Positron energy distribution in a factorized trident process

A. I. Titov, U. Hernandez Acosta, and B. Kämpfer, Phys Rev A **104**, 062811 (2021)

Notebook: Óscar Amaro, November 2022 @ GoLP-EPP

Introduction

In this notebook we reproduce some results from the paper.

In[∘]:=
```
Clear[ħ, c, ωeV]
ħ = 1.05 × 10^-34;(*[Js]*)
c = 3 × 10^8;(*[m/s]*)
ωeV = 1.55;(*[eV] laser photon energy *)
```
$$\frac{2\,\pi\,c}{\omega eV\,\frac{e}{\hbar}}\,10^{6}$$ `(*[μm] wavelength ~800nm *)`

Out[∘]=
```
0.798066
```

Eq 7 asymptotic expression for Bessel function

```
(* eq 7 asymptotic expression for Bessel function *)
(* this expression has 2 typos. the first power should be -
 1/2 and the argument of the exponent should not have the 2 factor. see for
  example Acosta_2021_New_J._Phys._23_095008 for the correct expression *)
Clear[Jnz7, n, z, a]
a = ArcTanh[Sqrt[1 - z^2 / n^2]];
Jnz7 = (2 π n Tanh[a])^(-1/2) Exp[-n (a - Tanh[a])]
D[Jnz7, z] // Simplify;
n = 2;
LogPlot[{BesselJ[n, z], Jnz7},
  {z, n / 100, n}, Frame → True, FrameLabel → {"z", ""},
  PlotLegends → {"Jn", "eq 7"}, PlotStyle → {Default, Dashed}] // Quiet
```

Out[◦]=
$$\frac{e^{-n\left(-\sqrt{1-\frac{z^2}{n^2}}+\text{ArcTanh}\left[\sqrt{1-\frac{z^2}{n^2}}\right]\right)}}{\sqrt{2\pi}\sqrt{n\sqrt{1-\frac{z^2}{n^2}}}}$$

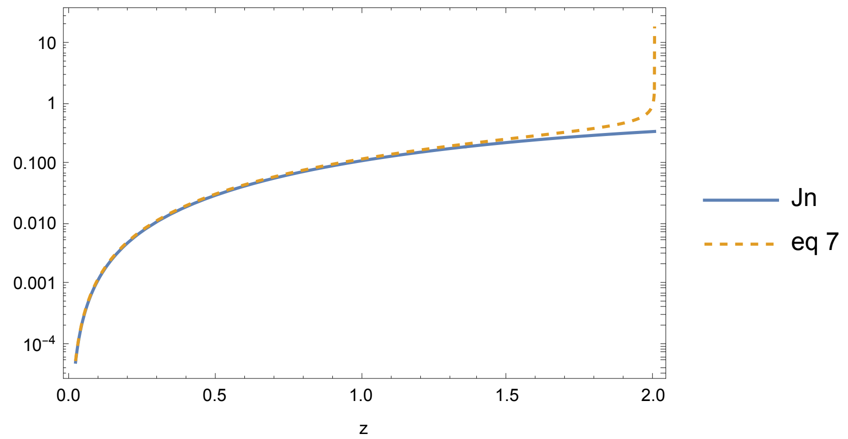
Out[◦]=


Figure 1: nCS

Increase nmaxmin to have better match with paper

```
Clear[z, u, χ, α, m, e, Ee, w, dΓnlCdωeq8, ωp,
 C1, Jnzeq7, dJnzeq7, Fneq3, Fneq7, uneq6, zeq5, nmaxmin]
Clear[Fig1aξ1full, Fig1aξ3full, Fig1aξ10full, Fig1aξ1, Fig1aξ3, Fig1aξ10,
 Fig1bξ1full, Fig1bξ3full, Fig1bξ10full, Fig1bξ1, Fig1bξ3, Fig1bξ10,
 Fig1cξ1full, Fig1cξ3full, Fig1cξ10full, Fig1cξ1, Fig1cξ3, Fig1cξ10]
```

