

Rajeev Atla

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A hexagonal grid of 19 cells, numbered 0 to 18, arranged in a honeycomb pattern. Each cell contains a red dot and a black number. The cells are surrounded by 21 green dots, numbered 19 to 39, arranged in a larger hexagonal pattern around the central cluster.

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.



$$N^{-1} = \begin{pmatrix} p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = \frac{1}{6} & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = \frac{1}{6} & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = \frac{1}{6} & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = \frac{1}{6} & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = \frac{1}{6} & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 1 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = 0 & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = \frac{1}{6} & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = \frac{1}{6} & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 1 & p_{10,11} = 0 & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = 0 & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 1 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 1 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = \frac{1}{6} & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14$$

$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = 45 \\ P_{0,1} = 34506 \\ P_{0,2} = 107714 \\ P_{0,3} = 249993 \\ P_{0,4} = 479997 \\ P_{0,5} = 799995 \\ P_{0,6} = 1199985 \\ P_{0,7} = 1599975 \\ P_{0,8} = 1999959 \\ P_{0,9} = 2399937 \\ P_{0,10} = 2799909 \\ P_{0,11} = 3199875 \\ P_{0,12} = 3599835 \\ P_{0,13} = 3999789 \\ P_{0,14} = 4399737 \\ P_{0,15} = 4799679 \\ P_{0,16} = 5199615 \\ P_{0,17} = 5599545 \\ P_{0,18} = 5999469 \\ P_{0,19} = 6399387 \\ P_{0,20} = 6799299 \\ P_{0,21} = 7199205 \\ P_{0,22} = 7599105 \\ P_{0,23} = 7999000 \\ P_{0,24} = 8398890 \\ P_{0,25} = 8798775 \\ P_{0,26} = 9198655 \\ P_{0,27} = 9598530 \\ P_{0,28} = 9998400 \\ P_{0,29} = 10398265 \\ P_{0,30} = 10798125 \\ P_{0,31} = 11197980 \\ P_{0,32} = 11597830 \\ P_{0,33} = 11997675 \\ P_{0,34} = 12397515 \\ P_{0,35} = 12797350 \\ P_{0,36} = 13197180 \\ P_{0,37} = 13597005 \\ P_{0,38} = 13996825 \\ P_{0,39} = 14396640 \\ P_{0,40} = 14796450 \\ P_{0,41} = 15196255 \\ P_{0,42} = 15596055 \\ P_{0,43} = 15995850 \\ P_{0,44} = 16395640 \\ P_{0,45} = 16795435 \\ P_{0,46} = 17195225 \\ P_{0,47} = 17595010 \\ P_{0,48} = 17994790 \\ P_{0,49} = 18394565 \\ P_{0,50} = 18794335 \\ P_{0,51} = 19194100 \\ P_{0,52} = 19593860 \\ P_{0,53} = 19993615 \\ P_{0,54} = 20393365 \\ P_{0,55} = 20793110 \\ P_{0,56} = 21192850 \\ P_{0,57} = 21592585 \\ P_{0,58} = 21992315 \\ P_{0,59} = 22392040 \\ P_{0,60} = 22791760 \\ P_{0,61} = 23191475 \\ P_{0,62} = 23591185 \\ P_{0,63} = 23990890 \\ P_{0,64} = 24390590 \\ P_{0,65} = 24790285 \\ P_{0,66} = 25189975 \\ P_{0,67} = 25589660 \\ P_{0,68} = 25989340 \\ P_{0,69} = 26389015 \\ P_{0,70} = 26788685 \\ P_{0,71} = 27188350 \\ P_{0,72} = 27588010 \\ P_{0,73} = 27987665 \\ P_{0,74} = 28387315 \\ P_{0,75} = 28786960 \\ P_{0,76} = 29186600 \\ P_{0,77} = 29586235 \\ P_{0,78} = 29985865 \\ P_{0,79} = 30385490 \\ P_{0,80} = 30785110 \\ P_{0,81} = 31184725 \\ P_{0,82} = 31584335 \\ P_{0,83} = 31983940 \\ P_{0,84} = 32383540 \\ P_{0,85} = 32783135 \\ P_{0,86} = 33182725 \\ P_{0,87} = 33582310 \\ P_{0,88} = 33981890 \\ P_{0,89} = 34381465 \\ P_{0,90} = 34781035 \\ P_{0,91} = 35180600 \\ P_{0,92} = 35580160 \\ P_{0,93} = 35979715 \\ P_{0,94} = 36379265 \\ P_{0,95} = 36778810 \\ P_{0,96} = 37178350 \\ P_{0,97} = 37577885 \\ P_{0,98} = 37977415 \\ P_{0,99} = 38376940 \\ P_{1,0} = 38776460 \\ P_{1,1} = 39175975 \\ P_{1,2} = 39575485 \\ P_{1,3} = 39974990 \\ P_{1,4} = 40374490 \\ P_{1,5} = 40773985 \\ P_{1,6} = 41173475 \\ P_{1,7} = 41572960 \\ P_{1,8} = 41972440 \\ P_{1,9} = 42371915 \\ P_{1,10} = 42771385 \\ P_{1,11} = 43170850 \\ P_{1,12} = 43570310 \\ P_{1,13} = 43969765 \\ P_{1,14} = 44369215 \\ P_{1,15} = 44768660 \\ P_{1,16} = 45168100 \\ P_{1,17} = 45567535 \\ P_{1,18} = 45966965 \\ P_{1,19} = 46366390 \\ P_{1,20} = 46765810 \\ P_{1,21} = 47165225 \\ P_{1,22} = 47564635 \\ P_{1,23} = 47964040 \\ P_{1,24} = 48363440 \\ P_{1,25} = 48762835 \\ P_{1,26} = 49162225 \\ P_{1,27} = 49561610 \\ P_{1,28} = 49960990 \\ P_{1,29} = 50360365 \\ P_{1,30} = 50759735 \\ P_{1,31} = 51159100 \\ P_{1,32} = 51558460 \\ P_{1,33} = 51957815 \\ P_{1,34} = 52357165 \\ P_{1,35} = 52756510 \\ P_{1,36} = 53155850 \\ P_{1,37} = 53555185 \\ P_{1,38} = 53954515 \\ P_{1,39} = 54353840 \\ P_{1,40} = 54753160 \\ P_{1,41} = 55152475 \\ P_{1,42} = 55551785 \\ P_{1,43} = 55951090 \\ P_{1,44} = 56350390 \\ P_{1,45} = 56749685 \\ P_{1,46} = 57148975 \\ P_{1,47} = 57548260 \\ P_{1,48} = 57947540 \\ P_{1,49} = 58346815 \\ P_{1,50} = 58746085 \\ P_{1,51} = 59145350 \\ P_{1,52} = 59544610 \\ P_{1,53} = 59943865 \\ P_{1,54} = 60343115 \\ P_{1,55} = 60742360 \\ P_{1,56} = 61141600 \\ P_{1,57} = 61540835 \\ P_{1,58} = 61940065 \\ P_{1,59} = 62339290 \\ P_{1,60} = 62738510 \\ P_{1,61} = 63137725 \\ P_{1,62} = 63536935 \\ P_{1,63} = 63936140 \\ P_{1,64} = 64335340 \\ P_{1,65} = 64734535 \\ P_{1,66} = 65133725 \\ P_{1,67} = 65532910 \\ P_{1,68} = 65932090 \\ P_{1,69} = 66331265 \\ P_{1,70} = 66730435 \\ P_{1,71} = 67129600 \\ P_{1,72} = 67528760 \\ P_{1,73} = 67927915 \\ P_{1,74} = 68327065 \\ P_{1,75} = 68726210 \\ P_{1,76} = 69125350 \\ P_{1,77} = 69524485 \\ P_{1,78} = 69923615 \\ P_{1,79} = 70322740 \\ P_{1,80} = 70721860 \\ P_{1,81} = 71120975 \\ P_{1,82} = 71520085 \\ P_{1,83} = 71919190 \\ P_{1,84} = 72318290 \\ P_{1,85} = 72717385 \\ P_{1,86} = 73116475 \\ P_{1,87} = 73515560 \\ P_{1,88} = 73914640 \\ P_{1,89} = 74313715 \\ P_{1,90} = 74712785 \\ P_{1,91} = 75111850 \\ P_{1,92} = 75510910 \\ P_{1,93} = 75909965 \\ P_{1,94} = 76309015 \\ P_{1,95} = 76708060 \\ P_{1,96} = 77107100 \\ P_{1,97} = 77506135 \\ P_{1,98} = 77905165 \\ P_{1,99} = 78304190 \\ P_{2,0} = 78703210 \\ P_{2,1} = 79102225 \\ P_{2,2} = 79501235 \\ P_{2,3} = 79900240 \\ P_{2,4} = 80299240 \\ P_{2,5} = 80698235 \\ P_{2,6} = 81097225 \\ P_{2,7} = 81496210 \\ P_{2,8} = 81895190 \\ P_{2,9} = 82294165 \\ P_{2,10} = 82693135 \\ P_{2,11} = 83092100 \\ P_{2,12} = 83491060 \\ P_{2,13} = 83890015 \\ P_{2,14} = 84288965 \\ P_{2,15} = 84687910 \\ P_{2,16} = 85086850 \\ P_{2,17} = 85485785 \\ P_{2,18} = 85884715 \\ P_{2,19} = 86283640 \\ P_{2,20} = 86682560 \\ P_{2,21} = 87081475 \\ P_{2,22} = 87480385 \\ P_{2,23} = 87879290 \\ P_{2,24} = 88278190 \\ P_{2,25} = 88677085 \\ P_{2,26} = 89075975 \\ P_{2,27} = 89474860 \\ P_{2,28} = 8$$

In order to get the expected number of steps, we find  $t_0$ , where

$$t \equiv N\mathbf{1}$$

Here,  $\mathbf{1}$  is a vector whose entries are all 1.

[illegible]

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$