Stochastic Processes

Homework #4

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Problem 1

Q.

Suppose $X(t)$ is a Gaussian process with zero mean and kernel $k(t_1, t_2) = 2exp(-\frac{|t_1-t_2|}{\alpha})$. Let $Y(t)$ be the result of the application of filter $H(w) = 1 + \frac{1}{2}e^{-2jw}$ on $X(t)$.

(a) Compute mean and auto-covariance of $Y(t)$

(b) Prove that $Y(t)$ is a Gaussian process.

A.

(a) Mean will be zero because $\mu_Y = \mu_X * h$ and μ_X is zero.

Auto-covariance:

$$R_X(t_1, t_2) = C_X(t_1, t_2) + \mu_X(t_1)\mu_X(t_2) = 2exp(-\frac{|t_1 - t_2|}{\alpha})$$

$$R_X(\tau) = 2exp(-\frac{|\tau|}{\alpha})$$

$$S_X(w) = \mathcal{F}\big(R_X(\tau)\big) = 4\frac{\frac{1}{\alpha}}{(\frac{1}{\alpha})^2 + w^2}$$

$$S_Y(w) = S_X(w)|H(w)|^2 = 4\frac{\frac{1}{\alpha}}{(\frac{1}{\alpha})^2 + w^2}(1.5)^2$$

$$R_Y(\tau) = 4.5exp(-\frac{|\tau|}{\alpha})$$

$$C_X(t_1, t_2) = R_Y(t_1, t_2) - \mu_Y(t_1)\mu_Y(t_2) = 4.5exp(-\frac{|t_1, t_2|}{\alpha})$$

(b) We have $h(t) = \delta(t) + \frac{1}{2}\delta(t-2)$. Since convolution is a linear function, $Y(t)$ is linear combination of Gaussian Process X, and thus, it is a Gaussian Process itself.

Problem 2

Q. Assume that K_1 and K_2 kernels be represented as $K_i(x,y) = \phi_i(x)^T\phi_i(y)$ for i = 1, 2 where ϕ_i is a mapping that maps input onto a higher dimensional space. Note that ϕ_i can have infinite dimensions. We call such kernels as valid kernels.

(a) Prove that $K_1 + K_2$ is also a valid kernel.

(b) Prove that $K_1 \times K_2$ is also a valid kernel.

(c) Prove that e^{K_1} is also valid.

A.

(a)

$$\begin{aligned}
K_1(x,y) + K_2(x,y) &= \phi_1(x)^T\phi_1(y) + \phi_2(x)^T\phi_2(y) \\
&= \sum_{i=1}^{\infty} \phi_{1i}(x)\phi_{1i}(y) + \sum_{i=1}^{\infty} \phi_{2i}(x)\phi_{2i}(y) \\
&= \sum_{i=1}^{\infty} \phi_{3i}(x)\phi_{3i}(y) \\
&= \phi_3(x)^T\phi_3(y)
\end{aligned}$$

In which we have defined $\phi_{3_i}(x)$ as follows:

$$\phi_{3_i}(x) = \begin{cases} \phi_{1\lceil \frac{i}{2} \rceil}(x) & i \equiv 0 \mod 2 \\ \phi_{2\lceil \frac{i}{2} \rceil}(x) & i \equiv 1 \mod 2 \end{cases}$$

(b)

$$\begin{aligned}
K_1(x,y) \times K_2(x,y) &= \phi_1(x)^T\phi_1(y)\phi_2(x)^T\phi_2(y) \\
&= \sum_{i=1}^{\infty} \phi_{1i}(x)\phi_{1i}(y) \times \sum_{i=1}^{\infty} \phi_{2i}(x)\phi_{2i}(y) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_{1i}(x)\phi_{1i}(y)\phi_{2j}(x)\phi_{2j}(y) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_{1i}(x)\phi_{2j}(x)\phi_{1i}(y)\phi_{2j}(y) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_{3ij}(x)\phi_{3ij}(y) \\
&= \phi_3(x)^T\phi_3(y)
\end{aligned}$$

In the above equation, we have defined ϕ_{3ij} as $\phi_{1i} \times \phi_{2j}$.

