PHYS 512 - Problem Set 5

5. a) We have $\sum_{x=0}^{N-1} \exp\left(-\frac{2\pi i k x}{N}\right)$. We know that:

$$\sum_{x=0}^{N-1} \exp\left(-\frac{2\pi i k x}{N}\right) = \sum_{x=0}^{N-1} \exp\left(-\frac{2\pi i k}{N}\right)^{x}$$

This corresponds to a geometric series with $r = \exp\left(-\frac{2\pi i k}{N}\right)$.
Then the sum is equal to:

$$\sum_{x=0}^{N-1} \exp\left(-\frac{2\pi i k}{N}\right)^{x} = \frac{\exp\left(-\frac{2\pi i k}{N}\right)^{0} - \exp\left(-\frac{2\pi i k}{N}\right)^{N}}{1 - \exp\left(-\frac{2\pi i k}{N}\right)}$$

$$\Leftrightarrow \sum_{x=0}^{N-1} \exp\left(-\frac{2\pi i k x}{N}\right) = \frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)}$$

b) We want to find $\lim_{k \to 0} \frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)}$, which gives us the undetermined $0/0$ if we plug in k. Then, using l'Hopital's rule, we have:

$$\lim_{k \to 0} \frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)} = \lim_{k \to 0} \frac{\frac{d}{dk}(1 - \exp(-2\pi i k))}{\frac{d}{dk}(1 - \exp(-2\pi i k/N))}$$

$$= \lim_{k \to 0} \frac{2\pi i \, e^{2\pi i k}}{\frac{2\pi i}{N} e^{2\pi i k/N}} = \underline{N}$$

I chose to prove numerically that $\sum \exp\left(-\frac{2\pi i k x}{N}\right)$ is zero for any integer k that is not a multiple of N.

c) Let $f(x) = \sin\left(\frac{2\pi k x}{N}\right)$ with k a non-integer and $x \in [0, N-1]$.
Then:

$$DFT(f(x)) = \sum_{x=0}^{N-1} \sin\left(\frac{2\pi k x}{N}\right) e^{-2\pi i k' x/N}$$

Using Euler's formula, we have $\sin\left(\frac{2\pi k x}{N}\right) = \frac{1}{2i}\left(e^{\frac{2\pi i k x}{N}} - e^{-\frac{2\pi i k x}{N}}\right)$

Then, $DFT(f(x)) = \sum_{x=0}^{N-1} \frac{1}{2i}\left(e^{\frac{2\pi i k x}{N}} - e^{-\frac{2\pi i k x}{N}}\right) e^{-\frac{2\pi i k' x}{N}}$

$$= \frac{1}{2i} \sum_{x=0}^{N-1} e^{\frac{2\pi i x (k-k')}{N}} - \frac{1}{2i} \sum_{x=0}^{N-1} e^{-\frac{2\pi i (k+k') x}{N}}$$

$$= \frac{1}{2i}\left(\frac{1-e^{2\pi i (k-k')}}{1-e^{2\pi i (k-k')/N}} - \frac{1-e^{-2\pi i (k+k')}}{1-e^{-2\pi i (k+k')/N}}\right)$$

where $k' \in [0, N-1]$.

6. a) Let $X(t)$ represent our random walk. Then, by definition at each t, $X(t)$ has a single constant and random value. Then $\frac{dX}{dt}$ is white noise, since $X(t)$ is random.
We also know that the power spectrum of white noise is constant. Let us call P the power spectrum, then $P(X(t)) = P_0$.
We also know that $P(f(x)) = |FT(f(x))|^2$.
Then, $P(\frac{dX}{dt}) = |FT(\frac{dX}{dt})|^2$.

$$FT\left(\frac{dX}{dt}\right) = \int_{-\infty}^{+\infty} \frac{dX}{dt} e^{-i\omega t} dt, \quad \text{with } \omega = \frac{2\pi k}{N}$$

$$= \left[X e^{-i\omega t}\right]_{-\infty}^{+\infty} - \omega i \int_{-\infty}^{+\infty} X e^{-i\omega t} = -\omega i \, FT(X)$$

Then $P(\frac{dX}{dt}) = |FT(\frac{dX}{dt})|^2 = \omega^2 |FT(X)|^2 = \omega^2 P(X(t))$

But we have $P(\frac{dX}{dt}) = P_0$, so $P(X(t)) = \frac{P_0}{\omega^2} \propto \frac{1}{\omega^2}$

We have shown that the power spectrum of a random walk is proportional to $\frac{1}{k^2}$.