Stochastic Processes

Homework #5

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Problem 1

Q. Use method of moments to estimate the parameters μ and σ for the density

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{(x-\mu)^2}{2\sigma^2}]$$

based on a random sample $X_1, ..., X_n$.

A.
First moment:

$$\mathrm{E}[x] = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Second moment:

$$\mathrm{E}[x^2] = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$
$$var(x) + \mathrm{E}[x]^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$
$$\sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$
$$\hat{\sigma^2} = (\frac{1}{n}\sum_{i=1}^{n} x_i^2) - \hat{\mu}^2$$
$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\frac{1}{n}\sum_{i=1}^{n} x_i)^2$$

Problem 2

Q. Find MLE estimator for the following pdfs. ($X_1, ..., X_n$ is seen)

(a) $f(x|\theta) = \frac{1}{\theta}$ if $0 \leq x \leq \theta$ else 0

(b) $f(x|\theta) = \frac{1}{\theta}$ if $0 < x < \theta$ else 0

(c) $f(x|\theta) = 1$ if $\theta \leq x \leq \theta + 1$ else 0

(d) $f(x|\theta) = \frac{1}{\theta_2 - \theta_1}$ if $\theta_1 < x < \theta_2$ else 0

A.

(a) We can write the likelihood function as follows: (I is the indicator function).

$$
\begin{aligned}
L(X|\theta) &= \prod_{i=1}^{n} \frac{1}{\theta} I(0 \leq x_i \leq \theta) \\
&= \frac{1}{\theta}^{n} I(0 \leq \min(x_i) \leq \max(x_i) \leq \theta)
\end{aligned}
$$

Thus, likelihood will be zero if the condition $0 \leq \min(x_i) \leq \max(x_i) \leq \theta$ does not hold. Otherwise, we can write the derivetive and set it to zero:

$$
\frac{\partial L(X|\theta)}{\partial \theta} = n \frac{1}{\theta}^{n-1} = 0
$$

However, there is no value of θ that satisfy this equation. The function $\frac{1}{\theta}^{n}$ is decreasing so the highest value will be at the point that θ is minimum. Since we had $\max(x_i) \leq \theta$, so the highest value of likelihood function will happen at $\theta = \max(x_i)$.

(b)

(c)

$$
\begin{aligned}
L(X|\theta) &= \prod_{i=1}^{n} 1 \times I(\theta \leq x_i \leq \theta + 1) \\
&= I(\theta \leq \min(x_i) \leq \max(x_i) \leq \theta + 1)
\end{aligned}
$$

So any value of θ that satisfies $\max(x_i) - 1 \leq \theta \leq \min(x_i)$ will result in maximum likelihood.

(d)

$$
\begin{aligned}
L(X|\theta) &= \prod_{i=1}^{n} \frac{1}{\theta_2 - \theta_1} I(\theta_1 \leq x_i \leq \theta_2) \\
&= (\frac{1}{\theta_2 - \theta_1})^{n} I(\theta_1 \leq \min(x_i) \leq \max(x_i) \leq \theta_2)
\end{aligned}
$$

This part is similar to part one. The maximum likelihood will happen at $\theta_1 = \min(x_i)$ and $\theta_2 = \max(x_i)$

Problem 3

Q. Consider n iid samples $x_1, ..., x_n$ drawn from

1. $f(x|a) = \frac{1}{a}$ for $x \in [0, a]$
2. $f(x|\eta) = \frac{1}{\eta}exp(-\frac{x}{\eta})$ for $x > 0$
3. $f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}exp[-\frac{(x-\mu)^2}{2\sigma^2}]$

(a) Derive MLE estimator for each of the pdfs.

(b) Show that each of $\hat{a}_{ML}, \hat{\eta}_{ML}, \hat{\mu}_{ML}$ is biased or unbiased.

(c) Show that $\hat{\sigma}^2_{ML}$ is biased and find the unbiased estimator.

(d) Show that MSE of biased estimator for σ^2 is lower than its unbiased one.

A.

(a) 1. We calculate MLE for this pdf in Problem 2: $\hat{a}_{ML} = \max(x_i)$.

2.

$$L(X|\eta) = \prod_{i=1}^{n} \frac{1}{\eta}exp(-\frac{x_i}{\eta})$$

$$\ln L(X|\eta) = \sum_{i=1}^{n} -\ln(\eta) - \frac{x_i}{\eta}$$

$$= -n\ln(\eta) - \frac{1}{\eta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial \ln L(X|\eta)}{\partial \eta} = 0$$

$$-\frac{n}{\eta} + \frac{1}{\eta^2}\sum_{i=1}^{n} x_i = 0$$

$$-n + \frac{1}{\eta}\sum_{i=1}^{n} x_i = 0$$

$$\hat{\eta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

3.

