Designing numerical methods for free-surface flows towards reliable wave-structure interactions

Sacha Cardonna, Fabien Marche & François Vilar

Institute of Mathematics Alexander Grothendieck, University of Montpellier

MIPS Colloquia – Semaine du pôle MIPS

Montpellier, France – June 2025

Overview of this talk

Designing numerical methods for **free-surface flows** towards reliable **wave–structure interactions**

Some keywords.

- **Numerical methods**: mathematical tools to simulate physical systems that can't be solved analytically;
- **Free-surface flows**: time-dependent flows (here we focus on water waves governed by nonlinear equations);
- **Wave–structure interactions**: modeling how waves impact or are affected by obstacles like rocks, walls, or offshore structures.

Some questions

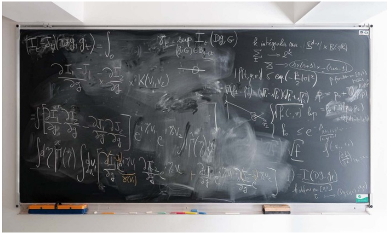
1. **Why do we model physical phenomena with PDEs?**
2. **What are the water waves equations?**
3. **How can we solve these equations?**
4. **What is the idea behind our approach?**
5. **How well does it perform and where is it going?**

Some questions

1. **Why do we model physical phenomena with PDEs?**
2. **What are the water waves equations?**
3. **How can we solve these equations?**
4. **What is the idea behind our approach?**
5. **How well does it perform and where is it going?**

From real-world phenomena to simulation and decisions

1. Real-world phenomena
(e.g. waves, heat, biology)



2. Mathematical modeling
(describe and study equations)

3. Numerical simulation
(solve equations approximately)

4. Prediction and/or decision
(risk, control, design)

Modeling time-dependant phenomenas

PDEs as a universal tool for modeling

Many physical, biological or engineering phenomena are governed by quantities that evolve in time and space.

↪ **Partial Differential Equations (PDEs)** allow us to describe how such quantities change, often based on conservation laws or empirical observations.

A few examples of PDE-based models

- **Physics:** the heat equation models thermal shielding and temperature diffusion in a rocket's body during reentry;
- **Chemistry:** reaction–diffusion equations model how two substances mix to form spatial patterns;
- **Biology/Medicine:** model how electrical signals propagate in the heart, helping to understand arrhythmias or defibrillation;
- **Finance:** the Black–Scholes equation models the price of options based on volatility and time to maturity.

Some definition and terminology

General form of a PDE

A partial differential equation (PDE) involves an **unknown function** $\mathbf{U}(t, \boldsymbol{x})$ defined over time $t \geq 0$ and space $\boldsymbol{x} \in \Omega \subset \mathbb{R}^d$, and relates it to its **partial derivatives**:

$$\mathscr{L}(\mathbf{U}, \partial_t \mathbf{U}, \nabla_{\boldsymbol{x}} \mathbf{U}, \nabla_{\boldsymbol{x}}^2 \mathbf{U}, \dots) = \mathbf{S}(t, \boldsymbol{x})$$

- $\mathscr{L}$ is a differential operator that may be **nonlinear**;
- $\mathbf{S}$ is a given **source term** that may depend on time and space;
- $\mathbf{U} : [0, T] \times \Omega \to \mathbb{R}^n$ is the **unknown** vector-valued function we aim to determine.

Three major types of PDEs

- **Elliptic:** describe steady-state problems — e.g. electrostatics, stationary temperature,...
- **Parabolic:** describe evolution with diffusion — e.g. heat conduction, chemical diffusion,...
- **Hyperbolic:** describe wave-like behavior — e.g. sound, seismic waves, water waves,...

Some questions

1. **Why do we model physical phenomena with PDEs?**
2. **What are the water waves equations?**
3. **How can we solve these equations?**
4. **What is the idea behind our approach?**
5. **How well does it perform and where is it going?**

The shallow water approximation

Modeling surface waves in shallow regions

In coastal and near-shore flows, vertical motion is **small** compared to horizontal one $\rightarrow$ **simplify the full 3D fluid equations** by averaging over depth!

Nonlinear shallow water (NSW) equations

$$\boxed{\partial_t \mathbf{U} + \nabla_{\boldsymbol{x}} \cdot \mathbb{F}(\mathbf{U}, b) = \mathbf{S}(\mathbf{U}, b)}$$

- $\mathbf{U}$: contains the data of the **water height** and **horizontal velocities**;
- b: describes the **bottom topography**;
- $\mathbb{F}$: represents the **wave transport** (nonlinear flux);
- $\mathbf{S}$: adds **physical effects** to the solution (e.g. friction, Coriolis effects...).

Why are these models important?

- Used to simulate **tsunamis**, **storm surges** and **flooding events**;
- Help to predict the **impact of waves** on infrastructure and coastal populations.

Some questions

1. Why do we model physical phenomena with PDEs?
2. What are the water waves equations?
3. **How can we solve these equations?**
4. What is the idea behind our approach?
5. How well does it perform and where is it going?

Complexity of solving PDEs

No analytical solution in general

PDEs like the shallow water equations usually cannot be solved **analytically**:

- shocks, dry zones and nonlinearities can also **prevent explicit solutions**;
- only **partial** theoretical results are available (e.g. existence or uniqueness under restrictive assumptions).

