

## $\mathbb{R}^n$ Bonus Problem #3

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### §1 Problem

~~Settlers of Catan~~ A board game is played on a hexagonal grid of 19 tiles. A 'traveler' token starts on the center tile. Each turn a die is rolled to determine what neighboring tile the traveler moves to (all six directions equally likely). The turn that the traveler leaves the board, the game ends. What is the expected number of turns of the game?

### §2 Diagram



### §3 Solution

We wish to find the expected value of the number of turns in the game, which we denote  $N$ .

$$\mathbb{E}(N) = \sum N \mathbb{P}(N)$$

The dice is truly random, so there is no upper bound on  $N$ . We note that this game is really akin to a Markov chain, in that it doesn't matter what the past states are.

Let  $X_i \in [0, 36]$  be the current state, or position of the traveler. The traveler always starts at position  $X_0 = 0$ . The final state must be  $X_N \in [19, 36]$ .

Now that we've defined some notation, we can write the transition matrix  $P$ . Because a  $37 \times 37$  matrix is cumbersome, we combine the states  $[19, 36]$  into a

$$P = \begin{pmatrix} p_{0,0} = 0 & p_{0,1} = \frac{1}{6} & p_{0,2} = \frac{1}{6} & p_{0,3} = \frac{1}{6} & p_{0,4} = \frac{1}{6} & p_{0,5} = \frac{1}{6} & p_{0,6} = \frac{1}{6} & p_{0,7} = 0 & p_{0,8} = 0 & p_{0,9} = 0 & p_{0,10} = 0 & p_{0,11} = 0 & p_{0,12} = 0 & p_{0,13} = 0 & p_{0,14} = 0 & p_{0,15} = 0 & p_{0,16} = 0 & p_{0,17} = 0 & p_{0,18} = 0 & p_{0,19} = 0 \\ p_{1,0} = \frac{1}{6} & p_{1,1} = 0 & p_{1,2} = \frac{1}{6} & p_{1,3} = 0 & p_{1,4} = 0 & p_{1,5} = 0 & p_{1,6} = \frac{1}{6} & p_{1,7} = \frac{1}{6} & p_{1,8} = \frac{1}{6} & p_{1,9} = \frac{1}{6} & p_{1,10} = 0 & p_{1,11} = 0 & p_{1,12} = 0 & p_{1,13} = 0 & p_{1,14} = 0 & p_{1,15} = 0 & p_{1,16} = 0 & p_{1,17} = 0 & p_{1,18} = 0 & p_{1,19} = 0 \\ p_{2,0} = 0 & p_{2,1} = \frac{1}{6} & p_{2,2} = 0 & p_{2,3} = \frac{1}{6} & p_{2,4} = 0 & p_{2,5} = 0 & p_{2,6} = 0 & p_{2,7} = 0 & p_{2,8} = 0 & p_{2,9} = \frac{1}{6} & p_{2,10} = \frac{1}{6} & p_{2,11} = \frac{1}{6} & p_{2,12} = 0 & p_{2,13} = 0 & p_{2,14} = 0 & p_{2,15} = 0 & p_{2,16} = 0 & p_{2,17} = 0 & p_{2,18} = 0 & p_{2,19} = 0 \\ p_{3,0} = \frac{1}{6} & p_{3,1} = 0 & p_{3,2} = \frac{1}{6} & p_{3,3} = 0 & p_{3,4} = \frac{1}{6} & p_{3,5} = 0 & p_{3,6} = 0 & p_{3,7} = 0 & p_{3,8} = 0 & p_{3,9} = 0 & p_{3,10} = 0 & p_{3,11} = \frac{1}{6} & p_{3,12} = \frac{1}{6} & p_{3,13} = \frac{1}{6} & p_{3,14} = 0 & p_{3,15} = 0 & p_{3,16} = 0 & p_{3,17} = 0 & p_{3,18} = 0 & p_{3,19} = 0 \\ p_{4,0} = \frac{1}{6} & p_{4,1} = 0 & p_{4,2} = 0 & p_{4,3} = \frac{1}{6} & p_{4,4} = 0 & p_{4,5} = \frac{1}{6} & p_{4,6} = 0 & p_{4,7} = 0 & p_{4,8} = 0 & p_{4,9} = 0 & p_{4,10} = 0 & p_{4,11} = 0 & p_{4,12} = 0 & p_{4,13} = \frac{1}{6} & p_{4,14} = \frac{1}{6} & p_{4,15} = \frac{1}{6} & p_{4,16} = 0 & p_{4,17} = 0 & p_{4,18} = 0 & p_{4,19} = 0 \\ p_{5,0} = 0 & p_{5,1} = 0 & p_{5,2} = 0 & p_{5,3} = 0 & p_{5,4} = \frac{1}{6} & p_{5,5} = 0 & p_{5,6} = \frac{1}{6} & p_{5,7} = 0 & p_{5,8} = 0 & p_{5,9} = 0 & p_{5,10} = 0 & p_{5,11} = 0 & p_{5,12} = 0 & p_{5,13} = 0 & p_{5,14} = 0 & p_{5,15} = \frac{1}{6} & p_{5,16} = \frac{1}{6} & p_{5,17} = \frac{1}{6} & p_{5,18} = 0 & p_{5,19} = 0 \\ p_{6,0} = \frac{1}{6} & p_{6,1} = \frac{1}{6} & p_{6,2} = 0 & p_{6,3} = 0 & p_{6,4} = 0 & p_{6,5} = \frac{1}{6} & p_{6,6} = 0 & p_{6,7} = \frac{1}{6} & p_{6,8} = 0 & p_{6,9} = 0 & p_{6,10} = 0 & p_{6,11} = 0 & p_{6,12} = 0 & p_{6,13} = 0 & p_{6,14} = 0 & p_{6,15} = 0 & p_{6,16} = 0 & p_{6,17} = \frac{1}{6} & p_{6,18} = \frac{1}{6} & p_{6,19} = 0 \\ p_{7,0} = 0 & p_{7,1} = \frac{1}{6} & p_{7,2} = 0 & p_{7,3} = 0 & p_{7,4} = 0 & p_{7,5} = 0 & p_{7,6} = \frac{1}{6} & p_{7,7} = 0 & p_{7,8} = \frac{1}{6} & p_{7,9} = 0 & p_{7,10} = 0 & p_{7,11} = 0 & p_{7,12} = 0 & p_{7,13} = 0 & p_{7,14} = 0 & p_{7,15} = 0 & p_{7,16} = 0 & p_{7,17} = 0 & p_{7,18} = \frac{1}{6} & p_{7,19} = \frac{1}{6} \\ p_{8,0} = 0 & p_{8,1} = \frac{1}{6} & p_{8,2} = 0 & p_{8,3} = 0 & p_{8,4} = 0 & p_{8,5} = 0 & p_{8,6} = 0 & p_{8,7} = \frac{1}{6} & p_{8,8} = 0 & p_{8,9} = \frac{1}{6} & p_{8,10} = 0 & p_{8,11} = 0 & 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& P_{15,3} = 0 & P_{15,4} = \frac{1}{6} & P_{15,5} = \frac{1}{6} & P_{15,6} = 0 & P_{15,7} = 0 & P_{1$$

