

On the distribution of the largest connected component size in random graphs with fixed edges set size

Robin Petit

March-April 2017

Contents

1	Introduction	1
1.1	Definitions	1
1.2	(Largest) Connected Components	2
1.3	Objectives	3
2	Preliminary Results	3
2.1	$ \Lambda_{k=1}(V, \cdot) $	3
2.2	Upper Boundary of m for $ \Lambda_k^m(V, \cdot) $	4
2.3	$ \Lambda_{k=2}(V, \cdot) $	5
3	Processing on Examples — Cardinality of $\Lambda_k^m(V, \cdot)$	5
3.1	Decomposing set $\Lambda_k(V)$	6
3.2	The Set $\Omega_{k,\alpha}^m(V)$	7
3.2.1	Special case of $\alpha = 1$	7
3.2.2	General Case	8
4	Counting connected graphs	9
4.1	Connected graphs of $ V $ vertices	9
4.2	Connected graphs of $ V $ vertices and m edges	10
5	Conclusion	10

1 Introduction

1.1 Definitions

Let's consider $V = \{v_1, \dots, v_{|V|}\}$ a set of vertices. We denote by $|V|$ the cardinality of the set V . Let's define the function:

$$X : \mathbb{N} \rightarrow \mathbb{N} : n \mapsto \frac{n(n-1)}{2}.$$

Definition 1.1. For $(\alpha, \beta) \in \mathbb{N}^2$, if $\beta \geq \alpha$, we define:

$$[\alpha, \beta] := \{n \in \mathbb{N} \text{ s.t. } \alpha \leq n \leq \beta\}.$$

Definition 1.2. An undirected graph Γ is denoted $\Gamma(V, E)$ for V its vertices set, and E its edges set, with $E = \{e_1, \dots, e_{|E|}\}$ and $\forall i \in [1, |E|] : e_i = \{v_{i1}, v_{i2}\}$ for $1 \leq i_1, i_2 \leq |V|$ with $i_1 \neq i_2$ (i.e. loops are not tolerated).

Remark. $|E|$ is usually denoted as m , and $|V|$ is sometimes denoted as n . Both these numbers are (non-strictly) positive integers.

Definition 1.3. The set of all the existing graphs having given vertices set V is denoted by $\Gamma(V, \cdot)$. We denote $\Gamma_m(V, \cdot)$ the subset of $\Gamma(V, \cdot)$ such that $|E| = m$.

Remark. We observe that:

$$\Gamma(V, \cdot) = \bigsqcup_{m \in \mathbb{N}} \Gamma_m(V, \cdot).$$

Definition 1.4. For every $n \in \mathbb{N}$, we define \mathcal{K}_n as the *complete graph* of size n .

Lemma 1.5. For a graph $\Gamma(V, E)$, we have $|E| \leq X(|V|)$.

Proof. We know that $\Gamma(V, E) \leq \mathcal{K}_{|V|}$, and $\mathcal{K}_{|V|}$ has exactly $X(|V|)$ edges (vertex v_i is connected to vertices v_{i+1} to $v_{|V|}$, so the number of edges is equal to $\sum_{i=1}^{|V|} (|V| - i) = \sum_{i=0}^{|V|-1} i = X(|V|)$. \square

Lemma 1.6. For given vertices set V and fixed number of edges $m \in \mathbb{N}$, we have:

$$|\Gamma_m(V, \cdot)| = \begin{cases} \binom{X(|V|)}{m} & \text{if } m \leq X(|V|) \\ 0 & \text{else} \end{cases}.$$

Proof. There are $X(|V|)$ edges $e = \{v_i, v_j\}$ which exist in $\mathcal{K}_{|V|}$. A graph $\Gamma(V, E)$ in $\Gamma_m(V, \cdot)$ has m of these $X(|V|)$ edges. The amount of such graphs is then $\binom{X(|V|)}{m}$. \square

Corollary 1.7. For given vertices set V , we have $|\Gamma(V, \cdot)| = 2^{X(|V|)}$.

Proof. Since $\Gamma(V, \cdot)$ is given by a disjoint union over m , its cardinality is equal to the sum of the individual cardinalities:

$$|\Gamma(V, \cdot)| = \sum_{m \in \mathbb{N}} |\Gamma_m(V, \cdot)| = \sum_{k=0}^{X(|V|)} |\Gamma_m(V, \cdot)| = \sum_{k=0}^{X(|V|)} \binom{X(|V|)}{k} = 2^{X(|V|)}.$$

\square

Definition 1.8. A graph $\Gamma(V, E)$ is said to be connected if for each $v, w \in V$, there exists a path between v and w . We denote by $\chi(V, \cdot)$ the set of all connected graphs having vertices set V . Again, for $m \in \mathbb{N}$, we denote by $\chi_m(V, \cdot) \subset \chi(V, \cdot)$ the set of connected graphs having m edges.

