Stochastic Processes

Homework #2

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Saba Hashemi - 97100581

Problem 1

Q. We define $X_0 = Y_0$ and $X_n = \lambda X_n 1 + Y_n$ for $n \geq 1$. Suppose $Y_0, Y_1, ...$ are uncorrelated random variables with $\mathrm{E}[Y_n] = 0$ and $\mathrm{Var}(Y_n) = \sigma^2/(1-\lambda^2)$ if $n = 0$ and σ^2 otherwise, where $\lambda < 1$. Find $Cov(X_n, X_{n+m})$. Is this process WSS? why?

A. If we expand X_n we will have:

$$\begin{aligned}
X_n &= \lambda X_{n-1} + Y_n \\
&= \lambda(\lambda X_{n-2} + Y_{n-1}) + Y_n \\
&... \\
&= \lambda^n Y_0 + \lambda^{n-1} Y_1 + ... + Y_n \\
&= \sum_{i=0}^{n} \lambda^{n-i} Y_i
\end{aligned}$$

For expectation we can write:

$$\begin{aligned}
\mathrm{E}[X_n] &= E[\sum_{i=0}^{n} \lambda^{n-i} Y_i] \\
&= \sum_{i=0}^{n} \lambda^{n-i} \mathrm{E}[Y_i] \\
&= \sum_{i=0}^{n} \lambda^{n-i} \times 0 \\
&= 0
\end{aligned}$$

For covariance we can write:

$$\begin{aligned}
\mathrm{Cov}(X_n, X_{n+m}) &= C_{XX}(n, n+m) \\
&= R_{XX}(n, n+m) - \mathrm{E}[X]^2 \\
&= R_{XX}(n, n+m) \\
&= \mathrm{E}[X_n X_{n+m}] \\
&= \mathrm{E}[(\sum_{i=0}^{n} \lambda^{n-i} Y_i)(\sum_{i=0}^{n+m} \lambda^{n+m-i} Y_i)] \\
&= \mathrm{E}[\sum_{i=0}^{n} \sum_{j=0}^{n+m} \lambda^{n-i} Y_i \lambda^{n+m-j} Y_j] \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n+m} \lambda^{2n+m-i-j} \mathrm{E}[Y_i Y_j]
\end{aligned}$$

Since Y_i are correlated we have $\mathrm{E}[Y_i Y_j] = \mathrm{E}[Y_i]\mathrm{E}[Y_j] = 0$ for $i \neq j$. So:

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$$\begin{aligned}
\mathrm{Cov}(X_n, X_{n+m}) &= \sum_{i=0}^{n} \lambda^{2n+m-2i}\mathrm{E}[Y_iY_i] \\
&= \lambda^{2n+m}\frac{\sigma^2}{(1-\lambda^2)} + \sum_{i=1}^{n} \lambda^{2n+m-2i}\sigma^2 \\
&= \lambda^{2n+m}\sigma^2\Big[\frac{1}{(1-\lambda^2)} + \sum_{i=1}^{n} \lambda^{-2i}\Big] \\
&= \lambda^{2n+m}\sigma^2\frac{1}{(1-\lambda^2)(\lambda^{2n})} \\
&= \frac{\lambda^m\sigma^2}{(1-\lambda^2)}
\end{aligned}$$

The autocorrelation function R_{XX} only depends on the difference of indexes, m. Also, as we see in previous part, $\mathrm{E}[X_n]$ is a constant. Thus, this process is wide-sense stationary.

Problem 2

Q. Prove that $Var(X(t+s)X(t)) = 2RX(0)2RX(s)$. Let $X(t)$ be a WSS.

A. We have:

$$\begin{aligned}
\mathrm{E}[(X(t+s)-X(t))^2] &= \mathrm{E}[X(t+s)^2 + X(t)^2 - 2X(t+s)X(t)] \\
&= \mathrm{E}[X(t+s)^2] + \mathrm{E}[X(t)^2] - 2\mathrm{E}[X(t+s)X(t)] \\
&= R_X(0) + R_X(0) - 2R_X(s) \\
&= 2R_X(0) - 2R_X(s)
\end{aligned}$$

$$\begin{aligned}
E[X(t+s)-X(t)]^2 &= \Big(\mathrm{E}[X(t+s)] - \mathrm{E}[X(t)]\Big)^2 \\
&= (\mu_X - \mu_X)^2 \\
&= 0
\end{aligned}$$

So:

$$\begin{aligned}
\mathrm{Var}(X(t+s)-X(t)) &= \mathrm{E}[(X(t+s)-X(t))^2] - E[X(t+s)-X(t)]^2 \\
&= 2R_X(0) - 2R_X(s)
\end{aligned}$$

Problem 3

Q. Let $X_i (i \in Z)$ be a process in which the X_is are i.i.d. Supposing CDF $F_{X_i}(x) = F(x)$, prove that this process is SSS.

