

$\mathbb{R}^n$  Bonus Problem #3

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## §1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

## §2 Diagram



## §3 Solution

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

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|                  |                    |                    |                    |                    |                   |                   |                |                |                |                 |                 |                 |                 |                 |                 |                 |                 |                 |
|------------------|--------------------|--------------------|--------------------|--------------------|-------------------|-------------------|----------------|----------------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $P_{0,0} = 45$   | $P_{0,1} = 16$     | $P_{0,2} = 3456$   | $P_{0,3} = 16$     | $P_{0,4} = 106714$ | $P_{0,5} = 16$    | $P_{0,6} = 17222$ | $P_{0,7} = 16$ | $P_{0,8} = 7$  | $P_{0,9} = 5$  | $P_{0,10} = 7$  | $P_{0,11} = 5$  | $P_{0,12} = 7$  | $P_{0,13} = 5$  | $P_{0,14} = 7$  | $P_{0,15} = 5$  | $P_{0,16} = 7$  | $P_{0,17} = 5$  | $P_{0,18} = 7$  |
| $P_{1,0} = 16$   | $P_{1,1} = 3456$   | $P_{1,2} = 16$     | $P_{1,3} = 106714$ | $P_{1,4} = 16$     | $P_{1,5} = 17222$ | $P_{1,6} = 16$    | $P_{1,7} = 7$  | $P_{1,8} = 5$  | $P_{1,9} = 7$  | $P_{1,10} = 5$  | $P_{1,11} = 7$  | $P_{1,12} = 5$  | $P_{1,13} = 7$  | $P_{1,14} = 5$  | $P_{1,15} = 7$  | $P_{1,16} = 5$  | $P_{1,17} = 7$  | $P_{1,18} = 5$  |
| $P_{2,0} = 3456$ | $P_{2,1} = 16$     | $P_{2,2} = 106714$ | $P_{2,3} = 16$     | $P_{2,4} = 17222$  | $P_{2,5} = 16$    | $P_{2,6} = 7$     | $P_{2,7} = 5$  | $P_{2,8} = 7$  | $P_{2,9} = 5$  | $P_{2,10} = 7$  | $P_{2,11} = 5$  | $P_{2,12} = 7$  | $P_{2,13} = 5$  | $P_{2,14} = 7$  | $P_{2,15} = 5$  | $P_{2,16} = 7$  | $P_{2,17} = 5$  | $P_{2,18} = 7$  |
| $P_{3,0} = 16$   | $P_{3,1} = 106714$ | $P_{3,2} = 16$     | $P_{3,3} = 17222$  | $P_{3,4} = 16$     | $P_{3,5} = 7$     | $P_{3,6} = 5$     | $P_{3,7} = 7$  | $P_{3,8} = 5$  | $P_{3,9} = 7$  | $P_{3,10} = 5$  | $P_{3,11} = 7$  | $P_{3,12} = 5$  | $P_{3,13} = 7$  | $P_{3,14} = 5$  | $P_{3,15} = 7$  | $P_{3,16} = 5$  | $P_{3,17} = 7$  | $P_{3,18} = 5$  |
| $P_{4,0} = 16$   | $P_{4,1} = 17222$  | $P_{4,2} = 7$      | $P_{4,3} = 5$      | $P_{4,4} = 7$      | $P_{4,5} = 5$     | $P_{4,6} = 7$     | $P_{4,7} = 5$  | $P_{4,8} = 7$  | $P_{4,9} = 5$  | $P_{4,10} = 7$  | $P_{4,11} = 5$  | $P_{4,12} = 7$  | $P_{4,13} = 5$  | $P_{4,14} = 7$  | $P_{4,15} = 5$  | $P_{4,16} = 7$  | $P_{4,17} = 5$  | $P_{4,18} = 7$  |
