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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A diagram showing a hexagonal grid of 19 cells, numbered 0 to 18, arranged in a roughly circular pattern. Each cell contains a red dot and a black number. The cells are surrounded by 21 green dots, numbered 19 to 39, arranged in a larger hexagonal pattern around the central cluster.

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

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|-------|---------------------------|-----------------------------------|-------------------------------------|--------------------------------------|-------------------------------------|-------------------------------------|--------------------------------------|-------------------------------------|--------------------------------------|--------------------------------------|---------------------------------------|---------------------------------------|---------------------------------------|---------------------------------------|--------------------------------------|---------------------------------------|---------------------------------------|--------------------------------------|--------------------------------------|
| $N =$ | $P_{0,0} = \frac{45}{16}$ | $P_{0,1} = \frac{15}{16}$ | $P_{0,2} = \frac{15}{16}$ | $P_{0,3} = \frac{15}{16}$ | $P_{0,4} = \frac{15}{16}$ | $P_{0,5} = \frac{15}{16}$ | $P_{0,6} = \frac{15}{16}$ | $P_{0,7} = \frac{7}{2}$ | $P_{0,8} = \frac{5}{4}$ | $P_{0,9} = \frac{7}{2}$ | $P_{0,10} = \frac{15}{16}$ | $P_{0,11} = \frac{7}{2}$ | $P_{0,12} = \frac{5}{4}$ | $P_{0,13} = \frac{7}{2}$ | $P_{0,14} = \frac{5}{4}$ | $P_{0,15} = \frac{7}{2}$ | $P_{0,16} = \frac{5}{4}$ | $P_{0,17} = \frac{7}{2}$ | $P_{0,18} = \frac{5}{4}$ |
| | $P_{1,0} = \frac{15}{16}$ | $P_{1,1} = \frac{10771}{16384}$ | $P_{1,2} = \frac{62605}{1048576}$ | $P_{1,3} = \frac{35363}{65536}$ | $P_{1,4} = \frac{19193}{131072}$ | $P_{1,5} = \frac{10193}{262144}$ | $P_{1,6} = \frac{5193}{524288}$ | $P_{1,7} = \frac{1313}{1048576}$ | $P_{1,8} = \frac{22965}{2097152}$ | $P_{1,9} = \frac{11363}{4194304}$ | $P_{1,10} = \frac{5681}{8388608}$ | $P_{1,11} = \frac{2841}{16777216}$ | $P_{1,12} = \frac{14203}{33554432}$ | $P_{1,13} = \frac{7101}{67108864}$ | $P_{1,14} = \frac{3551}{134217728}$ | $P_{1,15} = \frac{1776}{268435456}$ | $P_{1,16} = \frac{888}{536870912}$ | $P_{1,17} = \frac{444}{1073741824}$ | $P_{1,18} = \frac{222}{2147483648}$ |
| | $P_{2,0} = \frac{15}{16}$ | $P_{2,1} = \frac{61805}{1048576}$ | $P_{2,2} = \frac{309025}{16778240}$ | $P_{2,3} = \frac{154513}{268435456}$ | $P_{2,4} = \frac{77257}{429496704}$ | $P_{2,5} = \frac{38628}{687194752}$ | $P_{2,6} = \frac{19314}{1091511680}$ | $P_{2,7} = \frac{9657}{1746418624}$ | $P_{2,8} = \frac{48285}{2834270080}$ | $P_{2,9} = \frac{24143}{4534832384}$ | $P_{2,10} = \frac{12071}{7255731808}$ | $P_{2,11} = \frac{6036}{11609170816}$ | $P_{2,12} = \frac{3018}{18574673280}$ | $P_{2,13} = \frac{1509}{29719477248}$ | $P_{2,14} = \frac{754}{47551163584}$ | $P_{2,15} = \frac{377}{76081861760}$ | $P_{2,16} = \frac{189}{121730970752}$ | $P_{2,17} = \frac{94}{194769553280}$ | $P_{2,18} = \frac{47}{389539106560}$ |
