A Fast Approach to Minimum Curvature Raceline Planning via Probabilistic Inference

Salman Bari¹, Ahmad Schoha Haidari^{1,2} and Dirk Wollherr¹

Abstract—Finding a racing path that allows minimum lap time is the first and foremost step in overall autonomous race car planning. Minimum curvature path is the raceline that offers the highest cornering speed resulting in improved lap time for a given racetrack. Unfortunately, solving the geometrical optimization problem for finding a raceline is computationally expensive. In a competitive race, a few seconds difference in raceline computation could prove to be decisive. This paper presents a novel approach for finding the minimum curvature raceline via probabilistic inference. We leverage the tangential geometry and structure in the minimum curvature planning problem to formulate it on a factor graph, which is then solved as sparse non-linear least squares leading to a much faster algorithm. The proposed framework is evaluated for different racetracks and benchmark results are presented highlighting the computational efficiency, lap time and resulting raceline. Initial results indicate the viability of the proposed approach as it offers comparable lap time achievement with approximately four times faster computation time.

I. INTRODUCTION

In autonomous racing, the target is to travel the racetrack as fast as possible i.e. reach the goal in the shortest possible lap time. In order to achieve this, the first step is to find which path to follow that covers the racetrack as fast as possible. This path is often referred to as the raceline. The motion planning module in autonomous racing lies in the middle of general workflow architecture of racing [1]. The perception module perceives the environment around the race car and sends centerline of racetrack to the planning module, whereas, the planning module generates trajectory that is then tracked by the control module. Motion planning module is responsible for global planning and local planning. Global planning involves generation of offline global race trajectory. On the other hand, local planning covers the locallevel online decisions such as overtaking maneuvers. Global planning is further divided into two steps: finding the raceline that can offer minimum possible lap time and generation of velocity profile based on that raceline. Our work in this paper is relevant to finding the minimum curvature raceline given the centerline of the racetrack.

Ideally, solving the minimum time optimization problem will result in optimal lap time. However, it is quite computationally expensive and minimum curvature raceline is preferred as the lap time is close to minimum time optimization case with much less computation cost. We

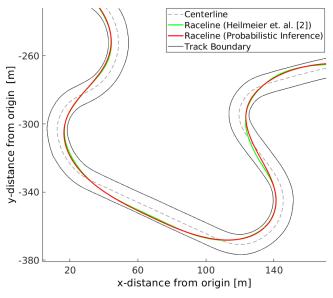


Fig. 1: A snippet of racelines generated for Modena E-Racetrack. QP optimization [2] took 4.28 sec as compared to 1.09 sec for the proposed probabilistic inference approach.

believe that minimum curvature raceline computation time can be further reduced by formulating it as a probabilistic inference problem. We opted the similar approach of minimizing the cumulative quadratic curvature [2], [3] to obtain optimal raceline by formulating the problem as probabilistic inference with factor graph representation. The planning as inference (PAI) problem formulation offers the utility of approximate inference algorithms for fast solution. The key contributions of this paper are:

- Theoretical formulation of minimum curvature raceline planning as probabilistic inference that results in fast computation time.
- A novel approach to represent raceline planning problem as factors to simplify the curvature minimization.
- Critical analysis of benchmark results against state of the art approach proposed in [2].

We discuss the related work in Section II. Basic definitions, notations and mathematical background is outlined in Section III. The core methodology and the theoretical formulation of PAI for raceline planning is described in Section IV. Implementation details, benchmark results and analysis are discussed in Section V, followed by Section VI which concludes the overall work of the paper.

¹Technical University of Munich, Germany; Chair of Automatic Control Engineering (LSR), Department of Electrical and Computer Engineering [s.bari|ahmad.haidari|dw]@tum.de

²ARRK Engineering GmbH, Munich, Germany ahmad-schoha.haidari@arrk-engineering.com

II. RELATED WORK

Lap time is the defining criterion in autonomous racing. The planning algorithm should generate a raceline that achieve the best possible lap time by generating the smooth racing path while keeping the computation time as minimum as possible. Finding the optimal raceline is not a straightforward problem as the shortest path does not offer minimum lap time for curved race tracks as the speed is reduced considerably at turns. For the racetrack curves, the path with minmum curvature offers more speed. Hence, the shortest path reduces lap time by space minimization but results in lower speed at turns while the minimum curvature path reduces lap time by offering higher speed at turns but results in longer path. Therefore, the optimal racing path is the compromise between the shortest path and the minimum curvature path.

