A General Introduction to Game Theory: An Interdisciplinary Approach*

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Abstract. Submissions to Problem Set 2 for COMPSCI/ECON 206 Computational Microeconomics, 2023 Spring Term (Seven Week - Second) instructed by Prof. Luyao Zhang at Duke Kunshan University.

Keywords: computational economics · game theory · innovative education.

1 Part I: Self-Introduction (2 points)

Yuchen Song is a sophomore student majoring in Computation and Design with tracks in Computer Science at DKU, who is interested in the intersection between Computer Science and other subjects. He knows the programming languages of Python, Java, and C++. He's now working in Prof. Ming-Chun Huang's Sensing and Interaction Lab. The research direction of the lab is the study of human brain waves. He is also a member of the DKU VEX team. In the team, he is responsible for robot programming with C and C++.

* Supported by Duke Kunshan University

2 Part II: Reflections on Game Theory (5 points)

– In the book "Theory of Games and Economic Behavior" by John von Neumann and Oskar Morgenstern (Neumann and Morgenstern [1], Neumann and Morgenstern, 1947) the authors introduce the fundamental of game theory, including utility theory. Later, John F. Nash showed the differences between cooperative and non-cooperative games and developed the Nash equilibrium for non-cooperative games (Nash Jr [2], Nash Jr, 1950). Then, Reinhard Selten first made some improvements to Nash Equilibrium by applying it to the analysis of dynamic strategic interaction (Selten and Bielefeld [3], Selten and Bielefeld, 1988). John C. Harsanyi introduced the analysis of incomplete information games, in which agents have no information about each other's strategies (Harsanyi [4], Harsanyi, 1968). In the late 1950s, Thomas C. Schelling developed game theory through its application in social science (Schelling [5], Schelling, 1980).

3 Part III: Bayesian Nash Equilibrium: Definition, Theorem, and Proof (3 points)

3.1 MULTIAGENT SYSTEMS Algorithmic, Game-Theoretic, and Logical Foundations: Definitions

Shoham and Leyton-Brown [6], here are the definitions.

Information Sets

Definition 1. *(Bayesian game: information sets) A Bayesian game is a tuple* (N, G, P, I) *where:*

· N *is a set of agents;*

· G *is a set of games with* N *agents each such that if* $g, g' \in G$ *then for each agent* $i \in N$ *the strategy space in* g *is identical to the strategy space in* g'*;*

· $P \in \prod(G)$ *is a common prior over games, where* $\prod(G)$ *is the set of all probability distributions over* G*; and*

· $I = (I_1, \cdots, I_N)$ *is a tuple of partitions of* G*, one for each agent (p.165).*

This is the definition of the information sets in the Bayesian game. From the definition, we can find that the features of the information sets in the Bayesian game are private information, incomplete information, uncertainty, and sequential decision-making.

Epistemic types

Definition 2. *(Bayesian game: types) A Bayesian game is a tuple* (N, A, Θ, p, u) *where:*

· N *is a set of agents;*

· $A = A_1 \times \cdots \times A_n$*, where* A_i *is the set of actions available to player* i*;*

- *$\Theta = \Theta_1 \times \cdots \Theta_n$, where Θ_i is the type space of player i;*
- *$p : \Theta \mapsto [0,1]$ is a common prior over types; and*
- *$u = (u_1, \cdots, u_n)$, where $u_i : A \times \Theta \mapsto \mathbb{R}$ is the utility function for player i. (p.167)*

This is the definition of an epistemic type of the Bayesian game, which is a way of defining uncertainty directly over a game's utility function.

Strategies and equilibria There are three definitions of expected utility in the Bayesian game, which are ex post, ex interim, and ex ante.

Definition 3. *(Ex post expected utility) Agent i's ex post expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by s and the agent' types are given by θ, is defined as*

$$EU_i(s, \theta) = \sum_{a \in A} (\prod_{j \in N} s_j(a_j|\theta_j)) u_i(a, \theta).$$

(p.168)

Definition 4. *(Ex interim expected utility) Agent i's ex interim expected utility in a Bayesian game (N, A, Θ, p, u), where i's type is θ_i and where the agents' strategies are given by the mixed-strategy profile s, is defined as*

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} (\prod_{j \in N} s_j(a_j|\theta_j)) u_i(a, \theta_{-i}, \theta_i),$$

or equivalently as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i(s, (\theta_i, \theta_{-i}))$$

(p.169)

Definition 5. *(Ex ante expected utility) Agent i's ex ante expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by the mixed-strategy profile s, is defined as*

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} (\prod_{j \in N} s_j(a_j|\theta_j)) u_i(a, \theta),$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta),$$

or again equivalently as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i),$$

(p.169)

Ex post expected utility refers to the expected utility of a decision after the realization of all relevant information, ex interim expected utility refers to the expected utility of a decision after some, but not all, the relevant information has been realized, and ex ante expected utility refers to the expected utility of a decision before any relevant information has been realized.