| | $P_{3,0} = \frac{15}{16}$ | $P_{3,1} = \frac{27181}{16384}$ | $P_{3,2} = \frac{13591}{262144}$ | $P_{3,3} = \frac{6795}{4194304}$ | $P_{3,4} = \frac{3398}{67108864}$ | $P_{3,5} = \frac{1699}{1073741824}$ | $P_{3,6} = \frac{849}{17179869184}$ | $P_{3,7} = \frac{425}{27487790720}$ | $P_{3,8} = \frac{2125}{43980465152}$ | $P_{3,9} = \frac{1063}{70368744256}$ | $P_{3,10} = \frac{531}{112590070784}$ | $P_{3,11} = \frac{266}{180144113216}$ | $P_{3,12} = \frac{133}{288230581248}$ | $P_{3,13} = \frac{67}{461168930048}$ | $P_{3,14} = \frac{33}{737870288096}$ | $P_{3,15} = \frac{17}{1180592460928}$ | $P_{3,16} = \frac{9}{1888947937792}$ | $P_{3,17} = \frac{4}{2982316700672}$ | $P_{3,18} = \frac{2}{4771706701056}$ |
| | $P_{4,0} = \frac{15}{16}$ | $P_{4,1} = \frac{12091}{16384}$ | $P_{4,2} = \frac{6045}{262144}$ | $P_{4,3} = \frac{3023}{4194304}$ | $P_{4,4} = \frac{1511}{67108864}$ | $P_{4,5} = \frac{756}{1073741824}$ | $P_{4,6} = \frac{378}{17179869184}$ | $P_{4,7} = \frac{189}{27487790720}$ | $P_{4,8} = \frac{945}{43980465152}$ | $P_{4,9} = \frac{473}{70368744256}$ | $P_{4,10} = \frac{236}{112590070784}$ | $P_{4,11} = \frac{118}{180144113216}$ | $P_{4,12} = \frac{59}{288230581248}$ | $P_{4,13} = \frac{29}{461168930048}$ | $P_{4,14} = \frac{15}{737870288096}$ | $P_{4,15} = \frac{7}{1180592460928}$ | $P_{4,16} = \frac{4}{1888947937792}$ | $P_{4,17} = \frac{2}{2982316700672}$ | $P_{4,18} = \frac{1}{4771706701056}$ |
| | $P_{5,0} = \frac{15}{16}$ | $P_{5,1} = \frac{5445}{16384}$ | $P_{5,2} = \frac{2723}{262144}$ | $P_{5,3} = \frac{1362}{4194304}$ | $P_{5,4} = \frac{681}{67108864}$ | $P_{5,5} = \frac{340}{1073741824}$ | $P_{5,6} = \frac{170}{17179869184}$ | $P_{5,7} = \frac{85}{27487790720}$ | $P_{5,8} = \frac{425}{43980465152}$ | $P_{5,9} = \frac{213}{70368744256}$ | $P_{5,10} = \frac{106}{112590070784}$ | $P_{5,11} = \frac{53}{180144113216}$ | $P_{5,12} = \frac{27}{288230581248}$ | $P_{5,13} = \frac{13}{461168930048}$ | $P_{5,14} = \frac{7}{737870288096}$ | $P_{5,15} = \frac{3}{1180592460928}$ | $P_{5,16} = \frac{2}{1888947937792}$ | $P_{5,17} = \frac{1}{2982316700672}$ | $P_{5,18} = \frac{1}{4771706701056}$ |
| | $P_{6,0} = \frac{15}{16}$ | $P_{6,1} = \frac{2723}{16384}$ | $P_{6,2} = \frac{1362}{262144}$ | $P_{6,3} = \frac{681}{4194304}$ | $P_{6,4} = \frac{340}{67108864}$ | $P_{6,5} = \frac{170}{1073741824}$ | $P_{6,6} = \frac{85}{17179869184}$ | $P_{6,7} = \frac{42}{27487790720}$ | $P_{6,8} = \frac{213}{43980465152}$ | $P_{6,9} = \frac{106}{70368744256}$ | $P_{6,10} = \frac{53}{112590070784}$ | $P_{6,11} = \frac{27}{180144113216}$ | $P_{6,12} = \frac{13}{288230581248}$ | $P_{6,13} = \frac{7}{461168930048}$ | $P_{6,14} = \frac{3}{737870288096}$ | P | | | |

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$