The planning algorithms can generally be defined into three categories, namely graph search [4]-[6], samplingbased [7], [8] and optimization-based methods [9], [10]. As the racing path is dependent on geometry of race track, there are several geometrical optimization approaches that have been proposed for planning optimal raceline. These approaches divide the global racing trajectory planning into two-step procedure; finding the minimum curvature path, and velocity profile generation. Braghin et. al. [3] introduces a race car driver model that uses geometrical optimization of the path along a racetrack to find the minimum curvature path. Kapania et. al. [11] adopted the similar approach and divided the trajectory generation into two sequential steps. First, a velocity profile is generated then the path curvature is minimized using convex optimization. Recently, Heilmeier et. el. [2] proposed the quadratic optimization problem (QP) formulation for finding the minimum curvature raceline. The QP optimization is the state of the art approach that has been tested for multiple racetracks. In order to improve the raceline, an iterative QP optimization strategy is adopted that results in further increase in computation time.

Recently, a PAI approach [12] has been proposed that offers fast solution. The approach fuses all the planning problem objectives which are represented as factors and solves non-linear least square optimization problem via numerical (Gauss-Newton or Levenberg-Marquardt) methods. This is the Bayesian view on planning and the basic idea stems from the trajectory optimization as probabilistic inference [13], [14]. The conceptualization of PAI offers structured representation and reasoning. We leverage the idea of PAI to formulate the minimum curvature raceline planning problem on factor graph corresponding to prior and likelihood function. Contrary to the QP optimization problem formulation [2], the minimum curvature path objective function is replaced by a joint distribution over coupled random variables using factor graph and the solution method is replaced by probabilistic inference. The proposed framework provides a new perspective on decomposing raceline planning problem and exploits the inherent structure in the coupling between random variables.

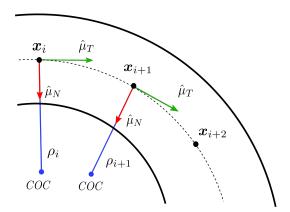


Fig. 2: An example of a racetrack representing the centerline and corresponding N-T coordinate system.

III. PRELIMINARIES

The input to our framework is the centerline and the corresponding track width representing the boundary of the racetrack. Centerline of racetrack consists of discritized way-points, hereafter, called states. We formally define centerline as,

Definition 1 (Centerline) The centerline is parametrized by states, $\mathbf{x} = \begin{bmatrix} \mathbf{x}_i & \cdots & \mathbf{x}_n \end{bmatrix}^T$ that are equidistant from the track boundaries \mathbf{b}_l and \mathbf{b}_r . Whereas, $\mathbf{x}_i = \begin{bmatrix} x_i & y_i \end{bmatrix} \in \mathbb{R}^2$.

The minimum curvature racing path is computed by minimizing the globally summed quadratic curvature. Minimum curvature raceline planning problem is then an optimization problem represented as

$$\min \sum_{i=1}^{n} k_i, \tag{1}$$

where, k_i represents the instantaneous curvature at x_i . The minimum curvature path x resulting from (1) is called raceline, which is formally defined as

Definition 2 (Raceline) Given the centerline x, the raceline is the minimum curvature path represented as $\overset{*}{x} = \begin{bmatrix} x \\ x_i & \cdots & x_n \end{bmatrix}^T$ resulting from (1).

In order to parameterize the instantaneous curvature k_i for the corresponding state \boldsymbol{x}_i , the racing car path is described in Normal-Tangential (N-T) coordinate system as shown in Figure 2. We assume that for the curvilinear motion of the racing car, the origin of the coordinate system is not fixed and located on the individual states along the curve. The N-axis is perpendicular to the T-axis with the positive direction towards the center of curvature (COC). The T-axis is instant tangent to the curve with the positive value in the direction of the motion of the car. The COC always lies on the concave side of the curve. The radius of curvature ρ is the perpendicular distance from the curve to the COC. The unit vectors $\hat{\mu}_N$ and $\hat{\mu}_T$ are always directed towards the positive direction of N-axis and T-axis, respectively. The amount of

curvature of an arc of a circle is determined by the radius ρ . There is an inverse relationship between curvature and radius such that the amount of bending of an arc will increase as the radius decreases.

