$\mathbb{R}^n$ Bonus Problem #3

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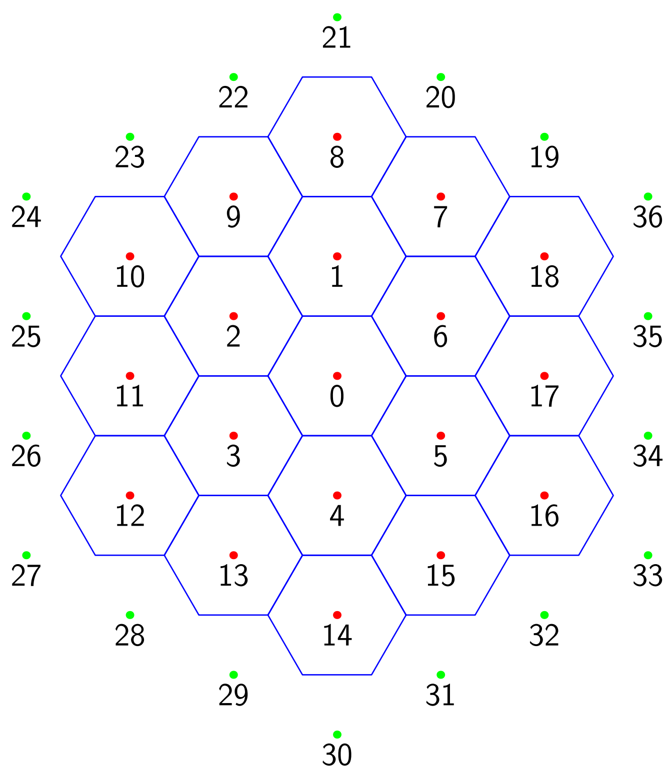
§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram



§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N.

$$\mathbb{E}(N) = \sum N \, \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N. We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

§3.1 Notation

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

§3.2 Transition Matrix

Now that we've defined some notation, we can write the transition matrix P Because a 37x37 matrix is cumbersome, we combine the states [19, 36] into a

