$\mathbb{R}^n$ Bonus Problem #3

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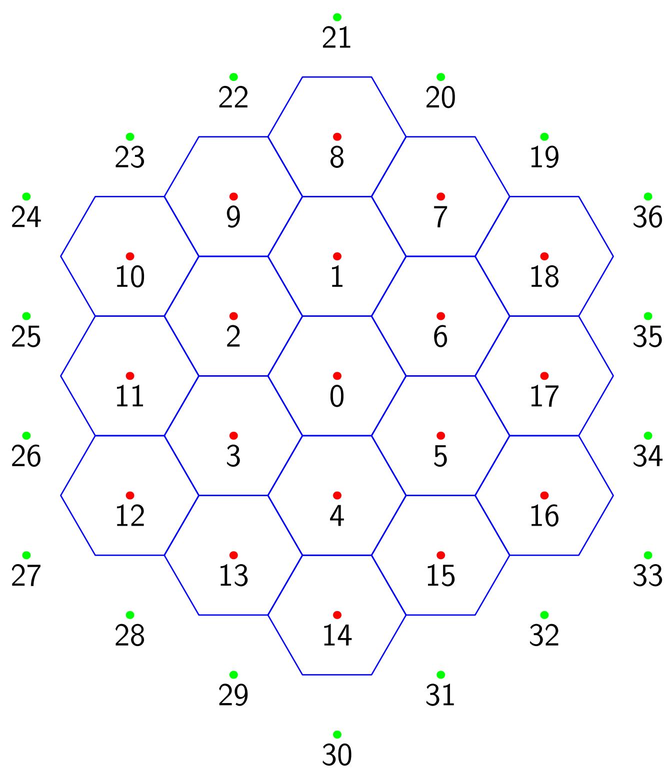
§1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

§2 Diagram



§3 Solution

We wish to find the expected value of the number of turns in the game, which we denote N.

$$\mathbb{E}(N) = \sum N \; \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on N. We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

§3.1 Notation

Let $X_i \in [0, 36]$ be the current state, or position of the traveler. The traveler always starts at position $X_0 = 0$. The final state must be $X_N \in [19, 36]$.

§3.2 Transition Matrix

Now that we've defined some notation, we can write the transition matrix P Because a 37x37 matrix is cumbersome, we combine the states [19, 36] into a

$$P = \begin{pmatrix}
p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 & p_{0,19}=0 \\
p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 & p_{1,19}=0 \\
p_{2,0}=\frac{1}{6} & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 & p_{2,19}=0 \\
p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 & p_{3,19}=0 \\
p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=0 & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 & p_{4,19}=0 \\
p_{5,0}=\frac{1}{6} & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 & p_{5,19}=0 \\
p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} & p_{6,19}=0 \\
p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=0 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} & p_{7,19}=\frac{1}{3} \\
p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=0 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 & p_{8,19}=\frac{1}{2} \\
p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=0 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & p_{9,14}=0 & p_{9,15}=0 & p_{9,16}=0 & p_{9,17}=0 & p_{9,18}=0 & p_{9,19}=\frac{1}{3} \\
p_{10,0}=0 & p_{10,1}=0 & p_{10,2}=\frac{1}{6} & p_{10,3}=0 & p_{10,4}=0 & p_{10,5}=0 & p_{10,6}=0 & p_{10,7}=0 & p_{10,8}=0 & p_{10,9}=\frac{1}{6} & p_{10,10}=0 & p_{10,11}=\frac{1}{6} & p_{10,12}=0 & p_{10,13}=0 & p_{10,14}=0 & p_{10,15}=0 & p_{10,16}=0 & p_{10,17}=0 & p_{10,18}=0 & p_{10,19}=\frac{1}{2} \\
p_{11,0}=0 & p_{11,1}=0 & p_{11,2}=\frac{1}{6} & p_{11,3}=\frac{1}{6} & p_{11,4}=0 & p_{11,5}=0 & p_{11,6}=0 & p_{11,7}=0 & p_{11,8}=0 & p_{11,9}=0 & p_{11,10}=\frac{1}{6} & p_{11,11}=0 & p_{11,12}=\frac{1}{6} & p_{11,13}=0 & p_{11,14}=0 & p_{11,15}=0 & p_{11,16}=0 & p_{11,17}=0 & p_{11,18}=0 & p_{11,19}=\frac{1}{3} \\
p_{12,0}=0 & p_{12,1}=0 & p_{12,2}=0 & p_{12,3}=\frac{1}{6} & p_{12,4}=0 & p_{12,5}=0 & p_{12,6}=0 & p_{12,7}=0 & p_{12,8}=0 & p_{12,9}=0 & p_{12,10}=0 & p_{12,11}=\frac{1}{6} & p_{12,12}=0 & p_{12,13}=\frac{1}{6} & p_{12,14}=0 & p_{12,15}=0 & p_{12,16}=0 & p_{12,17}=0 & p_{12,18}=0 & p_{12,19}=\frac{1}{2} \\
p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=0 & p_{13,12}=\frac{1}{6} & p_{13,13}=0 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 & p_{13,19}=\frac{1}{3} \\
p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=0 & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 & p_{14,19}=\frac{1}{2} \\
p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=0 & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=0 & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 & p_{15,19}=\frac{1}{3} \\
p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=0 & p_{16,11}=0 & p_{16,12}=0 & p_{16,13}=0 & p_{16,14}=0 & p_{16,15}=\frac{1}{6} & p_{16,16}=0 & p_{16,17}=\frac{1}{6} & p_{16,18}=0 & p_{16,19}=\frac{1}{2} \\
p_{17,0}=0 & p_{17,1}=0 & p_{17,2}=0 & p_{17,3}=0 & p_{17,4}=0 & p_{17,5}=\frac{1}{6} & p_{17,6}=\frac{1}{6} & p_{17,7}=0 & p_{17,8}=0 & p_{17,9}=0 & p_{17,10}=0 & p_{17,11}=0 & p_{17,12}=0 & p_{17,13}=0 & p_{17,14}=0 & p_{17,15}=0 & p_{17,16}=\frac{1}{6} & p_{17,17}=0 & p_{17,18}=\frac{1}{6} & p_{17,19}=\frac{1}{3} \\
p_{18,0}=0 & p_{18,1}=0 & p_{18,2}=0 & p_{18,3}=0 & p_{18,4}=0 & p_{18,5}=0 & p_{18,6}=\frac{1}{6} & p_{18,7}=\frac{1}{6} & p_{18,8}=0 & p_{18,9}=0 & p_{18,10}=0 & p_{18,11}=0 & p_{18,12}=0 & p_{18,13}=0 & p_{18,14}=0 & p_{18,15}=0 & p_{18,16}=0 & p_{18,17}=\frac{1}{6} & p_{18,18}=0 & p_{18,19}=\frac{1}{2} \\
p_{19,0}=0 & p_{19,1}=0 & p_{19,2}=0 & p_{19,3}=0 & p_{19,4}=0 & p_{19,5}=0 & p_{19,6}=0 & p_{19,7}=0 & p_{19,8}=0 & p_{19,9}=0 & p_{19,10}=0 & p_{19,11}=0 & p_{19,12}=0 & p_{19,13}=0 & p_{19,14}=0 & p_{19,15}=0 & p_{19,16}=0 & p_{19,17}=0 & p_{19,18}=0 & p_{19,19}=1
\end{pmatrix}$$

