}$$

$$\hat{\nu}_{\alpha\beta} = \left[\frac{1}{n_0 x_0^3}\right] \frac{1}{\hat{d}_{D,0}^4} \frac{\sqrt{2}\hat{n}_\beta \hat{q}_\alpha^2 \hat{q}_\beta^2 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\hat{m}_{\alpha\beta}\hat{m}_\alpha} \left[\left(\frac{2}{9\pi}\right)^{1/3} (\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta)\cdot(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta) + \frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta} \right]^{-3/2}$$

$$\hat{\nu}_{\alpha\beta} = \frac{1}{\hat{n}_0 \hat{d}_{D,0}^4} \frac{\sqrt{2}\hat{n}_\beta \hat{q}_\alpha^2 \hat{q}_\beta^2 \ln\Lambda_{\alpha\beta}}{12\pi^{3/2}\hat{m}_{\alpha\beta}\hat{m}_\alpha} \left[\left(\frac{2}{9\pi}\right)^{1/3} (\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta)\cdot(\hat{\boldsymbol{u}}_\alpha - \hat{\boldsymbol{u}}_\beta) + \frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta} \right]^{-3/2}$$

$$\hat{d}_{D,0} = \frac{1}{x_0}\sqrt{\frac{u_0^2 \epsilon_0 m_0}{n_0 q_0^2}}$$

$$\hat{d}_{D,0}^4 = \frac{u_0^4 \epsilon_0^2 m_0^2}{x_0^4 n_0^2 q_0^4}$$

$$\frac{n_0 q_0^4 x_0}{\epsilon_0^2 m_0^2 u_0^4} = \frac{1}{n_0 x_0^3}\frac{x_0^4 n_0^2 q_0^4}{u_0^4 \epsilon_0^2 m_0^2}$$

$$\frac{n_0 q_0^4 x_0}{\epsilon_0^2 m_0^2 u_0^4} = \frac{1}{\hat{n}_0 \hat{d}_{D,0}^4}$$

C.3.2. *Neutral.*

$$\nu_{\alpha\beta} = \frac{4}{3}\frac{n_\beta m_\beta \sigma_{\alpha\beta}}{m_\alpha + m_\beta}\sqrt{\frac{8 k_B T_\alpha}{\pi m_\alpha} + \frac{8 k_B T_\beta}{\pi m_\beta}}$$

$$\left[\frac{u_0}{x_0}\right] \hat{\nu}_{\alpha\beta} = \left[\frac{n_0 m_0 x_0^2}{m_0}\right] \frac{4}{3}\frac{\hat{n}_\beta \hat{m}_\beta \hat{\sigma}_{\alpha\beta}}{\hat{m}_\alpha + \hat{m}_\beta}\sqrt{[u_0^2]\frac{8}{\pi}\left(\frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta}\right)}$$

$$\left[\frac{u_0}{x_0}\right] \hat{\nu}_{\alpha\beta} = \left[\frac{n_0 x_0^2 u_0}{1}\right] \frac{4}{3}\frac{\hat{n}_\beta \hat{m}_\beta \hat{\sigma}_{\alpha\beta}}{\hat{m}_\alpha + \hat{m}_\beta}\sqrt{[u_0^2]\frac{8}{\pi}\left(\frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta}\right)}$$

$$\hat{\nu}_{\alpha\beta} = \left[\frac{n_0 x_0^3}{1}\right] \frac{4}{3}\frac{\hat{n}_\beta \hat{m}_\beta \hat{\sigma}_{\alpha\beta}}{\hat{m}_\alpha + \hat{m}_\beta}\sqrt{[u_0^2]\frac{8}{\pi}\left(\frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta}\right)}$$

$$\hat{\nu}_{\alpha\beta} = \frac{4}{3}\frac{\hat{n}_0 \hat{n}_\beta \hat{m}_\beta \hat{\sigma}_{\alpha\beta}}{\hat{m}_\alpha + \hat{m}_\beta}\sqrt{\frac{8}{\pi}\left(\frac{\hat{T}_\alpha}{\hat{m}_\alpha} + \frac{\hat{T}_\beta}{\hat{m}_\beta}\right)}$$

C.4. **Friction force.**

$$\boldsymbol{R}_\alpha^{\alpha\beta} = m_\alpha n_\alpha \nu_{\alpha\beta}(\boldsymbol{u}_\beta - \boldsymbol{u}_\alpha)$$

$$\boldsymbol{R}_\alpha^{\alpha\beta} = \left[\frac{m_0 n_0 u_0^2}{x_0}\right] \hat{m}_\alpha \hat{n}_\alpha \hat{\nu}_{\alpha\beta}(\hat{\boldsymbol{u}}_\beta - \hat{\boldsymbol{u}}_\alpha)$$

$$\hat{\boldsymbol{R}}_\alpha^{\alpha\beta} = \hat{m}_\alpha \hat{n}_\alpha \hat{\nu}_{\alpha\beta}(\hat{\boldsymbol{u}}_\beta - \hat{\boldsymbol{u}}_\alpha) = \boldsymbol{R}_\alpha^{\alpha\beta}\left[\frac{x_0}{\rho_0 u_0^2}\right]$$

Appendix D. Evolution equations

D.1. Mass.

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha) = S_\alpha$$
$$\left[\frac{\rho_0 u_0}{x_0}\right] \frac{\partial \hat{\rho}_\alpha}{\partial \hat{t}} + \left[\frac{\rho_0 u_0}{x_0}\right] \hat{\nabla} \cdot (\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha) = \left[\frac{\rho_0 u_0}{x_0}\right] S_\alpha$$
$$\frac{\partial \hat{\rho}_\alpha}{\partial \hat{t}} + \hat{\nabla} \cdot (\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha) = S_\alpha$$

D.2. Momentum.

$$\frac{\partial \rho_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha + \mathbb{P}_\alpha) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \sum_\beta \boldsymbol{R}_\alpha^{\alpha\beta} + \rho_\alpha g$$
$$= \frac{\rho_\alpha q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \sum_\beta \boldsymbol{R}_\alpha^{\alpha\beta} + \rho_\alpha g$$

$$\left[\frac{\rho_0 u_0^2}{x_0}\right] \frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \left[\frac{\rho_0 u_0^2}{x_0}\right] \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \left[\frac{\rho_0 q_0}{m_0}\right] \frac{\hat{\rho}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha} \left([cB_0]\, \hat{\mathbf{E}} + [u_0 B_0]\, \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \left[\frac{\rho_0 u_0^2}{x_0}\right] \left(\sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta}\right.$$

$$\left[\frac{\rho_0 u_0^2}{x_0}\right] \frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \left[\frac{\rho_0 u_0^2}{x_0}\right] \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \left[\frac{\rho_0 q_0}{m_0}\right] \frac{\hat{\rho}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha} \left([u_0 B_0]\, \hat{c}\hat{\mathbf{E}} + [u_0 B_0]\, \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \left[\frac{\rho_0 u_0^2}{x_0}\right] \left(\sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta}\right.$$

$$\left[\frac{\rho_0 u_0^2}{x_0}\right] \frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \left[\frac{\rho_0 u_0^2}{x_0}\right] \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \left[\frac{\rho_0 u_0 B_0 q_0}{m_0}\right] \frac{\hat{\rho}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha} \left(\hat{c}\hat{\mathbf{E}} + \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \left[\frac{\rho_0 u_0^2}{x_0}\right] \left(\sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta} + \hat{\rho}_\alpha \hat{g}\right)$$

$$\frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \left[\frac{x_0 \rho_0 u_0 B_0 q_0}{\rho_0 u_0^2 m_0}\right] \frac{\hat{\rho}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha} \left(\hat{c}\hat{\mathbf{E}} + \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta} + \hat{\rho}_\alpha \hat{g}$$

$$\frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \left[\frac{x_0 B_0 q_0}{u_0 m_0}\right] \frac{\hat{\rho}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha} \left(\hat{c}\hat{\mathbf{E}} + \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta} + \hat{\rho}_\alpha \hat{g}$$

$$\frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \frac{\hat{\rho}_\alpha \hat{q}_\alpha}{\hat{d}_{L,0} \hat{m}_\alpha} \left(\hat{c}\hat{\mathbf{E}} + \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta} + \hat{\rho}_\alpha \hat{g}$$

$$\frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \frac{\hat{\rho}_\alpha \hat{r}_\alpha}{\hat{d}_{L,0}} \left(\hat{c}\hat{\mathbf{E}} + \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta} + \hat{\rho}_\alpha \hat{g}$$

$$\frac{\partial \hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha}{\partial \hat{t}} + \hat{\nabla} \cdot \left(\hat{\rho}_\alpha \hat{\mathbf{u}}_\alpha \otimes \hat{\mathbf{u}}_\alpha + \hat{\mathbb{P}}_\alpha\right) = \sqrt{\frac{2}{\beta_0}} \frac{\hat{\rho}_\alpha \hat{r}_\alpha}{\hat{d}_{S,0}} \left(\hat{c}\hat{\mathbf{E}} + \hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) + \sum_\beta \hat{\boldsymbol{R}}_\alpha^{\alpha\beta} + \hat{\rho}_\alpha \hat{g},$$

where

$$\mathbb{P}_\alpha = \begin{bmatrix} b_x^2 \left(p^{\|} - p^{\perp}\right) + p^{\perp} & b_x b_y \left(p^{\|} - p^{\perp}\right) & b_x b_z \left(p^{\|} - p^{\perp}\right) \\ b_x b_y \left(p^{\|} - p^{\perp}\right) & b_y^2 \left(p^{\|} - p^{\perp}\right) + p^{\perp} & b_y b_z \left(p^{\|} - p^{\perp}\right) \\ b_x b_z \left(p^{\|} - p^{\perp}\right) & b_y b_z \left(p^{\|} - p^{\perp}\right) & b_z^2 \left(p^{\|} - p^{\perp}\right) + p^{\perp} \end{bmatrix}$$

