Graph Algorithms - Lecture 5

October 30, 2020

Table of contents

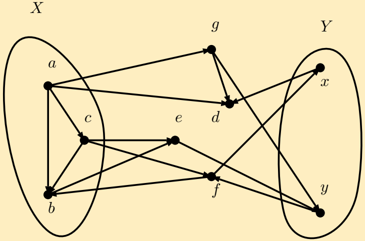
1. Connectivity
   - Menger's theorem
   - p-connectivity
   - König's theorem
   - Hall's theorem
   - Dirac's theorem
2. Trees
   - Basics
   - Generating all spanning trees
   - Counting spanning trees: Kirchhoff's theorem
3. Exercises for the next week seminar
4. (Partially) solved exercises

Definition 1

Let $G = (V, E)$ be a (di)graph and $X, Y \subseteq V$. An XY-path *is any path P in G from a vertex $x \in X$ to a vertex $y \in Y$ such that $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$.*

We denote by $\mathcal{P}(X, Y; G)$ the set of all XY-paths in G. Note that if $x \in X \cap Y$ then the path of length 0, $P = \{x\}$, is an XY-path.

Example



XY-paths: (b, e, y), (c, f, x), and (a, g, y); an YX-path: (y, f, b)

- We say that the paths P_1 and P_2 are (vertex) disjoint if $V(P_1) \cap V(P_2) = \varnothing$.
- Motivated by practical problems in communication networks, and also by the theoretical study of (di)graph connectivity, we are interested in finding a maximum cardinality set of disjoint XY-paths.
- We denote by $p(X, Y; G)$ the maximum number of disjoint XY-paths in G.
- The theorem which reveals this number was established by Menger (1927) and represents one of the fundamental results in Graph Theory.

Definition 2

Let $G = (V, E)$ be a (di)graph and $X, Y \subseteq V$. An XY-separating set *in G is any subset $Z \subseteq V$ such that*

$$V(P) \cap Z \neq \varnothing, \textit{ for each } P \in \mathcal{P}(X, Y; G).$$

We denote by

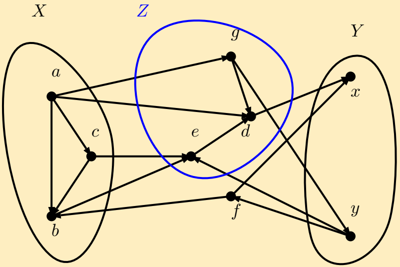
$$\mathrm{S}(X, Y; G) = \{Z \ : \ Z \textit{ is } XY - \textit{separating set in } G\} \textit{ and}$$

$$k(X, Y; G) = \min\{|Z| \ \ Z \in \mathrm{S}(X, Y; G)\}$$

From the definition we easily get:

- If $Z \in \mathrm{S}(X, Y; G)$, then $\mathcal{P}(X, Y; G \setminus Z) = \varnothing$.
- $X, Y \in \mathrm{S}(X, Y; G)$.

Example



A XY-separating set: $Z = \{g, e, d\}$

- If $Z \in \mathrm{S}(X, Y; G)$, then $A \in \mathrm{S}(X, Y; G)$, $\forall A$ such that $Z \subseteq A \subseteq V$.
- If $Z \in \mathrm{S}(X, Y; G)$ and $T \in \mathrm{S}(Z, Y; G)$, then $T \in \mathrm{S}(X, Y; G)$.

Theorem 1

Menger's theorem. Let $G = (V, E)$ be a (di)graph and $X, Y \subseteq V$. Then $p(X, Y; G) = k(X, Y; G)$.
(I. e., the maximum number of disjoint XY-paths = the minimum cardinality of an XY-separating set.)

Proof:
$k(X, Y; G) \geqslant p(X, Y; G) = p$. Let $P_1, \ldots, P_r$ disjoint XY-paths in G; $Z \cap V(P_i) \neq \varnothing$, $\forall Z \in \mathrm{S}(X, Y; G)$. Since P_i are disjoint ($i = \overline{1, r}$):

$$|Z| \geqslant \left| Z \cap \left(\bigcup_{i=1}^{r} V(P_i) \right) \right| = \sum_{i=1}^{r} |Z \cap V(P_i)| \geqslant \sum_{i=1}^{r} 1 = r.$$

Hence, $|Z| \geqslant r$, $\forall Z \in \mathrm{S}(X, Y; G)$; it follows that $k(X, Y; G) \geqslant r$.

$k(X, Y; G) \leqslant p(X, Y; G)$. Omitted. (We will later show that $\forall G = (V, E)$ and $\forall X, Y \subseteq V$, $\exists k(X, Y; G)$ disjoint XY-paths in G using flows in a certain network.) $\square$

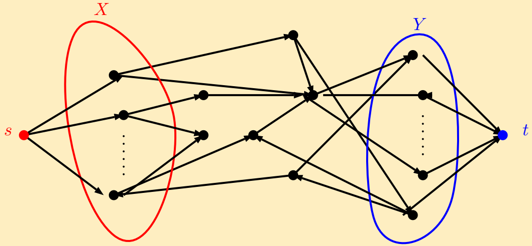
Menger (1927) enounced equivalently the above theorem, using internally-disjoint paths: $P_1, P_2 \in \mathcal{P}_{st}$ such that $V(P_1) \cap V(P_2) = \{s, t\}$:

Theorem 2

Let $G = (V, E)$ be a (di)graph and $s, t \in V$, such that $s \neq t$, $st \notin E$. There are k internally-disjoint paths from s to t in G if and only if there is at least one path from s to t in the (di)graph obtained from G by removing any set of $< k$ vertices different from s and t.