Remark. $\chi(V, \cdot) \subset \Gamma(V, \cdot)$, and:

$$\chi(V, \cdot) = \bigsqcup_{m \in \mathbb{N}} \chi_m(V, \cdot).$$

Lemma 1.9. For $m < |V| - 1$ or $m > X(|V|)$, we have $\chi_m(V, \cdot) = \emptyset$.

Proof. For $m > X(|V|)$, we know that $\Gamma_m(V, \cdot) = \emptyset$. As $\chi_m(V, \cdot) \subset \Gamma_m(V, \cdot)$, we know that $\chi_m(V, \cdot) = \emptyset$.

For $m < |V| - 1$, let's notice firstly that a cyclic graph needs at least $|V|$ edges: for a cyclic graph, each vertex's degree must be at least 2. We can deduce:

$$2|E| = \sum_{v \in V} \deg(v) \geq 2|V|.$$

Thus $|V| \geq |E|$ for a cyclic graph. Let's assume (*ad absurdum*) that there exists $\Gamma(V, E) \in \chi_m(V, \cdot)$. Therefore $\Gamma(V, E)$ is acyclic. By definition of $\chi_m(V, \cdot)$, we know that $\Gamma(V, E)$ is connected. But an acyclic and connected graph is a tree, and a tree has exactly $|V| - 1 \not\geq m$ edges, which is a contradiction. Therefore, $\chi_m(V, \cdot) = \emptyset$. \square

Definition 1.10. Let's define the function:

$$\mu : \mathcal{P}(V) \rightarrow \llbracket 1, |V| \rrbracket : W \mapsto \mu(W) := \inf \{i \in \llbracket 1, |V| \rrbracket \text{ s.t. } v_i \in W\}$$

representing the lowest index of a vertex present in a given subset of $W \subset V$.

Definition 1.11. For every $W \in \mathcal{P}(V)$, we define $\Delta_W : \Gamma(V, \cdot) \rightarrow \Gamma(W, \cdot) : \Gamma(V, E) \mapsto \Gamma'(W, E')$ such that:

$$E' = \{\{v_i, v_j\} \in E \text{ s.t. } v_i, v_j \in W\}.$$

Let's call Δ_W the *restriction to subset W operator*.

1.2 (Largest) Connected Components

Definition 1.12. We define the *connected component of vertex $v_i \in V$ in graph $\Gamma(V, E)$* by the biggest subset (in the sense of inclusion) W of V such that $v_i \in W$ and $\Delta_W(\Gamma(V, E)) \in \chi(W, \cdot)$.

For graph $\Gamma(V, E) \in \Gamma(V, \cdot)$, we define $|LCC(\Gamma(V, E))|$ by:

$$|LCC(\Gamma(V, E))| := \max_{W \in \mathcal{P}(V)} |W| \mathbb{I}_{[\Delta_W(\Gamma(V, E)) \in \chi(W, \cdot)]}.$$

We then define the *largest connected component of the graph $\Gamma(V, E)$* as:

$$LCC(\Gamma(V, E)) := \arg \min_{\substack{W \in \mathcal{P}(V) \\ |W| = |LCC(\Gamma(V, E))|}} \mu(W).$$

The set $\Lambda_k^m(V, \cdot)$ is then the set of all graphs $\Gamma(V, E) \in \Gamma(V, \cdot)$, such that $|E| = m$ and $|LCC(\Gamma(V, E))| = k$.

Remark. The notations here are consistent since $|LCC(\Gamma(V, E))|$ corresponds indeed to the cardinality of $LCC(\Gamma(V, E))$.

Furthermore, this definition of largest connected component allows to define uniquely the LCC, even though a graph $\Gamma(V, E)$ has several connected component of same size. For example, following graph has two connected component of size 2, i.e. $\{1, 2\}$ (in red) and $\{3, 4\}$ (in blue).

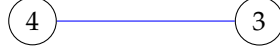
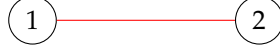


Figure 1: Graph $\Gamma(\llbracket 1, 4 \rrbracket, \{\{1, 2\}, \{3, 4\}\})$

Nevertheless, the LCC operator yields $\{1, 2\}$ since it minimizes the lowest id of element in connected component (1 for this graph).

Remark. Since $\Lambda_k(V, \cdot) = \bigsqcup_{m=0}^{X(\llbracket V \rrbracket)} \Lambda_k^m(V, \cdot)$ and:

$$\Gamma(V, \cdot) = \bigsqcup_{k=1}^{|V|} \bigsqcup_{m=0}^{X(\llbracket V \rrbracket)} \Lambda_k^m(V, \cdot),$$

we want to know what is $|\Lambda_k^m(V, \cdot)|$ equal to.