A. We will calculate joint CDF function and show that it is independent of indexes.

$$F(x_1, x_2, ..., x_n; k_1, ..., k_n) = P(X_{k_1} \leq x_1, ..., X_{k_n} \leq x_n)$$

Since X_is are i.i.d we can write:

$$\begin{aligned}
F(x_1, x_2, ..., x_n; k_1, ..., k_n) &= P(X_{k_1} \leq x_1, ..., X_{k_n} \leq x_n) \\
&= P(X_{k_1} \leq x_1) P(X_{k_2} \leq x_2) ... P(X_{k_n} \leq x_n) \\
&= \prod_{i=1}^{i=n} P(X_{k_i} \leq x_i) \\
&= \prod_{i=1}^{i=n} F_{X_{k_i}}(x_i) \\
&= \prod_{i=1}^{i=n} F(x_i)
\end{aligned}$$

So the joint CDF function is independent of indexes, and thus, this process is strict-sense stationary.

Problem 4

Q. Consider the process $X(t) = Y(t+T)$ in which $T \sim U(0, T_0)$ and $Y(t)$ is a periodic function with period T_0. Is this process SSS? Give reasons for your answer.

A. Consider the CDF of n-th order distribution function:

$$\begin{aligned} F(x_1, ..., x_n; t_1, ..., t_n) &= P(X(t_1) \leq x_1, ..., X(t_n) \leq x_n) \\ &= P(Y(t_1 + T) \leq x_1, ..., Y(t_n + T) \leq x_n) \end{aligned}$$

However, since $Y(t)$ is periodic and $T \sim U(0, T_0)$, the space that $Y(t_i + T) \leq x_i$ represents is the same as that of $Y(T) \leq x_i$; Because for every θ as a value of T there exists a $0 \leq \theta' \leq T_0$ where $Y(t_i + \theta) = Y(\theta')$. Specificaly, $\theta' = (t_i + \theta)\,\%\,T_0$.
An illustration of this is shown below:

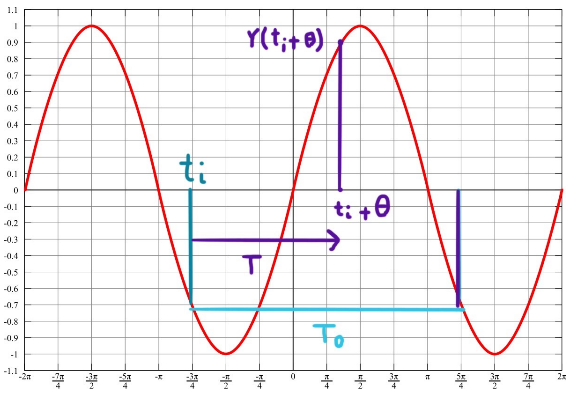

Figure 1: $Y(t_i + T)$

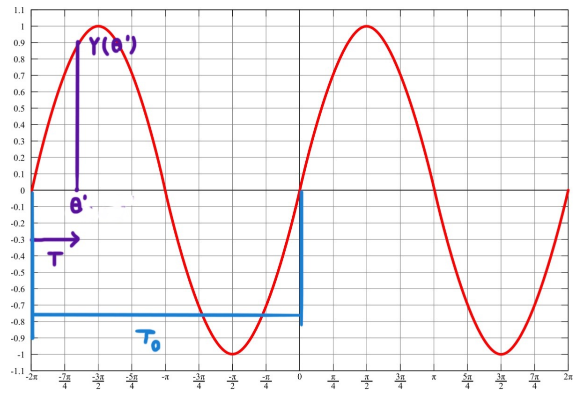

Figure 2: $Y(T)$

Hence, we can write:

$$F(x_1, ..., x_n; t_1, ..., t_n) = P(Y(T) \leq x_1, ..., Y(T) \leq x_n)$$

Thus, the n-th order distribution function is independent of indexes and this process is strict-sense stationary.

Problem 5

Q. Consider the process X(t) = Y + Zt in wich Y and Z are normal N(1,1) and independent random variables. For this process find the correlation function and covariance function.

A.

$$\begin{aligned}
R_{XX}(t,t+s) &= \mathrm{E}[X(t+s)X(t)] \\
&= \mathrm{E}[\big(Y+Zt\big)\big(Y+Z(t+s)\big)] \\
&= \mathrm{E}[Y^2+YZ(t+s)+YZt+Z^2t(t+s)] \\
&= \mathrm{E}[Y^2]+\mathrm{E}[YZ(2t+s)]+\mathrm{E}[Z^2t(t+s)] \\
&= \mathrm{E}[Y^2]+(2t+s)\mathrm{E}[Y]\mathrm{E}[Z]+t(t+s)\mathrm{E}[Z^2] \\
&= (\sigma^2+\mu^2)+(2t+s)(\mu^2)+t(t+s)(\sigma^2+\mu^2) \\
&= 2+(2t+s)+2t(t+s) \\
&= 2t^2+2st+2t+s+2
\end{aligned}$$