We further define the mean curvature to describe the relationship between instantaneous curvature and tangent angle.

Definition 3 (Mean Curvature) If a curve is parameterized by states $\begin{bmatrix} x_i & \cdots & x_n \end{bmatrix}^T$ then the mean curvature between two points x_i and x_{i+1} lying on that curve is defined as:

$$h_i = \lim_{\Delta t \to 0} \left| \frac{\Delta \Psi_i}{\Delta S_i} \right|,\tag{2}$$

where, the Ψ_i represents the angle along which the tangent unit vector rotates, Δt is time and ΔS_i shows the arc length.

Definition 3 clears out an alternative perspective on reducing the curvature of an arc by minimizing the tangential rotation angle Ψ_i , which leads us to the first proposition.

Proposition 1 For a car moving along a non-degenerate arc parameterized by centerline x, minimizing the $\Delta\Psi$ will result in a reduced curvature. Minimum curvature raceline planning problem can be equivalently described as,

$$\min \sum_{i=1}^{n} k_i \cong \min \sum_{i=1}^{n-1} \Delta \Psi_i. \tag{3}$$

Where, $\Delta \Psi_i$ represents the unit tangent vector rotation angle between \mathbf{x}_i and \mathbf{x}_{i+1} .

Proof: From definition 3, the rotation of tangents between two points characterizes the curvature. So, as the $\Delta\Psi\to 0$, the radius of curvature $\rho\to\infty$. The relation between the $\Delta\Psi_i$ and the curvature k_i can be observed form Figure 3. Since the unit vectors $\hat{\mu}_T$ and $\hat{\mu}_N$ are perpendicular at any pint of the arc, the angle $\Delta\Psi_i$ along which the tangent rotates is also the same angle along which the normal vector rotates. If the time interval Δt is very small, the two unit vectors meet at the same COC. By projecting the point x_{i+1} onto x_i , it can be observed that by reducing the $\Delta\Psi_i$, both the points will lie on the same straight line resulting in $\rho\to\infty$ and the minimum mean curvature h_i .

IV. RACELINE PLANNING VIA PROBABILISTIC INFERENCE

In this section, we discuss the explicit representation of minimum curvature raceline finding problem as PAI. Generating the raceline from the centerline needs to take care of two specific criteria. First, the obtained raceline should have minimum curvature. Second, the raceline should not go out of the boundaries of the racetrack. Considering the above mentioned criteria, we formulate the raceline planning problem via probabilistic inference by defining the prior and likelihood.

Given the centerline from the Definition 1, the prior p(x) limits the raceline to remain inside the boundary of racetrack.

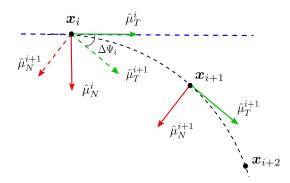


Fig. 3: Representation of normal-tangent unit vectors projection of x_{i+1} onto x_i indicating the relationship between mean curvature h_i and tangent angle rotation Ψ_i .

If the likelihood function $l(x; \mathbf{e})$ encourages the minimum curvature path, then the optimal raceline \mathbf{x} is found by the maximum a posteriori (MAP) estimation, which generates the raceline that maximizes the conditional posterior $p(\mathbf{x}|\mathbf{e})$,

$$\overset{*}{x} = \underset{x}{\operatorname{arg \, max}} \underbrace{p(x|e)}_{p(x)l(x;e)}. \tag{4}$$

The posterior distribution can be equivalently formulated as MAP inference problem on a factor graph $\mathcal{G} = (\mathcal{X}, \mathcal{F}, \mathcal{E})$ using conditional probability density functions (PDFs) $p(x_i|e)$ over the variables x_i , given the constraints e. The bipartite graph \mathcal{G} factorizes the conditional distribution over x as

$$p(\mathbf{x}|\mathbf{e}) \propto \prod_{m=1}^{M} f_m(\mathcal{X}_m),$$
 (5)

given a subset of M factor nodes $f_m \in \mathcal{F}$ on an adjacent subset of variable nodes $\mathcal{X}_m \in \mathcal{X}$, that are set in relation via the edges \mathcal{E} of the factor graph.

