# Harmonic map

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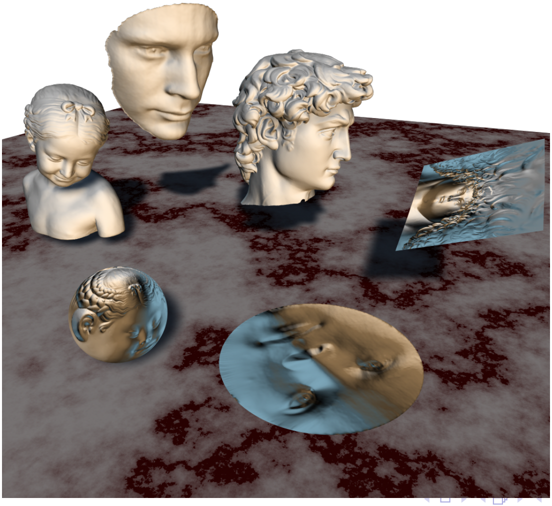
# Surface Parameterization

Map the surfaces onto canonical parameter domains



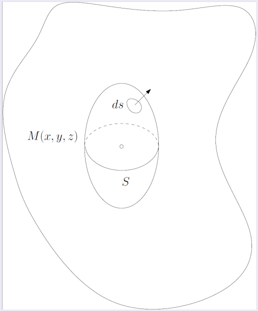
# Heat Diffusion

# Heat Flow

## Physics

Assume an object in $\mathbb{R}^3$ has temperature $u(x, y, z, t)$ at the point $M(x, y, z)$,
$k(x, y, z)$ represents the heat conductivity coefficient,
$c(x, y, z)$ represents the heat capacity ratio,
$\rho(x, y, z)$ represents the material density.

## Configuration



# Fourier's law of conduction

## Theorem (Fourier's law of conduction)

*Within a infinitesimal small time period $dt$, the heat transfer $dQ$ through an infinitesimal area $ds$ is proportional to the* ***negative*** *directional derivative of the temperature of the object $\frac{\partial u}{\partial \mathbf{n}} = \langle \nabla u, \mathbf{n} \rangle$,*

$$dQ = -k(x,y,z)\frac{\partial u}{\partial \mathbf{n}} ds dt,$$

*where $\mathbf{n}$ is the normal to the area element $ds$, the gradient $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$.*

The heat transfer always flows from the high temperature area to the low temperature area; the gradient always from low temperature area points to the hight temperature area.

# Fourier's law of conduction

Get a closed surface $S$ inside the object, the volume inside is denoted as $\Omega$, then from time $t_1$ to time $t_2$ the heat transfer through the surface $S$ entering $\Omega$ is

$$Q_1 = \int_{t_1}^{t_2} \int_S k \frac{\partial u}{\partial \mathbf{n}} dsdt,$$

According to Stokes theorem

$$Q_1 = \int_{t_1}^{t_2} \int_\Omega k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial z^2} \right) dvdt.$$

# Heat Transfer Equation

At every point inside $\Omega$, from $t_1$ to $t_2$ the temperature change needs the heat

$$Q = \int_{t_1}^{t_2} \int_{\Omega} c(x,y,z)\rho(x,y,z)\frac{\partial u}{\partial t} dvdt.$$

Because $\Omega$ and the time interval $[t_1, t_2]$ are arbitrary, let $a^2 = \frac{k}{c\rho}$, then we get the heat transfer equation

## Definition (Heat Diffusion Equation)

$$\frac{\partial u}{\partial t} = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

# Heat Transfer Equation

## Definition (Heat Diffusion Equation)

At steady state, $\frac{\partial u}{\partial t} = 0$, then the function $u$

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0.$$

then we say $u$ is a harmonic function.

