Stochastic Processes

Homework #3

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Problem 1

Q. Short answer:

(a) Is sum of two WSS processes always a WSS process?

(b) Assume a stochastic process such that $X(t) = At + b$. We know that b is a constant and $A \sim Normal(0, 1)$. Is this process mean ergodic?

(c) We have a stochastic process that is zero mean and $S_X(\omega) = \frac{5}{9+\omega^2}$. Is this process mean ergodic?

A.

(a) This is not true in the general case. Assume $X(t)$ and $Y(t)$ are WSS and $Z(t) = X(t) + Y(t)$. We know for the process Z to be WSS, its auto-correlation should be a function of the time difference:

$$\begin{aligned} R_Z(t_1, t_2) &= \mathrm{E}\big[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))^*\big] \\ &= R_X(t_1, t_2) + R_Y(t_1, t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2) \\ &= R_X(t_1 - t_2) + R_Y(t_1 - t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2) \end{aligned}$$

This will be independent of time indexes if only $R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2)$ be independent of time indexes which holds if only $X(t)$ and $Y(t)$ are jointly WSS.

(b)

$$\mathrm{E}[X(t)] = \mathrm{E}[At + b] = \mathrm{E}[A]\mathrm{E}[t] + \mathrm{E}[b] = b$$

$$\begin{aligned} < X(t) > &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} At + b\, dt \\ &= \lim_{T\to\infty} \frac{1}{2T} A \int_{-T}^{T} t\, dt + \frac{1}{2T} \int_{-T}^{T} b\, dt \\ &= \lim_{T\to\infty} 0 + b = b \end{aligned}$$

Since $\mathrm{E}[X(t)] =< X(t) >$, this process is mean-ergodic.

(c) We know that a process is mean-ergodic iff $\frac{1}{T}\int_0^T C(t)\, dt \xrightarrow[T\to\infty]{} 0$. (Reference: Papoulis 4th Ed., Equation 12-7)

So we need to calculate $C_X(t)$, which is equal to $R_X(t)$ since the process is zero mean:

$$\begin{aligned} C_X[t] &= R_X[t] \\ &= \frac{5}{6} e^{-3|t|} \end{aligned}$$

$$
\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\int_0^T C_X(t)\,dt = & \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{5}{6}e^{-3|t|}\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{5}{6}\int_0^T e^{-3t}\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{5}{6}[\frac{-1}{3}e^{-3t}]\Big|_0^T \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{-5}{18}[e^{-3T}-1] \\
&= 0
\end{aligned}
$$

So this process is mean-ergodic.

Problem 2

Q. Suppose we have an LTI system with the impulse response like below:

$$|H(f)| = \begin{cases} \sqrt{1+4\pi^2 f^2} & |f| < 2 \\ 0 & otherwise \end{cases}$$

Assume $X(t)$ is a zero mean WSS process such that:

$$R_X(\tau) = e^{-|\tau|}$$

If $X(t)$ is the input and $Y(t)$ is the output, calculate below parameters:

$$\mu_Y(t), R_Y(\tau), \mathrm{E}[Y^2(t)]$$

A.
For mean value we have:

$$\mu_Y(t) = \mu_X(t) * h(t) = 0$$

For auto-correlation, we first calculate power specterum of X using Fourier transform:

$$S_X(w) = \frac{2}{1+w^2}$$

Then we can calculate power specterum of Y for $|w| < 2$:

$$\begin{aligned} S_Y(w) &= S_X(w)|H(w)|^2 \\ &= \frac{2}{1+w^2}(1+4\pi^2 w^2) \\ &= \frac{2 - 8\pi^2 + 8\pi^2 + 8\pi^2 w^2}{1+w^2} \\ &= 8\pi^2 + (1-4\pi^2)\frac{2}{1+w^2} \end{aligned}$$

So:

$$S_Y(w) = \begin{cases} 8\pi^2 + (1-4\pi^2)\frac{2}{1+w^2} & |w| < 2 \\ 0 & otherwise \end{cases}$$