Definition 1.13. Let's declare a new random variable $\mathcal{G}(V, m)$, a graph uniformly distributed in $\Gamma_m(V, \cdot)$, thus such that:

$$\forall \Gamma(V, E) \in \Gamma_m(V, \cdot) : \mathbb{P}[\mathcal{G}(V, m) = \Gamma(V, E)] = \frac{1}{|\Gamma_m(V, \cdot)|} = \frac{1}{\binom{X(\llbracket V \rrbracket)}{m}}.$$

1.3 Objectives

The objective now is to find an expression for $|\Lambda_k(V, \cdot)|$ since we are looking for:

$$\mathbb{P}[|\text{LCC}(\mathcal{G}(V, m))| = k] = \frac{|\Lambda_k(V, \cdot)|}{|\Gamma(V, \cdot)|} = \frac{1}{|\Gamma(V, E)|} \sum_{m=0}^{X(\llbracket V \rrbracket)} |\Lambda_k^m(V, \cdot)|.$$

Let's denote this value $p_k := \mathbb{P}[|\text{LCC}(\mathcal{G}(V, m))| = k]$.

2 Preliminary Results

The general idea in order to determine $|\Lambda_k^m(V, \cdot)|$ is to insert a connected component of size k on vertices set V , and then to tally the configurations placing $m-k$ vertices without making a bigger connected component than the first one.

2.1 $|\Lambda_{k=1}(V, \cdot)|$

It is trivial to tell $|\Lambda_1^m(V, \cdot)| = \delta_0^{m1}$, i.e. equals one if $m = 0$ and equals zero if $m > 0$: a graph having at least one edge, cannot have a largest connected component of size 1 because if $e = \{v_i, v_j\}$ is an edge in E , then $\{v_i, v_j\} \subset V$ is a connected component of size 2.

¹ δ_i^j is the Kronecker delta operator.

2.2 Upper Boundary of m for $|\Lambda_k^m(V, \cdot)|$

Lemma 2.1 (Upper boundary of edges amount for $k = 2$). For $m > \frac{|V|}{2}$, we have $\Lambda_2^m(V, \cdot) = \emptyset$.

Proof. To have a largest connected component of size 2, each vertex must have degree 0 or 1. Take $m \in \mathbb{N}$ such that $m > \frac{|V|}{2}$. Take $\Gamma(V, E)$ such that $|E| = m$, and take $\mathcal{V}_1 := \{v \in V \text{ s.t. } \deg(v) \leq 1\} \subset V$. Take the restriction $\Gamma'(\mathcal{V}_1, E') = \Delta_{\mathcal{V}_1}(\Gamma(V, E))$.

Since in a graph, the sum of the degree of each vertex is equal to twice the amount of edges, when applied on Γ' , it follows that:

$$2|E'| = \sum_{v \in \mathcal{V}_1} \deg(v) \leq \sum_{v \in \mathcal{V}_1} 1 = |\mathcal{V}_1|.$$

We then deduce that $|E'| \leq \frac{|\mathcal{V}_1|}{2} \leq \frac{|V|}{2}$. Thus \mathcal{V}_1 must be *strictly* included in V , and then there must exist $v \in V$ such that $\deg(v) \geq 2$. Thus:

$$\forall m > \frac{|V|}{2} : \forall \Gamma(V, E) \in \Gamma_m(V, \cdot) : \Gamma(V, E) \notin \Lambda_2^m(V, \cdot).$$

□

Lemma 2.2. For $\Gamma(V, E) \in \Gamma(V, \cdot)$ a graph and $k \in \llbracket 1, |V| \rrbracket$, if there exists a vertex $v \in V$ such that $\deg(v) = k$, then $|\text{LCC}(\Gamma(V, E))| \geq k + 1$.

Proof. Take $v \in V$ such that $\deg(v) = k$. There exist $\{v_{i_1}, \dots, v_{i_k}\} \subset V$ such that:

$$\forall j \in \llbracket 1, k \rrbracket : \{v, v_{i_j}\} \in E.$$

Thus $\{v, v_{i_1}, \dots, v_{i_k}\}$ is a connected component of size $k + 1$. Thus the largest connected component must have size at least that big. □

Proposition 2.3 (Upper boundary of edges amount generalized). For $k \in \llbracket 1, |V| \rrbracket$, and $m > \frac{|V|(k-1)}{2}$, we have $\Lambda_k^m(V, \cdot) = \emptyset$.

Proof. Take $m > \frac{(k-1)|V|}{2}$, and $\Gamma(V, E) \in \Gamma_m(V, \cdot)$. Take $\mathcal{V}_k := \{v \in V \text{ s.t. } \deg(v) \leq k-1\}$. Let $\Gamma'(\mathcal{V}_k, E')$ be defined by $\Delta_{\mathcal{V}_k}(\Gamma(V, E))$. We know that:

$$2|E'| = \sum_{v \in \mathcal{V}_k} \deg(v) \leq (k-1)|\mathcal{V}_k| \leq (k-1)|V|.$$

We deduce that $|E'| \leq \frac{(k-1)|V|}{2} < m = |E|$. Thus $|E| \not\geq |E'|$, and this implies that there exists $v \in V$ such that $\deg(v) \geq k$. By previous lemma, largest connected component size must be at least $k + 1$. □

Remark. We can understand this upper boundary as $m > \frac{|V|(k-1)}{2} = \frac{|V|}{k} \frac{k(k-1)}{2} = \frac{|V|}{k} \cdot X(k)$. So in order to have a LCC of size k , edges can be distributed to make $\left\lfloor \frac{|V|}{k} \right\rfloor$ complete graphs having each $X(k)$ edges. The maximum amount of edges is then given by $\frac{|V|(k-1)}{2}$.

