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1 Introduction

Generally there are two points of view to analyze dynamical systems - Markov chain model and Differential equations in a probabilistic and deterministic context respectively. Analysis of differential equation is often more efficient and easy both mathematically and computationally. Therefore, the normal way to describe a dynamical system is to derive differential equations for describing the evolution of the system with respect to time and ensuring that the system follows those differential equations "closely" with high probability. In this paper we want to study the evolution of a graph where all nodes are Markov chains that interact with their neighbour. The method that we use to derive the differential equation is by approximating the rate of change of the proportion vector by the total *drift* as described by Darling and Norris in [1].

1.1 Example to illustrate approximations using drift

Drift is defined as the the product of total jump by the total rate, which may vary from state to state. When a number of different types of jump are present in the system the drift can be calculated by summing over the product of all the jumps and their corresponding rates. Once we have the drift, we can write the differential equation for the evolution of process as the rate of change of proportion of particles with a certain property will be equal to the drift.

Consider a system where $(X_t)_{t\geq 0}$ is a Markov chain that increases by 1 at rate λ and decreases by 1 at rate μ. Set $\mathcal{X}_t = X_t/N$. Note that $\mathcal{X}$ takes jumps of size 1/N at rate λN and $-1/N$ at rate μN. The drift is then $\lambda - \mu$ and the differential equation is

$$\dot{\mathcal{X}}_t = \lambda - \mu$$

If we take as initial state $\mathcal{X}_{\prime} = X_0 = 0$, then we may expect that $\mathcal{X}_t$ stay close to the solution $x_t = (\lambda - \mu)t$. This is a law of large numbers for the Poisson process and the error of approximation decreases with N which is the number of transitions that are possible within the system.

Now, we get back to our original setup of dynamic processes on networks and apply this method to obtain approximate master equations for the process.

2 Derivation of approximate master equations for SIS dynamics

We restrict ourselves to binary state dynamics, i.e.,$X_n = \{0, 1\}$ on undirected graphs. In order to perform the transformation, we have to define mappings. Let us denote the number of infected neighbours of a given vertex with m and also denote the rate of transition of a vertex, having k neighbours and m infected neighbours out of them, from a local state x to a local state y with $r_k^m(x, y)$. At a given time t, let s be the global state vector, i.e., the vector of 0's and 1's depending on the infection status of the nodes at a particular time t. Say for example, (0,1,0,0,1) is a global state vector for a graph on 5 vertices that are susceptible, infected, susceptible, susceptible and infected respectively. We define functions $\phi : V \times S \to N$ mapping a node index and a global state s to the number of neighbours of a given vertex in state 1 with respect to s. Further, we define the function $\psi : V \times S \to X$ that maps a node index and the global state $s \in S$, to its local state with respect to s. Let us denote the set of infected vertices for a given global state vector as Inf(s). So, the aforementioned state vector s=(0,1,0,0,1), the set Inf(s) will be {2,5}.

So, from the preceding literature and references we calculate the drift. We divide the drift into two parts according to the nature of the change that takes place:

- $Drift_{recovery}$
- $Drift_{infection}$

$$Drift_{recovery} = \mu \sum_{i \in Inf(s)} [e_{d(i),\phi(i,s'),\psi(i,s')} - e_{d(i),\phi(i,s),\psi(i,s)} + e_{g_{i,s}}] \tag{1}$$

Let us focus on the above expression term by term and try to simplify them individually. Since we are concerned with recovery rates, the state of vertex i in state s will be infected while that in s' will be susceptible. So $\psi(i, s')$ is 0 and $\psi(i, s)$ is 1. Thus the first term inside brackets becomes,

$$\sum_{i \in Inf(s)} e_{d(i),\phi(i,s'),\psi(i,s')} = \sum_{i \in Inf(s)} e_{d(i),\phi(i,s),0} = \sum_k \sum_m I_{k,m} e_{k,m,0}$$

The last equality follows since in the last term we sum the product of the number of infected vertices with k neighbours and m infected neighbours and the unit vector corresponding to degree k, number of infected neighbours m and infection status 0 over all possible degrees and all possible number of infected neighbours which precisely is the sum of unit vectors over all infected vertices in s. Moreover note that since only i changes its state but the neighbours of i don't change state, $\phi(i, s') = \phi(i, s)$.

