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Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch die Interesse, welches *Gauss* und *Dirichlet* demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von *Euler* gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int\limits_0^\infty e^{-nx} x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\zeta(s) = \int\limits_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

Betrachtet man nun das Integral

$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

von $+\infty$ bis $+\infty$ positiv um ein Grössengebiet erstreckt, welches den Werth 0, aber keinen andern Unstetigkeitswerth der Function unter dem Integralzeichen im Innern enthält, so ergiebt sich dieses leicht gleich

$$(e^{-\pi si} - e^{\pi si}) \int\limits_0^\infty \frac{x^{s-1}\,dx}{e^x-1},$$

vorausgesetzt, dass in der vieldeutigen Function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ der Logarithmus von $-x$ so bestimmt worden ist, dass er für ein negatives x reell wird. Man hat daher

$$2\sin\pi s\,\Pi(s-1)\zeta(s) = i\int\limits_{\infty}^{\infty}\frac{(-x)^{s-1}\,dx}{e^x-1},$$

das Integral in der eben angegebenen Bedeutung verstanden.

Diese Gleichung giebt nun den Werth der Function $\zeta(s)$ für jedes beliebige complexe s und zeigt, dass sie einwerthig und für alle endlichen Werthe von s, ausser 1, endlich ist, so wie auch, dass sie verschwindet, wenn s gleich einer negativen geraden Zahl ist.

Wenn der reelle Theil von s negativ ist, kann das Integral, statt positiv um das angegebene Grössengebiet auch negativ um das Grössengebiet, welches sämmtliche übrigen complexen Grössen enthält, erstreckt werden, da das Integral durch Werthe mit unendlich grossem Modul dann unendlich klein ist. Im Innern dieses Grössengebiets aber wird die Function unter dem Integralzeichen nur unstetig, wenn x gleich einem ganzen Vielfachen von $\pm 2\pi i$ wird und das Integral ist daher gleich der Summe der Integrale negativ um diese Werthe genommen. Das Integral um den Werth $n2\pi i$ aber ist $= (-n2\pi i)^{s-1}(-2\pi i))$, man erhält daher

$$2\sin\pi s\,\Pi(s-1)\zeta(s) = (2\pi)^s\sum n^{s-1}((-i)^{s-1}+i^{s-1}),$$

also eine Relation zwischen $\zeta(s)$ und $\zeta(1-s)$, welche sich mit Benutzung bekannter Eigenschaften der Function Π auch so ausdrücken lässt:

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)$$

bleibt ungeändert, wenn s in $1-s$ verwandelt wird.

This property of the function induced me to introduce, in place of $\Pi(s-1)$, the integral $\Pi\left(\frac{s}{2}-1\right)$ into the general term of the series $\sum\frac{1}{n^s}$, whereby one obtains a very convenient expression for the function $\zeta(s)$. In fact

$$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}} = \int\limits_0^\infty e^{-nn\pi x}x^{\frac{s}{2}-1}\,dx,$$

thus, if one sets

$$\sum_1^\infty e^{-nn\pi x} = \psi(x)$$

then

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s) = \int\limits_0^\infty \psi(x)x^{\frac{s}{2}-1}\,dx,$$

or since

$$2\psi(x)+1 = x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right)+1\right), \text{ (Jacobi, Fund. S. 184)}$$

$$\begin{aligned}\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s) &= \int_1^\infty \psi(x)x^{\frac{s}{2}-1}\,dx + \int_1^\infty \psi\left(\frac{1}{x}\right)x^{\frac{s-3}{2}}\,dx \\ &\quad +\tfrac{1}{2}\int_0^1 \left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1}\right)dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}}\right)dx.\end{aligned}$$

I now set $s=\frac{1}{2}+ti$ and

$$\Pi\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) = \xi(t),$$

so that

$$\xi(t) = \tfrac{1}{2} - (tt+\tfrac{1}{4})\int_1^\infty \psi(x)x^{-\frac{3}{4}}\cos\left(\tfrac{1}{2}t\log x\right)dx$$

or, in addition,

$$\xi(t) = 4\int_1^\infty \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx}x^{-\frac{1}{4}}\cos\left(\tfrac{1}{2}t\log x\right)dx.$$

This function is finite for all finite values of t, and allows itself to be developed in powers of tt as a very rapidly converging series. Since, for a value of s whose real part is greater than 1, $\log\zeta(s) = -\sum\log(1-p^{-s})$ remains finite, and since the same holds for the logarithms of the other factors of $\xi(t)$, it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$. The number of roots of $\xi(t)=0$, whose real parts lie between 0 and T is approximately

$$= \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi};$$

because the integral $\int d\log\xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$) equal to $\left(T\log\frac{T}{2\pi}-T\right)i$; this integral however is equal to the number of roots of $\xi(t)=0$ lying within in this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

