→ Robotics HW # 03

Question # 01
Solution

→ Part a

In order to obtain the transformation ${}^{s}T_b$ we assume a frame $\{a\}$ origin at the center of the disc and same as the orientation of frame $\{s\}$ at $t=0$ which is given

$$\Rightarrow {}^{s}T_b = {}^{s}T_a \, {}^{a}T_b$$

first we have to find ${}^{s}T_a$ and ${}^{a}T_b$ to determine the required transformation

⇒ For ${}^{s}T_a$:-

we have given that $\dot{\theta} = 1$ rad/s and the angle of rotation at time t is $\theta_1(t)$ which is express as "t" radians by limits and can be obtained by rotating frame $\{s\}$ about y-axis by $\theta_1(t)$ radians and translate along z-axis by L units and can be written in this form

$${}^{s}T_a = \begin{bmatrix} \cos t & 0 & \sin t & 0 \\ 0 & 1 & 0 & 0 \\ -\sin t & 0 & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ FoR ${}^{a}T_{b}$

similarly can determine this transformation by rotating $\theta_2(t)$ along z-axis and translation along x-axis by R and can be written in this form

$$
{}^{a}T_{b} = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & R \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

After Multipilying bothe transformation on MatLab we get the resultant

$$
{}^{s}T_{b} = \begin{bmatrix} \cos^2 t & -\cos t \sin t & \sin t & R\cos^2 t - L\sin t \\ \sin t & \cos t & 0 & R\sin t \\ -\cos t \sin t & \sin^2 t & \cos t & -L\cos t - R\cos t \sin t \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Part b

⇒ Angular Velocity ${}^{s}\omega_{sb}$

We can obtain by sum of successive frame angular velocities

$$\because \; {}^{s}\omega_{sb} = {}^{s}\omega_{sa} + {}^{s}R_{a}\, {}^{a}\omega_{ab}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin t \\ 1 \\ \cos t \end{bmatrix} \text{rad/s}$$

Part C

=> linear velocity ${}^sV_{sb}$

We can obtain by differentiating the 4th column of ${}^sT_b$ which we get in Part a

$$\begin{bmatrix} -L\cos t - R\sin 2t \\ R\cos t \\ L\sin t - R + 2R\sin^2 t \end{bmatrix}$$

$\therefore {}^sV_{sb} = {}^s\dot{O}_b$

Part d

$\Rightarrow$ Spatial velocity vector in body co-ordinates is obtained by changing the coordinates using ${}^{b}R_s$

$$\begin{bmatrix} {}^{b}\omega_{sb} \\ {}^{b}V_{sb} \end{bmatrix} = \begin{bmatrix} {}^{b}R_s & 0 \\ 0 & {}^{b}R_s \end{bmatrix} \begin{bmatrix} {}^{s}\omega_{sb} \\ {}^{s}V_{sb} \end{bmatrix}$$

$$= \begin{bmatrix} \sin t \\ \cos t \\ 1 \\ -L\cos t \\ R+L\sin t \\ -R\cos t \end{bmatrix}$$

Part e

$\Rightarrow$ linear velocity of the rider since rider is seated at the origin of the body frame, linear velocity is ${}^{b}V_{sb}$ in the body frame co-ordinates and ${}^{s}V_{sb}$ in the fixed frame coordinate.

$$\begin{bmatrix} -L\cos t \\ R+L\sin t \\ -R\cos t \end{bmatrix}$$

Question # 2

i - $R(a \times b) = Ra \times Rb$

$\Rightarrow$ Cross Product of $a \times b$ is characterized by the Property that

$$\det(x, a, b) = \langle x, a \times b \rangle \quad \forall x \in R^3$$

$\Rightarrow$ Now let $R \in SO(3)$

then by using the fact that $R^T = R^{-1}$ we get

$$\rightarrow \langle x, R(a \times b) \rangle = \langle R^T x, a \times b \rangle$$

$$= \langle R^{-1} x, a \times b \rangle$$

$$= \det(R^{-1} x, a, b)$$

then utilizing the assumption $\det(R) = 1$

then

$$\rightarrow \det(R) \det(R^{-1} x, a, b)$$

$$= \det(x, Ra, Rb)$$

$$= \langle x, Ra \times Rb \rangle$$

Finally

$$\langle x, R(a \times b) \rangle = \langle x, Ra \times Rb \rangle$$

holds for any $x \in R^3$.

