

\mathbb{R}^n Bonus Problem #3

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§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram



§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\ p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\ p_{2,0}=0 & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\ p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\ p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=\frac{1}{6} & p_{4,12}=\frac{1}{6} & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=0 & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\ p_{5,0}=0 & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\ p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & 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| $N =$ | $P_{0,0} = \frac{45}{16}$ | $P_{1,0} = \frac{15}{16}$ | $P_{2,0} = \frac{15}{16}$ | $P_{3,0} = \frac{15}{16}$ | $P_{4,0} = \frac{15}{16}$ | $P_{5,0} = \frac{15}{16}$ | $P_{6,0} = \frac{15}{16}$ | $P_{7,0} = \frac{7}{2}$ | $P_{8,0} = \frac{5}{4}$ | $P_{9,0} = \frac{7}{2}$ | $P_{10,0} = \frac{15}{16}$ | $P_{11,0} = \frac{7}{8}$ | $P_{12,0} = \frac{5}{4}$ | $P_{13,0} = \frac{7}{8}$ | $P_{14,0} = \frac{5}{4}$ | $P_{15,0} = \frac{7}{8}$ | $P_{16,0} = \frac{7}{8}$ | $P_{17,0} = \frac{7}{8}$ | $P_{18,0} = \frac{5}{4}$ |
| | $P_{1,1} = \frac{15}{16}$ | $P_{2,1} = \frac{15}{16}$ | $P_{3,1} = \frac{15}{16}$ | $P_{4,1} = \frac{15}{16}$ | $P_{5,1} = \frac{15}{16}$ | $P_{6,1} = \frac{15}{16}$ | $P_{7,1} = \frac{7}{2}$ | $P_{8,1} = \frac{5}{4}$ | $P_{9,1} = \frac{7}{2}$ | $P_{10,1} = \frac{15}{16}$ | $P_{11,1} = \frac{7}{8}$ | $P_{12,1} = \frac{5}{4}$ | $P_{13,1} = \frac{7}{8}$ | $P_{14,1} = \frac{5}{4}$ | $P_{15,1} = \frac{7}{8}$ | $P_{16,1} = \frac{7}{8}$ | $P_{17,1} = \frac{7}{8}$ | $P_{18,1} = \frac{5}{4}$ | $P_{19,1} = \frac{15}{16}$ |
| | $P_{2,2} = \frac{15}{16}$ | $P_{3,2} = \frac{15}{16}$ | $P_{4,2} = \frac{15}{16}$ | $P_{5,2} = \frac{15}{16}$ | $P_{6,2} = \frac{15}{16}$ | $P_{7,2} = \frac{7}{2}$ | $P_{8,2} = \frac{5}{4}$ | $P_{9,2} = \frac{7}{2}$ | $P_{10,2} = \frac{15}{16}$ | $P_{11,2} = \frac{7}{8}$ | $P_{12,2} = \frac{5}{4}$ | $P_{13,2} = \frac{7}{8}$ | $P_{14,2} = \frac{5}{4}$ | $P_{15,2} = \frac{7}{8}$ | $P_{16,2} = \frac{7}{8}$ | $P_{17,2} = \frac{7}{8}$ | $P_{18,2} = \frac{5}{4}$ | $P_{19,2} = \frac{15}{16}$ | $P_{20,2} = \frac{15}{16}$ |
| | $P_{3,3} = \frac{15}{16}$ | $P_{4,3} = \frac{15}{16}$ | $P_{5,3} = \frac{15}{16}$ | $P_{6,3} = \frac{15}{16}$ | $P_{7,3} = \frac{7}{2}$ | $P_{8,3} = \frac{5}{4}$ | $P_{9,3} = \frac{7}{2}$ | $P_{10,3} = \frac{15}{16}$ | $P_{11,3} = \frac{7}{8}$ | $P_{12,3} = \frac{5}{4}$ | $P_{13,3} = \frac{7}{8}$ | $P_{14,3} = \frac{5}{4}$ | $P_{15,3} = \frac{7}{8}$ | $P_{16,3} = \frac{7}{8}$ | $P_{17,3} = \frac{7}{8}$ | $P_{18,3} = \frac{5}{4}$ | $P_{19,3} = \frac{15}{16}$ | $P_{20,3} = \frac{15}{16}$ | $P_{21,3} = \frac{15}{16}$ |