$$P = \begin{pmatrix}
p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\
p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\
p_{2,0}=\frac{1}{6} & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\
p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\
p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=0 & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\
p_{5,0}=\frac{1}{6} & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\
p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} & p_{6,19}=0 \\
p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=0 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} & p_{7,19}=\frac{1}{3} \\
p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=0 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 & p_{8,19}=\frac{1}{2} \\
p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=0 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & p_{9,14}=0 & p_{9,15}=0 & p_{9,16}=0 & p_{9,17}=0 & p_{9,18}=0 & p_{9,19}=\frac{1}{3} \\
p_{10,0}=0 & p_{10,1}=0 & p_{10,2}=\frac{1}{6} & p_{10,3}=0 & p_{10,4}=0 & p_{10,5}=0 & p_{10,6}=0 & p_{10,7}=0 & p_{10,8}=0 & p_{10,9}=\frac{1}{6} & p_{10,10}=0 & p_{10,11}=\frac{1}{6} & p_{10,12}=0 & p_{10,13}=0 & p_{10,14}=0 & p_{10,15}=0 & p_{10,16}=0 & p_{10,17}=0 & p_{10,18}=0 & p_{10,19}=\frac{1}{2} \\
p_{11,0}=0 & p_{11,1}=0 & p_{11,2}=\frac{1}{6} & p_{11,3}=\frac{1}{6} & p_{11,4}=0 & p_{11,5}=0 & p_{11,6}=0 & p_{11,7}=0 & p_{11,8}=0 & p_{11,9}=0 & p_{11,10}=\frac{1}{6} & p_{11,11}=0 & p_{11,12}=\frac{1}{6} & p_{11,13}=0 & p_{11,14}=0 & p_{11,15}=0 & p_{11,16}=0 & p_{11,17}=0 & p_{11,18}=0 & p_{11,19}=\frac{1}{3} \\
p_{12,0}=0 & p_{12,1}=0 & p_{12,2}=0 & p_{12,3}=\frac{1}{6} & p_{12,4}=0 & p_{12,5}=0 & p_{12,6}=0 & p_{12,7}=0 & p_{12,8}=0 & p_{12,9}=0 & p_{12,10}=0 & p_{12,11}=\frac{1}{6} & p_{12,12}=0 & p_{12,13}=\frac{1}{6} & p_{12,14}=0 & p_{12,15}=0 & p_{12,16}=0 & p_{12,17}=0 & p_{12,18}=0 & p_{12,19}=\frac{1}{2} \\
p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=0 & p_{13,12}=\frac{1}{6} & p_{13,13}=0 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 & p_{13,19}=\frac{1}{3} \\
p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=0 & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 & p_{14,19}=\frac{1}{2} \\
p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=0 & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=0 & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 & p_{15,19}=\frac{1}{3} \\
p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=0 & p_{16,11}=0 & p_{16,12}=0 & p_{16,13}=0 & p_{16,14}=0 & p_{16,15}=\frac{1}{6} & p_{16,16}=0 & p_{16,17}=\frac{1}{6} & p_{16,18}=0 & p_{16,19}=\frac{1}{2} \\
p_{17,0}=0 & p_{17,1}=0 & p_{17,2}=0 & p_{17,3}=0 & p_{17,4}=0 & p_{17,5}=\frac{1}{6} & p_{17,6}=\frac{1}{6} & p_{17,7}=0 & p_{17,8}=0 & p_{17,9}=0 & p_{17,10}=0 & p_{17,11}=0 & p_{17,12}=0 & p_{17,13}=0 & p_{17,14}=0 & p_{17,15}=0 & p_{17,16}=\frac{1}{6} & p_{17,17}=0 & p_{17,18}=\frac{1}{6} & p_{17,19}=\frac{1}{3} \\
p_{18,0}=0 & p_{18,1}=0 & p_{18,2}=0 & p_{18,3}=0 & p_{18,4}=0 & p_{18,5}=0 & p_{18,6}=\frac{1}{6} & p_{18,7}=\frac{1}{6} & p_{18,8}=0 & p_{18,9}=0 & p_{18,10}=0 & p_{18,11}=0 & p_{18,12}=0 & p_{18,13}=0 & p_{18,14}=0 & p_{18,15}=0 & p_{18,16}=0 & p_{18,17}=\frac{1}{6} & p_{18,18}=0 & p_{18,19}=\frac{1}{2} \\
p_{19,0}=0 & p_{19,1}=0 & p_{19,2}=0 & p_{19,3}=0 & p_{19,4}=0 & p_{19,5}=0 & p_{19,6}=0 & p_{19,7}=0 & p_{19,8}=0 & p_{19,9}=0 & p_{19,10}=0 & p_{19,11}=0 & p_{19,12}=0 & p_{19,13}=0 & p_{19,14}=0 & p_{19,15}=0 & p_{19,16}=0 & p_{19,17}=0 & p_{19,18}=0 & p_{19,19}=1
\end{pmatrix}$$

We also write the matrix Q, which doesn't have any absorbing states.

