拉普拉斯变换的目的 求解微分方程 (和傅里叶变换一样)

本质: 对于不存在傅里叶变换的函数, 乘一个收敛因子使其可作傅里叶变换

研究对象: f(t), t>0 置 f(t)=0, t<0

记
$$p=0+i\omega$$
, $\overline{f}(p)=2\pi G(\omega)$, 则 $\overline{f}(p)=\int_0^\infty f(t)e^{-pt}dt$ f(t) 的拉普拉斯变换

逆变换 g(t)= \int_m G(w)e^{iwt} dw = 立 \int_m F(\sigma+iw)e^{iwt} dw

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\sigma + iw) e^{\sigma + iw} dw , \quad \text{th} \quad \sigma + iw = p \Rightarrow dw = \frac{1}{\epsilon} dp$$

$$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \overline{f}(p) e^{pt} dp$$

基本性质

$$\mathbb{Q}[Gf_{i}(t)+Gf_{i}(t)=G\overline{f_{i}}(p)+G\overline{f_{i}}(p)$$

(4)相似性定理 f(at)= a f(f)

(7)卷积定理 若f(t)=f(p),f(t)=f(p),则f(t)*f(t)=f(p)f(p), 其中f(t)*f(t)=f^tf(t)f(t-t)dt

例 傅里叶级数法求解非齐次方程

| Um-a²uxx = A cos Tex sin wt ν 右手不可傅里叶变换!于是做拉普拉斯变换

$$\begin{cases} u_{x}|_{x=0} = 0, \ u_{x}|_{x=1} = 0 \\ u_{t=0} = \varphi(x), \ u_{t}|_{t=0} = \psi(x), \ 0 < x < 1 \end{cases}$$

解: 显然 $u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos \frac{n\pi x}{L}$ 求解 $T_n(t)$

代入初始条件

$$\sum_{n=0}^{\infty} T_n(0) \cos \frac{n\pi x}{L} = \psi(x) = \sum_{n=0}^{\infty} \psi_n \cos \frac{n\pi x}{L}$$

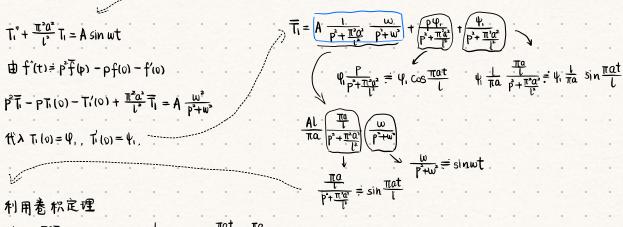
$$\sum_{n=0}^{\infty} T'_n(0) \cos \frac{n\pi x}{L} = \psi(x) = \sum_{n=0}^{\infty} \psi_n \cos \frac{n\pi x}{L}$$

得 $T_0(o) = \psi_0 = \frac{1}{L} \int_0^L \psi(\xi) d\xi$ $T_0'(o) = \psi_0 = \frac{1}{L} \int_0^L \psi(\xi) d\xi$

$$\begin{cases} T_n(0) = \psi_n = \frac{2}{l} \int_0^l \psi(\xi) \cos \frac{n\pi \xi}{l} d\xi \\ T_n'(0) = \psi_n = \frac{2}{l} \int_0^l \psi(\xi) \cos \frac{n\pi \xi}{l} d\xi \end{cases} \quad n \neq 0$$

$$T_{l}(t) = \frac{Al}{\pi a} \frac{1}{\omega^{2} - \pi^{2} a^{2}/l^{2}} \left(w \sin \frac{\pi a t}{l} - \frac{\pi a}{l} \sin w t \right) + \psi_{l} \cos \frac{\pi a t}{l} + \frac{l}{\pi a} \psi_{l} \sin w t$$

$$T_n(t) = \psi_n \cos \frac{n\pi a t}{l} + \frac{l}{n\pi a} \psi_n \sin \frac{n\pi a t}{l} \cdot (n \neq 0, 1)$$



$$\int_0^t \sin \frac{\pi a \tau}{l} \sin (t-\tau) \ d\tau = \frac{1}{w^2 - \frac{\pi a}{l^2}} \left(w \sin \frac{\pi a t}{l} - \frac{\pi a}{l} \sin w t \right)$$

$$\vec{\mathcal{T}} \stackrel{B}{\longleftarrow} T_i(t) = \frac{AL}{\pi a} \frac{1}{\omega^2 - \pi^2 a^2 / L^2} \left(w \sin \frac{\pi a t}{L} - \frac{\pi a}{L} \sin w t \right) + \psi_i \cos \frac{\pi a t}{L} + \frac{L}{\pi a} \psi_i \sin w t$$