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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

$$N^{-1} = \begin{pmatrix} p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = \frac{1}{6} & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = \frac{1}{6} & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = \frac{1}{6} & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = \frac{1}{6} & p_{3,6} = \frac{1}{6} & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = \frac{1}{6} & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 1 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = 0 & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = \frac{1}{6} & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = \frac{1}{6} & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = \frac{1}{6} & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = \frac{1}{6} & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 1 & p_{10,11} = 0 & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = 0 & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 1 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 1 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = \frac{1}{6} & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{1$$

$$N = \begin{array}{c} \begin{array}{c} P_{0,0} = 45 \\ P_{0,1} = 34506 \\ P_{0,2} = 10714 \\ P_{0,3} = 249999 \\ P_{0,4} = 519999 \\ P_{0,5} = 1039999 \\ P_{0,6} = 1559999 \\ P_{0,7} = 2079999 \\ P_{0,8} = 2599999 \\ P_{0,9} = 3119999 \\ P_{0,10} = 3639999 \\ P_{0,11} = 4159999 \\ P_{0,12} = 4679999 \\ P_{0,13} = 5199999 \\ P_{0,14} = 5719999 \\ P_{0,15} = 6239999 \\ P_{0,16} = 6759999 \\ P_{0,17} = 7279999 \\ P_{0,18} = 7799999 \\ P_{0,19} = 8319999 \\ P_{0,20} = 8839999 \\ P_{0,21} = 9359999 \\ P_{0,22} = 9879999 \\ P_{0,23} = 10399999 \\ P_{0,24} = 10919999 \\ P_{0,25} = 11439999 \\ P_{0,26} = 11959999 \\ P_{0,27} = 12479999 \\ P_{0,28} = 12999999 \\ P_{0,29} = 13519999 \\ P_{0,30} = 14039999 \\ P_{0,31} = 14559999 \\ P_{0,32} = 15079999 \\ P_{0,33} = 15599999 \\ P_{0,34} = 16119999 \\ P_{0,35} = 16639999 \\ P_{0,36} = 17159999 \\ P_{0,37} = 17679999 \\ P_{0,38} = 18199999 \\ P_{0,39} = 18719999 \\ P_{0,40} = 19239999 \\ P_{0,41} = 19759999 \\ P_{0,42} = 20279999 \\ P_{0,43} = 20799999 \\ P_{0,44} = 21319999 \\ P_{0,45} = 21839999 \\ P_{0,46} = 22359999 \\ P_{0,47} = 22879999 \\ P_{0,48} = 23399999 \\ P_{0,49} = 23919999 \\ P_{0,50} = 24439999 \\ P_{0,51} = 24959999 \\ P_{0,52} = 25479999 \\ P_{0,53} = 25999999 \\ P_{0,54} = 26519999 \\ P_{0,55} = 27039999 \\ P_{0,56} = 27559999 \\ P_{0,57} = 28079999 \\ P_{0,58} = 28599999 \\ P_{0,59} = 29119999 \\ P_{0,60} = 29639999 \\ P_{0,61} = 30159999 \\ P_{0,62} = 30679999 \\ P_{0,63} = 31199999 \\ P_{0,64} = 31719999 \\ P_{0,65} = 32239999 \\ P_{0,66} = 32759999 \\ P_{0,67} = 33279999 \\ P_{0,68} = 33799999 \\ P_{0,69} = 34319999 \\ P_{0,70} = 34839999 \\ P_{0,71} = 35359999 \\ P_{0,72} = 35879999 \\ P_{0,73} = 36399999 \\ P_{0,74} = 36919999 \\ P_{0,75} = 37439999 \\ P_{0,76} = 37959999 \\ P_{0,77} = 38479999 \\ P_{0,78} = 38999999 \\ P_{0,79} = 39519999 \\ P_{0,80} = 40039999 \\ P_{0,81} = 40559999 \\ P_{0,82} = 41079999 \\ P_{0,83} = 41599999 \\ P_{0,84} = 42119999 \\ P_{0,85} = 42639999 \\ P_{0,86} = 43159999 \\ P_{0,87} = 43679999 \\ P_{0,88} = 44199999 \\ P_{0,89} = 44719999 \\ P_{0,90} = 45239999 \\ P_{0,91} = 45759999 \\ P_{0,92} = 46279999 \\ P_{0,93} = 46799999 \\ P_{0,94} = 47319999 \\ P_{0,95} = 47839999 \\ P_{0,96} = 48359999 \\ P_{0,97} = 48879999 \\ P_{0,98} = 49399999 \\ P_{0,99} = 49919999 \end{array} & \begin{array}{c} P_{1,0} = 16 \\ P_{1,1} = 34506 \\ P_{1,2} = 10714 \\ P_{1,3} = 249999 \\ P_{1,4} = 519999 \\ P_{1,5} = 1039999 \\ P_{1,6} = 1559999 \\ P_{1,7} = 2079999 \\ P_{1,8} = 2599999 \\ P_{1,9} = 3119999 \\ P_{1,10} = 3639999 \\ P_{1,11} = 4159999 \\ P_{1,12} = 4679999 \\ P_{1,13} = 5199999 \\ P_{1,14} = 5719999 \\ P_{1,15} = 6239999 \\ P_{1,16} = 6759999 \\ P_{1,17} = 7279999 \\ P_{1,18} = 7799999 \\ P_{1,19} = 8319999 \\ P_{1,20} = 8839999 \\ P_{1,21} = 9359999 \\ P_{1,22} = 9879999 \\ P_{1,23} = 10399999 \\ P_{1,24} = 10919999 \\ P_{1,25} = 11439999 \\ P_{1,26} = 11959999 \\ P_{1,27} = 12479999 \\ P_{1,28} = 12999999 \\ P_{1,29} = 13519999 \\ P_{1,30} = 14039999 \\ P_{1,31} = 14559999 \\ P_{1,32} = 15079999 \\ P_{1,33} = 15599999 \\ P_{1,34} = 16119999 \\ P_{1,35} = 16639999 \\ P_{1,36} = 17159999 \\ P_{1,37} = 17679999 \\ P_{1,38} = 18199999 \\ P_{1,39} = 18719999 \\ P_{1,40} = 19239999 \\ P_{1,41} = 19759999 \\ P_{1,42} = 20279999 \\ P_{1,43} = 20799999 \\ P_{1,44} = 21319999 \\ P_{1,45} = 21839999 \\ P_{1,46} = 22359999 \\ P_{1,47} = 22879999 \\ P_{1,48} = 23399999 \\ P_{1,49} = 23919999 \\ P_{1,50} = 24439999 \\ P_{1,51} = 24959999 \\ P_{1,52} = 25479999 \\ P_{1,53} = 25999999 \\ P_{1,54} = 26519999 \\ P_{1,55} = 27039999 \\ P_{1,56} = 27559999 \\ P_{1,57} = 28079999 \\ P_{1,58} = 28599999 \\ P_{1,59} = 29119999 \\ P_{1,60} = 29639999 \\ P_{1,61} = 30159999 \\ P_{1,62} = 30679999 \\ P_{1,63} = 31199999 \\ P_{1,64} = 31719999 \\ P_{1,65} = 32239999 \\ P_{1,66} = 32759999 \\ P_{1,67} = 33279999 \\ P_{1,68} = 33799999 \\ P_{1,69} = 34319999 \\ P_{1,70} = 34839999 \\ P_{1,71} = 35359999 \\ P_{1,72} = 35879999 \\ P_{1,73} = 36399999 \\ P_{1,74} = 36919999 \\ P_{1,75} = 37439999 \\ P_{1,76} = 37959999 \\ P_{1,77} = 38479999 \\ P_{1,78} = 38999999 \\ P_{1,79} = 39519999 \\ P_{1,80} = 40039999 \\ P_{1,81} = 40559999 \\ P_{1,82} = 41079999 \\ P_{1,83} = 41599999 \\ P_{1,84} = 42119999 \\ P_{1,85} = 42639999 \\ P_{1,86} = 43159999 \\ P_{1,87} = 43679999 \\ P_{1,88} = 44199999 \\ P_{1,89} = 44719999 \\ P_{1,90} = 45239999 \\ P_{1,91} = 45759999 \\ P_{1,92} = 46279999 \\ P_{1,93} = 46799999 \\ P_{1,94} = 47319999 \\ P_{1,95} = 47839999 \\ P_{1,96} = 48359999 \\ P_{1,97} = 48879999 \\ P_{1,98} = 49399999 \\ P_{1,99} = 49919999 \end{array} & \begin{array}{c} P_{2,0} = 7 \\ P_{2,1} = 14325 \\ P_{2,2} = 22485 \\ P_{2,3} = 30645 \\ P_{2,4} = 38805 \\ P_{2,5} = 46965 \\ P_{2,6} = 55125 \\ P_{2,7} = 63285 \\ P_{2,8} = 71445 \\ P_{2,9} = 79605 \\ P_{2,10} = 87765 \\ P_{2,11} = 95925 \\ P_{2,12} = 104085 \\ P_{2,13} = 112245 \\ P_{2,14} = 120405 \\ P_{2,15} = 128565 \\ P_{2,16} = 136725 \\ P_{2,17} = 144885 \\ P_{2,18} = 153045 \\ P_{2,19} = 161205 \\ P_{2,20} = 169365 \\ P_{2,21} = 177525 \\ P_{2,22} = 185685 \\ P_{2,23} = 193845 \\ P_{2,24} = 202005 \\ P_{2,25} = 210165 \\ P_{2,26} = 218325 \\ P_{2,27} = 226485 \\ P_{2,28} = 234645 \\ P_{2,29} = 242805 \\ P_{2,30} = 250965 \\ P_{2,31} = 259125 \\ P_{2,32} = 267285 \\ P_{$$

In order to get the expected number of steps, we find t_0 , where

$$t \equiv N\mathbf{1}$$

Here, $\mathbf{1}$ is a vector whose entries are all 1.

[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$