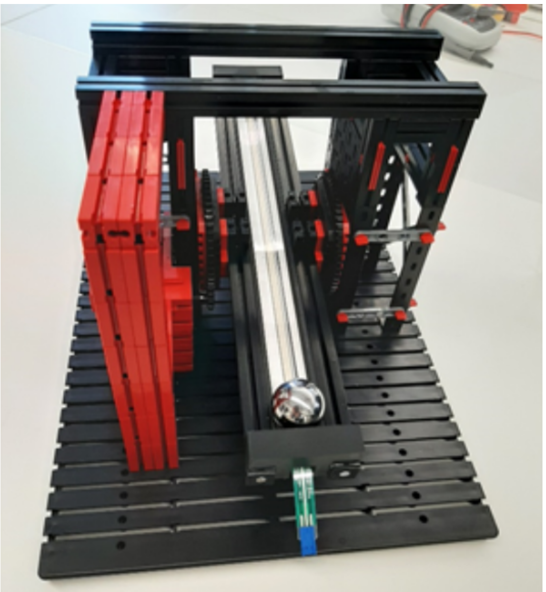
Mathematical Modelling and State-Space Control of "Ball-on-Beam" System

In this document a mathematical model of the "ball-on-beam" system illustrated below is derived and expressed in form of nonlinear state equations. To this end, Lagrange's approach for modelling mechanical systems is applied. Later different methods to control this model to balance the ball on the beam is applied.

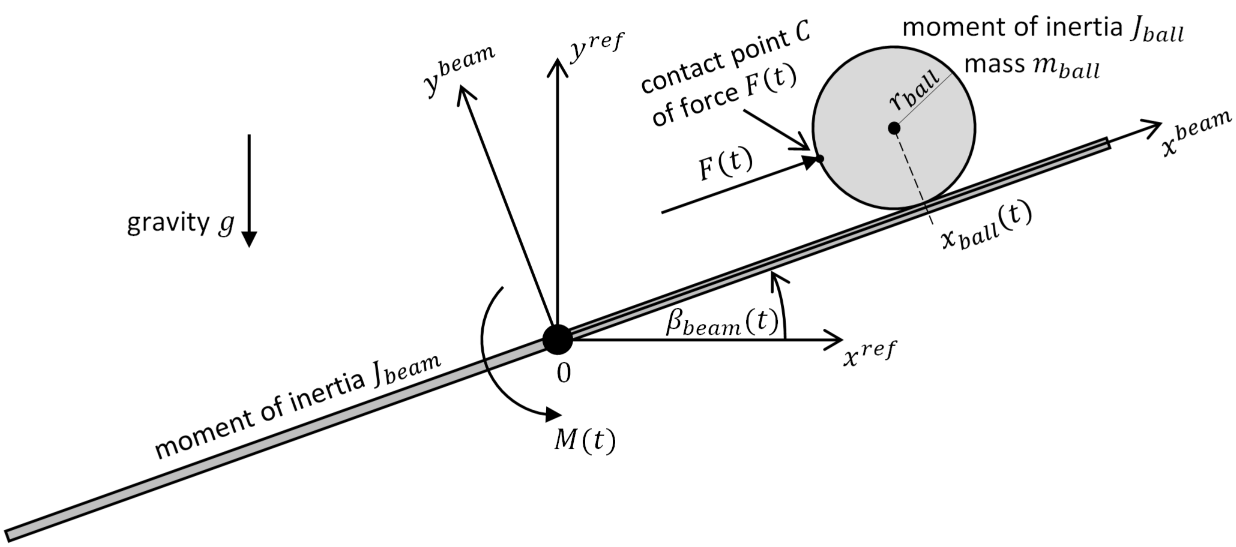


Definitions and Variable Declarations

```
clc;                                        % clear display
clearvars;                                  % reset variables
```

System description and definition of important quantities

The following schematic diagram defines coordinate systems and counting directions of important quantities.



In particular, the figure introduces the fixed Cartesian reference frame $(x^{\text{ref}}, y^{\text{ref}})$ and the beam-fixed coordinate system $(x^{\text{beam}}, y^{\text{beam}})$ which is rotated by the beam angle $\beta_{\text{beam}}(t)$ with respect to the reference frame. The beam is pivot-mounted at the common origin of these coordinate systems such that it can freely rotate around the z-axis. By means of an electric motor, which is of no further interest, the torque $M(t)$ can be applied to the beam, i.e. $M(t)$ represents the control input $u(t)$. A force $F(t)$ acting on the ball must be taken into account as disturbance input $z(t)$. It is assumed that the ball rolls on the beam without slipping at all times. All other friction forces and/or torques are neglected. The measured variables are the beam angle $\beta_{\text{beam}}(t)$ and the ball position on the beam $x_{\text{ball}}(t)$, which corresponds to the x-value of the ball's center of mass in beam-coordinates. The meaning of some other parameters and quantities of interest is described by the comments in the following variable declaration.

Declare symbolic variables

```
syms x_ball(t)                          % ball position on beam (in beam coordinate system)
syms beta_beam(t)                       % beam angle
syms M(t)                               % motor torque acting on beam
syms F(t)                               % disturbance force acting on ball
syms r_Ball                             % radius of ball
syms J_Ball                             % moment of inertia of ball with respect to its symmetr
syms m_Ball                             % mass of ball
syms G                                  % gravity of earth
syms J_Beam                             % moment of inertia of beam with respect to its fixed a
syms q1(t) q2(t)                        % generalized coordinates
syms x1(t) x2(t) x3(t) x4(t) u(t) z(t)  % state variables x, control input u, disturbance input
syms x1dot x2dot x3dot x4dot            % auxiliary variables needed to express the state diffe
```

Specify Generalized Coordinates

It is obvious that the position and orientation of the ball and the beam are completely defined if both x_{ball} and β_{beam} are known. Thus, the system has two degrees of freedom and the two measured variables can be defined as generalized coordinates:

```
x_ball = q1
```

x_ball(t) = $q_1(t)$

```
beta_beam = q2
```

beta_beam(t) = $q_2(t)$

Determine Total Kinetic Energy of the System

The "ball-on-beam" system consists of two rigid bodies: the ball and the beam. Consequently, the total kinetic energy can be expressed as the sum of the kinetic energy of the ball and the kinetic energy of the beam.