We also write the matrix Q, which doesn't have any absorbing states.

$$Q = \begin{pmatrix}
p_{0,0}=0 & p_{0,1}=\frac{1}{6} & p_{0,2}=\frac{1}{6} & p_{0,3}=\frac{1}{6} & p_{0,4}=\frac{1}{6} & p_{0,5}=\frac{1}{6} & p_{0,6}=\frac{1}{6} & p_{0,7}=0 & p_{0,8}=0 & p_{0,9}=0 & p_{0,10}=0 & p_{0,11}=0 & p_{0,12}=0 & p_{0,13}=0 & p_{0,14}=0 & p_{0,15}=0 & p_{0,16}=0 & p_{0,17}=0 & p_{0,18}=0 \\
p_{1,0}=\frac{1}{6} & p_{1,1}=0 & p_{1,2}=\frac{1}{6} & p_{1,3}=0 & p_{1,4}=0 & p_{1,5}=0 & p_{1,6}=\frac{1}{6} & p_{1,7}=\frac{1}{6} & p_{1,8}=\frac{1}{6} & p_{1,9}=\frac{1}{6} & p_{1,10}=0 & p_{1,11}=0 & p_{1,12}=0 & p_{1,13}=0 & p_{1,14}=0 & p_{1,15}=0 & p_{1,16}=0 & p_{1,17}=0 & p_{1,18}=0 \\
p_{2,0}=\frac{1}{6} & p_{2,1}=\frac{1}{6} & p_{2,2}=0 & p_{2,3}=\frac{1}{6} & p_{2,4}=0 & p_{2,5}=0 & p_{2,6}=0 & p_{2,7}=0 & p_{2,8}=0 & p_{2,9}=\frac{1}{6} & p_{2,10}=\frac{1}{6} & p_{2,11}=\frac{1}{6} & p_{2,12}=0 & p_{2,13}=0 & p_{2,14}=0 & p_{2,15}=0 & p_{2,16}=0 & p_{2,17}=0 & p_{2,18}=0 \\
p_{3,0}=\frac{1}{6} & p_{3,1}=0 & p_{3,2}=\frac{1}{6} & p_{3,3}=0 & p_{3,4}=\frac{1}{6} & p_{3,5}=0 & p_{3,6}=0 & p_{3,7}=0 & p_{3,8}=0 & p_{3,9}=0 & p_{3,10}=0 & p_{3,11}=\frac{1}{6} & p_{3,12}=\frac{1}{6} & p_{3,13}=\frac{1}{6} & p_{3,14}=0 & p_{3,15}=0 & p_{3,16}=0 & p_{3,17}=0 & p_{3,18}=0 \\
p_{4,0}=\frac{1}{6} & p_{4,1}=0 & p_{4,2}=0 & p_{4,3}=\frac{1}{6} & p_{4,4}=0 & p_{4,5}=\frac{1}{6} & p_{4,6}=0 & p_{4,7}=0 & p_{4,8}=0 & p_{4,9}=0 & p_{4,10}=0 & p_{4,11}=0 & p_{4,12}=0 & p_{4,13}=\frac{1}{6} & p_{4,14}=\frac{1}{6} & p_{4,15}=\frac{1}{6} & p_{4,16}=0 & p_{4,17}=0 & p_{4,18}=0 \\
p_{5,0}=\frac{1}{6} & p_{5,1}=0 & p_{5,2}=0 & p_{5,3}=0 & p_{5,4}=\frac{1}{6} & p_{5,5}=0 & p_{5,6}=\frac{1}{6} & p_{5,7}=0 & p_{5,8}=0 & p_{5,9}=0 & p_{5,10}=0 & p_{5,11}=0 & p_{5,12}=0 & p_{5,13}=0 & p_{5,14}=0 & p_{5,15}=\frac{1}{6} & p_{5,16}=\frac{1}{6} & p_{5,17}=\frac{1}{6} & p_{5,18}=0 \\
p_{6,0}=\frac{1}{6} & p_{6,1}=\frac{1}{6} & p_{6,2}=0 & p_{6,3}=0 & p_{6,4}=0 & p_{6,5}=\frac{1}{6} & p_{6,6}=0 & p_{6,7}=\frac{1}{6} & p_{6,8}=0 & p_{6,9}=0 & p_{6,10}=0 & p_{6,11}=0 & p_{6,12}=0 & p_{6,13}=0 & p_{6,14}=0 & p_{6,15}=0 & p_{6,16}=0 & p_{6,17}=\frac{1}{6} & p_{6,18}=\frac{1}{6} \\
p_{7,0}=0 & p_{7,1}=\frac{1}{6} & p_{7,2}=0 & p_{7,3}=0 & p_{7,4}=0 & p_{7,5}=0 & p_{7,6}=\frac{1}{6} & p_{7,7}=0 & p_{7,8}=\frac{1}{6} & p_{7,9}=0 & p_{7,10}=0 & p_{7,11}=0 & p_{7,12}=0 & p_{7,13}=0 & p_{7,14}=0 & p_{7,15}=0 & p_{7,16}=0 & p_{7,17}=0 & p_{7,18}=\frac{1}{6} \\
p_{8,0}=0 & p_{8,1}=\frac{1}{6} & p_{8,2}=0 & p_{8,3}=0 & p_{8,4}=0 & p_{8,5}=0 & p_{8,6}=0 & p_{8,7}=\frac{1}{6} & p_{8,8}=0 & p_{8,9}=\frac{1}{6} & p_{8,10}=0 & p_{8,11}=0 & p_{8,12}=0 & p_{8,13}=0 & p_{8,14}=0 & p_{8,15}=0 & p_{8,16}=0 & p_{8,17}=0 & p_{8,18}=0 \\