D.3. Pressure.

$$\frac{\partial p_\alpha^{\|}}{\partial t} + \nabla \cdot \left(p_\alpha^{\|} \boldsymbol{u}_\alpha\right) = -2 p_\alpha^{\|} \boldsymbol{b} \cdot (\boldsymbol{b} \cdot \nabla)\, \boldsymbol{u}_\alpha + \frac{p_\alpha - p_\alpha^{\|}}{\tau_\alpha},$$
$$\left[\frac{\rho_0 u_0^3}{x_0}\right] \frac{\partial \hat{p}_\alpha^{\|}}{\partial \hat{t}} + \left[\frac{\rho_0 u_0^3}{x_0}\right] \hat{\nabla} \cdot \left(\hat{p}_\alpha^{\|} \hat{\boldsymbol{u}}_\alpha\right) = -\left[\frac{\rho_0 u_0^3}{x_0}\right] 2 \hat{p}_\alpha^{\|} \boldsymbol{b} \cdot \left(\boldsymbol{b} \cdot \hat{\nabla}\right) \hat{\boldsymbol{u}}_\alpha + \left[\frac{\rho_0 u_0^3}{x_0}\right] \frac{\hat{p}_\alpha - \hat{p}_\alpha^{\|}}{\hat{\tau}_\alpha},$$
$$\frac{\partial \hat{p}_\alpha^{\|}}{\partial \hat{t}} + \hat{\nabla} \cdot \left(\hat{p}_\alpha^{\|} \hat{\boldsymbol{u}}_\alpha\right) = -2 \hat{p}_\alpha^{\|} \boldsymbol{b} \cdot \left(\boldsymbol{b} \cdot \hat{\nabla}\right) \hat{\boldsymbol{u}}_\alpha + \frac{\hat{p}_\alpha - \hat{p}_\alpha^{\|}}{\hat{\tau}_\alpha}$$

where

$$
\begin{aligned}
\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u} =& \boldsymbol{b}\cdot\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \\
\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u} =& \begin{pmatrix} b_x\frac{\partial u}{\partial x} + b_y\frac{\partial v}{\partial x} + b_z\frac{\partial w}{\partial x} \\ b_x\frac{\partial u}{\partial y} + b_y\frac{\partial v}{\partial y} + b_z\frac{\partial w}{\partial y} \\ b_x\frac{\partial u}{\partial z} + b_y\frac{\partial v}{\partial z} + b_z\frac{\partial w}{\partial z} \end{pmatrix} \\
\boldsymbol{b}\cdot\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u} =& b_x\left(b_x\frac{\partial u}{\partial x} + b_y\frac{\partial v}{\partial x} + b_z\frac{\partial w}{\partial x}\right) + b_y\left(b_x\frac{\partial u}{\partial y} + b_y\frac{\partial v}{\partial y} + b_z\frac{\partial w}{\partial y}\right) + b_z\left(b_x\frac{\partial u}{\partial z} + b_y\frac{\partial v}{\partial z} + b_z\frac{\partial w}{\partial z}\right)
\end{aligned}
$$

D.4. **Energy.**

$$
\begin{aligned}
Q_{\alpha\beta}^{\text{fric}} &= m_{\alpha\beta}n_\alpha\nu_{\alpha\beta}\left(\boldsymbol{u}_\alpha-\boldsymbol{u}_\beta\right)\cdot\left(\boldsymbol{u}_\alpha-\boldsymbol{u}_\beta\right), \\
\left[\frac{\rho_0u_0^3}{x_0}\right]\hat{Q}_{\alpha\beta}^{\text{fric}} &= \left[\frac{\rho_0u_0^3}{x_0}\right]\hat{m}_{\alpha\beta}\hat{n}_\alpha\hat{\nu}_{\alpha\beta}\left(\hat{\boldsymbol{u}}_\alpha-\hat{\boldsymbol{u}}_\beta\right)\cdot\left(\hat{\boldsymbol{u}}_\alpha-\hat{\boldsymbol{u}}_\beta\right), \\
\hat{Q}_{\alpha\beta}^{\text{fric}} &= \hat{m}_{\alpha\beta}\hat{n}_\alpha\hat{\nu}_{\alpha\beta}\left(\hat{\boldsymbol{u}}_\alpha-\hat{\boldsymbol{u}}_\beta\right)\cdot\left(\hat{\boldsymbol{u}}_\alpha-\hat{\boldsymbol{u}}_\beta\right),
\end{aligned}
$$

$$
\begin{aligned}
Q_{\alpha\beta}^{\text{eq}} &= 3m_\alpha n_\alpha\frac{\nu_{\alpha\beta}}{m_\alpha+m_\beta}\left(k_BT_\beta-k_BT_\alpha\right), \\
\left[\frac{\rho_0u_0^3}{x_0}\right]Q_{\alpha\beta}^{\text{eq}} &= \left[\frac{\rho_0u_0k_BT_0}{x_0m_0}\right]3\hat{m}_\alpha\hat{n}_\alpha\frac{\hat{\nu}_{\alpha\beta}}{\hat{m}_\alpha+\hat{m}_\beta}\left(\hat{T}_\beta-\hat{T}_\alpha\right), \\
\left[\frac{\rho_0u_0^3}{x_0}\right]Q_{\alpha\beta}^{\text{eq}} &= \left[\frac{\rho_0u_0^3}{x_0}\right]3\hat{m}_\alpha\hat{n}_\alpha\frac{\hat{\nu}_{\alpha\beta}}{\hat{m}_\alpha+\hat{m}_\beta}\left(\hat{T}_\beta-\hat{T}_\alpha\right), \\
Q_{\alpha\beta}^{\text{eq}} &= 3\hat{m}_\alpha\hat{n}_\alpha\frac{\hat{\nu}_{\alpha\beta}}{\hat{m}_\alpha+\hat{m}_\beta}\left(\hat{T}_\beta-\hat{T}_\alpha\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\epsilon_\alpha}{\partial t}+\nabla\cdot\left(\epsilon_\alpha\boldsymbol{u}_\alpha+\boldsymbol{u}_\alpha\cdot\mathbb{P}_\alpha+\boldsymbol{h}_\alpha\right) =& \boldsymbol{u}_\alpha\cdot\left(n_\alpha q_\alpha\mathbf{E}+\sum_{\beta\neq\alpha}\boldsymbol{R}_{\alpha\beta}+\rho_\alpha g\right)+\sum_{\beta\neq\alpha}Q_{\alpha\beta}, \\
\left[\frac{\rho_0u_0^3}{x_0}\right]\frac{\partial\hat{\epsilon}_\alpha}{\partial\hat{t}}+\left[\frac{\rho_0u_0^3}{x_0}\right]\hat{\nabla}\cdot\left(\hat{\epsilon}_\alpha\hat{\boldsymbol{u}}_\alpha+\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbb{P}}_\alpha+\hat{\boldsymbol{h}}_\alpha\right) =& \left[n_0q_0u_0^2B_0\right]\hat{n}_\alpha\hat{q}_\alpha\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbf{E}}+\left[\frac{\rho_0u_0^3}{x_0}\right]\hat{\boldsymbol{u}}_\alpha\cdot\left(\sum_{\beta\neq\alpha}\hat{\boldsymbol{R}}_{\alpha\beta}+\hat{\rho}_\alpha\hat{g}\right)+\Big[\\
\frac{\partial\hat{\epsilon}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\epsilon}_\alpha\hat{\boldsymbol{u}}_\alpha+\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbb{P}}_\alpha+\hat{\boldsymbol{h}}_\alpha\right) =& \left[\frac{x_0n_0q_0u_0^2B_0}{\rho_0u_0^3}\right]\hat{n}_\alpha\hat{q}_\alpha\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbf{E}}+\hat{\boldsymbol{u}}_\alpha\cdot\left(\sum_{\beta\neq\alpha}\hat{\boldsymbol{R}}_{\alpha\beta}+\hat{\rho}_\alpha\hat{g}\right)+\sum_{\beta\neq\alpha}\hat{Q} \\
\frac{\partial\hat{\epsilon}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\epsilon}_\alpha\hat{\boldsymbol{u}}_\alpha+\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbb{P}}_\alpha+\hat{\boldsymbol{h}}_\alpha\right) =& \frac{\hat{n}_\alpha\hat{q}_\alpha\hat{c}}{\hat{d}_{L,0}}\hat{n}_\alpha\hat{q}_\alpha\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbf{E}}+\hat{\boldsymbol{u}}_\alpha\cdot\left(\sum_{\beta\neq\alpha}\hat{\boldsymbol{R}}_{\alpha\beta}+\hat{\rho}_\alpha\hat{g}\right)+\sum_{\beta\neq\alpha}\hat{Q}_{\alpha\beta}, \\
\frac{\partial\hat{\epsilon}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\epsilon}_\alpha\hat{\boldsymbol{u}}_\alpha+\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbb{P}}_\alpha+\hat{\boldsymbol{h}}_\alpha\right) =& \hat{\boldsymbol{u}}_\alpha\cdot\left(\frac{\hat{\rho}_\alpha\hat{r}_\alpha\hat{c}}{\hat{d}_{L,0}}\hat{\mathbf{E}}+\sum_{\beta\neq\alpha}\hat{\boldsymbol{R}}_{\alpha\beta}+\hat{\rho}_\alpha\hat{g}\right)+\sum_{\beta\neq\alpha}\hat{Q}_{\alpha\beta}, \\
\frac{\partial\hat{\epsilon}_\alpha}{\partial\hat{t}}+\hat{\nabla}\cdot\left(\hat{\epsilon}_\alpha\hat{\boldsymbol{u}}_\alpha+\hat{\boldsymbol{u}}_\alpha\cdot\hat{\mathbb{P}}_\alpha+\hat{\boldsymbol{h}}_\alpha\right) =& \hat{\boldsymbol{u}}_\alpha\cdot\left(\sqrt{\frac{2}{\beta_0}}\frac{\hat{\rho}_\alpha\hat{r}_\alpha\hat{c}}{\hat{d}_{S,0}}\hat{\mathbf{E}}+\sum_{\beta\neq\alpha}\hat{\boldsymbol{R}}_{\alpha\beta}+\hat{\rho}_\alpha\hat{g}\right)+\sum_{\beta\neq\alpha}\hat{Q}_{\alpha\beta},
\end{aligned}
$$

