

## $\mathbb{R}^n$ Bonus Problem #3

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### §1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

### §2 Diagram



### §3 Solution

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let  $X_i \in [0, 36]$  be the current state, or position of the traveler. The traveler always starts at position  $X_0 = 0$ . The final state must be  $X_N \in [19, 36]$ .

### §3.2 Transition Matrix

Now that we've defined some notation, we can write the transition matrix  $P$ . Because a  $37 \times 37$  matrix is cumbersome, we combine the states [19, 36] into a

$$P = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = 0 & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 0 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 0 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 0 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 0 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 0 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = 0 & p_{9,8} = \frac{1}{6} & p_{9,9} = 0 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 1 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 0 & p_{10,11} = \frac{1}{6} & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = \frac{1}{6} & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 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0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} &amp$$

We also write the matrix  $Q$ , which doesn't have any absorbing states.

$$Q = \begin{pmatrix} P_{0,0} = 0 & P_{0,1} = \frac{1}{6} & P_{0,2} = \frac{1}{6} & P_{0,3} = \frac{1}{6} & P_{0,4} = \frac{1}{6} & P_{0,5} = \frac{1}{6} & P_{0,6} = \frac{1}{6} & P_{0,7} = 0 & P_{0,8} = 0 & P_{0,9} = 0 & P_{0,10} = 0 & P_{0,11} = 0 & P_{0,12} = 0 & P_{0,13} = 0 & P_{0,14} = 0 & P_{0,15} = 0 & P_{0,16} = 0 & P_{0,17} = 0 & P_{0,18} = 0 \\ P_{1,0} = \frac{1}{6} & P_{1,1} = 0 & P_{1,2} = \frac{1}{6} & P_{1,3} = 0 & P_{1,4} = 0 & P_{1,5} = 0 & P_{1,6} = \frac{1}{6} & P_{1,7} = \frac{1}{6} & P_{1,8} = \frac{1}{6} & P_{1,9} = \frac{1}{6} & P_{1,10} = 0 & P_{1,11} = 0 & P_{1,12} = 0 & P_{1,13} = 0 & P_{1,14} = 0 & P_{1,15} = 0 & P_{1,16} = 0 & P_{1,17} = 0 & P_{1,18} = 0 \\ P_{2,0} = \frac{1}{6} & P_{2,1} = \frac{1}{6} & P_{2,2} = 0 & P_{2,3} = \frac{1}{6} & P_{2,4} = 0 & P_{2,5} = 0 & P_{2,6} = 0 & P_{2,7} = 0 & P_{2,8} = 0 & P_{2,9} = \frac{1}{6} & P_{2,10} = \frac{1}{6} & P_{2,11} = \frac{1}{6} & P_{2,12} = 0 & P_{2,13} = 0 & P_{2,14} = 0 & P_{2,15} = 0 & P_{2,16} = 0 & P_{2,17} = 0 & P_{2,18} = 0 \\ P_{3,0} = \frac{1}{6} & P_{3,1} = 0 & P_{3,2} = \frac{1}{6} & P_{3,3} = 0 & P_{3,4} = \frac{1}{6} & P_{3,5} = 0 & P_{3,6} = 0 & P_{3,7} = 0 & P_{3,8} = 0 & P_{3,9} = 0 & P_{3,10} = 0 & P_{3,11} = \frac{1}{6} & P_{3,12} = \frac{1}{6} & P_{3,13} = \frac{1}{6} & P_{3,14} = 0 & P_{3,15} = 0 & P_{3,16} = 0 & P_{3,17} = 0 & P_{3,18} = 0 \\ P_{4,0} = \frac{1}{6} & P_{4,1} = 0 & P_{4,2} = 0 & P_{4,3} = \frac{1}{6} & P_{4,4} = 0 & P_{4,5} = \frac{1}{6} & P_{4,6} = 0 & P_{4,7} = 0 & P_{4,8} = 0 & P_{4,9} = 0 & P_{4,10} = 0 & P_{4,11} = 0 & P_{4,12} = 0 & P_{4,13} = \frac{1}{6} & P_{4,14} = \frac{1}{6} & P_{4,15} = \frac{1}{6} & P_{4,16} = 0 & P_{4,17} = 0 & P_{4,18} = 0 \\ P_{5,0} = \frac{1}{6} & P_{5,1} = 0 & P_{5,2} = 0 & P_{5,3} = 0 & P_{5,4} = \frac{1}{6} & P_{5,5} = 0 & P_{5,6} = \frac{1}{6} & P_{5,7} = 0 & P_{5,8} = 0 & P_{5,9} = 0 & P_{5,10} = 0 & P_{5,11} = 0 & P_{5,12} = 0 & P_{5,13} = 0 & P_{5,14} = 0 & P_{5,15} = \frac{1}{6} & P_{5,16} = \frac{1}{6} & 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\frac{1}{6} & P_{12,4} = 0 & P_{12,5} = 0 & P_{12,6} = 0 & P_{12,7} = 0 & P_{12,8} = 0 & P_{12,9} = 0 & P_{12,10} = 0 & P_{12,11} = \frac{1}{6} & P_{12,12} = 0 & P_{12,13} = \frac{1}{6} & P_{12,14} = 0 & P_{12,15} = 0 & P_{12,16} = 0 & P_{12,17} = 0 & P_{12,18} = 0 \\ P_{13,0} = 0 & P_{13,1} = 0 & P_{13,2} = 0 & P_{13,3} = \frac{1}{6} & P_{13,4} = \frac{1}{6} & P_{13,5} = 0 & P_{13,6} = 0 & P_{13,7} = 0 & P_{13,8} = 0 & P_{13,9} = 0 & P_{13,10} = 0 & P_{13,11} = 0 & P_{13,12} = \frac{1}{6} & P_{13,13} = 0 & P_{13,14} = \frac{1}{6} & P_{13,15} = 0 & P_{13,16} = 0 & P_{13,17} = 0 & P_{13,18} = 0 \\ P_{14,0} = 0 & P_{14,1} = 0 & P_{14,2} = 0 & P_{14,3} = 0 & P_{14,4} = \frac{1}{6} & P_{14,5} = 0 & P_{14,6} = 0 & P_{14,7} = 0 & P_{14,8} = 0 & P_{14,9} = 0 & P_{14,10} = 0 & P_{14,11} = 0 & P_{14,12} = 0 & P_{14,13} = \frac{1}{6} & P_{14,14} = \frac{1}{6} & P_{14,15} = \frac{1}{6} & P_{14,16} = 0 & P_{14,17} = 0 & P_{14,18} = 0 \\ P_{15,0} = 0 & P_{15,1} = 0 & P_{15,2} = 0 & P_{15,3} = 0 & P_{15,4} = \frac{1}{6} & P_{15,5} = 0 & P_{15,6} =$$