Proof of equivalence:

Theorem 1 $\Rightarrow$ Theorem 2: take $X = N^+_G(s)$ $(N_G(s))$ and $Y = N^-_G(t)$ $(N_G(t))$.



Theorem 2 $\Rightarrow$ Theorem 1: add two new vertices s and t to the (di)graph G, and all (directed) edges from s to any vertex in X and from any vertex in Y to t. $\square$

Applications: p-connectivity

- A graph G is p-connected ($p \in \mathbb{N}^*$) if either $G = K_p$, or $|G| > p$ and $G \setminus A$ is connected for any $A \subseteq V(G)$ with $|A| < p$.
- By Theorem 2, an equivalent characterization of the p-connectivity is:
  A graph G is p-connected ($p \in \mathbb{N}^*$) if either $G = K_p$, or $\forall st \in E(\overline{G})$ there are p internally-disjoint paths from s to t in G.
- The vertex connectivity number of the graph G, $k(G)$, is the maximum p, for which G is p-connected.
- It follows that, in order to compute $k(G)$, we must find

$$\min_{st \notin E(G)} p(\{s\}, \{t\}; G)$$

  which can be determined in polynomial time using network flows.

Applications: König's theorem

- A vertex-cover in a graph G is a set $X \subseteq V(G)$ of vertices such that $G - X$ is a null graph (each edge of G has at least one extremity in X).
- A special case of the Theorem 1, is obtained when G is a bipartite graph and $X = S$, $Y = T$ are the two bipartite classes of G:

Theorem 3

(König, 1931) Let $G = (S, T; E)$ be a bipartite graph. Then, the maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex-cover.

Proof: The maximum cardinality of a matching in G is $p(S, T; G) = k(S, T; G)$, by Theorem 1. Since a set of vertices is an ST-separating set if and only if is a vertex-cover, the Theorem 3 is proved. $\square$

Applications: Hall's theorem

- Let I and S be non-empty finite sets. A family of subsets of S (indexed by I) is a map $A : I \to 2^S$. We will denote $\mathcal{A} = (A_i)_{i\in I}$ and (using the functional notation) $\mathcal{A}(J) = \bigcup_{j\in J} A_j$ (for $J \subseteq I$).
- A representative function for the family $\mathcal{A} = (A_i)_{i\in I}$ is any function $r_{\mathcal{A}} : I \to S$ with the property $r_{\mathcal{A}}(i) \in A_i$, $\forall i \in I$; then, $(r_{\mathcal{A}}(i))_{i\in I}$ is called a system of representatives for $\mathcal{A}$.
- If the representative function, $r_{\mathcal{A}}$, is injective, then $r_{\mathcal{A}}(I)$ is a subset of S and is called a system of distinct representatives for $\mathcal{A}$, or a transversal of $\mathcal{A}$.
- The central problem in the Transversal Theory is to characterize the families that admit transversal (with some properties). Hall's Theorem (1935) is the first result of this type.

Theorem 4

Hall, 1935 *The family* $\mathcal{A} = (A_i)_{i \in I}$ *of subsets of* S *has a transversal if and only if*

(H) $$|\mathcal{A}(J)| \geqslant |J|, \forall J \subseteq I.$$

Proof: "$\Rightarrow$" If $r_{\mathcal{A}}$ is an injective representative function for $\mathcal{A}$, then $r_{\mathcal{A}}(J) \subseteq \mathcal{A}(J)$, $\forall J \subseteq I$. Hence, $r_{\mathcal{A}}$ being injective, $|\mathcal{A}(J)| \geqslant |r_{\mathcal{A}}(J)| = |J|$.

"$\Leftarrow$" Let $G_{\mathcal{A}} = (I, S; E)$ be the bipartite graph associated to $\mathcal{A}$ (if $I \cap S \neq \varnothing$, we can consider disjoint isomorphic copies): $E = \{is | i \in I, s \in S \cap A_i\}$. Note that $N_{G_{\mathcal{A}}}(i) = A_i$. Moreover, $\mathcal{A}$ has a transversal if and only if $G_{\mathcal{A}}$ has a matching of cardinality $|I|$.

Proof of Hall's Theorem (cont'd): We show that if the condition (H) holds, then any vertex-cover of $G_{\mathcal{A}}$ has at least $|I|$ vertices, and - by Konig's Theorem - $G_{\mathcal{A}}$ has a matching of cardinality $|I|$.

Let $X = I' \cup S' \subseteq I \cup S$ be a vertex cover in $G_{\mathcal{A}}$: it follows that $N_{G_{\mathcal{A}}}(I \setminus I') \subseteq S'$, that is, $\mathcal{A}(I \setminus I') \subseteq S'$. Then,

$$|X| = |I'| + |S'| \geqslant |I'| + |\mathcal{A}(I \setminus I')|.$$

Since (H) holds, it follows that

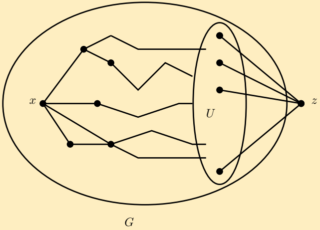
$$|X| \geqslant |I'| + |\mathcal{A}(I \setminus I')| \geqslant |I'| + |I \setminus I'| = |I|. \ \square$$

Applications: Dirac's theorem (p-connected graphs structure)

Lemma

Let $G = (V, E)$ be a p-connected graph of order $|G| \geqslant p + 1$, $U \subseteq V$, $|U| = p$ and $x \in V \setminus U$. Then there are p xU-paths such that any pair of them has x as the only common vertex.