Definition 6. *(Best response in a Bayesian game) The set of agent i's best responses to mixed-strategy profile s_{-i} are given by*

$$BR_i(s_{-i}) = \underset{s'_i \in S_i}{argmax} EU_i(s'_i, s_{-i})$$

(p.169)

Bayes–Nash equilibrium Then we can define the Bayes–Nash equilibrium.

Definition 7. *(Bayes–Nash equilibrium) A Bayes–Nash equilibrium is a mixed-strategy profile s that satisfies $\forall i \; s_i \in BR_i(s_{-i})$ (p.170)*

Ex post equilibrium

Definition 8. *(Ex post equilibrium) An ex post equilibrium is a mixed-strategy profile s that satisfies $\forall \theta, \forall i, s_i \in argmax_{s'_i \in S_i} EU_i(s'_i, s_{-i})$ (p.173)*

The ex post equilibrium is a definition stronger than the Bayes–Nash equilibrium.

3.2 Twenty lectures on algorithmic game theory

According to Roughgarden [7], here are the materials.

The Challenge of Revenue Maximization: Bayesian Analysis Considering the follow model for classical approach is to use average-case or Bayesian analysis.

Definition 9.

- *A single-parameter environment. We assume that there is a constant M such that $x_i \leq M$ for every i and feasible solution $(x_1, \cdots, x_n) \in X$.*
- *Independent distributions $F_1, \cdots, F_n$ with positive and continuous density functions $f_1, \cdots f_n$, We assume that the private valuation v_i of of participant i is drawn from the distribution F_i. We also assume that the support of every distribution F_i belongs to $[0, v_{max}]$ for some $v_{max} < \infty$. (p.57)*

The Challenge of Revenue Maximization: One Bidder and One Item, Revisited With the above Bayesian model, we can construct single-bidder single-item auctions.

Definition 10. *The expected revenue of a posted price r is simply*

$$\underbrace{r}_{revenue\ of\ a\ sale} \cdot \underbrace{(1 - F(r))}_{prbability of a sale}$$

(p.57)

This definition defines a way to solve the best posted price r, given a distribution F. And an optimal posted price is called a monopoly price of the distribution of F.

Characterization of Optimal DSIC Mechanisms: Preliminaries

Definition 11. *The expected revenue of a DSIC mechanism (x, p) is, by definition,*

$$E_{v \sim F}[\sum_{i=1}^{n} p_i(v)],$$

where the expectation is with respect to the distribution $F = F_1 \times \cdots \times F_n$ over agents' valuations (p.59).

Characterization of Optimal DSIC Mechanisms: Virtual Valuations The formula for expected revenue uses the concept of virtual valuations.

Definition 12. *For an agent i with valuation distribution F_i and valuation v_i, her virtual valuation is defined as*

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

(p.59)

The virtual valuation of an agent depends on her own valuation and distribution, and not on those of the other agents.

Characterization of Optimal DSIC Mechanisms: Expected Revenue Equals Expected Virtual Welfare

Lemma 1. *For every single-parameter environment with valuation distributions $F_1, \cdots F_n$, every DSIC mechanism (x, p), every agent i, and every value v_{-i} of the valuations of the other agents,*

$$E_{v_i \sim F_i}[p_i(v)] = E_{v_i \sim F_i}[\phi_i(v_i) \cdot x_i(v)] \quad (3.2.1)$$

(p.60)

In this lemma, the expected payment of an agent equals the expected virtual value earned by the agent.

Based on Lemma 5.1, we have the following important result.

