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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A hexagonal grid of 19 cells, numbered 0 to 18, arranged in a honeycomb pattern. Each cell contains a red dot and a number. The cells are surrounded by 21 green dots, numbered 19 to 39, arranged in a larger hexagonal pattern around the central cluster.

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

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$$N^{-1} = \begin{pmatrix} p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = 0 & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = \frac{1}{6} & p_{1,11} = \frac{1}{6} & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = 0 & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 1 &p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = 0 & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = \frac{1}{6} & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = \frac{1}{6} & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = 0 & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = \frac{1}{6} & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = \frac{1}{6} & p_{6,16} = \frac{1}{6} & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = 0 \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = 0 & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = 0 & p_{10,10} = 1 & p_{10,11} = \frac{1}{6} & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = \frac{1}{6} & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 1 & p_{11,12} = 0 & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = \frac{1}{6} & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = 0 & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = 0 & p_{14,4} = \frac{1}{6} & p_{14,5} = \frac{1}{6} & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{$$

$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = 45 \\ P_{0,1} = 34506 \\ P_{0,2} = 107714 \\ P_{0,3} = 249993 \\ P_{0,4} = 479997 \\ P_{0,5} = 799995 \\ P_{0,6} = 1119993 \\ P_{0,7} = 1439991 \\ P_{0,8} = 1759989 \\ P_{0,9} = 2079987 \\ P_{0,10} = 2399985 \\ P_{0,11} = 2719983 \\ P_{0,12} = 3039981 \\ P_{0,13} = 3359979 \\ P_{0,14} = 3679977 \\ P_{0,15} = 3999975 \\ P_{0,16} = 4319973 \\ P_{0,17} = 4639971 \\ P_{0,18} = 4959969 \\ P_{0,19} = 5279967 \\ P_{0,20} = 5599965 \\ P_{0,21} = 5919963 \\ P_{0,22} = 6239961 \\ P_{0,23} = 6559959 \\ P_{0,24} = 6879957 \\ P_{0,25} = 7199955 \\ P_{0,26} = 7519953 \\ P_{0,27} = 7839951 \\ P_{0,28} = 8159949 \\ P_{0,29} = 8479947 \\ P_{0,30} = 8799945 \\ P_{0,31} = 9119943 \\ P_{0,32} = 9439941 \\ P_{0,33} = 9759939 \\ P_{0,34} = 10079937 \\ P_{0,35} = 10399935 \\ P_{0,36} = 10719933 \\ P_{0,37} = 11039931 \\ P_{0,38} = 11359929 \\ P_{0,39} = 11679927 \\ P_{0,40} = 11999925 \\ P_{0,41} = 12319923 \\ P_{0,42} = 12639921 \\ P_{0,43} = 12959919 \\ P_{0,44} = 13279917 \\ P_{0,45} = 13599915 \\ P_{0,46} = 13919913 \\ P_{0,47} = 14239911 \\ P_{0,48} = 14559909 \\ P_{0,49} = 14879907 \\ P_{0,50} = 15199905 \\ P_{0,51} = 15519903 \\ P_{0,52} = 15839901 \\ P_{0,53} = 16159899 \\ P_{0,54} = 16479897 \\ P_{0,55} = 16799895 \\ P_{0,56} = 17119893 \\ P_{0,57} = 17439891 \\ P_{0,58} = 17759889 \\ P_{0,59} = 18079887 \\ P_{0,60} = 18399885 \\ P_{0,61} = 18719883 \\ P_{0,62} = 19039881 \\ P_{0,63} = 19359879 \\ P_{0,64} = 19679877 \\ P_{0,65} = 19999875 \\ P_{0,66} = 20319873 \\ P_{0,67} = 20639871 \\ P_{0,68} = 20959869 \\ P_{0,69} = 21279867 \\ P_{0,70} = 21599865 \\ P_{0,71} = 21919863 \\ P_{0,72} = 22239861 \\ P_{0,73} = 22559859 \\ P_{0,74} = 22879857 \\ P_{0,75} = 23199855 \\ P_{0,76} = 23519853 \\ P_{0,77} = 23839851 \\ P_{0,78} = 