$$Q = \begin{pmatrix}
p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 \\
p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 \\
p_{2,0}=\frac{1}{6} & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 \\
p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 \\
p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=0 & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 \\
p_{5,0}=\frac{1}{6} & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 \\
p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} \\
p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=0 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} \\
p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=0 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 \\
p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=0 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & p_{9,14}=0 & p_{9,15}=0 & p_{9,16}=0 & p_{9,17}=0 & p_{9,18}=0 \\
p_{10,0}=0 & p_{10,1}=0 & p_{10,2}=\frac{1}{6} & p_{10,3}=0 & p_{10,4}=0 & p_{10,5}=0 & p_{10,6}=0 & p_{10,7}=0 & p_{10,8}=0 & p_{10,9}=\frac{1}{6} & p_{10,10}=0 & p_{10,11}=\frac{1}{6} & p_{10,12}=0 & p_{10,13}=0 & p_{10,14}=0 & p_{10,15}=0 & p_{10,16}=0 & p_{10,17}=0 & p_{10,18}=0 \\
p_{11,0}=0 & p_{11,1}=0 & p_{11,2}=\frac{1}{6} & p_{11,3}=\frac{1}{6} & p_{11,4}=0 & p_{11,5}=0 & p_{11,6}=0 & p_{11,7}=0 & p_{11,8}=0 & p_{11,9}=0 & p_{11,10}=\frac{1}{6} & p_{11,11}=0 & p_{11,12}=\frac{1}{6} & p_{11,13}=0 & p_{11,14}=0 & p_{11,15}=0 & p_{11,16}=0 & p_{11,17}=0 & p_{11,18}=0 \\
p_{12,0}=0 & p_{12,1}=0 & p_{12,2}=0 & p_{12,3}=\frac{1}{6} & p_{12,4}=0 & p_{12,5}=0 & p_{12,6}=0 & p_{12,7}=0 & p_{12,8}=0 & p_{12,9}=0 & p_{12,10}=0 & p_{12,11}=\frac{1}{6} & p_{12,12}=0 & p_{12,13}=\frac{1}{6} & p_{12,14}=0 & p_{12,15}=0 & p_{12,16}=0 & p_{12,17}=0 & p_{12,18}=0 \\
p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=0 & p_{13,12}=\frac{1}{6} & p_{13,13}=0 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 \\
p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=0 & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 \\
p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=0 & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=0 & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 \\
p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=0 & p_{16,11}=0 & p_{16,12}=0 & p_{16,13}=0 & p_{16,14}=0 & p_{16,15}=\frac{1}{6} & p_{16,16}=0 & p_{16,17}=\frac{1}{6} & p_{16,18}=0 \\
p_{17,0}=0 & p_{17,1}=0 & p_{17,2}=0 & p_{17,3}=0 & p_{17,4}=0 & p_{17,5}=\frac{1}{6} & p_{17,6}=\frac{1}{6} & p_{17,7}=0 & p_{17,8}=0 & p_{17,9}=0 & p_{17,10}=0 & p_{17,11}=0 & p_{17,12}=0 & p_{17,13}=0 & p_{17,14}=0 & p_{17,15}=0 & p_{17,16}=\frac{1}{6} & p_{17,17}=0 & p_{17,18}=\frac{1}{6} \\
p_{18,0}=0 & p_{18,1}=0 & p_{18,2}=0 & p_{18,3}=0 & p_{18,4}=0 & p_{18,5}=0 & p_{18,6}=\frac{1}{6} & p_{18,7}=\frac{1}{6} & p_{18,8}=0 & p_{18,9}=0 & p_{18,10}=0 & p_{18,11}=0 & p_{18,12}=0 & p_{18,13}=0 & p_{18,14}=0 & p_{18,15}=0 & p_{18,16}=0 & p_{18,17}=\frac{1}{6} & p_{18,18}=0
\end{pmatrix}$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P.