Similarly the second term in equation (1) becomes,

$$\sum_{i \in Inf(s)} e_{d(i),\phi(i,s),\psi(i,s)} = \sum_k \sum_m I_{k,m} e_{k,m,1}$$

Now the third term in (1) needs more careful inspection than the previous two, which goes:

$$\sum_{i \in Inf(s)} e_{g_{i,s}} = \sum_{i \in Inf(s)} \sum_{j \in N(i)} [e_{d(j),\phi(j,s)-1,\psi(j,s)} - e_{d(j),\phi(j,s),\psi(j,s)}] \tag{2}$$

Let us sum the terms individually. So, the first term becomes,

$$\begin{aligned}
&\sum_{i \in Inf(s)} \sum_{j \in N(i)} e_{d(j),\phi(j,s)-1,\psi(j,s)} \\
&= \sum_{\substack{j \in [N] \\ j\ has\ at\ least\ 1 \\ infected\ neighbour}} \quad \sum_{\substack{i \in N(j) \\ i\ is\ infected}} e_{d(j),\phi(j,s)-1,\psi(j,s)} \\
&= \sum_{j \in [N]} \phi(j,s) e_{d(j),\phi(j,s)-1,\psi(j,s)} \\
&= \sum_k \sum_m (m\ s_{k,m} e_{k,m-1,0} + m\ i_{k,m} e_{k,m-1,1})
\end{aligned} \tag{3}$$

The first equality can be obtained by interchanging the sums (the second line is the sum of all vertices j such that it has at least one infected neighbour and then the sum of unit vectors is over all i which are neighbours of j). Also the last equality is obtained by conditioning on the state of j (susceptible or infected) $\psi(j,s)$ and summing over all possible degrees and all possible number of infected neighbours.

Similarly, the second term in (2) becomes,

$$\begin{aligned}
&\sum_{i \in Inf(s)} \sum_{j \in N(i)} e_{d(j),\phi(j,s),\psi(j,s)} \\
&= \sum_k \sum_m (m\ s_{k,m} e_{k,m,0} + m\ i_{k,m} e_{k,m,1})
\end{aligned} \tag{4}$$

Similarly we define

$$Drift_{infection} = \sum_{i \in sus(s)} \lambda \phi(i,s) [e_{d(i),\phi(i,s'),\psi(i,s')} - e_{d(i),\phi(i,s),\psi(i,s)} + e_{g_{i,s}}] \tag{5}$$

Again looking at the terms of equation (5) one by one,

$$\begin{aligned}
&\sum_{i\ \in\ sus(s)} \lambda \phi(i,s) e_{d(i),\phi(i,s'),\psi(i,s')} \\
&= \sum_k \sum_m \lambda\ m e_{k,m,1} s_{k,m}
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\sum_{i\ \in\ sus(s)} \lambda\phi(i,s)e_{d(i),\phi(i,s'),\psi(i,s')} \\
&= \sum_k \sum_m \lambda\ m e_{k,m,0} s_{k,m}
\end{aligned} \tag{7}$$

Note $\phi(i,s) = \phi(i,s')$ since the neighbours of i are not changing states.

And the last term becomes

$$\begin{aligned}
&\sum_{i\ \in\ sus(s)} \lambda\phi(i,s)e_{g_{i,s}} \\
&= \sum_{i\ \in\ sus(s)} \sum_{j \in\ N(i)} \lambda\phi(i,s)(e_{d(j),\phi(j,s)+1,\psi(j,s')} - e_{d(j),\phi(j,s),\psi(j,s)})
\end{aligned} \tag{8}$$

Again similarly we deal with the terms separately. Now, $\psi(j,s) = \psi(j,s')$ since i is changing state but j is not changing its state. The first term becomes,

$$= \sum_{\substack{j\in[N] \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} \lambda\phi(i,s)e_{d(j),\phi(j,s)+1,\psi(j,s)} \tag{9}$$

