

# **Local subcell monolithic DG/FV scheme for nonlinear shallow water equations with source terms on unstructured grids**

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**UNIVERSITÉ DE  
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Overview of this talk

## Local subcell monolithic DG/FV scheme for nonlinear shallow water equations with source terms on unstructured grids

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- ▶ **Local subcell monolithic DG/FV scheme:** combines DG accuracy with FV robustness for stabilization;
- ▶ **Nonlinear shallow water equations:** describe the water waves under the hydrostatic assumption;
- ▶ **Source terms:** account for geometry and physical effects (e.g., topography, friction, Coriolis force ...).

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Formulation

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## 4. Numerical results

## 5. Conclusion and perspectives



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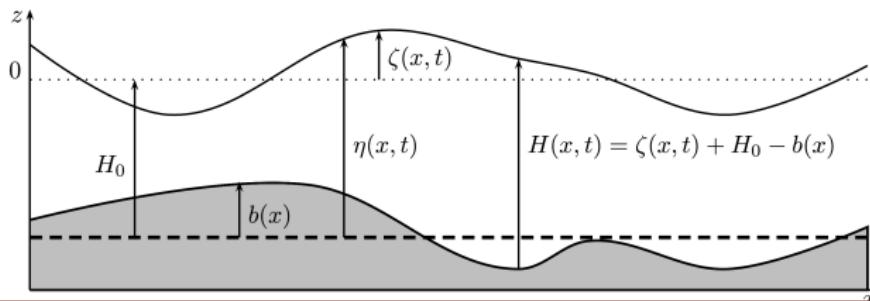
# Shallow water asymptotics

## Nonlinear shallow water (NSW) equations

$$\partial_t \mathbf{v} + \nabla_{\mathbf{x}} \cdot \mathbb{F}(\mathbf{v}, b) = \mathbf{B}[b](\mathbf{v})$$

$$\Leftrightarrow \begin{cases} \partial_t \eta + \nabla_{\mathbf{x}} \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \nabla_{\mathbf{x}} \cdot (\mathbf{u} \otimes \mathbf{q} + \frac{g\eta}{2}(\eta - 2b)\mathbb{I}_2) = -g\eta \nabla_{\mathbf{x}} b \end{cases}$$

- ▶  $b : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the **topography** parametrization;
- ▶  $\mathbf{v} : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathcal{H}_b^+$  is the vector gathering **total elevation**  $\eta$  and **discharge**  $(q_x, q_y)^T$ , with  $\mathcal{H}_b^+ = \{(\eta, q_x, q_y) \in \mathbb{R}^3 \mid H := \eta - b \geq 0\}$ ;
- ▶  $\mathbb{F} : \mathcal{H}_b^+ \times \mathbb{R} \rightarrow \mathcal{M}_{2 \times 3}(\mathbb{R})$  is the nonlinear **flux** tensor;
- ▶  $\mathbf{B} : \mathcal{H}_b^+ \rightarrow \mathbb{R}^3$  is the **source term** depending on topography.



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# Finite Volume & Discontinuous Galerkin frameworks

## Finite Volume (FV)

- ▶ Integral formulation over control volumes  $\omega_c \subset \Omega$  with  $\Omega = \bigcup \omega_c$ ;
- ▶ Piecewise constant approximation:

$$\mathbf{v}_h^c(t) \simeq \frac{1}{|\omega_c|} \int_{\omega_c} \mathbf{v}(\mathbf{x}, t) d\mathbf{x},$$

where  $\mathbf{v}$  is the exact solution;

- ▶ Numerical flux  $\mathbb{F}^*$  ensures conservation and stability.
- ✓ Robust and easy to implement, well-suited for nonlinear problems;
- ✗ Low-order accuracy unless polynomial reconstruction is applied.

## Discontinuous Galerkin (DG)

- ▶ Weak formulation on each element  $\omega_c \subset \Omega$  with  $\Omega = \bigcup \omega_c$ ;
- ▶ Piecewise polynomial approx.:

$$\mathbf{v}_h^c(\mathbf{x}, t) = \sum_{m=1}^{\dim \mathbb{P}^k} \mathbf{v}_m^c(t) \psi_m^c(\mathbf{x}),$$

with test functions in  $\mathbb{P}^k(\omega_c)$ ;

- ▶ Numerical flux  $\mathbb{F}^*$  to ensure local conservation.
- ✓ High-order accuracy with compact stencil, well-suited for parallelism;
- ✗ Less robust, more complex implementation and prone to oscillations.

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## Ideal setup for the NSW system

For our purposes, an ideal numerical scheme for the nonlinear shallow water (NSW) equations should be:

- ▶ **High-order accurate** to capture smooth solutions and small-scale features;
- ▶ **Shock-capturing** to handle discontinuities and strong nonlinearities;
- ▶ **Positivity-preserving** to ensure non-negative water height and physical admissibility (i.e. stays in  $\mathcal{H}_b^+$ );
- ▶ **Well-balanced** to exactly preserve lake at rest steady states;
- ▶ **Adaptable to source terms** such as bottom topography and friction effects;
- ▶ **Well-suited for unstructured meshes** to deal with complex geometries and realistic domains.

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## DG formulation through residuals

DG formulation for all  $\psi_p^c \in \mathbb{P}^k(\omega_c)$

$$\sum_{m=1}^{N_k} \frac{d\mathbf{v}_m^c}{dt} \int_{\omega_c} \psi_m^c \psi_p^c d\mathbf{x} - \int_{\omega_c} \mathbb{F} \cdot \nabla_{\mathbf{x}} \psi_p^c d\mathbf{x} + \int_{\partial\omega_c} \mathbb{F}^* \cdot \mathbf{n} \psi_p^c dS = \int_{\omega_c} \mathbf{B} \psi_p^c d\mathbf{x}$$

Residual DG formulation for any basis function  $\psi_m^c \in \mathbb{P}^k(\omega_c)$

$$\mathbb{M}_c \frac{d\mathbf{V}_c}{dt} = \Phi_c + \mathbf{S}_c$$

- ▶  $(\mathbf{V}_c)_m = \mathbf{v}_m^c(t)$  **solution moments**
- ▶  $(\mathbb{M}_c)_{mp} = \int_{\omega_c} \psi_m^c(\mathbf{x}) \psi_p^c(\mathbf{x}) d\mathbf{x}$  **local mass matrix**
- ▶  $(\Phi_c)_m = \int_{\partial\omega_c} \mathbb{F}^* \cdot \mathbf{n} \psi_p^c dS - \int_{\omega_c} \mathbb{F}(\mathbf{v}_h^c, b_h^c) \cdot \nabla_{\mathbf{x}} \psi_p^c d\mathbf{x}$  **DG residuals**
- ▶  $(\mathbf{S}_c)_m = \int_{\omega_c} \mathbf{B}(\mathbf{v}_h^c, \nabla_{\mathbf{x}} b_h^c) \psi_p^c d\mathbf{x}$  **source term**

## Stabilization principle

- ▶ **Classical stabilization:** apply limiters/a posteriori correction on the full cell
  - risks **discarding** a mostly accurate solution due to a **local failure**
- ▶ **Subcell approach:** partition each cell into finer subcells to reduce the correction scale
  - enabling a **surgical correction**, meaning only fix what's necessary, preserving as much of the high-order DG content as possible

Theory needed – Reformulation of DG as a subcell FV-like scheme

## A little state-of-the-art on subcell stabilization and monolithic schemes

❓ Generally, in subcell techniques, applying a **robust FV correction** on the **subgrid level** implies that **all the subcells** in the troubled cell are **impacted**.

Some pioneering references for this presentation

📄 **M. Sonntag & C. D. Munz**, *Shock capturing for discontinuous Galerkin methods using finite volume subcells*. Finite Volumes for Complex Applications VII, pp. 945–953. Springer, 2014.

📄 **M. Dumbser & R. Loubère**, *A simple robust and accurate a posteriori subcell finite volume limiter for the discontinuous Galerkin method on unstructured meshes*. J. Comp. Phys., 319:163–199, 2016.

📄 **A. Rueda-Ramírez, B. Bolm, D. Kuzmin & G. Gassner**, *Monolithic convex limiting for Legendre-Gauss-Lobatto Discontinuous Galerkin Spectral-Element methods*. Commun. Appl. Math., 2024

**Our approach:** thanks to the **reconstructed flux** formalism, we can activate the correction **locally** on **non-admissible subcells only!**

→ avoids wasting information in non-troubled regions and preserves accuracy.

