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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A hexagonal grid of 19 cells, each containing a red dot and a number from 0 to 18. The grid is surrounded by 20 green dots, each labeled with a number from 19 to 38. The grid is arranged in a larger hexagonal shape with 4 cells on the left and right sides, 3 cells on the top and bottom sides, and 2 cells on the top-left and bottom-right corners.

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.



$$N^{-1} = \begin{pmatrix} p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = \frac{1}{6} & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = \frac{1}{6} & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = \frac{1}{6} & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = \frac{1}{6} & p_{3,6} = \frac{1}{6} & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = \frac{1}{6} & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 1 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = 0 & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = \frac{1}{6} & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = \frac{1}{6} & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = \frac{1}{6} & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = \frac{1}{6} & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 0 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 1 & p_{10,11} = 0 & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = 0 & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 1 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 1 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = \frac{1}{6} & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{1$$

|                         |                 |                         |                 |                 |                         |                 |                 |                         |                 |                  |                  |                  |                  |                  |                  |                  |                  |                  |
|-------------------------|-----------------|-------------------------|-----------------|-----------------|-------------------------|-----------------|-----------------|-------------------------|-----------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $P_{0,0} = 45$          | $P_{0,1} = 16$  | $P_{0,2} = 16$          | $P_{0,3} = 16$  | $P_{0,4} = 16$  | $P_{0,5} = 16$          | $P_{0,6} = 16$  | $P_{0,7} = 7$   | $P_{0,8} = \frac{5}{2}$ | $P_{0,9} = 7$   | $P_{0,10} = 16$  | $P_{0,11} = 7$   | $P_{0,12} = 5$   | $P_{0,13} = 7$   | $P_{0,14} = 5$   | $P_{0,15} = 7$   | $P_{0,16} = 5$   | $P_{0,17} = 7$   | $P_{0,18} = 5$   |
| $P_{1,0} = 16$          | $P_{1,1} = 16$  | $P_{1,2} = 16$          | $P_{1,3} = 16$  | $P_{1,4} = 16$  | $P_{1,5} = 16$          | $P_{1,6} = 16$  | $P_{1,7} = 16$  | $P_{1,8} = 16$          | $P_{1,9} = 16$  | $P_{1,10} = 16$  | $P_{1,11} = 16$  | $P_{1,12} = 16$  | $P_{1,13} = 16$  | $P_{1,14} = 16$  | $P_{1,15} = 16$  | $P_{1,16} = 16$  | $P_{1,17} = 16$  | $P_{1,18} = 16$  |
| $P_{2,0} = 16$          | $P_{2,1} = 16$  | $P_{2,2} = 16$          | $P_{2,3} = 16$  | $P_{2,4} = 16$  | $P_{2,5} = 16$          | $P_{2,6} = 16$  | $P_{2,7} = 16$  | $P_{2,8} = 16$          | $P_{2,9} = 16$  | $P_{2,10} = 16$  | $P_{2,11} = 16$  | $P_{2,12} = 16$  | $P_{2,13} = 16$  | $P_{2,14} = 16$  | $P_{2,15} = 16$  | $P_{2,16} = 16$  | $P_{2,17} = 16$  | $P_{2,18} = 16$  |