Kinetic energy of the ball

The overall motion of the ball is rather complex, since it simultaneously performs a rotational and a translational movement. With respect to its center of mass, however, the kinetic energy of any rigid body can still be stated as a simple sum of a purely translational and a purely rotational movement in relation to a fixed and non-moving reference coordinate system. Thus, the total kinetic energy of the ball T_{ball} can be calculated as

$$T_{\text{ball}} = \frac{1}{2} J_{\text{ball}} |\omega_{\text{ball}}^{\text{ref}}|^2 + \frac{1}{2} m_{\text{ball}} |v_{\text{ball}}^{\text{ref}}|^2$$

with

$|\omega_{\text{ball}}^{\text{ref}}|$: absolute value of angular ball speed around its center of mass with respect to fixed reference coordinate system

$|v_{\text{ball}}^{\text{ref}}|$: absolute value of translational speed at center of mass with respect to fixed reference coordinate system

Calculation of the translational speed at the center of mass $|v_{\text{ball}}^{\text{ref}}|$

Position of the ball's center of mass in the fixed Cartesian reference frame as a function of the generalized coordinates:

```
xref_ball(t) =x_ball(t)*cos(beta_beam(t)) - r_Ball*sin(beta_beam(t))
```

xref_ball(t) = $\cos(q_2(t))\, q_1(t) - r_{\text{Ball}} \sin(q_2(t))$

```
yref_ball(t) =x_ball(t) * sin(beta_beam(t)) + r_Ball * cos(beta_beam(t))
```

yref_ball(t) = $r_{\text{Ball}} \cos(q_2(t)) + \sin(q_2(t))\, q_1(t)$

Absolute value of translational ball speed as a function of $q_{1/2}$ and $\dot{q}_{1/2}$ with respect to the fixed reference coordinate system:

```
vref_ball(t) = simplify(sqrt(diff(xref_ball(t),t)^2+diff(yref_ball(t),t)^2))
```

vref_ball(t) =

$$\sqrt{{r_{\text{Ball}}}^2 \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2 + q_1(t)^2 \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2 + \left(\frac{\partial}{\partial t}\, q_1(t)\right)^2 - 2\, r_{\text{Ball}} \frac{\partial}{\partial t}\, q_2(t) \frac{\partial}{\partial t}\, q_1(t)}$$

Calculation of the angular speed around the center of mass $\omega_{\text{ball}}^{\text{ref}}$

Applying the summation rule for angular velocities (see e.g. [1]), the total angular speed of the ball $\underline{\omega}_{\text{ball}}^{\text{ref}}$ can be stated as the sum of the angular speed of the ball with respect to the beam $\underline{\omega}_{\text{ball}}^{\text{beam}}$ and the angular speed of the beam with respect to the fixed reference coordinate sytem $\underline{\omega}_{\text{beam}}^{\text{ref}}$, i.e. $\underline{\omega}_{\text{ball}}^{\text{ref}} = \underline{\omega}_{\text{ball}}^{\text{beam}} + \underline{\omega}_{\text{beam}}^{\text{ref}}$. Since in case of the "ball-on-beam" system all angular speed vectors are parallel to the z-axis, this vector equation can be simplified to the scalar relationship $\omega_{\text{ball}}^{\text{ref}} = \omega_{\text{ball}}^{\text{beam}} + \omega_{\text{beam}}^{\text{ref}}$ where

```
omegaref_beam(t) = diff(q2(t),t)
```

omegaref_beam(t) =

$$\frac{\partial}{\partial t}\, q_2(t)$$

and $\underline{\omega}_{\text{ball}}^{\text{beam}}$ follows from the ideal rolling conditions as

```
omegabeam_ball(t) = -diff(x_ball,t)/r_Ball
```

```
omegabeam_ball(t) =
```

$$-\frac{\frac{\partial}{\partial t}\, q_1(t)}{r_{\text{Ball}}}$$

Note that there is a negative sign in the later equation. It is caused by the fact that a negative rotation of the ball against the "right-hand-rule" is needed to move the system into positive x^{beam} direction (see definition of coordinate systems and counting direction in the figure above).

Summing both terms up yields

```
omegaref_ball(t) = omegaref_beam(t) + omegabeam_ball(t)
```

```
omegaref_ball(t) =
```

$$\frac{\partial}{\partial t}\, q_2(t) - \frac{\frac{\partial}{\partial t}\, q_1(t)}{r_{\text{Ball}}}$$

Calculation of the kinetic energy of the ball T_{ball}

Using the above formula for the kinetic energy expressed with respect to the mass center point finally yields

```
T_ball(t) = simplify(1/2*m_Ball*vref_ball(t)^2+1/2*J_Ball*omegaref_ball(t)^2)
```

```
T_ball(t) =
```

$$\frac{J_{\text{Ball}} \left(\frac{\partial}{\partial t}\, q_2(t) - \frac{\frac{\partial}{\partial t}\, q_1(t)}{r_{\text{Ball}}}\right)^2}{2} + \frac{m_{\text{Ball}} \left(\left(\frac{\partial}{\partial t}\, q_1(t)\right)^2 + r_{\text{Ball}}{}^2 \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2 + q_1(t)^2 \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2 - 2\, r_{\text{Ball}} \frac{\partial}{\partial t}\, q_2(t) \frac{\partial}{\partial}\right.}{2}$$

