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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A hexagonal grid of 19 cells, numbered 0 to 18, arranged in a honeycomb pattern. Each cell contains a red dot and a number. The cells are surrounded by 21 green dots, numbered 19 to 39, arranged in a larger hexagonal pattern around the central cluster.

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

$$N^{-1} = \begin{pmatrix} p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = \frac{1}{6} & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = \frac{1}{6} & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = \frac{1}{6} & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = \frac{1}{6} & p_{3,6} = \frac{1}{6} & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = \frac{1}{6} & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 1 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = 0 & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = \frac{1}{6} & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = \frac{1}{6} & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = \frac{1}{6} & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = \frac{1}{6} & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 1 & p_{10,11} = 0 & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = 0 & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 1 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 1 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = \frac{1}{6} & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{1$$

$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = \frac{45}{16} \\ P_{0,1} = \frac{34506}{107144} \\ P_{0,2} = \frac{116568}{345060} \\ P_{0,3} = \frac{249990}{1071440} \\ P_{0,4} = \frac{345060}{1611160} \\ P_{0,5} = \frac{345060}{1611160} \\ P_{0,6} = \frac{1611160}{3450600} \\ P_{0,7} = \frac{1611160}{3450600} \\ P_{0,8} = \frac{3450600}{16111600} \\ P_{0,9} = \frac{16111600}{34506000} \\ P_{0,10} = \frac{34506000}{161116000} \\ P_{0,11} = \frac{161116000}{345060000} \\ P_{0,12} = \frac{345060000}{1611160000} \\ P_{0,13} = \frac{1611160000}{3450600000} \\ P_{0,14} = \frac{3450600000}{16111600000} \\ P_{0,15} = \frac{16111600000}{34506000000} \\ P_{0,16} = \frac{34506000000}{161116000000} \\ P_{0,17} = \frac{161116000000}{345060000000} \\ P_{0,18} = \frac{345060000000}{1611160000000} \\ P_{0,19} = \frac{1611160000000}{3450600000000} \\ P_{0,20} = \frac{3450600000000}{16111600000000} \\ P_{0,21} = \frac{16111600000000}{34506000000000} \\ P_{0,22} = \frac{34506000000000}{161116000000000} \\ P_{0,23} = \frac{161116000000000}{345060000000000} \\ P_{0,24} = \frac{345060000000000}{1611160000000000} \\ P_{0,25} = \frac{1611160000000000}{3450600000000000} \\ P_{0,26} = \frac{3450600000000000}{16111600000000000} \\ P_{0,27} = \frac{16111600000000000}{34506000000000000} \\ P_{0,28} = \frac{34506000000000000}{161116000000000000} \\ P_{0,29} = \frac{161116000000000000}{345060000000000000} \\ P_{0,30} = \frac{345060000000000000}{1611160000000000000} \\ P_{0,31} = \frac{1611160000000000000}{3450600000000000000} \\ P_{0,32} = \frac{3450600000000000000}{16111600000000000000} \\ P_{0,33} = \frac{16111600000000000000}{34506000000000000000} \\ P_{0,34} = \frac{34506000000000000000}{161116000000000000000} \\ P_{0,35} = \frac{161116000000000000000}{345060000000000000000} \\ P_{0,36} = \frac{345060000000000000000}{1611160000000000000000} \\ P_{0,37} = \frac{1611160000000000000000}{3450600000000000000000} \\ P_{0,38} = \frac{3450600000000000000000}{16111600000000000000000} \\ P_{0,39} = \frac{16111600000000000000000}{34506000000000000000000} \\ P_{0,40} = \frac{34506000000000000000000}{161116000000000000000000} \\ P_{0,41} = \frac{161116000000000000000000}{345060000000000000000000} \\ P_{0,42} = \frac{345060000000000000000000}{1611160000000000000000000} \\ P_{0,43} = \frac{1611160000000000000000000}{3450600000000000000000000} \\ P_{0,44} = \frac{3450600000000000000000000}{16111600000000000000000000} \\ P_{0,45} = \frac{16111600000000000000000000}{34506000000000000000000000} \\ P_{0,46} = \frac{34506000000000000000000000}{161116000000000000000000000} \\ P_{0,47} = \frac{161116000000000000000000000}{345060000000000000000000000} \\ P_{0,48} = \frac{345060000000000000000000000}{1611160000000000000000000000} \\ P_{0,49} = \frac{1611160000000000000000000000}{3450600000000000000000000000} \\ P_{0,50} = \frac{3450600000000000000000000000}{16111600000000000000000000000} \\ P_{0,51} = \frac{16111600000000000000000000000}{34506000000000000000000000000} \\ P_{0,52} = \frac{34506000000000000000000000000}{161116000000000000000000000000} \\ P_{0,53} = \frac{161116000000000000000000000000}{345060000000000000000000000000} \\ P_{0,54} = \frac{345060000000000000000000000000}{1611160000000000000000000000000} \\ P_{0,55} = \frac{1611160000000000000000000000000}{3450600000000000000000000000000} \\ P_{0,56} = \frac{3450600000000000000000000000000}{16111600000000000000000000000000} \\ P_{0,57} = \frac{16111600000000000000000000000000}{34506000000000000000000000000000} \\ P_{0,58} = \frac{34506000000000000000000000000000}{161116000000000000000000000000000} \\ P_{0,59} = \frac{161116000000000000000000000000000}{345060000000000000000000000000000} \\ P_{0,60} = \frac{345060000000000000000000000000000}{1611160000000000000000000000000000} \\ P_{0,61} = \frac{1611160000000000000000000000000000}{3450600000000000000000000000000000} \\ P_{0,62} = \frac{3450600000000000000000000000000000}{16111600000000000000000000000000000} \\ P_{0,63} = \frac{16111600000000000000000000000000000}{34506000000000000000000000000000000} \\ P_{0,64} = \frac{34506000000000000000000000000000000}{161116000000000000000000000000000000} \\ P_{0,65} = \frac{161116000000000000000000000000000000}{345060000000000000000000000000000000} \\ P_{0,66} = \frac{345060000000000000000000000000000000}{1611160000000000000000000000000000000} \\ P_{0,67} = \frac{1611160000000000000000000000000000000}{3450600000000000000000000000000000000} \\ P_{0,68} = \frac{3450600000000000000000000000000000000}{16111600000000000000000000000000000000} \\ P_{0,69} = \frac{16111600000000000000000000000000000000}{34506000000000000000000000000000000000} \\ P_{0,70} = \frac{34506000000000000000000000000000000000}{161116000000000000000000000000000000000} \\ P_{0,71} = \frac{161116000000000000000000000000000000000}{34506000000000000$$

In order to get the expected number of steps, we find t_0 , where

$$t = N\mathbf{1}$$

Here, $\mathbf{1}$ is a vector whose entries are all 1.

[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$