Proof: Let $G' = (V \cup \{z\}, E')$, where $E' = E \cup \{zu : u \in U\}$.



Applications: Dirac's theorem (p-connected graphs structure)

Then, G' is a p-connected graph. Indeed, let $A \subseteq V(G')$ with $|A| \leqslant p-1$. If $A \subseteq V(G)$, then $G' - A$ is connected (by the p-connectivity of G, $G - A$ is connected; since $|A| < p$, $\exists u \in U \setminus A$ and, hence, there exists the edge $zu \in E(G' - A)$. If $z \in A$, then $G' - A = G - A$ which is connected.

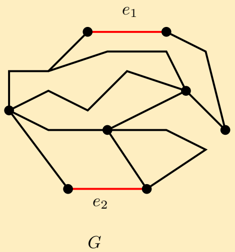
The lemma now follows by applying Theorem 2 to the graph G' and the pair x, z. $\square$

Proposition

Let $G = (V, E)$ be a p-connected graph, $p \geqslant 2$. Then, for every two edges e_1 and e_2 of G and for every, $x_1, \ldots, x_{p-2}$, $p-2$ vertices of G, there is a circuit in G containing all of them.

Proof: Induction on p.

For $p = 2$, we must prove that in a 2-connected graph, G, every two edges e_1 and e_2 belongs to a circuit. Let G' be the graph obtained from G by inserting one vertex a on e_1 and one vertex b on e_2:



G' is 2-connected (any deleted subgraph $G' - v$ is connected). Hence, there are two internally-disjoint paths from a to b, giving the circuit in G containing e_1 and e_2 (after removing a and b).

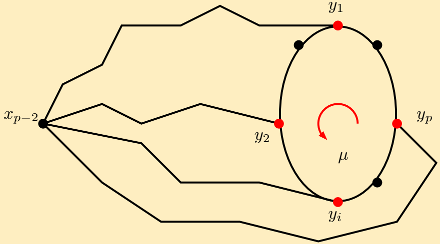
In the inductive step, let $p \geqslant 3$, suppose that the proposition holds for every p'-connected graph with $2 \leqslant p' < p$, and consider a p-connected graph G, two of its edges, e_1 and e_2 and a set of $p-2$ vertices $\{x_1, x_2, \ldots, x_{p-2}\}$.

We can suppose that no extremity v of e_1, e_2 belongs to the set $\{x_1, x_2, \ldots, x_{p-2}\}$ (otherwise, we can apply the induction hypothesis to infer that in the $(p-1)$-connected graph, G, there is a cycle C containing e_1, e_2 and the set of vertices $\{x_1, x_2, \ldots, x_{p-2}\} \setminus \{v\}$; but, clearly v is a vertex of C since e_1 and e_2 are edges of C).

The graph $G - x_{p-2}$ is $(p-1)$-connected. By the induction hypothesis, there exists a cycle μ containing $x_1, x_2, \ldots, x_{p-3}$, e_1 and e_2. Let Y the set of vertices of μ. Clearly, $|Y| \geqslant p$ (to the set of $p-3$ vertices $x_1, x_2, \ldots, x_{p-3}$, we add at least three extremities of the edges e_1 and e_2). By the Lemma, there are p $x_{p-2}Y$-paths such that any pair of them has x_{p-2} as the only common vertex.

Let $P_{x_{p-2}y_1}, P_{x_{p-2}y_2}, \ldots P_{x_{p-2}y_p}$ be these paths, where the ordering $y_1, \ldots, y_p$ is obtained by performing a traversal of μ.

The vertices $y_1, \ldots, y_p$ split the cycle μ in the paths $P_{y_1 y_2}$, $P_{y_2 y_3}, \ldots, P_{y_{p-1} y_p}$, $P_{y_p y_1}$:



At least one of the above paths doesn't contain any element from the set $x_1, x_2, \ldots, x_{p-3}$, e_1 and e_2 (by the pigeon hole principle).

Let $P_{y_1 y_2}$ be this path (otherwise, we appropriately change the ordering of y_i).

Then,

$$P_{x_{p-2}y_2}, P_{y_2y_3}, \ldots, P_{y_py_1}, P_{y_1x_{p-2}}$$

is the circuit in G containing $x_1, x_2, \ldots, x_{p-2}$, e_1 and e_2. $\square$

Theorem 5

(Dirac, 1953) *Through any $p \geqslant 2$ vertices of a p-connected graph passes a circuit.*

Proof: Let $G = (V, E)$ be a p-connected graph, $p \geqslant 2$. Let $x_1, x_2, \ldots, x_p$ be p vertices of G. Since G is connected, there exist the edges $e_1 = xx_{p-1}$ and $e_2 = yx_p$. Then, the theorem follows from the above Proposition. $\square$

A nice application of this theorem (and of the proof of the proposition) is the next Hamiltonian sufficient condition given by Erdös and Chvatal.

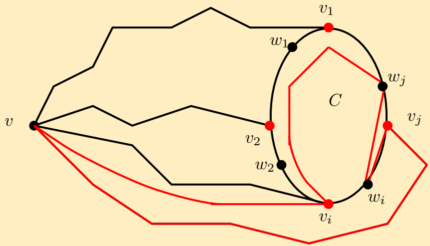
Theorem 6

(Erdös-Chvatal, 1972) *Let $G = (V, E)$ be a p-connected graph. If $\alpha(G) \leqslant p$ then G is a Hamiltonian graph.*

Proof: Suppose, by contradiction, that G is not Hamiltonian. Let C be the set of vertices of a longest circuit in G.