| $P_{3,0} = 16$          | $P_{3,1} = 16$  | $P_{3,2} = 16$          | $P_{3,3} = 16$  | $P_{3,4} = 16$  | $P_{3,5} = 16$          | $P_{3,6} = 16$  | $P_{3,7} = 16$  | $P_{3,8} = 16$          | $P_{3,9} = 16$  | $P_{3,10} = 16$  | $P_{3,11} = 16$  | $P_{3,12} = 16$  | $P_{3,13} = 16$  | $P_{3,14} = 16$  | $P_{3,15} = 16$  | $P_{3,16} = 16$  | $P_{3,17} = 16$  | $P_{3,18} = 16$  |
| $P_{4,0} = 16$          | $P_{4,1} = 16$  | $P_{4,2} = 16$          | $P_{4,3} = 16$  | $P_{4,4} = 16$  | $P_{4,5} = 16$          | $P_{4,6} = 16$  | $P_{4,7} = 16$  | $P_{4,8} = 16$          | $P_{4,9} = 16$  | $P_{4,10} = 16$  | $P_{4,11} = 16$  | $P_{4,12} = 16$  | $P_{4,13} = 16$  | $P_{4,14} = 16$  | $P_{4,15} = 16$  | $P_{4,16} = 16$  | $P_{4,17} = 16$  | $P_{4,18} = 16$  |
| $P_{5,0} = 16$          | $P_{5,1} = 16$  | $P_{5,2} = 16$          | $P_{5,3} = 16$  | $P_{5,4} = 16$  | $P_{5,5} = 16$          | $P_{5,6} = 16$  | $P_{5,7} = 16$  | $P_{5,8} = 16$          | $P_{5,9} = 16$  | $P_{5,10} = 16$  | $P_{5,11} = 16$  | $P_{5,12} = 16$  | $P_{5,13} = 16$  | $P_{5,14} = 16$  | $P_{5,15} = 16$  | $P_{5,16} = 16$  | $P_{5,17} = 16$  | $P_{5,18} = 16$  |
| $P_{6,0} = 16$          | $P_{6,1} = 16$  | $P_{6,2} = 16$          | $P_{6,3} = 16$  | $P_{6,4} = 16$  | $P_{6,5} = 16$          | $P_{6,6} = 16$  | $P_{6,7} = 16$  | $P_{6,8} = 16$          | $P_{6,9} = 16$  | $P_{6,10} = 16$  | $P_{6,11} = 16$  | $P_{6,12} = 16$  | $P_{6,13} = 16$  | $P_{6,14} = 16$  | $P_{6,15} = 16$  | $P_{6,16} = 16$  | $P_{6,17} = 16$  | $P_{6,18} = 16$  |
| $P_{7,0} = 7$           | $P_{7,1} = 16$  | $P_{7,2} = 7$           | $P_{7,3} = 16$  | $P_{7,4} = 16$  | $P_{7,5} = 7$           | $P_{7,6} = 16$  | $P_{7,7} = 7$   | $P_{7,8} = 16$          | $P_{7,9} = 7$   | $P_{7,10} = 16$  | $P_{7,11} = 7$   | $P_{7,12} = 16$  | $P_{7,13} = 16$  | $P_{7,14} = 16$  | $P_{7,15} = 16$  | $P_{7,16} = 16$  | $P_{7,17} = 7$   | $P_{7,18} = 16$  |
| $P_{8,0} = \frac{5}{2}$ | $P_{8,1} = 16$  | $P_{8,2} = \frac{5}{2}$ | $P_{8,3} = 16$  | $P_{8,4} = 16$  | $P_{8,5} = \frac{5}{2}$ | $P_{8,6} = 16$  | $P_{8,7} = 16$  | $P_{8,8} = 16$          | $P_{8,9} = 16$  | $P_{8,10} = 16$  | $P_{8,11} = 16$  | $P_{8,12} = 16$  | $P_{8,13} = 16$  | $P_{8,14} = 16$  | $P_{8,15} = 16$  | $P_{8,16} = 16$  | $P_{8,17} = 16$  | $P_{8,18} = 16$  |
| $P_{9,0} = 7$           | $P_{9,1} = 16$  | $P_{9,2} = 7$           | $P_{9,3} = 16$  | $P_{9,4} = 16$  | $P_{9,5} = 7$           | $P_{9,6} = 16$  | $P_{9,7} = 7$   | $P_{9,8} = 16$          | $P_{9,9} = 7$   | $P_{9,10} = 16$  | $P_{9,11} = 7$   | $P_{9,12} = 16$  | $P_{9,13} = 16$  | $P_{9,14} = 16$  | $P_{9,15} = 16$  | $P_{9,16} = 16$  | $P_{9,17} = 7$   | $P_{9,18} = 16$  |
| $P_{10,0} = 16$         | $P_{10,1} = 16$ | $P_{10,2} = 16$         | $P_{10,3} = 16$ | $P_{10,4} = 16$ | $P_{10,5} = 16$         | $P_{10,6} = 16$ | $P_{10,7} = 16$ | $P_{10,8} = 16$         | $P_{10,9} = 16$ | $P_{10,10} = 16$ | $P_{10,11} = 16$ | $P_{10,12} = 16$ | $P_{10,13} = 16$ | $P_{10,14} = 16$ | $P_{10,15} = 16$ | $P_{10,16} = 16$ | $P_{10,17} = 16$ | $P_{10,18} = 16$ |
| $P_{11,0} = 7$          | $P_{11,1} = 16$ | $P_{11,2} = 7$          | $P_{11,3} = 16$ | $P_{11,4} = 16$ | $P_{11,5} = 7$          | $P_{11,6} = 16$ |                 |                         |                 |                  |                  |                  |                  |                  |                  |                  |                  |                  |

In order to get the expected number of steps, we find  $t_0$ , where

$$t \equiv N\mathbf{1}$$

Here,  $\mathbf{1}$  is a vector whose entries are all 1.

$$t = \begin{pmatrix} 213 \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \end{pmatrix}$$

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$