Proofed

Answer

=> Another approach

- Express the cross Product as a skew symmetric matrix multiplication
$a \times b = [a] \times [b]$ where $[a]$ and $[b]$ are skew symmetric matrix corresponding to vector a, b.
Use the Property of matrix multiplication and fact that $R \in SO(3)$ to rewrite the left hand side of identity

$$\rightarrow R(a \times b) = R([a] \times [b])$$
$$\rightarrow \quad = R[a][b] - R[b][a]$$

Apply same Property on Right hand side

$$\rightarrow Ra \times Rb = [Ra][Rb]$$
$$= R[a][b] - R[b][a]$$

=> therefore the RHS expression is equal to LHS expression

$$\Rightarrow R(a \times b) = Ra \times Rb$$

which prooves the identity

Answer

Date: ___

M	T	W	T	F	S	S

Q2 b). Prove that:

$$R S(a) R^T = S(Ra)$$

Using the above given proof (Q2 part a)) : $R(a \times b) = Ra \times Rb$

and using another statement:

For any vectors a and p belonging to R^3:

$$S(a) p = a \times p$$

where $a \times p$ is the vector cross product.

We say that b is an arbitrary vector ~~be~~ belonging in R^3

Then following these:

$$R S(a) R^T b = R(a \times R^T b)$$

$$= (Ra) \times (R R^T b)$$

$$= Ra \times I b$$

$$= Ra \times b$$

$$R S(a) R^T b = S(Ra) b$$

Thus removing b,

$$R S(a) R^T = S(Ra)$$

Scanned with CamScanner

Q3 Part C

Using the Homogenous transformation given in the book:

```
%% transformations for cylindrical robot with spherical wrists
syms('d_1', 'theta_1', 'theta_2', 'theta_4', 'theta_5', 'theta_6', 'd_2', 'd_3',
'd_6')





%0_T_1
A_1 = [cos(theta_1) -sin(theta_1) 0   0;
       sin(theta_1) cos(theta_1)  0   0;
        0                 0       1   d_1;
        0                 0       0   1];




%1_T_2
A_2 = [1  0  0   0;
       0  0  1   0;
       0 -1  0  d_2;
       0  0  0   1];


%2_T_3
A_3 = [1   0   0   0;
       0   1   0   0;
       0   0   1  d_3;
       0   0   0   1];


%3_T_4
A_4 = [cos(theta_4) 0 -sin(theta_4) 0;
       sin(theta_4) 0 cos(theta_4) 0;
        0           -1     0       0;
        0           0      0       1];


%4_T_5
A_5 = [cos(theta_5) 0 sin(theta_5)  0;
       sin(theta_5) 0 -cos(theta_5) 0;
        0          -1     0         0;
        0           0     0         1];
```

```
%5_T_6
A_6 = [cos(theta_6) -sin(theta_6) 0    0;
       sin(theta_6) cos(theta_6)  0    0;
        0           0             1   d_6;
        0           0             0    1];


T_1 = A_1;
%0_T_2
T_2 = A_1*A_2;


%0_T_3
T_3 = A_1*A_2*A_3;


T_3_6 = A_4*A_5*A_6;


%0_T_4
T_4 = T_3*A_4;


%0_T_5
T_5 = T_4*A_5;


%0_T_6
T_6 = T_5*A_6
```