By Dirac's Theorem, $|C| \geqslant p$ and, by our assumption, there exists a vertex $v \in V(G) \setminus V(C) \neq \varnothing$.

Since $|C| \geqslant p$ we can repeat the argument in the proof of the Proposition from above to prove that there are P_{vv_1}, $P_{vv_2}, \ldots, P_{vv_p}$, p vC-paths pairwise meeting just in v and with extremities v_i numbered in the order they are reached by a traversal of the circuit.
Let us denote by w_i the successor vertex of v_i on the circuit.



Note that $vw_i \notin E$ (otherwise, the circuit $vw_i, w_i, C \setminus \{w_iv_i\}, P_{v_iv}$ is longer than C, contradiction).

Since $\alpha(G) \leqslant p$, the set $\{v, w_1, w_2, \ldots, w_p\}$ is not a stable set, and by the above remark, it follows that there is an edge $w_i w_j \in E$.

But then, P_{vv_i}, the converse of path from v_i to w_j on the circuit, the edge $w_j w_i$, the path from w_i to v_j on the circuit, and the path $P_{v_j v}$ give a cycle longer than C, contradiction). $\square$

A tree is a connected graph without cycles.

Theorem 7

Let $G = (V, E)$ be a graph. Then the following statements are equivalent:

(i) *G is a tree (is connected and has no circuit).*

(ii) *G is connected and it is minimal with this property.*

(iii) *G has no circuit and is maximal with this property.*

Proof: Is omitted. □

The minimality and maximality in the above statements are with respect to the order relation given by inclusion on the sets of edges. More precisely, the above (ii) and (iii) statements means:

(ii) G is connected and $\forall e \in E$, $G - e$ is not connected.

(iii) G has no circuit and $\forall e \notin E$, $G + e$ has a circuit.

Definition

Let $G = (V, E)$ be a (multi)graph. A spanning tree of G is a spanning subgraph of G, $T = (V, E')$ $(E' \subseteq E)$, which is a tree. We denote by $\mathcal{T}_G$ the set of all spanning trees of G.

Remarks

1. $\mathcal{T}_G \neq \varnothing$ if and only if G is connected. Indeed, if $\mathcal{T}_G \neq \varnothing$, then there is a spanning tree $T = (V, E')$ of G. T is connected, hence between any two vertices of G there is a path P in T. Since $E' \subseteq E$, P is a path in G, therefore G is connected.

Conversely, if G is connected, then let us consider the following algorithm:

$T \leftarrow G$;
while ($\exists e \in E(T)$ such that $T - e$ is connected) do
  $T \leftarrow T - e$;

By construction, T is a spanning subgraph of G, and the statement (ii) in the Theorem 7 is fulfilled, therefore it is a tree.

2. Another constructive proof that if G is connected then $\mathcal{T}_G \neq \varnothing$ is based on the remark that there exists a crossing-edge between the two classes of any bipartition of V: $\exists e = v_1v_2 \in E$ with $v_i \in V_i$, $i = \overline{1,2}$.
If $|V| = n > 0$ then the following algorithm constructs a spanning tree of the connected graph $G = (V, E)$:

$k \leftarrow 1$; $T_1 \leftarrow (\{v\}, \varnothing)$; // $v \in V$
while $(k < n)$ do
  let $xy \in E$ with $x \in V(T_k), y \in V \setminus V(T_k)$;
  // such an edge exists by the conectedness of G
  $V(T_{k+1}) \leftarrow V(T_k) \cup \{y\}$;
  $E(T_{k+1}) \leftarrow E(T_k) \cup \{xy\}$;
  $k++$;

Clearly, T_k is a tree $\forall k = \overline{1, n}$ (inductively, if T_k is a tree then, by construction, T_{k+1} is connected and has no circuit). Moreover, we have $|V(T_k)| = k$ and $|E(T_k)| = k - 1$, $\forall k = \overline{1, n}$.

3. If this construction is applied for a tree G with n vertices, we obtain that G has $n - 1$ edges. This property can be used to extends the Theorem 7 with other characterizations of the trees:

Theorem 8

The following statements are equivalent for a graph $G = (V, E)$ with n vertices:

(i) *G is a tree.*

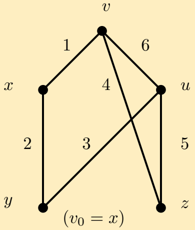
(ii) *G is connected and has $n - 1$ edges.*

(iii) *G has no circuits and has $n - 1$ edges.*

(iv) *$G = K_n$ for $n \in \{1, 2\}$ and $G \neq K_n$ for $n \geqslant 3$ and $G + e$ has exactly one circuit, for every edge $e \in E$.*

Proof: Omitted. □

- We describe a simple backtracking method to generate all spanning trees of a connected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$, $|E| = m$.
- The set of edges, E, will be represented as an array $E[1..2, 1..m]$ having entries from V, with the meaning: if $v = E[1, i]$ and $w = E[2, i]$, then vw is the edge i of G. Furthermore, we assume that the first $d_G(v_0)$ columns in the array E have v_0 in the row 1 ($E[1, i] = v_0, \forall i = \overline{1, d_G(v_0)}$), for $v_0 \in V$.