p_{9,0}=0 & p_{9,1}=\frac{1}{6} & p_{9,2}=\frac{1}{6} & p_{9,3}=0 & p_{9,4}=0 & p_{9,5}=0 & p_{9,6}=0 & p_{9,7}=0 & p_{9,8}=\frac{1}{6} & p_{9,9}=0 & p_{9,10}=\frac{1}{6} & p_{9,11}=0 & p_{9,12}=0 & p_{9,13}=0 & p_{9,14}=0 & p_{9,15}=0 & p_{9,16}=0 & p_{9,17}=0 & p_{9,18}=0 \\
p_{10,0}=0 & p_{10,1}=0 & p_{10,2}=\frac{1}{6} & p_{10,3}=0 & p_{10,4}=0 & p_{10,5}=0 & p_{10,6}=0 & p_{10,7}=0 & p_{10,8}=0 & p_{10,9}=\frac{1}{6} & p_{10,10}=0 & p_{10,11}=\frac{1}{6} & p_{10,12}=0 & p_{10,13}=0 & p_{10,14}=0 & p_{10,15}=0 & p_{10,16}=0 & p_{10,17}=0 & p_{10,18}=0 \\
p_{11,0}=0 & p_{11,1}=0 & p_{11,2}=\frac{1}{6} & p_{11,3}=\frac{1}{6} & p_{11,4}=0 & p_{11,5}=0 & p_{11,6}=0 & p_{11,7}=0 & p_{11,8}=0 & p_{11,9}=0 & p_{11,10}=\frac{1}{6} & p_{11,11}=0 & p_{11,12}=\frac{1}{6} & p_{11,13}=0 & p_{11,14}=0 & p_{11,15}=0 & p_{11,16}=0 & p_{11,17}=0 & p_{11,18}=0 \\
p_{12,0}=0 & p_{12,1}=0 & p_{12,2}=0 & p_{12,3}=\frac{1}{6} & p_{12,4}=0 & p_{12,5}=0 & p_{12,6}=0 & p_{12,7}=0 & p_{12,8}=0 & p_{12,9}=0 & p_{12,10}=0 & p_{12,11}=\frac{1}{6} & p_{12,12}=0 & p_{12,13}=\frac{1}{6} & p_{12,14}=0 & p_{12,15}=0 & p_{12,16}=0 & p_{12,17}=0 & p_{12,18}=0 \\
p_{13,0}=0 & p_{13,1}=0 & p_{13,2}=0 & p_{13,3}=\frac{1}{6} & p_{13,4}=\frac{1}{6} & p_{13,5}=0 & p_{13,6}=0 & p_{13,7}=0 & p_{13,8}=0 & p_{13,9}=0 & p_{13,10}=0 & p_{13,11}=0 & p_{13,12}=\frac{1}{6} & p_{13,13}=0 & p_{13,14}=\frac{1}{6} & p_{13,15}=0 & p_{13,16}=0 & p_{13,17}=0 & p_{13,18}=0 \\
p_{14,0}=0 & p_{14,1}=0 & p_{14,2}=0 & p_{14,3}=0 & p_{14,4}=\frac{1}{6} & p_{14,5}=0 & p_{14,6}=0 & p_{14,7}=0 & p_{14,8}=0 & p_{14,9}=0 & p_{14,10}=0 & p_{14,11}=0 & p_{14,12}=0 & p_{14,13}=\frac{1}{6} & p_{14,14}=0 & p_{14,15}=\frac{1}{6} & p_{14,16}=0 & p_{14,17}=0 & p_{14,18}=0 \\
p_{15,0}=0 & p_{15,1}=0 & p_{15,2}=0 & p_{15,3}=0 & p_{15,4}=\frac{1}{6} & p_{15,5}=\frac{1}{6} & p_{15,6}=0 & p_{15,7}=0 & p_{15,8}=0 & p_{15,9}=0 & p_{15,10}=0 & p_{15,11}=0 & p_{15,12}=0 & p_{15,13}=0 & p_{15,14}=\frac{1}{6} & p_{15,15}=0 & p_{15,16}=\frac{1}{6} & p_{15,17}=0 & p_{15,18}=0 \\
p_{16,0}=0 & p_{16,1}=0 & p_{16,2}=0 & p_{16,3}=0 & p_{16,4}=0 & p_{16,5}=\frac{1}{6} & p_{16,6}=0 & p_{16,7}=0 & p_{16,8}=0 & p_{16,9}=0 & p_{16,10}=0 & p_{16,11}=0 & p_{16,12}=0 & p_{16,13}=0 & p_{16,14}=0 & p_{16,15}=\frac{1}{6} & p_{16,16}=0 & p_{16,17}=\frac{1}{6} & p_{16,18}=0 \\
p_{17,0}=0 & p_{17,1}=0 & p_{17,2}=0 & p_{17,3}=0 & p_{17,4}=0 & p_{17,5}=\frac{1}{6} & p_{17,6}=\frac{1}{6} & p_{17,7}=0 & p_{17,8}=0 & p_{17,9}=0 & p_{17,10}=0 & p_{17,11}=0 & p_{17,12}=0 & p_{17,13}=0 & p_{17,14}=0 & p_{17,15}=0 & p_{17,16}=\frac{1}{6} & p_{17,17}=0 & p_{17,18}=\frac{1}{6} \\
p_{18,0}=0 & p_{18,1}=0 & p_{18,2}=0 & p_{18,3}=0 & p_{18,4}=0 & p_{18,5}=0 & p_{18,6}=\frac{1}{6} & p_{18,7}=\frac{1}{6} & p_{18,8}=0 & p_{18,9}=0 & p_{18,10}=0 & p_{18,11}=0 & p_{18,12}=0 & p_{18,13}=0 & p_{18,14}=0 & p_{18,15}=0 & p_{18,16}=0 & p_{18,17}=\frac{1}{6} & p_{18,18}=0
\end{pmatrix}$$

