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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

The diagram shows a hexagonal lattice structure. The internal nodes are numbered 0 to 18, and the external nodes are numbered 19 to 30. The nodes are arranged in a hexagonal pattern, with the central node being 0. The nodes are connected by blue lines forming hexagons. The external nodes are located at the vertices of the hexagonal lattice.

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = 0 & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 0 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = 0 & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 0 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} 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We also write the matrix Q , which doesn't have any absorbing states.

$$Q = \begin{pmatrix} P_{0,0} = 0 & P_{0,1} = \frac{1}{6} & P_{0,2} = \frac{1}{6} & P_{0,3} = \frac{1}{6} & P_{0,4} = \frac{1}{6} & P_{0,5} = \frac{1}{6} & P_{0,6} = \frac{1}{6} & P_{0,7} = 0 & P_{0,8} = 0 & P_{0,9} = 0 & P_{0,10} = 0 & P_{0,11} = 0 & P_{0,12} = 0 & P_{0,13} = 0 & P_{0,14} = 0 & P_{0,15} = 0 & P_{0,16} = 0 & P_{0,17} = 0 & P_{0,18} = 0 \\ P_{1,0} = \frac{1}{6} & P_{1,1} = 0 & P_{1,2} = 0 & P_{1,3} = 0 & P_{1,4} = 0 & P_{1,5} = 0 & P_{1,6} = \frac{1}{6} & P_{1,7} = \frac{1}{6} & P_{1,8} = \frac{1}{6} & P_{1,9} = \frac{1}{6} & P_{1,10} = 0 & P_{1,11} = 0 & P_{1,12} = 0 & P_{1,13} = 0 & P_{1,14} = 0 & P_{1,15} = 0 & P_{1,16} = 0 & P_{1,17} = 0 & P_{1,18} = 0 \\ P_{2,0} = \frac{1}{6} & P_{2,1} = \frac{1}{6} & P_{2,2} = 0 & P_{2,3} = \frac{1}{6} & P_{2,4} = 0 & P_{2,5} = 0 & P_{2,6} = 0 & P_{2,7} = 0 & P_{2,8} = 0 & P_{2,9} = \frac{1}{6} & P_{2,10} = \frac{1}{6} & P_{2,11} = \frac{1}{6} & P_{2,12} = 0 & P_{2,13} = 0 & P_{2,14} = 0 & P_{2,15} = 0 & P_{2,16} = 0 & P_{2,17} = 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P_{12,7} = 0 & P_{12,8} = 0 & P_{12,9} = 0 & P_{12,10} = 0 & P_{12,11} = \frac{1}{6} & P_{12,12} = 0 & P_{12,13} = \frac{1}{6} & P_{12,14} = 0 & P_{12,15} = 0 & P_{12,16} = 0 & P_{12,17} = 0 & P_{12,18} = 0 \\ P_{13,0} = 0 & P_{13,1} = 0 & P_{13,2} = 0 & P_{13,3} = \frac{1}{6} & P_{13,4} = \frac{1}{6} & P_{13,5} = 0 & P_{13,6} = 0 & P_{13,7} = 0 & P_{13,8} = 0 & P_{13,9} = 0 & P_{13,10} = 0 & P_{13,11} = 0 & P_{13,12} = \frac{1}{6} & P_{13,13} = 0 & P_{13,14} = \frac{1}{6} & P_{13,15} = 0 & P_{13,16} = 0 & P_{13,17} = 0 & P_{13,18} = 0 \\ P_{14,0} = 0 & P_{14,1} = 0 & P_{14,2} = 0 & P_{14,3} = 0 & P_{14,4} = \frac{1}{6} & P_{14,5} = 0 & P_{14,6} = 0 & P_{14,7} = 0 & P_{14,8} = 0 & P_{14,9} = 0 & P_{14,10} = 0 & P_{14,11} = 0 & P_{14,12} = 0 & P_{14,13} = \frac{1}{6} & P_{14,14} = \frac{1}{6} & P_{14,15} = \frac{1}{6} & P_{14,16} = 0 & P_{14,17} = 0 & P_{14,18} = 0 \\ P_{15,0} = 0 & P_{15,1} = 0 & P_{15,2} = 0 & P_{15,3} = 0 & P_{15,4} = \frac{1}{6} & P_{15,5} = \frac{1}{6} & P_{15,6} = 0 & P_{15,7} = 0 & P_{15,8} = 0 & P_{15,9} = 0 & P_{15,10} = 0 &$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P .

[illegible]

$$t = N\mathbf{1}$$
$$t = \begin{pmatrix} 213 \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{101} \end{pmatrix}$$

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$