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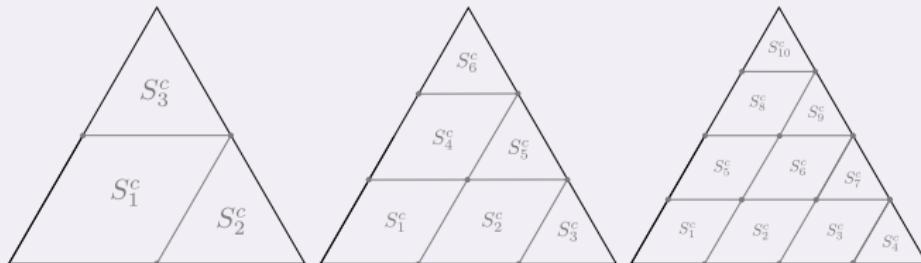
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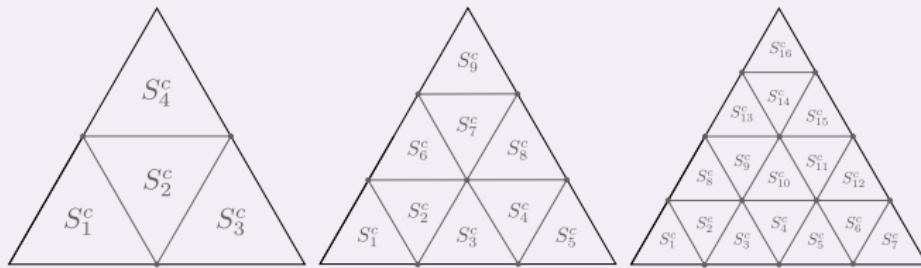
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# Mesh subdivision

Cell subdivision into  $N_s \geq N_k$  subcells

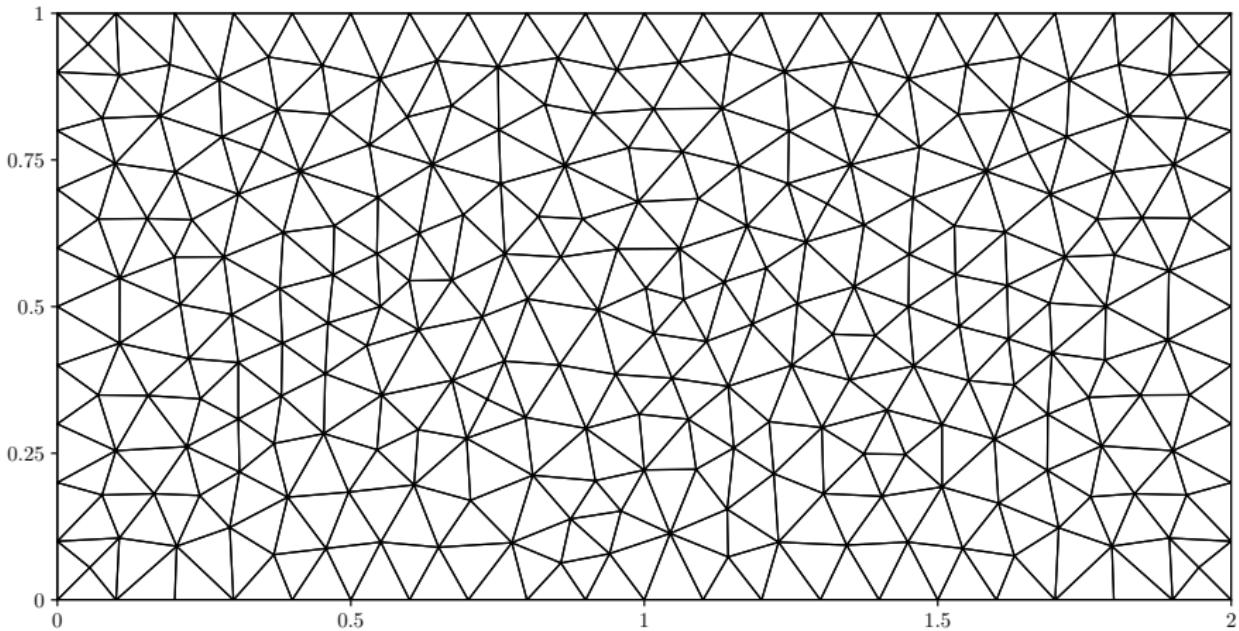


Cell  $\omega_c$  subdivided into  $N_s = N_k$  subcells for  $\mathbb{P}^1$  (left),  $\mathbb{P}^2$  (center) and  $\mathbb{P}^3$  (right) cases



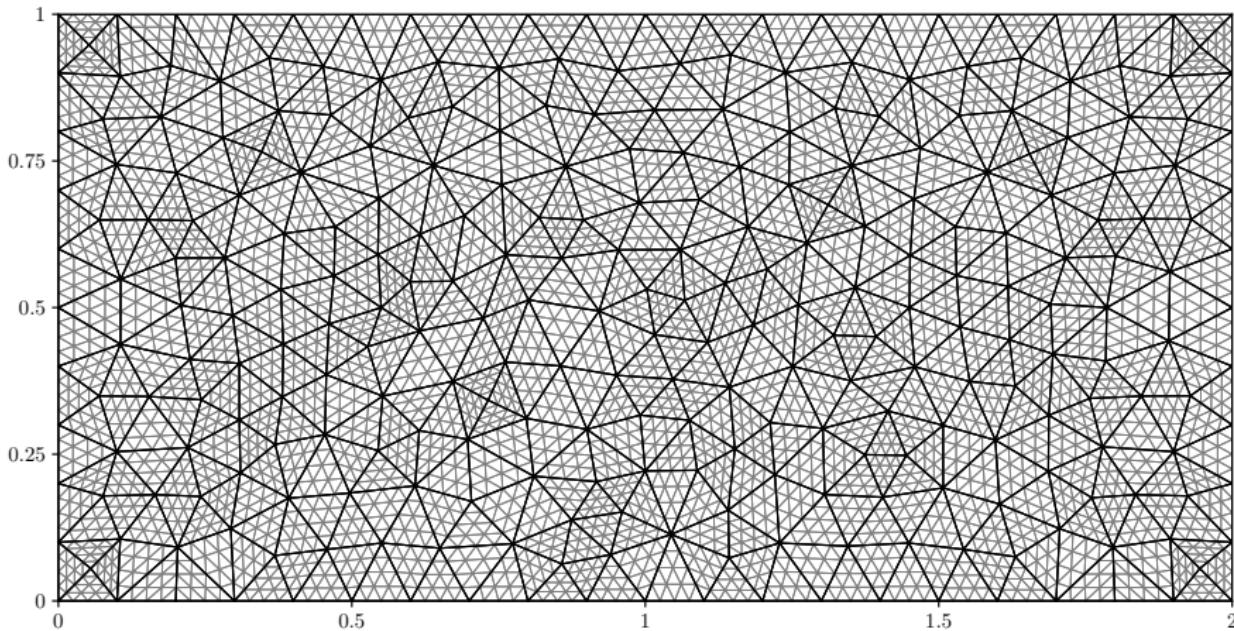
Cell  $\omega_c$  subdivided into  $N_s \geq N_k$  subcells for  $\mathbb{P}^1$  (left),  $\mathbb{P}^2$  (center) and  $\mathbb{P}^3$  (right) cases

A classical mesh ...



**Figure:** Unstructured simplicial mesh with  $n_{\text{el}} = 350$  cells.

... and its subdivision



**Figure:** Unstructured simplicial mesh  $\mathbb{P}^3$  subdivision onto triangles with  $n_{\text{el}} = 350$  cells.

## Subdivision and submean values

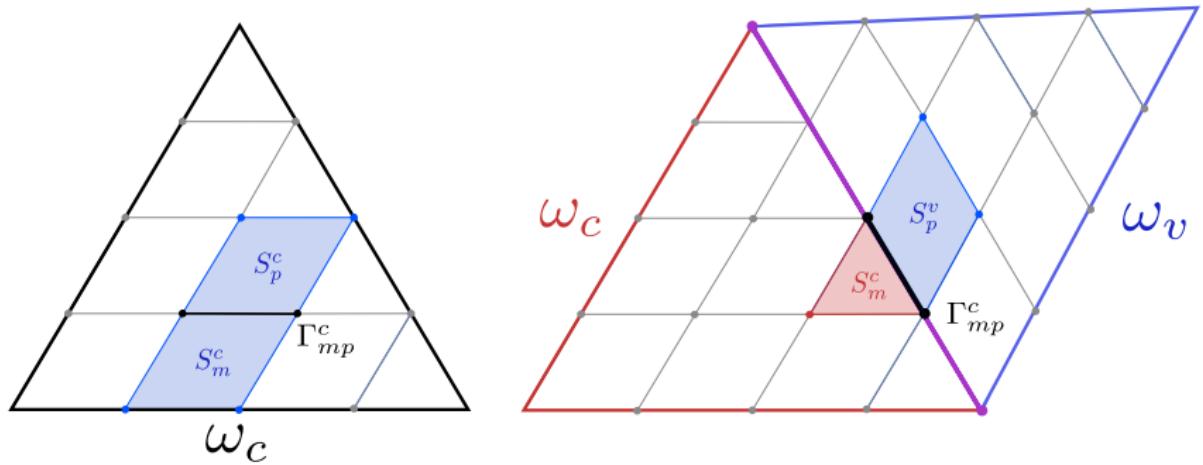
### Some notations

- ▶ For any element  $\omega_c \in \mathcal{T}_h$ , we define a sub-partition:

$$\mathcal{T}_{\omega_c} := \{S_1^c, \dots, S_{N_s}^c\}, \quad \overline{\omega}_c = \bigcup_{m=1}^{N_s} \overline{S}_m^c$$

