$\mathbb{R}^n$ **Bonus Problem #3**

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§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram

§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N.

$$\mathbb{E}(N) = \sum N \, \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N. We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

§3.1 Notation

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

§3.2 Transition Matrix

Now that we've defined some notation, we can write the transition matrix P Because a 37x37 matrix is cumbersome, we combine the states [19, 36] into a

$$P = \begin{pmatrix} p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & \cdots \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

We also write the matrix Q, which doesn't have any absorbing states.

$$Q = \begin{pmatrix} p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & \cdots \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P.

$$N^{-1} = \begin{pmatrix} \cdots \end{pmatrix}$$

(A 20×20 matrix with entries $p_{i,j}$. Row 0: $p_{0,0}=1,\ p_{0,1}=\tfrac{1}{6},\ p_{0,2}=\tfrac{1}{6},\ p_{0,3}=\tfrac{1}{6},\ p_{0,4}=\tfrac{1}{6},\ p_{0,5}=\tfrac{1}{6},\ p_{0,6}=\tfrac{1}{6},\ p_{0,7}=0,\ \ldots,\ p_{0,19}=0$. Remaining rows contain entries equal to 1, $\tfrac{1}{6}$, or 0.)

$$N = \begin{pmatrix} \cdots \end{pmatrix}$$

(A 20×20 matrix with rational entries, e.g. $p_{0,0}=\tfrac{45}{16},\ p_{0,1}=\tfrac{16}{29},\ p_{0,2}=\tfrac{16}{29},\ p_{0,3}=\tfrac{16}{29},\ p_{0,4}=\tfrac{16}{29},\ \ldots$)

In order to get the expected number of steps, we find t_0, where

$$t = N\mathbf{1}$$

Here, $\mathbf{1}$ is a vector whose entries are all 1.

$$t = \begin{pmatrix} \frac{213}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \end{pmatrix}$$

Finally, we see that $t_0 = \boxed{\dfrac{213}{29} \approx 7.345}$