D.5. **Electric field update.** Reference quantities $D_0=\varepsilon_0E_0$, $B_0=\mu_0H_0$, $E_0=c_0B_0$
$D_0={}^{H_0}/_{c_0}$
$\phi=\hat{\phi}H_0$, $\psi=\hat{\psi}E_0$

$$
\begin{aligned}
\frac{\partial \boldsymbol{D}}{\partial t} =& \nabla \times \boldsymbol{H} - \boldsymbol{J} - \nabla \phi, \\
\left[\frac{D_0 u_0}{x_0}\right] \frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \left[\frac{H_0}{x_0}\right] \hat{\nabla} \times \hat{\boldsymbol{H}} - \left[\frac{n_0 q_0 u_0}{1}\right] \hat{\boldsymbol{J}} - \left[\frac{H_0}{x_0}\right] \hat{\nabla} \hat{\phi}, \\
\left[\frac{H_0 u_0}{x_0 c_0}\right] \frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \left[\frac{H_0}{x_0}\right] \hat{\nabla} \times \hat{\boldsymbol{H}} - \left[\frac{n_0 q_0 u_0}{1}\right] \hat{\boldsymbol{J}} - \left[\frac{H_0}{x_0}\right] \hat{\nabla} \hat{\phi}, \\
\left[\frac{H_0}{x_0}\right] \frac{1}{\hat{c}_0} \frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \left[\frac{H_0}{x_0}\right] \hat{\nabla} \times \hat{\boldsymbol{H}} - \left[\frac{n_0 q_0 u_0}{1}\right] \hat{\boldsymbol{J}} - \left[\frac{H_0}{x_0}\right] \hat{\nabla} \hat{\phi}, \\
\frac{1}{\hat{c}_0} \frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \hat{\nabla} \times \hat{\boldsymbol{H}} - \left[\frac{n_0 q_0 u_0 x_0}{H_0}\right] \hat{\boldsymbol{J}} - \hat{\nabla} \hat{\phi}, \\
\frac{1}{\hat{c}_0} \frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \hat{\nabla} \times \hat{\boldsymbol{H}} - \left[\frac{n_0 q_0 x_0}{u_0 \varepsilon_0 B_0}\right] \frac{\hat{\boldsymbol{J}}}{\hat{c}_0^2} - \hat{\nabla} \hat{\phi}, \\
\frac{1}{\hat{c}_0} \frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \hat{\nabla} \times \hat{\boldsymbol{H}} - \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} \frac{\hat{\boldsymbol{J}}}{\hat{c}_0^2} - \hat{\nabla} \hat{\psi}_D, \\
\frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} =& \hat{c}_0 \hat{\nabla} \times \hat{\boldsymbol{H}} - \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} \frac{\hat{\boldsymbol{J}}}{\hat{c}_0} - \hat{c}_0 \hat{\nabla} \hat{\phi}, \\
\frac{\partial \hat{\boldsymbol{D}}}{\partial \hat{t}} - \hat{c}_0 \hat{\nabla} \times \hat{\boldsymbol{H}} =& - \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2 \hat{c}_0} \hat{\boldsymbol{J}} - \hat{c}_0 \hat{\nabla} \hat{\phi},
\end{aligned}
$$

D.6. **Magnetic field update.** $D_0 = \varepsilon_0 E_0$, $B_0 = \mu_0 H_0$, $E_0 = c_0 B_0$

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} =& - \nabla \psi \\
\left[\frac{B_0 u_0}{x_0}\right] \frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \left[\frac{E_0}{x_0}\right] \hat{\nabla} \times \hat{\mathbf{E}} =& - \left[\frac{E_0}{x_0}\right] \hat{\nabla} \hat{\psi} \\
\left[\frac{E_0 u_0}{x_0 c_0}\right] \frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \left[\frac{E_0}{x_0}\right] \hat{\nabla} \times \hat{\mathbf{E}} =& - \left[\frac{E_0}{x_0}\right] \hat{\nabla} \hat{\psi} \\
\left[\frac{E_0}{x_0}\right] \frac{1}{\hat{c}_0} \frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \left[\frac{E_0}{x_0}\right] \hat{\nabla} \times \hat{\mathbf{E}} =& - \left[\frac{E_0}{x_0}\right] \hat{\nabla} \hat{\psi} \\
\frac{1}{\hat{c}_0} \frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \hat{\nabla} \times \hat{\mathbf{E}} =& - \hat{\nabla} \hat{\psi} \\
\frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \hat{c}_0 \hat{\nabla} \times \hat{\mathbf{E}} =& - \hat{c}_0 \hat{\nabla} \hat{\psi}
\end{aligned}
$$

D.7. **Electric field divergence constraint.**

$$
\begin{aligned}
\nabla \cdot \boldsymbol{D} =& \rho_c, \\
\left[\frac{D_0}{x_0}\right] \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{n_0 q_0}{1}\right] \hat{\rho}_c, \\
\hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{x_0 n_0 q_0}{D_0}\right] \hat{\rho}_c, \\
\hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{x_0 n_0 q_0}{\varepsilon_0 E_0}\right] \hat{\rho}_c, \\
\hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{x_0 n_0 q_0}{\varepsilon_0 c_0 B_0}\right] \hat{\rho}_c, \\
\hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{x_0 n_0 q_0}{\varepsilon_0 u_0 B_0}\right] \frac{\hat{\rho}_c}{\hat{c}_0}, \\
\hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} \frac{\hat{\rho}_c}{\hat{c}_0}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} + c_h^2 \nabla \cdot \boldsymbol{D} =& c_h^2 \rho_c - \frac{c_h^2}{c_p^2} \phi, \\
\left[\frac{H_0 u_0}{x_0}\right] \frac{\partial \hat{\phi}}{\partial \hat{t}} + \left[\frac{u_0^2 D_0}{x_0}\right] \hat{c}_h^2 \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{u_0^2 n_0 q_0}{1}\right] \hat{c}_h^2 \hat{\rho}_c - \left[\frac{u_0^2 H_0}{u_0 x_0}\right] \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\left[\frac{H_0 u_0}{x_0}\right] \frac{\partial \hat{\phi}}{\partial \hat{t}} + \left[\frac{u_0^2 H_0}{x_0 c_0}\right] \hat{c}_h^2 \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{u_0^2 n_0 q_0}{1}\right] \hat{c}_h^2 \hat{\rho}_c - \left[\frac{u_0 H_0}{x_0}\right] \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\left[\frac{H_0 u_0}{x_0}\right] \frac{\partial \hat{\phi}}{\partial \hat{t}} + \left[\frac{u_0 H_0}{x_0}\right] \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{u_0^2 n_0 q_0}{1}\right] \hat{c}_h^2 \hat{\rho}_c - \left[\frac{u_0 H_0}{x_0}\right] \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{u_0^2 n_0 q_0 x_0}{H_0 u_0}\right] \hat{c}_h^2 \hat{\rho}_c - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{u_0^2 n_0 q_0 x_0 \mu_0}{B_0 u_0}\right] \hat{c}_h^2 \hat{\rho}_c - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{u_0^2 n_0 q_0 x_0}{B_0 u_0 c^2 \varepsilon_0}\right] \hat{c}_h^2 \hat{\rho}_c - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \left[\frac{n_0 q_0 x_0}{u_0 B_0 \varepsilon_0}\right] \frac{\hat{c}_h^2}{\hat{c}_0^2} \hat{\rho}_c - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi}, \\
\frac{\partial \hat{\phi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\boldsymbol{D}} =& \frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} \frac{\hat{c}_h^2}{\hat{c}_0^2} \hat{\rho}_c - \frac{\hat{c}_h^2}{\hat{c}_p^2} \hat{\phi},
\end{aligned}
$$

D.8. **Magnetic field divergence constraint.**

$$
\begin{aligned}
\nabla \cdot \boldsymbol{B} =& 0, \\
\left[\frac{B_0}{x_0}\right] \hat{\nabla} \cdot \hat{\boldsymbol{B}} =& 0, \\
\hat{\nabla} \cdot \hat{\boldsymbol{B}} =& 0,
\end{aligned}
$$

$$\frac{\partial \psi}{\partial t} + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2}\psi$$
$$\left[\frac{E_0 u_0}{x_0}\right]\frac{\partial \hat{\psi}}{\partial \hat{t}} + \left[\frac{u_0^2 B_0}{x_0}\right]\hat{c}_h^2 \hat{\nabla} \cdot \hat{\mathbf{B}} = -\left[\frac{u_0^2 E_0}{x_0 u_0}\right]\frac{\hat{c}_h^2}{\hat{c}_p^2}\hat{\psi}$$
$$\left[\frac{E_0 u_0}{x_0}\right]\frac{\partial \hat{\psi}}{\partial \hat{t}} + \left[\frac{u_0^2 E_0}{x_0 c_0}\right]\hat{c}_h^2 \hat{\nabla} \cdot \hat{\mathbf{B}} = -\left[\frac{E_0 u_0}{x_0}\right]\frac{\hat{c}_h^2}{\hat{c}_p^2}\hat{\psi}$$
$$\left[\frac{E_0 u_0}{x_0}\right]\frac{\partial \hat{\psi}}{\partial \hat{t}} + \left[\frac{u_0^2 E_0}{x_0 u_0}\right]\frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\mathbf{B}} = -\left[\frac{E_0 u_0}{x_0}\right]\frac{\hat{c}_h^2}{\hat{c}_p^2}\hat{\psi}$$
$$\left[\frac{E_0 u_0}{x_0}\right]\frac{\partial \hat{\psi}}{\partial \hat{t}} + \left[\frac{u_0 E_0}{x_0}\right]\frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\mathbf{B}} = -\left[\frac{E_0 u_0}{x_0}\right]\frac{\hat{c}_h^2}{\hat{c}_p^2}\hat{\psi}$$
$$\frac{\partial \hat{\psi}}{\partial \hat{t}} + \frac{\hat{c}_h^2}{\hat{c}_0} \hat{\nabla} \cdot \hat{\mathbf{B}} = -\frac{\hat{c}_h^2}{\hat{c}_p^2}\hat{\psi}$$

