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~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

A diagram of a hexagonal lattice with 37 cells. The central cell is labeled 0 and has a red dot. Cells 1 through 14 are labeled with black numbers and have red dots. Cells 15 through 36 are labeled with black numbers and have green dots. The lattice is surrounded by 21 green dots labeled 19 through 36, 38 through 40, and 42 through 44.

We wish to find the expected value of the number of turns in the game, which we denote N .

The dice is truly random, so there is no upper bound on N . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

Now that we've defined some notation, we can write the transition matrix P . Because a 37×37 matrix is cumbersome, we combine the states $[19, 36]$ into a

$$P = \begin{pmatrix} p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\ p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\ p_{2,0}=0 & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\ p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\ p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=\frac{1}{6} & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\ p_{5,0}=0 & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\ p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} & p_{6,19}=0 \\ p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=0 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} & p_{7,19}=\frac{1}{6} \\ p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=0 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 & p_{8,19}=\frac{1}{6} \\ p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=0 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & 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& p_{12,18}=0 & p_{12,19}=\frac{1}{6} \\ p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=\frac{1}{6} & p_{13,12}=\frac{1}{6} & p_{13,13}=0 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 & p_{13,19}=\frac{1}{6} \\ p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=\frac{1}{6} & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 & p_{14,19}=\frac{1}{6} \\ p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=0 & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=\frac{1}{6} & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 & p_{15,19}=\frac{1}{6} \\ p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{$$

We also write the matrix Q , which doesn't have any absorbing states.

$$Q = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 \\ p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = \frac{1}{6} & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = 0 & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = \frac{1}{6} & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 \\ p_{6,0} = 0 & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 0 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} \\ p_{7,0} = 0 & p_{7,1} = 0 & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 0 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = 0 & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 0 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 \\ p_{9,0} = 0 & p_{9,1} = 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\frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 0 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 \\ p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 0 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 \\ p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = 0 & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} & p_{14,14} = \frac{1}{6} & p_{14,15} = \frac{1}{6} & p_{14,16} = 0 & p_{14,17} = 0 & p_{14,18} = 0 \\ p_{15,0} = 0 & p_{15,1} = 0 & p_{15,2} = 0 & p_{15,3} = 0 & p_{15,4} = \frac{1}{6} & p_{15,5} = 0 & p_{15,6} = 0 & p_{15,7} = 0 & p_{15,8} = 0 & p_{15,$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P .