$\hookrightarrow$ To obtain actual solutions, we rely on **numerical analysis** framework.

Numerical analysis of partial differential equations

Building a **rigorous** and **consistent** approximation of the continuous problem. Indeed, we want to ensure:

- **Accuracy** — capturing the solution and its potential singularities;
- **Stability/Robustness** — prevent non-physical behavior, while being able to handle complex situations.

Most existing methods fail to meet all these goals at once!

Some questions

1. **Why do we model physical phenomena with PDEs?**
2. **What are the water waves equations?**
3. **How can we solve these equations?**
4. **What is the idea behind our approach?**
5. **How well does it perform and where is it going?**

Combining the best of both worlds

Objectives

Designing a numerical approximation that remains **mathematically consistent** to the PDE, achieving **high accuracy** in smooth regions and **robustness** in challenging flow conditions!

Coupling two complementary methods

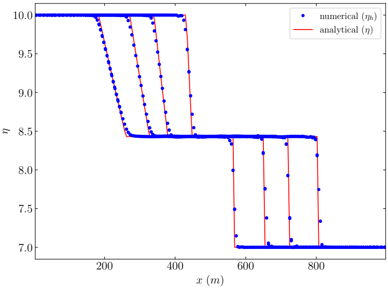
- **Discontinuous Galerkin (DG):** highest-order **accuracy**
  ↪ but may become **unstable** or oscillatory near discontinuities.
- **Finite Volume (FV): robust** in presence of nonlinearities or dry zones
  ↪ but usually limited to **low-order accuracy**.

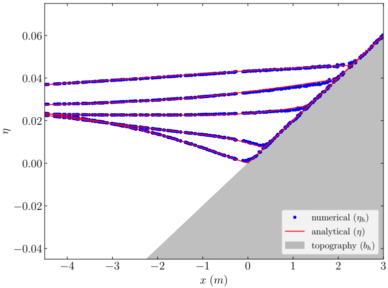
Combining **adaptively** between **DG** and **FV** frameworks
↪ Weighs **precision** against **robustness** as needed!

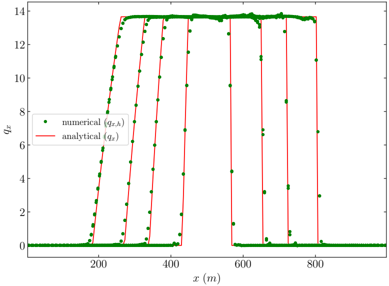
Some questions

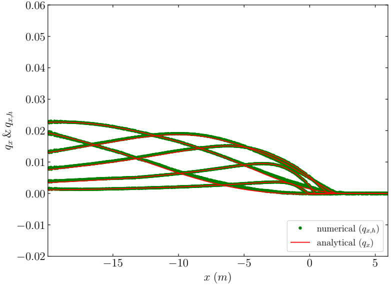
1. Why do we model physical phenomena with PDEs?
2. What are the water waves equations?
3. How can we solve these equations?
4. What is the idea behind our approach?
5. **How well does it perform and where is it going?**

Some numerical results with analytical solutions









Simulation of a rock-wave interaction

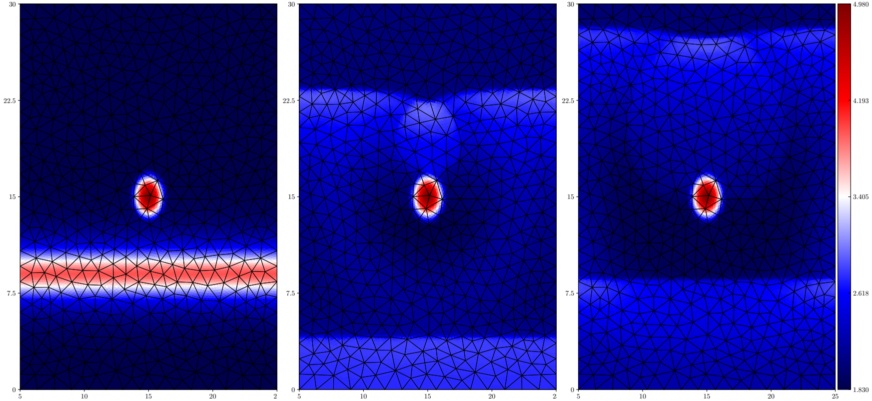
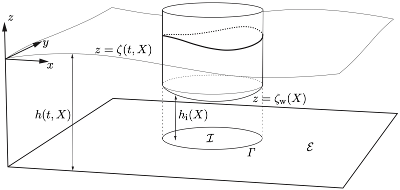


Figure: Snapshots of the water elevation at several times (and link to simulation).

Ph.D. objectives

We want an ideal scheme to solve the shallow water equations, such that we can then study:

$$\boxed{\textbf{wave-structure interactions}}$$



From the theory...

to its potential applications...

Thank you for your attention!

Figure: *The Great Wave of Kanagawa*, Hokusai, 1830.

E-mail: *sacha.cardonna@umontpellier.fr*
Website: *sachacardonna.github.io*