

Rajeev Atla

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

The diagram shows a hexagonal lattice structure. The central node is labeled 0. The nodes are arranged in a hexagonal pattern, with the central node being 0. The nodes are colored: internal nodes are red dots, and external nodes are green dots.

The internal nodes (red dots) are numbered 0 to 18. The external nodes (green dots) are numbered 19 to 36. The nodes are arranged in a hexagonal pattern, with the central node being 0.

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.



|       |                           |                                 |                                 |                                 |                                 |                           |                           |                           |                                 |                           |                            |                            |                                  |                            |                                  |                            |                            |                            |                            |
|-------|---------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------|---------------------------|---------------------------|---------------------------------|---------------------------|----------------------------|----------------------------|----------------------------------|----------------------------|----------------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| $N =$ | $P_{0,0} = \frac{45}{16}$ | $P_{0,1} = \frac{15}{16}$       | $P_{0,2} = \frac{15}{16}$       | $P_{0,3} = \frac{15}{16}$       | $P_{0,4} = \frac{15}{16}$       | $P_{0,5} = \frac{15}{16}$ | $P_{0,6} = \frac{15}{16}$ | $P_{0,7} = \frac{7}{2}$   | $P_{0,8} = \frac{5}{4}$         | $P_{0,9} = \frac{7}{2}$   | $P_{0,10} = \frac{15}{16}$ | $P_{0,11} = \frac{7}{2}$   | $P_{0,12} = \frac{5}{4}$         | $P_{0,13} = \frac{7}{2}$   | $P_{0,14} = \frac{5}{4}$         | $P_{0,15} = \frac{7}{2}$   | $P_{0,16} = \frac{5}{4}$   | $P_{0,17} = \frac{7}{2}$   | $P_{0,18} = \frac{5}{4}$   |
|       | $P_{1,0} = \frac{15}{16}$ | $P_{1,1} = \frac{10571}{16384}$ | $P_{1,2} = \frac{63005}{16384}$ | $P_{1,3} = \frac{63005}{16384}$ | $P_{1,4} = \frac{10571}{16384}$ | $P_{1,5} = \frac{15}{16}$ | $P_{1,6} = \frac{15}{16}$ | $P_{1,7} = \frac{15}{16}$ | $P_{1,8} = \frac{22965}{16384}$ | $P_{1,9} = \frac{15}{16}$ | $P_{1,10} = \frac{15}{16}$ | $P_{1,11} = \frac{15}{16}$ | $P_{1,12} = \frac{22965}{16384}$ | $P_{1,13} = \frac{15}{16}$ | $P_{1,14} = \frac{22965}{16384}$ | $P_{1,15} = \frac{15}{16}$ | $P_{1,16} = \frac{15}{16}$ | $P_{1,17} = \frac{15}{16}$ | $P_{1,18} = \frac{15}{16}$ |
|       | $P_{2,0} = \frac{15}{16}$ | $P_{2,1} = \frac{15}{16}$       | $P_{2,2} = \frac{15}{16}$       | $P_{2,3} = \frac{15}{16}$       | $P_{2,4} = \frac{15}{16}$       | $P_{2,5} = \frac{15}{16}$ | $P_{2,6} = \frac{15}{16}$ | $P_{2,7} = \frac{15}{16}$ | $P_{2,8} = \frac{22965}{16384}$ | $P_{2,9} = \frac{15}{16}$ | $P_{2,10} = \frac{15}{16}$ | $P_{2,11} = \frac{15}{16}$ | $P_{2,12} = \frac{15}{16}$       | $P_{2,13} = \frac{15}{16}$ | $P_{2,14} = \frac{15}{16}$       | $P_{2,15} = \frac{15}{16}$ | $P_{2,16} = \frac{15}{16}$ | $P_{2,17} = \frac{15}{16}$ | $P_{2,18} = \frac{15}{16}$ |
