

Rajeev Atla

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A hexagonal grid graph with 36 nodes. The central node is labeled 0. Nodes are arranged in a hexagonal pattern, with labels 1 through 35. Nodes 1 through 14 are red dots, and nodes 15 through 35 are green dots. The grid is surrounded by a larger grid of green dots, with labels 15 through 35.

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.



$N =$	$P_{0,0} = \frac{45}{16}$	$P_{0,1} = \frac{15}{16}$	$P_{0,2} = \frac{15}{16}$	$P_{0,3} = \frac{15}{16}$	$P_{0,4} = \frac{15}{16}$	$P_{0,5} = \frac{15}{16}$	$P_{0,6} = \frac{15}{16}$	$P_{0,7} = \frac{7}{2}$	$P_{0,8} = \frac{5}{4}$	$P_{0,9} = \frac{7}{2}$	$P_{0,10} = \frac{1}{16}$	$P_{0,11} = \frac{7}{2}$	$P_{0,12} = \frac{5}{4}$	$P_{0,13} = \frac{7}{2}$	$P_{0,14} = \frac{5}{4}$	$P_{0,15} = \frac{7}{2}$	$P_{0,16} = \frac{1}{16}$	$P_{0,17} = \frac{7}{2}$	$P_{0,18} = \frac{5}{4}$
	$P_{1,0} = \frac{15}{16}$	$P_{1,1} = \frac{10771}{16384}$	$P_{1,2} = \frac{62609}{1048576}$	$P_{1,3} = \frac{353609}{67108864}$	$P_{1,4} = \frac{201953}{402653184}$	$P_{1,5} = \frac{117193}{2415919104}$	$P_{1,6} = \frac{67193}{14495514624}$	$P_{1,7} = \frac{1}{16384}$	$P_{1,8} = \frac{22969}{268435456}$	$P_{1,9} = \frac{1337}{16777216}$	$P_{1,10} = \frac{1}{16384}$	$P_{1,11} = \frac{1}{16384}$	$P_{1,12} = \frac{1337}{16777216}$	$P_{1,13} = \frac{22969}{268435456}$	$P_{1,14} = \frac{22969}{268435456}$	$P_{1,15} = \frac{1}{16384}$	$P_{1,16} = \frac{22969}{268435456}$	$P_{1,17} = \frac{1}{16384}$	$P_{1,18} = \frac{1}{16384}$
	$P_{2,0} = \frac{15}{16}$	$P_{2,1} = \frac{62609}{1048576}$	$P_{2,2} = \frac{353609}{67108864}$	$P_{2,3} = \frac{201953}{402653184}$	$P_{2,4} = \frac{117193}{2415919104}$	$P_{2,5} = \frac{67193}{14495514624}$	$P_{2,6} = \frac{1}{16384}$	$P_{2,7} = \frac{1}{16384}$	$P_{2,8} = \frac{22969}{268435456}$	$P_{2,9} = \frac{1337}{16777216}$	$P_{2,10} = \frac{1}{16384}$	$P_{2,11} = \frac{1}{16384}$	$P_{2,12} = \frac{1337}{16777216}$	$P_{2,13} = \frac{22969}{268435456}$	$P_{2,14} = \frac{22969}{268435456}$	$P_{2,15} = \frac{1}{16384}$	$P_{2,16} = \frac{22969}{268435456}$	$P_{2,17} = \frac{1}{16384}$	$P_{2,18} = \frac{1}{16384}$