# Harmonic map for topological disks

# Harmonic map for topological disks

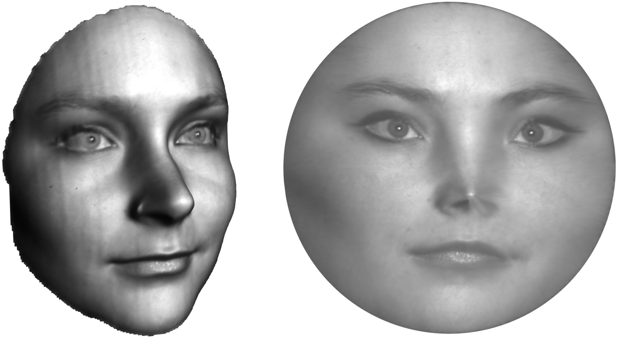


Figure: Harmonic map for topological disks

# Rado's theorem

**Theorem (Rado)**

*$f : \Omega \to D$ is a smooth map where $\Omega$ is a simply connected surface with boundary, domain $D \subset \mathbb{R}^2$ is convex and $f|_{\partial\Omega}$ is homeomorphism. If $f$ is harmonic, then $f$ is diffeomorphism.*

# Harmonic map for topological disks

**Basic Idea**:

We first map the boundary of the model to a unit circle, and then we evolve $\vec{f}$ until it becomes a harmonic map. The evolution of the map $\vec{f}$ is according to a iteration process or it can be gotten directly by using Newton's method.

# Harmonic map for topological disks

**Definition (String Energy)**

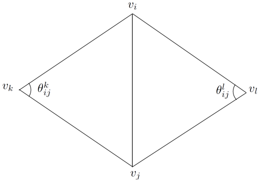
Suppose $f \in C^{PL}(M)$, the string energy is defined as

$$E(f) = \langle f, f \rangle = \sum_{[v_i, v_j] \in M} k_{ij}(f(v_j) - f(v_i))^2$$

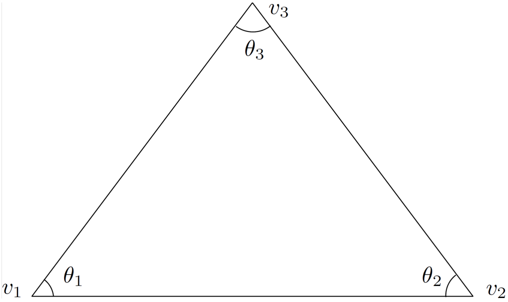
If the edge weight $k_{ij}$ is defined as:

$$k_{ij} = \begin{cases} \cot\theta_{ij}^k + \cot\theta_{ij}^l & \forall e_{ij} \notin \partial M \\ \cot\theta_{ij}^k & \forall e_{ij} \in \partial M \end{cases}$$

then the string energy is called the harmonic energy.



# Harmonic map for topological disks




$$\cot\theta_1 = \frac{(v_2 - v_1)\cdot(v_3 - v_1)}{|(v_2 - v_1)\times(v_3 - v_1)|}$$
$$\cot\theta_2 = \frac{(v_1 - v_2)\cdot(v_2 - v_2)}{|(v_1 - v_2)\times(v_3 - v_2)|}$$
$$\cot\theta_3 = \frac{(v_1 - v_3)\cdot(v_2 - v_3)}{|(v_1 - v_3)\times(v_2 - v_3)|}$$

# Harmonic map for topological disks

**Definition (Harmonic Energy of Maps)**

For a map $\vec{f}: M_1 \to \mathbb{R}^3, \vec{f} = (f_1, f_2, f_3), f_i \in \mathbb{C}^{PL}(M_1), i = 1,2,3$, we define the energy of $\vec{f}$ as the sum of the energies of $f_i$:

$$E(\vec{f}) = \sum_{i=1}^{3} E(f_i)$$

**Algorithm 1** Closed curve to unit circle

**Require:** A closed curve $L$

**Ensure:** A map $\vec{f}: L \to S^1$

1: Traverse the curve $L$, store its vertices ccwly to a list:

$$vlist = \{v_0, v_1, \dots, v_{n-1}\}$$

where $v_0$, $v_n$ are identical.

