

\mathbb{R}^n Bonus Problem #3

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§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram



§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

$$N^{-1} = \begin{pmatrix} p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = 0 & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 1 & p_{3,11} = \frac{1}{6} & p_{3,12} = 0 & p_{3,13} = 0 & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = \frac{1}{6} & p_{5,6} = 0 & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = 0 & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = 0 \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = 0 & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 1 & p_{10,11} = 0 & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = 0 & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = 0 & p_{11,11} = 1 & p_{11,12} = 0 & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 1 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = 0 & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = 0 & p_{14,4} = \frac{1}{6} & p_{14,5} = \frac{1}{6} & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} & p_{14,14} = 1 & p_{14,15} = \frac{1}{6} & p_{14,16}$$

$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = \frac{45}{16} \\ P_{0,1} = \frac{34506}{107144} \\ P_{0,2} = \frac{116568}{345060} \\ P_{0,3} = \frac{249996}{1071440} \\ P_{0,4} = \frac{479996}{2142880} \\ P_{0,5} = \frac{709996}{3214320} \\ P_{0,6} = \frac{939996}{4285760} \\ P_{0,7} = \frac{1169996}{5357200} \\ P_{0,8} = \frac{1399996}{6428640} \\ P_{0,9} = \frac{1629996}{7500080} \\ P_{0,10} = \frac{1859996}{8571520} \\ P_{0,11} = \frac{2089996}{9642960} \\ P_{0,12} = \frac{2319996}{10714400} \\ P_{0,13} = \frac{2549996}{11785840} \\ P_{0,14} = \frac{2779996}{12857280} \\ P_{0,15} = \frac{3009996}{13928720} \\ P_{0,16} = \frac{3239996}{15000160} \\ P_{0,17} = \frac{3469996}{16071600} \\ P_{0,18} = \frac{3699996}{17143040} \\ P_{0,19} = \frac{3929996}{18214480} \\ P_{0,20} = \frac{4159996}{19285920} \\ P_{0,21} = \frac{4389996}{20357360} \\ P_{0,22} = \frac{4619996}{21428800} \\ P_{0,23} = \frac{4849996}{22500240} \\ P_{0,24} = \frac{5079996}{23571680} \\ P_{0,25} = \frac{5309996}{24643120} \\ P_{0,26} = \frac{5539996}{25714560} \\ P_{0,27} = \frac{5769996}{26786000} \\ P_{0,28} = \frac{5999996}{27857440} \\ P_{0,29} = \frac{6229996}{28928880} \\ P_{0,30} = \frac{6459996}{30000320} \\ P_{0,31} = \frac{6689996}{31071760} \\ P_{0,32} = \frac{6919996}{32143200} \\ P_{0,33} = \frac{7149996}{33214640} \\ P_{0,34} = \frac{7379996}{34286080} \\ P_{0,35} = \frac{7609996}{35357520} \\ P_{0,36} = \frac{7839996}{36428960} \\ P_{0,37} = \frac{8069996}{37500400} \\ P_{0,38} = \frac{8299996}{38571840} \\ P_{0,39} = \frac{8529996}{39643280} \\ P_{0,40} = \frac{8759996}{40714720} \\ P_{0,41} = \frac{8989996}{41786160} \\ P_{0,42} = \frac{9219996}{42857600} \\ P_{0,43} = \frac{9449996}{43929040} \\ P_{0,44} = \frac{9679996}{45000480} \\ P_{0,45} = \frac{9909996}{46071920} \\ P_{0,46} = \frac{10139996}{47143360} \\ P_{0,47} = \frac{10369996}{48214800} \\ P_{0,48} = \frac{10599996}{49286240} \\ P_{0,49} = \frac{10829996}{50357680} \\ P_{0,50} = \frac{11059996}{51429120} \\ P_{0,51} = \frac{11289996}{52500560} \\ P_{0,52} = \frac{11519996}{53572000} \\ P_{0,53} = \frac{11749996}{54643440} \\ P_{0,54} = \frac{11979996}{55714880} \\ P_{0,55} = \frac{12209996}{56786320} \\ P_{0,56} = \frac{12439996}{57857760} \\ P_{0,57} = \frac{12669996}{58929200} \\ P_{0,58} = \frac{12899996}{60000640} \\ P_{0,59} = \frac{13129996}{61072080} \\ P_{0,60} = \frac{13359996}{62143520} \\ P_{0,61} = \frac{13589996}{63214960} \\ P_{0,62} = \frac{13819996}{64286400} \\ P_{0,63} = \frac{14049996}{65357840} \\ P_{0,64} = \frac{14279996}{66429280} \\ P_{0,65} = \frac{14509996}{67500720} \\ P_{0,66} = \frac{14739996}{68572160} \\ P_{0,67} = \frac{14969996}{69643600} \\ P_{0,68} = \frac{15199996}{70715040} \\ P_{0,69} = \frac{15429996}{71786480} \\ P_{0,70} = \frac{15659996}{72857920} \\ P_{0,71} = \frac{15889996}{73929360} \\ P_{0,72} = \frac{16119996}{75000800} \\ P_{0,73} = \frac{16349996}{76072240} \\ P_{0,74} = \frac{16579996}{77143680} \\ P_{0,75} = \frac{16809996}{78215120} \\ P_{0,76} = \frac{17039996}{79286560} \\ P_{0,77} = \frac{17269996}{80358000} \\ P_{0,78} = \frac{17499996}{81429440} \\ P_{0,79} = \frac{17729996}{82500880} \\ P_{0,80} = \frac{17959996}{83572320} \\ P_{0,81} = \frac{18189996}{84643760} \\ P_{0,82} = \frac{18419996}{85715200} \\ P_{0,83} = \frac{18649996}{86786640} \\ P_{0,84} = \frac{18879996}{87858080} \\ P_{0,85} = \frac{19109996}{88929520} \\ P_{0,86} = \frac{19339996}{90000960} \\ P_{0,87} = \frac{19569996}{91072400} \\ P_{0,88} = \frac{19799996}{92143840} \\ P_{0,89} = \frac{20029996}{93215280} \\ P_{0,90} = \frac{20259996}{94286720} \\ P_{0,91} = \frac{20489996}{95358160} \\ P_{0,92} = \frac{20719996}{96429600} \\ P_{0,93} = \frac{20949996}{97501040} \\ P_{0,94} = \frac{21179996}{98572480} \\ P_{0,95} = \frac{21409996}{99643920} \\ P_{0,96} = \frac{21639996}{100715360} \\ P_{0,97} = \frac{21869996}{101786800} \\ P_{0,98} = \frac{22099996}{102858240} \\ P_{0,99} = \frac{22329996}{103929680} \\ P_{0,100} = \frac{22559996}{105001120} \end{array} \end{array}$$

In order to get the expected number of steps, we find t_0 , where

$$t = N\mathbf{1}$$

Here, $\mathbf{1}$ is a vector whose entries are all 1.

$$t = \begin{pmatrix} 213 \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \end{pmatrix}$$

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$