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A Note on the Submodular Vertex Cover Problem with Submodular Penalties

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Abstract

In this paper, we prove that there exists a combinatorial 3-approximation algorithm for the submodular vertex cover problem with submodular penalties introduced by Xu, Wang, Du, and Wu.

Keywords: Submodular functions, the vertex cover problem, the set cover problem

1. Introduction

A real-valued function f on the family of subsets of a finite set N is called a *submodular function*, if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

for every pair of subsets X, Y of N. In this paper, we consider the submodular vertex cover problem with submodular penalties introduced by Xu, Wang, Du, and Wu [1] (see the next section for its formal definition). They gave a combinatorial (primal-dual) 4-approximation algorithm for this problem. In this paper, we prove that there exists a combinatorial (primal-dual) 3-approximation algorithm for the submodular vertex cover problem with submodular penalties. Our algorithm is based on the approximation algorithm of Iwata and Nagano [2] for the submodular cost set cover problem. It should be noted that in the paper [1], the authors claimed that we can obtain a non-combinatorial approximation algorithm with the ratio $(1 - e^{-1/2})^{-1} \leq 2.542$ by using the ellipsoid method and the techniques proposed in [3].

Throughout this paper, we denote by $\mathbb{R}_+$ the set of non-negative real numbers. For each positive integer i, we define $[i] := \{1, 2, \ldots, i\}$.

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2. The Submodular Vertex Cover Problem with Submodular Penalties

The submodular vertex cover problem with submodular penalties is defined as follows. In this problem, we are given a finite undirected graph $G = (V, E)$, and submodular functions $C\colon 2^V \to \mathbb{R}_+$ and $P\colon 2^E \to \mathbb{R}_+$ such that $C(\emptyset) = P(\emptyset) = 0$. Assume that we can evaluate the functions C, P in polynomial time (with respect to the size of G). A pair (U, F) of a subset U of V and a subset of F of E is called a *feasible solution*, if $\delta(U) \cup F = E$, where $\delta(U)$ is the set of edges in E incident to some vertex in U.[1] The cost of a feasible solution (U, F) is defined as $C(U) + P(F)$. Then, the goal of the submodular vertex cover problem with submodular penalties is to find a minimum-cost feasible solution. This problem was introduced by Xu, Wang, Du, and Wu [1], and they gave a combinatorial 4-approximation algorithm for this problem.

3. The Submodular Cost Set Cover Problem

In the next section, we prove that the submodular vertex cover problem with submodular penalties can be reduced to the submodular cost set cover problem introduce by Iwata and Nagano [2]. In this problem, we are given a ground set S, a family $S_1, S_2, \ldots, S_k$ of subsets of S, and a submodular function $\rho\colon 2^{[k]} \to \mathbb{R}_+$ such that $\rho(\emptyset) = 0$. A subset I of $[k]$ is called a *set cover*, if for every element s in S, there exists an integer i in I such that $s \in S_i$. We assume that there exists a set cover. Then, the goal of the submodular cost set cover problem is to find a set cover I minimizing $\rho(I)$ among all set covers. Define

$$\eta := \max_{s \in S} |\{i \in [k] \mid s \in S_i\}|.$$

The following result is known.

Theorem 1 (Iwata and Nagano [2, Theorem 8])**.** *There exists a combinatorial (primal-dual) η-approximation algorithm for the submodular cost set cover problem.*

4. Our Result

In this section, we prove that there exists a combinatorial 3-approximation algorithm for the submodular vertex cover problem with submodular penalties. We prove this by reducing the submodular vertex cover problem with submodular penalties to the submodular cost set cover problem. Assume that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. For each integer i in $[n]$,

[1] Notice that it is possible that $\delta(U) \cap F \neq \emptyset$. This is the same as in the paper [1] (see Equation (5) of [1]).

we define $X_i := \delta(\{v_i\})$. Furthermore, for each integer i in $[m]$, we define $X_{n+i} := \{e_i\}$. For each subset I of $[n+m]$, we define $V(I)$ and $E(I)$ by

$$\begin{aligned} V(I) &:= \{v_i \in V \mid i \in I\} \\ E(I) &:= \{e_i \in E \mid n+i \in I\}. \end{aligned}$$