$N = (I - Q)^{-1}$ is known as the fundamental matrix of P.

$$N^{-1} = \begin{pmatrix}
p_{0,0} = 1 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\
p_{1,0} = \frac{1}{6} & p_{1,1} = 1 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\
p_{2,0} = \frac{1}{6} & p_{2,1} = \frac{1}{6} & p_{2,2} = 1 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\
p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 1 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\
p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 1 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\
p_{5,0} = \frac{1}{6} & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 1 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\
p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 1 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\
p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 1 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{3} \\
p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 1 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & p_{8,12} = 0 & p_{8,13} = 0 & p_{8,14} = 0 & p_{8,15} = 0 & p_{8,16} = 0 & p_{8,17} = 0 & p_{8,18} = 0 & p_{8,19} = \frac{1}{2} \\
p_{9,0} = 0 & p_{9,1} = \frac{1}{6} & p_{9,2} = \frac{1}{6} & p_{9,3} = 0 & p_{9,4} = 0 & p_{9,5} = 0 & p_{9,6} = 0 & p_{9,7} = 0 & p_{9,8} = \frac{1}{6} & p_{9,9} = 1 & p_{9,10} = \frac{1}{6} & p_{9,11} = 0 & p_{9,12} = 0 & p_{9,13} = 0 & p_{9,14} = 0 & p_{9,15} = 0 & p_{9,16} = 0 & p_{9,17} = 0 & p_{9,18} = 0 & p_{9,19} = \frac{1}{3} \\
p_{10,0} = 0 & p_{10,1} = 0 & p_{10,2} = \frac{1}{6} & p_{10,3} = 0 & p_{10,4} = 0 & p_{10,5} = 0 & p_{10,6} = 0 & p_{10,7} = 0 & p_{10,8} = 0 & p_{10,9} = \frac{1}{6} & p_{10,10} = 1 & p_{10,11} = \frac{1}{6} & p_{10,12} = 0 & p_{10,13} = 0 & p_{10,14} = 0 & p_{10,15} = 0 & p_{10,16} = 0 & p_{10,17} = 0 & p_{10,18} = 0 & p_{10,19} = \frac{1}{2} \\
p_{11,0} = 0 & p_{11,1} = 0 & p_{11,2} = \frac{1}{6} & p_{11,3} = \frac{1}{6} & p_{11,4} = 0 & p_{11,5} = 0 & p_{11,6} = 0 & p_{11,7} = 0 & p_{11,8} = 0 & p_{11,9} = 0 & p_{11,10} = \frac{1}{6} & p_{11,11} = 1 & p_{11,12} = \frac{1}{6} & p_{11,13} = 0 & p_{11,14} = 0 & p_{11,15} = 0 & p_{11,16} = 0 & p_{11,17} = 0 & p_{11,18} = 0 & p_{11,19} = \frac{1}{3} \\
p_{12,0} = 0 & p_{12,1} = 0 & p_{12,2} = 0 & p_{12,3} = \frac{1}{6} & p_{12,4} = 0 & p_{12,5} = 0 & p_{12,6} = 0 & p_{12,7} = 0 & p_{12,8} = 0 & p_{12,9} = 0 & p_{12,10} = 0 & p_{12,11} = \frac{1}{6} & p_{12,12} = 1 & p_{12,13} = \frac{1}{6} & p_{12,14} = 0 & p_{12,15} = 0 & p_{12,16} = 0 & p_{12,17} = 0 & p_{12,18} = 0 & p_{12,19} = \frac{1}{2} \\