$$
N^{-1} = \begin{pmatrix}
p_{0,0}=1 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\
p_{1,0}=\frac{1}{6} & p_{1,1}=1 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\
p_{2,0}=\frac{1}{6} & p_{2,1}=\frac{1}{6} & p_{2,2}=1 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\
p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=1 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\
p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=1 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=0 & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\
p_{5,0}=\frac{1}{6} & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=1 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\
p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=1 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} & p_{6,19}=0 \\
p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=1 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} & p_{7,19}=\frac{1}{3} \\
p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=1 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 & p_{8,19}=\frac{1}{2} \\
p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=1 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & p_{9,14}=0 & p_{9,15}=0 & p_{9,16}=0 & p_{9,17}=0 & p_{9,18}=0 & p_{9,19}=\frac{1}{3} \\
p_{10,0}=0 & p_{10,1}=0 & p_{10,2}=\frac{1}{6} & p_{10,3}=0 & p_{10,4}=0 & p_{10,5}=0 & p_{10,6}=0 & p_{10,7}=0 & p_{10,8}=0 & p_{10,9}=\frac{1}{6} & p_{10,10}=1 & p_{10,11}=\frac{1}{6} & p_{10,12}=0 & p_{10,13}=0 & p_{10,14}=0 & p_{10,15}=0 & p_{10,16}=0 & p_{10,17}=0 & p_{10,18}=0 & p_{10,19}=\frac{1}{2} \\
p_{11,0}=0 & p_{11,1}=0 & p_{11,2}=\frac{1}{6} & p_{11,3}=\frac{1}{6} & p_{11,4}=0 & p_{11,5}=0 & p_{11,6}=0 & p_{11,7}=0 & p_{11,8}=0 & p_{11,9}=0 & p_{11,10}=\frac{1}{6} & p_{11,11}=1 & p_{11,12}=\frac{1}{6} & p_{11,13}=0 & p_{11,14}=0 & p_{11,15}=0 & p_{11,16}=0 & p_{11,17}=0 & p_{11,18}=0 & p_{11,19}=\frac{1}{3} \\
p_{12,0}=0 & p_{12,1}=0 & p_{12,2}=0 & p_{12,3}=\frac{1}{6} & p_{12,4}=0 & p_{12,5}=0 & p_{12,6}=0 & p_{12,7}=0 & p_{12,8}=0 & p_{12,9}=0 & p_{12,10}=0 & p_{12,11}=\frac{1}{6} & p_{12,12}=1 & p_{12,13}=\frac{1}{6} & p_{12,14}=0 & p_{12,15}=0 & p_{12,16}=0 & p_{12,17}=0 & p_{12,18}=0 & p_{12,19}=\frac{1}{2} \\
p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=0 & p_{13,12}=\frac{1}{6} & p_{13,13}=1 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 & p_{13,19}=\frac{1}{3} \\
p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=1 & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 & p_{14,19}=\frac{1}{2} \\
p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=0 & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=1 & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 & p_{15,19}=\frac{1}{3} \\
p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=0 & p_{16,11}=0 & p_{16,12}=0 & p_{16,13}=0 & p_{16,14}=0 & p_{16,15}=\frac{1}{6} & p_{16,16}=1 & p_{16,17}=\frac{1}{6} & p_{16,18}=0 & p_{16,19}=\frac{1}{2} \\
p_{17,0}=0 & p_{17,1}=0 & p_{17,2}=0 & p_{17,3}=0 & p_{17,4}=0 & p_{17,5}=\frac{1}{6} & p_{17,6}=\frac{1}{6} & p_{17,7}=0 & p_{17,8}=0 & p_{17,9}=0 & p_{17,10}=0 & p_{17,11}=0 & p_{17,12}=0 & p_{17,13}=0 & p_{17,14}=0 & p_{17,15}=0 & p_{17,16}=\frac{1}{6} & p_{17,17}=1 & p_{17,18}=\frac{1}{6} & p_{17,19}=\frac{1}{3} \\
p_{18,0}=0 & p_{18,1}=0 & p_{18,2}=0 & p_{18,3}=0 & p_{18,4}=0 & p_{18,5}=0 & p_{18,6}=\frac{1}{6} & p_{18,7}=\frac{1}{6} & p_{18,8}=0 & p_{18,9}=0 & p_{18,10}=0 & p_{18,11}=0 & p_{18,12}=0 & p_{18,13}=0 & p_{18,14}=0 & p_{18,15}=0 & p_{18,16}=0 & p_{18,17}=\frac{1}{6} & p_{18,18}=1 & p_{18,19}=\frac{1}{2} \\
p_{19,0}=0 & p_{19,1}=0 & p_{19,2}=0 & p_{19,3}=0 & p_{19,4}=0 & p_{19,5}=0 & p_{19,6}=0 & p_{19,7}=0 & p_{19,8}=0 & p_{19,9}=0 & p_{19,10}=0 & p_{19,11}=0 & p_{19,12}=0 & p_{19,13}=0 & p_{19,14}=0 & p_{19,15}=0 & p_{19,16}=0 & p_{19,17}=0 & p_{19,18}=0 & p_{19,19}=0
\end{pmatrix}
$$

