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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states [19, 36] into a

$$P = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = 0 & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 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| | P ₁₀ = .45 | P ₁₁ = .16 | P ₁₂ = .16 | P ₁₃ = .16 | P ₁₄ = .16 | P ₁₅ = .16 | P ₁₆ = .16 | P ₁₇ = .16 | P ₁₈ = .16 | P ₁₉ = .16 | P ₂₀ = .16 | P ₂₁ = .16 | P ₂₂ = .16 | P ₂₃ = .16 | P ₂₄ = .16 | P ₂₅ = .16 | P ₂₆ = .16 | P ₂₇ = .16 | P ₂₈ = .16 | P ₂₉ = .16 | P ₃₀ = .16 | P ₃₁ = .16 | P ₃₂ = .16 | P ₃₃ = .16 | P ₃₄ = .16 | P ₃₅ = .16 | P ₃₆ = .16 | P ₃₇ = .16 | P ₃₈ = .16 | P ₃₉ = .16 | P ₄₀ = .16 | P ₄₁ = .16 | P ₄₂ = .16 | P ₄₃ = .16 | P ₄₄ = .16 | P ₄₅ = .16 | P ₄₆ = .16 | P ₄₇ = .16 | P ₄₈ = .16 | P ₄₉ = .16 | P ₅₀ = .16 | P ₅₁ = .16 | P ₅₂ = .16 | P ₅₃ = .16 | P ₅₄ = .16 | P ₅₅ = .16 | P ₅₆ = .16 | P ₅₇ = .16 | P ₅₈ = .16 | P ₅₉ = .16 | P ₆₀ = .16 | P ₆₁ = .16 | P ₆₂ = .16 | P ₆₃ = .16 | P ₆₄ = .16 | P ₆₅ = .16 | P ₆₆ = .16 | P ₆₇ = .16 | P ₆₈ = .16 | P ₆₉ = .16 | P ₇₀ = .16 | P ₇₁ = .16 | P ₇₂ = .16 | P ₇₃ = .16 | P ₇₄ = .16 | P ₇₅ = .16 | P ₇₆ = .16 | P ₇₇ = .16 | P ₇₈ = .16 | P ₇₉ = .16 | P ₈₀ = .16 | P ₈₁ = .16 | P ₈₂ = .16 | P ₈₃ = .16 | P ₈₄ = .16 | P ₈₅ = .16 | P ₈₆ = .16 | P ₈₇ = .16 | P ₈₈ = .16 | P ₈₉ = .16 | P ₉₀ = .16 | P ₉₁ = .16 | P ₉₂ = .16 | P ₉₃ = .16 | P ₉₄ = .16 | P ₉₅ = .16 | P ₉₆ = .16 | P ₉₇ = .16 | P ₉₈ = .16 | P ₉₉ = .16 | P ₁₀₀ = .16 | P ₁₀₁ = .16 | P ₁₀₂ = .16 | P ₁₀₃ = .16 | P ₁₀₄ = .16 | P ₁₀₅ = .16 | P ₁₀₆ = .16 | P ₁₀₇ = .16 | P ₁₀₈ = .16 | P ₁₀₉ = .16 | P ₁₁₀ = .16 | P ₁₁₁ = .16 | P ₁₁₂ = .16 | P ₁₁₃ = .16 | P ₁₁₄ = .16 | P ₁₁₅ = .16 | P ₁₁₆ = .16 | P ₁₁₇ = .16 | P ₁₁₈ = .16 | P ₁₁₉ = .16 | P ₁₂₀ = .16 | P ₁₂₁ = .16 | P ₁₂₂ = .16 | P ₁₂₃ = .16 | P ₁₂₄ = .16 | P ₁₂₅ = .16 | P ₁₂₆ = .16 | P ₁₂₇ = .16 | P ₁₂₈ = .16 | P ₁₂₉ = .16 | P ₁₃₀ = .16 | P ₁₃₁ = .16 | P ₁₃₂ = .16 | P ₁₃₃ = .16 | P ₁₃₄ = .16 | P ₁₃₅ = .16 | P ₁₃₆ = .16 | P ₁₃₇ = .16 | P ₁₃₈ = .16 | P ₁₃₉ = .16 | P ₁₄₀ = .16 | P ₁₄₁ = .16 | P ₁₄₂ = .16 | P ₁₄₃ = .16 | P ₁₄₄ = .16 | P ₁₄₅ = .16 | P ₁₄₆ = .16 | P ₁₄₇ = .16 | P ₁₄₈ = .16 | P ₁₄₉ = .16 | P ₁₅₀ = .16 | P ₁₅₁ = .16 | P ₁₅₂ = .16 | P ₁₅₃ = .16 | P ₁₅₄ = .16 | P ₁₅₅ = .16 | P ₁₅₆ = .16 | P ₁₅₇ = .16 | P ₁₅₈ = .16 | P ₁₅₉ = .16 | P ₁₆₀ = .16 | P ₁₆₁ = .16 | P ₁₆₂ = .16 | P ₁₆₃ = .16 | P ₁₆₄ = .16 | P ₁₆₅ = .16 | P ₁₆₆ = .16 | P ₁₆₇ = .16 | P ₁₆₈ = .16 | P ₁₆₉ = .16 | P ₁₇₀ = .16 | P ₁₇₁ = .16 | P ₁₇₂ = .16 | P ₁₇₃ = .16 | P ₁₇₄ = .16 | P ₁₇₅ = .16 | P ₁₇₆ = .16 | P ₁₇₇ = .16 | P ₁₇₈ = .16 | P ₁₇₉ = .16 | P ₁₈₀ = .16 | P ₁₈₁ = .16 | P ₁₈₂ = .16 | P ₁₈₃ = .16 | P ₁₈₄ = .16 | P ₁₈₅ = .16 | P ₁₈₆ = .16 | P ₁₈₇ = .16 | P ₁₈₈ = .16 | P ₁₈₉ = .16 | P ₁₉₀ = .16 | P ₁₉₁ = .16 | P ₁₉₂ = .16 | P ₁₉₃ = .16 | P ₁₉₄ = .16 | P ₁₉₅ = .16 | P ₁₉₆ = .16 | P ₁₉₇ = .16 | P ₁₉₈ = .16 | P ₁₉₉ = .16 | P ₂₀₀ = .16 | P ₂₀₁ = .16 | P ₂₀₂ = .16 | P ₂₀₃ = .16 | P ₂₀₄ = .16 | P ₂₀₅ = .16 | P ₂₀₆ = .16 | P ₂₀₇ = .16 | P ₂₀₈ = .16 | P ₂₀₉ = .16 | P ₂₁₀ = .16 | P ₂₁₁ = .16 | P ₂₁₂ = .16 | P ₂₁₃ = .16 | P ₂₁₄ = .16 | P ₂₁₅ = .16 | P ₂₁₆ = .16 | P ₂₁₇ = .16 | P ₂₁₈ = .16 | P ₂₁₉ = .16 | P ₂₂₀ = .16 | P ₂₂₁ = .16 | P ₂₂₂ = .16 | P ₂₂₃ = .16 | P ₂₂₄ = .16 | P ₂₂₅ = .16 | P ₂₂₆ = .16 | P ₂₂₇ = .16 | P ₂₂₈ = .16 | P ₂₂₉ = .16 | P ₂₃₀ = .16 | P ₂₃₁ = .16 | P ₂₃₂ = .16 | P ₂₃₃ = .16 | P ₂₃₄ = .16 | P ₂₃₅ = .16 | P ₂₃₆ = .16 | P ₂₃₇ = .16 | P ₂₃₈ = .16 | P ₂₃₉ = .16 | P ₂₄₀ |
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$$t = N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$