A. Prior Bounding Factor

For PAI problems, a smoothness factor [12] is designed that put cost on deviation from the prior state or variable node value. However, the smoothness factor is not a reasonable choice for raceline planning. In autonomous racing, the vehicles are not usually driving along the centerline. Rather, more often than not, driving along the either side of the racetrack boundary. The smoothness factor will not work for raceline planning as it will penalizes any deviation from centerline hindering the generation of minimum curvature raceline.

In order to allow the minimum curvature factor to search for a valid racing state along the entire track width, a custom prior racetrack bounding factor f^b is designed. This is a unary factor and it is attached to the each state of the centerline. It bounds the states to a line along the normal of the centerline, rather than bounding it to a specific point. As a result, the state can move along the entire width of racetrack for a minimum curvature path without incurring a cost.

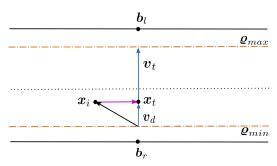


Fig. 4: State x_i projection to a normal vector v_t to bound it across the length of vector.

The error function for the prior bounding factor f^b is computed on the basis of the distance between the state and to the nearest point on the line between left and right track boundary. This distance is computed by first projecting the state onto the infinite line that goes through the left and right track boundary points and then clamping this value to the left or right track boundary, if it lies outside of the racetrack boundary.

Lets assume that there is a heading vector v_t pointing from the right to the left boundary of the racetrack¹. The heading vector lies inside the boundary limits $\begin{bmatrix} \varrho_{min} & \varrho_{max} \end{bmatrix}$ which are computed as a pre-processing step. This is the target vector on which the raceline points should move along to find the minimum curvature path. If x_i lies outside this vector, then we can find the projection of the given state onto the vector v_t . The length of vector v_d from right track boundary to target state x_t is,

$$v_d = (\boldsymbol{x}_i - \boldsymbol{b}_l) \,.\hat{v}_t. \tag{6}$$

However, this projection is actually on the infinite vector between the racetrack boundaries. It means that the optimized raceline could go out of the racetrack boundaries. In order to circumvent this issue, we use the clamping function that limits the target point to lie inside the racetrack boundary by following the boundary limits of $\begin{bmatrix} \varrho_{min} & \varrho_{max} \end{bmatrix}$. The clamping function is defined via

clamp
$$(v_d) = \max(0, \min(v_d, v_t)) \in [0, v_t].$$
 (7)

Figure 4 represents the design principle of projecting the state to a normal vector. The bounding prior factor is

$$f_i^b(\boldsymbol{x}_i) = \exp\left\{-\frac{1}{2} \|\boldsymbol{x} - \boldsymbol{x}_t\|_{\boldsymbol{\mathcal{K}}}^2\right\},$$
 (8)

The kernel K is an isotropic diagonal matrix that controls the cost by limiting the deviation. The prior function is then factorized for the complete graph as

$$p(\boldsymbol{x}) \propto \prod_{i=1}^{N} f_i^b(\boldsymbol{x}_i)$$
. (9)

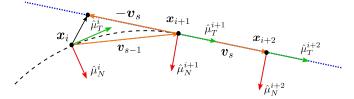


Fig. 5: Initial state x_i tangent angle rotation corresponding to next two states x_{i+1}, x_{i+2} and corresponding vectors v_{s-1}, v_s .

B. Minimum Curvature Factor

We leverage the insight from the Proposition 1 to design the minimum curvature factor f_i^{curv} that minimizes the mean curvature h_i of the centerline. The minimum curvature factor is attached to three consecutive states, in a manner that factor f_i^{curv} attached to x_i will also be attached to the x_{i+1} and x_{i+2} . We start with the assumption;

Assumption 1 For a factor $f_i^{curv}(\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \boldsymbol{x}_{i+2})$, the two consecutive states $\boldsymbol{x}_{i+1}, \boldsymbol{x}_{i+2}$ lie on the same straight line represented by a vector \boldsymbol{v}_s resulting in the tangential rotation angle $\Delta \Psi_i = 0$.

