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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A diagram of a hexagonal lattice with 37 cells. The central cell is labeled 0 and has a red dot. Cells 1 through 14 are labeled with black numbers and have red dots. Cells 15 through 36 are labeled with black numbers and have green dots. The lattice is surrounded by 21 green dots labeled 19 through 36, 38 through 40, and 42 through 44.

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\ p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\ p_{2,0}=0 & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\ p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\ p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=\frac{1}{6} & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\ p_{5,0}=0 & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\ p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & 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p_{15,19}=\frac{1}{6} \\ p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=$$

$$Q = \begin{pmatrix} P_{0,0} = 0 & P_{0,1} = \frac{1}{6} & P_{0,2} = \frac{1}{6} & P_{0,3} = \frac{1}{6} & P_{0,4} = \frac{1}{6} & P_{0,5} = \frac{1}{6} & P_{0,6} = \frac{1}{6} & P_{0,7} = 0 & P_{0,8} = 0 & P_{0,9} = 0 & P_{0,10} = 0 & P_{0,11} = 0 & P_{0,12} = 0 & P_{0,13} = 0 & P_{0,14} = 0 & P_{0,15} = 0 & P_{0,16} = 0 & P_{0,17} = 0 & P_{0,18} = 0 \\ P_{1,0} = \frac{1}{6} & P_{1,1} = 0 & P_{1,2} = 0 & P_{1,3} = 0 & P_{1,4} = 0 & P_{1,5} = 0 & P_{1,6} = \frac{1}{6} & P_{1,7} = \frac{1}{6} & P_{1,8} = \frac{1}{6} & P_{1,9} = \frac{1}{6} & P_{1,10} = 0 & P_{1,11} = 0 & P_{1,12} = 0 & P_{1,13} = 0 & P_{1,14} = 0 & P_{1,15} = 0 & P_{1,16} = 0 & P_{1,17} = 0 & P_{1,18} = 0 \\ P_{2,0} = \frac{1}{6} & P_{2,1} = \frac{1}{6} & P_{2,2} = 0 & P_{2,3} = \frac{1}{6} & P_{2,4} = 0 & P_{2,5} = 0 & P_{2,6} = 0 & P_{2,7} = 0 & P_{2,8} = 0 & P_{2,9} = \frac{1}{6} & P_{2,10} = \frac{1}{6} & P_{2,11} = \frac{1}{6} & P_{2,12} = 0 & P_{2,13} = 0 & P_{2,14} = 0 & P_{2,15} = 0 & P_{2,16} = 0 & P_{2,17} = 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& P_{12,4} = 0 & P_{12,5} = 0 & P_{12,6} = 0 & P_{12,7} = 0 & P_{12,8} = 0 & P_{12,9} = 0 & P_{12,10} = 0 & P_{12,11} = \frac{1}{6} & P_{12,12} = 0 & P_{12,13} = \frac{1}{6} & P_{12,14} = 0 & P_{12,15} = 0 & P_{12,16} = 0 & P_{12,17} = 0 & P_{12,18} = 0 \\ P_{13,0} = 0 & P_{13,1} = 0 & P_{13,2} = 0 & P_{13,3} = \frac{1}{6} & P_{13,4} = \frac{1}{6} & P_{13,5} = 0 & P_{13,6} = 0 & P_{13,7} = 0 & P_{13,8} = 0 & P_{13,9} = 0 & P_{13,10} = 0 & P_{13,11} = 0 & P_{13,12} = \frac{1}{6} & P_{13,13} = 0 & P_{13,14} = \frac{1}{6} & P_{13,15} = 0 & P_{13,16} = 0 & P_{13,17} = 0 & P_{13,18} = 0 \\ P_{14,0} = 0 & P_{14,1} = 0 & P_{14,2} = 0 & P_{14,3} = 0 & P_{14,4} = 0 & P_{14,5} = 0 & P_{14,6} = 0 & P_{14,7} = 0 & P_{14,8} = 0 & P_{14,9} = 0 & P_{14,10} = 0 & P_{14,11} = 0 & P_{14,12} = 0 & P_{14,13} = \frac{1}{6} & P_{14,14} = 0 & P_{14,15} = \frac{1}{6} & P_{14,16} = 0 & P_{14,17} = 0 & P_{14,18} = 0 \\ P_{15,0} = 0 & P_{15,1} = 0 & P_{15,2} = 0 & P_{15,3} = 0 & P_{15,4} = 0 & P_{15,5} = \frac{1}{6} & P_{15,6} = 0 & P_{15,7} = 0 & P_{15,8} = 0 & P_{15,9} = 0 & P_{15,10$$