$N = (I - Q)^{-1}$  is known as the fundamental matrix of  $P$ .

	P <sub>10</sub> = .45	P <sub>11</sub> = .16	P <sub>12</sub> = .16	P <sub>13</sub> = .16	P <sub>14</sub> = .16	P <sub>15</sub> = .16	P <sub>16</sub> = .16	P <sub>17</sub> = .16	P <sub>18</sub> = .16	P <sub>19</sub> = .16	P <sub>20</sub> = .16	P <sub>21</sub> = .16	P <sub>22</sub> = .16	P <sub>23</sub> = .16	P <sub>24</sub> = .16	P <sub>25</sub> = .16	P <sub>26</sub> = .16	P <sub>27</sub> = .16	P <sub>28</sub> = .16	P <sub>29</sub> = .16	P <sub>30</sub> = .16	P <sub>31</sub> = .16	P <sub>32</sub> = .16	P <sub>33</sub> = .16	P <sub>34</sub> = .16	P <sub>35</sub> = .16	P <sub>36</sub> = .16	P <sub>37</sub> = .16	P <sub>38</sub> = .16	P <sub>39</sub> = .16	P <sub>40</sub> = .16	P <sub>41</sub> = .16	P <sub>42</sub> = .16	P <sub>43</sub> = .16	P <sub>44</sub> = .16	P <sub>45</sub> = .16	P <sub>46</sub> = .16	P <sub>47</sub> = .16	P <sub>48</sub> = .16	P <sub>49</sub> = .16	P <sub>50</sub> = .16	P <sub>51</sub> = .16	P <sub>52</sub> = .16	P <sub>53</sub> = .16	P <sub>54</sub> = .16	P <sub>55</sub> = .16	P <sub>56</sub> = .16	P <sub>57</sub> = .16	P <sub>58</sub> = .16	P <sub>59</sub> = .16	P <sub>60</sub> = .16	P <sub>61</sub> = .16	P <sub>62</sub> = .16	P <sub>63</sub> = .16	P <sub>64</sub> = .16	P <sub>65</sub> = .16	P <sub>66</sub> = .16	P <sub>67</sub> = .16	P <sub>68</sub> = .16	P <sub>69</sub> = .16	P <sub>70</sub> = .16	P <sub>71</sub> = .16	P <sub>72</sub> = .16	P <sub>73</sub> = .16	P <sub>74</sub> = .16	P <sub>75</sub> = .16	P <sub>76</sub> = .16	P <sub>77</sub> = .16	P <sub>78</sub> = .16	P <sub>79</sub> = .16	P <sub>80</sub> = .16	P <sub>81</sub> = .16	P <sub>82</sub> = .16	P <sub>83</sub> = .16	P <sub>84</sub> = .16	P <sub>85</sub> = .16	P <sub>86</sub> = .16	P <sub>87</sub> = .16	P <sub>88</sub> = .16	P <sub>89</sub> = .16	P <sub>90</sub> = .16	P <sub>91</sub> = .16	P <sub>92</sub> = .16	P <sub>93</sub> = .16	P <sub>94</sub> = .16	P <sub>95</sub> = .16	P <sub>96</sub> = .16	P <sub>97</sub> = .16	P <sub>98</sub> = .16	P <sub>99</sub> = .16	P <sub>100</sub> = .16	P <sub>101</sub> = .16	P <sub>102</sub> = .16	P <sub>103</sub> = .16	P <sub>104</sub> = .16	P <sub>105</sub> = .16	P <sub>106</sub> = .16	P <sub>107</sub> = .16	P <sub>108</sub> = .16	P <sub>109</sub> = .16	P <sub>110</sub> = .16	P <sub>111</sub> = .16	P <sub>112</sub> = .16	P <sub>113</sub> = .16	P <sub>114</sub> = .16	P <sub>115</sub> = .16	P <sub>116</sub> = .16	P <sub>117</sub> = .16	P <sub>118</sub> = .16	P <sub>119</sub> = .16	P <sub>120</sub> = .16	P <sub>121</sub> = .16	P <sub>122</sub> = .16	P <sub>123</sub> = .16	P <sub>124</sub> = .16	P <sub>125</sub> = .16	P <sub>126</sub> = .16	P <sub>127</sub> = .16	P <sub>128</sub> = .16	