

\mathbb{R}^n Bonus Problem #3

Rajeev Atla

§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram



§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = 0 & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 0 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 0 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 0 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 0 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 0 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & 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p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} & p_{14$$

$$Q = \begin{pmatrix} P_{0,0} = 0 & P_{0,1} = \frac{1}{6} & P_{0,2} = \frac{1}{6} & P_{0,3} = \frac{1}{6} & P_{0,4} = \frac{1}{6} & P_{0,5} = \frac{1}{6} & P_{0,6} = \frac{1}{6} & P_{0,7} = 0 & P_{0,8} = 0 & P_{0,9} = 0 & P_{0,10} = 0 & P_{0,11} = 0 & P_{0,12} = 0 & P_{0,13} = 0 & P_{0,14} = 0 & P_{0,15} = 0 & P_{0,16} = 0 & P_{0,17} = 0 & P_{0,18} = 0 \\ P_{1,0} = \frac{1}{6} & P_{1,1} = 0 & P_{1,2} = 0 & P_{1,3} = 0 & P_{1,4} = 0 & P_{1,5} = 0 & P_{1,6} = \frac{1}{6} & P_{1,7} = \frac{1}{6} & P_{1,8} = \frac{1}{6} & P_{1,9} = \frac{1}{6} & P_{1,10} = 0 & P_{1,11} = 0 & P_{1,12} = 0 & P_{1,13} = 0 & P_{1,14} = 0 & P_{1,15} = 0 & P_{1,16} = 0 & P_{1,17} = 0 & P_{1,18} = 0 \\ P_{2,0} = \frac{1}{6} & P_{2,1} = \frac{1}{6} & P_{2,2} = 0 & P_{2,3} = \frac{1}{6} & P_{2,4} = 0 & P_{2,5} = 0 & P_{2,6} = 0 & P_{2,7} = 0 & P_{2,8} = 0 & P_{2,9} = \frac{1}{6} & P_{2,10} = \frac{1}{6} & P_{2,11} = \frac{1}{6} & P_{2,12} = 0 & P_{2,13} = 0 & P_{2,14} = 0 & P_{2,15} = 0 & P_{2,16} = 0 & P_{2,17} = 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& P_{12,2} = 0 & P_{12,3} = \frac{1}{6} & P_{12,4} = 0 & P_{12,5} = 0 & P_{12,6} = 0 & P_{12,7} = 0 & P_{12,8} = 0 & P_{12,9} = 0 & P_{12,10} = 0 & P_{12,11} = \frac{1}{6} & P_{12,12} = 0 & P_{12,13} = \frac{1}{6} & P_{12,14} = 0 & P_{12,15} = 0 & P_{12,16} = 0 & P_{12,17} = 0 & P_{12,18} = 0 \\ P_{13,0} = 0 & P_{13,1} = 0 & P_{13,2} = 0 & P_{13,3} = \frac{1}{6} & P_{13,4} = \frac{1}{6} & P_{13,5} = 0 & P_{13,6} = 0 & P_{13,7} = 0 & P_{13,8} = 0 & P_{13,9} = 0 & P_{13,10} = 0 & P_{13,11} = 0 & P_{13,12} = \frac{1}{6} & P_{13,13} = 0 & P_{13,14} = \frac{1}{6} & P_{13,15} = 0 & P_{13,16} = 0 & P_{13,17} = 0 & P_{13,18} = 0 \\ P_{14,0} = 0 & P_{14,1} = 0 & P_{14,2} = 0 & P_{14,3} = 0 & P_{14,4} = \frac{1}{6} & P_{14,5} = 0 & P_{14,6} = 0 & P_{14,7} = 0 & P_{14,8} = 0 & P_{14,9} = 0 & P_{14,10} = 0 & P_{14,11} = 0 & P_{14,12} = 0 & P_{14,13} = \frac{1}{6} & P_{14,14} = 0 & P_{14,15} = \frac{1}{6} & P_{14,16} = 0 & P_{14,17} = 0 & P_{14,18} = 0 \\ P_{15,0} = 0 & P_{15,1} = 0 & P_{15,2} = 0 & P_{15,3} = 0 & P_{15,4} = \frac{1}{6} & P_{15,5} = \frac{1}{6} & P_{15,6} = 0 & P_{15,7} = 0 & P_{1$$