The above sum can be separated conditioned on the state of j. So (9)

$$\begin{aligned}
&= \sum_{\substack{j\in[N] \\ j\ is\ infected \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} \lambda\phi(i,s)e_{d(j),\phi(j,s)+1,\psi(j,s)} \\
&+ \sum_{\substack{j\in[N] \\ j\ is\ susceptible \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} \lambda\phi(i,s)e_{d(j),\phi(j,s)+1,\psi(j,s)}
\end{aligned} \tag{10}$$

The first term of equation (10) becomes,

$$= \sum_{\substack{j\in[N] \\ j\ is\ susceptible \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \lambda\ e_{d(j),\phi(j,s)+1,\psi(j,s)} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} \phi(i,s) \tag{11}$$

Now we need $\sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} \phi(i,s)$ which for a fixed j, is the sum of number of infected neighbours of all susceptible neighbours of j where j is infected. The average number of infected neighbours of any vertex k which is susceptible and is a neighbour of j which is infected is

$$
\begin{aligned}
local\ average\ for\ j &= \frac{\sum_{\substack{i \in N(j) \\ i\ is\ susceptible}} \phi(i,s)}{susceptible\ neighbours\ of\ j} \\
&= \frac{\sum_{\substack{i \in N(j) \\ i\ is\ susceptible}} \phi(i,s)}{d(j) - \phi(j,s)}
\end{aligned}
$$

$$
\therefore \sum_{\substack{i \in N(j) \\ i\ is\ susceptible}} \phi(i,s) = (local\ average\ for\ j)\ *\ (d(j) - \phi(j,s))
$$

Since there is no closed form for this sum on the LHS, we can use an approximation for $\phi(i,s)$ and we replace the local average which is conditioned on j, by a global average, $m_{av}^{i,s}$. Here $m_{av}^{x,y}$ is defined as the average number of infected neighbours of any vertex i which has state y and is a neighbour of j which has state x. So (11) becomes,

$$
\begin{aligned}
&= \sum_{\substack{j \in [N] \\ j\ is\ infected \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \lambda\ e_{d(j),\phi(j,s)+1,\psi(j,s)}(d(j) - \phi(i,s)) m_{av}^{i,s} \\
&= \sum_k \sum_m I_{k,m} \lambda m_{av}^{i,s} (k-m) e_{k,m+1,1}
\end{aligned}
\tag{12}
$$

Similarly the second part of equation (10) can be expressed as

$$
\begin{aligned}
&= \sum_{\substack{j \in [N] \\ j\ is\ infected \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \lambda\ e_{d(j),\phi(j,s)+1,\psi(j,s)}(d(j) - \phi(i,s)) m_{av}^{s,s} \\
&= \sum_k \sum_m S_{k,m} \lambda m_{av}^{s,s} (k-m) e_{k,m+1,0}
\end{aligned}
\tag{13}
$$

Hence, the contribution of $e_{g_{i,s}}$ term to the sum in (1) becomes

$$
\sum_k \sum_m I_{k,m} \lambda m_{av}^{i,s} (k-m) e_{k,m+1,1} + \sum_k \sum_m S_{k,m} \lambda m_{av}^{s,s} (k-m) e_{k,m+1,0}
$$

Now, finally we combine all the terms derived above. If M_t denotes the proportion vector then M_t approximately follows the differential equation:

$$
\dot{M_t} = Drift
$$

$$
\begin{aligned}
\implies \dot{M}_t = \frac{\mu}{N}[\sum_k \sum_m (& I_{k,m}e_{k,m,0} - I_{k,m}e_{k,m,1} + mS_{k,m}e_{k,m-1,0} \\
&+ mI_{k,m}e_{k,m-1,1} - mS_{k,m}e_{k,m,0} - mI_{k,m}e_{k,m,1})] \\
&+ \lambda/N \sum_k \sum_m [mS_{k,m}e_{k,m,1} - mS_{k,m}e_{k,m,0} \\
&+ m_{av}^{s,s}(k-m)S_{k,m}e_{k,m+1,0} + m_{av}^{i,s}(k-m)I_{k,m}e_{k,m+1,1} \\
&- m_{av}^{s,s}(k-m)S_{k,m}e_{k,m,0} - m_{av}^{i,s}(k-m)I_{k,m}e_{k,m,1}]
\end{aligned} \tag{14}
$$