- ▶  $\Gamma_{mp}^c$ : interface between  $S_m^c$  and its neighbor  $S_p^v$
- ▶  $n_f^m$ : number of faces of subcell  $S_m^c$
- ▶  $\mathcal{F}_{S_m^c}$ : set of all faces of  $S_m^c$
- ▶  $n_f^c$ : total number of subcell faces inside element  $\omega_c$
- ▶  $\mathcal{V}_m^c$ : set of face-neighboring subcells of  $S_m^c$  (with  $|\mathcal{V}_m^c| = n_f^m$ )
- ▶  $\check{\mathcal{V}}_m^c$ : subset of  $\mathcal{V}_m^c$  containing only neighbors within the same element  $\omega_c$

# Subneighbors



**Figure:** Two cases: subneighbor  $S_p$  inside cell  $\omega_c$  (left), and subneighbor  $S_p$  inside neighbor cell  $\omega_v$  (right).

## Submean values and polynomial moments (1)

Mean value of a function over a subcell  $S_m^c \subset \omega_c$

For any  $f \in L^2(\omega_c)$ , its submean value is  $\bar{f}_m^c := \frac{1}{|S_m^c|} \int_{S_m^c} f(\mathbf{x}) d\mathbf{x}$ .

### Submean values and projection matrix

►  $(\bar{\mathbf{V}}_c)_m = \bar{\mathbf{v}}_m^c(t)$  **submean values**

►  $(\mathbb{P}_c)_{mp} = \frac{1}{|S_m^c|} \int_{S_m^c} \psi_p^c(\mathbf{x}) d\mathbf{x}$  **projection matrix**

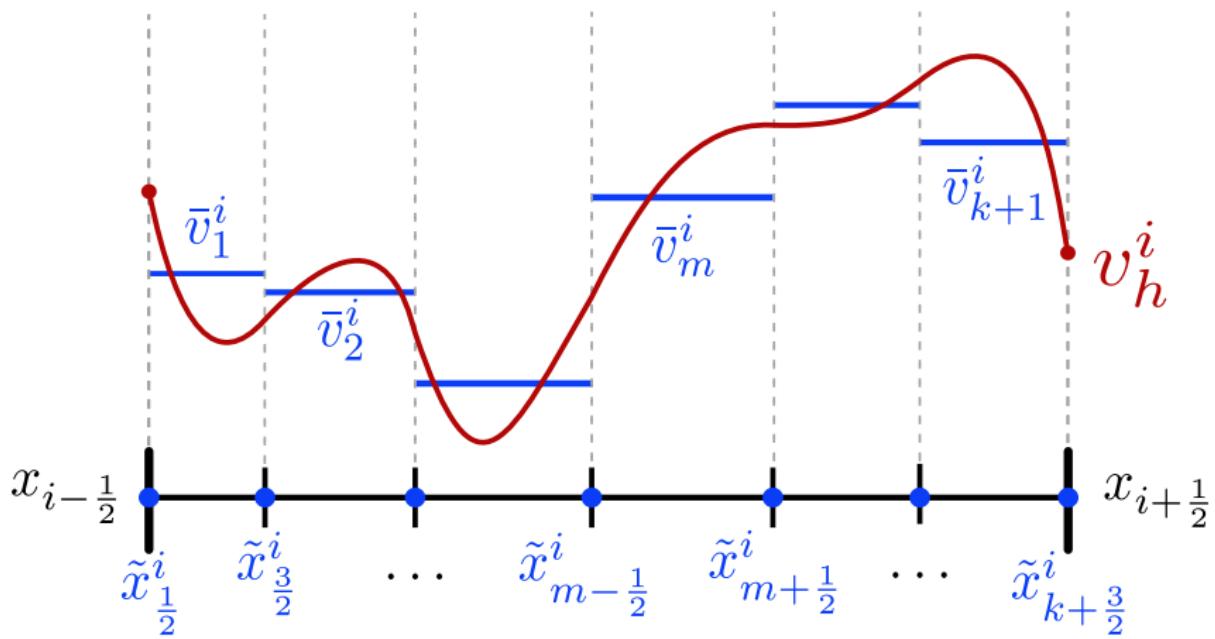
$$\bar{\mathbf{v}}_m^c(t) = \frac{1}{|S_m^c|} \sum_{q=1}^{N_k} \mathbf{v}_q^c(t) \int_{S_m^c} \psi_q^c(\mathbf{x}) d\mathbf{x} \implies \boxed{\bar{\mathbf{V}}_c = \mathbb{P}_c \mathbf{V}_c}$$

⚠  $\mathbb{P}_c^t \mathbb{P}_c$  has to be **non-singular**, so we use the least-square procedure:

$$\boxed{\mathbf{V}_c = (\mathbb{P}_c^t \mathbb{P}_c)^{-1} \mathbb{P}_c^t \bar{\mathbf{V}}_c}$$

If  $N_s = N_k$ , then  $\bar{\mathbf{V}}_c = \mathbb{P}_c \mathbf{V}_c \Leftrightarrow \mathbf{V}_c = \mathbb{P}_c^{-1} \bar{\mathbf{V}}_c$ .

## Submean values and polynomial moments (2)



**Figure:** Piecewise polynomial function  $v_h^i$  and associated sub-mean-values (1D case).

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## Reconstructed DG fluxes (1)

Submean values vector derivative

$$\text{Since } \mathbb{M}_c \frac{d\mathbf{V}_c}{dt} = \Phi_c + \mathbf{S}_c \text{ and } \bar{\mathbf{V}}_c = \mathbb{P}_c \mathbf{V}_c \implies \frac{d\bar{\mathbf{V}}_c}{dt} = \mathbb{P}_c \mathbb{M}_c^{-1} (\Phi_c + \mathbf{S}_c)$$

Flux reconstruction to get a FV-like scheme

Let us consider the DG reconstructed flux  $\widehat{\mathbb{F}}_n$  such that

$$\begin{aligned} \frac{d\bar{\mathbf{v}}_m^c}{dt} &= -\frac{1}{|S_m^c|} \int_{\partial S_m^c} \widehat{\mathbb{F}}_n(\mathbf{x}) d\mathbf{x} + (\mathbb{P}_c \mathbb{M}_c^{-1} \mathbf{S}_c)_m && \text{(FV-like scheme)} \\ &= -\frac{1}{|S_m^c|} \sum_{S_p^v \in \check{\mathcal{V}}_m^c} \int_{\Gamma_{mp}^c} \widehat{\mathbb{F}}_n(\mathbf{x}) d\mathbf{x} + (\mathbb{P}_c \mathbb{M}_c^{-1} \mathbf{S}_c)_m && \left( \partial S_m^c = \cup_{S_p^v \in \check{\mathcal{V}}_m^c} \Gamma_{mp}^c \right) \\ &= -\frac{1}{|S_m^c|} \left( \sum_{S_p^v \in \check{\mathcal{V}}_m^c} \int_{\Gamma_{mp}^c} \widehat{\mathbb{F}}_n(\mathbf{x}) d\mathbf{x} + \int_{\partial \omega_c \cap \partial S_m^c} \mathbb{F}_n^* d\mathbf{x} \right) + (\mathbb{P}_c \mathbb{M}_c^{-1} \mathbf{S}_c)_m \end{aligned}$$

under the hypothesis that  $\widehat{\mathbb{F}}_{n|\partial\omega} = \mathbb{F}^*$  for all  $\omega \in \mathcal{T}_h$ .

## Reconstructed DG fluxes (2)

### Interface reconstructed flux

We define  $\widehat{\mathbb{F}}_{mp}$  at interface  $\Gamma_{mp}^c$  as:  $\int_{\Gamma_{mp}^c} \widehat{\mathbb{F}}_n(\mathbf{x}) d\mathbf{x} = \varepsilon_{mp}^c \widehat{\mathbb{F}}_{mp},$

where subface orientation is carried through  $\varepsilon_{mp}^c$ , such that  $\varepsilon_{pm}^c = -\varepsilon_{mp}^c$ .