As from Proposition 1, if the three consecutive states x_i, x_{i+1}, x_{i+2} have tangential rotation angle $\Delta \Psi = 0$, then the curvature would be minimum. From Assumption 1, the x_i needs to be adjusted in order to lie on the straight line. Considering the central state x_{i+1} as the reference point, the v_{s-1} is the vector from x_i to x_{i+1} , then the factor function that will induce minimum curvature capability is written as

$$f_i^{curv}\left(x_i, x_{i+1}, x_{i+2}\right) = \exp\left\{-\frac{1}{2} \left\|-\left(v_{s-1} + v_s\right)\right\|_{\Sigma}^2\right\}.$$
(10)

Figure 5 represent the vectors formed among the x_i, x_{i+1}, x_{i+2} . Having defined the minimum curvature factor, we represent the likelihood function in terms of minimum curvature factors as

$$l\left(\boldsymbol{x};\mathbf{e}\right) \propto \prod_{i=1}^{N} f_{i}^{curv}\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{i+1}, \boldsymbol{x}_{i+2}\right).$$
 (11)

C. MAP Inference

An inference algorithm on the factor graph will compute the posterior distribution over all states fulfilling the constraints imposed through attached factors. Thus, finding the most probable values for x from a factor graph is,

$$x^* = \arg\max_{x} \sum_{m=1}^{M} f_m(\mathcal{X}_m). \tag{12}$$

Note that the factor graph will be consisting of the factors comprising of prior (9) and likelihood factors (11). The depiction of the designed factor graph structure is shown in the Figure 6. The prior bounding factor introduces the

¹The choice of defining the heading vector from left to right is arbitrary. Same equations hold for different direction of heading vector.

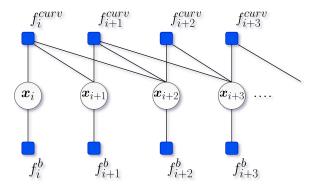


Fig. 6: An example factor graph for minimum curvature raceline planning.

non-linearity in the factor graph. The non-linear factor graph can be efficiently solved by exploiting the underlying sparse least squares. Several SLAM [15] and planning algorithms [12] have utilized this approach in which the sparse graphical model representation is converted into a least square optimization problem. Similarly, we formulate the MAP inference problem into least square optimization by taking the negative log of the posterior distribution (4) which results in

$$x^* = \arg\min_{x} \left[\frac{1}{2} \|x - x_t\|_{\mathcal{K}}^2 + \frac{1}{2} \|-(v_{s-1} + v_s)\|_{\Sigma}^2 \right].$$
 (13)

We then use the Levenberg–Marquardt algorithm to solve the least square problem in (13).

V. EVALUATION

We perform the evaluation of proposed framework in simulation for two racetracks, Berlin Formula E-racetrack and Modena E-racetrack. In our framework the inputs are the centerline and the track widths.

A. Implementation Details

We use GTSAM liberary [16] to implement and optimize the factor graph. Factors are implemented in C++. A MAT-LAB wrapper is used to observe the simulated results. Our implementation is available as an open source library². The prior bounding factor f^b cost is adjusted by the bounding weight parameter σ_b . Where,

$$\mathcal{K} = \sigma_b^2 \mathbf{I}.\tag{14}$$

The value of parameter σ_b is set to 1 for both the racetracks. Similarly, we set the value of parameter σ_{curv} equal to $6e^{-3}$ for Berlin E-racetrack and $2e^{-3}$ is found to be suitable for Modena E-racetrack. Where,

$$\Sigma = \sigma_{curv}^2 \boldsymbol{I}.\tag{15}$$

The boundary range $[\varrho_{min}, \varrho_{max}]$ to limit the states inside the racetrack is computed as a pre-processing step by simply

TABLE I: Benchmark results representing computation efficiency of proposed probabilistic inference-based framework.

	Proposed	Framework	Heilmeier et. el. [2]	
	Berlin	Modena	Berlin	Modena
Lap Time (s) Curvature (1/m) Distance (m) Computation Time (s)	81.60 12.07 2326.11 1.477	78.77 13.00 1995.06 1.099	81.77 11.05 2326.71 6.47	79.44 13.15 2000.69 4.28

adjusting the range according to the angle at that racetrack segment consisting of two consecutive states.