24159849 \\ P_{0,79} = 24479847 \\ P_{0,80} = 24799845 \\ P_{0,81} = 25119843 \\ P_{0,82} = 25439841 \\ P_{0,83} = 25759839 \\ P_{0,84} = 26079837 \\ P_{0,85} = 26399835 \\ P_{0,86} = 26719833 \\ P_{0,87} = 27039831 \\ P_{0,88} = 27359829 \\ P_{0,89} = 27679827 \\ P_{0,90} = 27999825 \\ P_{0,91} = 28319823 \\ P_{0,92} = 28639821 \\ P_{0,93} = 28959819 \\ P_{0,94} = 29279817 \\ P_{0,95} = 29599815 \\ P_{0,96} = 29919813 \\ P_{0,97} = 30239811 \\ P_{0,98} = 30559809 \\ P_{0,99} = 30879807 \\ P_{0,100} = 31199805 \\ P_{0,101} = 31519803 \\ P_{0,102} = 31839801 \\ P_{0,103} = 32159799 \\ P_{0,104} = 32479797 \\ P_{0,105} = 32799795 \\ P_{0,106} = 33119793 \\ P_{0,107} = 33439791 \\ P_{0,108} = 33759789 \\ P_{0,109} = 34079787 \\ P_{0,110} = 34399785 \\ P_{0,111} = 34719783 \\ P_{0,112} = 35039781 \\ P_{0,113} = 35359779 \\ P_{0,114} = 35679777 \\ P_{0,115} = 35999775 \\ P_{0,116} = 36319773 \\ P_{0,117} = 36639771 \\ P_{0,118} = 36959769 \\ P_{0,119} = 37279767 \\ P_{0,120} = 37599765 \\ P_{0,121} = 37919763 \\ P_{0,122} = 38239761 \\ P_{0,123} = 38559759 \\ P_{0,124} = 38879757 \\ P_{0,125} = 39199755 \\ P_{0,126} = 39519753 \\ P_{0,127} = 39839751 \\ P_{0,128} = 40159749 \\ P_{0,129} = 40479747 \\ P_{0,130} = 40799745 \\ P_{0,131} = 41119743 \\ P_{0,132} = 41439741 \\ P_{0,133} = 41759739 \\ P_{0,134} = 42079737 \\ P_{0,135} = 42399735 \\ P_{0,136} = 42719733 \\ P_{0,137} = 43039731 \\ P_{0,138} = 43359729 \\ P_{0,139} = 43679727 \\ P_{0,140} = 43999725 \\ P_{0,141} = 44319723 \\ P_{0,142} = 44639721 \\ P_{0,143} = 44959719 \\ P_{0,144} = 45279717 \\ P_{0,145} = 45599715 \\ P_{0,146} = 45919713 \\ P_{0,147} = 46239711 \\ P_{0,148} = 46559709 \\ P_{0,149} = 46879707 \\ P_{0,150} = 47199705 \\ P_{0,151} = 47519703 \\ P_{0,152} = 47839701 \\ P_{0,153} = 48159699 \\ P_{0,154} = 48479697 \\ P_{0,155} = 48799695 \\ P_{0,156} = 49119693 \\ P_{0,157} = 49439691 \\ P_{0,158} = 49759689 \\ P_{0,159} = 50079687 \\ P_{0,160} = 50399685 \\ P_{0,161} = 50719683 \\ P_{0,162} = 51039681 \\ P_{0,163} = 51359679 \\ P_{0,164} = 51679677 \\ P_{0,165} = 51999675 \\ P_{0,166} = 52319673 \\ P_{0,167} = 52639671 \\ P_{0,168} = 52959669 \\ P_{0,169} = 53279667 \\ P_{0,170} = 53599665 \\ P_{0,171} = 53919663 \\ P_{0,172} = 54239661 \\ P_{0,173} = 54559659 \\ P_{0,174} = 54879657 \\ P_{0,175} = 55199655 \\ P_{0,176} = 55519653 \\ P_{0,177} = 55839651 \\ P_{0,178} = 56159649 \\ P_{0,179} = 56479647 \\ P_{0,180} = 56799645 \\ P_{0,181} = 57119643 \\ P_{0,182} = 57439641 \\ P_{0,183} = 57759639 \\ P_{0,184} = 58079637 \\ P_{0,185} = 58399635 \\ P_{0,186} = 58719633 \\ P_{0,187} = 59039631 \\ P_{0,188} = 59359629 \\ P_{0,189} = 59679627 \\ P_{0,190} = 59999625 \\ P_{0,191} = 60319623 \\ P_{0,192} = 60639621 \\ P_{0,193} = 60959619 \\ P_{0,194} = 61279617 \\ P_{0,195} = 61599615 \\ P_{0,196} = 61919613 \\ P_{0,197} = 62239611 \\ P_{0,198} = 62559609 \\ P_{0,199} = 62879607 \\ P_{0,200} = 63199605 \\ P_{0,201} = 63519603 \\ P_{0,202} = 63839601 \\ P_{0,203} = 64159599 \\ P_{0,204} = 64479597 \\ P_{0,205} = 64799595 \\ P_{0,206} = 65119593 \\ P_{0,207} = 65439591 \\ P_{0,208} = 65759589 \\ P_{0,209} = 66079587 \\ P_{0,210} = 66399585 \\ P_{0,211} = 66719583 \\ P_{0,212} = 67039581 \\ P_{0,213} = 67359579 \\ P_{0,214} = 67679577 \\ P_{0,215} = 67999575 \\ P_{0,216} = 68319573 \\ P_{0,217} = 68639571 \\ P_{0,218} = 68959569 \\ P_{0,219} = 69279567 \\ P_{0,220} = 69599565 \\$$

In order to get the expected number of steps, we find  $t_0$ , where

$$t = N\mathbf{1}$$

Here,  $\mathbf{1}$  is a vector whose entries are all 1.

[illegible]

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$