Kinetic energy of the beam

The beam performs a purely rotational movement around the fixed axis of rotation at the center of the coordinate system (see figure above). Thus, using J_{beam} as the moment of inertia of the beam with respect to that axis, the total kinetic energy of the beam is

```
T_beam(t) = 1/2*J_Beam*omegaref_beam^2
```

```
T_beam(t) =
```

$$\frac{J_{\text{Beam}} \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2}{2}$$

Total kinetic energy of the system

The total kinetic energy of the system results from the sum of the kinetic energy of the ball and of the beam as

```
T(t) = T_ball(t) + T_beam(t)
```

T(t) =

$$\frac{J_{\mathrm{Ball}}\left(\frac{\partial}{\partial t}\, q_2(t) - \frac{\frac{\partial}{\partial t}\, q_1(t)}{r_{\mathrm{Ball}}}\right)^2}{2} + \frac{m_{\mathrm{Ball}}\left(\left(\frac{\partial}{\partial t}\, q_1(t)\right)^2 + r_{\mathrm{Ball}}{}^2\,\sigma_1 + q_1(t)^2\,\sigma_1 - 2\, r_{\mathrm{Ball}}\frac{\partial}{\partial t}\, q_2(t)\frac{\partial}{\partial t}\, q_1(t)\right)}{2} + \frac{J_{\mathrm{Beam}}\,\sigma_1}{2}$$

where

$$\sigma_1 = \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2$$

Determine Total Potential Energy of the System

The impact of gravity can be expressed using a potential energy term. As the beam's center of mass does not move, however, the total potential energy of the system is solely determined by the ball, i.e. $V = V_{\mathrm{ball}}$. Moreover, the ball's potential energy V_{ball} only depends on its y-coordinate in the fixed Cartesian reference coordinate system. Assuming without loss of generality that $V = V_{\mathrm{ball}} = 0$ would hold if the ball's center of mass was at the origin of the coordinate system, V can be expressed as

```
V(t) = G*m_Ball*yref_ball(t)
```

V(t) = $G\, m_{\mathrm{Ball}}\,(r_{\mathrm{Ball}}\cos(q_2(t)) + \sin(q_2(t))\, q_1(t))$

Lagrangian

The Lagrangian

$$L = T - V$$

can now easily be stated as a function of $q_{1/2}$ and $\dot{q}_{1/2}$:

```
L(t) = simplify(T(t) - V(t))
```

L(t) =

$$\frac{J_{\mathrm{Ball}}\left(\frac{\partial}{\partial t}\, q_2(t) - \frac{\frac{\partial}{\partial t}\, q_1(t)}{r_{\mathrm{Ball}}}\right)^2}{2} + \frac{m_{\mathrm{Ball}}\left(\left(\frac{\partial}{\partial t}\, q_1(t)\right)^2 + r_{\mathrm{Ball}}{}^2\,\sigma_1 + q_1(t)^2\,\sigma_1 - 2\, r_{\mathrm{Ball}}\frac{\partial}{\partial t}\, q_2(t)\frac{\partial}{\partial t}\, q_1(t)\right)}{2} + \frac{J_{\mathrm{Beam}}\,\sigma_1}{2}$$

where

$$\sigma_1 = \left(\frac{\partial}{\partial t}\, q_2(t)\right)^2$$

Non-conservative Generalized Forces

In general, the non-conservative generalized forces $Q_{i,\mathrm{nc}}$ of a system with N degrees of freedom are given as [1]

$$Q_{i,\mathrm{nc}} = \left(\sum_{k=1}^{K} \underline{F}_{k,\mathrm{nc}}^{T} \frac{\partial \underline{v}_k}{\partial \dot{q}_i} \right) + \left(\sum_{l=1}^{L} \underline{\underline{M}}_{l,\mathrm{nc}}^{T} \frac{\partial \underline{\omega}_l}{\partial \dot{q}_i} \right), \; i = 1, 2, \ldots, N$$

where $\underline{F}_{k,\mathrm{nc}}$, $k = 1, 2, \ldots, K$, and $\underline{\underline{M}}_{l,\mathrm{nc}}$, $l = 1, 2, \ldots, L$, represent all non-constraint and non-conservative force and torque vectors acting on the system and $\underline{v}_k$, $k = 1, 2, \ldots, K$, as well as $\underline{\omega}_l$, $l = 1, 2, \ldots, L$, are the speed and angular speed vectors of the corresponding contact points.

Applied to the "ball-and-beam" system, the number of degrees of freedom is $N = 2$ and the external force $F(t)$ as well as the motor torque $M(t)$ (see figure above) represent the only non-conservative quantities acting on the mechanics. Thus, expressed in reference coordinates the formula above simplifies to the two non-conservative generalized force terms

$$Q_{1,\mathrm{nc}} = \left(\underline{F}^{\mathrm{ref}}\right)^{T} \frac{\partial \underline{v}_C^{\mathrm{ref}}}{\partial \dot{q}_1} + \left(\underline{\underline{M}}^{\mathrm{ref}}\right)^{T} \frac{\partial \underline{\omega}_{\mathrm{beam}}^{\mathrm{ref}}}{\partial \dot{q}_1}$$

$$Q_{2,\mathrm{nc}} = \left(\underline{F}^{\mathrm{ref}}\right)^{T} \frac{\partial \underline{v}_C^{\mathrm{ref}}}{\partial \dot{q}_2} + \left(\underline{\underline{M}}^{\mathrm{ref}}\right)^{T} \frac{\partial \underline{\omega}_{\mathrm{beam}}^{\mathrm{ref}}}{\partial \dot{q}_2}$$

Force vector $\underline{F}^{\mathrm{ref}}$ and corresponding speed vector of the contact point $\underline{v}_C^{\mathrm{ref}}$ in reference coordinates