### Reconstructed flux system

$$-\mathbb{A}_c \widehat{\mathbb{F}}_c = \mathbb{D}_c \frac{d\overline{\mathbf{V}}_c}{dt} + \partial \mathbb{F}_c$$

- ▶  $(\widehat{\mathbb{F}}_c)_{mp} = \ell_{mp} \widehat{\mathbb{F}}_{mp}$  **interior subfaces fluxes**
- ▶  $(\mathbb{A}_c)_{mp} = \varepsilon_{mp}^c$  **adjacency matrix**
- ▶  $(\mathbb{D}_c)_m = |S_m^c|$  **subvolume matrix**
- ▶  $(\partial \mathbb{F}_c)_m = \int_{\partial \omega_c \cap \partial S_m^c} \mathbb{F}_n^* d\mathbf{x}$  **cell boundary contribution**

⚠ Since  $\ker \mathbb{A}_c \neq \{\mathbf{0}\}$ , we use a *Graph Laplacian technique*

## Reconstructed DG fluxes (3)

Residual definition of reconstructed fluxes

$$\widehat{\mathbb{F}}_c = -\mathbb{A}_c^t \mathcal{L}_c^{-1} (\mathbb{D}_c \mathbb{P}_c \mathbb{M}_c^{-1} \Phi_c + \partial \mathbb{F}_c)$$

where  $\mathcal{L}_c^{-1}$  is the gen. inverse of  $\mathbb{L}_c := \mathbb{A}_c \mathbb{A}_c^t$  on the orthogonal of its kernel:

$$\mathcal{L}_c^{-1} = (\mathbb{L}_c + \lambda \Pi)^{-1} - \frac{1}{\lambda} \Pi, \quad \Pi = \frac{1}{N_s} (1 \otimes 1) \in \mathcal{M}_{N_k}, \quad \forall \lambda \neq 0$$

 **R. Abgrall**, *Some Remarks about Conservation for Residual Distribution Schemes*. Methods Appl. Math., 18:327-351, 2018.

Few remarks

- ▶ **Source term** is excluded in the definition since only flux-dependent integrals are considered in reconstruction;
- ▶ **Implementation**: only  $\Phi_c$  and boundary terms  $\partial \mathbb{F}_c$  depend on time, but all the other terms are precomputable;
- ▶ **Alternative expression**: using spanning set of subresolution functions  $\phi_m^c = p_{\omega_c}^k(\mathbb{1}_m^c)$ , where  $p_{\omega_c}^k$  is the  $L^2$ -projector on cell  $\omega_c$ .

## DG schemes ≡ Subcell FV schemes

Theorem (equivalence of DG and subcell FV schemes)

The NSW-DG residual scheme  $\frac{d\bar{\mathbf{V}}_c}{dt} = \mathbb{M}_c^{-1}(\Phi_c + \mathbf{S}_c)$  can be recast into  $N_s$  FV-like subcell schemes as

$$\boxed{\frac{d\bar{\mathbf{V}}_c}{dt} = -\mathbb{D}_c^{-1} \left( \mathbb{A}_c \widehat{\mathbb{F}}_c + \partial \mathbb{F}_c \right) + \bar{\mathbf{S}}_c}$$

where  $\bar{\mathbf{S}}_c := \mathbb{P}_c \mathbb{M}_c^{-1} \mathbf{S}_c$  contains the submean values of source term projection, i.e.

$$\bar{\mathbf{B}}_m^c := \frac{1}{|S_m^c|} \int_{S_m^c} p_{\omega_c}^k (\mathbf{B}(\mathbf{v}_h, \nabla_{\mathbf{x}} b_h)) d\mathbf{x}.$$

DG equiv. discrete scheme with FE time integration for all subcell  $S_m^c \subset \omega_c$

$$\bar{\mathbf{v}}_m^{c,n+1} = \bar{\mathbf{v}}_m^{c,n} - \frac{\Delta t^n}{|S_m^c|} \sum_{S_p^c \in \mathcal{V}_m^c} \ell_{mp} \widehat{\mathbb{F}}_{mp} + \Delta t^n \bar{\mathbf{B}}_m^{c,n}, \quad \forall m \in [1, N_s]$$

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# Combining DG and FV frameworks (1)

Finite Volume scheme

👍 robustness   👎 1<sup>st</sup> order accuracy

+

Discontinuous Galerkin scheme

👍  $k^{th}$ -order accuracy   👎 robustness

⇓

Monolithic DG-FV subcell scheme

👍  $k^{th}$ -order accuracy & robustness

## Combining DG and FV frameworks (2)

Our numerical solution should satisfy the following properties:

- ▶ **Accuracy:** high-order precision can be required  
    → natural in DG schemes; requires mesh refinement in FV schemes
- ▶ **Physical admissibility:** in NSW context, the solution should stay in  $\mathcal{H}_b^+$   
    → automatic in FV schemes; requires dedicated techniques in DG schemes
- ▶ **Stability / No spurious oscillations:** satisfy a discrete maximum principle  
    → guaranteed in FV schemes; not ensured by DG schemes (limiters needed)

Idea – blending DG reconstructed fluxes and FV fluxes at subcell scale

## Combining DG and FV frameworks (3)

### Blended fluxes and blending coefficient

For every face  $\Gamma_{mp}^c \in \mathcal{F}_{S_m^c}$ , the high-order DG reconstructed flux  $\widehat{\mathbb{F}}_{mp}$  and a first-order FV flux  $\mathbb{F}_{mp}^{*,\text{FV}}$  are assembled in a convex way:

$$\widetilde{\mathbb{F}}_{mp} = \mathbb{F}_{mp}^{*,\text{FV}} + \Theta_{mp} \left( \widehat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*,\text{FV}} \right) = \mathbb{F}_{mp}^{*,\text{FV}} + \Theta_{mp} \Delta \mathbb{F}_{mp}$$

**A** The **blending coefficient**  $\Theta_{mp} \in [0, 1]$  is:

- ▶ computed *a priori* on each  $\Gamma_{mp}^c$ , at each time step (or RK stage);
- ▶ uniquely defined *i.e.*  $\Theta_{mp} = \Theta_{pm}$ , for all  $S_p^v \in \mathcal{V}_m^c$ .

### Monolithic DG-FV subcell scheme with FE time integration

$$\overline{\mathbf{v}}_m^{c,n+1} = \overline{\mathbf{v}}_m^{c,n} - \frac{\Delta t^n}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \widetilde{\mathbb{F}}_{mp} + \Delta t^n \overline{\mathbf{B}}_m^{c,n}, \quad \forall m \in \llbracket 1, N_s \rrbracket$$

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## Source term treatment

### Flowchart of the discretization

⌚ Dealing with both **polynomial DOFs** and **subcell-averaged values**

- Subcell averages:** compute  $\bar{b}_m^c$  and  $\bar{\eta}_m^c$  on each subcell, then reconstruct  $b_h$  and  $\eta_h$  via projection matrix  $\mathbb{P}_c$ ;
- Projection:** evaluate  $\mathbf{B}(\mathbf{v}_h, \nabla_{\mathbf{x}} b_h)$  at quadrature nodes, then apply an  $L^2$  projection onto  $\mathbb{P}^k$ ;
- Integration:** compute the mean value of the projected source over each subcell:

$$\bar{\mathbf{B}}_m^c := \frac{1}{|S_m^c|} \int_{S_m^c} \mathbf{B}_h \, d\mathbf{x}$$

### Implementation remark

Formally corresponds to multiplying the DG source integral by  $\mathbb{P}_c \mathbb{M}_c^{-1}$ :

$$\bar{\mathbf{B}}_m^c = \mathbb{P}_c \mathbb{M}_c^{-1} \left( \int_{\omega_c} \mathbf{B}_h \varphi_h \, d\mathbf{x} \right)$$

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## Reformulation as a Godunov-like scheme

Solution at  $t^{n+1}$  as a convex combination of quantities defined at  $t^n$

$$\begin{aligned}\bar{\mathbf{v}}_m^{c,n+1} &= \bar{\mathbf{v}}_m^{c,n} - \frac{\Delta t^n}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \tilde{\mathbb{F}}_{mp} + \Delta t^n \bar{\mathbf{B}}_m^{c,n} \\ &\quad + \frac{\Delta t^n}{|S_m^c|} \mathbb{F} \left( \bar{\mathbf{v}}_m^{c,n}, \bar{b}_m^c \right) \cdot \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \mathbf{n}_{mp} \pm \frac{\sigma \Delta t^n}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \bar{\mathbf{v}}_m^{c,n} \\ &= \left( 1 - \frac{\sigma \Delta t^n}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \right) \bar{\mathbf{v}}_m^{c,n} + \frac{\sigma \Delta t^n}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \tilde{\mathbf{v}}_{mp}^{*, -} + \Delta t^n \bar{\mathbf{B}}_m^{c,n}\end{aligned}$$

- $\tilde{\mathbf{v}}_{mp}^{*, -}$  are the interior **blended Riemann intermediate states**

$$\tilde{\mathbf{v}}_{mp}^{*, -} := \bar{\mathbf{v}}_m^{c,n} - \frac{\tilde{\mathbb{F}}_{mp} - \mathbb{F} \left( \bar{\mathbf{v}}_m^{c,n}, \bar{b}_m^c \right) \cdot \mathbf{n}_{mp}}{\sigma} = \mathbf{v}_{mp}^{*, -} - \Theta_{mp} \left( \frac{\hat{\mathbb{F}}_{mp} - \mathbb{F}_{mp}^{*, FV}}{\sigma} \right);$$

- $\mathbf{v}_{mp}^{*, -}$  are the **1<sup>st</sup>-order FV Riemann intermediate states**.