p_{13,0} = 0 & p_{13,1} = 0 & p_{13,2} = 0 & p_{13,3} = \frac{1}{6} & p_{13,4} = \frac{1}{6} & p_{13,5} = 0 & p_{13,6} = 0 & p_{13,7} = 0 & p_{13,8} = 0 & p_{13,9} = 0 & p_{13,10} = 0 & p_{13,11} = 0 & p_{13,12} = \frac{1}{6} & p_{13,13} = 1 & p_{13,14} = \frac{1}{6} & p_{13,15} = 0 & p_{13,16} = 0 & p_{13,17} = 0 & p_{13,18} = 0 & p_{13,19} = \frac{1}{3} \\
p_{14,0} = 0 & p_{14,1} = 0 & p_{14,2} = 0 & p_{14,3} = 0 & p_{14,4} = \frac{1}{6} & p_{14,5} = 0 & p_{14,6} = 0 & p_{14,7} = 0 & p_{14,8} = 0 & p_{14,9} = 0 & p_{14,10} = 0 & p_{14,11} = 0 & p_{14,12} = 0 & p_{14,13} = \frac{1}{6} & p_{14,14} = 1 & p_{14,15} = \frac{1}{6} & p_{14,16} = 0 & p_{14,17} = 0 & p_{14,18} = 0 & p_{14,19} = \frac{1}{2} \\
p_{15,0} = 0 & p_{15,1} = 0 & p_{15,2} = 0 & p_{15,3} = 0 & p_{15,4} = \frac{1}{6} & p_{15,5} = \frac{1}{6} & p_{15,6} = 0 & p_{15,7} = 0 & p_{15,8} = 0 & p_{15,9} = 0 & p_{15,10} = 0 & p_{15,11} = 0 & p_{15,12} = 0 & p_{15,13} = 0 & p_{15,14} = \frac{1}{6} & p_{15,15} = 1 & p_{15,16} = \frac{1}{6} & p_{15,17} = 0 & p_{15,18} = 0 & p_{15,19} = \frac{1}{3} \\
p_{16,0} = 0 & p_{16,1} = 0 & p_{16,2} = 0 & p_{16,3} = 0 & p_{16,4} = 0 & p_{16,5} = \frac{1}{6} & p_{16,6} = 0 & p_{16,7} = 0 & p_{16,8} = 0 & p_{16,9} = 0 & p_{16,10} = 0 & p_{16,11} = 0 & p_{16,12} = 0 & p_{16,13} = 0 & p_{16,14} = 0 & p_{16,15} = \frac{1}{6} & p_{16,16} = 1 & p_{16,17} = \frac{1}{6} & p_{16,18} = 0 & p_{16,19} = \frac{1}{2} \\
p_{17,0} = 0 & p_{17,1} = 0 & p_{17,2} = 0 & p_{17,3} = 0 & p_{17,4} = 0 & p_{17,5} = \frac{1}{6} & p_{17,6} = \frac{1}{6} & p_{17,7} = 0 & p_{17,8} = 0 & p_{17,9} = 0 & p_{17,10} = 0 & p_{17,11} = 0 & p_{17,12} = 0 & p_{17,13} = 0 & p_{17,14} = 0 & p_{17,15} = 0 & p_{17,16} = \frac{1}{6} & p_{17,17} = 1 & p_{17,18} = \frac{1}{6} & p_{17,19} = \frac{1}{3} \\
p_{18,0} = 0 & p_{18,1} = 0 & p_{18,2} = 0 & p_{18,3} = 0 & p_{18,4} = 0 & p_{18,5} = 0 & p_{18,6} = \frac{1}{6} & p_{18,7} = \frac{1}{6} & p_{18,8} = 0 & p_{18,9} = 0 & p_{18,10} = 0 & p_{18,11} = 0 & p_{18,12} = 0 & p_{18,13} = 0 & p_{18,14} = 0 & p_{18,15} = 0 & p_{18,16} = 0 & p_{18,17} = \frac{1}{6} & p_{18,18} = 1 & p_{18,19} = \frac{1}{2} \\
p_{19,0} = 0 & p_{19,1} = 0 & p_{19,2} = 0 & p_{19,3} = 0 & p_{19,4} = 0 & p_{19,5} = 0 & p_{19,6} = 0 & p_{19,7} = 0 & p_{19,8} = 0 & p_{19,9} = 0 & p_{19,10} = 0 & p_{19,11} = 0 & p_{19,12} = 0 & p_{19,13} = 0 & p_{19,14} = 0 & p_{19,15} = 0 & p_{19,16} = 0 & p_{19,17} = 0 & p_{19,18} = 0 & p_{19,19} = 0
\end{pmatrix}$$