$$
N = \begin{pmatrix}
p_{0,0}=\frac{45}{29} & p_{0,1}=\frac{16}{29} & p_{0,2}=\frac{16}{29} & p_{0,3}=\frac{16}{29} & p_{0,4}=\frac{16}{29} & p_{0,5}=\frac{16}{29} & p_{0,6}=\frac{16}{29} & p_{0,7}=\frac{7}{29} & p_{0,8}=\frac{5}{29} & p_{0,9}=\frac{7}{29} & p_{0,10}=\frac{5}{29} & p_{0,11}=\frac{7}{29} & p_{0,12}=\frac{5}{29} & p_{0,13}=\frac{7}{29} & p_{0,14}=\frac{5}{29} & p_{0,15}=\frac{7}{29} & p_{0,16}=\frac{5}{29} & p_{0,17}=\frac{7}{29} & p_{0,18}=\frac{5}{29} \\
p_{1,0}=\frac{16}{29} & p_{1,1}=\frac{992666}{674685} & p_{1,2}=\frac{348808}{674685} & p_{1,3}=\frac{191222}{674685} & p_{1,4}=\frac{160714}{674685} & p_{1,5}=\frac{191222}{674685} & p_{1,6}=\frac{348808}{674685} & p_{1,7}=\frac{6133}{14355} & p_{1,8}=\frac{87176}{224895} & p_{1,9}=\frac{6133}{14355} & p_{1,10}=\frac{14056}{74965} & p_{1,11}=\frac{173}{957} & p_{1,12}=\frac{21752}{224895} & p_{1,13}=\frac{1667}{14355} & p_{1,14}=\frac{5878}{74965} & p_{1,15}=\frac{1667}{14355} & p_{1,16}=\frac{21752}{224895} & p_{1,17}=\frac{173}{957} & p_{1,18}=\frac{14056}{74965} \\
p_{2,0}=\frac{16}{29} & p_{2,1}=\frac{348808}{674685} & p_{2,2}=\frac{992666}{674685} & p_{2,3}=\frac{348808}{674685} & p_{2,4}=\frac{191222}{674685} & p_{2,5}=\frac{160714}{674685} & p_{2,6}=\frac{191222}{674685} & p_{2,7}=\frac{173}{957} & p_{2,8}=\frac{14056}{74965} & p_{2,9}=\frac{6133}{14355} & p_{2,10}=\frac{87176}{224895} & p_{2,11}=\frac{6133}{14355} & p_{2,12}=\frac{14056}{74965} & p_{2,13}=\frac{173}{957} & p_{2,14}=\frac{21752}{224895} & p_{2,15}=\frac{1667}{14355} & p_{2,16}=\frac{5878}{74965} & p_{2,17}=\frac{1667}{14355} & p_{2,18}=\frac{21752}{224895} \\
p_{3,0}=\frac{16}{29} & p_{3,1}=\frac{191222}{674685} & p_{3,2}=\frac{348808}{674685} & p_{3,3}=\frac{992666}{674685} & p_{3,4}=\frac{348808}{674685} & p_{3,5}=\frac{191222}{674685} & p_{3,6}=\frac{160714}{674685} & p_{3,7}=\frac{1667}{14355} & p_{3,8}=\frac{21752}{224895} & p_{3,9}=\frac{173}{957} & p_{3,10}=\frac{14056}{74965} & p_{3,11}=\frac{6133}{14355} & p_{3,12}=\frac{87176}{224895} & p_{3,13}=\frac{6133}{14355} & p_{3,14}=\frac{14056}{74965} & p_{3,15}=\frac{173}{957} & p_{3,16}=\frac{21752}{224895} & p_{3,17}=\frac{1667}{14355} & p_{3,18}=\frac{5878}{74965} \\
p_{4,0}=\frac{16}{29} & p_{4,1}=\frac{160714}{674685} & p_{4,2}=\frac{191222}{674685} & p_{4,3}=\frac{348808}{674685} & p_{4,4}=\frac{992666}{674685} & p_{4,5}=\frac{348808}{674685} & p_{4,6}=\frac{191222}{674685} & p_{4,7}=\frac{1667}{14355} & p_{4,8}=\frac{5878}{74965} & p_{4,9}=\frac{1667}{14355} & p_{4,10}=\frac{21752}{224895} & p_{4,11}=\frac{173}{957} & p_{4,12}=\frac{14056}{74965} & p_{4,13}=\frac{6133}{14355} & p_{4,14}=\frac{87176}{224895} & p_{4,15}=\frac{6133}{14355} & p_{4,16}=\frac{14056}{74965} & p_{4,17}=\frac{173}{957} & p_{4,18}=\frac{21752}{224895} \\