2.3 $|\Lambda_{k=2}(V, \cdot)|$

Example of size 2 is a bit more complicated:

$$\forall m \in \left[1, \left\lfloor \frac{|V|}{2} \right\rfloor\right] : |\Lambda_2^m(V, \cdot)| = \begin{cases} \frac{1}{m!} \prod_{k=0}^{m-1} \binom{|V|-2k}{2} & \text{if } m \leq \frac{|V|}{2} \\ 0 & \text{else} \end{cases}.$$

Proof. For $m > \frac{|V|}{2}$, result is shown in Lemma 2.1. The part $\prod_{k=0}^{m-1} \binom{|V|-2k}{2}$ corresponds to the choice of m edges without making a connected component of size ≥ 3 .

$\binom{|V|-2\cdot 0}{2}$ is the choice of the first edge (two vertices) among $|V|$ vertices, $\binom{|V|-2}{2}$ is the choice of the second edge (two vertices) among the $|V| - 2$ vertices left, etc. At step ℓ , only $|V| - 2(\ell - 1)$ vertices are available because two are selected per step, and a selected vertex cannot be used again, otherwise its degree would be ≥ 2 , and then the largest component size would be ≥ 3 .

The $\frac{1}{m!}$ comes from the fact that the order the edges are selected doesn't matter (so for each choice of m edges, there are $m!$ permutations of these). \square

Remark. This can also be expressed as:

$$|\Lambda_2^m(V, \cdot)| = \frac{1}{m!} \frac{|V|!}{2^m (|V| - 2m)!},$$

by simplification of the product.

3 Processing on Examples — Cardinality of $\Lambda_k^m(V, \cdot)$

$$\begin{aligned} |\Lambda_3^0(V, \cdot)| &= |\Lambda_3^1(V, \cdot)| = 0 \\ |\Lambda_3^2(V, \cdot)| &= \binom{|V|}{3} \binom{3}{2} \\ |\Lambda_3^3(V, \cdot)| &= \binom{|V|}{3} \binom{3}{3} + \binom{|V|}{3} \binom{3}{2} \binom{|V|-3}{2} \\ |\Lambda_3^4(V, \cdot)| &= \binom{|V|}{3} \binom{3}{3} \binom{|V|-3}{2} + \binom{|V|}{3} \binom{3}{2} \binom{|V|-3}{3} \binom{3}{2} + \binom{|V|}{3} \binom{3}{2} \binom{|V|-3}{2} \binom{|V|-5}{2}. \end{aligned}$$

Definition 3.1. Let's denote equally $|\Lambda_k^m(n)| = \Lambda_k^m(n) \equiv |\Lambda_k^m(V, \cdot)|$ for V such that $|V| = n$.

This notation allows to lighten the expressions.

Remark. Current explorations tend to a formula looking something like:

$$|\Lambda_k^m(V)| = \binom{|V|}{k} \sum_{\ell=k-1}^{\min(m, X(k))} |\Lambda_k^\ell(k)| \sum_{p=1}^k |\Lambda_p^{m-\ell}(|V| - k)| \beta_{p\ell}(m, k, |V|),$$

with $\beta_{p\ell}(m, k, |V|)$, a coefficient.

The idea behind this formula is explained in introduction of Section 2: to find the amount of graphs having n vertices, m edges and a largest connected component of size k , let's place a connected component of size k somewhere in the graph (so choose k in $|V|$ vertices), and then multiply this by the amount of possible graphs of largest connected component of size $p \in \{1, \dots, k\}$ (so lower or equal to k).

Idea of proof to be extended later on. In order to prove the equality of the cardinalities, let's find a bijective function Ω between $\Lambda_k^m(V, \cdot)$ and a set like:

$$\Omega_k^m(V) := \bigsqcup_{\substack{W \in \mathcal{P}(V) \\ |W|=k}} \bigsqcup_{\ell=k-1}^{\min(m, X(k))} \Lambda_k^\ell(W, \cdot) \times \left(\bigsqcup_{p=1}^k \Lambda_p^{m-\ell}(V \setminus W, \cdot) \right).$$

□

Lemma 3.2. *The sets $\chi_\ell(V, \cdot)$ and $\Lambda_{|V|}^\ell(V, \cdot)$ are equal.*

Proof. A graph $\Gamma(V, E)$ is connected if and only if its largest connected component contains all its vertices, i.e. $\text{LCC}(\Gamma(V, E)) = V$.

