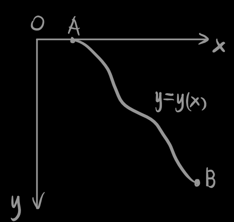
变量J取值由 $y=y(x)$ 决定 $\Rightarrow$ J是$y(x)$的泛函.

记作 $J[y(x)]$.

求泛函的极值问题 $\rightarrow$ 变分问题

例1：从A自由下滑至B，轨迹为 $y=y(x)$，问所需时间J最短？



$$J=\int dt=\int\frac{ds}{v}=\int\frac{ds}{\sqrt{2gy}}$$
$$=\int_A^B\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

变分问题所必须满足的必要条件——欧拉方程

把变分问题转化为求解微分方程

$$J[y(x)]=\int_a^b F(x,y,y')dx$$

假设 $y\rightarrow y+\delta y$ ← 函数$y(x)$的变分

$$J[y+\delta y]-J[y]=\int_a^b[F(x,y+\delta y,y'+\delta y')-F(x,y,y')]dx$$
$$\approx\int_a^b\left[\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right]dx$$
$$=\delta J[y]$$

把 $\delta y(x)$ 记作 $\epsilon\eta(x)$ （ϵ很小）

由函数极值必要条件

$$\frac{\partial J[y+\epsilon\eta]}{\partial\epsilon}\bigg|_{\epsilon=0}=0\ \Rightarrow\ \int_a^b\left[\frac{\partial F}{\partial y}\eta+\frac{\partial F}{\partial y'}\eta'\right]dx=0$$

$$\overset{\times\epsilon}{\Leftrightarrow}\ \int_a^b\left[\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right]dx=0$$ 即 $\delta J[y]=0$

使积分号下只出现 δy

$$\int_a^b\frac{\partial F}{\partial y'}\delta y'dx=\int_a^b\frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)dx$$
$$=\left[\frac{\partial F}{\partial y'}\delta y\right]_a^b-\int_a^b\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\,dx$$

$$\delta J[y]=0\Leftrightarrow\int_a^b\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx+\underline{\left[\frac{\partial F}{\partial y'}\delta y\right]_a^b}=0$$

轨道总通过A与B：$\delta y|_{x=a,b}=0$

对任意 δy 成立 $\Rightarrow$ $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$ 欧拉方程 (1^{st})

又 $F=F(x,y,y')\Rightarrow\frac{dF}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial y'}\frac{dy'}{dx}$

$$\frac{\partial F}{\partial x}=\frac{dF}{dx}-\frac{\partial F}{\partial y}\frac{dy}{dx}-\frac{\partial F}{\partial y'}\frac{dy'}{dx}$$
$$=\frac{d}{dx}\left(F-y'\frac{\partial F}{\partial y'}\right)-y'\underbrace{\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]}_{=0}$$

$\Rightarrow\frac{\partial F}{\partial x}+\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}-F\right)=0$ 欧拉方程 (2^{nd})

F不显含x时，$y'\frac{\partial F}{\partial y'}-F=$ 常数

例1：$F(x,y,y')=\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$ 代入 $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

$$y'\frac{\partial F}{\partial y'}-F=-\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}=-\frac{1}{\sqrt{4gC_1}}\ (-个\mathrm{const})$$
$$\Downarrow$$
$$y'=\sqrt{\frac{2C_1-y}{y}}\qquad\frac{\sqrt{y}\,dy}{\sqrt{2C_1-y}}=dx$$

$$x=-\sqrt{2C_1y-y^2}+C_1\underbrace{\arccos\frac{C_1-y}{C_1}}_{\theta}+C_2$$

$$\begin{cases}x=C_1(\theta-\sin\theta)+C_2\\ y=C_1(1-\cos\theta)\end{cases}$$

推广1：$J=J[y_1,y_2,\cdots,y_s]$ $\quad y_\alpha=y_\alpha(x)$ $(\alpha=1,2,\cdots,s)$

欧拉方程 $\frac{\partial F}{\partial y_\alpha}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'_\alpha}\right)=0$ $\quad(\alpha=1,2,\cdots,s)$

推广2：约束条件下的变分问题

$$\delta J=\delta\int_a^b F(x,y_1,y_2,\cdots,y_s,y_1',y_2',\cdots,y_s')dx=0$$

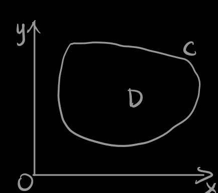
但存在 $k\ (k<s)$ 个约束条件

$$\int_a^b G_j(x,y_1,y_2,\cdots,y_s,y_1',y_2',\cdots,y_s')dx=C_j\quad(j=1,2,\cdots,k)$$

$$\int_a^b\sum_{\alpha=1}^s\left\{\frac{\partial F}{\partial y_\alpha}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'_\alpha}\right)+\sum_{j=1}^k\lambda_j\left[\frac{\partial G_j}{\partial y_\alpha}-\frac{d}{dx}\left(\frac{\partial G_j}{\partial y'_\alpha}\right)\right]\right\}\delta y_\alpha\,dx=0$$

$$\frac{\partial}{\partial y_\alpha}\left[F+\sum_{j=1}^k\lambda_jG_j\right]-\frac{d}{dx}\frac{\partial}{\partial y'_\alpha}\left[F+\sum_{j=1}^k\lambda_jG_j\right]=0\quad(\alpha=1,2,\cdots,s)$$

例2：设平面上封闭曲线长度为l，求围最大面积.



设曲线 $\begin{cases}x=x(\xi)\\ y=y(\xi)\end{cases}$ $\quad 0\le\xi\le\xi_0$

封闭条件 $\begin{cases}x(0)=x(\xi_0)\\ y(0)=y(\xi_0)\end{cases}$

$$l = \oint_c \sqrt{(dx)^2 + (dy)^2} = \int_0^{\xi_0} \sqrt{x'^2 + y'^2}\, d\xi$$

$$J = \iint_D dx\,dy = \frac{1}{2}\oint_c (x\,dy - y\,dx) = \frac{1}{2}\int_0^{\xi_0} (xy' - yx')\, d\xi$$

$$\bar{F} = \frac{1}{2}(xy' - yx') + \lambda\sqrt{x'^2 + y'^2}$$

$$\begin{cases} \dfrac{\partial \bar{F}}{\partial x} - \dfrac{d}{d\xi}\dfrac{\partial \bar{F}}{\partial x'} = 0 \\ \dfrac{\partial \bar{F}}{\partial y} - \dfrac{d}{d\xi}\dfrac{\partial \bar{F}}{\partial y'} = 0 \end{cases} \Rightarrow \begin{cases} x' + \lambda \dfrac{d}{d\xi}\left(\dfrac{y'}{\sqrt{x'^2+y'^2}}\right) = 0 \\ y' - \lambda \dfrac{d}{d\xi}\left(\dfrac{x'}{\sqrt{x'^2+y'^2}}\right) = 0 \end{cases} \Rightarrow \begin{cases} x + \dfrac{\lambda y'}{\sqrt{x'^2+y'^2}} = C_1 \\ y - \dfrac{\lambda x'}{\sqrt{x'^2+y'^2}} = C_2 \end{cases}$$

消 $\lambda \Rightarrow (x - C_1)x' + (y - C_2)y' = 0$

$$(x - C_1)dx + (y - C_2)dy = 0$$

$$(x - C_1)^2 + (y - C_2)^2 = C_3^2$$

$$l = 2\pi C_3$$

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2 = C_3^2$$