T_6 =

$$\begin{pmatrix} \cos(\theta_6)\,\sigma_2 + \cos(\theta_1)\sin(\theta_4)\sin(\theta_6) & \cos(\theta_1)\cos(\theta_6)\sin(\theta_4) - \sin(\theta_6)\,\sigma_2 & \sigma_4 - \sigma_5 & -d_6\,(\sigma_5 - \sigma_4) - d_3\sin(\theta_1) \\ \sin(\theta_1)\sin(\theta_4)\sin(\theta_6) - \cos(\theta_6)\,\sigma_1 & \sin(\theta_6)\,\sigma_1 + \cos(\theta_6)\sin(\theta_1)\sin(\theta_4) & \sigma_3 & d_6\,\sigma_3 + d_3\cos(\theta_1) \\ \cos(\theta_4)\sin(\theta_6) - \cos(\theta_5)\cos(\theta_6)\sin(\theta_4) & \cos(\theta_4)\cos(\theta_6) + \cos(\theta_5)\sin(\theta_4)\sin(\theta_6) & -\sin(\theta_4)\sin(\theta_5) & d_1 + d_2 - d_6\sin(\theta_4)\sin(\theta_5) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\sigma_1 = \cos(\theta_1)\sin(\theta_5) - \cos(\theta_4)\cos(\theta_5)\sin(\theta_1)$$

$$\sigma_2 = \sin(\theta_1)\sin(\theta_5) + \cos(\theta_1)\cos(\theta_4)\cos(\theta_5)$$

$$\sigma_3 = \cos(\theta_1)\cos(\theta_5) + \cos(\theta_4)\sin(\theta_1)\sin(\theta_5)$$

$$\sigma_4 = \cos(\theta_1)\cos(\theta_4)\sin(\theta_5)$$

$$\sigma_5 = \cos(\theta_5)\sin(\theta_1)$$

Since the second and third joint are prismatic, the angular velocities are zero and since the first column is just z0 in zeroth frame, we write (1,0,0)

```
w = [0 0 0 T_3(1, 3) T_4(1, 3) T_5(1, 3)
     0 0 0 T_3(2, 3) T_4(2, 3) T_5(2, 3)
     1 0 0 T_3(3, 3) T_4(3, 3) T_5(3, 3)]
```

w =

$$\begin{pmatrix} 0 & 0 & 0 & -\sin(\theta_1) & -\cos(\theta_1)\sin(\theta_4) & \cos(\theta_1)\cos(\theta_4)\sin(\theta_5)-\cos(\theta_5)\sin(\theta_1) \\ 0 & 0 & 0 & \cos(\theta_1) & -\sin(\theta_1)\sin(\theta_4) & \cos(\theta_1)\cos(\theta_5)+\cos(\theta_4)\sin(\theta_1)\sin(\theta_5) \\ 1 & 0 & 0 & 0 & -\cos(\theta_4) & -\sin(\theta_4)\sin(\theta_5) \end{pmatrix}$$

Q3 Part A

fx, fy and fz are the position vector of the 0_T_6 matrix which will then be differentiated with respect to the respective theta and d. Since the 2nd third joints are prismatic, thus they are differentiated with respect to d2 and d3.

```
fx = (T_6(1,4))
```

fx = $-d_6\ (\cos(\theta_5)\sin(\theta_1)-\cos(\theta_1)\cos(\theta_4)\sin(\theta_5))-d_3\sin(\theta_1)$

```
fy = (T_6(2,4))
```

fy = $d_6\ (\cos(\theta_1)\cos(\theta_5)+\cos(\theta_4)\sin(\theta_1)\sin(\theta_5))+d_3\cos(\theta_1)$

```
fz = (T_6(3,4))
```

fz = $d_1+d_2-d_6\sin(\theta_4)\sin(\theta_5)$

```
v = [diff(fx, theta_1) diff(fx, d_2) diff(fx, d_3) diff(fx, theta_4) diff(fx, theta_5) diff(fx, theta_6);
     diff(fy, theta_1) diff(fy, d_2) diff(fy, d_3) diff(fy, theta_4) diff(fy, theta_5) diff(fy, theta_6);
     diff(fz, theta_1) diff(fz, d_2) diff(fz, d_3) diff(fz, theta_4) diff(fz, theta_5) diff(fz, theta_6)]
```