$$L(X|\mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}}exp[-\frac{(x_i-\mu)^2}{2\sigma^2}]$$

$$L(X|\mu, \sigma) = \sum_{i=1}^{n} -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x_i-\mu)^2}{2\sigma^2}$$

$$= -\frac{n}{2}\ln(2\pi) - nln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial \ln L(X|\mu,\sigma)}{\partial \mu} = 0$$

$$-\frac{1}{2\sigma^2}\sum_{i=1}^{n} -2(x_i - \mu) = 0$$

$$(\sum_{i=1}^{n} x_i) - n\mu = 0$$

$$\hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\frac{\partial \ln L(X|\mu,\sigma)}{\partial \sigma} = 0$$

$$-\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 = 0$$

$$\begin{aligned} \hat{\sigma^2}_{ML} &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j)^2 \end{aligned}$$

(b) 1. We need to calculate $\mathrm{E}[\hat{a}_{ML}] = \mathrm{E}[max(x_i)]$. For the n-th order statistic we now the distribution is:

$$f_{max(X)}(x) = nF_X(x)^{n-1}f(x)$$

Now we can calculate expected value of estimator:

$$\begin{aligned} \mathrm{E}[\hat{a}_{ML}] =& \mathrm{E}[max(x_i)] \\ &= \int_0^a xnF_X(x)^{n-1}f(x)dx \\ &= \int_0^a xn(\frac{x}{a})^{n-1}\frac{1}{a}dx \\ &= n(\frac{1}{a})^n\int_0^a x^n dx \\ &= n(\frac{1}{a})^n(\frac{1}{n+1}x^{n+1}\Big|_0^a) \\ &= n(\frac{1}{a})^n(\frac{1}{n+1}a^{n+1}) \\ &= \frac{n}{n+1}a \end{aligned}$$

So this estimator is biased.

2.

$$\begin{aligned} \mathrm{E}[\hat{\eta}_{ML}] &= \mathrm{E}[\frac{1}{n}\sum_{i=1}^{n} x_i] \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}[x_i] \\ &= \frac{1}{n}\sum_{i=1}^{n}\eta \\ &= \eta \end{aligned}$$

This estimator is unbiased.

3.

$$
\begin{aligned}
\mathrm{E}[\hat{\mu}_{ML}] &= \mathrm{E}[\frac{1}{n}\sum_{i=1}^{n} x_i] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu \\
&= \mu
\end{aligned}
$$

The estimator for μ is unbiased.

(c)

$$
\begin{aligned}
\mathrm{E}[\hat{\sigma^2}_{ML}] &= \mathrm{E}[\frac{1}{n}\sum_{i=1}^{n}(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j)^2] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j)^2] \\
&= \frac{1}{n}\sum_{i=1}^{n} \Big[\mathrm{E}[x_i^2] + \mathrm{E}[(\frac{1}{n}\sum_{j=1}^{n} x_j)^2] - 2\mathrm{E}[x_i \frac{1}{n}\sum_{j=1}^{n} x_j]\Big] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i^2] + \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[(\frac{1}{n}\sum_{j=1}^{n} x_j)^2] - 2\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i \frac{1}{n}\sum_{j=1}^{n} x_j] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i^2] + \mathrm{E}[(\frac{1}{n}\sum_{j=1}^{n} x_j)^2] - 2\mathrm{E}[(\frac{1}{n}\sum_{i=1}^{n} x_i)^2] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i^2] - \mathrm{E}[(\frac{1}{n}\sum_{i=1}^{n} x_i)^2] \\
&= \frac{1}{n}\sum_{i=1}^{n} \big(\mathrm{E}[x_i]^2 + var(x_i)\big) - \big(\mathrm{E}[\frac{1}{n}\sum_{i=1}^{n} x_i)]^2 + var(\frac{1}{n}\sum_{i=1}^{n} x_i)\big) \\
&= (\mu^2 + \sigma^2) - \big((\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[x_i])^2 + \frac{1}{n^2}\sum_{i=1}^{n} var(x_i)\big) \\
&= (\mu^2 + \sigma^2) - \big(\mu^2 + \frac{1}{n}\sigma^2\big) \\
&= \sigma^2 - \frac{1}{n}\sigma^2 \\
&= \frac{n-1}{n}\sigma^2
\end{aligned}
$$

Since $\mathrm{E}[\hat{\sigma^2}_{ML}] \neq \sigma^2$, this estimator is biased.

In order to find an unbiased estimator, we can scale the previous estimator by $\frac{n}{n-1}$. So our new estimator will be:

$$
S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j)^2
$$

Problem 3 continued on next page...