If one denotes by α all the roots of the equation $\xi(\alpha) = 0$, one can express $\log \xi(t)$ as

$$\sum \log\left(1 - \frac{tt}{\alpha\alpha}\right) + \log \xi(0);$$

for, since the density of the roots of the quantity t grows with t only as $\log \frac{t}{2\pi}$, it follows that this expression converges and becomes for an infinite t only infinite as $t \log t$; thus it differs from $\log \xi(t)$ by a function of tt, that for a finite t remains continuous and finite and, when divided by tt, becomes infinitely small for infinite t. This difference is consequently a constant, whose value can be determined through setting $t = 0$.

With the assistance of these methods, the number of prime numbers that are smaller than x can now be determined.

Let $F(x)$ be equal to this number when x is not exactly equal to a prime number; but let it be greater by $\frac{1}{2}$ when x is a prime number, so that, for any x at which there is a jump in the value in $F(x)$,

$$F(x) = \frac{F(x+0) + F(x-0)}{2}.$$

If in the identity

$$\log \zeta(s) = -\sum \log\left(1 - p^{-s}\right) = \sum p^{-s} + \tfrac{1}{2}\sum p^{-2s} + \tfrac{1}{3}\sum p^{-3s} + \cdots$$

one now replaces

$$p^{-s} \text{ by } s\int_p^\infty x^{-s-1}\,ds, \quad p^{-2s} \text{ by } s\int_{p^2}^\infty x^{-s-1}\,ds, \ldots,$$

one obtains

$$\frac{\log \zeta(s)}{s} = \int_1^\infty f(x) x^{-s-1}\,dx,$$

if one denotes

$$F(x) + \tfrac{1}{2}F(x^{\frac{1}{2}}) + \tfrac{1}{3}F(x^{\frac{1}{3}}) + \cdots$$

by $f(x)$.

This equation is valid for each complex value $a + bi$ of s for which $a > 1$. If, though, the equation

$$g(s) = \int_0^\infty h(x) x^{-s}\,d\log x$$

holds within this range, then, by making use of *Fourier*'s theorem, one can express the function h in terms of the function g. The equation decomposes, if $h(x)$ is real and

$$g(a+bi) = g_1(b) + ig_2(b),$$

into the two following:

$$g_1(b) = \int_0^\infty h(x)x^{-a}\cos(b\log x)\,d\log x,$$

$$ig_2(b) = -i\int_0^\infty h(x)x^{-a}\sin(b\log x)\,d\log x.$$

If one multiplies both equations with

$$(\cos(b\log y) + i\sin(b\log y))\,db$$

and integrates them from $-\infty$ to $+\infty$, then one obtains $\pi h(y)y^{-\alpha}$ on the right hand side in both, on account of *Fourier*'s theorems; thus, if one adds both equations and multiplies them by iy^α, one obtains

$$2\pi i h(y) = \int_{a-\infty i}^{a+\infty i} g(s)y^s\,ds,$$

where the integration is carried out so that the real part of s remains constant.

For a value of y at which there is a jump in the value of $h(y)$, the integral takes on the mean of the values of the function h on either side of the jump. From the manner in which the function f was defined, we see that it has the same property, and hence in full generality

$$f(y) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} \frac{\log\zeta(s)}{s}y^s\,ds.$$

One can substitute for $\log\zeta$ the expression

$$\frac{s}{2}\log\pi - \log(s-1) - \log\Pi\left(\frac{s}{2}\right) + \sum\nolimits^\alpha \log\left(1 + \frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right) + \log\xi(0)$$

found earlier; however the integrals of the individual terms of this expression do not converge, when extended to infinity, for which reason it is appropriate to convert the previous equation by means of integration by parts into

$$f(x) = -\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i} \frac{d\dfrac{\log\zeta(s)}{s}}{ds}x^s\,ds$$

Since

$$-\log\Pi\left(\frac{s}{2}\right)=\lim\left(\sum_{n=1}^{n=m}\log\left(1+\frac{s}{2n}\right)-\frac{s}{2}\log m\right),$$

for $m=\infty$ and therefore

$$-\frac{d\frac{1}{s}\log\Pi\left(\frac{s}{2}\right)}{ds}=\sum_{1}^{\infty}\frac{d\frac{1}{s}\log\left(1+\frac{s}{2n}\right)}{ds},$$

it then follows that all the terms of the expression for $f(x)$, with the exception of