1	2	3	4	5	6
x	x	y	z	z	u
v	y	u	v	u	v

- A spanning tree $T \in \mathcal{T}_G$ will be represented as an set of $n-1$ indexes (in increasing order) of the rows in the array E (designating its edges).
- During the generation, we maintain a vector $T[1..n-1]$ with entries from $\{1, \ldots, m\}$ and a flag $i \in \{1, \ldots, n\}$ with the following meaning:
  We are searching for all spanning trees of G, with the property that the smallest $i-1$ edges are: $T[1] < T[2] < \ldots < T[i-1]$.
- In the above example, if $i = 2$, $T[1] = 1$, and $T[2] = 2$, then the trees which must be found are $\{1, 2, 3\}$, $\{1, 2, 5\}$, and $\{1, 2, 6\}$. If $i = 2$, $T[1] = 3$, and $T[2] = 5$ then the tree which must be found is $\{3, 5, 6\}$. But, if $i = 2$, $T[1] = 1$, and $T[2] = 6$ then no tree will be found.

ALL-ST-Gen(i)

//we generate all spanning trees of G , with the smallest $i-1$ edges: $T[1], \ldots, T[i-1]$

if $(i = n)$ then

  // $\{T[1], \ldots, T[n-1]\}$ is a spanning tree

  $process(T)$; // print, store etc

else

  if $(i = 1)$ then

    for $(j = \overline{1, d_G(v_0)})$ do

      $T[i] \leftarrow j$; A All-ST-Gen$(i+1)$; B

  else

    for $(j = \overline{T[i-1]+1, m-(n-1)+i})$ do

      if $(\langle \{T[1], \ldots, T[i-1]\} \cup \{j\} \rangle_G$ has no circuit) then

        $T[i] \leftarrow j$; A All-ST-Gen$(i+1)$; B

Trees - Generating $\mathcal{T}_G$

- By the call All-ST-Gen(1) we obtain $\mathcal{T}_G$.
- To test if the graph $\langle\{T[1], \ldots, T[i-1]\} \cup \{j\}\rangle_G$ has no circuit, let us observe that, by construction,

$$\langle\{T[1], \ldots, T[i-1]\}\rangle_G$$

  has no circuit, hence it is a forest (each connected component is a tree).
- Let $root[1..n]$ be a (global) vector with entries from V and the meaning: $root[v] =$ the root of the connected component containing v (one of its vertices).
- Before the call All-ST-Gen(1), the vector *root* is initialized to satisfy this property: $root[v] \leftarrow v$ $(\forall v \in V)$ (since then, $\{T[1], \ldots, T[i-1]\} = \varnothing$).

- During the recursive calls, when we test if the edge j can be added to the set $\{T[1], \ldots, T[i-1]\}$ without creating a circuit, let $v = E[1, j]$ and $w = E[2, j]$. Then, $\langle\{T[1], \ldots, T[i-1]\} \cup \{j\}\rangle_G$ has no circuit if and only if v and w are in different connected components of the forest, i.e., $root[v] \neq root[w]$.
- In order to maintain the vector root, in the places marked A and B in the algorithm, we must make the following changes.
- Instead of A:

  $S \leftarrow \varnothing$; $x \leftarrow root[v]$;
  for $(u \in V)$ do
      if $(root[u] = x)$ then
          $S \leftarrow S \cup \{u\}$; $root[u] \leftarrow root[w]$;

- In other words all vertices in the tree with the root x are added to the tree with the root $root[w]$; these vertices are saved in the set S.
- After the call All-ST-Gen$(i+1)$, the vector $root$ must be set again to the value before the call, and this can be done by replacing B with:

  for $(u \in S)$ do

  $\quad root[u] = x;$

Let $G = (V, E)$ be a multi-graph with $V = \{1, 2, \ldots, n\}$, and adjacency matrix $A = (a_{ij})_{n\times n}$ (a_{ij} = multiplicity of edge ij if $ij \in E$, 0 otherwise). Let

$$\mathrm{D} = \mathrm{diag}(\mathrm{d_G(1), d_G(2), \ldots, d_G(n)}) = \begin{pmatrix} d_G(1) & 0 & \ldots & 0 \\ 0 & d_G(2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_G(n) \end{pmatrix}.$$

The Laplacian matrix of G is defined as:

$$\mathrm{L[G] = D - A.}$$

Note that the sum of the entries in $L[G]$ on every row and every column is 0. We denote by $L[G]_{ij}$ the minor of the matrix $L[G]$ obtained by removing the ith row and jth column.

Theorem 9

Matrix-tree Theorem (Kirchoff-Trent). *Let G be a (multi)graph with vertex set $\{1, \ldots, n\}$ and Laplacian matrix $L[G]$. Then, the number of spanning trees of G is:* $|\mathcal{T}_G| = det(L[G]_{ii}), \forall 1 \leqslant i \leqslant n$.

Proof: *Omitted.* $\square$

Corollary

(Cayley's formula). $|\mathcal{T}_{K_n}| = n^{n-2}$.

Proof:

$$L[K_n] = \begin{pmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{pmatrix}.$$

Hence:

$$det(L[K_n]_{11}) = \begin{vmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{vmatrix}$$

If we add all the lines to the first one we get

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{vmatrix} = n^{n-2}. \ (\textit{Why?})$$

$\square$

Exercise 1. Let $G = (V, E)$ be a connected graph and $v \in V$ such that $N_G(v) \neq V \setminus \{v\}$. For $X \subseteq V$ we denote $N_G(X) = \left(\bigcup_{v \in X} N_G(v)\right) \setminus X$.