| | $P_{4,4} = \frac{15}{16}$ | $P_{5,4} = \frac{15}{16}$ | $P_{6,4} = \frac{15}{16}$ | $P_{7,4} = \frac{7}{2}$ | $P_{8,4} = \frac{5}{4}$ | $P_{9,4} = \frac{7}{2}$ | $P_{10,4} = \frac{15}{16}$ | $P_{11,4} = \frac{7}{8}$ | $P_{12,4} = \frac{5}{4}$ | $P_{13,4} = \frac{7}{8}$ | $P_{14,4} = \frac{5}{4}$ | $P_{15,4} = \frac{7}{8}$ | $P_{16,4} = \frac{7}{8}$ | $P_{17,4} = \frac{7}{8}$ | $P_{18,4} = \frac{5}{4}$ | $P_{19,4} = \frac{15}{16}$ | $P_{20,4} = \frac{15}{16}$ | $P_{21,4} = \frac{15}{16}$ | $P_{22,4} = \frac{15}{16}$ |
| | $P_{5,5} = \frac{15}{16}$ | $P_{6,5} = \frac{15}{16}$ | $P_{7,5} = \frac{7}{2}$ | $P_{8,5} = \frac{5}{4}$ | $P_{9,5} = \frac{7}{2}$ | $P_{10,5} = \frac{15}{16}$ | $P_{11,5} = \frac{7}{8}$ | $P_{12,5} = \frac{5}{4}$ | $P_{13,5} = \frac{7}{8}$ | $P_{14,5} = \frac{5}{4}$ | $P_{15,5} = \frac{7}{8}$ | $P_{16,5} = \frac{7}{8}$ | $P_{17,5} = \frac{7}{8}$ | $P_{18,5} = \frac{5}{4}$ | $P_{19,5} = \frac{15}{16}$ | $P_{20,5} = \frac{15}{16}$ | $P_{21,5} = \frac{15}{16}$ | $P_{22,5} = \frac{15}{16}$ | $P_{23,5} = \frac{15}{16}$ |
| | $P_{6,6} = \frac{15}{16}$ | $P_{7,6} = \frac{7}{2}$ | $P_{8,6} = \frac{5}{4}$ | $P_{9,6} = \frac{7}{2}$ | $P_{10,6} = \frac{15}{16}$ | $P_{11,6} = \frac{7}{8}$ | $P_{12,6} = \frac{5}{4}$ | $P_{13,6} = \frac{7}{8}$ | $P_{14,6} = \frac{5}{4}$ | $P_{15,6} = \frac{7}{8}$ | $P_{16,6} = \frac{7}{8}$ | $P_{17,6} = \frac{7}{8}$ | $P_{18,6} = \frac{5}{4}$ | $P_{19,6} = \frac{15}{16}$ | $P_{20,6} = \frac{15}{16}$ | $P_{21,6} = \frac{15}{16}$ | $P_{22,6} = \frac{15}{16}$ | $P_{23,6} = \frac{15}{16}$ | $P_{24,6} = \frac{15}{16}$ |
| | $P_{7,7} = \frac{7}{2}$ | $P_{8,7} = \frac{5}{4}$ | $P_{9,7} = \frac{7}{2}$ | $P_{10,7} = \frac{15}{16}$ | $P_{11,7} = \frac{7}{8}$ | $P_{12,7} = \frac{5}{4}$ | $P_{13,7} = \frac{7}{8}$ | $P_{14,7} = \frac{5}{4}$ | $P_{15,7} = \frac{7}{8}$ | $P_{16,7} = \frac{7}{8}$ | $P_{17,7} = \frac{7}{8}$ | $P_{18,7} = \frac{5}{4}$ | $P_{19,7} = \frac{15}{16}$ | $P_{20,7} = \frac{15}{16}$ | $P_{21,7} = \frac{15}{16}$ | $P_{22,7} = \frac{15}{16}$ | $P_{23,7} = \frac{15}{16}$ | $P_{24,7} = \frac{15}{16}$ | $P_{25,7} = \frac{15}{16}$ |
| | $P_{8,8} = \frac{5}{4}$ | $P_{9,8} = \frac{7}{2}$ | $P_{10,8} = \frac{15}{16}$ | $P_{11,8} = \frac{7}{8}$ | $P_{12,8} = \frac{5}{4}$ | $P_{13,8} = \frac{7}{8}$ | $P_{14,8} = \frac{5}{4}$ | $P_{15,8} = \frac{7}{8}$ | $P_{16,8} = \frac{7}{8}$ | $P_{17,8} = \frac{7}{8}$ | $P_{18,8} = \frac{5}{4}$ | $P_{19,8} = \frac{15}{16}$ | $P_{20,8} = \frac{15}{16}$ | $P_{21,8} = \frac{15}{16}$ | $P_{22,8} = \frac{15}{16}$ | $P_{23,8} = \frac{15}{16}$ | $P_{24,8} = \frac{15}{16}$ | $P_{25,8} = \frac{15}{16}$ | $P_{26,8} = \frac{15}{16}$ |
| | $P_{9,9} = \frac{7}{2}$ | $P_{10,9} = \frac{15}{16}$ | $P_{11,9} = \frac{7}{8$ | | | | | | | | | | | | | | | | |

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$