Conjecture

The conjecture states in a general sense that there exists a recurrence relation in order to determine the cardinality of $\Lambda_k^m(V,\cdot)$. Furthermore, this recurrence relation looks something like:

$$\left|\Lambda_k^m(V,\cdot)\right| = \binom{|V|}{k} \sum_{\ell=1}^{\min(m,X(k))} \sum_{p=1}^{k} \left|\chi_\ell(W)\right| \left|\Lambda_p^{m-\ell}(V\setminus W,\cdot)\right|.$$

The idea is that in order to have a graph with largest connected component of given size $k \in \mathbb{N}$, one requires to be able to separate (unambiguously) the graph's LCC and its complement.

The conjecture is *probably* (yeah. . .) true for $p \lneqq k$ because LCC is clearly uniquely defined (only one connected component has maximum size), but this does not fit graphs having several connected components with same size (i.e. k) because several graphs are counted too many times. For instance, graph

1—2

4—3

is counted twice: once for $W = \{1,2\} \subset V = \{1,2,3,4\}$, and once for $W = \{3,4\} \subset V$.

To remove this redundance, two options are possible:

- find if a given proportion is redundant, thus divide the cardinality of $\chi_\ell(W,\cdot) \times \Lambda_k^{m-\ell}(V\setminus W,\cdot)$,
- or change the expression in order to isolate the case where $p = k$, and find the right expression (would something like $\chi_\ell(W) \times \mathcal{L}_k^{m-\ell}(V,W)$, for:

  $$\mathcal{L}_k^{m-\ell}(V,W) \coloneqq \Lambda_k^{m-\ell}(V\setminus W,\cdot) \setminus \mathfrak{L}_k^{m-\ell}(V,W),$$

  for:

  $$\mathfrak{L}_k^{m-\ell}(V,W) \coloneqq \left\{\Gamma(V,E) \in \Lambda_k^{m-\ell}(V\setminus W,\cdot) \text{ s.t. } \mathrm{LCC}(\Gamma(V,E)) \subset \left\{v_1,\dots,v_{\mu(W)}\right\}\right\}$$

  work knowing that:

  $$\mu(W) \coloneqq \max_{i=1,\dots,|V|} i\mathbb{I}_{[v_i\in W]}$$

  ?)

Function to prove cardinality equality

To prove that two sets have equal cardinality, a bijective function must be determined between these two. If $\mathfrak{Q}_k^m(V)$ is the set having the right cardinality, the function will be:

$$\Omega : \Lambda_k^m(V) \to \mathfrak{Q}_k^m(V) : \Gamma(V,E) \mapsto \Big(\Delta_{\mathrm{LCC}(\Gamma(V,E))}(\Gamma(V,E)), \Delta_{V\setminus \mathrm{LCC}(\Gamma(V,E))}(\Gamma(V,E))\Big).$$

This Ω function is obviously injective. Now, the right set $\mathfrak{Q}_k^m(V)$ needs to be found in order to be surjective (the hard point is on graphs having more than one connected component of maximum size.)

04/13

Since the set $\Lambda_l^m(V,\cdot)$ has been split again into a disjoint union of $\Lambda_{k,\alpha}^m(V,\cdot)$, it has been proven that the conjecture stands for $\alpha = 1$ and the β coefficients equal to 1.

Yet, the formula has to be proven and arranged for $\alpha > 1$. Something like the follwing could work:

$$\left|\Lambda_{k,\alpha}^m(V,\cdot)\right| = \left|\mathfrak{Q}_{k,\alpha}^m(V)\right|,$$

for:

$$\mathfrak{Q}_{k,\alpha}^m(V) \coloneqq \bigsqcup_{(W_1,\dots,W_\alpha)\in\mathcal{P}_{k,\alpha}(V)} \bigsqcup_{\substack{i_1,\dots,i_\alpha \\ \sum_{j=1}^\alpha i_j \le m}} \left[\left(\prod_{j=1}^{\alpha} \chi_{i_j}(W_j)\right) \times \left(\bigsqcup_{p=1}^{k-1} \Lambda_p^{m-\sum_{j=1}^\alpha i_j}(V,\cdot)\right)\right].$$

with:

$$\mathcal{P}_{k,\alpha}(V) \coloneqq \left\{(W_1,\dots,W_\alpha)\in\mathcal{P}(V) \text{ s.t. } \left\{\begin{array}{l} \forall i \in \{1,\dots,\alpha\} : |W_i| = k \\ \forall (i,j)\in\{1,\dots,\alpha\}^2 : i \neq j \Leftrightarrow W_i \cap W_j = \emptyset \end{array}\right.\right\}$$

