

\mathbb{R}^n Bonus Problem #3

Rajeev Atla

§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram



§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

§3.2 Transition Matrix

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\ p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\ p_{2,0}=0 & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\ p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\ p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=\frac{1}{6} & p_{4,12}=\frac{1}{6} & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=0 & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\ p_{5,0}=0 & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\ p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} & p_{6,19}=0 \\ p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=0 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} & p_{7,19}=\frac{1}{6} \\ p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=0 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 & p_{8,19}=\frac{1}{6} \\ p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=0 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & p_{9,14}=0 & p_{9,15}=0 & p_{9,16}=0 & p_{9,17}=0 & p_{9,18}=0 & p_{9,19}=\frac{1}{6} \\ p_{10,0}=0 & p_{10,1}=0 & p_{10,2}=\frac{1}{6} & p_{10,3}=0 & p_{10,4}=0 & p_{10,5}=0 & p_{10,6}=0 & p_{10,7}=0 & p_{10,8}=0 & p_{10,9}=\frac{1}{6} & p_{10,10}=0 & p_{10,11}=\frac{1}{6} & p_{10,12}=0 & p_{10,13}=0 & p_{10,14}=0 & p_{10,15}=0 & p_{10,16}=0 & p_{10,17}=0 & p_{10,18}=0 & p_{10,19}=0 \\ p_{11,0}=0 & p_{11,1}=0 & p_{11,2}=\frac{1}{6} & p_{11,3}=\frac{1}{6} & p_{11,4}=0 & p_{11,5}=0 & p_{11,6}=0 & p_{11,7}=0 & p_{11,8}=0 & p_{11,9}=0 & p_{11,10}=\frac{1}{6} & p_{11,11}=0 & p_{11,12}=\frac{1}{6} & p_{11,13}=0 & p_{11,14}=0 & p_{11,15}=0 & p_{11,16}=0 & p_{11,17}=0 & p_{11,18}=0 & p_{11,19}=\frac{1}{6} \\ p_{12,0}=0 & p_{12,1}=0 & p_{12,2}=0 & p_{12,3}=\frac{1}{6} & p_{12,4}=0 & p_{12,5}=0 & p_{12,6}=0 & p_{12,7}=0 & p_{12,8}=0 & p_{12,9}=0 & p_{12,10}=\frac{1}{6} & p_{12,11}=\frac{1}{6} & p_{12,12}=0 & p_{12,13}=\frac{1}{6} & p_{12,14}=0 & p_{12,15}=0 & p_{12,16}=0 & p_{12,17}=0 & p_{12,18}=0 & p_{12,19}=\frac{1}{6} \\ p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=0 & p_{13,12}=\frac{1}{6} & p_{13,13}=0 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 & p_{13,19}=\frac{1}{6} \\ p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=0 & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 & p_{14,19}=\frac{1}{6} \\ p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=\frac{1}{6} & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=\frac{1}{6} & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 & p_{15,19}=\frac{1}{6} \\ p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=0 & p_{$$

We also write the matrix Q , which doesn't have any absorbing states.

$$Q = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = \frac{1}{6} & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = 0 & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = \frac{1}{6} & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = \frac{1}{6} & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 \\ p_{6,0} = 0 & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 0 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = 0 & p_{6,18} = \frac{1}{6} \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 0 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = 0 & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 0 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = 0 & p_{9,8} = \frac{1}{6} & p_{9,9} = 0 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 0 & p_{10,11} = \frac{1}{6} & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = \frac{1}{6} & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = 0 & p_{11,11} = 0 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 0 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 0 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = 0 & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} & p_{14,14} = 0 & p_{14,15} = \frac{1}{6} & p_{14,16} = 0 & p_{14,17} = 0 & p_{14,18} = 0 \\ p_{15,0} = 0 & p_{15,1} = 0 & p_{15,2} = 0 & p_{15,3} = 0 & p_{15,4} = \frac{1}{6} & p_{15,5} = \frac{1}{6} & p_{15,6} = 0 & p_{15,7} = 0 & p_{15,8} = 0 &$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P .