Now, by equating the entries in a particular row on both sides of the equation we get the approximate master equations for proportions of susceptible vertices with degree k and m infected neighbours

$$
\begin{aligned}
\frac{d}{dt}s_{k,m} = \mu i_{k,m} + \mu(m+1)s_{k,m+1} - \mu m s_{k,m} - \lambda m s_{k,m} \\
+ \lambda m_{av}^{s,s}(k-m+1)s_{k,m-1} - \lambda m_{av}^{s,s}(k-m)s_{k,m}
\end{aligned} \tag{15}
$$

Similarly, we get the equations for the infected vertices also,

$$
\begin{aligned}
\frac{d}{dt}i_{k,m} = -\mu i_{k,m} + \mu(m+1)i_{k,m+1} - \mu m i_{k,m} + \lambda m s_{k,m} \\
+ \lambda(k-m+1)m_{av}^{i,s}i_{k,m-1} - \lambda m_{av}^{i,s}(k-m)i_{k,m}
\end{aligned} \tag{16}
$$

Here, it is important to note that to describe the system we need appropriate estimates of $m_{av}^{s,s}$ and $m_{av}^{i,s}$. Gleeson takes the estimates for $m_{av}^{s,s}$ and $m_{av}^{i,s}$ as β^s and β^i respectively.[2]

3 Derivation of approximate master equations for two state dynamics on networks

$$
\begin{aligned}
Drift_{recovery} &= \frac{1}{N}\sum_{i\in Inf(s)} R_{d(i),\phi(i,s)}[e_{d(i),\phi(i,s'),\psi(i,s')} - e_{d(i),\phi(i,s),\psi(i,s)} + e_{g_{i,s}}] \\
&= \frac{1}{N}\sum_{i\in Inf(s)} R_{d(i),\phi(i,s)}[e_{d(i),\phi(i,s'),0} - e_{d(i),\phi(i,s),1} + e_{g_{i,s}}]
\end{aligned} \tag{17}
$$

Looking only at the sum of the first term,

$$
\sum_{i\in Inf(s)} R_{d(i),\phi(i,s)}e_{d(i),\phi(i,s'),0} = \sum_k \sum_m R_{k,m}I_{k,m}e_{k,m,0} \tag{18}
$$

Similarly the second term becomes,

$$
\sum_{i\in Inf(s)} R_{d(i),\phi(i,s)}e_{d(i),\phi(i,s'),1} = \sum_k \sum_m R_{k,m}I_{k,m}e_{k,m,1} \tag{19}
$$

and the last term becomes:

$$\begin{aligned}&\sum_{i\in Inf(s)} R_{d(i),\phi(i,s)} e_{g_{i,s}}\\&=\sum_{i\in Inf(s)}\sum_{j\in N(i)} R_{d(i),\phi(i,s)}[e_{d(j),\phi(j,s)-1,\psi(j,s)} - e_{d(j),\phi(j,s),\psi(j,s)}]\end{aligned}\tag{20}$$

Looking further into the first term in (20)