This is equivalent to say that $|\text{LCC}(\Gamma(V, E))| = |V|$ since $\forall W \in \mathcal{P}(V) : |W| = |V| \Rightarrow V = W$:

$$\forall W \in \mathcal{P}(V) : \left| \left\{ \widetilde{W} \in \mathcal{P}(V) \text{ s.t. } |W| = |\widetilde{W}| \right\} \right| = \binom{|V|}{|W|},$$

and $\binom{|V|}{|V|} = 1$, thus $\{W \in \mathcal{P}(V) \text{ s.t. } |W| = |V|\} = \{V\}$.

□

3.1 Decomposing set $\Lambda_k(V)$

Definition 3.3. For $k \in \mathbb{N}$, and $\alpha \in \mathbb{N}$, we define:

$$\Lambda_{k,\alpha}(V, \cdot) := \left\{ \Gamma(V, E) \in \Lambda_k(V, \cdot) \text{ s.t. } \left| \left\{ W \in \mathcal{P}(V) \text{ s.t. } \Delta_W(\Gamma(V, E)) \in \chi(W) \text{ and } |W| = |\text{LCC}(\Gamma(V, E))| \right\} \right| = \alpha \right\},$$

the class of all graphs in $\Lambda_k(V, \cdot)$ having exactly α connected components of maximum size.

Again, for $m \in \llbracket 1, X(|V|) \rrbracket$, we define $\Lambda_{k,\alpha}^m(V, \cdot)$ by $\Lambda_k^m(V, \cdot) \cap \Lambda_{k,\alpha}(V, \cdot)$.

Remark. Even though several connected components of maximum size do exist in a graph, *the one* LCC is still defined unambiguously!

Lemma 3.4.

1. For $k > |V|$ or $k = 0$, we have: $\forall \alpha \in \mathbb{N} : \Lambda_{k,\alpha}(V, \cdot) = \emptyset$.
2. For $k \in \llbracket 1, |V| \rrbracket$ and $\alpha > \left\lfloor \frac{|V|}{k} \right\rfloor$, we have $\Lambda_{k,\alpha}(V, \cdot) = \emptyset$.

Proof.

1. For $k > |V|$ or $k = 0$, it is obvious that: $\Lambda_k(V, \cdot) = \emptyset$ (and then $\Lambda_{k,\alpha}(V, \cdot)$).
2. Take such k and α . Assume (*ad absurdum*) that there exists $\Gamma(V, E) \in \Lambda_{k,\alpha}(V, \cdot)$. We have then $L_1, \dots, L_\alpha \in \mathcal{P}(V)$ such that $\forall i \in \llbracket 1, \alpha \rrbracket : |L_i| = k$. Also, since the L_i 's are connected component, they are disjoint, i.e. $\forall (i, j) \in \llbracket 1, \alpha \rrbracket^2 : i \neq j \Rightarrow L_i \cap L_j = \emptyset$.

Thus $\bigcup_{i=1}^\alpha L_i \subset V$, and $\sum_{i=1}^\alpha |L_i| \leq |V|$. But:

$$\sum_{i=1}^\alpha |L_i| = \alpha k > \left\lfloor \frac{|V|}{k} \right\rfloor k \geq \frac{|V|}{k} k = |V|,$$

which leads a contradiction: $|V| > |V|$. We deduce that $\Lambda_{k,\alpha}(V, \cdot) = \emptyset$.

□

Corollary 3.5.

$$\forall k < |V| : \Lambda_k(V, \cdot) = \bigsqcup_{m=k-1}^{X(|V|)} \bigsqcup_{\alpha=1}^{\lfloor \frac{|V|}{k} \rfloor} \Lambda_{k,\alpha}^m(V, \cdot).$$

Proof. Unions are trivially disjointed.

Now show the equality. The right-hand side is trivially included in $\Lambda_k(V, \cdot)$ (by definition of $\Lambda_{k,\alpha}^m(V, \cdot)$).

Now take $\Gamma(V, E) \in \Lambda_k(V, \cdot)$. We know that $\Lambda_k^{|E|}(V, \cdot)$ with $|E| \leq X(|V|)$. As well, we know that the amount of connected components of size $|LCC(\Gamma(V, E))| = k$ is at least 1 (because $\Gamma(V, E) \in \Lambda_k(V, \cdot)$), and lower or equal to $\lfloor \frac{|V|}{k} \rfloor$ by previous Lemma. □

3.2 The Set $\Omega_{k,\alpha}^m(V)$

3.2.1 Special case of $\alpha = 1$

From now on, let's write:

$$\Omega_{k,1}^m(V) := \bigsqcup_{\substack{W \in \mathcal{P}(V) \\ |W|=k}} \bigsqcup_{\ell=k-1}^{\min(m, X(k))} \chi_\ell(W, \cdot) \times \left(\bigsqcup_{p=1}^{k-1} \Lambda_p^{m-\ell}(V, \cdot) \right).$$

Proposition 3.6. For $k \in \llbracket 1, |V| \rrbracket$ and $m \in \llbracket 1, X(|V|) \rrbracket$, we have:

$$\Lambda_{k,1}^m(V, \cdot) \cong \Omega_{k,1}^m(V),$$

i.e. there exists a bijection between these two sets.