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$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = 45 \\ P_{0,1} = 34506 \\ P_{0,2} = 107714 \\ P_{0,3} = 241658 \\ P_{0,4} = 459993 \\ P_{0,5} = 767993 \\ P_{0,6} = 1109993 \\ P_{0,7} = 1489993 \\ P_{0,8} = 1879993 \\ P_{0,9} = 2269993 \\ P_{0,10} = 2659993 \\ P_{0,11} = 3049993 \\ P_{0,12} = 3439993 \\ P_{0,13} = 3829993 \\ P_{0,14} = 4219993 \\ P_{0,15} = 4609993 \\ P_{0,16} = 4999993 \\ P_{0,17} = 5389993 \\ P_{0,18} = 5779993 \\ P_{0,19} = 6169993 \\ P_{0,20} = 6559993 \\ P_{0,21} = 6949993 \\ P_{0,22} = 7339993 \\ P_{0,23} = 7729993 \\ P_{0,24} = 8119993 \\ P_{0,25} = 8509993 \\ P_{0,26} = 8899993 \\ P_{0,27} = 9289993 \\ P_{0,28} = 9679993 \\ P_{0,29} = 10069993 \\ P_{0,30} = 10459993 \\ P_{0,31} = 10849993 \\ P_{0,32} = 11239993 \\ P_{0,33} = 11629993 \\ P_{0,34} = 12019993 \\ P_{0,35} = 12409993 \\ P_{0,36} = 12799993 \\ P_{0,37} = 13189993 \\ P_{0,38} = 13579993 \\ P_{0,39} = 13969993 \\ P_{0,40} = 14359993 \\ P_{0,41} = 14749993 \\ P_{0,42} = 15139993 \\ P_{0,43} = 15529993 \\ P_{0,44} = 15919993 \\ P_{0,45} = 16309993 \\ P_{0,46} = 16699993 \\ P_{0,47} = 17089993 \\ P_{0,48} = 17479993 \\ P_{0,49} = 17869993 \\ P_{0,50} = 18259993 \\ P_{0,51} = 18649993 \\ P_{0,52} = 19039993 \\ P_{0,53} = 19429993 \\ P_{0,54} = 19819993 \\ P_{0,55} = 20209993 \\ P_{0,56} = 20599993 \\ P_{0,57} = 20989993 \\ P_{0,58} = 21379993 \\ P_{0,59} = 21769993 \\ P_{0,60} = 22159993 \\ P_{0,61} = 22549993 \\ P_{0,62} = 22939993 \\ P_{0,63} = 23329993 \\ P_{0,64} = 23719993 \\ P_{0,65} = 24109993 \\ P_{0,66} = 24499993 \\ P_{0,67} = 24889993 \\ P_{0,68} = 25279993 \\ P_{0,69} = 25669993 \\ P_{0,70} = 26059993 \\ P_{0,71} = 26449993 \\ P_{0,72} = 26839993 \\ P_{0,73} = 27229993 \\ P_{0,74} = 27619993 \\ P_{0,75} = 28009993 \\ P_{0,76} = 28399993 \\ P_{0,77} = 28789993 \\ P_{0,78} = 29179993 \\ P_{0,79} = 29569993 \\ P_{0,80} = 29959993 \\ P_{0,81} = 30349993 \\ P_{0,82} = 30739993 \\ P_{0,83} = 31129993 \\ P_{0,84} = 31519993 \\ P_{0,85} = 31909993 \\ P_{0,86} = 32299993 \\ P_{0,87} = 32689993 \\ P_{0,88} = 33079993 \\ P_{0,89} = 33469993 \\ P_{0,90} = 33859993 \\ P_{0,91} = 34249993 \\ P_{0,92} = 34639993 \\ P_{0,93} = 35029993 \\ P_{0,94} = 35419993 \\ P_{0,95} = 35809993 \\ P_{0,96} = 36199993 \\ P_{0,97} = 36589993 \\ P_{0,98} = 36979993 \\ P_{0,99} = 37369993 \\ P_{0,100} = 37759993 \\ P_{0,101} = 38149993 \\ P_{0,102} = 38539993 \\ P_{0,103} = 38929993 \\ P_{0,104} = 39319993 \\ P_{0,105} = 