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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

The diagram shows a hexagonal lattice structure. The nodes are numbered as follows:

- Internal nodes (red dots): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.
- External nodes (green dots): 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36.

The nodes are arranged in a honeycomb pattern, with 19 nodes forming the outer boundary and 30 nodes forming the inner core.

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

§3.2 Transition Matrix

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = 0 & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 0 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 0 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 0 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 0 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 0 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = 0 \\ p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = 0 & p_{9,8} = \frac{1}{6} & p_{9,9} = 0 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = 1 \\ p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 0 & p_{10,11} = \frac{1}{6} & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = 0 \\ p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = \frac{1}{6} & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = 0 & p_{11,11} = 0 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = 0 \\ p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 0 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = \frac{1}{6} & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = 0 & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} & p_{14$$

We also write the matrix Q , which doesn't have any absorbing states.

$$Q = \begin{matrix} & \begin{matrix} P_{0,0} = 0 \\ P_{1,0} = \frac{1}{6} \\ P_{2,0} = \frac{1}{6} \\ P_{3,0} = \frac{1}{6} \\ P_{4,0} = \frac{1}{6} \\ P_{5,0} = \frac{1}{6} \\ P_{6,0} = \frac{1}{6} \\ P_{7,0} = 0 \\ P_{8,0} = 0 \\ P_{9,0} = 0 \end{matrix} & \begin{matrix} P_{0,1} = \frac{1}{6} \\ P_{1,1} = 0 \\ P_{2,1} = \frac{1}{6} \\ P_{3,1} = 0 \\ P_{4,1} = 0 \\ P_{5,1} = 0 \\ P_{6,1} = \frac{1}{6} \\ P_{7,1} = \frac{1}{6} \\ P_{8,1} = 0 \\ P_{9,1} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,2} = \frac{1}{6} \\ P_{1,2} = 0 \\ P_{2,2} = 0 \\ P_{3,2} = \frac{1}{6} \\ P_{4,2} = 0 \\ P_{5,2} = 0 \\ P_{6,2} = 0 \\ P_{7,2} = 0 \\ P_{8,2} = 0 \\ P_{9,2} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,3} = \frac{1}{6} \\ P_{1,3} = 0 \\ P_{2,3} = \frac{1}{6} \\ P_{3,3} = 0 \\ P_{4,3} = 0 \\ P_{5,3} = 0 \\ P_{6,3} = 0 \\ P_{7,3} = 0 \\ P_{8,3} = 0 \\ P_{9,3} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,4} = \frac{1}{6} \\ P_{1,4} = 0 \\ P_{2,4} = 0 \\ P_{3,4} = \frac{1}{6} \\ P_{4,4} = 0 \\ P_{5,4} = 0 \\ P_{6,4} = 0 \\ P_{7,4} = 0 \\ P_{8,4} = 0 \\ P_{9,4} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,5} = \frac{1}{6} \\ P_{1,5} = 0 \\ P_{2,5} = 0 \\ P_{3,5} = 0 \\ P_{4,5} = \frac{1}{6} \\ P_{5,5} = 0 \\ P_{6,5} = \frac{1}{6} \\ P_{7,5} = \frac{1}{6} \\ P_{8,5} = 0 \\ P_{9,5} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,6} = \frac{1}{6} \\ P_{1,6} = 0 \\ P_{2,6} = 0 \\ P_{3,6} = 0 \\ P_{4,6} = 0 \\ P_{5,6} = \frac{1}{6} \\ P_{6,6} = 0 \\ P_{7,6} = 0 \\ P_{8,6} = \frac{1}{6} \\ P_{9,6} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,7} = 0 \\ P_{1,7} = \frac{1}{6} \\ P_{2,7} = 0 \\ P_{3,7} = 0 \\ P_{4,7} = 0 \\ P_{5,7} = 0 \\ P_{6,7} = \frac{1}{6} \\ P_{7,7} = 0 \\ P_{8,7} = \frac{1}{6} \\ P_{9,7} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,8} = 0 \\ P_{1,8} = \frac{1}{6} \\ P_{2,8} = 0 \\ P_{3,8} = 0 \\ P_{4,8} = 0 \\ P_{5,8} = 0 \\ P_{6,8} = 0 \\ P_{7,8} = \frac{1}{6} \\ P_{8,8} = 0 \\ P_{9,8} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,9} = 0 \\ P_{1,9} = \frac{1}{6} \\ P_{2,9} = \frac{1}{6} \\ P_{3,9} = 0 \\ P_{4,9} = 0 \\ P_{5,9} = 0 \\ P_{6,9} = 0 \\ P_{7,9} = 0 \\ P_{8,9} = \frac{1}{6} \\ P_{9,9} = 0 \end{matrix} & \begin{matrix} P_{0,10} = 0 \\ P_{1,10} = \frac{1}{6} \\ P_{2,10} = \frac{1}{6} \\ P_{3,10} = 0 \\ P_{4,10} = 0 \\ P_{5,10} = 0 \\ P_{6,10} = 0 \\ P_{7,10} = 0 \\ P_{8,10} = 0 \\ P_{9,10} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,11} = 0 \\ P_{1,11} = 0 \\ P_{2,11} = \frac{1}{6} \\ P_{3,11} = \frac{1}{6} \\ P_{4,11} = 0 \\ P_{5,11} = 0 \\ P_{6,11} = 0 \\ P_{7,11} = 0 \\ P_{8,11} = 0 \\ P_{9,11} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,12} = 0 \\ P_{1,12} = 0 \\ P_{2,12} = 0 \\ P_{3,12} = \frac{1}{6} \\ P_{4,12} = \frac{1}{6} \\ P_{5,12} = 0 \\ P_{6,12} = 0 \\ P_{7,12} = 0 \\ P_{8,12} = 0 \\ P_{9,12} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,13} = 0 \\ P_{1,13} = 0 \\ P_{2,13} = 0 \\ P_{3,13} = \frac{1}{6} \\ P_{4,13} = \frac{1}{6} \\ P_{5,13} = 0 \\ P_{6,13} = 0 \\ P_{7,13} = 0 \\ P_{8,13} = 0 \\ P_{9,13} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,14} = 0 \\ P_{1,14} = 0 \\ P_{2,14} = 0 \\ P_{3,14} = 0 \\ P_{4,14} = \frac{1}{6} \\ P_{5,14} = 0 \\ P_{6,14} = 0 \\ P_{7,14} = 0 \\ P_{8,14} = 0 \\ P_{9,14} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,15} = 0 \\ P_{1,15} = 0 \\ P_{2,15} = 0 \\ P_{3,15} = 0 \\ P_{4,15} = \frac{1}{6} \\ P_{5,15} = \frac{1}{6} \\ P_{6,15} = 0 \\ P_{7,15} = 0 \\ P_{8,15} = 0 \\ P_{9,15} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,16} = 0 \\ P_{1,16} = 0 \\ P_{2,16} = 0 \\ P_{3,16} = 0 \\ P_{4,16} = 0 \\ P_{5,16} = \frac{1}{6} \\ P_{6,16} = 0 \\ P_{7,16} = 0 \\ P_{8,16} = 0 \\ P_{9,16} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,17} = 0 \\ P_{1,17} = 0 \\ P_{2,17} = 0 \\ P_{3,17} = 0 \\ P_{4,17} = 0 \\ P_{5,17} = \frac{1}{6} \\ P_{6,17} = 0 \\ P_{7,17} = 0 \\ P_{8,17} = 0 \\ P_{9,17} = \frac{1}{6} \end{matrix} & \begin{matrix} P_{0,18} = 0 \\ P_{1,18} = 0 \\ P_{2,18} = 0 \\ P_{3,18} = 0 \\ P_{4,18} = 0 \\ P_{5,18} = \frac{1}{6} \\ P_{6,18} = 0 \\ P_{7,18} = 0 \\ P_{8,18} = 0 \\ P_{9,18} = \frac{1}{6} \end{matrix} \end{matrix}$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P .

[illegible]

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$