Clearly, the set $A = \{v\}$ satisfies the following conditions

(i) $v \in A$ and $[A]_G$ is connected.

(ii) $N = N_G(A) \neq \varnothing$.

(iii) $R = V \setminus (A \cup N) \neq \varnothing$.

(a) Show that, if $A \subseteq V$ is any maximal (w.r.t "$\subseteq$") set of vertices satisfying (i) - (iii), then $\forall x \in R$ and $\forall y \in N$ we have $xy \in E$.

(b) Prove that if A is as in (a) and G is $\{C_k\}_{k \geqslant 4}$-free, then N is a clique in G.

(c) Deduce that K_n $(n \in \mathbb{N}^*)$ are the only regular, chordal connected graphs.

Exercise 2. A graph of order at least three is called *confidentially connected* if, for every three distinct vertices a, b and c, it exists a path from a to b such that c differs and is not adjacent with the internal nodes (if any) of this path. (An example of confidential connected graph is the complete graph K_n, with $n \geqslant 3$.)
Show that a connected, incomplete graph, $G = (V, E)$, with at least three vertices is confidential connected if and only if:

(i) for every $v \in V$, $N_{\overline{G}}(v) \neq \varnothing$ and induces a connected subgraph;

(ii) any edge of G is part of an induced C_4 or is a mid edge of an induced P_4.

Exercise 3. Prove that a connected p-regular bipartite graph is 2-connected.

Exercise 4. Let $G = (V, E)$ be a digraph. Prove that:

(a) G is strongly connected if and only if for every $S \subsetneq V$, $S \neq \varnothing$, there exists an arc leaving S.

(b) If G is strongly connected and can be disconnected by removing at most p arcs (i. e., $\exists A \subseteq E$, $|A| \leqslant p$ such that $G - A$ is not strongly connected), then G can be disconnected by reversing at most p arcs (that is $\exists B \subseteq E$, $|B| \leqslant p$ such that $G' = (V, (E \setminus B) \cup \{uv : vu \in B\})$ is not strongly connected).

Exercise 5. Let G be a 2-edge-connected graph ($G - e$ is connected, $\forall e \in E(G)$). Define the following binary relation $e \asymp f$ if $e = f$ or $G - \{e, f\}$ is not connected.

(a) Prove that $e \asymp f$ if and only if e and f belong to the same circuits.

(b) Show that an equivalence class $[e]_{\asymp}$ is included in a circuit.

(c) Removing the edges of an entire equivalence class $[e]_{\asymp}$, the connected components of the remaining graph is 2-edge-connected.

Exercise 6.

(a) Let G be a graph with at least 3 vertices. If G is 2-connected, then we can orient its edges in such a way that the resulting oriented graph is strongly connected.

(b) Is the converse true?

Exercise 7.

(a) Let G be an incomplete 2-connected graph and $xy \in E(G)$. Prove that $G - xy$ or $G|xy$ is 2-connected.

(b) Give examples of graphs G and edges $xy \in E(G)$ such that: (b1) $G - xy$ and $G|xy$ are both 2-connected; (b2) $G - xy$ is not 2-connected and $G|xy$ is 2-connected; (b3) $G - xy$ is 2-connected and $G|xy$ is not 2-connected;

Exercise 8. For a given connected graph G we perform the following algorithm:

$\mathcal{Q} \leftarrow \{G\}$; // $\mathcal{Q}$ is a queue;
while ($\mathcal{Q} \neq \varnothing$) }
  $H \leftarrow$ pop($\mathcal{Q}$);
  let $A \subseteq V(H)$ a minimal cut-set in H;
  let $G_1, \ldots, G_k$ the connected components of $H - A$;
  for ($j = 1$ to k)
    push($\mathcal{Q}, [A \cup V(G_j)]_G$);
  }

Observe that if G is a complete graph, then in $\mathcal{Q}$ we do not push any other graph.

a) Show that any graph which is pushed in $\mathcal{Q}$ is a connected one.

b) Prove that the total number of graphs pushed in the queue $\mathcal{Q}$ is at most $|G|^2$.

Exercise 9. Let $G = (V, E)$ be a connected graph and T_1, T_2 be two spanning trees of G ($T_1, T_2 \in \mathcal{T}_G$).

(a) Prove that T_1 can be transformed into T_2 by repeatedly applying the following operation: remove an edge and add another edge to the current tree.

(b) If, in addition, G is 2-connected show that T_1 can be transformed into T_2 by repeatedly applying the following operation: remove an edge uv and add another edge uw to the current tree.

Exercise 10. Prove that the set of edges of a complete graph K_n ($n \geqslant 2$) can be partitioned in $\lceil n/2 \rceil$ subsets each representing the set of edges of a tree (subgraph in K_n).

Exercises for the next week seminar

Exercise 11. Let n be a positive integer and $G_n = (V, E)$ the graph with:

- $V = \{(i, j) : 1 \leqslant i, j \leqslant n\}$;
- $(i, j)(k, l) \in E$ (for $(i, j) \neq (k, l)$ from V) if and only if $i = l$ or $j = k$.

Show that G_n is universal for the trees of order n: for any tree T of order n, $\exists A \subseteq V$ such that $T \cong [A]_{G_n}$.

Exercise 12. Prove that a tournament is strongly connected if and only if it contains a Hamiltonian cycle.

Exercise 13. We consider the street network of a given city. Prove that if we can remove all the cycles in this network by creating at most p blockings (blocking means obstructing one way of a street), then we can remove all the cycles in the city network by reversing one way of at most p streets.
(Reversing one way of a given two ways street means to transform it into a one-way-street; reversing an one-way-street means to transform it into the other one-way-street.)