P <sub>129</sub> = .16	P <sub>130</sub> = .16	P <sub>131</sub> = .16	P <sub>132</sub> = .16	P <sub>133</sub> = .16	P <sub>134</sub> = .16	P <sub>135</sub> = .16	P <sub>136</sub> = .16	P <sub>137</sub> = .16	P <sub>138</sub> = .16	P <sub>139</sub> = .16	P <sub>140</sub> = .16	P <sub>141</sub> = .16	P <sub>142</sub> = .16	P <sub>143</sub> = .16	P <sub>144</sub> = .16	P <sub>145</sub> = .16	P <sub>146</sub> = .16	P <sub>147</sub> = .16	P <sub>148</sub> = .16	P <sub>149</sub> = .16	P <sub>150</sub> = .16	P <sub>151</sub> = .16	P <sub>152</sub> = .16	P <sub>153</sub> = .16	P <sub>154</sub> = .16	P <sub>155</sub> = .16	P <sub>156</sub> = .16	P <sub>157</sub> = .16	P <sub>158</sub> = .16	P <sub>159</sub> = .16	P <sub>160</sub> = .16	P <sub>161</sub> = .16	P <sub>162</sub> = .16	P <sub>163</sub> = .16	P <sub>164</sub> = .16	P <sub>165</sub> = .16	P <sub>166</sub> = .16	P <sub>167</sub> = .16	P <sub>168</sub> = .16	P <sub>169</sub> = .16	P <sub>170</sub> = .16	P <sub>171</sub> = .16	P <sub>172</sub> = .16	P <sub>173</sub> = .16	P <sub>174</sub> = .16	P <sub>175</sub> = .16	P <sub>176</sub> = .16	P <sub>177</sub> = .16	P <sub>178</sub> = .16	P <sub>179</sub> = .16	P <sub>180</sub> = .16	P <sub>181</sub> = .16	P <sub>182</sub> = .16	P <sub>183</sub> = .16	P <sub>184</sub> = .16	P <sub>185</sub> = .16	P <sub>186</sub> = .16	P <sub>187</sub> = .16	P <sub>188</sub> = .16	P <sub>189</sub> = .16	P <sub>190</sub> = .16	P <sub>191</sub> = .16	P <sub>192</sub> = .16	P <sub>193</sub> = .16	P <sub>194</sub> = .16	P <sub>195</sub> = .16	P <sub>196</sub> = .16	P <sub>197</sub> = .16	P <sub>198</sub> = .16	P <sub>199</sub> = .16	P <sub>200</sub> = .16	P <sub>201</sub> = .16	P <sub>202</sub> = .16	P <sub>203</sub> = .16	P <sub>204</sub> = .16	P <sub>205</sub> = .16	P <sub>206</sub> = .16	P <sub>207</sub> = .16	P <sub>208</sub> = .16	P <sub>209</sub> = .16	P <sub>210</sub> = .16	P <sub>211</sub> = .16	P <sub>212</sub> = .16	P <sub>213</sub> = .16	P <sub>214</sub> = .16	P <sub>215</sub> = .16	P <sub>216</sub> = .16	P <sub>217</sub> = .16	P <sub>218</sub> = .16	P <sub>219</sub> = .16	P <sub>220</sub> = .16	P <sub>221</sub> = .16	P <sub>222</sub> = .16	P <sub>223</sub> = .16	P <sub>224</sub> = .16	P <sub>225</sub> = .16	P <sub>226</sub> = .16	P <sub>227</sub> = .16	P <sub>228</sub> = .16	P <sub>229</sub> = .16	P <sub>230</sub> = .16	P <sub>231</sub> = .16	P <sub>232</sub> = .16	P <sub>233</sub> = .16	P <sub>234</sub> = .16	P <sub>235</sub> = .16	P <sub>236</sub> = .16	P <sub>237</sub> = .16	P <sub>238</sub> = .16	P <sub>239</sub> = .16	P <sub>240</sub>
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$$t = N\mathbf{1}$$
[illegible]

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$