2: Calculate $L$'s length:

$$s = \sum_{i=0}^{i=n-1} l_{v_i, v_{i+1}}$$

where $l_{v_i, v_{i+1}}$ is the length of the edge $[v_i, v_{i+1}]$

3: **for all** $v_i \in vlist$ **do**

4: $\quad s_i = \sum_{j=1}^{i} l_{v_{j-1}, v_j}$

5: $\quad \theta_i = 2\pi \frac{s_i}{s}$

6: $\quad \vec{f}(v_i) = (\cos\theta_i, \sin\theta_i)$

7: **end for**

8: **return** $\vec{f}$

# Harmonic map for topological disks

**Algorithm 2** Harmonic map for topological disks

**Require:** An open Mesh $M$, energy difference threshold $\delta E$

**Ensure:** A harmonic map $\vec{f}: M \to \mathbb{D}$

Map $\partial M$, the boundary of $M$, to a unit circle using Alg 1.

Initialize $\vec{f} = (0,0)$ for all inner vertices of $M$.

Compute the initial harmonic energy $E$

**repeat**

    $E_0 = E$

    **for all** vertex $v_i \notin \partial M$ **do**

        $\vec{f}(v_i) = \sum_{[v_i,v_j]\in M} \frac{k_{ij}\vec{f}(v_j)}{\sum_j k_{ij}}$

    **end for**

    Recalculate the harmonic energy $E$

**until** Harmonic energy difference $|E - E_0|$ is less than $\delta E$

**return** $\vec{f}$

# Harmonic map for topological disks

## Definition (Laplace Operator)

(1) The piecewise Laplace is a linear operator

$$\Delta_{PL} : C^{PL}(M) \to C^{PL}(M)$$

on the space of piecewise linear functions on M, defined by the formula

$$\Delta_{PL} f(v_i) = \sum_{[v_i, v_j] \in M} k_{ij}(f(v_j) - f(v_i))$$

(2) For a map $\vec{f} : M_1 \to \mathbb{R}^3$, the piecewise Laplacian of $\vec{f}$ is

$$\Delta_{PL}\vec{f} = (\Delta_{PL} f_1, \Delta_{PL} f_2, \Delta_{PL} f_3)$$

# Harmonic map for topological disks

**Algorithm II: Optimize the harmonic energy by solving linear equation systems**
Input: A simply connected mesh $M = (V, E, F)$ with boundary map $g$
output: A harmonic map $f$, mapping the mesh to the unit disk with $f|_{\partial M} = g$

1. Construct a index map to reindex the **inner** vertices:

$$index : v \mapsto i$$

   where $v$ is the i-th inner vertex in $M$.

# Harmonic map for topological disks

2. Construct coefficient matrix $A$ and column vectors $b_x$, $b_y$ of $\Delta f = 0$:
   - Initialize matrix and vectors: $A = 0, b_x = 0, b_y = 0$
   - **For** all edge $\{v, w\} \in M$, do

     **if** both v and w are inner vertices

     $A(index(v), index(w)) = k_{v,w}$
     $A(index(w), index(v)) = k_{v,w}$
     $A(index(v), index(v))$ -= $k_{v,w}$
     $A(index(w), index(w))$ -= $k_{v,w}$

     **else if** v is boundary vertex and w is inner vertex

     $A(index(w), index(w))$ -= $k_{v,w}$
     $b_x(index(w))$ -= $k_{v,w} g(v)[0]$
     $b_y(index(w))$ -= $k_{v,w} g(v)[1]$

     **else if** v is inner vertex and w is boundary vertex

     $A(index(v), index(v))$ -= $k_{v,w}$
     $b_x(index(v))$ -= $k_{v,w} g(w)[0]$
     $b_y(index(v))$ -= $k_{v,w} g(w)[1]$