(c) We will use taylor expansion for e^x:

$$e^{K_1(x,y)} = \sum_{n=0}^{\infty} \frac{K_1(x,y)^n}{n!}$$

Problem 2 continued on next page...

We know $K_1(x,y)^n$ is valid from the section (b). The multiplication of a number and a valid kernel is also a valid kernel because:

$$\begin{aligned} c \times K_1(x,y) &= c \times \phi_1(x)^T \phi_1(y) \\ &= \left(\sqrt{c}\phi_1(x)\right)^T \left(\sqrt{c}\phi_1(y)\right) \end{aligned}$$

Thus, $\frac{K_1(x,y)^n}{n!}$ is also valid for each n. We also know that sum of valid kernels are valid from the section (a). So, $\sum_{n=0}^{\infty} \frac{K_1(x,y)^n}{n!}$ is a valid kernel and therefore, $e^{K_1(x,y)}$ is a valid kernel.

Problem 3

Q. Suppose $N(t)$ is a Poisson process with parameter λ.

(a) Calculate auto-covariance function. Is $N(t)$ W.S.S?

(b) Calculate p.d.f of $(N(t_1), N(t_2))$.

(c) For $t_0, t_1, t_2, ..., t_n$ find the p.d.f of $(N(t_0), ..., N(t_n)) = f(a_0, ..., a_n)$.

(d) From the previous question, for $t_i = t + in$ find:

$$argmax_{a_1,...,a_{n-1}} f(a, a_1, ..., a_n, a + kn)$$

(e) Find the probability that $N(4) < 3$ and $N(5) - N(3) > 3$.

A.

(a) We will assume $t \leq s$:
Since $\mathrm{Cov}(N(t), N(s)) = \mathrm{E}[N(t)N(s)] - \mathrm{E}[N(t)][N(s)]$, we will first calculate these two terms:

$$\begin{aligned}
\mathrm{E}[N(t)N(s)] &= \mathrm{E}[N(0,t)(N(0,t) + N(t,s))] \\
&= \mathrm{E}[N(0,t)N(0,t)] + \mathrm{E}[N(0,t)N(t,s)] \\
&= \mathrm{E}[N(0,t)^2] + \mathrm{E}[N(0,t)]\mathrm{E}[N(t,s)] \\
&= (\mathrm{E}[N(0,t)]^2 + Var(N(0,t))) + \mathrm{E}[N(0,t)]\mathrm{E}[N(t,s)] \\
&= ((\lambda t)^2 + \lambda t) + \lambda t\lambda(s - t) \\
&= \lambda^2 t^2 + \lambda t + \lambda^2 ts - \lambda^2 t^2 \\
&= \lambda^2 ts + \lambda t
\end{aligned}$$

$$\mathrm{E}[N(t)][N(s)] = \lambda^2 ts$$

Thus:

$$\begin{aligned}
\mathrm{Cov}(N(t), N(s)) &= \mathrm{E}[N(t)N(s)] - \mathrm{E}[N(t)][N(s)] \\
&= \lambda^2 ts + \lambda t - \lambda^2 ts \\
&= \lambda t
\end{aligned}$$

Similarly, if we assume $s \leq t$, the autocovariance will be equal to λs. So:

$$\mathrm{Cov}(N(t), N(s)) = \lambda min(t, s)$$

Also, $N(t)$ is not W.S.S since its mean is not constant (λt), and its autocorrelation is not a function of time difference.

Problem 3 continued on next page...

(b) We will assume $t_1 \leq t_2$:

$$\begin{aligned}
P(N(t_1) = k, N(t_2) = s) &= P(N(0, t_1) = k, N(t_1, t_2) = s - k) \\
&= P(N(0, t_1) \leq k) \times P(N(t_1, t_2) \leq s - k) \\
&= e^{-\lambda t_1} \frac{(\lambda t_1)^k}{k!} \times e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^{s-k}}{(s-k)!} \\
&= e^{-\lambda t_2} \frac{\lambda^k t_1^k (t_2 - t_1)^{s-k}}{k!(s-k)!}
\end{aligned}$$