D.9. **Electric scalar potential.** Let reference voltage ($kgm^2s^{-2}C^{-1}$) be given by $V_0 = x_0 E_0 = x_0 c_0 B_0$.

$$d_{D,0} = \sqrt{\frac{u_0^2 \epsilon_0 m_0}{n_0 q_0^2}},$$
$$d_{L,0} = \frac{u_0 m_0}{q_0 B_0},$$

$$\frac{\hat{d}_{L,0}}{\hat{d}_{D,0}^2} = \frac{x_0 n_0 q_0}{u_0 \epsilon_0 B_0}$$

$$\nabla^2 \phi = -\frac{\rho_c}{\epsilon_0 \epsilon_r},$$
$$\nabla^2 \phi = -\frac{1}{\epsilon_0 \epsilon_r}\sum \rho_\alpha \frac{q_\alpha}{m_\alpha},$$
$$\left[\frac{V_0}{x_0^2}\right]\hat{\nabla}^2 \hat{\phi} = -\left[\frac{\rho_0 q_0}{\epsilon_0 m_0}\right]\frac{1}{\epsilon_r}\sum \hat{\rho}_\alpha \frac{\hat{q}_\alpha}{\hat{m}_\alpha},$$
$$\left[\frac{x_0 c_0 B_0}{x_0^2}\right]\hat{\nabla}^2 \hat{\phi} = -\left[\frac{\rho_0 q_0}{\epsilon_0 m_0}\right]\frac{1}{\epsilon_r}\sum \hat{\rho}_\alpha \frac{\hat{q}_\alpha}{\hat{m}_\alpha},$$
$$\left[\frac{u_0 B_0}{x_0}\right]\hat{c}\hat{\nabla}^2 \hat{\phi} = -\left[\frac{\rho_0 q_0}{\epsilon_0 m_0}\right]\frac{1}{\epsilon_r}\sum \hat{\rho}_\alpha \frac{\hat{q}_\alpha}{\hat{m}_\alpha},$$
$$\hat{\nabla}^2 \hat{\phi} = -\left[\frac{x_0 \rho_0 q_0}{\epsilon_0 m_0 u_0 B_0}\right]\frac{1}{c\hat{\epsilon}_r}\sum \hat{\rho}_\alpha \frac{\hat{q}_\alpha}{\hat{m}_\alpha},$$
$$\hat{\nabla}^2 \hat{\phi} = -\frac{\hat{d}_{L,0}}{\hat{c}\epsilon_r \hat{d}_{D,0}^2}\sum \hat{\rho}_\alpha \frac{\hat{q}_\alpha}{\hat{m}_\alpha},$$
$$\hat{\nabla}^2 \epsilon_r \hat{\phi} = -\frac{\hat{d}_{L,0}}{\hat{c}\hat{d}_{D,0}^2}\sum \hat{\rho}_\alpha \frac{\hat{q}_\alpha}{\hat{m}_\alpha},$$

$D_0 = \varepsilon_0 E_0,\ B_0 = \mu_0 H_0,\ E_0 = c_0 B_0$

$$
\begin{aligned}
\boldsymbol{E} =& - \nabla \phi \\
\frac{\boldsymbol{D}}{\epsilon_0 \epsilon_r} =& - \nabla \phi \\
\left[\frac{D_0}{\epsilon_0 \epsilon_r} \right] \hat{\boldsymbol{D}} =& - \left[\frac{V_0}{x_0} \right] \hat{\nabla} \hat{\phi} \\
\left[\frac{\varepsilon_0 c_0 B_0}{\epsilon_0 \epsilon_r} \right] \hat{\boldsymbol{D}} =& - \left[\frac{x_0 c_0 B_0}{x_0} \right] \hat{\nabla} \hat{\phi} \\
\frac{\hat{\boldsymbol{D}}}{\epsilon_r} =& - \hat{\nabla} \hat{\phi} \\
\hat{\boldsymbol{D}} =& - \hat{\nabla} \epsilon_r \hat{\phi}
\end{aligned}
$$

potential from continuous charge

$$
\begin{aligned}
\phi\left(\vec{x}\right) =& \frac{1}{4\pi\epsilon_0} \int \frac{\varrho\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} \mathrm{d}\vec{r}' \\
\left[V_0\right] \hat{\phi}\left(\hat{\vec{x}}\right) =& \left[\frac{1}{4\pi\epsilon_0} \frac{n_0 q_0 x_0^3}{x_0} \right] \int \frac{\hat{\varrho}\left(\hat{\vec{x}}'\right)}{\left|\hat{\vec{x}} - \hat{\vec{x}}'\right|} \mathrm{d}\hat{\vec{x}}' \\
\left[x_0 c_0 B_0\right] \hat{\phi}\left(\hat{\vec{x}}\right) =& \left[\frac{n_0 q_0 x_0^3}{4\pi\epsilon_0 x_0} \right] \int \frac{\hat{\varrho}\left(\hat{\vec{x}}'\right)}{\left|\hat{\vec{x}} - \hat{\vec{x}}'\right|} \mathrm{d}\hat{\vec{x}}' \\
\hat{\phi}\left(\hat{\vec{x}}\right) =& \left[\frac{n_0 q_0 x_0}{4\pi\epsilon_0 c_0 B_0} \right] \int \frac{\hat{\varrho}\left(\hat{\vec{x}}'\right)}{\left|\hat{\vec{x}} - \hat{\vec{x}}'\right|} \mathrm{d}\hat{\vec{x}}' \\
\hat{\phi}\left(\hat{\vec{x}}\right) =& \left[\frac{n_0 q_0 x_0}{\epsilon_0 u_0 B_0} \right] \frac{1}{4\pi\hat{c}} \int \frac{\hat{\varrho}\left(\hat{\vec{x}}'\right)}{\left|\hat{\vec{x}} - \hat{\vec{x}}'\right|} \mathrm{d}\hat{\vec{x}}' \\
\hat{\phi}\left(\hat{\vec{x}}\right) =& \frac{\hat{d}_{L,0}}{4\pi\hat{c}_0 \hat{d}_{D,0}^2} \int \frac{\hat{\varrho}\left(\hat{\vec{x}}'\right)}{\left|\hat{\vec{x}} - \hat{\vec{x}}'\right|} \mathrm{d}\hat{\vec{x}}'
\end{aligned}
$$

D.10. **Magnetic vector potential.**

$$
\begin{aligned}
d_{D,0}^2 =& \frac{u_0^2 m_0}{c^2 \mu_0 n_0 q_0^2} = \frac{m_0}{\hat{c}^2 \mu_0 n_0 q_0^2} \\
d_{L,0} =& \frac{u_0 m_0}{q_0 B_0},
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2 \boldsymbol{A} &= -\, \mu_0 \mu_r \boldsymbol{J}, \\
\nabla^2 \boldsymbol{A} &= -\, \mu_0 \mu_r \sum \frac{\rho_\alpha \boldsymbol{u}_\alpha q_\alpha}{m_\alpha}, \\
\left[\frac{B_0 x_0}{x_0^2}\right] \hat{\nabla}^2 \hat{\boldsymbol{A}} &= -\, \left[\frac{\mu_0 \rho_0 u_0 q_0}{m_0}\right] \mu_r \sum \frac{\hat{\rho}_\alpha \hat{\boldsymbol{u}}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha}, \\
\hat{\nabla}^2 \hat{\boldsymbol{A}} &= -\, \left[\frac{x_0 \mu_0 \rho_0 u_0 q_0}{B_0 m_0}\right] \mu_r \sum \frac{\hat{\rho}_\alpha \hat{\boldsymbol{u}}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha}, \\
\hat{\nabla}^2 \hat{\boldsymbol{A}} &= -\, \left[\frac{q_0 d_{L,0}}{u_0 m_0} \frac{x_0 \mu_0 \rho_0 u_0 q_0}{m_0}\right] \mu_r \sum \frac{\hat{\rho}_\alpha \hat{\boldsymbol{u}}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha}, \\
\hat{\nabla}^2 \hat{\boldsymbol{A}} &= -\, \left[\frac{d_{L,0} x_0}{1} \frac{\mu_0 n_0 q_0^2}{m_0}\right] \mu_r \sum \frac{\hat{\rho}_\alpha \hat{\boldsymbol{u}}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha}, \\
\hat{\nabla}^2 \hat{\boldsymbol{A}} &= -\, \left[\frac{d_{L,0} x_0}{d_{D,0}^2}\right] \frac{\mu_r}{\hat{c}^2} \sum \frac{\hat{\rho}_\alpha \hat{\boldsymbol{u}}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha}, \\
\hat{\nabla}^2 \hat{\boldsymbol{A}} &= -\, \frac{\mu_r \hat{d}_{L,0}}{\hat{c}^2 \hat{d}_{D,0}^2} \sum \frac{\hat{\rho}_\alpha \hat{\boldsymbol{u}}_\alpha \hat{q}_\alpha}{\hat{m}_\alpha},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B} &= \nabla \times \boldsymbol{A} \\
\left[\frac{B_0}{1}\right] \hat{\boldsymbol{B}} &= \left[\frac{B_0 x_0}{x_0}\right] \hat{\nabla} \times \hat{\boldsymbol{A}} \\
\hat{\boldsymbol{B}} &= \hat{\nabla} \times \hat{\boldsymbol{A}}
\end{aligned}
$$