B. Raceline Planning Benchmark

We benchmark our proposed framework against the state of the art QP optimization algorithm proposed by Heilmeier et. el. [2]. We evaluated both algorithms by running tests for both the racetracks to note cumulative curvature, total distance, lap time and the computation time. We adopt the same approach to compute the lap time as proposed in [2] for proper comparison. However, we use the sum of absolute curvature (16) values to compute the total curvature of the raceline in order to avoid the curvature value cancellation at opposite curves/turns.

$$k_{total} = \sum_{i=1}^{n} |k_i|$$
 (16)

All the tests are performed on an Intel i5-7200 CPU @ 2.5 GHz x4 system with 8 GB of RAM.

C. Analysis

Table I summarizes the benchmark results. Computation time for the proposed probabilistic inference-based framework is approximately four times less than the QP optimization approach [2]. Computational efficiency of the proposed approach is mainly due to the two reasons. First, the factor graph formulation of minimum curvature planning problem allows to simplify the overall problem by using basic vector geometry computations. Secondly, The resulting factor graph is a sparse probabilistic graphical model that results in sparse non-linear least squares which can be solved with quadratic convergence guarantees. The time given in the Table I for proposed framework includes the pre-processing step, factor graph formation and optimization. The pre-processing only involves the sampling of centerine and boundary limits calculation for prior bounding factor which roughly takes 0.02s. The resulting centerline consists of 1183 states for Berlin racetrack and 995 states for Modena racetrack.

Figure 7 represents the raceline generated from probabilistic approach and QP optimization method for the Berlin Eracetrack. It can be seen that the raceline is almost identical which is also evident from the recorded parameter values in Table I. The lap time, curvature and distance values for both the approaches are very close to each other. Similar results have been recorded for the Modena E-racetrack as depicted in Figure 1.

²Available at https://github.com/Ashn-1/pi-racing

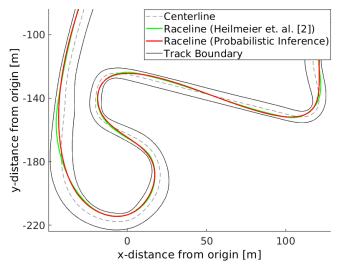


Fig. 7: A snippet of racelines generated via Probabilistic inference approach and QP optimization for Berlin E-racetrack.

We also recorded the computation time of 21.66s for Berlin E-racetrack and 13.71s for Modena E-racetrack for iterative QP optimization approach [2]. We believe that probabilistic inference approach circumvent not only the iterative improvement but also produce smooth raceline. As the factor graph formulation allows us to induce smoothness by considering the joint distribution over coupled random variables. A drawback of the proposed approach is its inability to handle hard constraints. The pre-processing step of calculating the boundary limits is introduced to circumvent this issue so that the states do not go out of the boundaries of racetrack. However, a better approach could be to investigate, in future, the formulation of chance constraint on factor graph for prior bounding factor that can offer better results without the need of pre-processing step.

VI. CONCLUSIONS

This work covers the theoretical formulation of minimum curvature raceline planning problem via probabilistic inference. The novel framework describes a new perspective on decomposing raceline planning problem and exploiting the inherent structure among random variables. The minimum curvature path objective function is designed as a joint distribution over variables using factor graph. The factor graph formulation simplifies the overall problem and dispense the availability of fast inference algorithms. Initial simulation results verifies the computation efficiency of the proposed framework that outperforms the state of the art QP optimization approach while keeping the quality of generated raceline intact. The proposed approach is, however, limited in terms of dealing with hard constraints. The racetrack boundary limit is formulated as soft constraint which needs an additional pre-processing step. Future investigation into constraints formulation for factor graphs could also pave the way to develop local planning algorithms via probabilistic inference.

ACKNOWLEDGMENT

The research leading to these results has partially received funding from the Horizon 2020 research and innovation programme under grant agreement 820742 of the project "HR-Recycler - Hybrid Human-Robot RECYcling plant for electriCal and eLEctRonic equipment".