v =

$$\begin{pmatrix} -d_6\ (\cos(\theta_1)\cos(\theta_5)+\cos(\theta_4)\sin(\theta_1)\sin(\theta_5))-d_3\cos(\theta_1) & 0 & -\sin(\theta_1) & -d_6\cos(\theta_1)\sin(\theta_4)\sin(\theta_5) & d_6\ (\sin(\theta_1)\sin(\theta_5)+\cos(\theta_1)\cos(\theta_4)\cos(\theta_5)) & 0 \\ -d_6\ (\cos(\theta_5)\sin(\theta_1)-\cos(\theta_1)\cos(\theta_4)\sin(\theta_5))-d_3\sin(\theta_1) & 0 & \cos(\theta_1) & -d_6\sin(\theta_1)\sin(\theta_4)\sin(\theta_5) & -d_6\ (\cos(\theta_1)\sin(\theta_5)-\cos(\theta_4)\cos(\theta_5)\sin(\theta_1)) & 0 \\ 0 & 1 & 0 & -d_6\cos(\theta_4)\sin(\theta_5) & -d_6\cos(\theta_5)\sin(\theta_4) & 0 \end{pmatrix}$$

Q3 Part B

```
o0 = [0;
      0;
      0];


o1 = T_1(1:3, 4);
o2 = T_2(1:3, 4);
o3 = T_3(1:3, 4);
o4 = T_4(1:3, 4);
o5 = T_5(1:3, 4);
o6 = T_6(1:3, 4);


v = simplify(expand([cross(w(:,1),(o6 - o0)) T_1([1,2,3], 3) T_2([1,2,3], 3)
cross(w(:,4),(o6 - o3)) cross(w(:,5),(o6 - o4)) cross(w(:,6),(o6 - o5))]))
```

v =

$$\begin{pmatrix} -d_3\cos(\theta_1) - d_6\cos(\theta_1)\cos(\theta_5) - d_6\cos(\theta_4)\sin(\theta_1)\sin(\theta_5) & 0 & -\sin(\theta_1) & -d_6\cos(\theta_1)\sin(\theta_4)\sin(\theta_5) & d_6\sin(\theta_1)\sin(\theta_5) + d_6\cos(\theta_1)\cos(\theta_4)\cos(\theta_5) & 0 \\ d_6\cos(\theta_1)\cos(\theta_4)\sin(\theta_5) - d_6\cos(\theta_5)\sin(\theta_1) - d_3\sin(\theta_1) & 0 & \cos(\theta_1) & -d_6\sin(\theta_1)\sin(\theta_4)\sin(\theta_5) & d_6\cos(\theta_4)\cos(\theta_5)\sin(\theta_1) - d_6\cos(\theta_1)\sin(\theta_5) & 0 \\ 0 & 1 & 0 & -d_6\cos(\theta_4)\sin(\theta_5) & -d_6\cos(\theta_5)\sin(\theta_4) & 0 \end{pmatrix}$$