$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = 45 \\ P_{0,1} = 34506 \\ P_{0,2} = 10714 \\ P_{0,3} = 249999 \\ P_{0,4} = 57876 \\ P_{0,5} = 11792 \\ P_{0,6} = 18354 \\ P_{0,7} = 24792 \\ P_{0,8} = 30792 \\ P_{0,9} = 36792 \\ P_{0,10} = 42792 \\ P_{0,11} = 48792 \\ P_{0,12} = 54792 \\ P_{0,13} = 60792 \\ P_{0,14} = 66792 \\ P_{0,15} = 72792 \\ P_{0,16} = 78792 \\ P_{0,17} = 84792 \\ P_{0,18} = 90792 \\ P_{0,19} = 96792 \\ P_{0,20} = 102792 \\ P_{0,21} = 108792 \\ P_{0,22} = 114792 \\ P_{0,23} = 120792 \\ P_{0,24} = 126792 \\ P_{0,25} = 132792 \\ P_{0,26} = 138792 \\ P_{0,27} = 144792 \\ P_{0,28} = 150792 \\ P_{0,29} = 156792 \\ P_{0,30} = 162792 \\ P_{0,31} = 168792 \\ P_{0,32} = 174792 \\ P_{0,33} = 180792 \\ P_{0,34} = 186792 \\ P_{0,35} = 192792 \\ P_{0,36} = 198792 \\ P_{0,37} = 204792 \\ P_{0,38} = 210792 \\ P_{0,39} = 216792 \\ P_{0,40} = 222792 \\ P_{0,41} = 228792 \\ P_{0,42} = 234792 \\ P_{0,43} = 240792 \\ P_{0,44} = 246792 \\ P_{0,45} = 252792 \\ P_{0,46} = 258792 \\ P_{0,47} = 264792 \\ P_{0,48} = 270792 \\ P_{0,49} = 276792 \\ P_{0,50} = 282792 \\ P_{0,51} = 288792 \\ P_{0,52} = 294792 \\ P_{0,53} = 300792 \\ P_{0,54} = 306792 \\ P_{0,55} = 312792 \\ P_{0,56} = 318792 \\ P_{0,57} = 324792 \\ P_{0,58} = 330792 \\ P_{0,59} = 336792 \\ P_{0,60} = 342792 \\ P_{0,61} = 348792 \\ P_{0,62} = 354792 \\ P_{0,63} = 360792 \\ P_{0,64} = 366792 \\ P_{0,65} = 372792 \\ P_{0,66} = 378792 \\ P_{0,67} = 384792 \\ P_{0,68} = 390792 \\ P_{0,69} = 396792 \\ P_{0,70} = 402792 \\ P_{0,71} = 408792 \\ P_{0,72} = 414792 \\ P_{0,73} = 420792 \\ P_{0,74} = 426792 \\ P_{0,75} = 432792 \\ P_{0,76} = 438792 \\ P_{0,77} = 444792 \\ P_{0,78} = 450792 \\ P_{0,79} = 456792 \\ P_{0,80} = 462792 \\ P_{0,81} = 468792 \\ P_{0,82} = 474792 \\ P_{0,83} = 480792 \\ P_{0,84} = 486792 \\ P_{0,85} = 492792 \\ P_{0,86} = 498792 \\ P_{0,87} = 504792 \\ P_{0,88} = 510792 \\ P_{0,89} = 516792 \\ P_{0,90} = 522792 \\ P_{0,91} = 528792 \\ P_{0,92} = 534792 \\ P_{0,93} = 540792 \\ P_{0,94} = 546792 \\ P_{0,95} = 552792 \\ P_{0,96} = 558792 \\ P_{0,97} = 564792 \\ P_{0,98} = 570792 \\ P_{0,99} = 576792 \end{array} & \begin{array}{c} P_{1,0} = 16 \\ P_{1,1} = 34506 \\ P_{1,2} = 10714 \\ P_{1,3} = 249999 \\ P_{1,4} = 57876 \\ P_{1,5} = 11792 \\ P_{1,6} = 18354 \\ P_{1,7} = 24792 \\ P_{1,8} = 30792 \\ P_{1,9} = 36792 \\ P_{1,10} = 42792 \\ P_{1,11} = 48792 \\ P_{1,12} = 54792 \\ P_{1,13} = 60792 \\ P_{1,14} = 66792 \\ P_{1,15} = 72792 \\ P_{1,16} = 78792 \\ P_{1,17} = 84792 \\ P_{1,18} = 90792 \\ P_{1,19} = 96792 \\ P_{1,20} = 102792 \\ P_{1,21} = 108792 \\ P_{1,22} = 114792 \\ P_{1,23} = 120792 \\ P_{1,24} = 126792 \\ P_{1,25} = 132792 \\ P_{1,26} = 138792 \\ P_{1,27} = 144792 \\ P_{1,28} = 150792 \\ P_{1,29} = 156792 \\ P_{1,30} = 162792 \\ P_{1,31} = 168792 \\ P_{1,32} = 174792 \\ P_{1,33} = 180792 \\ P_{1,34} = 186792 \\ P_{1,35} = 192792 \\ P_{1,36} = 198792 \\ P_{1,37} = 204792 \\ P_{1,38} = 210792 \\ P_{1,39} = 216792 \\ P_{1,40} = 222792 \\ P_{1,41} = 228792 \\ P_{1,42} = 234792 \\ P_{1,43} = 240792 \\ P_{1,44} = 246792 \\ P_{1,45} = 252792 \\ P_{1,46} = 258792 \\ P_{1,47} = 264792 \\ P_{1,48} = 270792 \\ P_{1,49} = 276792 \\ P_{1,50} = 282792 \\ P_{1,51} = 288792 \\ P_{1,52} = 294792 \\ P_{1,53} = 300792 \\ P_{1,54} = 306792 \\ P_{1,55} = 312792 \\ P_{1,56} = 318792 \\ P_{1,57} = 324792 \\ P_{1,58} = 330792 \\ P_{1,59} = 336792 \\ P_{1,60} = 342792 \\ P_{1,61} = 348792 \\ P_{1,62} = 354792 \\ P_{1,63} = 360792 \\ P_{1,64} = 366792 \\ P_{1,65} = 372792 \\ P_{1,66} = 378792 \\ P_{1,67} = 384792 \\ P_{1,68} = 390792 \\ P_{1,69} = 396792 \\ P_{1,70} = 402792 \\ P_{1,71} = 408792 \\ P_{1,72} = 414792 \\ P_{1,73} = 420792 \\ P_{1,74} = 426792 \\ P_{1,75} = 432792 \\ P_{1,76} = 438792 \\ P_{1,77} = 444792 \\ P_{1,78} = 450792 \\ P_{1,79} = 456792 \\ P_{1,80} = 462792 \\ P_{1,81} = 468792 \\ P_{1,82} = 474792 \\ P_{1,83} = 480792 \\ P_{1,84} = 486792 \\ P_{1,85} = 492792 \\ P_{1,86} = 498792 \\ P_{1,87} = 504792 \\ P_{1,88} = 510792 \\ P_{1,89} = 516792 \\ P_{1,90} = 522792 \\ P_{1,91} = 528792 \\ P_{1,92} = 534792 \\ P_{1,93} = 540792 \\ P_{1,94} = 546792 \\ P_{1,95} = 552792 \\ P_{1,96} = 558792 \\ P_{1,97} = 564792 \\ P_{1,98} = 570792 \\ P_{1,99} = 576792 \end{array} & \begin{array}{c} P_{2,0} = 16 \\ P_{2,1} = 34506 \\ P_{2,2} = 10714 \\ P_{2,3} = 249999 \\ P_{2,4} = 57876 \\ P_{2,5} = 11792 \\ P_{2,6} = 18354 \\ P_{2,7} = 24792 \\ P_{2,8} = 30792 \\ P_{2,9} = 36792 \\ P_{2,10} = 42792 \\ P_{2,11} = 48792 \\ P_{2,12} = 54792 \\ P_{2,13} = 60792 \\ P_{2,14} = 66792 \\ P_{2,15} = 72792 \\ P_{2,16} = 78792 \\ P_{2,17} = 84792 \\ P_{2,18} = 90792 \\ P_{2,19} = 96792 \\ P_{2,20} = 102792 \\ P_{2,21} = 108792 \\ P_{2,22} = 114792 \\ P_{2,23} = 120792 \\ P_{2,24} = 126792 \\ P_{2,25} = 132792 \\ P_{2,26} = 138792 \\ P_{2,27} = 144792 \\ P_{2,28} = 150792 \\ P_{2,29} = 156792 \\ P_{2,30} = 162792 \\ P_{2,31} = 168792 \\ P_{2,32} = 174792 \\ P_{2,33} = 180792 \\ P_{2,34} = 186792 \\ P_{2,35} = 192792 \\ P_{2,36} = 198792 \\ P_{2,37} = 204792 \\ P_{2,38} = 210792 \\ P_{2,39} = 216792 \\ P_{2,40} = 222792 \\ P_{2,41} = 228792 \\ P_{2,42} = 234792 \\ P_{2,43} = 240792 \\ P_{2,44} = 246792 \\ P_{2,45} = 252792 \\ P_{2,46} = 258792 \\ P_{2,47} = 264792 \\ P_{2,48} = 270792 \\ P_{2,49} = 276792 \\ P_{2,50} = 282792 \\ P_{2,51} = 288792 \\ P_{2,52} = 294792 \\ P_{2,53} = 300792 \\ P_{2,54} = 306792 \\ P_{2,55} = 312792 \\ P_{2,56} = 318792 \\$$

$$t \equiv N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$