Theorem 1. *(Exp. Revenue Equals Exp. Virtual Welfare)*
For every single-parameter environment with valuation distributions $F_1, \cdots, F_n$ and every DSIC mechanism (x, p),

$$\underbrace{E_{v\sim F}}_{expected\ revenue} = \underbrace{E_{v\sim F}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v)]}_{expected\ virtual\ welfare} \quad (3.2.2)$$

(p.60)

Proof. Taking the expectation, with respect to $v_{-i} \sim F$ of both sides (1.1) of we obtain

$$E_{v\sim F}[p_i(v)] = E_{v\sim F}[\phi_i(v_i) \cdot x_i(v)]$$

Applying the linearity of expectation (twice) then gives

$$\begin{aligned} E_{v\sim F}[\sum_{i=1}^{n} p_i(v)] &= \sum_{i=1}^{n} E_{v\sim F}[p_i(v)] \\ &= \sum_{i=1}^{n} E_{v\sim F}[\phi_i(v_i) \cdot x_i(v)] \\ &= E_{v\sim F}[\sum_{i=1}^{n} \phi_i(v_i) \times x_i(v)], \end{aligned}$$

as desired. (p.61)

The theorem refers that maximizing expected revenue over the space of DSIC mechanisms reduces to maximizing expected virtual welfare over the same space.

Characterization of Optimal DSIC Mechanisms: Regular Distributions the following definition identifies a sufficient condition for monotonicity.

Definition 13. *A distribution F is regular if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is not decreasing. (p.62)*

Theorem 2. *(Virtual Welfare Maximizers Are Optimal)*
For every single-parameter environment and regular distributions $F_1, \cdots, F_n$, the corresponding virtual welfare maximizer is a DSIC mechanism with the maximum-possible expected revenue. (p.63)

Proof of Lemma 1

Proof. **Step 1:** Fix an agent i By Myerson's payment formula, we can write the expected (over $v_i \sim F_i$) payment by i for a given value of v_{-i} as

$$\begin{aligned} E_{v_i \sim F_i}[p_i(v)] &= \int_0^{v_{max}} p_i(v) f_i(v_i) dv_i \\ &= \int_0^{v_{max}} [\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz] f_i(v_i) dv_i. \end{aligned}$$

This step is to rewrite the expected payment in terms of the allocation rule.
Step 2: Reversing the order of integration in

$$\int_0^{v_{max}} [\int_0^{v_i} z \cdot x_i'(z, v_{-i} dz] f_i(v_i) dv_i$$

yields

$$\int_0^{v_{max}} [\int_0^{v_i} f_i(v_i) dv_i] z \cdot z \cdot x_i'(z, v_{-i}) dz,$$

which simplifies to

$$\int_0^{v_{max}} (1 - F_i(z)) \cdot z \cdot x_i'(z, v_{-i}) dz,$$

suggesting that we're on the right track.
Step 3: Continuing simplifications:

$$\int_0^{v_{max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x_i'(z, v_{-i} dz)}_{h'(z)}$$

$$\underbrace{= (1 - F_i(z)) \cdot z \cdot x_i(z, v_{-i}|_0^{v_{max}})}_{=0-0} - \int_0^{v_{max}} \underbrace{z - \frac{1 - F_i(z)}{f_i(z)}}_{\phi_i(z)} x_i(z, v_{-i}) f_i(z) dz.$$

$$= \int_0^{v_{max}} \underbrace{z - \frac{1 - F_i(z)}{f_i(z)}}_{\phi_i(z)} x_i(z, v_{-i}) f_i(z) dz \quad (3.1.3)$$

Step 4: interpreting (3.1.3) as an expected value, with z drawn from the distribution F_i. Recalling

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

which is the virtual valuations, the expectation is $E_{v_i \sim F_i}[\phi_i(v_i) \cdot x_i(v)]$. Summarizing, we have

$$E_{v_i \sim F_i}[p_i(v)] = E_{v_i \sim F_i}[\phi_i(v_i) \cdot x_i(v)],$$

as desired. (p.66)

4 Part IV: Game Theory Glossary Tables (5 points)

Glossary	Definition	Sources
Perfect Bayesian equilibrium	A refinement of Bayesian Nash equilibrium that takes into account the possibility of a player learning new information during the game.	Myerson [8]
Regret minimization	A strategy learning algorithm improves strategy by minimizing player's regret for choosing different strategies.	Hart and Mas-Colell [9]
Repeated game	In a game, a player can play the same or similar game multiple times, which may affect the player's choice of strategy.	Nash Jr [2]
Deterministic strategy	A strategy that specifies a single action for each possible information set in a game.	Neumann and Morgenstern [1]
Cooperative game	A cooperative game is a game in which players can form coalitions and work together to achieve a common goal.	Neumann and Morgenstern [1]

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