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{1}{ss}\log\xi(0)x^s\,ds=\log\xi(0),$$

take the form

$$\pm\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{ds}x^s\,ds.$$

But now

$$\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{d\beta}=\frac{1}{(\beta-s)\beta},$$

and, if the real part of s is larger than the real part of β,

$$-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{x^s\,ds}{(\beta-s)\beta}=\frac{x^\beta}{\beta}=\int_{\infty}^{x}x^{\beta-1}\,dx,$$

or

$$=\int_{0}^{x}x^{\beta-1}\,dx,$$

depending on whether the real part of β is negative or positive. One has as a result

$$\begin{aligned}&\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{ds}x^s\,ds\\ =\quad&-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)x^s\,ds\\ =\quad&\int_{\infty}^{x}\frac{x^{\beta-1}}{\log x}\,dx+\text{ const.}\end{aligned}$$

in the first, and

$$= \int_0^x \frac{x^{\beta-1}}{\log x}\,dx + \text{ const.}$$

in the second case.

In the first case the constant of integration is determined if one lets the real part of β become infinitely negative; in the second case the integral from 0 to x takes on values separated by $2\pi i$, depending on whether the integration is taken through complex values with positive or negative argument, and becomes infinitely small, for the former path, when the coefficient of i in the value of β becomes infinitely positive, but for the latter, when this coefficient becomes infinitely negative. From this it is seen how on the left hand side $\log\left(1-\frac{s}{\beta}\right)$ is to be determined in order that the constants of integration disappear.

Through the insertion of these values in the expression for $f(x)$ one obtains

$$f(x) = Li(x) - \sum\nolimits^{\alpha}\left(Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)\right) + \int_x^{\infty} \frac{1}{x^2-1}\frac{dx}{x\log x} + \log\xi(0),$$

if in $\sum^{\alpha}$ one substitutes for α all positive roots (or roots having a positive real part) of the equation $\xi(\alpha) = 0$, ordered by their magnitude. It may easily be shown, by means of a more thorough discussion of the function ξ, that with this ordering of terms the value of the series

$$\sum\left(Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)\right)\log x$$

agrees with the limiting value to which

$$\frac{1}{2\pi i}\int_{a-bi}^{a+bi} \frac{d\frac{1}{s}\sum\log\left(1+\frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right)}{ds}x^s\,ds$$

converges as the quantity b increases without bound; however when re-ordered it can take on any arbitrary real value.

From $f(x)$ one obtains $F(x)$ by inversion of the relation

$$f(x) = \sum\frac{1}{n}F\left(x^{\frac{1}{n}}\right),$$

to obtain the equation

$$F(x) = \sum(-1)^{\mu}\frac{1}{m}f\left(x^{\frac{1}{m}}\right),$$

in which one substitutes for m the series consisting of those natural numbers that are not divisible by any square other than 1, and in which μ denotes the number of prime factors of m.

If one restricts $\sum^\alpha$ to a finite number of terms, then the derivative of the expression for $f(x)$ or, up to a part diminishing very rapidly with growing x,

$$\frac{1}{\log x} - 2\sum\nolimits^{\alpha} \frac{\cos(\alpha \log x) x^{-\frac{1}{2}}}{\log x}$$

gives an approximating expression for the density of the prime number + half the density of the squares of the prime numbers + a third of the density of the cubes of the prime numbers etc. at the magnitude x.

The known approximating expression $F(x) = Li(x)$ is therefore valid up to quantities of the order $x^{\frac{1}{2}}$ and gives somewhat too large a value; because the non-periodic terms in the expression for $F(x)$ are, apart from quantities that do not grow infinite with x:

$$Li(x) - \tfrac{1}{2}Li(x^{\frac{1}{2}}) - \tfrac{1}{3}Li(x^{\frac{1}{3}}) - \tfrac{1}{5}Li(x^{\frac{1}{5}}) + \tfrac{1}{6}Li(x^{\frac{1}{6}}) - \tfrac{1}{7}Li(x^{\frac{1}{7}}) + \cdots$$

Indeed, in the comparison of $Li(x)$ with the number of prime numbers less than x, undertaken by *Gauss* and *Goldschmidt* and carried through up to $x =$ three million, this number has shown itself out to be, in the first hundred thousand, always less than $Li(x)$; in fact the difference grows, with many fluctuations, gradually with x. But also the increase and decrease in the density of the primes from place to place that is dependent on the periodic terms has already excited attention, without however any law governing this behaviour having been observed. In any future count it would be interesting to keep track of the influence of the individual periodic terms in the expression for the density of the prime numbers. A more regular behaviour than that of $F(x)$ would be exhibited by the function $f(x)$, which already in the first hundred is seen very distinctly to agree on average with $Li(x) + \log \xi(0)$.