(c)

$$\begin{aligned}
f_{N(t_0),...,N(t_n)}(a_0, ..., a_n) &= f_{N(t_0)}(a_0) f_{N(t_1 - t_0)}(a_1 - a_0) f_{N(t_2 - t_1)}(a_2 - a_1) ... f_{N(t_n - t_{n-1})}(a_n - a_{n-1}) \\
&= f_{N(t_0)}(a_0) \prod_{i=1}^{n} f_{N(t_i - t_{i-1})}(a_i - a_{i-1}) \\
&= e^{-\lambda t_0} \frac{(\lambda t_0)^{a_0}}{a_0!} \prod_{i=1}^{n} e^{-\lambda(t_i - t_{i-1})} \frac{(\lambda(t_i - t_{i-1}))^{a_i - a_{i-1}}}{(a_i - a_{i-1})!} \\
&= e^{-\lambda t_n} \lambda^{a_n} \frac{t_0^{a_0}}{a_0!} \prod_{i=1}^{n} \frac{(t_i - t_{i-1})^{a_i - a_{i-1}}}{(a_i - a_{i-1})!}
\end{aligned}$$

(d) If we put $t_i = t + in$ in the above equation we have:

$$\begin{aligned}
f_{N(t),...,N(t+in)}(a_0, ..., a_n) &= e^{-\lambda(t+n^2)} \lambda^{(a+kn)} \frac{t^a}{a!} \prod_{i=1}^{n} \frac{n^{a_i - a_{i-1}}}{(a_i - a_{i-1})!} \\
&= e^{-\lambda(t+n^2)} \lambda^{(a+kn)} n^{kn} \frac{t^a}{a!} \prod_{i=1}^{n} \frac{1}{(a_i - a_{i-1})!}
\end{aligned}$$

We have $a_1, ..., a_{n-1}$ only in the product. So if we want to calculate $argmax_{a_1,...,a_{n-1}} f$, we need to maximize this product. It will be maximum if each of $a_i - a_{i-1}$ is as close as they can be to zero; since $a_0 = a$ and $a_n = a + kn$, the closest $a_i - a_{i-1}$ can get to zero is for all of them to be equal to each other, resulting in $a_i - a_{i-1} = k$ and thus, $a_i = a + in$.

(e)

$$\begin{aligned}
P(N(4) < 3, N(5) - N(3) > 3) &= P(N(0, 4) < 3, N(3, 5) > 3) \\
&= P(N(0, 3) + N(3, 4) < 3, N(3, 4) + N(4, 5) > 3) \\
&= P(N(3, 4) = 0) P(N(0, 3) < 3) P(N(4, 5) > 3) \\
&+ P(N(3, 4) = 1) P(N(0, 3) < 2) P(N(4, 5) > 2) \\
&+ P(N(3, 4) = 2) P(N(0, 3) < 1) P(N(4, 5) > 1) \\
&= e^{-\lambda} \frac{\lambda^0}{0!} e^{-3\lambda} [\frac{(3\lambda)^0}{0!} + \frac{(3\lambda)^1}{1!} + \frac{(3\lambda)^2}{2!}] \big(1 - e^{-\lambda} [\frac{\lambda^0}{0!} - \frac{\lambda^1}{1!} - \frac{\lambda^2}{2!}]\big) \\
&+ e^{-\lambda} \frac{\lambda^1}{1!} e^{-3\lambda} [\frac{(3\lambda)^0}{0!} + \frac{(3\lambda)^1}{1!}] \big(1 - e^{-\lambda} [\frac{\lambda^0}{0!} - \frac{\lambda^1}{1!}]\big) \\
&+ e^{-\lambda} \frac{\lambda^2}{2!} e^{-3\lambda} \frac{(3\lambda)^0}{0!} \big(1 - e^{-\lambda} \frac{\lambda^0}{0!}\big) \\
&= e^{-4\lambda} [1 + 3\lambda + \frac{(3\lambda)^2}{2!}] \big(1 - e^{-\lambda} [1 - \lambda - \frac{\lambda^2}{2!}]\big) \\
&+ e^{-4\lambda} \lambda [1 + 3\lambda] \big(1 - e^{-\lambda} [1 - \lambda]\big) \\
&+ e^{-4\lambda} \frac{\lambda^2}{2!} \big(1 - e^{-\lambda}\big)
\end{aligned}$$

Problem 4

Q. Suppose $X(t)$ is a Gaussian process, with $X(0) = 0$ with probability 1. Suppose that $X(t) + X(s) \sim N(0, \sqrt{|t-s|})$.