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$N =$	$P_{0,0} = \frac{45}{16}$	$P_{0,1} = \frac{15}{16}$	$P_{0,2} = \frac{15}{16}$	$P_{0,3} = \frac{15}{16}$	$P_{0,4} = \frac{15}{16}$	$P_{0,5} = \frac{15}{16}$	$P_{0,6} = \frac{15}{16}$	$P_{0,7} = \frac{7}{2}$	$P_{0,8} = \frac{5}{4}$	$P_{0,9} = \frac{7}{2}$	$P_{0,10} = \frac{1}{16}$	$P_{0,11} = \frac{7}{2}$	$P_{0,12} = \frac{5}{4}$	$P_{0,13} = \frac{7}{2}$	$P_{0,14} = \frac{5}{4}$	$P_{0,15} = \frac{7}{2}$	$P_{0,16} = \frac{5}{4}$	$P_{0,17} = \frac{7}{2}$	$P_{0,18} = \frac{5}{4}$
	$P_{1,0} = \frac{15}{16}$	$P_{1,1} = \frac{10771}{16384}$	$P_{1,2} = \frac{62609}{1048576}$	$P_{1,3} = \frac{353609}{67108864}$	$P_{1,4} = \frac{201953}{402653440}$	$P_{1,5} = \frac{117193}{2415920640}$	$P_{1,6} = \frac{67193}{14495523840}$	$P_{1,7} = \frac{1}{16384}$	$P_{1,8} = \frac{22969}{268435456}$	$P_{1,9} = \frac{1337}{16777216}$	$P_{1,10} = \frac{1}{16384}$	$P_{1,11} = \frac{1}{16384}$	$P_{1,12} = \frac{22969}{268435456}$	$P_{1,13} = \frac{1337}{16777216}$	$P_{1,14} = \frac{22969}{268435456}$	$P_{1,15} = \frac{1}{16384}$	$P_{1,16} = \frac{22969}{268435456}$	$P_{1,17} = \frac{1}{16384}$	$P_{1,18} = \frac{1}{16384}$
	$P_{2,0} = \frac{15}{16}$	$P_{2,1} = \frac{62609}{1048576}$	$P_{2,2} = \frac{353609}{67108864}$	$P_{2,3} = \frac{201953}{402653440}$	$P_{2,4} = \frac{117193}{2415920640}$	$P_{2,5} = \frac{67193}{14495523840}$	$P_{2,6} = \frac{1}{16384}$	$P_{2,7} = \frac{1}{16384}$	$P_{2,8} = \frac{22969}{268435456}$	$P_{2,9} = \frac{1337}{16777216}$	$P_{2,10} = \frac{1}{16384}$	$P_{2,11} = \frac{1}{16384}$	$P_{2,12} = \frac{22969}{268435456}$	$P_{2,13} = \frac{1337}{16777216}$	$P_{2,14} = \frac{22969}{268435456}$	$P_{2,15} = \frac{1}{16384}$	$P_{2,16} = \frac{22969}{268435456}$	$P_{2,17} = \frac{1}{16384}$	$P_{2,18} = \frac{1}{16384}$
	$P_{3,0} = \frac{15}{16}$	$P_{3,1} = \frac{201953}{402653440}$	$P_{3,2} = \frac{117193}{2415920640}$	$P_{3,3} = \frac{67193}{14495523840}$	$P_{3,4} = \frac{1}{16384}$	$P_{3,5} = \frac{1}{16384}$	$P_{3,6} = \frac{1}{16384}$	$P_{3,7} = \frac{1}{16384}$	$P_{3,8} = \frac{22969}{268435456}$	$P_{3,9} = \frac{1337}{16777216}$	$P_{3,10} = \frac{1}{16384}$	$P_{3,11} = \frac{1}{16384}$	$P_{3,12} = \frac{22969}{268435456}$	$P_{3,13} = \frac{1337}{16777216}$	$P_{3,14} = \frac{22969}{268435456}$	$P_{3,15} = \frac{1}{16384}$	$P_{3,16} = \frac{22969}{268435456}$	$P_{3,17} = \frac{1}{16384}$	$P_{3,18} = \frac{1}{16384}$