|       | $P_{3,0} = \frac{15}{16}$ | $P_{3,1} = \frac{15}{16}$       | $P_{3,2} = \frac{15}{16}$       | $P_{3,3} = \frac{15}{16}$       | $P_{3,4} = \frac{15}{16}$       | $P_{3,5} = \frac{15}{16}$ | $P_{3,6} = \frac{15}{16}$ | $P_{3,7} = \frac{15}{16}$ | $P_{3,8} = \frac{15}{16}$       | $P_{3,9} = \frac{15}{16}$ | $P_{3,10} = \frac{15}{16}$ | $P_{3,11} = \frac{15}{16}$ | $P_{3,12} = \frac{15}{16}$       | $P_{3,13} = \frac{15}{16}$ | $P_{3,14} = \frac{15}{16}$       | $P_{3,15} = \frac{15}{16}$ | $P_{3,16} = \frac{15}{16}$ | $P_{3,17} = \frac{15}{16}$ | $P_{3,18} = \frac{15}{16}$ |
|       | $P_{4,0} = \frac{15}{16}$ | $P_{4,1} = \frac{15}{16}$       | $P_{4,2} = \frac{15}{16}$       | $P_{4,3} = \frac{15}{16}$       | $P_{4,4} = \frac{15}{16}$       | $P_{4,5} = \frac{15}{16}$ | $P_{4,6} = \frac{15}{16}$ | $P_{4,7} = \frac{15}{16}$ | $P_{4,8} = \frac{15}{16}$       | $P_{4,9} = \frac{15}{16}$ | $P_{4,10} = \frac{15}{16}$ | $P_{4,11} = \frac{15}{16}$ | $P_{4,12} = \frac{15}{16}$       | $P_{4,13} = \frac{15}{16}$ | $P_{4,14} = \frac{15}{16}$       | $P_{4,15} = \frac{15}{16}$ | $P_{4,16} = \frac{15}{16}$ | $P_{4,17} = \frac{15}{16}$ | $P_{4,18} = \frac{15}{16}$ |
|       | $P_{5,0} = \frac{15}{16}$ | $P_{5,1} = \frac{15}{16}$       | $P_{5,2} = \frac{15}{16}$       | $P_{5,3} = \frac{15}{16}$       | $P_{5,4} = \frac{15}{16}$       | $P_{5,5} = \frac{15}{16}$ | $P_{5,6} = \frac{15}{16}$ | $P_{5,7} = \frac{15}{16}$ | $P_{5,8} = \frac{15}{16}$       | $P_{5,9} = \frac{15}{16}$ | $P_{5,10} = \frac{15}{16}$ | $P_{5,11} = \frac{15}{16}$ | $P_{5,12} = \frac{15}{16}$       | $P_{5,13} = \frac{15}{16}$ | $P_{5,14} = \frac{15}{16}$       | $P_{5,15} = \frac{15}{16}$ | $P_{5,16} = \frac{15}{16}$ | $P_{5,17} = \frac{15}{16}$ | $P_{5,18} = \frac{15}{16}$ |
|       | $P_{6,0} = \frac{15}{16}$ | $P_{6,1} = \frac{15}{16}$       | $P_{6,2} = \frac{15}{16}$       | $P_{6,3} = \frac{15}{16}$       | $P_{6,4} = \frac{15}{16}$       | $P_{6,5} = \frac{15}{16}$ | $P_{6,6} = \frac{15}{16}$ | $P_{6,7} = \frac{15}{16}$ | $P_{6,8} = \frac{15}{16}$       | $P_{6,9} = \frac{15}{16}$ | $P_{6,10} = \frac{15}{16}$ | $P_{6,11} = \frac{15}{16}$ | $P_{6,12} = \frac{15}{16}$       | $P_{6,13} = \frac{15}{16}$ | $P_{6,14} = \frac{15}{16}$       | $P_{6,15} = \frac{15}{16}$ | $P_{6,16} = \frac{15}{16}$ | $P_{6,17} = \frac{15}{16}$ | $P_{6,18} = \frac{15}{16}$ |
|       | $P_{7,0} = \frac{7}{2}$   | $P_{7,1} = \frac{15}{16}$       | $P_{7,2} = \frac{15}{16}$       | $P_{7,3} = \frac{15}{16}$       | $P_{7,4} = \frac{15}{16}$       | $P_{7,5} = \frac{7}{2}$   | $P_{7,6} = \frac{15}{16}$ | $P_{7,7} = \frac{15}{16}$ | $P_{7,8} = \frac{15}{16}$       | $P_{7,9} = \frac{15}{16}$ | $P_{7,10} = \frac{15}{16}$ | $P_{7,11} = \frac{15}{16}$ | $P_{7,12} = \frac{15}{16}$       | $P_{7,13} = \frac{15}{16}$ | $P_{7,14} = \frac{15}{16}$       | $P_{7,15} = \frac{15}{16}$ | $P_{7,16} = \frac{15}{16}$ | $P_{7,17} = \frac{15}{16}$ | $P_{7,18} = \frac{15}{16}$ |
|       | $P_{8,0} = \frac{5}{4}$   | $P_{8,1} = \frac{15}{16}$       | $P_{8,2} = \frac{15}{16}$       | $P_{8,3} = \frac{15}{16}$       | $P_{8,4} = \frac{15}{16}$       | $P_{8,5} = \frac{5}{4}$   | $P_{8,6} = \frac{15}{16}$ | $P_{8,7} = \frac{15}{16}$ | $P_{8,8} = \frac{15}{16}$       | $P_{8,9} = \frac{15}{16}$ | $P_{8,10} = \frac{15}{16}$ | $P_{8,11} = \frac{15}{16}$ | $P_{8,12} = \frac{15}{16}$       | $P_{8,13} = \frac{15}{16}$ | $P_{8,14} = \frac{15}{1$         |                            |                            |                            |                            |

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$