$$\begin{aligned}&=\sum_{i\in Inf(s)}\sum_{j\in N(i)} R_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)-1,\psi(j,s)}\\&=\sum_{\substack{j\in[N]\\ j\ has\ at\ least\ 1\\ infected\ neighbour}}\ \sum_{\substack{i\in N(j)\\ i\ is\ infected}} R_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)-1,\psi(j,s)}\\&=\sum_{\substack{j\in[N]\\ j\ is\ infected\\ j\ has\ at\ least\ 1\\ infected\ neighbour}}\ \sum_{\substack{i\in N(j)\\ i\ is\ infected}} R_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)-1,\psi(j,s)}\\&+\sum_{\substack{j\in[N]\\ j\ is\ susceptible\\ j\ has\ at\ least\ 1\\ infected\ neighbour}}\ \sum_{\substack{i\in N(j)\\ i\ is\ infected}} R_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)-1,\psi(j,s)}\\&=\sum_{\substack{j\in[N]\\ j\ is\ infected}} e_{d(j),\phi(j,s)-1,1} \sum_{\substack{i\in N(j)\\ i\ is\ infected}} R_{d(i),\phi(i,s)}\\&+\sum_{\substack{j\in[N]\\ j\ is\ susceptible}} e_{d(j),\phi(j,s)-1,0} \sum_{\substack{i\in N(j)\\ i\ is\ infected}} R_{d(i),\phi(i,s)}\end{aligned}\tag{21}$$

Now we don't have the rates $R_{k,m}$, so we can instead approximate the sum $\sum_{\substack{i\in N(j)\\ i\ is\ infected}} R_{d(i),\phi(i,s)}$ (which is the local average rate of transition of the infected neighbours of j) by the global average rate of transition which we denote by $w_{av}^{x,y}$ (which is the average rate of transition of the vertex i which is in state y and has a neighbour j in state x)

So, rewriting (21), we get,

$$\begin{aligned}&=\sum_{\substack{j\in[N]\\ j\ is\ infected}} e_{d(j),\phi(j,s)-1,1}\phi(j,s)w_{av}^{i,i} + \sum_{\substack{j\in[N]\\ j\ is\ susceptible}} e_{d(j),\phi(j,s)-1,1}\phi(j,s)w_{av}^{s,i}\\&=\sum_k\sum_m(m\ I_{k,m}\ e_{k,m-1,1}\ w_{av}^{i,i} + m\ S_{k,m}\ e_{k,m-1,0}\ w_{av}^{s,i})\end{aligned}\tag{22}$$

Similarly the second term in (20) would be,

$$\begin{aligned}
&\sum_{i\in Inf(s)} \sum_{j\in N(i)} R_{d(i),\phi(i,s)} e_{d(j),\phi(j,s),\psi(j,s)} \\
&= \sum_k \sum_m (m\ S_{k,m}\ e_{k,m,0}\ w_{av}^{s,i} + m\ I_{k,m}\ e_{k,m,1}\ w_{av}^{i,i})
\end{aligned} \tag{23}$$

So, the final expression for $Drift_{recovery}$ is

$$\begin{aligned}
Drift_{recovery} = \frac{1}{N}[&\sum_k \sum_m R_{k,m} I_{k,m} e_{k,m,0} - \sum_k \sum_m R_{k,m} I_{k,m} e_{k,m,1} \\
&+ \sum_k \sum_m m\ S_{k,m}\ e_{k,m-1,0}\ w_{av}^{s,i} + \sum_k \sum_m m\ I_{k,m}\ e_{k,m-1,1}\ w_{av}^{i,i} \\
&- \sum_k \sum_m m\ S_{k,m}\ e_{k,m,0}\ w_{av}^{s,i} - \sum_k \sum_m m\ I_{k,m}\ e_{k,m,1}\ w_{av}^{i,i}]
\end{aligned} \tag{24}$$

For $Drift_{infection}$, we do a similar calculation

$$\begin{aligned}
Drift_{infection} &= \frac{1}{N} \sum_{i\in sus(s)} F_{d(i),\phi(i,s)} [e_{d(i),\phi(i,s'),\psi(i,s')} - e_{d(i),\phi(i,s),\psi(i,s)} + e_{g_{i,s}}] \\
&= \frac{1}{N} \sum_{i\in sus(s)} F_{d(i),\phi(i,s)} [e_{d(i),\phi(i,s'),1} - e_{d(i),\phi(i,s),0} + e_{g_{i,s}}]
\end{aligned} \tag{25}$$

Looking only at the sum of the first term,

$$\sum_{i\in sus(s)} F_{d(i),\phi(i,s)} e_{d(i),\phi(i,s'),1} = \sum_k \sum_m F_{k,m} S_{k,m} e_{k,m,1} \tag{26}$$