Appendix E. Electromagnetic fluxes

Start with system of equations

$$
\frac{\partial}{\partial t}\begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \\ \phi \\ \psi \end{pmatrix} + \nabla \cdot \begin{pmatrix} \phi & -\frac{B_z}{\mu_0\mu_r} & \frac{B_y}{\mu_0\mu_r} \\ \frac{B_z}{\mu_0\mu_r} & \phi & -\frac{B_x}{\mu_0\mu_r} \\ -\frac{B_y}{\mu_0\mu_r} & \frac{B_x}{\mu_0\mu_r} & \phi \\ \psi & \frac{D_z}{\varepsilon_0\varepsilon_r} & -\frac{D_y}{\varepsilon_0\varepsilon_r} \\ -\frac{D_z}{\varepsilon_0\varepsilon_r} & \psi & \frac{D_x}{\varepsilon_0\varepsilon_r} \\ \frac{D_y}{\varepsilon_0\varepsilon_r} & -\frac{D_x}{\varepsilon_0\varepsilon_r} & \psi \\ c_h^2 D_x & c_h^2 D_y & c_h^2 D_z \\ c_h^2 B_x & c_h^2 B_y & c_h^2 B_z \end{pmatrix} = \begin{pmatrix} -j_x \\ -j_y \\ -j_z \\ 0 \\ 0 \\ 0 \\ c_h^2 \rho_c - \frac{c_h^2}{c_p^2}\phi \\ -\frac{c_h^2}{c_p^2}\psi \end{pmatrix}
$$

only consider the $x-$ component and disregard sources.

$$
\frac{\partial}{\partial t}\begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \\ \phi \\ \psi \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \phi \\ \frac{B_z}{\mu_0\mu_r} \\ -\frac{B_y}{\mu_0\mu_r} \\ \psi \\ -\frac{D_z}{\varepsilon_0\varepsilon_r} \\ \frac{D_y}{\varepsilon_0\varepsilon_r} \\ c_h^2 D_x \\ c_h^2 B_x \end{pmatrix} = \boldsymbol{0}
$$

Notice that we have a set of four independent system of equations

$$
\frac{\partial \boldsymbol{U}}{\partial t} + \mathbf{A}\frac{\partial \boldsymbol{U}}{\partial x} = \boldsymbol{0}
$$

$$\frac{\partial}{\partial t}\begin{pmatrix} D_x \\ \phi \end{pmatrix} = -\frac{\partial}{\partial x}\begin{pmatrix} \phi \\ c_h^2 D_x \end{pmatrix}, \quad \frac{\partial}{\partial t}\begin{pmatrix} D_y \\ B_z \end{pmatrix} = -\frac{\partial}{\partial x}\begin{pmatrix} \frac{B_z}{\mu_0\mu_r} \\ \frac{D_y}{\varepsilon_0\varepsilon_r} \end{pmatrix}, \quad \frac{\partial}{\partial t}\begin{pmatrix} D_z \\ B_y \end{pmatrix} = -\frac{\partial}{\partial x}\begin{pmatrix} -\frac{B_y}{\mu_0\mu_r} \\ -\frac{D_z}{\varepsilon_0\varepsilon_r} \end{pmatrix}, \quad \frac{\partial}{\partial t}\begin{pmatrix} B_x \\ \psi_B \end{pmatrix}$$

$$\boldsymbol{U}_1 = \begin{pmatrix} D_x \\ \phi \end{pmatrix}, \qquad \boldsymbol{U}_2 = \begin{pmatrix} D_y \\ B_z \end{pmatrix}, \qquad \boldsymbol{U}_3 = \begin{pmatrix} D_z \\ B_y \end{pmatrix}, \qquad \boldsymbol{U}_4$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ c_h^2 & 0 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 0 & \frac{1}{\mu_0\mu_r} \\ \frac{1}{\varepsilon_0\varepsilon_r} & 0 \end{pmatrix}, \qquad \mathbf{A}_3 = \begin{pmatrix} 0 & \frac{-1}{\mu_0\mu_r} \\ \frac{-1}{\varepsilon_0\varepsilon_r} & 0 \end{pmatrix}, \qquad \mathbf{A}_4$$

$$\lambda_1 = \pm c_h, \qquad \lambda_2 = \pm c_0 c_r, \qquad \lambda_3 = \pm c_0 c_r, \qquad \lambda_4$$

Apply Rankine-Hugoniot conditions ($\lambda\Delta\boldsymbol{U} = \Delta\boldsymbol{F} = \mathbf{A}\Delta\boldsymbol{U}$) to resulting waves along with continuity conditions (ρ_s= surface charge density, $\boldsymbol{K}$= surface current, $\mathbf{n}$= interface normal). Note that continuity conditions can be deduced from the fact we have all non-zero eigenvalues, that is we do not have a discontinuity at the contact surface post evolution.

$$\mathbf{n}\cdot\left(\boldsymbol{D}^R - \boldsymbol{D}^L\right) = \rho_s\,(=0)\,, \qquad \mathbf{n}\cdot\left((\varepsilon\boldsymbol{E})^R - (\varepsilon\boldsymbol{E})^L\right) = \rho_s\,(=0)\,,$$

$$\mathbf{n}\cdot\left(\boldsymbol{B}^R - \boldsymbol{B}^L\right) = 0, \qquad \mathbf{n}\cdot\left((\mu\boldsymbol{H})^R - (\mu\boldsymbol{H})^L\right) = 0,$$

$$\mathbf{n}\times\left(\boldsymbol{E}^R - \boldsymbol{E}^L\right) = 0, \qquad \mathbf{n}\times\left(\left(\frac{\boldsymbol{D}}{\varepsilon}\right)^R - \left(\frac{\boldsymbol{D}}{\varepsilon}\right)^L\right) = 0,$$

$$\mathbf{n}\times\left(\boldsymbol{H}^R - \boldsymbol{H}^L\right) = \boldsymbol{K}\,(=0)\,, \qquad \mathbf{n}\times\left(\left(\frac{\boldsymbol{B}}{\mu}\right)^R - \left(\frac{\boldsymbol{B}}{\mu}\right)^L\right) = \boldsymbol{K}\,(=0)\,,$$

Define $c_L^2 = \frac{1}{\varepsilon_L\mu_L}$, $c_R^2 = \frac{1}{\varepsilon_R\mu_R}$.

E.1. **I & IV.** Likewise for F_{B_x} and F_{ψ_B}.

$$\begin{aligned} c_h\left(D_x^R - D_x^{*R}\right) &= \phi^R - \phi_D^{*R} \\ -c_h\left(D_x^{*L} - D_x^L\right) &= \phi^{*L} - \phi^L \end{aligned} \qquad \begin{aligned} \phi^{*L} - \phi^L &= -c_h\left(D_x^{*L} - D_x^L\right) \\ \phi^{*L} &= \phi^L - c_h\left(D_x^{*L} - D_x^L\right) \end{aligned}$$

$$\begin{aligned} D_x^{*L} - D_x^L &= -\frac{1}{c_h}\left(\phi^{*L} - \phi^L\right) \\ D_x^{*L} &= D_x^L - \frac{1}{c_h}\left(\phi^{*L} - \phi^L\right) \end{aligned} \qquad \begin{aligned} c_h\left(D_x^R - D_x^{*R}\right) &= \phi^R - \phi^{*R} \\ D_x^R - D_x^{*R} &= \frac{\phi^R - \phi^{*R}}{c_h} \\ D_x^{*R} &= D_x^R - \frac{\phi^R - \phi^{*R}}{c_h} \end{aligned}$$

$$\begin{aligned} \phi^R - \phi^{*R} &= c_h\left(D_x^R - D_x^{*R}\right) \\ \phi^{*R} &= \phi^R - c_h\left(D_x^R - D_x^{*R}\right) \\ \phi^{*R} &= \phi^R - c_h\left(D_x^R - D_x^L + \frac{1}{c_h}\left(\phi^{*L} - \phi^L\right)\right) \\ \phi^{*R} &= \phi^R - c_h\left(D_x^R - D_x^L\right) - \phi^{*L} + \phi^L \\ \phi^{*R} + \phi^{*L} &= \phi^L + \phi^R - c_h\left(D_x^R - D_x^L\right) \\ \phi^{*R} &= \frac{\phi^L + \phi^R}{2} - \frac{c_h}{2}\left(D_x^R - D_x^L\right) \\ F_D &= \phi^{*R} = \frac{\phi^L + \phi^R}{2} - \frac{c_h}{2}\left(D_x^R - D_x^L\right) \end{aligned}$$

$$\begin{aligned} D_x^{*R} &= D_x^R - \frac{1}{c_h}\left(\phi^R - \phi^L + c_h\left(D_x^{*L} - D_x^L\right)\right) \\ D_x^{*R} &= D_x^R - \frac{1}{c_h}\left(\phi^R - \phi^L\right) - D_x^{*L} + D_x^L \\ D_x^{*R} + D_x^{*L} &= D_x^L + D_x^R - \frac{1}{c_h}\left(\phi^R - \phi^L\right) \\ D_x^{*R} &= \frac{D_x^L + D_x^R}{2} - \frac{1}{2c_h}\left(\phi^R - \phi^L\right) \\ F_\phi &= c_h^2 D_x^{*R} \\ F_\phi &= c_h^2\left(\frac{D_x^L + D_x^R}{2} - \frac{1}{2c_h}\left(\phi^R - \phi^L\right)\right) \\ F_\phi &= c_h^2\left(\frac{D_x^L + D_x^R}{2} - \frac{\phi^R - \phi^L}{2c_h}\right) \end{aligned}$$

Non-dimensionalisation (note that $c_r^2 = \frac{1}{\varepsilon_r\mu_r}$ with ε_r and μ_r already nondimensional):