p_{5,0}=\frac{16}{29} & p_{5,1}=\frac{191222}{674685} & p_{5,2}=\frac{160714}{674685} & p_{5,3}=\frac{191222}{674685} & p_{5,4}=\frac{348808}{674685} & p_{5,5}=\frac{992666}{674685} & p_{5,6}=\frac{348808}{674685} & p_{5,7}=\frac{173}{957} & p_{5,8}=\frac{21752}{224895} & p_{5,9}=\frac{1667}{14355} & p_{5,10}=\frac{5878}{74965} & p_{5,11}=\frac{1667}{14355} & p_{5,12}=\frac{21752}{224895} & p_{5,13}=\frac{173}{957} & p_{5,14}=\frac{14056}{74965} & p_{5,15}=\frac{6133}{14355} & p_{5,16}=\frac{87176}{224895} & p_{5,17}=\frac{6133}{14355} & p_{5,18}=\frac{14056}{74965} \\
p_{6,0}=\frac{16}{29} & p_{6,1}=\frac{348808}{674685} & p_{6,2}=\frac{191222}{674685} & p_{6,3}=\frac{160714}{674685} & p_{6,4}=\frac{191222}{674685} & p_{6,5}=\frac{348808}{674685} & p_{6,6}=\frac{992666}{674685} & p_{6,7}=\frac{6133}{14355} & p_{6,8}=\frac{14056}{74965} & p_{6,9}=\frac{173}{957} & p_{6,10}=\frac{21752}{224895} & p_{6,11}=\frac{1667}{14355} & p_{6,12}=\frac{5878}{74965} & p_{6,13}=\frac{1667}{14355} & p_{6,14}=\frac{21752}{224895} & p_{6,15}=\frac{173}{957} & p_{6,16}=\frac{14056}{74965} & p_{6,17}=\frac{6133}{14355} & p_{6,18}=\frac{87176}{224895} \\
p_{7,0}=\frac{7}{29} & p_{7,1}=\frac{6133}{14355} & p_{7,2}=\frac{173}{957} & p_{7,3}=\frac{1667}{14355} & p_{7,4}=\frac{1667}{14355} & p_{7,5}=\frac{173}{957} & p_{7,6}=\frac{6133}{14355} & p_{7,7}=\frac{375124}{301455} & p_{7,8}=\frac{10246}{33495} & p_{7,9}=\frac{49367}{301455} & p_{7,10}=\frac{460}{6699} & p_{7,11}=\frac{20338}{301455} & p_{7,12}=\frac{438}{11165} & p_{7,13}=\frac{15611}{301455} & p_{7,14}=\frac{438}{11165} & p_{7,15}=\frac{20338}{301455} & p_{7,16}=\frac{460}{6699} & p_{7,17}=\frac{49367}{301455} & p_{7,18}=\frac{10246}{33495} \\
p_{8,0}=\frac{5}{29} & p_{8,1}=\frac{87176}{224895} & p_{8,2}=\frac{14056}{74965} & p_{8,3}=\frac{21752}{224895} & p_{8,4}=\frac{5878}{74965} & p_{8,5}=\frac{21752}{224895} & p_{8,6}=\frac{14056}{74965} & p_{8,7}=\frac{10246}{33495} & p_{8,8}=\frac{1836491}{1574265} & p_{8,9}=\frac{10246}{33495} & p_{8,10}=\frac{147473}{1574265} & p_{8,11}=\frac{460}{6699} & p_{8,12}=\frac{53687}{1574265} & p_{8,13}=\frac{438}{11165} & p_{8,14}=\frac{41159}{1574265} & p_{8,15}=\frac{438}{11165} & p_{8,16}=\frac{53687}{1574265} & p_{8,17}=\frac{460}{6699} & p_{8,18}=\frac{147473}{1574265} \\
p_{9,0}=\frac{7}{29} & p_{9,1}=\frac{6133}{14355} & p_{9,2}=\frac{6133}{14355} & p_{9,3}=\frac{173}{957} & p_{9,4}=\frac{1667}{14355} & p_{9,5}=\frac{1667}{14355} & p_{9,6}=\frac{173}{957} & p_{9,7}=\frac{49367}{301455} & p_{9,8}=\frac{10246}{33495} & p_{9,9}=\frac{375124}{301455} & p_{9,10}=\frac{10246}{33495} & p_{9,11}=\frac{49367}{301455} & p_{9,12}=\frac{460}{6699} & p_{9,13}=\frac{20338}{301455} & p_{9,14}=\frac{438}{11165} & p_{9,15}=\frac{15611}{301455} & p_{9,16}=\frac{438}{11165} & p_{9,17}=\frac{20338}{301455} & p_{9,18}=\frac{460}{6699} \\