Similarly the second term becomes,

$$\sum_{i\in sus(s)} F_{d(i),\phi(i,s)} e_{d(i),\phi(i,s'),0} = \sum_k \sum_m F_{k,m} S_{k,m} e_{k,m,0} \tag{27}$$

and the last term becomes:

$$\begin{aligned}
&\sum_{i\in sus(s)} F_{d(i),\phi(i,s)} e_{g_{i,s}} \\
&= \sum_{i\in sus(s)} \sum_{j\in N(i)} F_{d(i),\phi(i,s)} [e_{d(j),\phi(j,s)+1,\psi(j,s)} - e_{d(j),\phi(j,s),\psi(j,s)}]
\end{aligned} \tag{28}$$

Now, looking at the first term in (28)

$$\begin{aligned}
&\sum_{i\in sus(s)}\sum_{j\in N(i)} F_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)+1,\psi(j,s)} \\
&= \sum_{\substack{j\in[N] \\ j\ has\ at\ least\ 1 \\ susceptible\ neighbour}} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} F_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)+1,\psi(j,s)} \\
&= \sum_{\substack{j\in[N] \\ j\ is\ infected \\ j\ has\ at\ least\ 1 \\ infected\ neighbour}} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} F_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)+1,\psi(j,s)} \\
&+ \sum_{\substack{j\in[N] \\ j\ is\ susceptible \\ j\ has\ at\ least\ 1 \\ infected\ neighbour}} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} F_{d(i),\phi(i,s)} e_{d(j),\phi(j,s)+1,\psi(j,s)} \\
&= \sum_{\substack{j\in[N] \\ j\ is\ infected}} e_{d(j),\phi(j,s)+1,1} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} F_{d(i),\phi(i,s)} \\
&+ \sum_{\substack{j\in[N] \\ j\ is\ susceptible}} e_{d(j),\phi(j,s)+1,0} \sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} F_{d(i),\phi(i,s)}
\end{aligned} \tag{29}$$

Now we don't have the rates $F_{k,m}$, so we can instead approximate the sum $\sum_{\substack{i\in N(j) \\ i\ is\ susceptible}} F_{d(i),\phi(i,s)}$ (which is the local average rate of transition of the neighbours of j) by the global average rate of transition which we denote by $w_{av}^{x,y}$ (which is the average rate of transition of the vertex i which is in state y and has a neighbour j in state x) as we did before in the case of the recovery drift.

So, rewriting (29), we get,

$$\begin{aligned}
&= \sum_{\substack{j\in[N] \\ j\ is\ infected}} e_{d(j),\phi(j,s)+1,1}\phi(j,s) w_{av}^{i,s} + \sum_{\substack{j\in[N] \\ j\ is\ susceptible}} e_{d(j),\phi(j,s)+1,1}\phi(j,s) w_{av}^{s,s} \\
&= \sum_k \sum_m ((k-m)\ I_{k,m}\ e_{k,m+1,1}\ w_{av}^{i,s} + (k-m)\ S_{k,m}\ e_{k,m+1,0}\ w_{av}^{s,s})
\end{aligned} \tag{30}$$

Similarly the second term in (28) would be,

$$\begin{aligned}
&\sum_{i\in sus(s)}\sum_{j\in N(i)} F_{d(i),\phi(i,s)} e_{d(j),\phi(j,s),\psi(j,s)} \\
&= \sum_k \sum_m ((k-m)\ S_{k,m}\ e_{k,m,0}\ w_{av}^{s,s} + (k-m)\ I_{k,m}\ e_{k,m,1}\ w_{av}^{i,s})
\end{aligned} \tag{31}$$