| $P_{5,0} = 7$    | $P_{5,1} = 5$      | $P_{5,2} = 7$      | $P_{5,3} = 5$      | $P_{5,4} = 7$      | $P_{5,5} = 5$     | $P_{5,6} = 7$     | $P_{5,7} = 5$  | $P_{5,8} = 7$  | $P_{5,9} = 5$  | $P_{5,10} = 7$  | $P_{5,11} = 5$  | $P_{5,12} = 7$  | $P_{5,13} = 5$  | $P_{5,14} = 7$  | $P_{5,15} = 5$  | $P_{5,16} = 7$  | $P_{5,17} = 5$  | $P_{5,18} = 7$  |
| $P_{6,0} = 5$    | $P_{6,1} = 7$      | $P_{6,2} = 5$      | $P_{6,3} = 7$      | $P_{6,4} = 5$      | $P_{6,5} = 7$     | $P_{6,6} = 5$     | $P_{6,7} = 7$  | $P_{6,8} = 5$  | $P_{6,9} = 7$  | $P_{6,10} = 5$  | $P_{6,11} = 7$  | $P_{6,12} = 5$  | $P_{6,13} = 7$  | $P_{6,14} = 5$  | $P_{6,15} = 7$  | $P_{6,16} = 5$  | $P_{6,17} = 7$  | $P_{6,18} = 5$  |
| $P_{7,0} = 7$    | $P_{7,1} = 5$      | $P_{7,2} = 7$      | $P_{7,3} = 5$      | $P_{7,4} = 7$      | $P_{7,5} = 5$     | $P_{7,6} = 7$     | $P_{7,7} = 5$  | $P_{7,8} = 7$  | $P_{7,9} = 5$  | $P_{7,10} = 7$  | $P_{7,11} = 5$  | $P_{7,12} = 7$  | $P_{7,13} = 5$  | $P_{7,14} = 7$  | $P_{7,15} = 5$  | $P_{7,16} = 7$  | $P_{7,17} = 5$  | $P_{7,18} = 7$  |
| $P_{8,0} = 5$    | $P_{8,1} = 7$      | $P_{8,2} = 5$      | $P_{8,3} = 7$      | $P_{8,4} = 5$      | $P_{8,5} = 7$     | $P_{8,6} = 5$     | $P_{8,7} = 7$  | $P_{8,8} = 5$  | $P_{8,9} = 7$  | $P_{8,10} = 5$  | $P_{8,11} = 7$  | $P_{8,12} = 5$  | $P_{8,13} = 7$  | $P_{8,14} = 5$  | $P_{8,15} = 7$  | $P_{8,16} = 5$  | $P_{8,17} = 7$  | $P_{8,18} = 5$  |
| $P_{9,0} = 7$    | $P_{9,1} = 5$      | $P_{9,2} = 7$      | $P_{9,3} = 5$      | $P_{9,4} = 7$      | $P_{9,5} = 5$     | $P_{9,6} = 7$     | $P_{9,7} = 5$  | $P_{9,8} = 7$  | $P_{9,9} = 5$  | $P_{9,10} = 7$  | $P_{9,11} = 5$  | $P_{9,12} = 7$  | $P_{9,13} = 5$  | $P_{9,14} = 7$  | $P_{9,15} = 5$  | $P_{9,16} = 7$  | $P_{9,17} = 5$  | $P_{9,18} = 7$  |
| $P_{10,0} = 5$   | $P_{10,1} = 7$     | $P_{10,2} = 5$     | $P_{10,3} = 7$     | $P_{10,4} = 5$     | $P_{10,5} = 7$    | $P_{10,6} = 5$    | $P_{10,7} = 7$ | $P_{10,8} = 5$ | $P_{10,9} = 7$ | $P_{10,10} = 5$ | $P_{10,11} = 7$ | $P_{10,12} = 5$ | $P_{10,13} = 7$ | $P_{10,14} = 5$ | $P_{10,15} = 7$ | $P_{10,16} = 5$ | $P_{10,17} = 7$ | $P_{10,18} = 5$ |
| $P_{11,0} = 7$   | $P_{11,1} = 5$     | $P_{11,2} = 7$     | $P_{11,3} = 5$     | $P_{11,4} = 7$     | $P_{11,5} = 5$    | $P_{11,6} = 7$    | $P_{11,7} = 5$ | $P_{11,8} = 7$ | $P_{11,9} = 5$ | $P_{11,10} = 7$ | $P_{11,11} = 5$ | $P_{11,12} = 7$ | $P_{11,13} = 5$ | $P_{$           |                 |                 |                 |                 |

$$t = N\mathbf{1}$$
$$t = \begin{pmatrix} 213 \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{101} \end{pmatrix}$$

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$