In beam coordinates the force $F(t)$ acts in positive x-direction. Consequently, with respect to the fixed reference coordinate system, the force vector can be expressed as

```
Fref_vec(t) = F(t)*[cos(q2);sin(q2);0]
```

Fref_vec(t) =

$$\begin{pmatrix} \cos(q_2(t))\, F(t) \\ \sin(q_2(t))\, F(t) \\ 0 \end{pmatrix}$$

The position of the corresponding contact point C (see figure) of the force $F(t)$ in beam coordinates is

```
xbeam_C(t) = x_ball(t)-r_Ball
```

xbeam_C(t) = $q_1(t) - r_{\mathrm{Ball}}$

```
ybeam_C = r_Ball
```

ybeam_C = r_{Ball}

Transformed into the fixed Cartesian reference frame, the contact point C is described by

```
xref_C(t) = xbeam_C(t)*cos(beta_beam(t))-r_Ball*sin(beta_beam(t))
```

```
xref_C(t) =
```
$$-r_{\text{Ball}}\sin(q_2(t))-\cos(q_2(t))\,(r_{\text{Ball}}-q_1(t))$$

```
yref_C(t) = xbeam_C(t)*sin(beta_beam(t))+r_Ball*cos(beta_beam(t))
```

```
yref_C(t) =
```
$$r_{\text{Ball}}\cos(q_2(t))-\sin(q_2(t))\,(r_{\text{Ball}}-q_1(t))$$

Calculating the time derivative yields the speed vector $\underline{v}_C$ of the contact point C in reference coordinates:

```
vref_Cvec(t)=[simplify(diff(xref_C(t),t));simplify(diff(yref_C(t),t));0]
```

```
vref_Cvec(t) =
```
$$\begin{pmatrix} \cos(q_2(t))\frac{\partial}{\partial t}\,q_1(t)+\sin(q_2(t))\,(r_{\text{Ball}}-q_1(t))\frac{\partial}{\partial t}\,q_2(t)-r_{\text{Ball}}\cos(q_2(t))\frac{\partial}{\partial t}\,q_2(t) \\ \sin(q_2(t))\frac{\partial}{\partial t}\,q_1(t)-\cos(q_2(t))\,(r_{\text{Ball}}-q_1(t))\frac{\partial}{\partial t}\,q_2(t)-r_{\text{Ball}}\sin(q_2(t))\frac{\partial}{\partial t}\,q_2(t) \\ 0 \end{pmatrix}$$

Torque vector $\underline{\underline{M}}^{\text{ref}}$ and corresponding angular speed vector $\underline{\omega}^{\text{ref}}_{\text{beam}}$ in reference coordinates

The motor torque $M(t)$ acts around the z-axis which represents the fixed axis of rotation for the beam. Thus, we can state the associated torque vector according to

```
Mref_vec(t) = M(t)*[0;0;1]
```

```
Mref_vec(t) =
```
$$\begin{pmatrix} 0 \\ 0 \\ M(t) \end{pmatrix}$$

and the corresponding angular speed vector as

```
omegaref_beamvec(t) = omegaref_beam(t)*[0;0;1]
```

```
omegaref_beamvec(t) =
```
$$\begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial t}\,q_2(t) \end{pmatrix}$$

with respect to the Cartesian reference frame and considering the counting directions specified in the figure above.

Non-conservative generalized force $Q_{1,\text{nc}}$ associated to q_1

Evaluating

$$Q_{1,\text{nc}}=\left(\underline{F}^{\text{ref}}\right)^T\frac{\partial \underline{v}^{\text{ref}}_C}{\partial \dot{q}_1}+\left(\underline{\underline{M}}^{\text{ref}}\right)^T\frac{\partial \underline{\omega}^{\text{ref}}_{\text{beam}}}{\partial \dot{q}_1}$$

results in

```
Q1_nc(t) = simplify(transpose(Fref_vec(t))*(diff(vref_Cvec(t),diff(q1(t),t)))+transpose(Mref_ve
```

Q1_nc(t) = $F(t)$

Non-conservative generalized force $Q_{2,\mathrm{nc}}$ associated to q_2

Evaluating

$$Q_{2,\mathrm{nc}} = \left(\underline{F}^{\mathrm{ref}}\right)^T \frac{\partial \underline{v}_C^{\mathrm{ref}}}{\partial \dot{q}_2} + \left(\underline{M}^{\mathrm{ref}}\right)^T \frac{\partial \underline{\omega}_{\mathrm{beam}}^{\mathrm{ref}}}{\partial \dot{q}_2}$$

results in

```
Q2_nc(t) = simplify(transpose(Fref_vec(t))*(diff(vref_Cvec(t),diff(q2(t),t)))+transpose(Mref_ve
```

Q2_nc(t) = $M(t) - r_{\mathrm{Ball}}\, F(t)$

Equations of Motion in Generalized Coordinates

In general, using the Lagrangian L and the non-conservative generalized forces $Q_{i,\mathrm{nc}}$, the equations of motion in the generalized coordinates are obtained from the N Lagrange's equations (of 2nd kind)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_{i,\mathrm{nc}} \;, \quad i = 1, 2, \ldots, N$$

Applied to the "ball-on-beam" system with two degrees of freedom ($N = 2$), the evaluation of