## Analytical formula to ensure water height positivity

Relying on 1<sup>st</sup>-order FV Riemann intermediate states

Proof of the natural **preservation of water-height positivity** for 1<sup>st</sup>-order elevation Riemann FV states  $\eta_{mp}^{*,\pm}$

↪ allows us to rely on the **robustness of FV framework** to ensure the properties we want

## Physical admissibility detector

$$\Theta_{mp}^{\mathcal{H}_b^+} := \min \left( \Theta_{mp}^{\mathcal{H}_b^+, -}, \Theta_{mp}^{\mathcal{H}_b^+, +} \right)$$

- ▶  $\Theta_{mp}^{\mathcal{H}_b^+, -} := \frac{\sigma \left( \eta_{mp}^{*, -} - \bar{b}_m^c \right)}{\Delta F_{mp}}$     if  $\Delta F_{mp} > 0$ ,     $\Theta_{mp}^{\mathcal{H}_b^+, -} = 1$     else;
- ▶  $\Theta_{mp}^{\mathcal{H}_b^+, +} := \frac{\sigma \left( \bar{b}_p^v - \eta_{mp}^{*, +} \right)}{\Delta F_{pm}}$     if  $\Delta F_{pm} < 0$ ,     $\Theta_{mp}^{\mathcal{H}_b^+, +} = 1$     else.

## Analytical formula to prevent spurious oscillations

Mimicking a local maximum principle

$$\alpha_m^c := \min_{S_p^v \in \mathcal{N}(S_m^c)} \bar{\eta}_p^{v,n} \leq \bar{\eta}_m^{c,n+1} \leq \max_{S_p^v \in \mathcal{N}(S_m^c)} \bar{\eta}_p^{v,n} =: \beta_m^c$$

where  $\mathcal{P}_m^c$  is the set of vertices  $x_p$  of subcell  $S_m^c$  and

$$\mathcal{N}(S_m^c) := \bigcup_{x_p \in \mathcal{P}_m^c} \{S_q \mid x_p \in S_q\}$$

Subcell numerical admissibility detector

$$\Theta_{mp}^{\text{SubNAD}} := \min \left( 1, \left| \frac{\sigma}{\Delta F_{mp}} \right| \begin{cases} \min (\beta_p^v - \eta_{mp}^{*,+}, \eta_{mp}^{*,-} - \alpha_m^c) & \text{if } \Delta F_{mp} > 0 \\ \min (\beta_m^c - \eta_{mp}^{*,-}, \eta_{mp}^{*,+} - \alpha_p^v) & \text{if } \Delta F_{mp} < 0 \end{cases} \right)$$

**A** For NSW, no local maximum principle for the conserved variable!

→ needs to be **relaxed** in the presence of **smooth extrema**

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## Test 1 – Order of accuracy assessment

Steady vortex with  $\mathcal{C}^\infty$  topography

- ▶ **Domain:**  $\Omega = [-5, 5]^2$    **Degree:**  $k = 1, 2, 3$    **Mesh:**  $n_{\text{el}} = 200 \rightarrow 12800$
- ▶ **Goal:** convergence of the scheme on a smooth solution with a consistent discretization of the topography source term

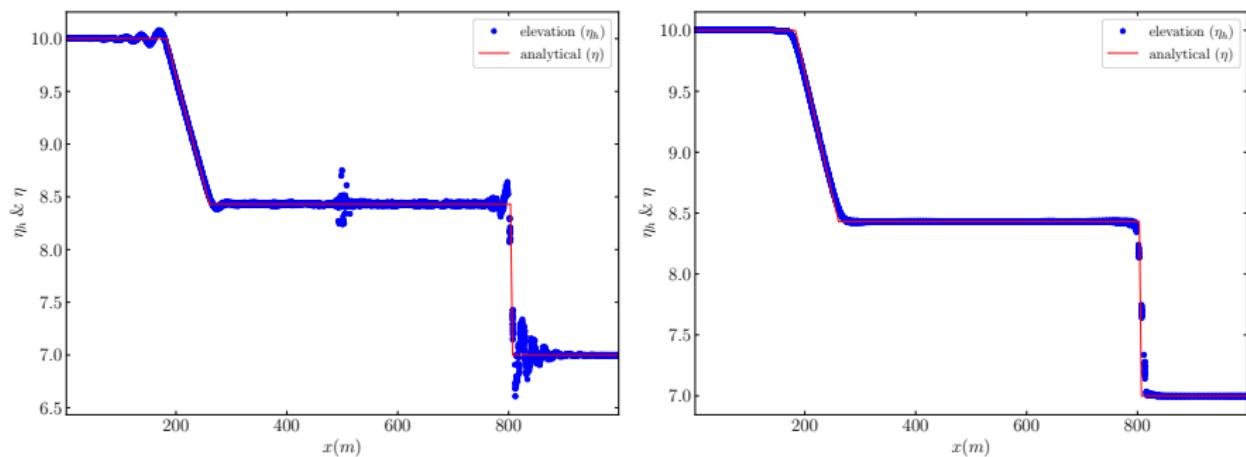
$k$	1		2		3	
$h$	$E_{L^2}^\eta$	$q_{L^2}^\eta$	$E_{L^2}^\eta$	$q_{L^2}^\eta$	$E_{L^2}^\eta$	$q_{L^2}^\eta$
1	9.445E-2	2.35	1.529E-2	2.91	4.580E-3	4.19
$\frac{1}{2}$	1.854E-2	2.16	2.039E-3	3.03	2.505E-4	4.10
$\frac{1}{4}$	4.158E-3	2.07	2.491E-4	2.97	1.465E-5	4.00
$\frac{1}{8}$	9.923E-4	—	3.187E-5	—	9.165E-7	—

**Figure:**  $L^2$ -errors between numerical and analytical solutions and convergence rates for  $\eta$  at time  $t = 0.1$  sec.