$$N = \begin{pmatrix}
p_{0,0} = \frac{45}{29} & p_{0,1} = \frac{16}{29} & p_{0,2} = \frac{16}{29} & p_{0,3} = \frac{16}{29} & p_{0,4} = \frac{16}{29} & p_{0,5} = \frac{16}{29} & p_{0,6} = \frac{16}{29} & p_{0,7} = \frac{7}{29} & p_{0,8} = \frac{5}{29} & p_{0,9} = \frac{7}{29} & p_{0,10} = \frac{5}{29} & p_{0,11} = \frac{7}{29} & p_{0,12} = \frac{5}{29} & p_{0,13} = \frac{7}{29} & p_{0,14} = \frac{5}{29} & p_{0,15} = \frac{7}{29} & p_{0,16} = \frac{5}{29} & p_{0,17} = \frac{7}{29} & p_{0,18} = \frac{5}{29} \\
p_{1,0} = \frac{16}{29} & p_{1,1} = \frac{992666}{674685} & p_{1,2} = \frac{348808}{674685} & p_{1,3} = \frac{191222}{674685} & p_{1,4} = \frac{160714}{674685} & p_{1,5} = \frac{191222}{674685} & p_{1,6} = \frac{348808}{674685} & p_{1,7} = \frac{6133}{14355} & p_{1,8} = \frac{87176}{224895} & p_{1,9} = \frac{6133}{14355} & p_{1,10} = \frac{14056}{74965} & p_{1,11} = \frac{173}{957} & p_{1,12} = \frac{21752}{224895} & p_{1,13} = \frac{1667}{14355} & p_{1,14} = \frac{5878}{74965} & p_{1,15} = \frac{1667}{14355} & p_{1,16} = \frac{21752}{224895} & p_{1,17} = \frac{173}{957} & p_{1,18} = \frac{14056}{74965} \\
p_{2,0} = \frac{16}{29} & p_{2,1} = \frac{348808}{674685} & p_{2,2} = \frac{992666}{674685} & p_{2,3} = \frac{348808}{674685} & p_{2,4} = \frac{191222}{674685} & p_{2,5} = \frac{160714}{674685} & p_{2,6} = \frac{191222}{674685} & p_{2,7} = \frac{173}{957} & p_{2,8} = \frac{14056}{74965} & p_{2,9} = \frac{6133}{14355} & p_{2,10} = \frac{87176}{224895} & p_{2,11} = \frac{6133}{14355} & p_{2,12} = \frac{14056}{74965} & p_{2,13} = \frac{173}{957} & p_{2,14} = \frac{21752}{224895} & p_{2,15} = \frac{1667}{14355} & p_{2,16} = \frac{5878}{74965} & p_{2,17} = \frac{1667}{14355} & p_{2,18} = \frac{21752}{224895} \\
p_{3,0} = \frac{16}{29} & p_{3,1} = \frac{191222}{674685} & p_{3,2} = \frac{348808}{674685} & p_{3,3} = \frac{992666}{674685} & p_{3,4} = \frac{348808}{674685} & p_{3,5} = \frac{191222}{674685} & p_{3,6} = \frac{160714}{674685} & p_{3,7} = \frac{1667}{14355} & p_{3,8} = \frac{21752}{224895} & p_{3,9} = \frac{173}{957} & p_{3,10} = \frac{14056}{74965} & p_{3,11} = \frac{6133}{14355} & p_{3,12} = \frac{87176}{224895} & p_{3,13} = \frac{6133}{14355} & p_{3,14} = \frac{14056}{74965} & p_{3,15} = \frac{173}{957} & p_{3,16} = \frac{21752}{224895} & p_{3,17} = \frac{1667}{14355} & p_{3,18} = \frac{5878}{74965} \\
p_{4,0} = \frac{16}{29} & p_{4,1} = \frac{160714}{674685} & p_{4,2} = \frac{191222}{674685} & p_{4,3} = \frac{348808}{674685} & p_{4,4} = \frac{992666}{674685} & p_{4,5} = \frac{348808}{674685} & p_{4,6} = \frac{191222}{674685} & p_{4,7} = \frac{1667}{14355} & p_{4,8} = \frac{5878}{74965} & p_{4,9} = \frac{1667}{14355} & p_{4,10} = \frac{21752}{224895} & p_{4,11} = \frac{173}{957} & p_{4,12} = \frac{14056}{74965} & p_{4,13} = \frac{6133}{14355} & p_{4,14} = \frac{87176}{224895} & p_{4,15} = \frac{6133}{14355} & p_{4,16} = \frac{14056}{74965} & p_{4,17} = \frac{173}{957} & p_{4,18} = \frac{21752}{224895} \\
p_{5,0} = \frac{16}{29} & p_{5,1} = \frac{191222}{674685} & p_{5,2} = \frac{160714}{674685} & p_{5,3} = \frac{191222}{674685} & p_{5,4} = \frac{348808}{674685} & p_{5,5} = \frac{992666}{674685} & p_{5,6} = \frac{348808}{674685} & p_{5,7} = \frac{173}{957} & p_{5,8} = \frac{21752}{224895} & p_{5,9} = \frac{1667}{14355} & p_{5,10} = \frac{5878}{74965} & p_{5,11} = \frac{1667}{14355} & p_{5,12} = \frac{21752}{224895} & p_{5,13} = \frac{173}{957} & p_{5,14} = \frac{14056}{74965} & p_{5,15} = \frac{6133}{14355} & p_{5,16} = \frac{87176}{224895} & p_{5,17} = \frac{6133}{14355} & p_{5,18} = \frac{14056}{74965} \\
p_{6,0} = \frac{16}{29} & p_{6,1} = \frac{348808}{674685} & p_{6,2} = \frac{191222}{674685} & p_{6,3} = \frac{160714}{674685} & p_{6,4} = \frac{191222}{674685} & p_{6,5} = \frac{348808}{674685} & p_{6,6} = \frac{992666}{674685} & p_{6,7} = \frac{6133}{14355} & p_{6,8} = \frac{14056}{74965} & p_{6,9} = \frac{173}{957} & p_{6,10} = \frac{21752}{224895} & p_{6,11} = \frac{1667}{14355} & p_{6,12} = \frac{5878}{74965} & p_{6,13} = \frac{1667}{14355} & p_{6,14} = \frac{21752}{224895} & p_{6,15} = \frac{173}{957} & p_{6,16} = \frac{14056}{74965} & p_{6,17} = \frac{6133}{14355} & p_{6,18} = \frac{87176}{224895} \\