REFERENCES

- [1] J. Betz, A. Wischnewski, A. Heilmeier, F. Nobis, L. Hermansdorfer, T. Stahl, T. Herrmann, and M. Lienkamp, "A software architecture for the dynamic path planning of an autonomous racecar at the limits of handling," in 2019 IEEE International Conference on Connected Vehicles and Expo, ICCVE 2019, Graz, Austria, November 4-8, 2019. IEEE, 2019, pp. 1–8.
- [2] A. Heilmeier, A. Wischnewski, L. Hermansdorfer, J. Betz, M. Lienkamp, and B. Lohmann, "Minimum curvature trajectory planning and control for an autonomous race car," *Vehicle System Dynamics*, vol. 58, no. 10, pp. 1497–1527, 2020.
- [3] F. Braghin, F. Cheli, S. Melzi, and E. Sabbioni, "Race driver model," *Computers & Structures*, vol. 86, no. 13, pp. 1503–1516, 2008.
- [4] P. E. Hart, N. J. Nilsson, and B. Raphael, "A formal basis for the heuristic determination of minimum cost paths," *IEEE Trans. Syst. Sci. Cybern.*, vol. 4, no. 2, pp. 100–107, 1968.
- [5] A. Stentz, "Optimal and efficient path planning for partially-known environments," in *Proceedings of the 1994 International Conference* on Robotics and Automation, San Diego, CA, USA, May 1994. IEEE Computer Society, 1994, pp. 3310–3317.
- [6] —, "The focussed d* algorithm for real-time replanning," in Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence, IJCAI 95, Montréal Québec, Canada, August 20-25 1995, 2 Volumes. Morgan Kaufmann, 1995, pp. 1652–1659.
- [7] L. E. Kavraki, P. Svestka, J. Latombe, and M. H. Overmars, "Probabilistic roadmaps for path planning in high-dimensional configuration spaces," *IEEE Trans. Robotics Autom.*, vol. 12, no. 4, pp. 566–580, 1996.
- [8] J. J. K. Jr. and S. M. LaValle, "RRT-connect: An efficient approach to single-query path planning," in *Proceedings of the 2000 IEEE International Conference on Robotics and Automation, ICRA 2000, San Francisco, CA, USA, April 24-28, 2000.* IEEE, 2000, pp. 995–1001
- [9] N. D. Ratliff, M. Zucker, J. A. Bagnell, and S. S. Srinivasa, "CHOMP: gradient optimization techniques for efficient motion planning," in 2009 IEEE International Conference on Robotics and Automation, ICRA 2009, Kobe, Japan, May 12-17, 2009. IEEE, 2009, pp. 489–494.
- [10] M. Kalakrishnan, S. Chitta, E. A. Theodorou, P. Pastor, and S. Schaal, "STOMP: stochastic trajectory optimization for motion planning," in IEEE International Conference on Robotics and Automation, ICRA 2011, Shanghai, China, 9-13 May 2011. IEEE, 2011, pp. 4569–4574.
- [11] N. R. Kapania, J. Subosits, and J. Christian Gerdes, "A Sequential Two-Step Algorithm for Fast Generation of Vehicle Racing Trajectories," *Journal of Dynamic Systems, Measurement, and Control*, vol. 138, no. 9, 06 2016.
- [12] M. Mukadam, J. Dong, X. Yan, F. Dellaert, and B. Boots, "Continuoustime Gaussian process motion planning via probabilistic inference," *Int. J. Robotics Res.*, vol. 37, no. 11, pp. 1319–1340, 2018.
- [13] M. Toussaint and C. Goerick, "A bayesian view on motor control and planning," in *From Motor Learning to Interaction Learning in Robots*, ser. Studies in Computational Intelligence, O. Sigaud and J. Peters, Eds. Springer, 2010, vol. 264, pp. 227–252.
- [14] H. Attias, "Planning by probabilistic inference," in Proceedings of the Ninth International Workshop on Artificial Intelligence and Statistics, AISTATS 2003, Key West, Florida, USA, January 3-6, 2003, C. M. Bishop and B. J. Frey, Eds. Society for Artificial Intelligence and Statistics, 2003.
- [15] F. Dellaert and M. Kaess, "Square root SAM: simultaneous localization and mapping via square root information smoothing," *Int. J. Robotics Res.*, vol. 25, no. 12, pp. 1181–1203, 2006.
- [16] F. Dellaert, "Factor graphs and gtsam: A hands-on introduction." Technical report, Georgia Tech Technical Report, GT-RIM-CP&R-2012-002, 2012.