p_{10,0}=\frac{5}{29} & p_{10,1}=\frac{14056}{74965} & p_{10,2}=\frac{87176}{224895} & p_{10,3}=\frac{14056}{74965} & p_{10,4}=\frac{21752}{224895} & p_{10,5}=\frac{5878}{74965} & p_{10,6}=\frac{21752}{224895} & p_{10,7}=\frac{460}{6699} & p_{10,8}=\frac{147473}{1574265} & p_{10,9}=\frac{10246}{33495} & p_{10,10}=\frac{1836491}{1574265} & p_{10,11}=\frac{10246}{33495} & p_{10,12}=\frac{147473}{1574265} & p_{10,13}=\frac{460}{6699} & p_{10,14}=\frac{53687}{1574265} & p_{10,15}=\frac{438}{11165} & p_{10,16}=\frac{41159}{1574265} & p_{10,17}=\frac{438}{11165} & p_{10,18}=\frac{53687}{1574265} \\
p_{11,0}=\frac{7}{29} & p_{11,1}=\frac{173}{957} & p_{11,2}=\frac{6133}{14355} & p_{11,3}=\frac{6133}{14355} & p_{11,4}=\frac{173}{957} & p_{11,5}=\frac{1667}{14355} & p_{11,6}=\frac{1667}{14355} & p_{11,7}=\frac{20338}{301455} & p_{11,8}=\frac{460}{6699} & p_{11,9}=\frac{49367}{301455} & p_{11,10}=\frac{10246}{33495} & p_{11,11}=\frac{375124}{301455} & p_{11,12}=\frac{10246}{33495} & p_{11,13}=\frac{49367}{301455} & p_{11,14}=\frac{460}{6699} & p_{11,15}=\frac{20338}{301455} & p_{11,16}=\frac{438}{11165} & p_{11,17}=\frac{15611}{301455} & p_{11,18}=\frac{438}{11165} \\
p_{12,0}=\frac{5}{29} & p_{12,1}=\frac{21752}{224895} & p_{12,2}=\frac{14056}{74965} & p_{12,3}=\frac{87176}{224895} & p_{12,4}=\frac{14056}{74965} & p_{12,5}=\frac{21752}{224895} & p_{12,6}=\frac{5878}{74965} & p_{12,7}=\frac{438}{11165} & p_{12,8}=\frac{53687}{1574265} & p_{12,9}=\frac{460}{6699} & p_{12,10}=\frac{147473}{1574265} & p_{12,11}=\frac{10246}{33495} & p_{12,12}=\frac{1836491}{1574265} & p_{12,13}=\frac{10246}{33495} & p_{12,14}=\frac{147473}{1574265} & p_{12,15}=\frac{460}{6699} & p_{12,16}=\frac{53687}{1574265} & p_{12,17}=\frac{438}{11165} & p_{12,18}=\frac{41159}{1574265} \\
p_{13,0}=\frac{7}{29} & p_{13,1}=\frac{1667}{14355} & p_{13,2}=\frac{173}{957} & p_{13,3}=\frac{6133}{14355} & p_{13,4}=\frac{6133}{14355} & p_{13,5}=\frac{173}{957} & p_{13,6}=\frac{1667}{14355} & p_{13,7}=\frac{15611}{301455} & p_{13,8}=\frac{438}{11165} & p_{13,9}=\frac{20338}{301455} & p_{13,10}=\frac{460}{6699} & p_{13,11}=\frac{49367}{301455} & p_{13,12}=\frac{10246}{33495} & p_{13,13}=\frac{375124}{301455} & p_{13,14}=\frac{10246}{33495} & p_{13,15}=\frac{49367}{301455} & p_{13,16}=\frac{460}{6699} & p_{13,17}=\frac{20338}{301455} & p_{13,18}=\frac{438}{11165} \\
p_{14,0}=\frac{5}{29} & p_{14,1}=\frac{5878}{74965} & p_{14,2}=\frac{21752}{224895} & p_{14,3}=\frac{14056}{74965} & p_{14,4}=\frac{87176}{224895} & p_{14,5}=\frac{14056}{74965} & p_{14,6}=\frac{21752}{224895} & p_{14,7}=\frac{438}{11165} & p_{14,8}=\frac{41159}{1574265} & p_{14,9}=\frac{438}{11165} & p_{14,10}=\frac{53687}{1574265} & p_{14,11}=\frac{460}{6699} & p_{14,12}=\frac{147473}{1574265} & p_{14,13}=\frac{10246}{33495} & p_{14,14}=\frac{1836491}{1574265} & p_{14,15}=\frac{10246}{33495} & p_{14,16}=\frac{147473}{1574265} & p_{14,17}=\frac{460}{6699} & p_{14,18}=\frac{53687}{1574265} \\