In[75]:=
```
α = 1 / 137; (*[]*)
m = 9.1 × 10^-31; (*[Kg]*)
e = 1.6 × 10^-19; (*[C]*)

(* number of harmonics to include *)
```

```
nmaxmin = 30;

(* "global" multiplicative constant for
 normalization to match the figure in the paper *)
C1 = 10^33.9; (*[]*)

(* eq8: large -ξ approximation [34]*)
dΓnlCdωeq8[x_, ξ_, EeGeV_] := Module[{z, u, w, ξs, χ, γe, Ee, ωpGeV},
  ωpGeV = x EeGeV;

  (* The laser frequency is chosen as ω=1.55 eV... 0.8μm *)
  ξs = 4.12 × 10^5 × 0.8; (*[]*)

  γe = EeGeV / (0.511 × 10^-3); (* [] *)
  Ee = EeGeV e 10^9; (*[J]*)

  (* chi*)
  χ = 2 ξ γe / ξs;

  (* eq 6 *)
  u = ωpGeV / (EeGeV - ωpGeV);
  (* eq 5 *)
  z = (u / χ)^(2/3);
  w = 1 + u^2 / (2 × (1 + u));

  Return[-C1 (α m^2 / Ee^2) (NIntegrate[AiryAi[y], {y, z, ∞}] + 2/z w AiryAiPrime[z])]
 ]


(* eq3: includes full Bessel function (and its derivative) *)
Fneq3[z_, u_, ξ_, n_] := -BesselJ[n, z]^2 + ξ^2 (1 + u^2 / (2 × (1 + u)))
   ((n^2/z^2 - 1) BesselJ[n, z]^2 + (1/2 (BesselJ[-1 + n, z] - BesselJ[1 + n, z]))^2)

(* eq 6: auxiliary functions *)
uneq6[n_, χ_, ξ_] := 2 n χ / (ξ (1 + ξ^2));

zeq5[n_, χ_, ξ_, u_] := (ξ^2 Sqrt[1 + ξ^2] / χ) Sqrt[u (uneq6[n, χ, ξ] - u)];
```

```
(* eq 2: rate with full Bessel functions *)
dΓnlCdωeq2[x_, ξ_, EeGeV_] :=
 Module[{nmin, sum, ωpGeV, ξs, γe, Ee, χ, u, ωGeV, mGeV},
   ωpGeV = x EeGeV;
   mGeV = 0.511 × 10^-3; (*[GeV]*)
   ωGeV = 1.55 × 10^-9; (*[GeV]*)

   (* The laser frequency is chosen as ω=1.55 eV... 0.8μm *)
   ξs = 4.12 × 10^5 × 0.8; (*[]*)

   γe = EeGeV / (0.511 × 10^-3); (* [] *)
   Ee = EeGeV e 10^9; (*[J]*)

   (* chi*)
   χ = 2 ξ γe / ξs;

   (* eq 6 *)
   u = ωpGeV / (EeGeV - ωpGeV);

   nmin = Ceiling[1 + (mGeV^2 (1 + ξ^2) ωpGeV) / (4 ωGeV EeGeV (EeGeV - ωpGeV))];

   sum = Sum[Fneq3[zeq5[n, χ, ξ, u], u, ξ, n], {n, nmin, nmin + nmaxmin, 1}];
   Return[C1 (α m^2)/(Ee^2) sum]
  ] // Quiet
```

```
(* eq 7 *)
```

$$\text{Jnzeq7[n\_, z\_] :=} \frac{e^{-n\left(-\sqrt{1-\frac{z^2}{n^2}}+\text{ArcTanh}\left[\sqrt{1-\frac{z^2}{n^2}}\right]\right)}}{\sqrt{2\pi}\sqrt{n\sqrt{1-\frac{z^2}{n^2}}}}$$

$$\text{dJnzeq7[n\_, z\_] :=} \frac{e^{n\sqrt{1-\frac{z^2}{n^2}}-n\,\text{ArcTanh}\left[\sqrt{1-\frac{z^2}{n^2}}\right]}\left(2n^3\sqrt{1-\frac{z^2}{n^2}}+z^2\left(1-2n\sqrt{1-\frac{z^2}{n^2}}\right)\right)}{2\sqrt{2\pi}\,z\left(n^2-z^2\right)\sqrt{n\sqrt{1-\frac{z^2}{n^2}}}}$$

```
(* eq3 using approximation from eq 7 *)
Fneq7[z_, u_, ξ_, n_] :=
```