Q3 Part D

```
j = [v; w]
```

j =

$$\begin{pmatrix} -d_3\cos(\theta_1) - d_6\cos(\theta_1)\cos(\theta_5) - d_6\cos(\theta_4)\sin(\theta_1)\sin(\theta_5) & 0 & -\sin(\theta_1) & -d_6\cos(\theta_1)\sin(\theta_4)\sin(\theta_5) & d_6\sin(\theta_1)\sin(\theta_5) + d_6\cos(\theta_1)\cos(\theta_4)\cos(\theta_5) & 0 \\ d_6\cos(\theta_1)\cos(\theta_4)\sin(\theta_5) - d_6\cos(\theta_5)\sin(\theta_1) - d_3\sin(\theta_1) & 0 & \cos(\theta_1) & -d_6\sin(\theta_1)\sin(\theta_4)\sin(\theta_5) & d_6\cos(\theta_4)\cos(\theta_5)\sin(\theta_1) - d_6\cos(\theta_1)\sin(\theta_5) & 0 \\ 0 & 1 & 0 & -d_6\cos(\theta_4)\sin(\theta_5) & -d_6\cos(\theta_5)\sin(\theta_4) & 0 \\ 0 & 0 & 0 & -\sin(\theta_1) & -\cos(\theta_1)\sin(\theta_4) & \cos(\theta_1)\cos(\theta_4)\sin(\theta_5) - \cos(\theta_5)\sin(\theta_1) \\ 0 & 0 & 0 & \cos(\theta_1) & -\sin(\theta_1)\sin(\theta_4) & \cos(\theta_1)\cos(\theta_5) + \cos(\theta_4)\sin(\theta_1)\sin(\theta_5) \\ 1 & 0 & 0 & 0 & -\cos(\theta_4) & -\sin(\theta_4)\sin(\theta_5) \end{pmatrix}$$

Q3 Part E

```
theta_1 = pi/2; %rad
d_2 = 0.2; %m
d_3 = 0.3; %m
theta_4 = 0;
theta_5 = 0;
theta_6 = 0;


v = [- d_3*cos(theta_1) - d_6*cos(theta_1)*cos(theta_5) -
d_6*cos(theta_4)*sin(theta_1)*sin(theta_5) 0 -sin(theta_1) -
```

```
d_6*cos(theta_1)*sin(theta_4)*sin(theta_5) d_6*sin(theta_1)*sin(theta_5) +
d_6*cos(theta_1)*cos(theta_4)*cos(theta_5) 0;
  d_6*cos(theta_1)*cos(theta_4)*sin(theta_5) - d_6*cos(theta_5)*sin(theta_1) -
d_3*sin(theta_1) 0  cos(theta_1) -d_6*sin(theta_1)*sin(theta_4)*sin(theta_5)
d_6*cos(theta_4)*cos(theta_5)*sin(theta_1) - d_6*cos(theta_1)*sin(theta_5) 0;

0 1             0              -d_6*cos(theta_4)*sin(theta_5)
-d_6*cos(theta_5)*sin(theta_4) 0]
```

v =

$$\begin{pmatrix} -\frac{4967757600021511\, d_6}{81129638414606681695789005144064} - \frac{5961309120025813}{32451855365842672678315602057625 6} & 0 & -1 & 0 & \frac{4967757600021511\, d_6}{81129638414606681695789005144064} & 0 \\ -d_6 - \frac{3}{10} & 0 & \frac{4967757600021511}{81129638414606681695789005144064} & 0 & d_6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
theta_1_dot = 0.1; %rad/s
d_2_dot = 0.25; %m/s
d_3_dot = -0.05;
theta_4_dot = 0;
theta_5_dot = 0;
theta_6_dot = 0;


q_dot = [theta_1_dot; d_2_dot; d_3_dot; theta_4_dot; theta_5_dot; theta_6_dot];


v_lin = simplify(v*q_dot)
```

v_lin =

$$\begin{pmatrix} \frac{3245185536584267148605377805246 3}{64903710731685345356631204115251 2} - \frac{4967757600021511\, d_6}{811296384146066816957890051440640} \\ -\frac{d_6}{10} - \frac{243388915243820069926155015539747}{811296384146066816957890051440640 0} \\ \frac{1}{4} \end{pmatrix}$$