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_{1,\mathrm{nc}}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = Q_{2,\mathrm{nc}}$$

results in the following two differential equations in the generalized coordinates:

```
deq1 = simplify(diff(diff(L(t),diff(q1(t),t)),t) - diff(L(t),q1(t))) == Q1_nc(t)
```

deq1 =

$$G\, m_{\mathrm{Ball}} \sin(q_2(t)) - \frac{m_{\mathrm{Ball}} \left(2\, r_{\mathrm{Ball}} \frac{\partial^2}{\partial t^2}\, q_2(t) - 2 \frac{\partial^2}{\partial t^2}\, q_1(t)\right)}{2} + \frac{J_{\mathrm{Ball}} \left(\frac{\frac{\partial^2}{\partial t^2}\, q_1(t)}{r_{\mathrm{Ball}}} - \frac{\partial^2}{\partial t^2}\, q_2(t)\right)}{r_{\mathrm{Ball}}} - m_{\mathrm{Ball}}\, q_1(t) \left(\frac{\partial}{\partial t}\, q_2\right.$$

```
deq2 = simplify(diff(diff(L(t),diff(q2(t),t)),t) - diff(L(t),q2(t))) == Q2_nc(t)
```

deq2 =

$$J_{\mathrm{Beam}}\,\sigma_1 + J_{\mathrm{Ball}}\,\sigma_1 + m_{\mathrm{Ball}}\,q_1(t)^2\,\sigma_1 - m_{\mathrm{Ball}}\,r_{\mathrm{Ball}}\,\sigma_2 - \frac{J_{\mathrm{Ball}}\,\sigma_2}{r_{\mathrm{Ball}}} + m_{\mathrm{Ball}}\,{r_{\mathrm{Ball}}}^2\,\sigma_1 + 2\,m_{\mathrm{Ball}}\,q_1(t)\,\frac{\partial}{\partial t}\,q_2(t)\,\frac{\partial}{\partial t}\,q_1(t) +$$

where

$$\sigma_1 = \frac{\partial^2}{\partial t^2}\,q_2(t)$$

$$\sigma_2 = \frac{\partial^2}{\partial t^2}\,q_1(t)$$

Derive (Nonlinear) State Equations

Define state variables

In order to obtain a state space description of the "ball-on-beam" system from the equations of motion, the generalized coordinates and their first-order time derivatives are introduced as state variables:

```
x1_def = q1(t) == x1(t)                 % x1 --> ball position on beam
```

x1_def = $q_1(t) = x_1(t)$

```
x2_def = diff(q1(t),t) == x2(t)         % x2 --> translational ball speed relative to beam
```

x2_def =

$$\frac{\partial}{\partial t}\,q_1(t) = x_2(t)$$

```
x3_def = q2(t) == x3(t)                 % x3 --> beam angle
```

x3_def = $q_2(t) = x_3(t)$

```
x4_def = diff(q2(t),t) == x4(t)         % x4 --> angular speed of beam
```

x4_def =

$$\frac{\partial}{\partial t}\,q_2(t) = x_4(t)$$

Define first-order time derivatives of the states

In order to solve automatically for the first time derivatives of the states, the later must also formally be introduced as variables:

```
x1dot_def = x2 == x1dot                        % x1_dot --> x2
```

x1dot_def(t) = $x_2(t) = \mathrm{x1dot}$

```
x2dot_def = diff(q1(t),t,2) == x2dot           % x2_dot --> 2nd time derivative of q_1
```

x2dot_def =

$$\frac{\partial^2}{\partial t^2} q_1(t) = \mathrm{x2dot}$$

```
x3dot_def = x4 == x3dot                    % x3_dot --> x4
```

x3dot_def(t) = $x_4(t) = \mathrm{x3dot}$

```
x4dot_def = diff(q2(t),t,2) == x4dot       % x4_dot --> 2nd time derivative of q_2
```

x4dot_def =

$$\frac{\partial^2}{\partial t^2} q_2(t) = \mathrm{x4dot}$$

Define control input and disturbance input

The motor torque $M(t)$ is defined as control input $u(t)$:

```
udef = M(t) == u(t)
```

udef = $M(t) = u(t)$

The force $F(t)$ is interpreted as disturbance input $z(t)$:

```
zdef = F(t) == z(t)
```

zdef = $F(t) = z(t)$

Substitute states, control input and disturbance input in equations of motion

Note that the substitutions of the states must be applied in the correct order:

1. Substitute the acceleration variables $\ddot{q}_1$ and $\ddot{q}_2$ first.
2. Substitute the speed variables $\dot{q}_1$ and $\dot{q}_2$ second.
3. Substitute the position variables q_1 and q_2 last.

```
% Substitute state variables in equations of motion
deq1_in_xuz = deq1;
deq2_in_xuz = deq2;
% substitute q1_2dot --> x2dot
deq1_in_xuz = subs(deq1_in_xuz,lhs(x2dot_def),rhs(x2dot_def));
deq2_in_xuz = subs(deq2_in_xuz,lhs(x2dot_def),rhs(x2dot_def));
% substitute q1_dot --> x2
deq1_in_xuz = subs(deq1_in_xuz,lhs(x2_def),rhs(x2_def));
deq2_in_xuz = subs(deq2_in_xuz,lhs(x2_def),rhs(x2_def));
% substitute q1 --> x1
deq1_in_xuz = subs(deq1_in_xuz,lhs(x1_def),rhs(x1_def));
deq2_in_xuz = subs(deq2_in_xuz,lhs(x1_def),rhs(x1_def));
% substitute q2_2dot --> x4dot
deq1_in_xuz = subs(deq1_in_xuz,lhs(x4dot_def),rhs(x4dot_def));
deq2_in_xuz = subs(deq2_in_xuz,lhs(x4dot_def),rhs(x4dot_def));
% substitute q2_dot --> x4
```

```
deq1_in_xuz = subs(deq1_in_xuz,lhs(x4_def),rhs(x4_def));
deq2_in_xuz = subs(deq2_in_xuz,lhs(x4_def),rhs(x4_def));
% substitute q2 --> x3
deq1_in_xuz = subs(deq1_in_xuz,lhs(x3_def),rhs(x3_def));
deq2_in_xuz = subs(deq2_in_xuz,lhs(x3_def),rhs(x3_def));
% Substitute control input
deq1_in_xuz = subs(deq1_in_xuz,lhs(udef),rhs(udef));
deq2_in_xuz = subs(deq2_in_xuz,lhs(udef),rhs(udef));
% Substitute disturbance input
deq1_in_xuz = subs(deq1_in_xuz,lhs(zdef),rhs(zdef));
deq2_in_xuz = subs(deq2_in_xuz,lhs(zdef),rhs(zdef));
```