Finally we calculate auto-coorelation of Y using inverse Fourier transform:

$$R_Y(t) = 8\pi^2\delta(t) + (1-4\pi^2)e^{-|t|}$$

Since $\mathrm{E}[Y^2(t)] = R_Y(0)$ we can write:

$$\begin{aligned} \mathrm{E}[Y^2(t)] &= R_Y(0) \\ &= 8\pi^2 + (1-4\pi^2) \\ &= 4\pi^2 + 1 \end{aligned}$$

Problem 3

Q. Suppose $X(t)$ is a WSS stochastic process and we know that:

$$\mathrm{E}[X(t)] = \mu_X$$
$$R_X(\tau) = e^{-|\tau|}$$

Assume that A is an independent Gaussian random variable such that:

$$A \sim Normal(\mu_A, \sigma_A^2)$$

If we define Y(t) as below:

$$Y(t) = X(t) + A$$

Prove that :

$$\sigma_A^2 = \mu_X^2 \iff Y(t) \text{ is mean-ergodic}$$

A.
We know that a process is mean-ergodic iff $\frac{1}{T}\int_0^T C(t)\,dt \xrightarrow[T\to\infty]{} 0$.
So we need to calculate $C_{YY}(t)$:

$$\begin{aligned}\mathrm{E}[Y(t)] &= \mathrm{E}[X(t) + A] \\ &= \mathrm{E}[X(t)] + \mathrm{E}[A] \\ &= \mu_X + \mu_A\end{aligned}$$

$$\begin{aligned}C_{YY}(s) &= R_{YY}(s) - \mu_Y^2 \\ &= R_{YY}(s) - (\mu_X + \mu_A)^2 \\ &= E[(X(t) + A)(X(t+s) + A)] - (\mu_X + \mu_A)^2 \\ &= E[X(t)X(t+s)] + E[A]E[X(t+s)] + E[A]E[X(t)] + E[A^2] - (\mu_X + \mu_A)^2 \\ &= R_{XX}(s) + \mu_X\mu_A + \mu_X\mu_A + E[A^2] - (\mu_X^2 + \mu_A^2 + 2\mu_x\mu_A) \\ &= R_{XX}(s) - \mu_X^2 + E[A^2] - \mu_A^2 \\ &= R_{XX}(s) - \mu_X^2 + \sigma_A^2 \\ &= e^{-|s|} - \mu_X^2 + \sigma_A^2\end{aligned}$$

$e^{-|s|}$ goes to zero as s goes to infinity. So, for the above condition to satisfy, we should have $\mu_X^2 = \sigma_A^2$.

Problem 4

Q. Let $x(t)$ be a real valued, continuous time, zero mean WSS random process with correlation function $R_{XX}(\tau)$ and power spectrum $S_{XX}(w)$. Suppose $x(t)$ is the input to two real valued LTI systems as depicted below, producing two new processes $y_1(t)$ and $y_2(t)$. Find $C_{y_1y_2}(\tau)$ and $S_{y_1y_2}(w)$.

A.
First of all, we calculate cross-correlation of x and y_2:

$$\begin{aligned}
R_{xy_2}(\tau) &= \mathrm{E}[x(t)y_2(t-\tau)] \\
&= \mathrm{E}\Big[x(t)[\int_{-\infty}^{\infty} h_2(s)x(t-\tau-s)\,ds]\Big] \\
&= \mathrm{E}\Big[\int_{-\infty}^{\infty} x(t)x(t-\tau-s)h_2(s)\,ds\Big] \\
&= \int_{-\infty}^{\infty} \mathrm{E}[x(t)x(t-\tau-s)]h_2(s)\,ds\Big] \\
&= \mathrm{E}\Big[\int_{-\infty}^{\infty} R_{xx}(\tau+s)h_2(s)\,ds\Big] \\
&= \mathrm{E}\Big[\int_{-\infty}^{\infty} R_{xx}(-\tau-s)h_2(s)\,ds\Big] \\
&= R_{xx}(-\tau) * h_2(-\tau) \\
&= R_{xx}(\tau) * h_2(-\tau)
\end{aligned}$$