(a) Calculate the auto-covariance function.

(b) Calculate the distribution of $(X(t_1), X(t_2), ..., X(t_n))$

(c) Prove that such a process doesn't exists.

A.

(a) Since $X(0) = 0$ with probability 1, we have: $X(t) = X(t) + X(0) \sim N(0, \sqrt{|t|})$

$$\begin{aligned}
\mathrm{E}[(X(t) + X(s))^2] &= \mathrm{E}[X(t)^2] + \mathrm{E}[X(s)^2] + 2E[X(t)X(s)] \\
\sqrt{|t-s|} &= \sqrt{|t|} + \sqrt{|s|} + 2E[X(t)X(s)] \\
E[X(t)X(s)] &= \frac{1}{2}(\sqrt{|t-s|} - \sqrt{|t|} - \sqrt{|s|}) \\
Cov(X(t), X(s)) = E[X(t)X(s)] &= \frac{1}{2}(\sqrt{|t-s|} - \sqrt{|t|} - \sqrt{|s|})
\end{aligned}$$

(b) Since X is a GP, for every $t_1, ...t_n$ for each $n > 0$ we know that $(X(t_1), X(t_2), ..., X(t_n))$ are jointly Gaussian. Thus, their distribution will come from zero-mean Normal distribution with covariance matrix Σ where $\Sigma_{ij} = \frac{1}{2}(\sqrt{|t_i - t_j|} - \sqrt{|t_i|} - \sqrt{|t_j|})$.

(c) From the equation in part (a), we get that $\mathrm{E}[X(t)^2] = \sqrt{|t|}$. On the other hand, from the covariance function we got $Cov(X(t), X(t)) = E[X(t)X(t)] = \frac{1}{2}(\sqrt{|t-t|} - \sqrt{|t|} - \sqrt{|t|}) = -\sqrt{|t|}$ which contradicts with the previous value we got for $\mathrm{E}[X(t)^2]$.

Problem 5

Q. A machine needs frequent maintenance to stay on. The maintenance times occur as a Poisson process with rate μ. Once the machine receives no maintenance for a time interval of length h, it breaks down. It then needs to be repaired, which takes an Exponential(λ) time, after which it goes back on.

(a) After the machine is started, find the probability that the machine will break down before receiving its first maintenance.

(b) Find the expected time for the first breakdown.

(c) Find the proportion of time the machine is on.

A.

(a) The first maintance time follows Exponential distribution with rate μ. So we need to calculate the probability of it being greater than h. We will show first maintance time random variable with X:

$$\begin{aligned}
P(X > h) &= 1 - P(X \leq h) \\
&= 1 - (1 - e^{-\mu h}) \\
&= e^{-\mu h}
\end{aligned}$$

(b) Assume we show the expected time for the first breakdown with B. X is the R.V. showing first maintance time from the previous part. We have:

$$\begin{aligned}
B &= P(X > h) \times h + \int_0^h P(X = t) \times (t + B)\, dt \\
&= hP(X > h) + \int_0^h P(X = t)t\, dt + B \int_0^h P(X = t)\, dt \\
&= hP(X > h) + \int_0^h P(X = t)t\, dt + BP(X \leq h) \\
&= he^{-\mu h} + B(1 - e^{-\mu h}) + \int_0^h P(X = t)t\, dt \\
&= he^{-\mu h} + B(1 - e^{-\mu h}) + \int_0^h \mu e^{-\mu t} t\, dt \\
&= he^{-\mu h} + B(1 - e^{-\mu h}) - (te^{-\mu t} + \frac{1}{\mu} e^{-\mu t})\Big|_0^h \\
&= he^{-\mu h} + B(1 - e^{-\mu h}) - (he^{-\mu h} + \frac{1}{\mu} e^{-\mu h} - (0 + \frac{1}{\mu})) \\
&= he^{-\mu h} + B(1 - e^{-\mu h}) - he^{-\mu h} - \frac{1}{\mu} e^{-\mu h} + \frac{1}{\mu} \\
&= B(1 - e^{-\mu h}) + \frac{1}{\mu}(1 - e^{-\mu h})
\end{aligned}$$

$$\begin{aligned}
B - B(1 - e^{-\mu h}) &= \frac{1}{\mu}(1 - e^{-\mu h}) \\
Be^{-\mu h} &= \frac{1}{\mu}(1 - e^{-\mu h}) \\
B &= \frac{1 - e^{-\mu h}}{\mu e^{-\mu h}}
\end{aligned}$$

Problem 5 continued on next page...