	$P_{4,0} = \frac{15}{16}$	$P_{4,1} = \frac{67193}{14495523840}$	$P_{4,2} = \frac{1}{16384}$	$P_{4,3} = \frac{1}{16384}$	$P_{4,4} = \frac{1}{16384}$	$P_{4,5} = \frac{1}{16384}$	$P_{4,6} = \frac{1}{16384}$	$P_{4,7} = \frac{1}{16384}$	$P_{4,8} = \frac{22969}{268435456}$	$P_{4,9} = \frac{1337}{16777216}$	$P_{4,10} = \frac{1}{16384}$	$P_{4,11} = \frac{1}{16384}$	$P_{4,12} = \frac{22969}{268435456}$	$P_{4,13} = \frac{1337}{16777216}$	$P_{4,14} = \frac{22969}{268435456}$	$P_{4,15} = \frac{1}{16384}$	$P_{4,16} = \frac{22969}{268435456}$	$P_{4,17} = \frac{1}{16384}$	$P_{4,18} = \frac{1}{16384}$
	$P_{5,0} = \frac{15}{16}$	$P_{5,1} = \frac{1}{16384}$	$P_{5,2} = \frac{1}{16384}$	$P_{5,3} = \frac{1}{16384}$	$P_{5,4} = \frac{1}{16384}$	$P_{5,5} = \frac{1}{16384}$	$P_{5,6} = \frac{1}{16384}$	$P_{5,7} = \frac{1}{16384}$	$P_{5,8} = \frac{22969}{268435456}$	$P_{5,9} = \frac{1337}{16777216}$	$P_{5,10} = \frac{1}{16384}$	$P_{5,11} = \frac{1}{16384}$	$P_{5,12} = \frac{22969}{268435456}$	$P_{5,13} = \frac{1337}{16777216}$	$P_{5,14} = \frac{22969}{268435456}$	$P_{5,15} = \frac{1}{16384}$	$P_{5,16} = \frac{22969}{268435456}$	$P_{5,17} = \frac{1}{16384}$	$P_{5,18} = \frac{1}{16384}$
	$P_{6,0} = \frac{15}{16}$	$P_{6,1} = \frac{1}{16384}$	$P_{6,2} = \frac{1}{16384}$	$P_{6,3} = \frac{1}{16384}$	$P_{6,4} = \frac{1}{16384}$	$P_{6,5} = \frac{1}{16384}$	$P_{6,6} = \frac{1}{16384}$	$P_{6,7} = \frac{1}{16384}$	$P_{6,8} = \frac{22969}{268435456}$	$P_{6,9} = \frac{1337}{16777216}$	$P_{6,10} = \frac{1}{16384}$	$P_{6,11} = \frac{1}{16384}$	$P_{6,12} = \frac{22969}{268435456}$	$P_{6,13} = \frac{1337}{16777216}$	$P_{6,14} = \frac{22969}{268435456}$	$P_{6,15} = \frac{1}{16384}$	$P_{6,16} = \frac{22969}{268435456}$	$P_{6,17} = \frac{1}{16384}$	$P_{6,18} = \frac{1}{16384}$
	$P_{7,0} = \frac{7}{2}$	$P_{7,1} = \frac{1}{16384}$	$P_{7,2} = \frac{1}{16384}$	$P_{7,3} = \frac{1}{16384}$	$P_{7,4} = \frac{1}{16384}$	$P_{7,5} = \frac{1}{16384}$	$P_{7,6} = \frac{1}{16384}$	$P_{7,7} = \frac{1}{16384}$	$P_{7,8} = \frac{22969}{268435456}$	$P_{7,$									

$$t = N\mathbf{1}$$
[illegible]

Finally, we see that  $t_0 = \boxed{\frac{213}{29} \approx 7.345}$