Now we can calculate cross-correlation of y_1 and y_2:

$$\begin{aligned}
R_{y_1y_2}(\tau) &= \mathrm{E}[y_1(t)y_2(t-\tau)] \\
&= \mathrm{E}\Big[[\int_{-\infty}^{\infty} x(t-s)h_1(s)\,ds]y_2(t-\tau)\Big] \\
&= \mathrm{E}\Big[\int_{-\infty}^{\infty} x(t-s)y_2(t-\tau)h_1(s)\,ds\Big] \\
&= \int_{-\infty}^{\infty} \mathrm{E}\Big[x(t-s)y_2(t-\tau)\Big]h_1(s)\,ds \\
&= \int_{-\infty}^{\infty} R_{xy_2}(\tau-s)h_1(s)\,ds \\
&= R_{xy_2}(\tau) * h_1(\tau) \\
&= R_{xx}(\tau) * h_2(-\tau) * h_1(\tau)
\end{aligned}$$

We also know that:

$$\begin{aligned}
\mathrm{E}[Y_1(t)] &= \mathrm{E}[X(t)] * h(_1t) = 0 \\
\mathrm{E}[Y_2(t)] &= \mathrm{E}[X(t)] * h_2(t) = 0
\end{aligned}$$

So:

$$C_{y_1y_2}(\tau) = R_{y_1y_2}(\tau) = R_{xx}(\tau) * h_2(-\tau) * h_1(\tau)$$

In additon, since power spectrum is Fourier transform of cross-correlation function we can write:

$$S_{y_1y_2}(w) = S_{xx}(w)H_2(-w)H_1(w)$$

Problem 5

Q. Consider an LTI system with system function:

$$H(s) = \frac{1}{s^2 + 4s + 13}$$

The input to this system is a WSS process $X(t)$ with $E[X^2(t)] = 10$. Find $S_X(w)$ such that the average power of output is maximum.

A.

$$\begin{aligned}
H(w) &= \frac{1}{(jw)^2 + 4(jw) + 13} \\
&= \frac{1}{-w^2 + 4jw + 13} \\
|H(w)|^2 &= \frac{1}{(-w^2 + 13)^2 + (4w)^2} \\
&= \frac{1}{w^4 + 13^2 - 26w^2 + 16w^2} \\
&= \frac{1}{w^4 - 10w^2 + 13^2} \\
&= \frac{1}{(w^2 - 5)^2 - 25 + 13^2}
\end{aligned}$$

$|H(w)|^2$ will be at maximum when the denominator will be minimum which happens at $w = \sqrt(5)$. At this point $|H(w)|^2$ will be $\frac{1}{144}$.

$$\begin{aligned}
\text{average power of output} &= E[Y(t)^2] \\
&= R_Y(0) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(w)\, dw \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(w)|H(w)|^2\, dw \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(w) \frac{1}{144}\, dw \\
&= \frac{1}{144} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(w)\, dw \\
&= \frac{1}{144} R_X(0) \\
&= \frac{1}{144} \mathrm{E}[X^2(t)] \\
&= \frac{10}{144}
\end{aligned}$$

So $S_X(w)$ should be maximum at $w = \sqrt{5}$ and the maximum average power of ouput will be $\frac{10}{144}$.

Problem 6

Q. Consider a WSS process y(t) satisfying the equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

where $x(t)$ is a zero-mean WSS process with covariance function:

$$C_{XX}(\tau) = \delta(\tau) + 4e^{-|\tau|}$$

Also assume that transformation from $x(t)$ to $y(t)$ is LTI. Determine $R_{XY}(\tau), S_{XY}(w)$.

A.
From the equation we get:

$$H(w) = \frac{1}{2 + jw}$$

So the impulse response will be:

$$h(t) = e^{-2t}u(t)$$

We can now compute cross-correlation:

$$\begin{aligned}
R_{XY}(\tau) &= R_{XX}(\tau) * h(-\tau) \\
&= (\delta(\tau) + 4e^{-|\tau|}) * (e^{2\tau}u(-\tau)) \\
&= e^{2\tau}u(-\tau) + 4e^{-|\tau|} * (e^{2\tau}u(-\tau)) \\
&= e^{2\tau}u(-\tau) + \int_{-\infty}^{\infty} 4e^{-|\tau-t|}e^{2t}u(-t)\,dt \\
&= e^{2\tau}u(-\tau) + \int_{-\infty}^{0} 4e^{-|\tau-t|}e^{2t}\,dt
\end{aligned}$$