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|-------|----------------------------------|----------------------------------|---------------------------------|---------------------------------|---------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|------------------------------|------------------------------|---------------------------|
| $N =$ | $P_{0,0} = \frac{45}{16}$ | $P_{1,0} = \frac{15}{16}$ | $P_{2,0} = \frac{15}{16}$ | $P_{3,0} = \frac{15}{16}$ | $P_{4,0} = \frac{15}{16}$ | $P_{5,0} = \frac{15}{16}$ | $P_{6,0} = \frac{15}{16}$ | $P_{7,0} = \frac{7}{2}$ | $P_{8,0} = \frac{5}{16}$ | $P_{9,0} = \frac{7}{2}$ | $P_{10,0} = \frac{5}{16}$ | $P_{11,0} = \frac{7}{2}$ | $P_{12,0} = \frac{5}{16}$ | $P_{13,0} = \frac{7}{2}$ | $P_{14,0} = \frac{5}{16}$ | $P_{15,0} = \frac{7}{2}$ | $P_{16,0} = \frac{5}{16}$ | $P_{17,0} = \frac{7}{2}$ | $P_{18,0} = \frac{5}{16}$ |
| | $P_{1,1} = \frac{10771}{16384}$ | $P_{2,1} = \frac{62895}{16384}$ | $P_{3,1} = \frac{31447}{16384}$ | $P_{4,1} = \frac{15723}{16384}$ | $P_{5,1} = \frac{7861}{16384}$ | $P_{6,1} = \frac{3931}{16384}$ | $P_{7,1} = \frac{1965}{16384}$ | $P_{8,1} = \frac{982}{16384}$ | $P_{9,1} = \frac{491}{16384}$ | $P_{10,1} = \frac{245}{16384}$ | $P_{11,1} = \frac{122}{16384}$ | $P_{12,1} = \frac{61}{16384}$ | $P_{13,1} = \frac{30}{16384}$ | $P_{14,1} = \frac{15}{16384}$ | $P_{15,1} = \frac{7}{16384}$ | $P_{16,1} = \frac{3}{16384}$ | $P_{17,1} = \frac{1}{16384}$ | $P_{18,1} = \frac{1}{16384}$ | |
| | $P_{1,2} = \frac{318585}{16384}$ | $P_{2,2} = \frac{159292}{16384}$ | $P_{3,2} = \frac{79646}{16384}$ | $P_{4,2} = \frac{39823}{16384}$ | $P_{5,2} = \frac{19911}{16384}$ | $P_{6,2} = \frac{9955}{16384}$ | $P_{7,2} = \frac{4978}{16384}$ | $P_{8,2} = \frac{2489}{16384}$ | $P_{9,2} = \frac{1244}{16384}$ | $P_{10,2} = \frac{622}{16384}$ | $P_{11,2} = \frac{311}{16384}$ | $P_{12,2} = \frac{155}{16384}$ | $P_{13,2} = \frac{77}{16384}$ | $P_{14,2} = \frac{39}{16384}$ | $P_{15,2} = \frac{19}{16384}$ | $P_{16,2} = \frac{10}{16384}$ | $P_{17,2} = \frac{5}{16384}$ | $P_{18,2} = \frac{2}{16384}$ | |
| | $P_{1,3} = \frac{89799}{16384}$ | $P_{2,3} = \frac{44899}{16384}$ | $P_{3,3} = \frac{22449}{16384}$ | $P_{4,3} = \frac{11224}{16384}$ | $P_{5,3} = \frac{5612}{16384}$ | $P_{6,3} = \frac{2806}{16384}$ | $P_{7,3} = \frac{1403}{16384}$ | $P_{8,3} = \frac{701}{16384}$ | $P_{9,3} = \frac{350}{16384}$ | $P_{10,3} = \frac{175}{16384}$ | $P_{11,3} = \frac{87}{16384}$ | $P_{12,3} = \frac{44}{16384}$ | $P_{13,3} = \frac{22}{16384}$ | $P_{14,3} = \frac{11}{16384}$ | $P_{15,3} = \frac{5}{16384}$ | $P_{16,3} = \frac{2}{16384}$ | $P_{17,3} = \frac{1}{16384}$ | $P_{18,3} = \frac{1}{16384}$ | |