Reference quantities $D_0 = \varepsilon_0 E_0$, $B_0 = \mu_0 H_0$, $E_0 = c_0 B_0$
$D_0 = {}^{H_0}/_{c_0}$
$\phi = \hat{\phi} H_0$, $\psi = \hat{\psi} E_0$

$$
\begin{aligned}
F_D =& \frac{\phi^L + \phi^R}{2} - \frac{c_h}{2}\left(D_x^R - D_x^L\right) \\
F_D =& \left[\frac{H_0}{1}\right] \frac{\hat{\phi}^L + \hat{\phi}^R}{2} - \left[\frac{u_0 D_0}{1}\right] \frac{\hat{c}_h}{2}\left(D_x^R - D_x^L\right) \\
F_D =& \left[\frac{H_0}{1}\right] \frac{\hat{\phi}^L + \hat{\phi}^R}{2} - \left[\frac{u_0 D_0}{1}\right] \frac{\hat{c}_h}{2}\left(D_x^R - D_x^L\right) \\
F_D =& \left[\frac{c_0 D_0}{1}\right] \frac{\hat{\phi}^L + \hat{\phi}^R}{2} - \left[\frac{u_0 D_0}{1}\right] \frac{\hat{c}_h}{2}\left(D_x^R - D_x^L\right) \\
F_D =& \left[\frac{u_0 D_0}{1}\right] \hat{c}_0 \frac{\hat{\phi}^L + \hat{\phi}^R}{2} - \left[\frac{u_0 D_0}{1}\right] \frac{\hat{c}_h}{2}\left(D_x^R - D_x^L\right) \\
\left[\frac{1}{D_0 u_0}\right] F_D = \hat{F}_D =& \frac{\hat{c}_0}{2}\left(\hat{\phi}^L + \hat{\phi}^R\right) - \frac{\hat{c}_h}{2}\left(D_x^R - D_x^L\right)
\end{aligned}
$$

$$
\begin{aligned}
F_\phi =& c_h^2 \left(\frac{D_x^L + D_x^R}{2} - \frac{\phi^R - \phi^L}{2 c_h}\right) \\
F_\phi =& \left[\frac{u_0^2}{1}\right] \hat{c}_h^2 \left(\left[\frac{D_0}{1}\right] \frac{\hat{D}_x^L + \hat{D}_x^R}{2} - \left[\frac{H_0}{u_0}\right] \frac{\hat{\phi}^R - \hat{\phi}^L}{2\hat{c}_h}\right) \\
F_\phi =& \left[\frac{u_0^2}{1}\right] \hat{c}_h^2 \left(\left[\frac{H_0}{u_0}\right] \frac{\hat{D}_x^L + \hat{D}_x^R}{\hat{c}_0 2} - \left[\frac{H_0}{u_0}\right] \frac{\hat{\phi}^R - \hat{\phi}^L}{2\hat{c}_h}\right) \\
F_\phi =& \left[\frac{H_0 u_0}{1}\right] \hat{c}_h^2 \left(\frac{\hat{D}_x^L + \hat{D}_x^R}{\hat{c}_0 2} - \frac{\hat{\phi}^R - \hat{\phi}^L}{2\hat{c}_h}\right) \\
\left[\frac{1}{H_0 u_0}\right] F_\phi = \hat{F}_\phi =& \frac{\hat{c}_h^2}{2\hat{c}_0}\left(\hat{D}_x^L + \hat{D}_x^R\right) - \frac{\hat{c}_h}{2}\left(\hat{\phi}^R - \hat{\phi}^L\right)
\end{aligned}
$$

E.2. **II & III.** For F_{D_z} and F_{B_y} we set $\mu_0 = -\mu_0$ and $\varepsilon_0 = -\varepsilon_0$ (but only where they explicitly appear, not in c_0!)

E.2.1. *D*.

$$
\begin{aligned}
c_0 c_R \left(D_y^R - D_y^{*R}\right) =& \frac{1}{\mu_0}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^{*R}}{\mu^R}\right) \qquad -c_0 c_L \left(D_y^{*L} -\right.\\
D_y^R - D_y^{*R} =& \frac{1}{c_0 c_R \mu_0}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^{*R}}{\mu^R}\right) \qquad -\mu_0 c_0 c_L \left(D_y^{*L} -\right.\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^{*R}}{\mu^R}\right)\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^{*R}}{\mu^R}\right)\\
\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \left(\frac{B_z^L}{\mu^L} - \mu_0 c_0 c_L \left(D_y^{*L} - D_y^L\right)\right)\right)\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L} + \mu_0 c_0 c_L \left(D_y^{*L} - D_y^L\right)\right)\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L}\right) - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\mu_0 c_0 c_L \left(D_y^{*L} - D_y^L\right)\right)\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L}\right) - \frac{c_L}{c_R \varepsilon^R}\left(D_y^{*L} - D_y^L\right)\\
\frac{D_y^{*R}}{\varepsilon^R} + \frac{c_L \varepsilon_L}{c_R \varepsilon^R}\frac{D_y^{*L}}{\varepsilon_L} =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L}\right) + \frac{c_L \varepsilon_L}{c_R \varepsilon^R}\frac{D_y^L}{\varepsilon_L}\\
\frac{D_y^{*R}}{\varepsilon^R}\left(1 + \frac{c_L \varepsilon_L}{c_R \varepsilon^R}\right) =& \frac{D_y^R}{\varepsilon^R} - \frac{1}{c_0 c_R \mu_0 \varepsilon^R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L}\right) + \frac{c_L \varepsilon_L}{c_R \varepsilon^R}\frac{D_y^L}{\varepsilon_L}\\
\left(1 + \frac{c_L \varepsilon_L}{c_R \varepsilon^R}\right)^{-1} =& \frac{c_R \varepsilon_R}{c_R \varepsilon_R + c_L \varepsilon_L}\\
\frac{D_y^{*R}}{\varepsilon^R} =& \left(\frac{c_R \varepsilon_R}{c_R \varepsilon_R + c_L \varepsilon_L}\right)\left(\frac{D_y^R}{\varepsilon^R} + \frac{c_L \varepsilon_L}{c_R \varepsilon_R}\frac{D_y^L}{\varepsilon_L} - \frac{1}{c_0 \mu_0}\frac{1}{c_R \varepsilon_R}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L}\right)\right)\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{c_R \varepsilon_R}{c_R \varepsilon_R + c_L \varepsilon_L}\frac{D_y^R}{\varepsilon^R} + \frac{c_L \varepsilon_L}{c_R \varepsilon^R + c_L \varepsilon_L}\frac{D_y^L}{\varepsilon_L} - \frac{1}{c_R \varepsilon_R + c_L \varepsilon_L}\left(\frac{1}{c_0 \mu_0}\left(\frac{B_z^R}{\mu^R} - \frac{B_z^L}{\mu^L}\right)\right)\\
\frac{D_y^{*R}}{\varepsilon^R} =& \frac{1}{c_R \varepsilon_R + c_L \varepsilon_L}\left(c_R D_y^R + c_L D_y^L + \frac{1}{\mu_0 c_0}\left(\frac{B_z^L}{\mu^L} - \frac{B_z^R}{\mu^R}\right)\right)\\
\\
F_{B_z} =& \frac{D_y^{*R}}{\epsilon_0 \varepsilon^R}\\
F_{B_z} =& \frac{1}{c_R \varepsilon_R + c_L \varepsilon_L}\left(\frac{1}{\varepsilon_0}\left(c_R D_y^R + c_L D_y^L\right) + c_0\left(\frac{B_z^L}{\mu^L} - \frac{B_z^R}{\mu^R}\right)\right)
\end{aligned}
$$

Non-dimensionalisation (note that $c_r^2 = \frac{1}{\varepsilon_r \mu_r}$ with ε_r and μ_r already nondimensional):
Reference quantities $D_0 = \varepsilon_0 E_0$, $B_0 = \mu_0 H_0$, $E_0 = c_0 B_0$
$D_0 = {}^{H_0}/_{c_0}$
$\psi_D = \hat{\psi}_D H_0$, $\psi_B = \hat{\psi}_B E_0$

$$
\begin{aligned}
F_B =& \frac{1}{c_R\varepsilon_R + c_L\varepsilon_L}\left(\frac{s}{\varepsilon_0}\left(c_R D_y^R + c_L D_y^L\right) + c_0\left(\frac{B_z^L}{\mu_L} - \frac{B_z^R}{\mu_R}\right)\right)\\
F_B =& \frac{1}{c_R\varepsilon_R + c_L\varepsilon_L}\left(\left[\frac{D_0}{\varepsilon_0}\right] s\left(c_R \hat{D}_y^R + c_L \hat{D}_y^L\right) + \left[\frac{B_0 c_0}{1}\right]\left(\frac{\hat{B}_z^L}{\mu_L} - \frac{\hat{B}_z^R}{\mu_R}\right)\right)\\
F_B =& \frac{1}{c_R\varepsilon_R + c_L\varepsilon_L}\left(\left[\frac{D_0}{\varepsilon_0}\right] s\left(c_R \hat{D}_y^R + c_L \hat{D}_y^L\right) + \left[\frac{B_0 u_0}{1}\right]\hat{c}_0\left(\frac{\hat{B}_z^L}{\mu_L} - \frac{\hat{B}_z^R}{\mu_R}\right)\right)\\
\frac{D_0}{\varepsilon_0} =& \frac{\varepsilon_0 E_0}{\varepsilon_0} = c_0 B_0 = B_0 u_0 \hat{c}_0\\
F_B =& \frac{1}{c_R\varepsilon_R + c_L\varepsilon_L}\left(\left[\frac{B_0 u_0}{1}\right] s\hat{c}_0\left(c_R \hat{D}_y^R + c_L \hat{D}_y^L\right) + \left[\frac{B_0 u_0}{1}\right]\hat{c}_0\left(\frac{\hat{B}_z^L}{\mu_L} - \frac{\hat{B}_z^R}{\mu_R}\right)\right)\\
\frac{F_B}{B_0 u_0} = \hat{F}_B =& \frac{\hat{c}_0}{c_R\varepsilon_R + c_L\varepsilon_L}\left(s\left(c_R \hat{D}_y^R + c_L \hat{D}_y^L\right) + \left(\frac{\hat{B}_z^L}{\mu_L} - \frac{\hat{B}_z^R}{\mu_R}\right)\right)
\end{aligned}
$$