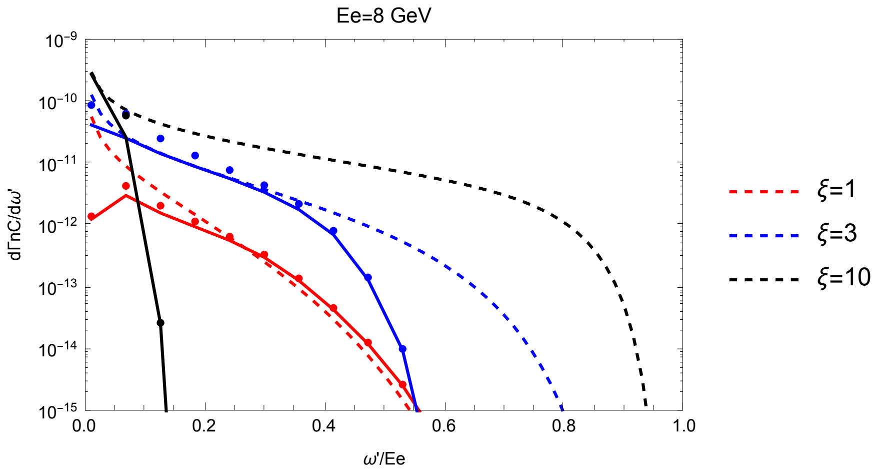
$$-\text{Jnzeq7[n, z]}^2+\xi^2\left(1+\frac{u^2}{2\times(1+u)}\right)\left(\left(\frac{n^2}{z^2}-1\right)\text{Jnzeq7[n, z]}^2+\text{dJnzeq7[n, z]}^2\right)$$

```
dΓnlCdωeq7[x_, ξ_, EeGeV_] :=
 Module[{nmin, sum, ωpGeV, ξs, γe, Ee, χ, u, ωGeV, mGeV},
   ωpGeV = x EeGeV;
   mGeV = 0.511 × 10^-3; (*[GeV]*)
   ωGeV = 1.55 × 10^-9; (*[GeV]*)

   (* The laser frequency is chosen as ω=1.55 eV... 0.8μm *)
   ξs = 4.12 × 10^5 × 0.8; (*[]*)

   γe = EeGeV / (0.511 × 10^-3); (* [] *)
   Ee = EeGeV e 10^9; (*[J]*)

   (* chi*)
   χ = 2 ξ γe / ξs;

   (* eq 6 *)
   u = ωpGeV / (EeGeV - ωpGeV);

   nmin = Ceiling[1 + (mGeV^2 (1 + ξ^2) ωpGeV) / (4 ωGeV EeGeV (EeGeV - ωpGeV))];

   sum = Sum[Fneq7[zeq5[n, χ, ξ, u], u, ξ, n], {n, nmin, nmin + nmaxmin, 1}];
   Return[C1 (α m^2)/(Ee^2) sum]
  ] // Quiet



xmin = 0.01;
xmax = 0.99;
xdim = 18;

(*a*)
Fig1aξ1full =
  ParallelTable[{x, dΓnlCdωeq2[x, 1, 8]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1aξ3full = ParallelTable[{x, dΓnlCdωeq2[x, 3, 8]},
   {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1aξ10full = ParallelTable[{x, dΓnlCdωeq2[x, 10, 8]},
   {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
```