```
v_lin = [(1/20-6.1232e-18*d_6); -(d_6/10-3/100); 1/4]
```

v_lin =

$$\begin{pmatrix} \frac{1}{20} - \frac{3974184015522557\, d_6}{649037107316853453566312041152512} \\ \frac{3}{100} - \frac{d_6}{10} \\ \frac{1}{4} \end{pmatrix}$$

With units of linear velocity as m/s.

Q3 Part F

For arms with spherical wrists, we can decouple the problem into arm singularities and wrist singularities: J = [JP JO]

We can assign frames such that o3 = o4 = o5 = o6 = o and thus for JO we will get the linear velocities to be zero. Here JO is the Ji column where i=3,4,5.

J = [J11 O;

J21 J22]

where J22 = [z3 z4 z5]

det J = det J11 det J22, det J11 = 0 gives arm singularities and det J22 = 0 gives wrist singularities.

Singularity occurs when z3, z4, and z5 are linearly dependent. This happens when z3 and z5 are colinear or θ5 = 0, π

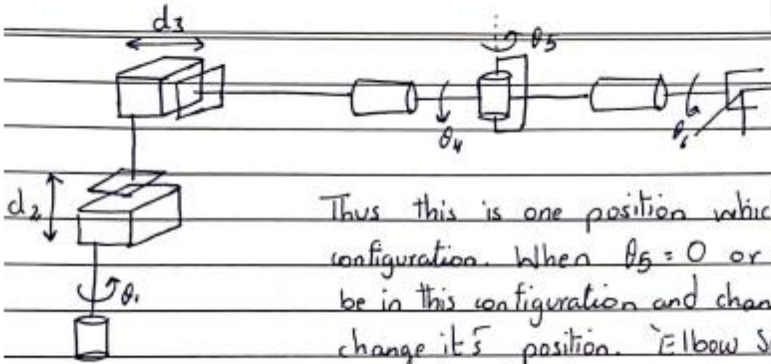
Finding the determinant of J11:

```
syms('d_1', 'theta_1', 'theta_2', 'theta_4', 'theta_5', 'theta_6', 'd_2', 'd_3', 'd_6')
d = det([- d_6*(cos(theta_1)*cos(theta_5) + cos(theta_4)*sin(theta_1)*sin(theta_5)) - d_3*cos(theta_1) 0 -sin(theta_1);
         - d_6*(cos(theta_5)*sin(theta_1) - cos(theta_1)*cos(theta_4)*sin(theta_5)) - d_3*sin(theta_1) 0  cos(theta_1);
          0 1             0])
```

$$d = d_3 \sin(\theta_1)^2 + d_3 \cos(\theta_1)^2 + d_6 \cos(\theta_1)^2 \cos(\theta_5) + d_6 \cos(\theta_5) \sin(\theta_1)^2$$

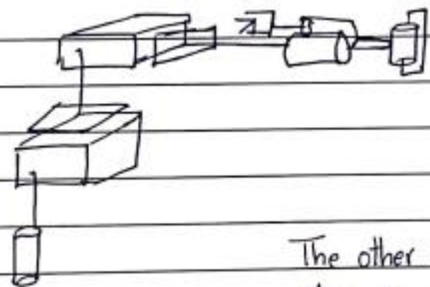
Thus we have singularities if $d_3 = 0$ and $d_6 = 0$.

Q3 Part F



(when $\theta_5 = 0$)

Thus this is one position which is a singular configuration. When $\theta_5 = 0$ or π The arm will always be in this configuration and changing other angles will not change it's position. 'Elbow Singularity'
When it is turned to π, it is bent the other way round.



when $\theta_5 = \pi$

The other singularity as computed above can arrive when $d_3 = 0$ ~~such that~~

Rida: This hw took me a total of 7 to 8 hours to complete and my part was to complete question 3 and question 2 part 2. The Jacobean part was easy although it took some time to fully understand and complete. The part with singularities was a bit tough and I am afraid I still am confused about this topic a little bit. The confusion I faced was mainly in finding the determinants after equating the origins equal to zero and finding how will the singularities be achieved.

Hussain: I spent 5 hours completing this homework and this was difficult according to me. I did question 1 and part 1 of question 2. The proof in question 2 was challenging but question 1 was tougher and I had to watch some videos to understand and revise the concepts studied in class. Overall, this homework served as a good practice and learning for me and the upcoming mid.