We can also write the above result in following form if we assume $t_1 = t$ and $t_2 = t+s$:

$$\begin{aligned}
R_{XX}(t,t+s) &= (\sigma^2+\mu^2)+(t_1+t_2)(\mu^2)+t_1t_2(\sigma^2+\mu^2) \\
&= 2t_1t_2+t_1+t_2+2
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[X(t)] &= \mathrm{E}[\big(Y+Zt\big)] \\
&= \mathrm{E}[Y]+t\mathrm{E}[Z] \\
&= (1+t)\mu \\
&= t+1
\end{aligned}$$

$$\begin{aligned}
C_{XX}(t,t+s) &= R_{XX}(t,t+s)-\mathrm{E}[X(t)]\mathrm{E}[X(t+s)] \\
&= 2t^2+2st+2t+s+2-(t+1)(t+s+1) \\
&= 2t^2+2st+2t+s+2-t^2-ts-2t-s-1 \\
&= t^2+ts+1
\end{aligned}$$

We can also write the above result in following form if we assume $t_1 = t$ and $t_2 = t+s$:

$$C_{XX}(t,t+s) = t_1t_2+1$$

Problem 6

Q. Define X(t) = A cos(wt) + B sin(wt), where A and B are independent unit normal random variables and w is constant . Show that X(t) is a WSS.

A. First we consider expected value:

$$\begin{aligned}\mathrm{E}[X(t)] &= \mathrm{E}[A\cos(wt) + B\sin(wt)] \\ &= \cos(wt)\mathrm{E}[A] + \sin(wt)\mathrm{E}[B] \\ &= 0\end{aligned}$$

So first-order moment is a constant and independent of t.

For second-order moment, we first calculate $\mathrm{E}[A^2]$, $\mathrm{E}[B^2]$, and $\mathrm{E}[AB]$:

$$\mathrm{E}[B^2] = \mathrm{E}[A^2] = var(A) + E[A]^2 = 1$$

$$\mathrm{E}[AB] = E[A]E[B] = 0$$

Then we calculate auto-correlation:

$$\begin{aligned}R_{XX}[t, t+s] =& \mathrm{E}[X(t)X(t+s)] \\ =& \mathrm{E}\Big[\big(A\cos(wt) + B\sin(wt)\big)\big(A\cos(w(t+s)) + B\sin(w(t+s))\big)\Big] \\ =& \mathrm{E}[A^2(wt)\cos(w(t+s)) + B^2\sin(wt)\sin(w(t+s)) + AB\cos(wt)\sin(w(t+s)) + AB\sin(wt)\cos(w(t+s)] \\ =& \cos(wt)\cos(w(t+s))\mathrm{E}[A^2] + \sin(wt)\sin(w(t+s))\mathrm{E}[B^2] \\ &+ \Big(\cos(wt)\sin(w(t+s)) + \sin(wt)\cos(w(t+s))\Big)\mathrm{E}[AB] \\ =& \cos(wt)\cos(w(t+s)) + \sin(wt)\sin(w(t+s)) \\ =& \cos(wt - w(t+s)) \\ =& \cos(ws)\end{aligned}$$

Second-order moment is a function of the difference of two indexes. Thus, this process is wide-sense stationary

Problem 7

Q. Suppose $X_1, X_2, ...$ are i.i.d with $\mathrm{E}[X_i] = 0$ and $\mathrm{Var}(X_i) = 4$. For the process $y(n) = X_1+X_2+...+X_n (n \in N)$ find the mean and covariance.

A.
Mean:

$$\begin{aligned} \mathrm{E}[Y_n] &= \mathrm{E}[X_1 + ... + X_n] \\ &= \sum_{i=1}^{n} \mathrm{E}[X_i] \\ &= 0 \end{aligned}$$

Covariance:

$$\begin{aligned} R_{YY}(n, n+m) &= \mathrm{E}[Y_n Y_{n+m}] \\ &= \mathrm{E}\Big[\big(\sum_{i=1}^{n} X_i\big)\big(\sum_{j=1}^{n+m} X_j\big)\Big] \\ &= \mathrm{E}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n+m} X_i X_j\Big] \\ &= \sum_{i=1}^{n} \mathrm{E}[X_i^2] + \sum_{i=1}^{n}\sum_{j=1, i\neq j}^{n+m} \mathrm{E}[X_i X_j] \\ &= \sum_{i=1}^{n} \mathrm{E}[X_i^2] + \sum_{i=1}^{n}\sum_{j=1, i\neq j}^{n+m} \mathrm{E}[X_i]\mathrm{E}[X_j] \\ &= \sum_{i=1}^{n} \mathrm{E}[X_i^2] \\ &= \sum_{i=1}^{n} (\mathrm{Var}[X_i] + \mathrm{E}[X_i]^2) \\ &= \sum_{i=1}^{n} 4 \\ &= 4n \end{aligned}$$