Furthermore, we define a function $\rho_{\mathrm{VC}} \colon 2^{[n+m]} \to \mathbb{R}_+$ by

$$\rho_{\mathrm{VC}}(I) := C(V(I)) + P(E(I)).$$

Lemma 2. *The function ρ_{VC} is a submodular function.*

Proof. Let I, J be subsets of $[n+m]$. It is not difficult to see that

$$\begin{aligned} V(I) \cup V(J) = V(I \cup J),\ V(I) \cap V(J) = V(I \cap J), \\ E(I) \cup E(J) = E(I \cup J),\ E(I) \cap E(J) = E(I \cap J). \end{aligned}$$

This and the submodularity of P, C imply that

$$\begin{aligned} \rho_{\mathrm{VC}}(I) + \rho_{\mathrm{VC}}(J) &= C(V(I)) + P(E(I)) + C(V(J)) + P(E(J)) \\ &\geq C(V(I) \cup V(J)) + P(E(I) \cup E(J)) \\ &\qquad + C(V(I) \cap V(J)) + P(E(I) \cap E(J)) \\ &= C(V(I \cup J)) + P(E(I \cup J)) \\ &\qquad + C(V(I \cap J)) + P(E(I \cap J)) \\ &= \rho_{\mathrm{VC}}(I \cup J) + \rho_{\mathrm{VC}}(I \cap J). \end{aligned}$$

This completes the proof. □

Let P be the submodular cost set cover problem in which we are given E as a ground set, the family $X_1, X_2, \ldots, X_{n+m}$, and the submodular function ρ_{VC}. Then, it is not difficult to see that the size of P is bounded by a polynomial in the size of the submodular vertex cover problem with submodular penalties. Furthermore, we can evaluate the function ρ_{VC} in polynomial time. In what follows, we will prove that these two problems are equivalent.

Lemma 3. *Assume that we are given a feasible solution (U, F) of the submodular vertex cover problem with submodular penalties. Then, there exists a set cover I of the problem P such that $\rho_{\mathrm{VC}}(I)$ is equal to the cost of (U, F).*

Proof. Define a subset $I_{U,F}$ of $[n+m]$ by

$$I_{U,F} := \{i \in [n] \mid v_i \in U\} \cup \{i \in [m+n] \setminus [n] \mid e_{i-n} \in F\}.$$

Then, it is not difficult to see that $\rho_{\mathrm{VC}}(I_{U,F}) = C(U) + P(F)$. Furthermore, since (U, F) is a feasible solution of the submodular vertex cover problem with submodular penalties, $I_{U,F}$ is clearly a set cover of the problem P. □

Lemma 4. *Assume that we are given a set cover I of the problem P. Then, there exists a feasible solution (U, F) of the submodular vertex cover problem with submodular penalties such that the cost of (U, F) is equal to $\rho_{\mathrm{VC}}(I)$.*

Proof. Define $U_I := V(I)$ and $F_I := E(I)$. Then, it is not difficult to see that $\rho_{\mathrm{VC}}(I) = C(U_I) + P(F_I)$. Furthermore, since I is a set cover of the problem P, (U_I, F_I) is clearly a feasible solution of the submodular vertex cover problem with submodular penalties. □

We are now ready to prove our main result.

Theorem 5. *There exists a combinatorial* 3*-approximation algorithm for the submodular vertex cover problem with submodular penalties.*

Proof. Lemmas 3 and 4 imply that the submodular vertex cover problem with submodular penalties and the problem P are equivalent. For every edge $e_i = \{v_p, v_q\}$, we have $e_i \in X_{n+i}$, $e_i \in X_p$, and $e_i \in X_q$. Furthermore, these are all subsets containing e_i among $X_1, X_2, \ldots, X_{n+m}$. Thus, we have

$$\max_{e \in E} |\{i \in [n+m] \mid e \in X_i\}| \leq 3.$$

This and Theorem 1 complete the proof. □

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