E.2.2. *B*.

$$
c_0 c_R\left(B_z^R - B_z^{*R}\right) = \frac{1}{\varepsilon_0}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^{*R}}{\varepsilon_R}\right) \qquad -c_0 c_L\left(B_z^{*L} - B_z^L\right)
$$

$$
\frac{B_z^{*R}}{\mu_R} = \frac{B_z^R}{\mu_R} - \frac{1}{\varepsilon_0 c_0 c_R \mu_R}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^{*R}}{\varepsilon_R}\right) \qquad \frac{D_y^{*L}}{\varepsilon_L}
$$

$$
\begin{aligned}
\frac{B_z^{*R}}{\mu_R} =& \frac{B_z^R}{\mu_R} - \frac{1}{\varepsilon_0 c_0 c_R \mu_R}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L} + \varepsilon_0 c_0 c_L \mu_L\left(\frac{B_z^{*L}}{\mu_L} - \frac{B_z^L}{\mu_L}\right)\right)\\
\frac{B_z^{*R}}{\mu_R} =& \frac{B_z^R}{\mu_R} - \frac{1}{\varepsilon_0 c_0 c_R \mu_R}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right) - \frac{c_L\mu_L}{c_R\mu_R}\left(\frac{B_z^{*L}}{\mu_L} - \frac{B_z^L}{\mu_L}\right)\\
\frac{B_z^{*R}}{\mu_R} + \frac{c_L\mu_L}{c_R\mu_R}\frac{B_z^{*L}}{\mu_L} =& \frac{B_z^R}{\mu_R} + \frac{c_L\mu_L}{c_R\mu_R}\frac{B_z^L}{\mu_L} - \frac{1}{\varepsilon_0 c_0 c_R \mu_R}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right)\\
\frac{B_z^{*R}}{\mu_R}\left(1 + \frac{c_L\mu_L}{c_R\mu_R}\right) =& \frac{B_z^R}{\mu_R} + \frac{c_L\mu_L}{c_R\mu_R}\frac{B_z^L}{\mu_L} - \frac{1}{\varepsilon_0 c_0 c_R \mu_R}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right)\\
\left(1 + \frac{c_L\mu_L}{c_R\mu_R}\right)^{-1} =& \frac{c_R\mu_R}{c_R\mu_R + c_L\mu_L}\\
\frac{B_z^{*R}}{\mu_R} =& \frac{c_R}{c_R\mu_R + c_L\mu_L}B_z^R + \frac{c_L}{c_R\mu_R + c_L\mu_L}B_z^L - \frac{1}{\varepsilon_0 c_0}\frac{1}{c_R\mu_R + c_L\mu_L}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right)\\
\frac{B_z^{*R}}{\mu_R} =& \frac{1}{c_R\mu_R + c_L\mu_L}\left(c_R B_z^R + c_L B_z^L - \frac{1}{\varepsilon_0 c_0}\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
F_{D_y} =& \frac{B_z^{*R}}{\mu_0\mu_R}\\
F_{D_y} =& \frac{1}{c_R\mu_R + c_L\mu_L}\left(\frac{1}{\mu_0}\left(c_R B_z^R + c_L B_z^L\right) - c_0\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right)\right)
\end{aligned}
$$

Non-dimensionalisation (note that $c_r^2 = \frac{1}{\varepsilon_r\mu_r}$ with ε_r and μ_r already nondimensional):
Reference quantities $D_0 = \varepsilon_0 E_0$, $B_0 = \mu_0 H_0$, $E_0 = c_0 B_0$
$D_0 = {}^{H_0}/_{c_0}$

$\psi_D = \hat{\psi}_D H_0$, $\psi_B = \hat{\psi}_B E_0$

$$
\begin{aligned}
F_D =& \frac{1}{c_R\mu_R + c_L\mu_L}\left(\frac{s}{\mu_0}\left(c_R B_z^R + c_L B_z^L\right) - c_0\left(\frac{D_y^R}{\varepsilon_R} - \frac{D_y^L}{\varepsilon_L}\right)\right)\\
F_D =& \frac{1}{c_R\mu_R + c_L\mu_L}\left(\left[\frac{B_0}{\mu_0}\right] s\left(c_R \hat{B}_z^R + c_L \hat{B}_z^L\right) - \left[\frac{D_0 c_0}{1}\right]\left(\frac{\hat{D}_y^R}{\varepsilon_R} - \frac{\hat{D}_y^L}{\varepsilon_L}\right)\right)\\
F_D =& \frac{1}{c_R\mu_R + c_L\mu_L}\left(\left[\frac{B_0}{\mu_0}\right] s\left(c_R \hat{B}_z^R + c_L \hat{B}_z^L\right) - \left[\frac{D_0 u_0}{1}\right]\hat{c}_0\left(\frac{\hat{D}_y^R}{\varepsilon_R} - \frac{\hat{D}_y^L}{\varepsilon_L}\right)\right)\\
\frac{B_0}{\mu_0} =& \frac{\mu_0 H_0}{\mu_0} = c_0 D_0 = D_0 u_0 \hat{c}_0\\
F_D =& \frac{1}{c_R\mu_R + c_L\mu_L}\left(\left[\frac{D_0 u_0}{1}\right] s\hat{c}_0\left(c_R \hat{B}_z^R + c_L \hat{B}_z^L\right) - \left[\frac{D_0 u_0}{1}\right]\hat{c}_0\left(\frac{\hat{D}_y^R}{\varepsilon_R} - \frac{\hat{D}_y^L}{\varepsilon_L}\right)\right)\\
\frac{F_D}{D_0 u_0} = \hat{F}_D =& \frac{\hat{c}_0}{c_R\mu_R + c_L\mu_L}\left(s\left(c_R \hat{B}_z^R + c_L \hat{B}_z^L\right) + \left(\frac{\hat{D}_y^L}{\varepsilon_L} - \frac{\hat{D}_y^R}{\varepsilon_R}\right)\right)
\end{aligned}
$$

Appendix F. Accelerations

Acceleration of any species may be recovered from the momentum equation according to $\mathbf{a} = \frac{d\mathbf{u}}{dt}$.

$$
\begin{aligned}
\frac{\partial \rho_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot\left(\rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{I}\right) =& n_\alpha q_\alpha\left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right)\\
\left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha\cdot\nabla\right)\mathbf{u}_\alpha + \frac{\nabla p_\alpha}{n_\alpha m_\alpha} =& \frac{q_\alpha}{m_\alpha}\left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right)
\end{aligned}
$$

Now

$$
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla
$$

So

$$
\begin{aligned}
\frac{d\mathbf{u}_\alpha}{dt} + \frac{\nabla p_\alpha}{n_\alpha m_\alpha} =& \frac{q_\alpha}{m_\alpha}\left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right)\\
\mathbf{a} =& \frac{q_\alpha}{m_\alpha}\left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right) - \frac{\nabla p_\alpha}{n_\alpha m_\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{u_0^2}{x_0}\right]\hat{\mathbf{a}} =& \left[\frac{q_0 u_0 B_0}{m_0}\right]\hat{r}_\alpha \hat{c}\mathbf{E} + \left[\frac{q_0 u_0 B_0}{m_0}\right]\hat{r}_\alpha\left(\hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right) - \left[\frac{u_0^2}{x_0}\right]\frac{\hat{\nabla}\hat{p}_\alpha}{\hat{n}_\alpha \hat{m}_\alpha}\\
\hat{\mathbf{a}} =& \left[\frac{x_0 q_0 B_0}{u_0 m_0}\right]\hat{r}_\alpha\left(\hat{c}\mathbf{E} + \left(\hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right)\right) - \frac{\hat{\nabla}\hat{p}_\alpha}{\hat{n}_\alpha \hat{m}_\alpha}\\
\hat{\mathbf{a}} =& \frac{\hat{r}_\alpha}{\hat{d}_{L,0}}\left(\hat{c}\mathbf{E} + \left(\hat{\mathbf{u}}_\alpha \times \hat{\mathbf{B}}\right)\right) - \frac{\hat{\nabla}\hat{p}_\alpha}{\hat{\rho}_\alpha}
\end{aligned}
$$

Appendix G. Vorticity equation

Starting from the non-dimensional momentum equation.