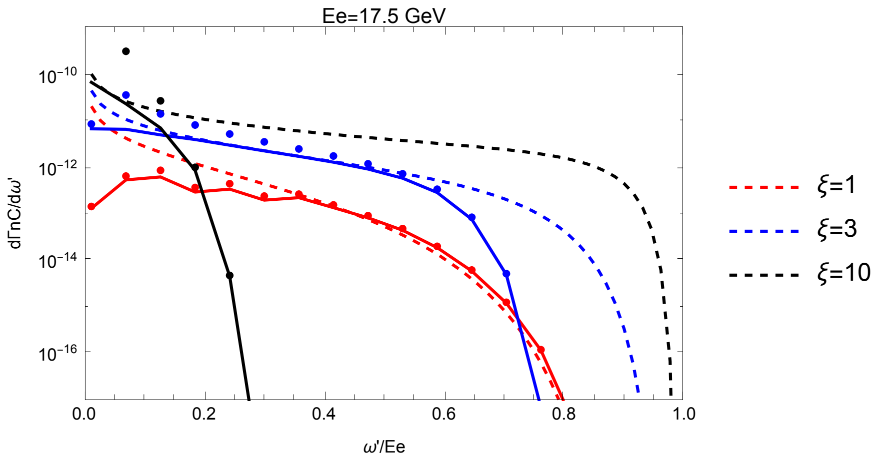
```
Fig1aξ1 = ParallelTable[{x, dΓnlCdωeq7[x, 1, 8]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1aξ3 = ParallelTable[{x, dΓnlCdωeq7[x, 3, 8]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1aξ10 = ParallelTable[{x, dΓnlCdωeq7[x, 10, 8]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];

(*b*)
Fig1bξ1full =
  ParallelTable[{x, dΓnlCdωeq2[x, 1, 17.5]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1bξ3full = ParallelTable[{x, dΓnlCdωeq2[x, 3, 17.5]},
   {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1bξ10full = ParallelTable[{x, dΓnlCdωeq2[x, 10, 17.5]},
   {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1bξ1 = ParallelTable[{x, dΓnlCdωeq7[x, 1, 17.5]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1bξ3 = ParallelTable[{x, dΓnlCdωeq7[x, 3, 17.5]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1bξ10 =
  ParallelTable[{x, dΓnlCdωeq7[x, 10, 17.5]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];

(*c*)
Fig1cξ1full =
  ParallelTable[{x, dΓnlCdωeq2[x, 1, 45]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1cξ3full = ParallelTable[{x, dΓnlCdωeq2[x, 3, 45]},
   {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1cξ10full = ParallelTable[{x, dΓnlCdωeq2[x, 10, 45]},
   {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1cξ1 = ParallelTable[{x, dΓnlCdωeq7[x, 1, 45]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1cξ3 = ParallelTable[{x, dΓnlCdωeq7[x, 3, 45]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];
Fig1cξ10 = ParallelTable[{x, dΓnlCdωeq7[x, 10, 45]}, {x, xmin, xmax, (xmax - xmin)/(xdim - 1)}];

Show[{LogPlot[{dΓnlCdωeq8[x, 1, 8], dΓnlCdωeq8[x, 3, 8], dΓnlCdωeq8[x, 10, 8]},
    {x, 0.01, 0.99}, Frame → True, FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    PlotLegends → {"ξ=1", "ξ=3", "ξ=10"},
```

```
    PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
    PlotLabel → "Ee=8 GeV", PlotPoints → 2, PlotRange → {{0, 1}, {10^-15, 10^-9}}],
  ListLogPlot[{Fig1aξ1full, Fig1aξ3full, Fig1aξ10full},
    Frame → True, FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    Joined → True, PlotStyle → {{Red}, {Blue}, {Black}},
    PlotLabel → "Ee=8 GeV", PlotRange → {10^-15, 10^-9}],
  ListLogPlot[{Fig1aξ1, Fig1aξ3, Fig1aξ10}, Frame → True,
    FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
    PlotLabel → "Ee=8 GeV", PlotRange → {10^-15, 10^-9}]
 }]

Show[{LogPlot[{dΓnlCdωeq8[x, 1, 17.5], dΓnlCdωeq8[x, 3, 17.5],
     dΓnlCdωeq8[x, 10, 17.5]}, {x, 0.01, 0.99}, Frame → True,
    FrameLabel → {"ω'/Ee", "dΓnC/dω'"}, PlotLegends → {"ξ=1", "ξ=3", "ξ=10"},
    PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
    PlotLabel → "Ee=17.5 GeV", PlotPoints → 2, PlotRange → {{0, 1}, {10^-17, 10^-9}}],
  ListLogPlot[{Fig1bξ1full, Fig1bξ3full, Fig1bξ10full},
    Frame → True, FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    Joined → True, PlotStyle → {{Red}, {Blue}, {Black}},
    PlotLabel → "Ee=17.5 GeV", PlotRange → {10^-17, 10^-9}],
  ListLogPlot[{Fig1bξ1, Fig1bξ3, Fig1bξ10}, Frame → True,
    FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
    PlotLabel → "Ee=17.5 GeV", PlotRange → {10^-17, 10^-9}]
 }]

Show[{LogPlot[{dΓnlCdωeq8[x, 1, 45], dΓnlCdωeq8[x, 3, 45], dΓnlCdωeq8[x, 10, 45]},
    {x, 0.01, 0.99}, Frame → True, FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    PlotLegends → {"ξ=1", "ξ=3", "ξ=10"},
    PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
    PlotLabel → "Ee=45 GeV", PlotPoints → 2, PlotRange → {{0, 1}, {10^-16, 10^-10}}],
  ListLogPlot[{Fig1cξ1full, Fig1cξ3full, Fig1cξ10full},
    Frame → True, FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    Joined → True, PlotStyle → {{Red}, {Blue}, {Black}},
    PlotLabel → "Ee=45 GeV", PlotRange → {10^-16, 10^-10}],
  ListLogPlot[{Fig1cξ1, Fig1cξ3, Fig1cξ10}, Frame → True,
    FrameLabel → {"ω'/Ee", "dΓnC/dω'"},
    PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
    PlotLabel → "Ee=45 GeV", PlotRange → {10^-16, 10^-10}]
 }]
```