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|-----------------|-----------------|-------------------|-----------------|---------------------|-----------------|--------------------|-----------------|----------------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $P_{0,0} = 45$ | $P_{0,1} = 16$ | $P_{0,2} = 3456$ | $P_{0,3} = 16$ | $P_{0,4} = 106714$ | $P_{0,5} = 16$ | $P_{0,6} = 10722$ | $P_{0,7} = 16$ | $P_{0,8} = 7$ | $P_{0,9} = 5$ | $P_{0,10} = 7$ | $P_{0,11} = 5$ | $P_{0,12} = 7$ | $P_{0,13} = 5$ | $P_{0,14} = 7$ | $P_{0,15} = 5$ | $P_{0,16} = 7$ | $P_{0,17} = 5$ | $P_{0,18} = 7$ |
| $P_{1,0} = 16$ | $P_{1,1} = 16$ | $P_{1,2} = 3456$ | $P_{1,3} = 16$ | $P_{1,4} = 106714$ | $P_{1,5} = 16$ | $P_{1,6} = 10722$ | $P_{1,7} = 16$ | $P_{1,8} = 7$ | $P_{1,9} = 5$ | $P_{1,10} = 7$ | $P_{1,11} = 5$ | $P_{1,12} = 7$ | $P_{1,13} = 5$ | $P_{1,14} = 7$ | $P_{1,15} = 5$ | $P_{1,16} = 7$ | $P_{1,17} = 5$ | $P_{1,18} = 7$ |
| $P_{2,0} = 16$ | $P_{2,1} = 16$ | $P_{2,2} = 3456$ | $P_{2,3} = 16$ | $P_{2,4} = 106714$ | $P_{2,5} = 16$ | $P_{2,6} = 10722$ | $P_{2,7} = 16$ | $P_{2,8} = 7$ | $P_{2,9} = 5$ | $P_{2,10} = 7$ | $P_{2,11} = 5$ | $P_{2,12} = 7$ | $P_{2,13} = 5$ | $P_{2,14} = 7$ | $P_{2,15} = 5$ | $P_{2,16} = 7$ | $P_{2,17} = 5$ | $P_{2,18} = 7$ |
| $P_{3,0} = 16$ | $P_{3,1} = 16$ | $P_{3,2} = 3456$ | $P_{3,3} = 16$ | $P_{3,4} = 106714$ | $P_{3,5} = 16$ | $P_{3,6} = 10722$ | $P_{3,7} = 16$ | $P_{3,8} = 7$ | $P_{3,9} = 5$ | $P_{3,10} = 7$ | $P_{3,11} = 5$ | $P_{3,12} = 7$ | $P_{3,13} = 5$ | $P_{3,14} = 7$ | $P_{3,15} = 5$ | $P_{3,16} = 7$ | $P_{3,17} = 5$ | $P_{3,18} = 7$ |
| $P_{4,0} = 16$ | $P_{4,1} = 16$ | $P_{4,2} = 3456$ | $P_{4,3} = 16$ | $P_{4,4} = 106714$ | $P_{4,5} = 16$ | $P_{4,6} = 10722$ | $P_{4,7} = 16$ | $P_{4,8} = 7$ | $P_{4,9} = 5$ | $P_{4,10} = 7$ | $P_{4,11} = 5$ | $P_{4,12} = 7$ | $P_{4,13} = 5$ | $P_{4,14} = 7$ | $P_{4,15} = 5$ | $P_{4,16} = 7$ | $P_{4,17} = 5$ | $P_{4,18} = 7$ |
| $P_{5,0} = 16$ | $P_{5,1} = 16$ | $P_{5,2} = 3456$ | $P_{5,3} = 16$ | $P_{5,4} = 106714$ | $P_{5,5} = 16$ | $P_{5,6} = 10722$ | $P_{5,7} = 16$ | $P_{5,8} = 7$ | $P_{5,9} = 5$ | $P_{5,10} = 7$ | $P_{5,11} = 5$ | $P_{5,12} = 7$ | $P_{5,13} = 5$ | $P_{5,14} = 7$ | $P_{5,15} = 5$ | $P_{5,16} = 7$ | $P_{5,17} = 5$ | $P_{5,18} = 7$ |
| $P_{6,0} = 16$ | $P_{6,1} = 16$ | $P_{6,2} = 3456$ | $P_{6,3} = 16$ | $P_{6,4} = 106714$ | $P_{6,5} = 16$ | $P_{6,6} = 10722$ | $P_{6,7} = 16$ | $P_{6,8} = 7$ | $P_{6,9} = 5$ | $P_{6,10} = 7$ | $P_{6,11} = 5$ | $P_{6,12} = 7$ | $P_{6,13} = 5$ | $P_{6,14} = 7$ | $P_{6,15} = 5$ | $P_{6,16} = 7$ | $P_{6,17} = 5$ | $P_{6,18} = 7$ |
| $P_{7,0} = 16$ | $P_{7,1} = 16$ | $P_{7,2} = 3456$ | $P_{7,3} = 16$ | $P_{7,4} = 106714$ | $P_{7,5} = 16$ | $P_{7,6} = 10722$ | $P_{7,7} = 16$ | $P_{7,8} = 7$ | $P_{7,9} = 5$ | $P_{7,10} = 7$ | $P_{7,11} = 5$ | $P_{7,12} = 7$ | $P_{7,13} = 5$ | $P_{7,14} = 7$ | $P_{7,15} = 5$ | $P_{7,16} = 7$ | $P_{7,17} = 5$ | $P_{7,18} = 7$ |
| $P_{8,0} = 16$ | $P_{8,1} = 16$ | $P_{8,2} = 3456$ | $P_{8,3} = 16$ | $P_{8,4} = 106714$ | $P_{8,5} = 16$ | $P_{8,6} = 10722$ | $P_{8,7} = 16$ | $P_{8,8} = 7$ | $P_{8,9} = 5$ | $P_{8,10} = 7$ | $P_{8,11} = 5$ | $P_{8,12} = 7$ | $P_{8,13} = 5$ | $P_{8,14} = 7$ | $P_{8,15} = 5$ | $P_{8,16} = 7$ | $P_{8,17} = 5$ | $P_{8,18} = 7$ |
| $P_{9,0} = 16$ | $P_{9,1} = 16$ | $P_{9,2} = 3456$ | $P_{9,3} = 16$ | $P_{9,4} = 106714$ | $P_{9,5} = 16$ | $P_{9,6} = 10722$ | $P_{9,7} = 16$ | $P_{9,8} = 7$ | $P_{9,9} = 5$ | $P_{9,10} = 7$ | $P_{9,11} = 5$ | $P_{9,12} = 7$ | $P_{9,13} = 5$ | $P_{9,14} = 7$ | $P_{9,15} = 5$ | $P_{9,16} = 7$ | $P_{9,17} = 5$ | $P_{9,18} = 7$ |
| $P_{10,0} = 16$ | $P_{10,1} = 16$ | $P_{10,2} = 3456$ | $P_{10,3} = 16$ | $P_{10,4} = 106714$ | $P_{10,5} = 16$ | $P_{10,6} = 10722$ | $P_{10,7} = 16$ | $P_{10,8} = 7$ | $P_{10,9} = 5$ | $P_{10,10} = 7$ | $P_{10,11} = 5$ | $P_{10,12} = 7$ | $P_{10,13} = 5$ | $P_{10,14} = 7$ | $P_{10,15} = 5$ | $P_{10,16} = 7$ | $P_{10,17} = 5$ | $P_{10,18} = 7$ |
| $P_{11,0} = 16$ | $P_{11,1} = 16$ | $P_{11,2} = 3456$ | $P_{11,3} = 16$ | $P_{11,4} = 106714$ | $P_{11,5} = 16$ | $P_{11,6} = 10722$ | | | | | | | | | | | | |

$$t \equiv N\mathbf{1}$$
$$t = \begin{pmatrix} 213 \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{184} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{124} \\ \frac{29}{101} \\ \frac{29}{101} \end{pmatrix}$$

Finally, we see that $t_0 = \boxed{\frac{213}{29} \approx 7.345}$