$$
\begin{aligned}
\frac{\partial \rho_\alpha \mathbf{U}_\alpha}{\partial t} + \nabla\cdot\left(\rho_\alpha \mathbf{U}_\alpha \mathbf{U}_\alpha + p_\alpha \mathbf{I}\right) =& \frac{\rho_\alpha r_\alpha}{d_L}\left(c\mathbf{E} + \mathbf{U}_\alpha \times \mathbf{B}\right)\\
\frac{\partial \mathbf{U}_\alpha}{\partial t} + \left(\mathbf{U}_\alpha\cdot\nabla\right)\mathbf{U}_\alpha + \frac{1}{\rho_\alpha}\nabla p_\alpha =& \frac{r_\alpha}{d_L}\left(c\mathbf{E} + \mathbf{U}_\alpha \times \mathbf{B}\right)
\end{aligned}
$$

using the identity $(\mathbf{A}\cdot\nabla)\,\mathbf{A} = \frac{1}{2}\nabla\,(\mathbf{A}\cdot\mathbf{A}) - \mathbf{A}\times(\nabla\times\mathbf{A})$ we get $(\mathbf{U}_\alpha\cdot\nabla)\,\mathbf{U}_\alpha = \frac{1}{2}\nabla\,(\mathbf{U}_\alpha\cdot\mathbf{U}_\alpha) - \mathbf{U}_\alpha\times\omega_\alpha$ and thus

$$\begin{aligned}
\frac{\partial\mathbf{U}_\alpha}{\partial t} + \frac{1}{2}\nabla\,(\mathbf{U}_\alpha\cdot\mathbf{U}_\alpha) - \mathbf{U}_\alpha\times\omega_\alpha + \frac{1}{\rho_\alpha}\nabla p_\alpha &= \frac{r_\alpha}{d_L}\,(c\mathbf{E} + \mathbf{U}_\alpha\times\mathbf{B}) \\
\nabla\times\left(\frac{\partial\mathbf{U}_\alpha}{\partial t} + \frac{1}{2}\nabla\,(\mathbf{U}_\alpha\cdot\mathbf{U}_\alpha) - \mathbf{U}_\alpha\times\omega_\alpha + \frac{1}{\rho_\alpha}\nabla p_\alpha\right) &= \nabla\times\left(\frac{r_\alpha}{d_L}\,(c\mathbf{E} + \mathbf{U}_\alpha\times\mathbf{B})\right)
\end{aligned}$$

now $\nabla\times\nabla\phi = 0$ such that $\nabla\times\left(\frac{1}{2}\nabla\,(\mathbf{U}_\alpha\cdot\mathbf{U}_\alpha)\right) = 0$ and we get

$$\frac{\partial\omega_\alpha}{\partial t} - \nabla\times(\mathbf{U}_\alpha\times\omega_\alpha) + \nabla\times\left(\frac{1}{\rho_\alpha}\nabla p_\alpha\right) = \nabla\times\left(\frac{r_\alpha}{d_L}\,(c\mathbf{E} + \mathbf{U}_\alpha\times\mathbf{B})\right)$$

now $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$ such that

$$\nabla\times(\mathbf{U}_\alpha\times\omega_\alpha) = \mathbf{U}_\alpha\left(\cancelto{0}{\nabla\cdot\omega_\alpha}\right) - \omega_\alpha\,(\nabla\cdot\mathbf{U}_\alpha) + (\omega_\alpha\cdot\nabla)\,\mathbf{U}_\alpha - (\mathbf{U}_\alpha\cdot\nabla)\,\omega_\alpha$$

giving

$$\frac{\partial\omega_\alpha}{\partial t} + \omega_\alpha\,(\nabla\cdot\mathbf{U}_\alpha) - (\omega_\alpha\cdot\nabla)\,\mathbf{U}_\alpha + (\mathbf{U}_\alpha\cdot\nabla)\,\omega_\alpha + \nabla\times\left(\frac{1}{\rho_\alpha}\nabla p_\alpha\right) = \nabla\times\left(\frac{r_\alpha}{d_L}\,(c\mathbf{E} + \mathbf{U}_\alpha\times\mathbf{B})\right)$$

also $\nabla\times(\psi\nabla\phi) = \nabla\psi\times\nabla\phi$ and $\nabla\left(\frac{\psi}{\phi}\right) = \frac{\phi\nabla\psi - (\nabla\phi)\psi}{\phi^2}$ such that

$$\begin{aligned}
\nabla\times\left(\frac{1}{\rho_\alpha}\nabla p_\alpha\right) &= \nabla\left(\frac{1}{\rho_\alpha}\right)\times\nabla p_\alpha \\
&= \left(-\frac{1}{\rho_\alpha^2}\nabla\rho_\alpha\right)\times\nabla p_\alpha \\
&= -\frac{1}{\rho_\alpha^2}\,(\nabla\rho_\alpha\times\nabla p_\alpha)
\end{aligned}$$

resulting in

$$\begin{aligned}
\frac{\partial\omega_\alpha}{\partial t} + \omega_\alpha\,(\nabla\cdot\mathbf{U}_\alpha) - (\omega_\alpha\cdot\nabla)\,\mathbf{U}_\alpha + (\mathbf{U}_\alpha\cdot\nabla)\,\omega_\alpha - \frac{1}{\rho_\alpha^2}\,(\nabla\rho_\alpha\times\nabla p_\alpha) &= \nabla\times\left(\frac{r_\alpha}{d_L}\,(c\mathbf{E} + \mathbf{U}_\alpha\times\mathbf{B})\right) \\
&= \nabla\times\left(\sqrt{\frac{2}{\beta}}\frac{r_\alpha}{d_S}\,(c\mathbf{E} + \mathbf{U}_\alpha\times\mathbf{B})\right)
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial\omega_\alpha}{\partial t} = \underbrace{(\omega_\alpha\cdot\nabla)\,\mathbf{U}_\alpha}_{\tau_v} - \underbrace{(\mathbf{U}_\alpha\cdot\nabla)\,\omega_\alpha}_{\tau_c} - \underbrace{\omega_\alpha\,(\nabla\cdot\mathbf{U}_\alpha)}_{\tau_s} \\
+ \underbrace{\frac{1}{\rho_\alpha^2}\,(\nabla\rho_\alpha\times\nabla p_\alpha)}_{\tau_b} + \underbrace{\sqrt{\frac{2}{\beta}}\frac{r_\alpha c}{d_S}\,(\nabla\times\mathbf{E})}_{\tau_E} + \underbrace{\sqrt{\frac{2}{\beta}}\frac{r_\alpha}{d_S}\,(\nabla\times(\mathbf{U}_\alpha\times\mathbf{B}))}_{\tau_B}
\end{aligned}$$

- τ_v: Stretch due to velocity
- τ_c: Convective term
- τ_s: Stretch due to compressibility

Appendix H. Self generation of magnetic fields

The non-dimensional Maxwell's equations (neglecting Lagrange multiplier) give us $-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = \nabla\times\mathbf{E}$.

$$
\begin{aligned}
\frac{\partial \hat{\mathbf{E}}}{\partial \hat{t}} - \hat{c}\hat{\nabla} \times \hat{\mathbf{B}} &= -\frac{\hat{\vartheta}_0}{\hat{\lambda}_0^2 \hat{c}} \sum_{\alpha} \hat{\rho}_\alpha \hat{r}_\alpha \hat{\mathbf{U}}_\alpha, \\
\frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} + \hat{c}\hat{\nabla} \times \hat{\mathbf{E}} &= 0,
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\partial \omega_\alpha}{\partial t} - \nabla \times (\mathbf{U}_\alpha \times \omega_\alpha) - \frac{1}{\rho_\alpha^2}(\nabla \rho_\alpha \times \nabla p_\alpha) &= \nabla \times \left(\frac{r_\alpha}{d_L} (c\mathbf{E} + \mathbf{U}_\alpha \times \mathbf{B}) \right) \\
&= \frac{r_\alpha c}{d_L} \nabla \times \mathbf{E} + \frac{r_\alpha}{d_L} \nabla \times (\mathbf{U}_\alpha \times \mathbf{B}) \\
\frac{\partial \omega_\alpha}{\partial t} - \nabla \times (\mathbf{U}_\alpha \times \omega_\alpha) - \frac{1}{\rho_\alpha^2}(\nabla \rho_\alpha \times \nabla p_\alpha) &= -\frac{r_\alpha}{d_L} \frac{\partial \mathbf{B}}{\partial t} + \frac{r_\alpha}{d_L} \nabla \times (\mathbf{U}_\alpha \times \mathbf{B})
\end{aligned}
$$

$$
\begin{aligned}
\frac{r_\alpha}{d_L} \frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{\rho_\alpha^2}(\nabla \rho_\alpha \times \nabla p_\alpha) + \frac{r_\alpha}{d_L} \nabla \times (\mathbf{U}_\alpha \times \mathbf{B}) - \frac{\partial \omega_\alpha}{\partial t} + \nabla \times (\mathbf{U}_\alpha \times \omega_\alpha) \\
\frac{\partial \mathbf{B}}{\partial t} &= \frac{d_L}{r_\alpha \rho_\alpha^2}(\nabla \rho_\alpha \times \nabla p_\alpha) + \nabla \times (\mathbf{U}_\alpha \times \mathbf{B}) - \frac{d_L}{r_\alpha} \left(\frac{\partial \omega_\alpha}{\partial t} - \nabla \times (\mathbf{U}_\alpha \times \omega_\alpha) \right) \\
\frac{\partial \mathbf{B}}{\partial t} &= \frac{d_L}{q_\alpha m_\alpha n_\alpha^2}(\nabla \rho_\alpha \times \nabla p_\alpha) + \nabla \times (\mathbf{U}_\alpha \times \mathbf{B}) - \frac{d_L m_\alpha}{q_\alpha} \left(\frac{\partial \omega_\alpha}{\partial t} - \nabla \times (\mathbf{U}_\alpha \times \omega_\alpha) \right)
\end{aligned}
$$

H.1. **Backwards Euler.**

$$
\begin{aligned}
f\left(\boldsymbol{u}^{n+1}\right) &= \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{h} \\
f\left(\boldsymbol{u}^{n+1}\right) &= f\left(\boldsymbol{u}^n\right) + \frac{\partial f}{\partial \boldsymbol{u}^n}\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) \\
f\left(\boldsymbol{u}^{n+1}\right) &= f\left(\boldsymbol{u}^n\right) + h\frac{\partial f}{\partial \boldsymbol{u}^n} f\left(\boldsymbol{u}^{n+1}\right) \\
f\left(\boldsymbol{u}^n\right) &= f\left(\boldsymbol{u}^{n+1}\right) - h\frac{\partial f}{\partial \boldsymbol{u}^n} f\left(\boldsymbol{u}^{n+1}\right) \\
f\left(\boldsymbol{u}^n\right) &= \left(\boldsymbol{I} - h\frac{\partial f}{\partial \boldsymbol{u}^n}\right) f\left(\boldsymbol{u}^{n+1}\right) \\
\boldsymbol{b} &= \boldsymbol{A}\boldsymbol{x}
\end{aligned}
$$