Final equations of motions after substitutions:

```
deq1_in_xuz
```

```
deq1_in_xuz =
```

$$\frac{m_{\mathrm{Ball}}\ (2\ \mathrm{x2dot} - 2\ r_{\mathrm{Ball}}\ \mathrm{x4dot})}{2} - m_{\mathrm{Ball}}\, x_1(t)\, x_4(t)^2 + G\, m_{\mathrm{Ball}} \sin(x_3(t)) - \frac{J_{\mathrm{Ball}} \left(\mathrm{x4dot} - \frac{\mathrm{x2dot}}{r_{\mathrm{Ball}}}\right)}{r_{\mathrm{Ball}}} = z(t)$$

```
deq2_in_xuz
```

```
deq2_in_xuz =
```

$$J_{\mathrm{Beam}}\ \mathrm{x4dot} + J_{\mathrm{Ball}}\ \mathrm{x4dot} + m_{\mathrm{Ball}}\, {r_{\mathrm{Ball}}}^2\ \mathrm{x4dot} + m_{\mathrm{Ball}}\ \mathrm{x4dot}\ x_1(t)^2 - m_{\mathrm{Ball}}\, r_{\mathrm{Ball}}\ \mathrm{x2dot} - \frac{J_{\mathrm{Ball}}\ \mathrm{x2dot}}{r_{\mathrm{Ball}}} + G\, m_{\mathrm{Ball}}\ \mathrm{c}$$

Solve for time derivatives of the states and determine state differential equation

These two resulting equations of motion and the two trivial state differential equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_3 = x_4$$

together form a system of four equations. Solving this system of equations for the time derivatives $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and $\dot{x}_4$ results in the vector function $\underline{f}(\underline{x}, u, z)$ which defines the right-hand side of the state differential equation $\underline{\dot{x}} = \underline{f}(\underline{x}, u, z)$:

```
% solve system of equations for first-order time derivatives of the states
aux = solve([x1dot_def,deq1_in_xuz,x3dot_def,deq2_in_xuz],[x1dot,x2dot,x3dot,x4dot]);
% combine solutions to the vector function f(x,u,z)
f = simplify([aux.x1dot;aux.x2dot;aux.x3dot;aux.x4dot])
```

```
f =
```

$$\left(\frac{r_{\mathrm{Ball}}\ \left(J_{\mathrm{Ball}}\, u(t) + J_{\mathrm{Beam}}\, r_{\mathrm{Ball}}\, z(t) + m_{\mathrm{Ball}}\, r_{\mathrm{Ball}}{}^2\, u(t) + m_{\mathrm{Ball}}{}^2\, r_{\mathrm{Ball}}\, x_1(t)^3\, x_4(t)^2 + m_{\mathrm{Ball}}{}^2\, r_{\mathrm{Ball}}{}^3\, x_1(t)\, x_4(t)^2 + m_{\mathrm{Ball}}\, r\right.}{}\right.$$

where

$$\sigma_1 = m_{\mathrm{Ball}}\, x_1(t)^2 + J_{\mathrm{Beam}}$$

Define measured signals as output variables

The ball position on the beam x_{ball} and the beam angle β_{beam} are measured such that these two signals are specified as outputs y_1 and y_2 of the system. Since the states x_1 and x_3 are defined according to

$$x_1 = q_1 = x_{\mathrm{ball}}$$
$$x_3 = q_2 = \beta_{\mathrm{beam}}$$

the output variables can easily be expressed as functions of the states:

$$y_1 = x_1$$
$$y_2 = x_3$$

With the vector function

```
h = [x1(t);x3(t)]
```

h =

$$\begin{pmatrix} x_1(t) \\ x_3(t) \end{pmatrix}$$

the output equation can alternatively be written in vector form as $\underline{y} = \underline{h}(\underline{x}, u, z) = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$.

Summary: Final State Equations of the Ball-on-beam System

```
displayFormula('diff(x(t),t)==f')
```

$$\frac{\partial}{\partial t}\, x(t) = \left(\frac{r_{\mathrm{Ball}}\ \left(J_{\mathrm{Ball}}\, u(t) + J_{\mathrm{Beam}}\, r_{\mathrm{Ball}}\, z(t) + m_{\mathrm{Ball}}\, {r_{\mathrm{Ball}}}^2\, u(t) + {m_{\mathrm{Ball}}}^2\, r_{\mathrm{Ball}}\, x_1(t)^3\, x_4(t)^2 + {m_{\mathrm{Ball}}}^2\, {r_{\mathrm{Ball}}}^3\, x_1(t)\, x_4(t)\right.}{} \right.$$

where

$$\sigma_1 = m_{\mathrm{Ball}}\, x_1(t)^2 + J_{\mathrm{Beam}}$$

```
displayFormula('y(t)==h')
```

$$y(t) = \begin{pmatrix} x_1(t) \\ x_3(t) \end{pmatrix}$$

RHS Functions without Explicit Statement of Time Dependence

```
syms x1_ x2_ x3_ x4_ u_ z_                    % declare auxiliary variables needed for substituti
f_wo_t = subs(f,[x1(t),x2(t),x3(t),x4(t),u(t),z(t)],[x1_,x2_,x3_,x4_,u_,z_])
```

