Deriving EOMs for Triple Pendulum and its Chaos Simulation

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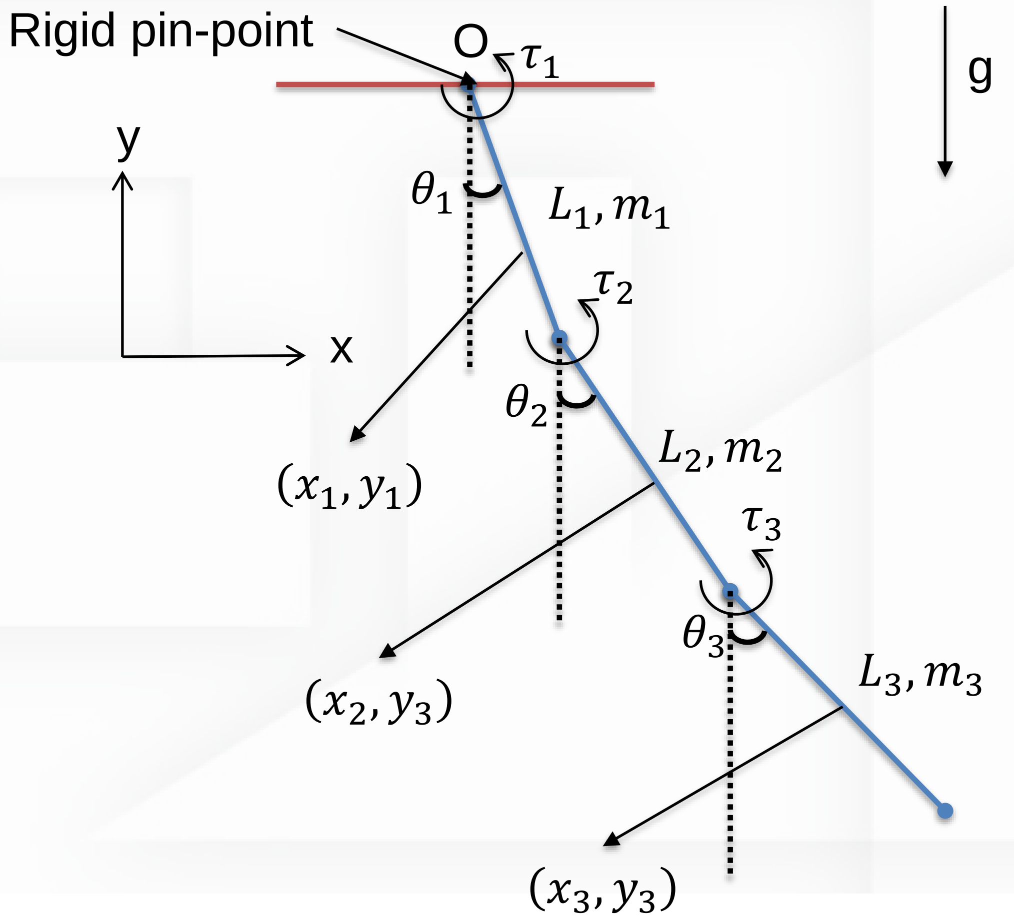
Triple Pendulum – Problem Design

- Three limbs connected to each other end to end
- One end is pinned at a rigid point
- Each limb is identified as $i = 1,2,3$
- Coordinate system
  - ❑ X-Y, origin at the rigid pin-point (O)
- DOF calculation (each at CG of limbs)
  - ❑ $No.of\ coordinates\ = 6, (x_i,\ y_i)$
  - ❑ $No.of\ constraints\ = 3, (L_i)$
  - ❑ $DOF\ = 6 - 2\ = 3$
- Thus, three generalized coordinates (q_i) identified:
  - ❑ $q_i = \theta_i$, angle subtended by each limb w.r.t vertical



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Position and Velocity of Cart and Pendulum

- The position of CG of each of the limbs are:

$$\boldsymbol{P}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{L_1}{2}\sin\theta_1 \\ \frac{L_1}{2}\cos\theta_1 \end{bmatrix}$$

$$\boldsymbol{P}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_1\sin\theta_1 + \frac{L_2}{2}\sin\theta_2 \\ L_1\cos\theta_1 + \frac{L_2}{2}\sin\theta_2 \end{bmatrix}$$

$$\boldsymbol{P}_3 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} L_1\sin\theta_1 + L_2\sin\theta_2 + \frac{L_3}{2}\sin\theta_3 \\ L_1\cos\theta_1 + L_2\cos\theta_2 + \frac{L_3}{2}\cos\theta_3 \end{bmatrix}$$

- The velocity of each of the CG points are:

$$\boldsymbol{v}_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} \frac{L_1}{2}\cos\theta_1\,\dot{\theta}_1 \\ -\frac{L_1}{2}\sin\theta_1\,\dot{\theta}_1 \end{bmatrix}$$

$$\boldsymbol{v}_2 = \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} L_1\cos\theta_1\,\dot{\theta}_1 + \frac{L_2}{2}\cos\theta_2\dot{\theta}_2 \\ -L_1\sin\theta_1\,\dot{\theta}_1 - \frac{L_2}{2}\sin\theta_2\,\dot{\theta}_2 \end{bmatrix}$$

$$\boldsymbol{v}_3 = \begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} L_1\cos\theta_1\,\dot{\theta}_1 + L_2\cos\theta_2\,\dot{\theta}_2 + \frac{L_3}{2}\cos\theta_3\,\dot{\theta}_3 \\ -L_1\sin\theta_1\,\dot{\theta}_1 - L_2\sin\theta_2\,\dot{\theta}_2 - \frac{L_3}{2}\sin\theta_3\,\dot{\theta}_3 \end{bmatrix}$$