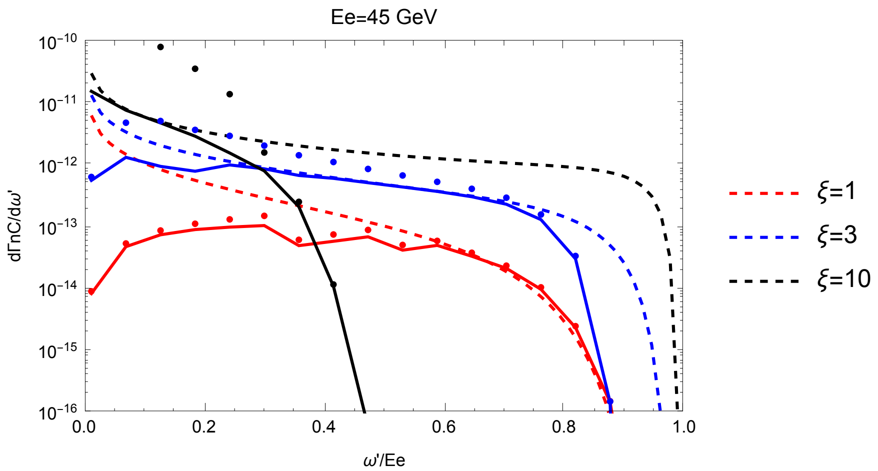
Out[109]=



Out[110]=



Out[111]=



Eq 14 auxiliary function using approximation to Bessel function

In[756]:=
```
(* because the aproximation in equation 7 has a typo,
equation 14 does not match the expansion applied to equation 10 *)
Clear[Jnz7, dJnz7, 𝒥n, n, z, a, ξ]
a = ArcTanh[Sqrt[1 - z^2 / n^2]];
Jnz7 = (2 π n Tanh[a])^(-1/2) Exp[-n (a - Tanh[a])];
dJnz7 = D[Jnz7, z] // Simplify;
𝒥n = (Jnz7^2 + ξ^2 (2 u - 1) ((n^2/z^2 - 1) Jnz7^2 + dJnz7^2)) // FullSimplify
```