p_{15,0}=\frac{7}{29} & p_{15,1}=\frac{1667}{14355} & p_{15,2}=\frac{1667}{14355} & p_{15,3}=\frac{173}{957} & p_{15,4}=\frac{6133}{14355} & p_{15,5}=\frac{6133}{14355} & p_{15,6}=\frac{173}{957} & p_{15,7}=\frac{20338}{301455} & p_{15,8}=\frac{438}{11165} & p_{15,9}=\frac{15611}{301455} & p_{15,10}=\frac{438}{11165} & p_{15,11}=\frac{20338}{301455} & p_{15,12}=\frac{460}{6699} & p_{15,13}=\frac{49367}{301455} & p_{15,14}=\frac{10246}{33495} & p_{15,15}=\frac{375124}{301455} & p_{15,16}=\frac{10246}{33495} & p_{15,17}=\frac{49367}{301455} & p_{15,18}=\frac{460}{6699} \\
p_{16,0}=\frac{5}{29} & p_{16,1}=\frac{21752}{224895} & p_{16,2}=\frac{5878}{74965} & p_{16,3}=\frac{21752}{224895} & p_{16,4}=\frac{14056}{74965} & p_{16,5}=\frac{87176}{224895} & p_{16,6}=\frac{14056}{74965} & p_{16,7}=\frac{460}{6699} & p_{16,8}=\frac{53687}{1574265} & p_{16,9}=\frac{438}{11165} & p_{16,10}=\frac{41159}{1574265} & p_{16,11}=\frac{438}{11165} & p_{16,12}=\frac{53687}{1574265} & p_{16,13}=\frac{460}{6699} & p_{16,14}=\frac{147473}{1574265} & p_{16,15}=\frac{10246}{33495} & p_{16,16}=\frac{1836491}{1574265} & p_{16,17}=\frac{10246}{33495} & p_{16,18}=\frac{147473}{1574265} \\
p_{17,0}=\frac{7}{29} & p_{17,1}=\frac{173}{957} & p_{17,2}=\frac{1667}{14355} & p_{17,3}=\frac{1667}{14355} & p_{17,4}=\frac{173}{957} & p_{17,5}=\frac{6133}{14355} & p_{17,6}=\frac{6133}{14355} & p_{17,7}=\frac{49367}{301455} & p_{17,8}=\frac{460}{6699} & p_{17,9}=\frac{20338}{301455} & p_{17,10}=\frac{438}{11165} & p_{17,11}=\frac{15611}{301455} & p_{17,12}=\frac{438}{11165} & p_{17,13}=\frac{20338}{301455} & p_{17,14}=\frac{460}{6699} & p_{17,15}=\frac{49367}{301455} & p_{17,16}=\frac{10246}{33495} & p_{17,17}=\frac{375124}{301455} & p_{17,18}=\frac{10246}{33495} \\
p_{18,0}=\frac{5}{29} & p_{18,1}=\frac{14056}{74965} & p_{18,2}=\frac{21752}{224895} & p_{18,3}=\frac{5878}{74965} & p_{18,4}=\frac{21752}{224895} & p_{18,5}=\frac{14056}{74965} & p_{18,6}=\frac{87176}{224895} & p_{18,7}=\frac{10246}{33495} & p_{18,8}=\frac{147473}{1574265} & p_{18,9}=\frac{460}{6699} & p_{18,10}=\frac{53687}{1574265} & p_{18,11}=\frac{438}{11165} & p_{18,12}=\frac{41159}{1574265} & p_{18,13}=\frac{438}{11165} & p_{18,14}=\frac{53687}{1574265} & p_{18,15}=\frac{460}{6699} & p_{18,16}=\frac{147473}{1574265} & p_{18,17}=\frac{10246}{33495} & p_{18,18}=\frac{1836491}{1574265}
\end{pmatrix}
$$

In order to get the expected number of steps, we find t_0, where

$$
\boldsymbol{t} = N\mathbf{1}
$$

Here, $\mathbf{1}$ is a vector whose entries are all 1.

$$
\boldsymbol{t} = \begin{pmatrix}
\frac{213}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29}
\end{pmatrix}
$$

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$