Remark: Since tuples in $\mathcal{P}_{k,\alpha}(V)$ are sensitive to order (i.e. $(W_1,W_2) \neq (W_2,W_1)$), we have that:

$$\left|\mathcal{P}_{k,\alpha}(V)\right| = \alpha! \prod_{i=1}^{\alpha} \binom{|V| - i(k-1)}{k} = \alpha! \frac{|V|!}{(k!)^\alpha (|V| - k\alpha)!}.$$

04/21

Formula in report.pdf reduces problem of largest connected component size to problem of counting connected graphs having n vertices and k edges. A result (recursive form) by Marko Riedel can be found on MSE (#689526) but only works for $n \le 11$... Let's then try to use the same type of reasonment as he did in order to find a correct formula:

$$
\begin{aligned}
\log\left(1+\sum_{m=1}^{n}(1+u)^{X(m)}\frac{z^m}{m!}\right) &= \sum_{q\geq 1}(-1)^{q+1}\frac{1}{q}\left[\sum_{m=1}^{n}(1+u)^{X(m)}\frac{z^m}{m!}\right]^q \\
&\simeq \sum_{q=1}^{n}(-1)^{q+1}\frac{1}{q}\sum_{|\alpha|=q}\binom{q}{\alpha}\prod_{m=1}^{n}\left((1+u)^{X(m)}\frac{z^m}{m!}\right)^{\alpha_m} \\
&=: G(z,u).
\end{aligned}
$$

Therefore, we find:

$$
\begin{aligned}
[u^k]G(z,u) &= \sum_{q=1}^{n}(-1)^{q+1}\frac{1}{q}\sum_{|\alpha|=q}\binom{q}{\alpha}\left(\prod_{m=1}^{n}\frac{z^{m\cdot\alpha_m}}{(m!)^{\alpha_m}}\right)[u^k](1+u)^{\sum_{m=1}^{n}X(m)\alpha_m} \\
&= \sum_{q=1}^{n}(-1)^{q+1}\frac{1}{q}\sum_{|\alpha|=q}\binom{q}{\alpha}\left(\prod_{m=1}^{n}\frac{z^{m\cdot\alpha_m}}{(m!)^{\alpha_m}}\right)\binom{\sum_{m=1}^{n}X(m)\alpha_m =: \beta_\alpha}{k},
\end{aligned}
$$

and thus, for Θ_q lthe set of all vectors α in $\mathbb{N}^n$ such that $\sum_{m=1}^{n} m\alpha_m = q$:

$$
\begin{aligned}
[u^k][z^n]G(z,u) &= \sum_{q=1}^{n}(-1)^{q+1}\frac{1}{q}\sum_{|\alpha|=q}\binom{q}{\alpha}[z^n]\left(\prod_{m=1}^{n}\frac{z^{m\cdot\alpha_m}}{(m!)^{\alpha_m}}\right)\binom{\beta_\alpha}{k} \\
&= \sum_{q=1}^{n}(-1)^{q+1}\frac{1}{q}\sum_{\alpha\in\Theta_q}\binom{q}{\alpha}\binom{\beta_\alpha}{k}\prod_{m=1}^{n}\frac{1}{(m!)^{\alpha_m}}
\end{aligned}
$$

04/23

Riedel's formula:

$$
q_{n,k} = \binom{X(n)}{k} - \sum_{m=0}^{n-2}\binom{n-1}{m}\sum_{p=0}^{k}\binom{X(n-m-1)}{p}q_{m+1,k-p}
$$

is indeed correct, and thus complete formula works. Last step to do is to write about Riedel's solution in mathematical report.