When $\tau \geq 0$ the integral will be:

$$\begin{aligned}
\int_{-\infty}^{0} 4e^{-|\tau-t|}e^{2t}\,dt &= \int_{-\infty}^{0} 4e^{-\tau+t}e^{2t}\,dt \\
&= \int_{-\infty}^{0} 4e^{-\tau+3t}\,dt \\
&= \int_{-\infty}^{0} 4e^{-\tau+3t}\,dt \\
&= \left[\frac{4}{3}e^{-\tau+3t}\right]\Big|_{-\infty}^{0} \\
&= \frac{4}{3}e^{-\tau}
\end{aligned}$$

Otherwise, i.e. if $\tau < 0$ the integral will be:

$$\begin{aligned}
\int_{-\infty}^{0} 4e^{-|\tau-t|}e^{2t}\,dt &= \int_{-\infty}^{\tau} 4e^{-\tau+t}e^{2t}\,dt + \int_{\tau}^{0} 4e^{\tau-t}e^{2t}\,dt \\
&= \int_{-\infty}^{\tau} 4e^{-\tau+3t}\,dt + \int_{\tau}^{0} 4e^{\tau+t}\,dt \\
&= \left[\frac{4}{3}e^{-\tau+3t}\right]\Big|_{-\infty}^{\tau} + \left[4e^{\tau+t}\right]\Big|_{\tau}^{0} \\
&= \frac{4}{3}e^{2\tau} + 4e^{\tau} - 4e^{2\tau} \\
&= -\frac{8}{3}e^{2\tau} + 4e^{\tau}
\end{aligned}$$

Problem 6 continued on next page...

Finaly we can write R_{XY} as one equation utlizing step function:

$$
\begin{aligned}
R_{XY}(\tau) &= e^{2\tau}u(-\tau) + \frac{4}{3}e^{-\tau}u(\tau) + (-\frac{8}{3}e^{2\tau} + 4e^{\tau})u(-\tau) \\
&= u(-\tau)\big[e^{-2\tau} - \frac{8}{3}e^{2\tau} + 4e^{\tau}\big] + u(\tau)\frac{4}{3}e^{-\tau} \\
&= u(-\tau)\big[e^{2\tau} - \frac{8}{3}e^{2\tau} + 4e^{\tau}\big] + u(\tau)\frac{4}{3}e^{-\tau} \\
&= u(-\tau)\big[-\frac{5}{3}e^{2\tau} + 4e^{\tau}\big] + u(\tau)\big[\frac{4}{3}e^{-\tau}\big] \\
&= -\frac{5}{3}e^{2\tau}u(-\tau) + 4e^{\tau}u(-\tau) + \frac{4}{3}e^{-\tau}u(\tau)
\end{aligned}
$$

And we can calculate cross power spectrum by calculating Fourier transform of R_{XY} :

$$
S_{XY}(\tau) = -\frac{5}{3(2-jw)} + \frac{4}{1-jw} + \frac{4}{3(1+jw)}
$$

Problem 7

Q. The process $x(t)$ is WSS with $\mathrm{E}[x(t)] = 5$ and $R_{xx}(\tau) = 25 + 4e^{-2|\tau|}$. If $y(t) = 2x(t) + 3x'(t)$, find $S_y(w)$.

A.
From the equation we get:

$$H(w) = 2 + 3jw$$

We can calculate the power spectrum of input using Fourier transform:

$$S_x(w) = 50\pi\delta(w) + \frac{16}{4 + w^2}$$

Now we can calculate the power spectrum of output:

$$\begin{aligned} S_y(w) &= S_x(w)|H(w)|^2 \\ &= (50\pi\delta(w) + \frac{16}{4 + w^2})(2 + 3w)^2 \\ &= 50\pi(2 + 3w)^2\delta(w) + \frac{16(2 + 3w)^2}{4 + w^2} \end{aligned}$$

Problem 8

Q. We have an LTI system. The output of system is defined as:

$$y[n] = x[n] + x[n-2] - \frac{1}{2}x[n-3] + \frac{1}{2}y[n-1]$$

(a) Find the impulse response of the system in the time and frequency domains.