Exercise 14. How many spanning trees has a complete bipartite graph $K_{n,n}$? Same question for $K_{p,q}$.

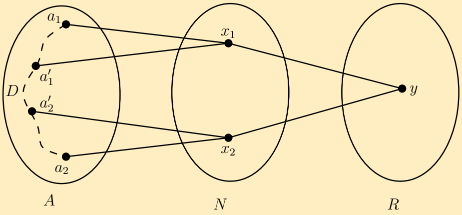
Exercise 1. Solution.

(a) Suppose on the contrary that exist two non-adjacent vertices $x \in N$ and $y \in R$.

- let $A' = A \cup \{x\}$; $[A']_G$ is a connected subgraph (why?);
- $N_G(A') \neq \varnothing$ because G is connected and $N_G(A') \neq V$ since $y \notin N_G(A')$;
- $y \in V \setminus (A' \cup N_G(A'))$;

A' has the properties (i)-(iii) and strictly contains A - contradiction (why?).

(b) Suppose on the contrary that N is not a clique: there exist two not adjacent vertices $x_1, x_2 \in N$.



- An induced cycle of length at least 4 (where?) can be detected in the figure above - contradiction.

(c) Suppose on the contrary that G is not a complete graph, then there exists a vertex $v \in V$ with $N_G(v) \neq V \setminus \{v\}$, and a subset $A \subset V$ maximal having the properties (i)-(iii) (why?).

- $d_G(y) \leqslant |N| + |R| - 1$ (why?).
- $d_G(x) \geqslant |N| + |R|$ (why?.
- $|N| + |R| - 1 \geqslant d_G(y) = d_G(x) \geqslant |N| + |R|$ - contradiction.

Exercise 3. Solution.

- Let $G = (S, T, E)$ be a bipartite p-regular connected graph and $u \in S$.
- Suppose on the contrary that $G - u$ is not connected, then there exists a bipartition (V_1, V_2) of $V(G) \setminus \{u\}$ such that there are no edges connecting V_1 and V_2 (why?).
- Let $G_1 = [V_1 \cup \{u\}]_G$ which is a bipartite graph (why?),
- Let $S_1 = S \cap V(G_1)$ and $T_1 = T \cap V(G_1)$, suppose that $u \in S_1$, then

$$\sum_{x \in S_1} d_{G_1}(x) = \sum_{y \in T_1} d_{G_1}(y) \Rightarrow p \cdot (|S_1| - 1) + d_{G_1}(u) =$$

$$p \cdot |T_1| \Rightarrow p | d_{G_1}(x) \Rightarrow d_{G_1}(x) = p \text{ - why?}$$

- This is a contradiction. Why?

Exercise 4. Solution.

(a) "$\Longrightarrow$" If G is not strongly connected we cannot reach a vertex from $V(G) \setminus S \neq \varnothing$ from a vertex in $S \neq \varnothing$ (why?)- contradiction.

- "$\Longleftarrow$'" Let $u, v \in V(G)$; it is enough to prove that there exists an uv-path in G.
- Let $u = u_1$, there exists an arc from $\{v_1\}$ to a vertex $v_2 \in V(G) \setminus \{v_1\}$ (why?). If we already defined the sequence $v_1, v_2, \ldots, v_p \in V(G)$ such that there are $v_1 v_i$-paths, $\forall 2 \leqslant i \leqslant p$, then there is an arc $v_j w$ with $w \notin \{v_1, v_2, \ldots, v_p\}$ and $1 \leqslant j \leqslant p$ (why?).
- We can define $v_{p+1} = w$.
- Repeat this procedure until $v_p = v$.

(Partially) solved exercises

- Let $A \subseteq E$, a set of minimum cardinality such that $G - A$ is not strongly connected (note that $|A| \leqslant p$).
- There exists $S \subsetneq V(G) \setminus A$, $S \neq \varnothing$ such that there are no arcs from S to $V(G - A) \setminus S$ (why?).
- Since A is minimal (why?), any arc $e \in A$ links a vertex from S to one of $V(G - A) \setminus S$ - otherwise by removing from G the arcs from $A \setminus \{e\}$ we get a digraph that is not strongly connected - contradiction.
- Let A' be the set of reversed arcs from A. Obviously, $G - A'$ is not strongly connected (why?) and $|A'| = |A| \leqslant p$.

Exercise 5. Solution.

(a) "$\Longrightarrow$" Let $e \asymp f$, $e \neq f$ and C be a cycle containing e.

- If $f \notin E(C)$, then $G - e$ has a cycle through f (why?), hence $G - \{e, f\}$ is connected, contradiction.
- "$\Longleftarrow$" Let $e \neq f$; $G - e$ is connected.
- If we suppose that $G - \{e, f\}$ is connected also, then there must exists a cycle C in $G - e$ containing f (why?), hence C doesn't contain, both, e and f - contradiction.

(b) Use (a). How?

(c) Let $e \in E(G)$, $F = [e]_{\asymp}$ and H be a connected component of $G - F$.

- Let $f \in E(H) \subseteq E(G) \setminus F$.
- $G - e$ and $G - \{e, f\}$ are connected (why?), hence, there exists a cycle C in $G - e$ containing f.
- $E(C) \cap F = \varnothing$ (why?), therefore C is a cycle in $G - F$ and in H, i. e., $H - f$ is connected.

Exercise 6. Solution.

(a) Let $G = (V, E)$ be a 2-connected graph.