p_{7,0} = \frac{7}{29} & p_{7,1} = \frac{6133}{14355} & p_{7,2} = \frac{173}{957} & p_{7,3} = \frac{1667}{14355} & p_{7,4} = \frac{1667}{14355} & p_{7,5} = \frac{173}{957} & p_{7,6} = \frac{6133}{14355} & p_{7,7} = \frac{375124}{301455} & p_{7,8} = \frac{10246}{33495} & p_{7,9} = \frac{49367}{301455} & p_{7,10} = \frac{460}{6699} & p_{7,11} = \frac{20338}{301455} & p_{7,12} = \frac{438}{11165} & p_{7,13} = \frac{15611}{301455} & p_{7,14} = \frac{438}{11165} & p_{7,15} = \frac{20338}{301455} & p_{7,16} = \frac{460}{6699} & p_{7,17} = \frac{49367}{301455} & p_{7,18} = \frac{10246}{33495} \\
p_{8,0} = \frac{5}{29} & p_{8,1} = \frac{87176}{224895} & p_{8,2} = \frac{14056}{74965} & p_{8,3} = \frac{21752}{224895} & p_{8,4} = \frac{5878}{74965} & p_{8,5} = \frac{21752}{224895} & p_{8,6} = \frac{14056}{74965} & p_{8,7} = \frac{10246}{33495} & p_{8,8} = \frac{1836491}{1574265} & p_{8,9} = \frac{10246}{33495} & p_{8,10} = \frac{147473}{1574265} & p_{8,11} = \frac{460}{6699} & p_{8,12} = \frac{53687}{1574265} & p_{8,13} = \frac{438}{11165} & p_{8,14} = \frac{41159}{1574265} & p_{8,15} = \frac{438}{11165} & p_{8,16} = \frac{53687}{1574265} & p_{8,17} = \frac{460}{6699} & p_{8,18} = \frac{147473}{1574265} \\
p_{9,0} = \frac{7}{29} & p_{9,1} = \frac{6133}{14355} & p_{9,2} = \frac{6133}{14355} & p_{9,3} = \frac{173}{957} & p_{9,4} = \frac{1667}{14355} & p_{9,5} = \frac{1667}{14355} & p_{9,6} = \frac{173}{957} & p_{9,7} = \frac{49367}{301455} & p_{9,8} = \frac{10246}{33495} & p_{9,9} = \frac{375124}{301455} & p_{9,10} = \frac{10246}{33495} & p_{9,11} = \frac{49367}{301455} & p_{9,12} = \frac{460}{6699} & p_{9,13} = \frac{20338}{301455} & p_{9,14} = \frac{438}{11165} & p_{9,15} = \frac{15611}{301455} & p_{9,16} = \frac{438}{11165} & p_{9,17} = \frac{20338}{301455} & p_{9,18} = \frac{460}{6699} \\
p_{10,0} = \frac{5}{29} & p_{10,1} = \frac{14056}{74965} & p_{10,2} = \frac{87176}{224895} & p_{10,3} = \frac{14056}{74965} & p_{10,4} = \frac{21752}{224895} & p_{10,5} = \frac{5878}{74965} & p_{10,6} = \frac{21752}{224895} & p_{10,7} = \frac{460}{6699} & p_{10,8} = \frac{147473}{1574265} & p_{10,9} = \frac{10246}{33495} & p_{10,10} = \frac{1836491}{1574265} & p_{10,11} = \frac{10246}{33495} & p_{10,12} = \frac{147473}{1574265} & p_{10,13} = \frac{460}{6699} & p_{10,14} = \frac{53687}{1574265} & p_{10,15} = \frac{438}{11165} & p_{10,16} = \frac{41159}{1574265} & p_{10,17} = \frac{438}{11165} & p_{10,18} = \frac{53687}{1574265} \\
p_{11,0} = \frac{7}{29} & p_{11,1} = \frac{173}{957} & p_{11,2} = \frac{6133}{14355} & p_{11,3} = \frac{6133}{14355} & p_{11,4} = \frac{173}{957} & p_{11,5} = \frac{1667}{14355} & p_{11,6} = \frac{1667}{14355} & p_{11,7} = \frac{20338}{301455} & p_{11,8} = \frac{460}{6699} & p_{11,9} = \frac{49367}{301455} & p_{11,10} = \frac{10246}{33495} & p_{11,11} = \frac{375124}{301455} & p_{11,12} = \frac{10246}{33495} & p_{11,13} = \frac{49367}{301455} & p_{11,14} = \frac{460}{6699} & p_{11,15} = \frac{20338}{301455} & p_{11,16} = \frac{438}{11165} & p_{11,17} = \frac{15611}{301455} & p_{11,18} = \frac{438}{11165} \\
p_{12,0} = \frac{5}{29} & p_{12,1} = \frac{21752}{224895} & p_{12,2} = \frac{14056}{74965} & p_{12,3} = \frac{87176}{224895} & p_{12,4} = \frac{14056}{74965} & p_{12,5} = \frac{21752}{224895} & p_{12,6} = \frac{5878}{74965} & p_{12,7} = \frac{438}{11165} & p_{12,8} = \frac{53687}{1574265} & p_{12,9} = \frac{460}{6699} & p_{12,10} = \frac{147473}{1574265} & p_{12,11} = \frac{10246}{33495} & p_{12,12} = \frac{1836491}{1574265} & p_{12,13} = \frac{10246}{33495} & p_{12,14} = \frac{147473}{1574265} & p_{12,15} = \frac{460}{6699} & p_{12,16} = \frac{53687}{1574265} & p_{12,17} = \frac{438}{11165} & p_{12,18} = \frac{41159}{1574265} \\