f_wo_t =

$$\left(\frac{r_{\mathrm{Ball}}\ \left(J_{\mathrm{Ball}}\, \mathrm{u\_} + m_{\mathrm{Ball}}\, {r_{\mathrm{Ball}}}^2\, \mathrm{u\_} + J_{\mathrm{Beam}}\, r_{\mathrm{Ball}}\, \mathrm{z\_} + {m_{\mathrm{Ball}}}^2\, r_{\mathrm{Ball}}\, {x_{1\_}}^3\, {x_{4\_}}^2 + {m_{\mathrm{Ball}}}^2\, {r_{\mathrm{Ball}}}^3\, x_{1\_}\, {x_{4\_}}^2 + m_{\mathrm{Ball}}\, r_{\mathrm{Ball}}\, {x_{1\_}}^2\, \mathrm{z\_}\right.}{} \right.$$

where

$$\sigma_1 = m_{\mathrm{Ball}}\, {x_{1\_}}^2 + J_{\mathrm{Beam}}$$

```
h_wo_t = subs(h,[x1(t),x2(t),x3(t),x4(t),u(t),z(t)],[x1_,x2_,x3_,x4_,u_,z_])
```

h_wo_t =

$$\begin{pmatrix} x_{1\_} \\ x_{3\_} \end{pmatrix}$$

Generate Simulink Function Blocks for Nonlinear Plant Model

```
% open_system('ball_on_beam_nonlinear_model_reference')
% matlabFunctionBlock('ball_on_beam_nonlinear_model_reference/actual_model_x_ball',f_wo_t,'vars
%
% matlabFunctionBlock('ball_on_beam_nonlinear_model_reference/actual_model_beta_beam',h_wo_t,..
%                      'vars',{'x1_','x2_','x3_','x4_','u_','z_','r_ball','J_ball','m_ball','g'
```

Define Parameters for Simulink Simulation

```
r_ball = 0.0125
```

```
r_ball = 0.0125
```

```
J_ball = 4*10^-6
```

```
J_ball = 4.0000e-06
```

```
m_ball = 0.064
```

```
m_ball = 0.0640
```

```
g = 9.81
```

```
g = 9.8100
```

```
J_beam = 2.9*10^-3
```

```
J_beam = 0.0029
```

Linearization Around Equilibrium Point

Verify that:

($\underline{x}$ = 0, u = 0, z = 0)

represents an equilibrium point of the system

```
% Define equilibrium point with time functions:
op = [x1; x2; x3; x4; u; z] == [0; 0; 0; 0; 0; 0];

% Define equilibrium point without time functions:
op_wo_t = [x1_; x2_; x3_; x4_; u_; z_] == [0; 0; 0; 0; 0; 0];

% Verify that f(x_op, u_op, z_op) = 0 holds:
f_op = simplify(subs(f, lhs(op), rhs(op)));
f_op_wo_t = simplify(subs(f, lhs(op_wo_t), rhs(op_wo_t)));
f_op
```

f_op =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Linearize the system around the equilibrium point

```
A = simplify( subs( jacobian(f_wo_t,[x1_,x2_,x3_,x4_]), lhs(op_wo_t),rhs(op_wo_t) ) )
```

A =

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{G\, m_{\mathrm{Ball}}\, r_{\mathrm{Ball}}}{J_{\mathrm{Beam}}} & 0 & -\frac{G\, m_{\mathrm{Ball}}\, {r_{\mathrm{Ball}}}^2}{m_{\mathrm{Ball}}\, {r_{\mathrm{Ball}}}^2 + J_{\mathrm{Ball}}} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{G\, m_{\mathrm{Ball}}}{J_{\mathrm{Beam}}} & 0 & 0 & 0 \end{pmatrix}$$

```
b_u = simplify( subs( jacobian(f_wo_t,[u_]), lhs(op_wo_t),rhs(op_wo_t) ) )
```

b_u =

$$\begin{pmatrix} 0 \\ \frac{r_{\mathrm{Ball}}}{J_{\mathrm{Beam}}} \\ 0 \\ \frac{1}{J_{\mathrm{Beam}}} \end{pmatrix}$$

```
b_z = simplify( subs( jacobian(f_wo_t,[z_]), lhs(op_wo_t),rhs(op_wo_t) ) )
```

b_z =

$$\begin{pmatrix} 0 \\ \frac{{r_{\mathrm{Ball}}}^2}{m_{\mathrm{Ball}}\, {r_{\mathrm{Ball}}}^2 + J_{\mathrm{Ball}}} \\ 0 \\ 0 \end{pmatrix}$$

```
C = simplify(subs( jacobian(h_wo_t,[x1_,x2_,x3_,x4_]), lhs(op_wo_t),rhs(op_wo_t) ))
```

C =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Substitute Numerical Parameter Values and Convert Matrices and Vectors to Standard Numerical Data Type

```
A_num=double(subs(A,[r_Ball,J_Ball,m_Ball,G,J_Beam],...
 [r_ball,J_ball,m_ball,g,J_beam]))
```

```
A_num = 4×4
         0    1.0000         0         0
   -2.7062         0   -7.0071         0
         0         0         0    1.0000
 -216.4966         0         0         0
```

```
b_u_num=double(subs(b_u,[r_Ball,J_Ball,m_Ball,G,J_Beam],...
 [r_ball,J_ball,m_ball,g,J_beam]))
```

```
b_u_num = 4×1
         0
    4.3103
         0
  344.8276
```

```
b_z_num=double(subs(b_z,[r_Ball,J_Ball,m_Ball,G,J_Beam],...
 [r_ball,J_ball,m_ball,g,J_beam]))
```

```
b_z_num = 4×1
         0
   11.1607
         0
         0
```

```
C_num=double(subs(C,[r_Ball,J_Ball,m_Ball,G,J_Beam],...
 [r_ball,J_ball,m_ball,g,J_beam]))
```

```
C_num = 2×4
     1     0     0     0
     0     0     1     0
```

```
C_num_1 = C_num(1,:)
```

```
C_num_1 = 1×4
     1     0     0     0
```