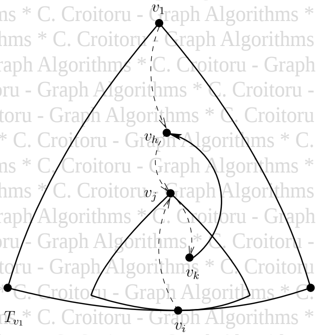
- We perform a *dfs* traversal on G retaining the visiting order of vertices; the traversed edges are oriented from the lower to the higher order vertex.
- The remaining untraversed edges are oriented the other way.
- The *dfs* procedure:

dfs(v) {
k++; $v_k \leftarrow v$; $order[v] \leftarrow k$;
for ($u \in \mathcal{A}(v)$)
  $E' \leftarrow E' \cup \{vu\}$;
  if ($order[u] == 0$) {
    dfs(u);
  }
    $E' \leftarrow E' \cup \{uv\}$;
}

- The main algorithm is:

$k \leftarrow 0$; $E' \leftarrow \varnothing$;
for ($v \in V$)
  $order[v] \leftarrow 0$;
dfs(r);

- We have no cross edges after this *dfs* traversal, i. e. every arc $v_i v_j$, with $i > j$, points towards an ancestor of v_i. We denote by T_v the *dfs* subtree rooted in v.



- One can prove that, from every leaf v_i there exists a path to v_1 in $\vec{G} = (V, E')$. How?

(b) No. $G = (\{x, y, z, t, u\}, \{xy, yz, zx, zt, tu, uz\})$ is strongly connected (why?), but $G - z$ is not connected.

Exercise 7. Solution.

(a) Suppose that $G - xy$ and $G|xy$ are not 2-connected.

- Since $G - x$ is connected, and $G - x \subseteq G - xy, G|xy$, it follows that $G - xy$ and $G|xy$ are connected too (why?).
- Let a be the vertex which replaces x and y in G after the contraction.
- Let z be a vertex such that $G|xy - z$ is not connected, then $z = a$, otherwise $G - z$ being connected, $G|xy - z = (G - z)|xy$ would be connected too (why?).
- Hence $G|xy$ can be disconnected only by deleting a.

- Let $u \in V(G)$ be a vertex such that $G - xy - u$ is not connected.
- It follows that $u \notin \{x, y\}$, and if G_1, G_2 are the two connected components in $G - xy - u$, then xy is the only edge between G_1 and G_2 in $G - u$ (why?).
- If we contract xy and then we delete it, u insures the connectivity of the remaining graph.
- Therefore $G|xy - a$ cannot be disconnected.

(b) (b1) K_4; (b2) $C_4 + e$ and $xy \neq e$; (b3) $C_4 + e$ and $xy = e$.

Exercise 9. Solution.

(a) Let $G = (V, E)$, $|G| = n$ and $T_1, T_2 \in \mathcal{T}_G$, $T_1 \neq T_2$, $T_i = (V, E_i)$.

- We use induction on $2k = |E_1 \Delta E_2|$.
- It is easy to verify the required property if $|E_1 \Delta E_2| = 2$ ($\Leftrightarrow T_1 = T_2$).
- The inductive step: suppose that, for every two spanning trees T_1' and T_2' with $|E_1' \Delta E_2'| = 2k \geqslant 2$, the above properties are true.

- Let $T_1, T_2 \in \mathcal{T}_G$, $|E_1 \Delta E_2| = 2k + 2$. Choose an edge $e_1 \in E_1 \setminus E_2$.
- $T_2 + e_1$ contains only one cycle C (why?).
- Since $E(C) \subsetneq E_1$, it will exist an edge $e_2 \in E(C) \cap (E_2 \setminus E_1)$.
- $T' = T_2 + e_1 - e_2 \in \mathcal{T}_G$, $|E_2 \Delta E(T')| = 2$ (why?) and $|E_1 \Delta E(T')| = 2k$.
- By the induction hypothesis we can transform T_1 into T' by using the procedure from above a certain number of times and, than, transform T' into T_2 by using it once again.

Exercise 11. Solution.

- Let $T =$ be a tree of order n; we *bfs* traverse T and label the vertices like this:
  - first vertex (the root) receives the label $(1, 1)$;
  - any other vertex v receives the label (i, j), where j is the *bfs* position of v, and i is the *bfs* position of its parent.

Obviously $T \simeq [\Lambda]_{G_n}$, where Λ is the set of labels constructed above. Why?

Exercise 13. Solution. Rephrasing: Let $G = (V, E)$ be a digraph without loops. If we can transform G into a dag by removing at most p arcs (that is $\exists A \subseteq E$, $|A| \leqslant p$ such that $G - A$ has no cycles), then we can also transform G into a dag by reversing at most p arcs (i. e., $\exists B \subseteq E$, $|B| \leqslant p$ such that $G' = (V, (E \setminus B) \cup \{uv : vu \in B\})$ has no cycles).

- Let $A \subseteq E$, a set of minimum cardinality such that $G - A$ is a dag ($|A| \leqslant p$ - why?).
- $G - A$ being a dag we can find a topological ordering of $V(G - A) = V(G)$: $v_1, v_2, \ldots, v_n$ such that $v_i v_j \in E(G - A)$ implies $i < j$.
- For each arc $v_k v_l \in A$ the digraph $G - A + v_k v_l$ contains cycles, i. e., $k > l$ (why?).
- Therefore all the arcs from A are of the same type, but by reversing them we keep the topological ordering in $(G - A) \cup \{uv : vu \in A\}$.