Out[760]=

$$\left(e^{2 n \sqrt{1-\frac{z^2}{n^2}}-2 n \operatorname{ArcTanh}\left[\sqrt{1-\frac{z^2}{n^2}}\right]} \left(4 \left(-n^2 z+z^3\right)^2+(-1+2 u)\times \left(8 n^6-24 n^4 z^2+z^4+24 n^2 z^4-8 z^6+4 n^3 z^2 \sqrt{1-\frac{z^2}{n^2}}-4 n z^4 \sqrt{1-\frac{z^2}{n^2}}\right) \xi^2\right)\right) \Big/ \left(8 n \pi z^2 \left(n^2-z^2\right)^2 \sqrt{1-\frac{z^2}{n^2}}\right)$$

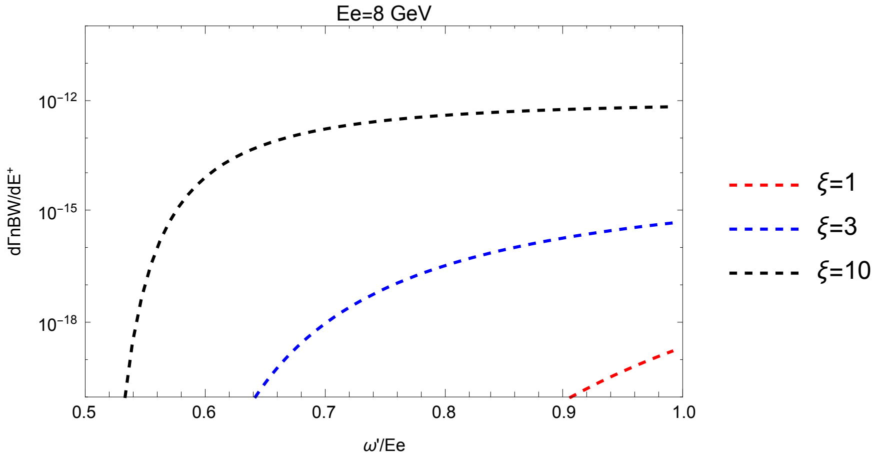
Figure 2: nBW

For “fixed positron energy E+ = Ee/2”