(d) Since sum of squares of k independent, standard normal variable, follows chi-squared distribution with k degree of freedom, we have:

$$Y = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} \sim \chi_{n-1}$$

The mean and variance of this distribution with k degree of freedom is k and $2k$ respectively. Thus:

$$\begin{aligned}
\hat{\sigma^2}_{ML} &= \frac{\sigma^2}{n} Y \\
MSE(\hat{\sigma^2}_{ML}) &= var(\hat{\sigma^2}_{ML}) + bias(\hat{\sigma^2}_{ML})^2 \\
&= \frac{\sigma^4}{n^2} var(Y) + \frac{1}{n^2}\sigma^4 \\
&= \frac{2(n-1)\sigma^4}{n^2} + \frac{1}{n^2}\sigma^4 \\
&= \frac{(2n-1)\sigma^4}{n^2}
\end{aligned}$$

$$\begin{aligned}
S^2 &= \frac{\sigma^2}{n-1} Y \\
MSE(S^2) &= var(\hat{\sigma^2}_{ML}) + bias(S^2)^2 \\
&= \frac{\sigma^4}{(n-1)^2} var(Y) \\
&= \frac{2\sigma^4}{(n-1)}
\end{aligned}$$

Now we can compare these two values:

$$MSE(S^2) = \frac{2\sigma^4}{n-1} > \frac{2\sigma^4}{n} = \frac{2n\sigma^4}{n^2} > \frac{(2n-1)\sigma^4}{n^2} = MSE(\hat{\sigma^2}_{ML})$$

Problem 4

Q.
Let $X_1, ..., X_n$ be iid with pdf $f(x|\theta) = \frac{1}{2\theta}$ with $-\theta \leq x \leq \theta$. Find the best unbiased estimator of θ.

A.
Gauss: $\hat{\theta} = \max(|X_i|)$
Since $|X_i|$ are iid with pdf $f(x|\theta) = \frac{1}{\theta}$ with $0 \leq x \leq \theta$, we can see from problem 3 part b-1 that the expected value of $\hat{\theta}$ is $\frac{n}{n+1}\theta$ and it is biased. So we will change our estimator to:

$$T = \frac{n+1}{n}\max(|X_i|)$$

Expected value of this estimator is θ and thus it is unbiased.
This estimator is also complete because we can factorize the likelihood function into g and h:

$$\begin{aligned} L(x|\theta) &= \prod_{i=1}^{n} \frac{1}{2\theta} I(-\theta \leq X_i \ \leq \theta) \\ &= (\frac{1}{2\theta})^n I(\max(|X_i|) \leq \theta) \\ h(x_1, ..., x_n) = 1 \quad , \quad g(\theta, T) &= (\frac{1}{2\theta})^n I(\frac{n}{n+1}T) \leq \theta) \end{aligned}$$

Thus, by the Lehmann–Scheffé theorem, this estimator is the UMVUE estimator.

Problem 5

Q. Let $X_1, ..., X_n$ be iid from below distributions. Is there a function of θ which there exists an unbiased estimator whose variance attains Cramer-Rao lower bound?

(a) $f(x|\theta) = \theta x^{\theta-1} \quad 0 \leq x \leq 1, \theta > 0$

(b) $f(x|\theta) = \frac{\log(\theta)}{\theta-1} x^{\theta} \quad 0 \leq x \leq 1, \theta > 0$

A.

(a) First, we will write the likelihood function:

$$\begin{aligned} L(x|\theta) &= \prod_{i=1}^{n} \theta x_i^{\theta-1} \\ \ln L(x|\theta) &= \sum_{i=1}^{n} \ln\theta + (\theta-1)\ln(x_i) \\ &= n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln(x_i) \end{aligned}$$

The equality condition for CRLB is $\frac{\partial}{\partial\theta}L(x|\theta) = k(\theta)(T(X) - g(\theta))$. We can write $\frac{\partial}{\partial\theta}L(x|\theta)$ as follows:

$$\frac{\partial \ln L(x|\theta)}{\partial\theta} = -n(-\frac{1}{\theta} - \frac{1}{n}\sum_{i=1}^{n}\ln(x_i))$$

Where $k(\theta) = -n$, $g(\theta) = \frac{1}{\theta}$, and $T(X) = -\frac{1}{n}\sum_{i=1}^{n}\ln(x_i)$. So the function of θ we are looking for is $g(\theta)$ and the estimator of which whose variance attains CRLB is $T(X)$.

We can now check the expectation. First, we will calculate the distribution of $y = -ln(x)$:

$$\begin{aligned} f_Y(y|\theta) &= f_X(e^{-y}|\theta)|\frac{de^{-y}}{dy}| \\ &= \theta e^{-y(\theta-1)}|e^{-y}| \\ &= \theta e^{-y\theta} \end{aligned}$$

We can see that $y \sim Exp(\theta)$. So $\sum_{i=1}^{n} - \ln(x_i) \sim Erlang(n, \theta)$. So:

We can also check that this estimator is unbiased:

$$\begin{aligned} \mathrm{E}[T(X)] &= \mathrm{E}[\frac{1}{n}\sum_{i=1}^{n} - \ln(x_i)] \\ &= \frac{1}{n}\frac{n}{\theta} \\ &= \frac{1}{\theta} \end{aligned}$$

(b) Likelihood:

$$
\begin{aligned}
L(x|\theta) &= \prod_{i=1}^{n} \frac{\log(\theta)}{\theta - 1} x_i^{\theta} \\
\ln L(x|\theta) &= \sum_{i=1}^{n} \ln\log(\theta) - \ln(\theta - 1) + \theta \ln(x_i) \\
&= n \ln\log(\theta) - n\ln(\theta - 1) + \theta \sum_{i=1}^{n} \ln(x_i)
\end{aligned}
$$