Proof. Consider the function defined by:

$$\Omega : \Lambda_{k,1}^m(V, \cdot) \rightarrow \Omega_{k,1}^m(V) : \Gamma(V, E) \mapsto \left(\Delta_{LCC(\Gamma(V, E))}(\Gamma(V, E)), \Delta_{V \setminus LCC(\Gamma(V, E))}(\Gamma(V, E)) \right).$$

We know that $\Omega(\Gamma(V, E))$ gives a partition of $\Gamma(V, E)$ because there doesn't exist $e = \{v_{i_1}, v_{i_2}\} \in E$ such that $v_{i_1} \in LCC(\Gamma(V, E))$ and $v_{i_2} \in V \setminus LCC(\Gamma(V, E))$ by definition of connected component. Thus if:

$$\left(\Gamma_1(LCC(\Gamma(V, E))), E_{LCC}, \Gamma_2(V \setminus LCC(\Gamma(V, E))), E_{V \setminus LCC} \right) = \Omega(\Gamma(V, E)),$$

we have $E_{LCC} \sqcup E_{V \setminus LCC} = E$.

Let's prove that Ω is injective. Take $\Gamma_1(V, E_1)$ and $\Gamma_2(V, E_2)$ in $\Lambda_k^m(V, \cdot)$. Assume that $\Omega(\Gamma_1(V, E_1)) = \Omega(\Gamma_2(V, E_2))$. This implies that $LCC(\Gamma_1(V, E_1)) = LCC(\Gamma_2(V, E_2))$ (and thus that their complementary in V are equal as well) and that the restriction of $\Gamma_1(V, E_1)$ and $\Gamma_2(V, E_2)$ to their common LCC are equal. Their restriction to the complementary of the common LCC in V are equal as well.

Thus $\Gamma_1(V, E_1)$ and $\Gamma_2(V, E_2)$ have the same partition and are then equal (i.e. $E_1 = E_2$).

Now, let's prove that Ω is surjective. For $W \in \mathcal{P}(V)$, take:

$$(\Gamma_1(W, E_1), \Gamma_2(V \setminus W, E_2)) \in \Omega_{k,1}^m(V).$$

We deduce that if $\ell := |E_1|$, we have $|E_2| = m - \ell$. Also, $|\text{LCC}(\Gamma_2(V \setminus W, E_2))| \leq k$ and $\Gamma_1(W, E_1) \in \chi_\ell(W)$.

For $\Gamma(V, E) := \Gamma(V \setminus W \sqcup W, E_1 \sqcup E_2)$, we have indeed that $\Gamma(V, E) \in \Lambda_k^m(V, \cdot)$ and $\Omega(\Gamma(V, E)) = (\Gamma_1, \Gamma_2)$ because $|\text{LCC}(\Gamma(V, E))| = |W| = k$ since W is the only connected component of size k . \square

3.2.2 General Case

Definition 3.7. For $k, \alpha \in \mathbb{N}^*$, let's define:

$$\mathcal{P}_{k,\alpha}(V) := \left\{ \{W_1, \dots, W_\alpha\} \in \mathcal{P}(\mathcal{P}(V)) \text{ s.t. } \begin{cases} \forall i \in \llbracket 1, \alpha \rrbracket : |W_i| = k \\ \forall (i, j) \in \llbracket 1, \alpha \rrbracket^2 : i \neq j \Leftrightarrow W_i \cap W_j = \emptyset \end{cases} \right\},$$

thus $\mathcal{P}_{k,\alpha}(V)$ is the set of all sets containing α subsets of V which are disjoint and of size k .

Remark. We can tell:

$$|\mathcal{P}_{k,\alpha}(V)| = \frac{1}{\alpha!} \prod_{\beta=0}^{\alpha-1} \binom{|V| - k\beta}{k} = \frac{1}{\alpha!} \frac{|V|!}{(k!)^\alpha (|V| - k\alpha)!}.$$

Definition 3.8. For $k \in \llbracket 1, |V| \rrbracket$, $m \in \llbracket 0, X(|V|) \rrbracket$, and $\alpha \in \llbracket 2, \lfloor \frac{|V|}{k} \rrbracket \rrbracket$, let's define:

$$\begin{aligned} \Omega_{k,\alpha}^m(V) &:= \bigsqcup_{\substack{\{W_1, \dots, W_\alpha\} \in \mathcal{P}_{k,\alpha}(V) \\ \mu(W_1) < \dots < \mu(W_\alpha)}} \bigsqcup_{\substack{(i_1, \dots, i_\alpha) \in \llbracket k-1, X(k) \rrbracket^\alpha \\ \text{s.t. } \sum_{j=1}^\alpha i_j \leq \min(m, \alpha X(k))}} \left[\left(\prod_{j=1}^\alpha \chi_{i_j}(W_j, \cdot) \right) \times \left(\bigsqcup_{p=1}^{k-1} \Lambda_p^{m - \sum_{j=1}^\alpha i_j} \left(V \setminus \bigsqcup_{j=1}^\alpha W_j, \cdot \right) \right) \right] \\ &= \bigsqcup_{\substack{\{W_1, \dots, W_\alpha\} \in \mathcal{P}_{k,\alpha}(V) \\ \mu(W_1) < \dots < \mu(W_\alpha)}} \bigsqcup_{\substack{\Sigma = \alpha(k-1) \\ \text{s.t. } \sum_{j=1}^\alpha i_j = \Sigma}}^{\min(m, \alpha X(k))} \bigsqcup_{\substack{(i_1, \dots, i_\alpha) \in \llbracket k-1, X(k) \rrbracket^\alpha \\ \text{s.t. } \sum_{j=1}^\alpha i_j = \Sigma}} \left[\left(\prod_{j=1}^\alpha \chi_{i_j}(W_j, \cdot) \right) \times \left(\bigsqcup_{p=1}^{k-1} \Lambda_p^{m - \Sigma} \left(V \setminus \bigsqcup_{j=1}^\alpha W_j, \cdot \right) \right) \right] \end{aligned}$$

Theorem 3.9. For $(k, m) \in \llbracket 1, |V| \rrbracket \times \llbracket 0, X(|V|) \rrbracket$ and $\alpha \in \llbracket 1, \lfloor \frac{|V|}{k} \rrbracket \rrbracket$, we have:

$$\Lambda_{k,\alpha}^m(V, \cdot) \cong \Omega_{k,\alpha}^m(V).$$

Proof. For $\alpha = 1$, the theorem is proven by Proposition 3.6.²

Now, for $\alpha \geq 2$, we have the function:

$$\begin{aligned} \Omega_\alpha : \Lambda_{k,\alpha}^m(V, \cdot) &\rightarrow \Omega_{k,\alpha}^m(V) : \\ \Gamma(V, E) &\mapsto \left(\Delta_{W_1}(\Gamma(V, E)), \dots, \Delta_{W_\alpha}(\Gamma(V, E)), \Delta_{V \setminus \bigsqcup_{j=1}^\alpha W_j}(\Gamma(V, E)) \right), \end{aligned}$$

for W_1, \dots, W_α the subsets of V two by two disjoint, such that $\forall i \in \llbracket 1, \alpha \rrbracket : |W_i| = k$, and that:

$$\forall i \in \llbracket 2, \alpha \rrbracket : \mu(W_{i-1}) \leq \mu(W_i).$$

²This Proposition is not exactly expressed the same way because $\Omega_{k,1}^m(V)$ does not consider a set $\{W_1\} \in \mathcal{P}(\mathcal{P}(V))$ but only a subset $W \in \mathcal{P}(V)$. Yet, the proof is equivalent.

We know that W_1, \dots, W_α are the only connected components of size k because $\Gamma(V, E) \in \Lambda_{k,\alpha}^m(V, \cdot)$. And also, values of $\mu(W_j)$ can't be equal for different indices by definition of connected components. This implies that function Ω_α is properly defined.

Now, prove that Ω_α is bijective.

Injective Take $\Gamma_1(V, E_1), \Gamma_2(V, E_2) \in \Lambda_{k,\alpha}^m(V, \cdot)$. Let's assume that:

$$\Omega_\alpha(\Gamma_1(V, E_1)) = \Omega_\alpha(\Gamma_2(V, E_2)).$$

We can deduce that Γ_1 and Γ_2 have the same connected components, and that their restrictions to these connected components are equal as well. By similar argument than in proof of Proposition 3.6, we find that Γ_1 and Γ_2 must be equal.

Surjective Take:

$$(\Gamma_1(W_1, E_1), \dots, \Gamma_\alpha(W_\alpha, E_\alpha), \Gamma(V, E)) \in \Omega_{k,\alpha}^m(V),$$

and prove that there exists a graph $\hat{\Gamma}(V, \hat{E}) \in \Lambda_{k,\alpha}^m(V, \cdot)$ such that:

$$\Omega_\alpha(\hat{\Gamma}(V, \hat{E})) = (\Gamma_1(W_1, E_1), \dots, \Gamma_\alpha(W_\alpha, E_\alpha), \Gamma(V, E)).$$

We know that $\{W_1, \dots, W_\alpha\} \in \mathcal{P}_{k,\alpha}(V)$ by definition of $\Omega_{k,\alpha}^m(V)$, and that:

$$V = W \sqcup \bigsqcup_{j=1}^{\alpha} W_j.$$

As well, by definition of $\Omega_{k,\alpha}^m(V)$, we know that if $\hat{E} := E \sqcup \bigsqcup_{j=1}^{\alpha} E_j$, then $|\hat{E}| = m$. So $\hat{\Gamma}(V, \hat{E})$, the graph created by *assembling* the different components Γ_1 to Γ_α and Γ , is indeed in $\Lambda_{k,\alpha}^m$ because:

$$\forall j \in [1, \alpha] : |\text{LCC}(\Gamma_j(W_j, E_j))| = k \quad \text{and} \quad |\text{LCC}(\Gamma(V, E))| \leq k$$

by definition of $\Omega_{k,\alpha}^m(V)$.