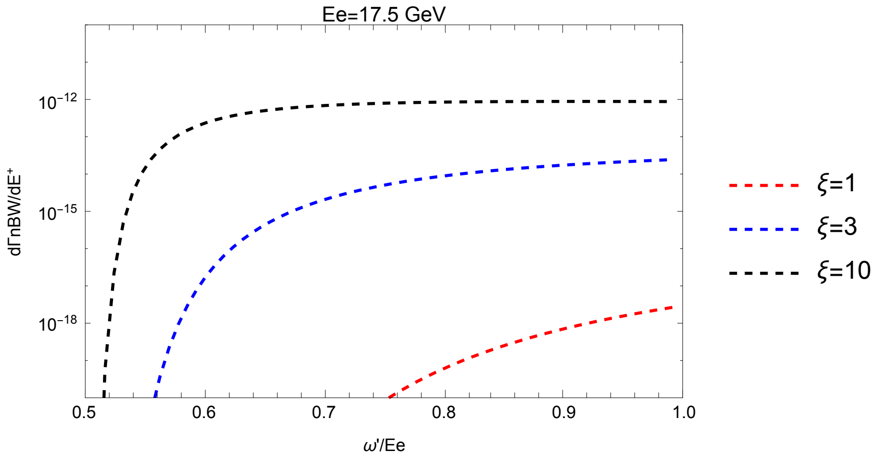
In[1245]:=
```
Clear[z, u, χ, α, m, e, Ee, w, dΓnlBWdωeq15, ωp, C1,
 Jnzeq7, dJnzeq7, Fneq3, Fneq7, uneq6, zeq5, dΓnlBWdEeq9, mGeV]
Clear[Fig2aξ1full, Fig2aξ3full, Fig2aξ10full, Fig2aξ1, Fig2aξ3, Fig2aξ10,
 Fig2bξ1full, Fig2bξ3full, Fig2bξ10full, Fig2bξ1, Fig2bξ3, Fig2bξ10, Fig2cξ1full,
 Fig2cξ3full, Fig2cξ10full, Fig2cξ1, Fig2cξ3, Fig2cξ10, 𝒥neq10, zeq12, uneq13]

α = 1 / 137;(*[]*)
m = 9.1 × 10^-31;(*[Kg]*)
e = 1.6 × 10^-19;(*[C]*)

(* number of harmonics to include *)
nmaxmin = 3;

C1 = 10^33.9;(*[]*)

(* large -ξ approximation *)
dΓnlBWdωeq15[x_, ξ_, EeGeV_] :=
 Module[{z, u, w, ξs, χγ, γe, Ee, ωpGeV, Ep, ωp, EpGeV, γγ},
   ωpGeV = x EeGeV; (*[GeV]*)
```

```
ωp = ωpGeV e 10^9; (*[J]*)

(* fixed positron energy *)
EpGeV = EeGeV / 2; (*[GeV]*)
Ep = EpGeV e 10^9; (*[J]*)

(* The laser frequency is chosen as ω=1.55 eV... 0.8μm *)
ξs = 4.12 × 10^5 × 0.8; (*[]*)

γγ = ωpGeV / (0.511 × 10^-3); (* [] *)
γe = EeGeV / (0.511 × 10^-3); (* [] *)
Ee = EeGeV e 10^9; (*[J]*)

(* chi*)
χγ = 2 ξ γγ / ξs;

(* eq 13 *)
u = ωpGeV^2 / (4 EpGeV (ωpGeV - EpGeV)); (*[]*)

z = (4 u / χγ)^(2/3); (*[]*)

Return[C1 (α m^2)/(ωp^2) (NIntegrate[AiryAi[y], {y, z, ∞}] - 2/z (2 u - 1) AiryAiPrime[z])]
] // Quiet

LogPlot[{10^5 dΓnlBWdωeq15[x, 1, 8], dΓnlBWdωeq15[x, 3, 8], dΓnlBWdωeq15[x, 10, 8]},
 {x, 0.501, 0.99}, Frame → True, FrameLabel → {"ω'/Ee", "dΓnBW/dE⁺"},
 PlotLegends → {"ξ=1", "ξ=3", "ξ=10"},
 PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
 PlotLabel → "Ee=8 GeV", PlotPoints → 2, PlotRange → {{0.5, 1}, {10^-20, 10^-10}}]

LogPlot[{dΓnlBWdωeq15[x, 1, 17.5], dΓnlBWdωeq15[x, 3, 17.5],
  dΓnlBWdωeq15[x, 10, 17.5]}, {x, 0.501, 0.99}, Frame → True,
 FrameLabel → {"ω'/Ee", "dΓnBW/dE⁺"}, PlotLegends → {"ξ=1", "ξ=3", "ξ=10"},
 PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
 PlotLabel → "Ee=17.5 GeV", PlotPoints → 2, PlotRange → {{0.5, 1}, {10^-20, 10^-10}}]

LogPlot[{dΓnlBWdωeq15[x, 1, 45], dΓnlBWdωeq15[x, 3, 45], dΓnlBWdωeq15[x, 10, 45]},
 {x, 0.501, 0.99}, Frame → True, FrameLabel → {"ω'/Ee", "dΓnBW/dE⁺"},
 PlotLegends → {"ξ=1", "ξ=3", "ξ=10"},
 PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Black, Dashed}},
 PlotLabel → "Ee=45 GeV", PlotPoints → 2, PlotRange → {{0.5, 1}, {10^-17, 10^-11}}]
```

Out[1253]=



Out[1254]=



Out[1255]=