(c) Assume we show the proportion time the machine is on with M. This process can be viewed in periods in which each ends with the machine breaking down and having to be repaired again. In each of these periods, the expected time of the machine being on is B from the previous part, and the expected time of it being off is $\frac{1}{\lambda}$. So:

$$M = \frac{B}{B + \frac{1}{\lambda}}$$

Problem 6

Q. A car wash has two stations, 1 and 2, with Exponential (λ_1) and Exponential (λ_2) service times. A car enters at station 1. Upon completing the service at station 1, the car then proceeds to station 2, provided station 2 is free; otherwise, the car has to wait at station 1, blocking the entrance of other cars. The car exits the wash after service at station 2 is completed. When you arrive at the wash there is a single car at station 1. Compute your expected time before you exit.

A. We show the expected time with $\mathrm{E}[T]$, the time that our service at station 1 will take with X_1, and the time that our service at station 2 will take with X_2, the time that previous car's service at station 1 will take with Y_1, and the its service time at station 2 will take with Y_2.

$$
\begin{aligned}
\mathrm{E}[T] &= \mathrm{E}[Y_1] + P(X_1 > Y_2) \times (\mathrm{E}[X_1] + \mathrm{E}[X_2]) + P(X_1 \leq Y_2) \times (\mathrm{E}[Y_2] + \mathrm{E}[X_2]) \\
&= \mathrm{E}[Y_1] + P(X_1 > Y_2) \times \mathrm{E}[X_1] + P(X_1 \leq Y_2) \times \mathrm{E}[Y_2] + \mathrm{E}[X_2] \\
&= \frac{1}{\lambda_1} + P(X_1 > Y_2) \times \frac{1}{\lambda_1} + P(X_1 \leq Y_2) \times \frac{1}{\lambda_2} + \frac{1}{\lambda_2} \\
&= \frac{1}{\lambda_1} + \frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} \times \frac{1}{\lambda_1} + \frac{\frac{1}{\lambda_2}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} \times \frac{1}{\lambda_2} + \frac{1}{\lambda_2} \\
&= \frac{1}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \times \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \times \frac{1}{\lambda_2} + \frac{1}{\lambda_2} \\
&= \frac{1}{\lambda_1} + \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)} + \frac{1}{\lambda_2} \\
&= \frac{2}{\lambda_1} + \frac{-2}{\lambda_1 + \lambda_2} + \frac{2}{\lambda_2}
\end{aligned}
$$

Problem 7

Q. Let (N_t) a Poisson process with intensity λ. We define $G_t = t - T_{N_t}$, $D_t = T_{N_t+1} - t$.

(a) For a given t and $0 < x \leq t, y \geq 0$, show that $(G_t < x, D_t \leq y) = (N_{t-x} < N_t < N_{t+y})$. Find $P(G_t < x, D_t \leq y)$.

(b) For a given t and $y > 0$, show that $(G_t = t, D_t \leq y) = (N_t = 0, N_{t+y} > 0)$. Find $P(G_t = t, D_t \leq y)$.

(c) For a given t and $y > 0$, calculate $(D_t \leq y)$ and find the distribution of D_t.

(d) Calculate the cumulative distribution function of G_t.

(e) Calculate $P(min(T_1, t) > x)$ for all $x \in R$. Deduce that G_t has the same distribution as $min(T_1, t)$.

(f) Show that $G_t$and D_t are independent.

(g) Calculate $E[G_t]$. Deduce $E[G_t + D_t]$. What do you think about this result?

A.