So, the final expression for $Drift_{infection}$ is

$$
\begin{aligned}
Drift_{infection} =& \frac{1}{N}[\sum_k \sum_m F_{k,m} S_{k,m} e_{k,m,1} - \sum_k \sum_m F_{k,m} S_{k,m} e_{k,m,0} \\
&+ \sum_k \sum_m ((k-m) S_{k,m} e_{k,m+1,0} w_{av}^{s,s} + (k-m) I_{k,m} e_{k,m+1,1} w_{av}^{i,s}) \\
&- \sum_k \sum_m ((k-m)\ S_{k,m}\ e_{k,m,0}\ w_{av}^{s,s} + (k-m)\ I_{k,m}\ e_{k,m,1}\ w_{av}^{i,s})]
\end{aligned} \tag{32}
$$

Finally we combine all the terms derived above. So, M_t which is the proportion vector, approximately follows the differential equation:

$$\dot{M}_t = Drift$$

$$
\begin{aligned}
\implies \dot{M}_t = & \frac{1}{N} \sum_k \sum_m (R_{k,m} I_{k,m} e_{k,m,0} - R_{k,m} I_{k,m} e_{k,m,1} + w_{av}^{si}\ m\ S_{k,m} e_{k,m-1,0} \\
&+ w_{av}^{ii}\ m\ I_{k,m} e_{k,m-1,1} - w_{av}^{si}\ m\ S_{k,m} e_{k,m,0} - w_{av}^{ii}\ m\ I_{k,m} e_{k,m,1}) \\
&+ \frac{1}{N} \sum_k \sum_m (F_{k,m} S_{k,m} e_{k,m,1} - F_{k,m} S_{k,m} e_{k,m,0} \\
&+ w_{av}^{s,s}(k-m) S_{k,m} e_{k,m+1,0} + w_{av}^{i,s}(k-m) I_{k,m} e_{k,m+1,1} \\
&- w_{av}^{s,s}(k-m) S_{k,m} e_{k,m,0} - w_{av}^{i,s}(k-m) I_{k,m} e_{k,m,1})
\end{aligned} \tag{33}
$$

Now, we compare the row on both sides of the equation same as before and obtain the approximate master equations as follows

$$
\begin{aligned}
\frac{d}{dt} s_{k,m} = & R_{k,m} i_{k,m} + w_{av}^{si}(m+1) s_{k,m+1} - w_{av}^{si} m s_{k,m} - F_{k,m} s_{k,m} \\
&+ w_{av}^{s,s}(k-m+1) s_{k,m-1} - w_{av}^{s,s}(k-m) s_{k,m}
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\frac{d}{dt} i_{k,m} = & -R_{k,m} i_{k,m} + w_{av}^{i,i}(m+1) i_{k,m+1} - w_{av}^{i,i} m i_{k,m} + F_{k,m} s_{k,m} \\
&+ w_{av}^{i,s}(k-m+1) i_{k,m-1} - w_{av}^{i,s}(k-m) i_{k,m}
\end{aligned} \tag{35}
$$

For any particular dynamic process all terms except $w_{av}^{x,y}$ are known. To obtain meaningful equations we need appropriate estimates of these quantities. Gleeson provides some estimates for $w_{av}^{x,y}$ in [2] which are:

$$
\begin{aligned}
\widehat{w_{av}^{s,i}} =& W(\sigma_i' = Skm \widehat{\to \sigma_i} = Sk(m-1))/m = \gamma^s \\
\widehat{w_{av}^{i,i}} =& W(\sigma_i' = Ikm \widehat{\to \sigma_i} = Ik(m-1))/m = \gamma^i \\
\widehat{w_{av}^{s,s}} =& W(\sigma_i' = Skm \widehat{\to \sigma_i} = Sk(m+1)/(k-m) = \beta^s \\
\widehat{w_{av}^{i,s}} =& W(\sigma_i' = Ikm \widehat{\to \sigma_i} = Ik(m+1))/(k-m) = \beta^i
\end{aligned}
$$

References

[1] R.W.R. Darling and J.R. Norris *Differential equation approximations for Markov chains* https://arxiv.org/pdf/0710.3269

[2] James P. Gleeson *Binary-State Dynamics on Complex Networks: Pair Approximation and Beyond* https://pdfs.semanticscholar.org/d7df/8ffdfb64efb0d9281767f1ef9dc538c05674.pdf