p_{13,0} = \frac{7}{29} & p_{13,1} = \frac{1667}{14355} & p_{13,2} = \frac{173}{957} & p_{13,3} = \frac{6133}{14355} & p_{13,4} = \frac{6133}{14355} & p_{13,5} = \frac{173}{957} & p_{13,6} = \frac{1667}{14355} & p_{13,7} = \frac{15611}{301455} & p_{13,8} = \frac{438}{11165} & p_{13,9} = \frac{20338}{301455} & p_{13,10} = \frac{460}{6699} & p_{13,11} = \frac{49367}{301455} & p_{13,12} = \frac{10246}{33495} & p_{13,13} = \frac{375124}{301455} & p_{13,14} = \frac{10246}{33495} & p_{13,15} = \frac{49367}{301455} & p_{13,16} = \frac{460}{6699} & p_{13,17} = \frac{20338}{301455} & p_{13,18} = \frac{438}{11165} \\
p_{14,0} = \frac{5}{29} & p_{14,1} = \frac{5878}{74965} & p_{14,2} = \frac{21752}{224895} & p_{14,3} = \frac{14056}{74965} & p_{14,4} = \frac{87176}{224895} & p_{14,5} = \frac{14056}{74965} & p_{14,6} = \frac{21752}{224895} & p_{14,7} = \frac{438}{11165} & p_{14,8} = \frac{41159}{1574265} & p_{14,9} = \frac{438}{11165} & p_{14,10} = \frac{53687}{1574265} & p_{14,11} = \frac{460}{6699} & p_{14,12} = \frac{147473}{1574265} & p_{14,13} = \frac{10246}{33495} & p_{14,14} = \frac{1836491}{1574265} & p_{14,15} = \frac{10246}{33495} & p_{14,16} = \frac{147473}{1574265} & p_{14,17} = \frac{460}{6699} & p_{14,18} = \frac{53687}{1574265} \\
p_{15,0} = \frac{7}{29} & p_{15,1} = \frac{1667}{14355} & p_{15,2} = \frac{1667}{14355} & p_{15,3} = \frac{173}{957} & p_{15,4} = \frac{6133}{14355} & p_{15,5} = \frac{6133}{14355} & p_{15,6} = \frac{173}{957} & p_{15,7} = \frac{20338}{301455} & p_{15,8} = \frac{438}{11165} & p_{15,9} = \frac{15611}{301455} & p_{15,10} = \frac{438}{11165} & p_{15,11} = \frac{20338}{301455} & p_{15,12} = \frac{460}{6699} & p_{15,13} = \frac{49367}{301455} & p_{15,14} = \frac{10246}{33495} & p_{15,15} = \frac{375124}{301455} & p_{15,16} = \frac{10246}{33495} & p_{15,17} = \frac{49367}{301455} & p_{15,18} = \frac{460}{6699} \\
p_{16,0} = \frac{5}{29} & p_{16,1} = \frac{21752}{224895} & p_{16,2} = \frac{5878}{74965} & p_{16,3} = \frac{21752}{224895} & p_{16,4} = \frac{14056}{74965} & p_{16,5} = \frac{87176}{224895} & p_{16,6} = \frac{14056}{74965} & p_{16,7} = \frac{460}{6699} & p_{16,8} = \frac{53687}{1574265} & p_{16,9} = \frac{438}{11165} & p_{16,10} = \frac{41159}{1574265} & p_{16,11} = \frac{438}{11165} & p_{16,12} = \frac{53687}{1574265} & p_{16,13} = \frac{460}{6699} & p_{16,14} = \frac{147473}{1574265} & p_{16,15} = \frac{10246}{33495} & p_{16,16} = \frac{1836491}{1574265} & p_{16,17} = \frac{10246}{33495} & p_{16,18} = \frac{147473}{1574265} \\
p_{17,0} = \frac{7}{29} & p_{17,1} = \frac{173}{957} & p_{17,2} = \frac{1667}{14355} & p_{17,3} = \frac{1667}{14355} & p_{17,4} = \frac{173}{957} & p_{17,5} = \frac{6133}{14355} & p_{17,6} = \frac{6133}{14355} & p_{17,7} = \frac{49367}{301455} & p_{17,8} = \frac{460}{6699} & p_{17,9} = \frac{20338}{301455} & p_{17,10} = \frac{438}{11165} & p_{17,11} = \frac{15611}{301455} & p_{17,12} = \frac{438}{11165} & p_{17,13} = \frac{20338}{301455} & p_{17,14} = \frac{460}{6699} & p_{17,15} = \frac{49367}{301455} & p_{17,16} = \frac{10246}{33495} & p_{17,17} = \frac{375124}{301455} & p_{17,18} = \frac{10246}{33495} \\
p_{18,0} = \frac{5}{29} & p_{18,1} = \frac{14056}{74965} & p_{18,2} = \frac{21752}{224895} & p_{18,3} = \frac{5878}{74965} & p_{18,4} = \frac{21752}{224895} & p_{18,5} = \frac{14056}{74965} & p_{18,6} = \frac{87176}{224895} & p_{18,7} = \frac{10246}{33495} & p_{18,8} = \frac{147473}{1574265} & p_{18,9} = \frac{460}{6699} & p_{18,10} = \frac{53687}{1574265} & p_{18,11} = \frac{438}{11165} & p_{18,12} = \frac{41159}{1574265} & p_{18,13} = \frac{438}{11165} & p_{18,14} = \frac{53687}{1574265} & p_{18,15} = \frac{460}{6699} & p_{18,16} = \frac{147473}{1574265} & p_{18,17} = \frac{10246}{33495} & p_{18,18} = \frac{1836491}{1574265}
\end{pmatrix}$$

In order to get the expected number of steps, we find t_0, where

$$\boldsymbol{t} = N\mathbf{1}$$

Here, $\mathbf{1}$ is a vector whose entries are all 1.

$$\boldsymbol{t} = \begin{pmatrix} \frac{213}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{184}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \\ \frac{124}{29} \\ \frac{101}{29} \end{pmatrix}$$

Finally, we see that $t_0 = \boxed{\dfrac{213}{29} \approx 7.345}$