(a) Here is a visualization:

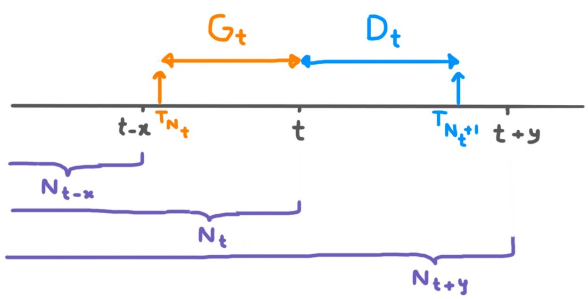

Figure 1: Poisson Process

We can see that if G_t wants to be less than x, at least one event should have occurred between $t - x$ and t, meaning the $N_{t-x} < N_t$. Same applies for D_t and N_{t+y}.

$$\begin{aligned}
P(G_t < x, D_t \leq y) &= P(N_{t-x} < N_t, N_t < N_{t+y}) \\
&= P(N_{(t-x,t)} \geq 1)P(N_{(t,t+y)} \geq 1) \qquad \text{(independent intervals)} \\
&= \big(1 - P(N_{(t-x,t)} = 0)\big)\big(1 - P(N_{(t,t+y)} = 0)\big) \\
&= (1 - e^{-x\lambda})(1 - e^{-y\lambda})
\end{aligned}$$

(b) Since G_t is equal to the difference of t and the previous event happened, if we want it to be equal to t, then no event should have happened in $(0, t)$, thus the $N_t = 0$. On the other hand, $D_t \leq y$ is equivalant to $N_t < N_{t+y}$ as we have from last part. Since $N_t = 0$, this is equal to $0 < N_{t+y}$.

$$\begin{aligned}
P(G_t = t, D_t \leq y) &= P(N_t = 0, 0 < N_{t+y}) \\
&= P(N_t = 0)P(N_{t+y} > 0) \\
&= e^{-t\lambda}(1 - e^{-y\lambda})
\end{aligned}$$

(c) D_t comes from Exponential distribution with rate λ.

$$\begin{aligned} P(D_t \leq y) &= P(N_{(t,t+y)} \geq 1) \\ &= (1 - e^{-y\lambda}) \end{aligned}$$

(d) If $x < t$:

$$\begin{aligned} P(G_t \leq x) &= P(G_{(t-x,t)} \geq 1) \\ &= 1 - P(G_{(t-x,t)} = 1) \\ &= (1 - e^{-x\lambda}) \end{aligned}$$

Otherwise if $x < 0$ then $P(G_t \leq x) = 0$ and if $x > t$ then $P(G_t \leq x) = 1$.

(e) If $x < 0$ then $P(min(T_1, t) > x) = 1$ and if $x > t$ then $P(min(T_1, t) > x) = 0$. Otherwise:

$$\begin{aligned} P(min(T_1, t) > x | t > x > 0) &= P(T_1 > x) \\ &= e^{-x\lambda} \end{aligned}$$

The distribution of $min(T_1, t)$ will be $1 - P(min(T_1, t) > x)$ which is equal to the distribution of G_t.

(f)

$$\begin{aligned} P(G_t \leq x, D_t \leq y) &= (N_{t-x} < N_t, N_t < N_{t+y}) \\ &= P(N_{(t-x,t)} \geq 1)P(N_{(t,t+y)} \geq 1) \qquad \text{(independent intervals)} \\ &= P(G_t \leq x)(D_t \leq y) \end{aligned}$$

(g)

$$\begin{aligned} \mathrm{E}[G_t] &= \int_0^t xP(G_t = x)\,dx \\ &= \int_0^t P(G_t \geq x)\,dx \\ &= \int_0^t e^{-x\lambda}\,dx \\ &= \frac{-1}{\lambda} e^{-x\lambda}\Big|_0^t \\ &= \frac{-1}{\lambda} e^{-t\lambda} + \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} \mathrm{E}[G_t + D_t] &= \mathrm{E}[G_t] + \mathrm{E}[D_t] \\ &= \frac{-1}{\lambda} e^{-t\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda} \\ &= \frac{1}{\lambda}\big(- e^{-t\lambda} + 2\big) \end{aligned}$$

As t goes to infinity, $\mathrm{E}[G_t + D_t]$ goes to $\frac{1}{\lambda}$ which is the same as expected value of Exponential(λ). It is reasonable since the interval between two events follows this distribution.