(b) Plot the impulse response in time domain from $n = 0$ to $n = 3$.

(c) If input is a zero-mean WSS process with autocorrelation defined as:

$$R_{XX}[m,n] = R_{XX}[k] = \delta[k] + \delta[|k|-1]$$

Find the R_{XY}.

A.

(a) First, we will simplify $y[n]$:

$$\begin{aligned}
y[n] &= x[n] + x[n-2] - \frac{1}{2}x[n-3] + \frac{1}{2}y[n-1] \\
&= x[n] + x[n-2] - \frac{1}{2}x[n-3] \\
&+ \frac{1}{2}x[n-1] + \frac{1}{2}x[n-3] - \frac{1}{4}x[n-4] \\
&+ \frac{1}{4}x[n-2] + \frac{1}{4}x[n-4] - \frac{1}{8}x[n-5] \\
&+ \frac{1}{8}x[n-3] + \frac{1}{8}x[n-5] - \frac{1}{16}x[n-6] \\
&+ ... \\
&= x[n] + x[n-2] \\
&+ \frac{1}{2}x[n-1] \\
&+ \frac{1}{4}x[n-2] \\
&+ \frac{1}{8}x[n-3] \\
&+ ... \\
&= x[n-2] + \sum_{i=0}^{\infty} \frac{1}{2^i}x[n-i]
\end{aligned}$$

We can calculate impulse response of system by setting $x[n] = \delta[n]$:

$$\begin{aligned}
h[n] &= \delta[n-2] + \sum_{i=0}^{\infty} \frac{1}{2^i}\delta[n-i] \\
&= \delta[n-2] + \frac{1}{2^n}u[n]
\end{aligned}$$

And we can calculate impulse response in frequency domain by using inverse Fourier transform:

$$X[e^{jw}] = e^{-2jw} + \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

(b) Check Figure 1.

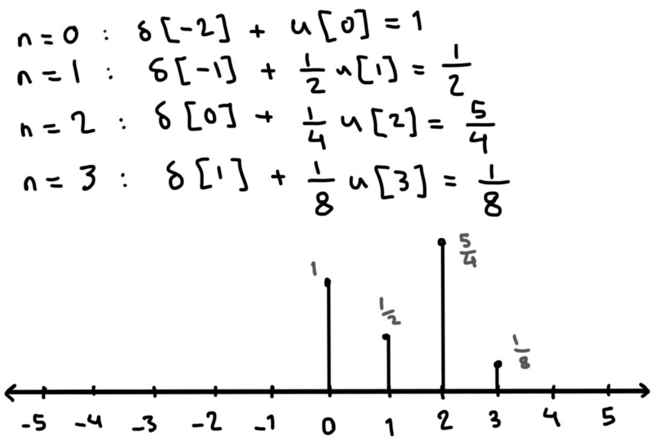

Figure 1: $h[n]$

(c)

$$
\begin{aligned}
R_{XY}[k] &= R_{XX}[k] * h[k] \\
&= (\delta[k] + \delta[|k| - 1]) * (\delta[k-2] + \frac{1}{2^k}u[k]) \\
&= \delta[k] * \delta[k-2] + \delta[|k| - 1] * \delta[k-2] + \delta[k] * \frac{1}{2^k}u[k] + \delta[|k| - 1] * \frac{1}{2^k}u[k]
\end{aligned}
$$

If $k \geq 0$:

$$
\begin{aligned}
R_{XY}[k] &= (\delta[k] + \delta[k-1]) * (\delta[k-2] + \frac{1}{2^k}u[k]) \\
&= \delta[k-2] + \frac{1}{2^k}u[k] + \delta[k-1] + \frac{1}{2^{k+1}}u[k+1]
\end{aligned}
$$

If $k < 0$:

$$
\begin{aligned}
R_{XY}[k] &= (\delta[k] + \delta[-(k+1)]) * (\delta[k-2] + \frac{1}{2^k}u[k]) \\
&= \delta[k-2] + \frac{1}{2^k}u[k] + \delta[-k+3] + \frac{1}{2^{-k+1}}u[-k+1]
\end{aligned}
$$