     **end**

     **end**

# Harmonic map for topological disks

3. Solve the linear equation systems $Ax = b_x$ and $Ay = b_y$ by using conjugate gradient solver in the library **Eigen.**
   Here is a typical and general example in the **documentation**
4. **For** each inner vertex $v \in M$, do
   f(v) = (x(index(v)), y(index(v)))
   **end**
5. **For** each boundary vertex $v \in M$, do
   f(v) = g(v)
   **end**

# Harmonic Map for Topological Quadrilateral

# Harmonic Map for Topological quadrilateral

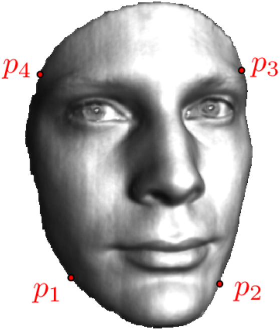



Figure: Harmonic map for topological quadrilateral.

# Harmonic Map for Topological Quadrilateral

**Algorithm 3** Harmonic Map for Topological Quadrilateral

**Require:** An open mesh $M$ with genus zero and one boundary, a threshold $\varepsilon$.

**Ensure:** A harmonic map $\vec{f}$ from $M$ to a rectangle $R$

1. Choose 4 points $v_1$, $v_2$, $v_3$ and $v_4$ ccwly from the boundary $\partial M$. Denote $\gamma_1 = v_2 - v_1$, $\gamma_2 = v_3 - v_2$, $\gamma_3 = v_4 - v_3$ and $\gamma_4 = v_1 - v_4$.
2. Map the boundary $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ to a unit square.
3. Calculate the initial harmonic energy $E$
4. **repeat**
5. $\quad E_0 = E$
6. $\quad \vec{f}(v_i) = \sum_{[v_i,v_j]\in M} \frac{k_{ij}\vec{f}(v_j)}{\sum_j k_{ij}} \quad v_i \notin \partial M$
7. $\quad$ Recalculate the harmonic energy $E$
8. **until** Harmonic energy difference $|E - E_0|$ is less than $\varepsilon$
9. **return** $\vec{f}$

# Harmonic Map for Topological Annulus

# Harmonic Map for Topological Annulus

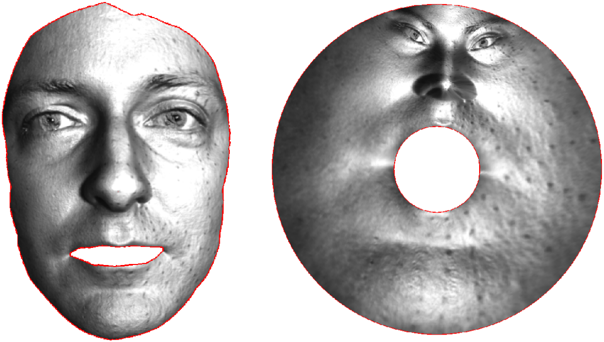


Figure: Topological annulus.

**Algorithm 4** Harmonic Map for Topological Annulus

**Require:** An open mesh $M$ with genus zero and two boundary, a threshold $\varepsilon$.

**Ensure:** A harmonic map $\vec{f}$ from $M$ to a annulus $\hat{M}$

1: Denote one boundary as $\gamma_1$, and the other boundary as $\gamma_2$.

2: Compute a path $\gamma_3 = p_1 p_2$, where $p_1 \in \gamma_1$, $p_2 \in \gamma_2$. Slice $M$ alone $\gamma_3$ to get a topological Quadrilateral $\hat{M}$.

3: Compute the harmonic map $F$ from $\hat{M}$ to planar quadrilateral such that $F_y(\gamma_1) = 0$, $F_y(\gamma_2) = 1$, $F_x(\gamma_3) = 1$ and $F_x(\gamma_3^{-1}) = 2$.

4: Compute the exponential map $\varphi_x(p) = p_x \cos(2\pi \cdot p_y)$, $\varphi_y(p) = p_x \sin(2\pi \cdot p_y)$, which maps the quadrilateral to planar annulus.