	$P_{3,0} = \frac{15}{16}$	$P_{3,1} = \frac{201953}{402653184}$	$P_{3,2} = \frac{117193}{2415919104}$	$P_{3,3} = \frac{67193}{14495514624}$	$P_{3,4} = \frac{1}{16384}$	$P_{3,5} = \frac{1}{16384}$	$P_{3,6} = \frac{1}{16384}$	$P_{3,7} = \frac{1}{16384}$	$P_{3,8} = \frac{22969}{268435456}$	$P_{3,9} = \frac{1337}{16777216}$	$P_{3,10} = \frac{1}{16384}$	$P_{3,11} = \frac{1}{16384}$	$P_{3,12} = \frac{1337}{16777216}$	$P_{3,13} = \frac{22969}{268435456}$	$P_{3,14} = \frac{22969}{268435456}$	$P_{3,15} = \frac{1}{16384}$	$P_{3,16} = \frac{22969}{268435456}$	$P_{3,17} = \frac{1}{16384}$	$P_{3,18} = \frac{1}{16384}$
	$P_{4,0} = \frac{15}{16}$	$P_{4,1} = \frac{67193}{14495514624}$	$P_{4,2} = \frac{1}{16384}$	$P_{4,3} = \frac{1}{16384}$	$P_{4,4} = \frac{1}{16384}$	$P_{4,5} = \frac{1}{16384}$	$P_{4,6} = \frac{1}{16384}$	$P_{4,7} = \frac{1}{16384}$	$P_{4,8} = \frac{22969}{268435456}$	$P_{4,9} = \frac{1337}{16777216}$	$P_{4,10} = \frac{1}{16384}$	$P_{4,11} = \frac{1}{16384}$	$P_{4,12} = \frac{1337}{16777216}$	$P_{4,13} = \frac{22969}{268435456}$	$P_{4,14} = \frac{22969}{268435456}$	$P_{4,15} = \frac{1}{16384}$	$P_{4,16} = \frac{22969}{268435456}$	$P_{4,17} = \frac{1}{16384}$	$P_{4,18} = \frac{1}{16384}$
	$P_{5,0} = \frac{15}{16}$	$P_{5,1} = \frac{1}{16384}$	$P_{5,2} = \frac{1}{16384}$	$P_{5,3} = \frac{1}{16384}$	$P_{5,4} = \frac{1}{16384}$	$P_{5,5} = \frac{1}{16384}$	$P_{5,6} = \frac{1}{16384}$	$P_{5,7} = \frac{1}{16384}$	$P_{5,8} = \frac{22969}{268435456}$	$P_{5,9} = \frac{1337}{16777216}$	$P_{5,10} = \frac{1}{16384}$	$P_{5,11} = \frac{1}{16384}$	$P_{5,12} = \frac{1337}{16777216}$	$P_{5,13} = \frac{22969}{268435456}$	$P_{5,14} = \frac{22969}{268435456}$	$P_{5,15} = \frac{1}{16384}$	$P_{5,16} = \frac{22969}{268435456}$	$P_{5,17} = \frac{1}{16384}$	$P_{5,18} = \frac{1}{16384}$
	$P_{6,0} = \frac{15}{16}$	$P_{6,1} = \frac{1}{16384}$	$P_{6,2} = \frac{1}{16384}$	$P_{6,3} = \frac{1}{16384}$	$P_{6,4} = \frac{1}{16384}$	$P_{6,5} = \frac{1}{16384}$	$P_{6,6} = \frac{1}{16384}$	$P_{6,7} = \frac{1}{16384}$	$P_{6,8} = \frac{22969}{268435456}$	$P_{6,9} = \frac{1337}{16777216}$	$P_{6,10} = \frac{1}{16384}$	$P_{6,11} = \frac{1}{16384}$	$P_{6,12} = \frac{1337}{16777216}$	$P_{6,13} = \frac{22969}{268435456}$	$P_{6,14} = \frac{22969}{268435456}$	$P_{6,15} = \frac{1}{16384}$	$P_{6,16} = \frac{22969}{268435456}$	$P_{6,17} = \frac{1}{16384}$	$P_{6,18} = \frac{1}{16384}$
	$P_{7,0} = \frac{7}{2}$	$P_{7,1} = \frac{1}{16384}$	$P_{7,2} = \frac{1}{16384}$	$P_{7,3} = \frac{1}{16384}$	$P_{7,4} = \frac{1}{16384}$	$P_{7,5} = \frac{1}{16384}$	$P_{7,6} = \frac{1}{16384}$	$P_{7,7} = \frac{1}{16384}$	$P_{7,8} = \frac{22969}{268435456}$	$P_{7$									

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$