We know that for the ratio-of-uniforms method, we have $0 \leq u \leq \sqrt{PDF(\frac{v}{u})}$.
PDF is the PDF of the exponential distribution we are considering, equal to e^{-x}. We also know that $0 \leq u \leq 1$, so:

$$0 \leq u \leq \sqrt{e^{-\frac{v}{u}}} \iff 0 \leq u^2 \leq e^{-\frac{v}{u}}$$

$$\iff \ln(0) = -\infty \leq 2\ln(u) \leq -\frac{v}{u}$$

$$\iff +\infty \geq -2u\ln(u) \geq v$$

$$\iff +\infty \geq -2u\ln(u) \times \frac{1}{v} \geq 1$$

$$\iff +\infty \geq \frac{1}{v} \geq -\frac{1}{2u\ln(u)}$$, because $\ln(u) < 0$ so $-\ln(u) > 0$

$$\iff \boxed{0 \leq v \leq -2u\ln(u)}$$

Let us now find u that maximizes $-2u\ln(u)$:

$$\frac{d}{du}(-2u\ln(u)) = 0 \iff -2u \times \frac{1}{u} - 2\ln(u) = 0$$

$$\iff -2 - 2\ln(u) = 0$$

$$\iff \ln(u) = -1$$

$$\iff u = e^{-1}$$, which is between 0 and 1.

So $\boxed{0 \leq v \leq \frac{2}{e}}$