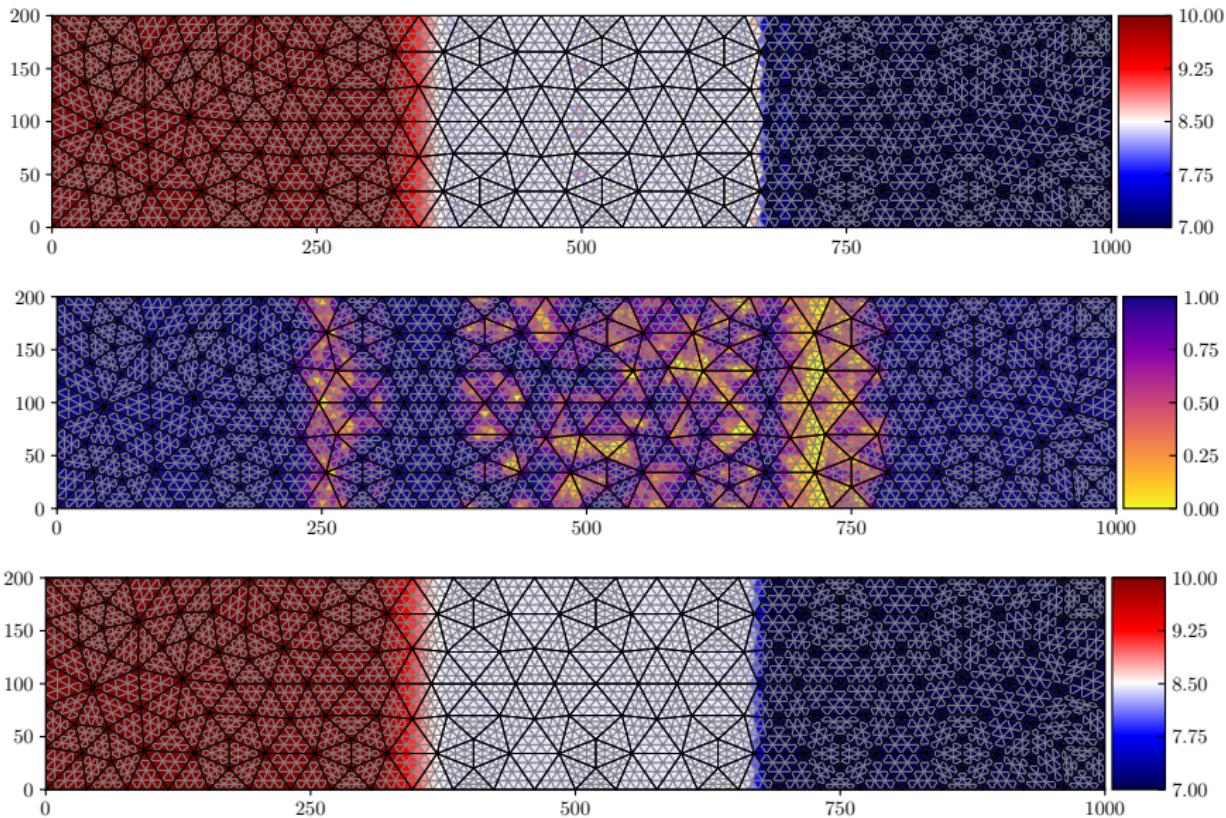
## Test 2 – Riemann problems (1)

### Dam-break on a wet bed

- ▶ **Domain:**  $\Omega = [0, 1000] \times [0, 200]$    **Degree:**  $k = 4$    **Mesh:**  $n_{\text{el}} = 350$
- ▶ **Goal:** handling shock waves and rarefaction fronts



**Figure:** At  $t = 32$  sec,  $\mathbb{P}^4$  pure DG elevation (left) and monolithic DG/FV subcells elevation (right).

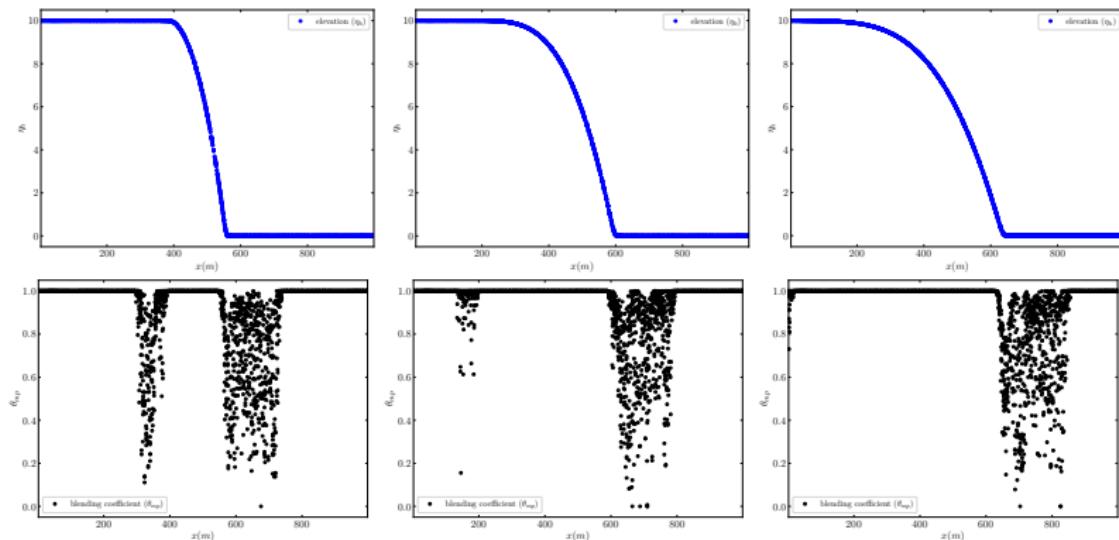


**Figure:** At  $t = 18$  sec,  $\mathbb{P}^4$  unlimited DG elevation (top), map of blending coefficient means per subcell (center) and monolithic DG/FV subcells elevation (bottom).

## Test 3 – Riemann problems (2)

Dam-break on a dry bed with friction

- **Domain:**  $\Omega = [0, 1000] \times [0, 200]$    **Degree:**  $k = 3$    **Mesh:**  $n_{\text{el}} = 350$
- **Goal:** treating wet/dry interfaces, supplemented with friction

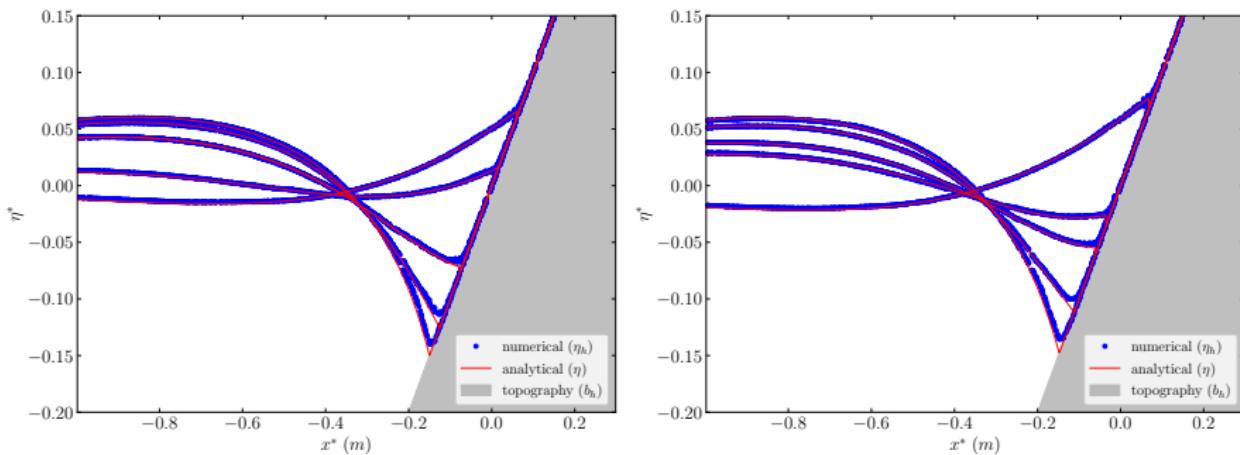


**Figure:** Snapshots of  $\mathbb{P}^3$  free surface elevation and blending density profiles for  $t \in [10, 60]$  sec for  $k_f = 0.5$ .

## Test 4 – Periodic run-up/run-down

### Carrier & Greenspan periodic solution

- **Domain:**  $\Omega = [-20, 6] \times [0, 4]$    **Degree:**  $k = 2$    **Mesh:**  $n_{\text{el}} = 1092$
- **Goal:** assessing the ability of the scheme to capture periodic solutions with no phase shift

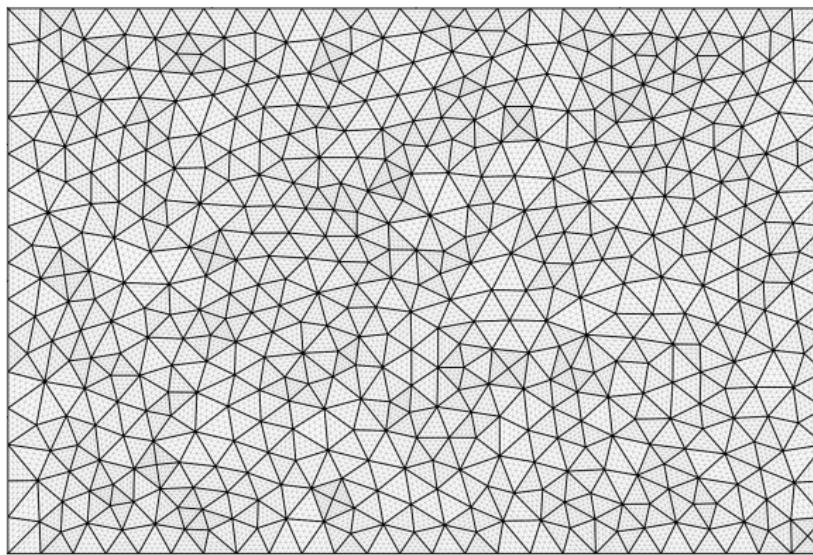


**Figure:** Snapshots of  $\mathbb{P}^2$  elevation for  $t \in [80, 86]$  sec (left) and for  $t \in [154, 160]$  sec (right).