Summary: Linearized State Equations

```
displayFormula('diff(Delta*x(t),t)==A_num*Delta*x(t)+b_u_num*(Delta*u(t)-0.1)+b_z_num*z(t)')
```

$$\frac{\partial}{\partial t}\ (\Delta\ x(t)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1962}{725} & 0 & -\frac{981}{140} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{31392}{145} & 0 & 0 & 0 \end{pmatrix} \Delta\ x(t) + \begin{pmatrix} 0 \\ \frac{125}{29} \\ 0 \\ \frac{10000}{29} \end{pmatrix} (\Delta\ u(t) - 0.1) + \begin{pmatrix} 0 \\ \frac{625}{56} \\ 0 \\ 0 \end{pmatrix} z(t)$$

```
displayFormula('Delta*y(t)==C_num*Delta*x(t)')
```

$$\Delta\ y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Delta\ x(t)$$

Linearized Plant Model Analysis

1. Stability

```
eigen_values = eig(A_num)
```

```
eigen_values = 4×1 complex
```

```
  -6.1335 + 0.0000i
  -0.0000 + 6.3502i
  -0.0000 - 6.3502i
   6.1335 + 0.0000i
```

We can observe here that our system is not asmptotically stable, this is because the eigenvalues of matrix A has a positive real part.

2. Controlability

```
Q_c = ctrb(A_num, b_u_num)
```

```
Q_c = 4×4
10^3 ×
         0    0.0043         0   -2.4279
    0.0043         0   -2.4279         0
         0    0.3448         0   -0.9332
    0.3448         0   -0.9332         0
```

```
d_Q_c = det(Q_c)
```

```
d_Q_c = 6.9421e+11
```

The system is completely controlable, because det(Qc != 0)

3. Observability

```
Q_o = obsv(A_num, C_num)
```

```
Q_o = 8×4
    1.0000         0         0         0
         0         0    1.0000         0
         0    1.0000         0         0
         0         0         0    1.0000
   -2.7062         0   -7.0071         0
 -216.4966         0         0         0
         0   -2.7062         0   -7.0071
         0 -216.4966         0         0
```

```
unob = length(A)-rank(Q_o)
```

```
unob = 0
```

The system does not have any unobservabilities. Hence, it is completely observable.

Linear State-Feedback Controller Design

```
p = [-70, -58, -1.6, -1.4]
```

```
p = 1×4
  -70.0000  -58.0000   -1.6000   -1.4000
```

```
k_fb = place(A_num, b_u_num, p)
```

```
k_fb = 1×4
   -4.3917   -5.1595   12.9411    0.4444
```

Luenberger Observer Design

```
p_lo = [-60,-50,-1.75,-1.45]
```

```
p_lo = 1×4
  -60.0000  -50.0000   -1.7500   -1.4500
```

```
k_lo = place(A_num', C_num_1', p_lo)
```

```
k_lo = 1×4
10^3 ×
    0.1132    3.3518   -1.4099   -1.3029
```

```
l_T = transpose(k_lo)
```

```
l_T = 4×1
10^3 ×
    0.1132
    3.3518
   -1.4099
   -1.3029
```

Luenberger disturbance observer

```
z1 = -9+12j
```

```
z1 = -9.0000 + 12.0000i
```

```
z2 = -9-12j
```

```
z2 = -9.0000 - 12.0000i
```

```
z3 = -12+6j
```

```
z3 = -12.0000 + 6.0000i
```

```
z4 = -12-6j
```

```
z4 = -12.0000 - 6.0000i
```

```
p_ldo = [-100 z1 z2 z3 z4]
```

```
p_ldo = 1×5 complex
10^2 ×
  -1.0000 + 0.0000i  -0.0900 + 0.1200i  -0.0900 - 0.1200i  -0.1200 + 0.0600i ⋯
```

```
A_e = zeros(5,5)
```

```
A_e = 5×5
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
```

```
A_e(1:4, 1:4) = A_num
```

```
A_e = 5×5
```

```
         0    1.0000         0         0         0
   -2.7062         0   -7.0071         0         0
         0         0         0    1.0000         0
 -216.4966         0         0         0         0
         0         0         0         0         0
```

```
A_e(1:4, 5) = b_u_num
```

```
A_e = 5×5
         0    1.0000         0         0         0
   -2.7062         0   -7.0071         0    4.3103
         0         0         0    1.0000         0
 -216.4966         0         0         0  344.8276
         0         0         0         0         0
```

```
B_e = zeros(5,1)
```

```
B_e = 5×1
     0
     0
     0
     0
     0
```

```
B_e(1:4,1) = b_u_num
```

```
B_e = 5×1
         0
    4.3103
         0
  344.8276
         0
```

```
C_e=zeros(1,5)
```

```
C_e = 1×5
     0     0     0     0     0
```

```
C_e(1,1:4)=C_num_1
```

```
C_e = 1×5
     1     0     0     0     0
```

```
l_e = place(A_e', C_e', p_ldo)
```

l_e = 1×5
10^5 ×
```
    0.0014    0.0503   -0.1421   -1.2930   -0.0168
```

Conclusion:

- **Successful control of the model was achieved.**
- **Luenberger Observer has been used to estimate the states and implement a state-feedback Controller.**
- **Later on the Observer was extended to observe the continuous disturbance on the input and reduced the estimation errors.**

- **Saturation Block was used as an anti-windup measure in order to avoid negative impacts when the requested control signal exceeds its feasible range resulting from the restricted motor power of the laboratory system.**

References

1. Kamman, James, "An Introduction to Three-Dimensional Rigid Body Dynamics", ebook, https://kamman-dynamics-control.org/3d-dynamics-ebook/ (accessed November 6, 2023)