39709993 \\ P_{0,106} = 40099993 \\ P_{0,107} = 40489993 \\ P_{0,108} = 40879993 \\ P_{0,109} = 41269993 \\ P_{0,110} = 41659993 \\ P_{0,111} = 42049993 \\ P_{0,112} = 42439993 \\ P_{0,113} = 42829993 \\ P_{0,114} = 43219993 \\ P_{0,115} = 43609993 \\ P_{0,116} = 43999993 \\ P_{0,117} = 44389993 \\ P_{0,118} = 44779993 \\ P_{0,119} = 45169993 \\ P_{0,120} = 45559993 \\ P_{0,121} = 45949993 \\ P_{0,122} = 46339993 \\ P_{0,123} = 46729993 \\ P_{0,124} = 47119993 \\ P_{0,125} = 47509993 \\ P_{0,126} = 47899993 \\ P_{0,127} = 48289993 \\ P_{0,128} = 48679993 \\ P_{0,129} = 49069993 \\ P_{0,130} = 49459993 \\ P_{0,131} = 49849993 \\ P_{0,132} = 50239993 \\ P_{0,133} = 50629993 \\ P_{0,134} = 51019993 \\ P_{0,135} = 51409993 \\ P_{0,136} = 51799993 \\ P_{0,137} = 52189993 \\ P_{0,138} = 52579993 \\ P_{0,139} = 52969993 \\ P_{0,140} = 53359993 \\ P_{0,141} = 53749993 \\ P_{0,142} = 54139993 \\ P_{0,143} = 54529993 \\ P_{0,144} = 54919993 \\ P_{0,145} = 55309993 \\ P_{0,146} = 55699993 \\ P_{0,147} = 56089993 \\ P_{0,148} = 56479993 \\ P_{0,149} = 56869993 \\ P_{0,150} = 57259993 \\ P_{0,151} = 57649993 \\ P_{0,152} = 58039993 \\ P_{0,153} = 58429993 \\ P_{0,154} = 58819993 \\ P_{0,155} = 59209993 \\ P_{0,156} = 59599993 \\ P_{0,157} = 59989993 \\ P_{0,158} = 60379993 \\ P_{0,159} = 60769993 \\ P_{0,160} = 61159993 \\ P_{0,161} = 61549993 \\ P_{0,162} = 61939993 \\ P_{0,163} = 62329993 \\ P_{0,164} = 62719993 \\ P_{0,165} = 63109993 \\ P_{0,166} = 63499993 \\ P_{0,167} = 63889993 \\ P_{0,168} = 64279993 \\ P_{0,169} = 64669993 \\ P_{0,170} = 65059993 \\ P_{0,171} = 65449993 \\ P_{0,172} = 65839993 \\ P_{0,173} = 66229993 \\ P_{0,174} = 66619993 \\ P_{0,175} = 67009993 \\ P_{0,176} = 67399993 \\ P_{0,177} = 67789993 \\ P_{0,178} = 68179993 \\ P_{0,179} = 68569993 \\ P_{0,180} = 68959993 \\ P_{0,181} = 69349993 \\ P_{0,182} = 69739993 \\ P_{0,183} = 70129993 \\ P_{0,184} = 70519993 \\ P_{0,185} = 70909993 \\ P_{0,186} = 71299993 \\ P_{0,187} = 71689993 \\ P_{0,188} = 72079993 \\ P_{0,189} = 72469993 \\ P_{0,190} = 72859993 \\ P_{0,191} = 73249993 \\ P_{0,192} = 73639993 \\ P_{0,193} = 74029993 \\ P_{0,194} = 74419993 \\ P_{0,195} = 74809993 \\ P_{0,196} = 75199993 \\ P_{0,197} = 75589993 \\ P_{0,198} = 75979993 \\ P_{0,199} = 76369993 \\ P_{0,200} = 76759993 \\ P_{0,201} = 77149993 \\ P_{0,202} = 77539993 \\ P_{0,203} = 77929993 \\ P_{0,204} = 78319993 \\ P_{0,205} = 78709993 \\ P_{0,206} = 79099993 \\ P_{0,207} = 79489993 \\ P_{0,208} = 79879993 \\ P_{0,209} = 80269993 \\ P_{0,210} = 80659993 \\ P_{0,211} = 81049993 \\ P_{0,212} = 81439993 \\ P_{0,213} = 81829993 \\ P_{0,214} = 82219993 \\ P_{0,215} = 82609993 \\ P_{0,216} = 82999993 \\ P_{0,217} = 83389993 \\ P_{0,218} = 83779993 \\ P_{0,219} = 84169993 \\ P_{0,220} = 8455$$

$$t \equiv N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$