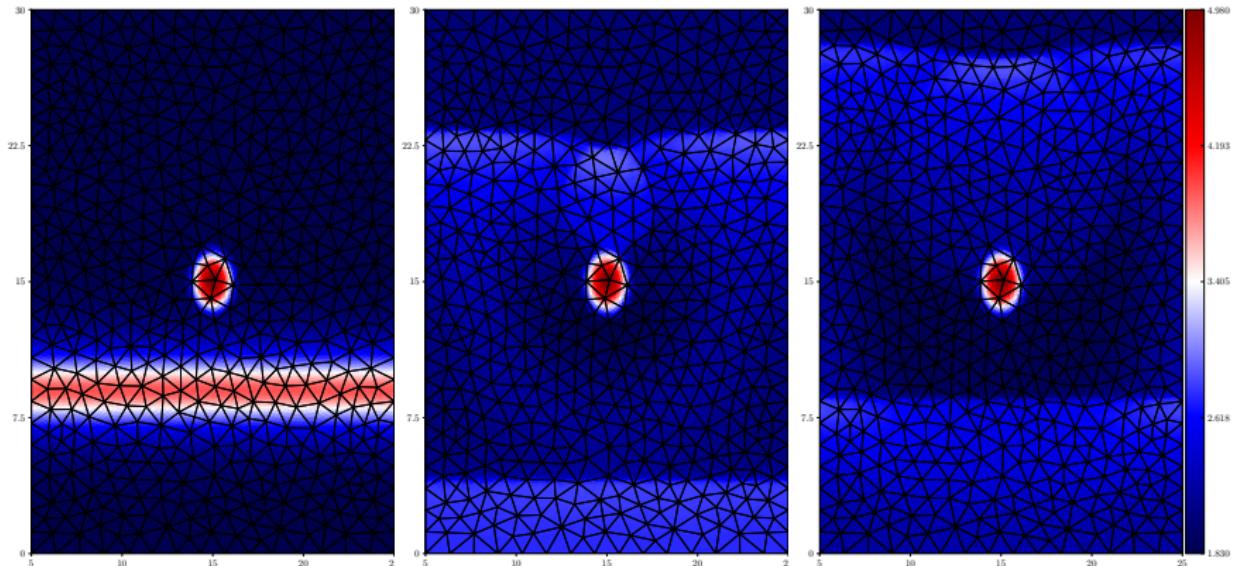
## Test 5 – Rock-wave interactions

Single wave collapsing on a Gaussian rock

- **Domain:**  $\Omega = [5, 25] \times [0, 30]$    **Degree:**  $k = 6$    **Mesh:**  $n_{\text{el}} = 584$
- **Goal:** assessing robustness and correct shock-capturing in challenging case



**Figure:** Unstructured simplicial mesh  $\mathbb{P}^6$  subdivision onto triangles with  $n_{\text{el}} = 584$  cells.



**Figure:** Snapshots of  $\mathbb{P}^6$  elevation at several times (and link to simulation).

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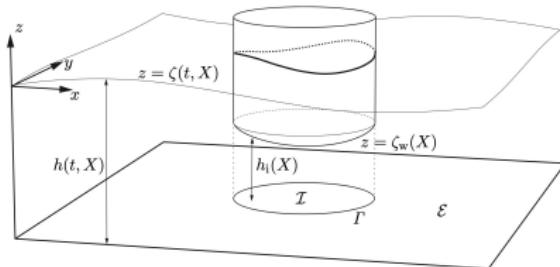


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## Ph.D. objectives

We want an ideal scheme to solve the nonlinear shallow water (NSW) equations, such that we can then study:

### wave-structure interactions



From the theory...



... to its potential applications

## Ongoing and upcoming work

### What has been done...

- 📄 **S.C., A. Haidar, F. Marche & F. Vilar**, *Monolithic DG-FV subcell schemes for nonlinear hyperbolic system with source terms. Applications to shallow water asymptotics.* In preparation. 2025.
- 📄 **S.C., F. Marche & F. Vilar**, *Local monolithic DG-FV subcell scheme for 2D NSW on unstructured grids.* In preparation. 2025.

### ... and what are the plans for the future!

- ▶ Designing a mixed **HHO/DG-FV subcells** method for **wave-structure interactions**;
- ▶ Adaptation of the method to **moving** or **deforming** meshes via an **ALE framework**;
- ▶ Extension to **dispersive water-waves equations** in 2D case;
- ▶ Generalization to PDEs posed on **surfaces**.

# ~ Thank you for your attention! ~

## Special acknowledgments

### ► Organizing committee of MS133

→ *W. Boscheri, F. Fambri, M. Han Veiga, R. Loubère & V. Perrier*

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### ► Supportive professors

→ *A. Duran, D. Lannes, B. Mohammadi, P. Azerad & D. Le Roux*

### ► Colleagues and friends

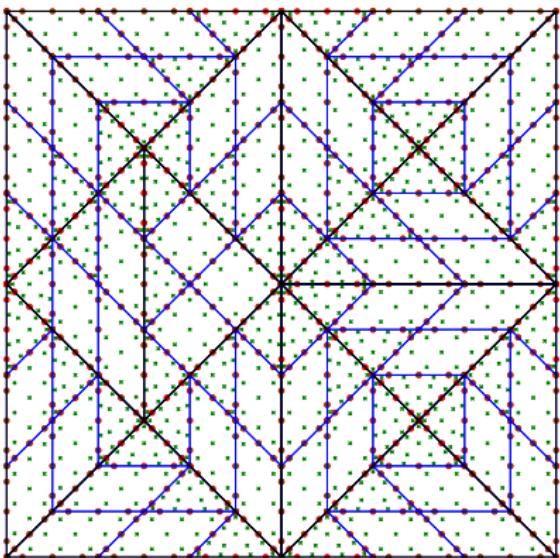
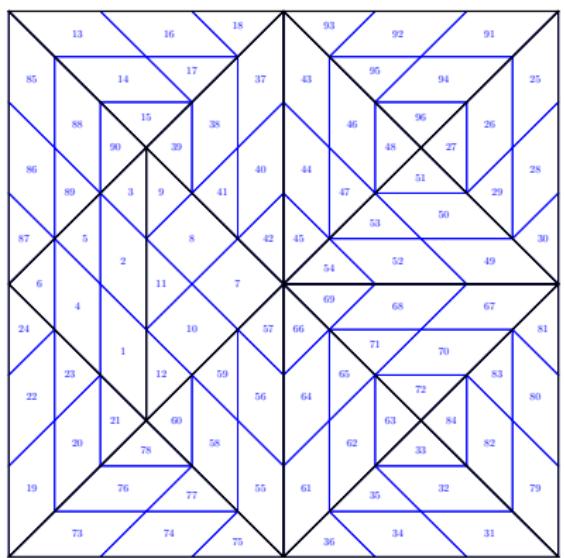
→ *A. Haidar & M. Hanot*

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## Quadrature on subcells



**Figure:** Subdivision of a coarse mesh into subcells with their global numbering (left), alongside the quadrature points for subcell interiors and faces (right).

# Initialization

## Initialization strategy

Initialization is performed via **subcell averages** followed by projection using  $\mathbb{P}_c$ , instead of  $L^2$  projection or interpolation as usually done in DG schemes

↪ this guarantees  $\mathbf{v}_h \in \mathcal{H}_b^+$  at  $t = 0$ , and enforces  $\eta_h = b_h$  in dry zones

⚠ Since  $b_h$  is discontinuous across cells, **hydrostatic reconstruction** is applied to both DG and subcell FV fluxes.

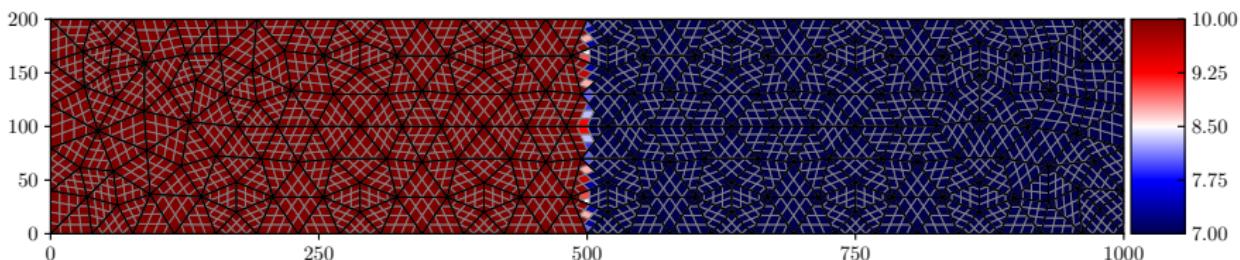


Figure:  $\mathbb{P}^3$  dam-break problem initialization.

