非线性振动，势能 $V=\frac{1}{2}kx^2+\frac{1}{4}bx^2$.

$$H=\frac{p^2}{2m}+\frac{1}{2}kx^2+\frac{1}{4}bx^4=H_0+\frac{1}{4}bx^4.\quad 其中\ H_0=\frac{p^2}{2m}+\frac{1}{2}kx^2$$

H_0对应的无微扰解：

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{2}kx^2=0.\quad 于是$$

$$S=-\alpha t+\sqrt{2m}\int\sqrt{\alpha-\frac{1}{2}kx^2}\,dx.$$

$$\begin{cases} p=\dfrac{\partial S}{\partial x}=\sqrt{2m}\sqrt{\alpha-\frac{1}{2}kx^2} \\ \beta=\dfrac{\partial S}{\partial \alpha}=-t+\dfrac{1}{\omega_0}\arcsin\dfrac{x}{\sqrt{\frac{2\alpha}{k}}} \end{cases}\quad 其中\ \omega_0=\sqrt{\frac{k}{m}}.\quad 于是$$

$$\begin{cases} x=\sqrt{\dfrac{2\alpha}{k}}\sin(\omega_0(t+\beta)) \\ p=\sqrt{2m\alpha}\cos(\omega_0(t+\beta)) \end{cases}$$

现在考虑微扰，α和β不再是常数.

$$\begin{aligned}\varepsilon H'=\frac{1}{4}bx^4&=\frac{\alpha^2 b}{k^2}\sin^4(\omega_0(t+\beta))\\&=\frac{\alpha^2 b}{8k^2}[3-4\cos(2\omega_0(t+\beta))+\cos(4\omega_0(t+\beta))]\end{aligned}$$

$$\begin{cases} \dot\beta=\dfrac{\partial(\varepsilon H')}{\partial\alpha} & \Rightarrow \dot\beta=\dfrac{\alpha b}{4k^2}[3-4\cos(2\omega_0(t+\beta))+\cos(4\omega_0(t+\beta))] \\ \dot\alpha=-\dfrac{\partial(\varepsilon H')}{\partial\beta} & \Rightarrow \dot\alpha=-\dfrac{\alpha^2 b\omega_0}{2k^2}[2\sin(2\omega_0(t+\beta))-\sin(4\omega_0(t+\beta))] \end{cases}$$

保留α项，忽略α^2项.

$$\begin{cases} \alpha_1=\alpha_0=\text{const} \\ \beta_1=\dfrac{\alpha_0 b}{4k^2}\displaystyle\int[3-4\cos(2\omega_0(t+\beta))+\cos(4\omega_0(t+\beta))]\,dt. \end{cases}$$

为方便，假定 $t=0$ 时 $x=0$. $\Rightarrow$ $t=0$ 时 $\beta=0$. $\Rightarrow \beta_1=\frac{\alpha_0 b}{4k^2}\left[3t-\frac{2}{\omega_0}\sin(2\omega_0(t+\beta))+\frac{1}{4\omega_0}\sin(4\omega_0(t+\beta))\right]$

长时间中，只有第一项起主要作用 $\Rightarrow \beta_1=\frac{3}{4}\frac{\alpha_0^2 b}{k^2}t$.

$$x=\sqrt{\frac{2\alpha_0}{k}}\sin\left(\omega_0\left(1+\frac{3}{4}\frac{\alpha_0 b}{k^2}\right)t\right)$$

则在一级近似下，频率修正为 $\omega_1=\omega_0\left(1+\frac{3}{4}\frac{\alpha_0 b}{k^2}\right)$

考虑到振幅 $A=\sqrt{\frac{2\alpha}{k}}\approx\sqrt{\frac{2\alpha_0}{k}}$， $\omega_1=\omega_0\left(1+\frac{3}{4}\frac{b}{k}A^2\right)$.