| | $P_{1,4} = \frac{25479}{16384}$ | $P_{2,4} = \frac{12739}{16384}$ | $P_{3,4} = \frac{6369}{16384}$ | $P_{4,4} = \frac{3184}{16384}$ | $P_{5,4} = \frac{1592}{16384}$ | $P_{6,4} = \frac{796}{16384}$ | $P_{7,4} = \frac{398}{16384}$ | $P_{8,4} = \frac{199}{16384}$ | $P_{9,4} = \frac{99}{16384}$ | $P_{10,4} = \frac{49}{16384}$ | $P_{11,4} = \frac{24}{16384}$ | $P_{12,4} = \frac{12}{16384}$ | $P_{13,4} = \frac{6}{16384}$ | $P_{14,4} = \frac{3}{16384}$ | $P_{15,4} = \frac{1}{16384}$ | $P_{16,4} = \frac{1}{16384}$ | $P_{17,4} = \frac{1}{16384}$ | $P_{18,4} = \frac{1}{16384}$ | |
| | $P_{1,5} = \frac{7187}{16384}$ | $P_{2,5} = \frac{3593}{16384}$ | $P_{3,5} = \frac{1797}{16384}$ | $P_{4,5} = \frac{898}{16384}$ | $P_{5,5} = \frac{449}{16384}$ | $P_{6,5} = \frac{224}{16384}$ | $P_{7,5} = \frac{112}{16384}$ | $P_{8,5} = \frac{56}{16384}$ | $P_{9,5} = \frac{28}{16384}$ | $P_{10,5} = \frac{14}{16384}$ | $P_{11,5} = \frac{7}{16384}$ | $P_{12,5} = \frac{3}{16384}$ | $P_{13,5} = \frac{1}{16384}$ | $P_{14,5} = \frac{1}{16384}$ | $P_{15,5} = \frac{1}{16384}$ | $P_{16,5} = \frac{1}{16384}$ | $P_{17,5} = \frac{1}{16384}$ | $P_{18,5} = \frac{1}{16384}$ | |
| | $P_{1,6} = \frac{2052}{16384}$ | $P_{2,6} = \frac{1026}{16384}$ | $P_{3,6} = \frac{513}{16384}$ | $P_{4,6} = \frac{256}{16384}$ | $P_{5,6} = \frac{128}{16384}$ | $P_{6,6} = \frac{64}{16384}$ | $P_{7,6} = \frac{32}{16384}$ | $P_{8,6} = \frac{16}{16384}$ | $P_{9,6} = \frac{8}{16384}$ | $P_{10,6} = \frac{4}{16384}$ | $P_{11,6} = \frac{2}{16384}$ | $P_{12,6} = \frac{1}{16384}$ | $P_{13,6} = \frac{1}{16384}$ | $P_{14,6} = \frac{1}{16384}$ | $P_{15,6} = \frac{1}{16384}$ | $P_{16,6} = \frac{1}{16384}$ | $P_{17,6} = \frac{1}{16384}$ | $P_{18,6} = \frac{1}{16384}$ | |
| | $P_{1,7} = \frac{588}{16384}$ | $P_{2,7} = \frac{294}{16384}$ | $P_{3,7} = \frac{147}{16384}$ | $P_{4,7} = \frac{73}{16384}$ | $P_{5,7} = \frac{37}{16384}$ | $P_{6,7} = \frac{19}{16384}$ | $P_{7,7} = \frac{9}{16384}$ | $P_{8,7} = \frac{4}{16384}$ | $P_{9,7} = \frac{2}{16384}$ | $P_{10,7} = \frac{1}{16384}$ | $P_{11,7} = \frac{1}{16384}$ | $P_{12,7} = \frac{1}{16384}$ | $P_{13,7} = \frac{1}{16384}$ | $P_{14,7} = \frac{1}{16384}$ | $P_{15,7} = \frac{1}{16384}$ | $P_{16,7} = \frac{1}{16384}$ | $P_{17,7} = \frac{1}{16384}$ | $P_{18,7} = \frac{1}{16384}$ | |
| | $P_{1,8} = \frac{168}{16384}$ | $P_{2,8} = \frac{84}{16384}$ | $P_{3,8} = \frac{42}{16384}$ | $P_{4,8} = \frac{21}{16384}$ </ | | | | | | | | | | | | | | | |

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$