## Hydrostatic reconstruction

### Assuring both WB and positivity in numerical fluxes

?

Hydrostatic reconstruction framework used on both DG and subcell FV fluxes  
 ↳ ensures **positivity** of the water height, even for discontinuous topography

At each interface  $\Gamma_{cv(k)}$  (resp. subinterface  $\Gamma_{mp(k)}$ ), reconstructed values are defined:

- ▶ Topography rec.:  $\tilde{b}_k = \max(b_k^-, b_k^+)$ ,  $\check{b}_k = \tilde{b}_k - \max(0, \tilde{b}_k - \eta_k^-)$
- ▶ Water height/elevation rec.:  $\check{H}_k^\pm = \max(0, \eta_k^\pm - \tilde{b}_k)$ ,  $\check{\eta}_k^\pm = \check{H}_k^\pm + \tilde{b}_k$
- ▶ Modified states:  $\check{\mathbf{v}}_k^\pm = \left( \check{\eta}_k^\pm, \frac{\check{H}_k^\pm}{H_k^\pm} \mathbf{q}_k^\pm \right)^t$

These are then used in a Lax-Friedrichs-type flux  $\mathbb{F}^*$ , completed by a correction term  $\check{\mathbb{F}}_{cv(k)}$  to ensure well-balancing:

$$\mathbb{F}_{cv(k)}^* = \mathbb{F}^*(\check{\mathbf{v}}_k^-, \check{\mathbf{v}}_k^+, \check{b}_k, \check{b}_k, \mathbf{n}_{cv(k)}) + \check{\mathbb{F}}_{cv(k)}$$

## Source term treatment

### Alternative discretization of the source term

$$\bar{\mathbf{B}}_m^c = \bar{\mathbf{B}}_m^{c,FV} + \Theta_m^c \left( \bar{\mathbf{B}}_m^{c,DG} - \bar{\mathbf{B}}_m^{c,FV} \right)$$

►  $\Theta_m^c = \frac{1}{\#\mathcal{V}_m^c} \sum_{S_p^v \in \mathcal{V}_m^c} \Theta_{mp}$  **subcell global blending**

►  $\bar{\mathbf{B}}_m^{c,DG} = \frac{1}{|S_m^c|} \int_{S_m^c} \mathbf{B}_h \, d\mathbf{x}$  **DG source term**

►  $\bar{\mathbf{B}}_m^{c,FV} = \frac{1}{|S_m^c|} \int_{S_m^c} \mathbf{B}(\bar{\mathbf{v}}_m^c, \nabla_{\mathbf{x}} b_h^c) \, d\mathbf{x}$  **FV source term**

💬 No significant difference in results → we keep  $\bar{\mathbf{B}}_m^c = \bar{\mathbf{B}}_m^{c,DG}$

## Generalization to algebraic/geometric source terms

### Topography and (nonlinear) friction effects

$$\mathbf{S}(\mathbf{v}, b) := \mathbf{B}(\mathbf{v}, \nabla_x b) + \mathbf{Fr}(\mathbf{v}, b)$$

►  $\mathbf{B}(\mathbf{v}, \nabla_x b) = (0, -g\eta \nabla_x b)^t$  **Topography source term**

►  $\mathbf{Fr}(\mathbf{v}, b) = \begin{cases} (0, -k_f^2 \mathbf{q})^t, k_f > 0 & \text{Linear friction law} \\ \left(0, -n_f^2 \frac{\mathbf{q} \|\mathbf{q}\|}{(\eta - b)^\gamma}\right)^t, n_f, \gamma > 0 & \text{Manning friction law} \end{cases}$

? Handled the same way as previously → **easily generalizable**

### Applications to Serre–Green–Naghdi (SGN) equations

#### **Reformulation: Elliptic problem + NSW with dispersive source term**

1. Elliptic problem solved *independently*, using a finite element method;
2. Resulting dispersive source term discretized within the NSW framework.

## Blending relaxation

### Smoothness detector on subcells

⚠ Must ensure to **relax correction** close to **smooth extrema**!

1. **Linearized reconstructions** of  $\partial_x \eta_h$  and  $\partial_y \eta_h$  are built inside subcell, i.e.

$$\mathfrak{E}_x^m(\mathbf{x}) := \overline{\partial_x \eta_h^c}^m + \overline{\nabla_{\mathbf{x}} (\partial_x \eta_h^c)}^m \cdot (\mathbf{x} - \mathbf{x}_m^c),$$

$$\mathfrak{E}_y^m(\mathbf{x}) := \overline{\partial_y \eta_h^c}^m + \overline{\nabla_{\mathbf{x}} (\partial_y \eta_h^c)}^m \cdot (\mathbf{x} - \mathbf{x}_m^c),$$

2. **Checking these indicators values at vertices**: if their values remain within the local extrema over a stencil, the solution is considered smooth over subcell  $S_m^c$ ;
3. **If two neighbors  $S_m^c$  and  $S_p^v$  are considered smooth**, the blending coefficient on  $\Gamma_{mp}$  is relaxed to  $\Theta_{mp} = 1 \rightarrow$  **full DG accuracy**

### When to use the smooth extrema detector?

- ▶ **Order  $\leq 2$** : No detector needed.
- ▶ **Order  $= 3$** : Detector required at the **cell** level.
- ▶ **Order  $\geq 4$** : Detector required at the **subcell** level.

## Blending smoothening

Why smoothening blending coefficient?

A **sharp switch** between low and high-order fluxes (i.e.,  $\Theta_{mp} = 0$  vs.  $\Theta_{mp} = 1$ ) may cause **local oscillations**

→ blending smoothers designed to **mitigate abrupt transitions**

► **Mean-value smoother** (default in experiments):

$$\Theta_m^c := \frac{1}{\#\mathcal{V}_m^c} \sum_{S_p^v \in \mathcal{V}_m^c} \Theta_{mp}, \quad \tilde{\Theta}_{mp} := \min \left( \Theta_{mp}, \frac{1}{\#\mathcal{V}_{mp}} \sum_{S_q^v \in \mathcal{V}_{mp}} \Theta_q^v \right)$$

→ Less diffusive, smoother transitions

► **Minimum-value smoother**:

$$\Theta_m^c := \min_{S_p^v \in \mathcal{V}_m^c} \Theta_{mp}, \quad \tilde{\Theta}_{mp} := \min \left( \Theta_{mp}, \min_{S_q^v \in \mathcal{V}_{mp}} \Theta_q^v \right)$$

→ Stronger damping near discontinuities

## Preservation of steady-states (1)

Why does it matter ?

- ▶ **Preserves lake at rest steady states exactly**, avoiding spurious motions;
- ▶ **Reduces numerical errors** near equilibrium, especially when small perturbations are present;
- ▶ **Essential for wet/dry interfaces**, where small oscillations can destabilize the scheme.

Well-balancing (WB) property

Providing that the integrals of discrete formulation are exactly computed, we have the following result:

$$\forall n \in \mathbb{N}, \quad \forall \eta^e \in \mathbb{R}, \quad (\eta_h^n = \eta^e \text{ and } \mathbf{q}_h^n = \mathbf{0}) \implies (\eta_h^{n+1} = \eta^e \text{ and } \mathbf{q}_h^{n+1} = \mathbf{0})$$

## Preservation of steady-states (2)

### Sketch of proof

**Objective:** showing that numerical fluxes are cancelling the source term *i.e.*

$$\frac{1}{|S_m^c|} \sum_{S_p^v \in \mathcal{V}_m^c} \ell_{mp} \tilde{\mathbb{F}}_{mp} = \bar{\mathbf{B}}_m^{c,n} \quad \text{s.t.} \quad \bar{\mathbf{v}}_m^{c,n+1} = \bar{\mathbf{v}}_m^{c,n}.$$

- ▶ Exact integration required  $\rightarrow$  natural with high-order quadrature;
- ▶ Under well-balanced assumptions:

$$\nabla_{\mathbf{x}} \cdot \mathbb{F}(\mathbf{v}_c, b_c) = \mathbf{B}(\mathbf{v}_c, \nabla_{\mathbf{x}} b_c), \quad \forall \omega_c \in \mathcal{T}_h;$$

- ▶ Fluxes  $\hat{\mathbb{F}}_{mp}$  and  $\mathbb{F}_{mp}^{*,\text{FV}}$  match the continuous flux  $\mathbb{F}_h^c \cdot \mathbf{n}_{mp}$  under equilibrium;
- ▶  $\tilde{\mathbb{F}}_{mp}$  is built as a convex combination of these well-balanced fluxes  
 $\hookrightarrow$  preserves equilibrium as well !