Finally, $\Omega_\alpha(\hat{\Gamma}(V, \hat{E}))$ yields indeed $(\Gamma_1, \dots, \Gamma_\alpha, \Gamma)$ since connected components are ordered according to the μ function defined in Definition 1.10 \square

Remark. The problem of the largest connected component size has been reduced to a problem of connected graphs and recursive combinatorics.

Recursive values like these ones can be computed pretty efficiently thanks to dynamic programming.

4 Counting connected graphs

4.1 Connected graphs of $|V|$ vertices

Harary and Palmer proposed a solution in [1] in 1973 to the number of connected graphs of n vertices, no matter the number of edges.

Definition 4.1. For notations to be consistent with [1], let's define:

$$\forall p \in \mathbb{N} : C_p := |\chi(p)|,$$

the number of connected graphs having p vertices.

Let's state the following theorem from [1], pages 7-8.

Theorem 4.2 (Harary and Palmer). *For all $p \in \mathbb{N}^*$, the number of connected graphs of p vertices is given by:*

$$C_p = \sum_{k=1}^{p-1} \binom{p-2}{k-1} (2^k - 1) C_k C_{p-k}.$$

The second equality stands as well:

$$C_p = 2^{X(p)} - \frac{1}{p} \sum_{k=1}^{p-1} k \binom{p}{k} 2^{X(p-k)} C_k.$$

4.2 Connected graphs of $|V|$ vertices and m edges

Yet, in definition of set $\mathfrak{Q}_{k,\alpha}^m(V)$, it is the cardinality of $\chi_\ell(n)$ that is needed, i.e. C_p is not sufficient.

Definition 4.3. Again, in order to stay consistent with cited references, let's denote:

$$\forall n \in \mathbb{N}^* : \forall k \in \llbracket 0, X(n) \rrbracket : q_{n,k} := |\chi_k(n)|.$$

Starting from generating function equality given by Bona and Noy in [2], namely:

$$\sum_{n \geq 0} \sum_{k \geq 0} q_{n,k} u^k \frac{z^n}{n!} = \log \left(\sum_{n \geq 0} (1+u)^{X(n)} \frac{z^n}{n!} \right),$$

one can find the recursion relation given by Marko Riedel [ADD SOURCE!]:

$$q_{n,k} = \binom{X(n)}{k} - \sum_{m=0}^{n-2} \binom{n-1}{m} \sum_{p=0}^k \binom{X(n-m-1)}{p} q_{m+1,k-p}.$$

5 Conclusion

We have then proved that the cardinality of set $\Lambda_k^m(V, \cdot)$ is equal to:

$$\begin{aligned} |\Lambda_k^m(V, \cdot)| &= \left| \bigcup_{\alpha=1}^{\lfloor \frac{|V|}{k} \rfloor} \Lambda_{k,\alpha}^m(V, \cdot) \right| = \sum_{\alpha=1}^{\lfloor \frac{|V|}{k} \rfloor} |\Lambda_{k,\alpha}^m(V, \cdot)| = \sum_{\alpha=1}^{\lfloor \frac{|V|}{k} \rfloor} |\mathfrak{Q}_{k,\alpha}^m(V)| \\ &= \sum_{\alpha=1}^{\lfloor \frac{|V|}{k} \rfloor} |\mathcal{P}_{k,\alpha}(V)| \sum_{\Sigma=\alpha(k-1)}^{\min(m, \alpha X(k))} \sum_{\substack{(i_1, \dots, i_\alpha) \in \llbracket k-1, X(k) \rrbracket^\alpha \\ \sum_{j=1}^\alpha i_j = \Sigma}} \left(\prod_{j=1}^\alpha q_{i_j, k} \times \sum_{p=1}^{k-1} |\Lambda_p^{m-\Sigma}(|V| - k\alpha)| \right), \end{aligned}$$

from which we eventually deduce:

$$\forall k \in \llbracket 1, |V| \rrbracket : \mathbb{P} \left[\left| \text{LCC}(\mathcal{G}(V, \mathfrak{m})) \right| = k \right] = \frac{|\Lambda_k^{\mathfrak{m}}(V, \cdot)|}{|\Gamma_{\mathfrak{m}}(V, \cdot)|}.$$

References

- [1] H. F. and P. E., *Graphical Enumeration*. New York and London: Academic Press, 1973.
- [2] B. M. and N. M., *Handbook of Enumerative Combinatorics*. CRC Press, 2015.