Kinetic Energies and Potential Energies

- Total Kinetic Energy (T) is the sum of the Translational KE (TKE) and Rotational KE (RKE) of each of the limbs

$$TKE = \sum_{i}^{3} \frac{1}{2} m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i$$

$$RKE = \sum_{i}^{3} \frac{1}{2} I_i \dot{\theta}_i$$

$$T = TKE + RKE$$

- Total Potential Energy (V) is the sum of the gravitational PE of each of the limbs

$$V = \sum_{i}^{3} -m_i \boldsymbol{g} \cdot \boldsymbol{P}_i$$

$$= \sum_{i}^{3} -m_i \begin{bmatrix} 0 \\ -g \end{bmatrix} \cdot \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \sum_{i}^{3} m_i g y_i$$

KE (T) and PE (V) final relations

$$
\begin{aligned}
T = {} & \left(I_1\dot{\theta}_1^2\right)/2 + \left(I_2\dot{\theta}_2^2\right)/2 + \left(I_3\dot{\theta}_3^2\right)/2 + \left(L_1^2 m_1\dot{\theta}_1^2\right)/8 + \left(L_1^2 m_2\dot{\theta}_1^2\right)/2 + \left(L_1^2 m_3\dot{\theta}_1^2\right)/2 \\
& + \left(L_2^2 m_2\dot{\theta}_2^2\right)/8 + \left(L_2^2 m_3\dot{\theta}_2^2\right)/2 + \left(L_3^2 m_3\dot{\theta}_3^2\right)/8 + \left(L_1L_2m_2\dot{\theta}_1\dot{\theta}_2 cos(\theta_1 - \theta_2)\right)/2 \\
& + L_1L_2m_3\dot{\theta}_1\dot{\theta}_2 cos(\theta_1 - \theta_2) + \left(L_1L_3m_3\dot{\theta}_1\dot{\theta}_3 cos(\theta_1 - \theta_3)\right)/2 \\
& + \left(L_2L_3m_3\dot{\theta}_2\dot{\theta}_3 cos(\theta_2 - \theta_3)\right)/2
\end{aligned}
$$

$$
\begin{aligned}
V = {} & -m_3 g\left(L_1 cos(\theta_1) + L_2 cos(\theta_2) + \left(L_3 cos(\theta_3)\right)/2\right) - m_2 g\left(L_1 cos(\theta_1) + \left(L_2 cos(\theta_2)\right)/2\right) \\
& - \left(L_1 m_1 g cos(\theta_1)\right)/2
\end{aligned}
$$

Note: MATLAB Symbolic Toolbox was used to derive the above equations

Lagrangian and Application of Euler-Lagrange

- Firstly, we calculate the Lagrangian (L_g) as follows:

$$L_g = T - V$$

- Now we know, Euler-Lagrange Equation is as follows, where we should note the action of other forces/torques (Q_{q_i}) at respective generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L_g}{\partial \dot{q}_i}\right) - \frac{\partial L_g}{\partial q_i} = Q_{q_i} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_{q_i}$$

Effect of friction/damping – Rayleigh Dissipation

- Ideally, there is some friction at each of the joints in the real world scenario. Here, the damping force is given by Rayleigh Dissipation function:

$$D = \sum_{i}^{3} \frac{1}{2} k_i \dot{\theta}_i^2$$

- This effect is realized in Lagrange's Equation as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_{q_i} = \tau_i$$

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Generalized Equations of Motion

- For $q_1 = \theta_1$, Euler-Lagrange Equation gives us:

$$\tau_1 = I_1\ddot{\theta}_1 + k_1\dot{\theta}_1 + (L_1^2M_1\ddot{\theta}_1)/4 + L_1^2M_2\ddot{\theta}_1 + L_1^2M_3\ddot{\theta}_1 + (L_1M_1gsin(\theta_1))/2 + L_1M_2gsin(\theta_1) + L_1M_3gsin(\theta_1) + \left(L_1L_2M_2\dot{\theta}_2^2sin(\theta_1-\theta_2)\right)/2 + L_1L_2M_3\dot{\theta}_2^2sin(\theta_1-\theta_2) + \left(L_1L_3M_3\dot{\theta}_3^2sin(\theta_1-\theta_3)\right)/2 + \left(L_1L_2M_2\ddot{\theta}_2cos(\theta_1-\theta_2)\right)/2 + L_1L_2M_3\ddot{\theta}_2cos(\theta_1-\theta_2) + \left(L_1L_3M_3\ddot{\theta}_3cos(\theta_1-\theta_3)\right)/2$$

- For $q_2 = \theta_2$, Euler-Lagrange Equation gives us:

$$\tau_2 = I_2\ddot{\theta}_2 + k_2\dot{\theta}_2 + (L_2^2M_2\ddot{\theta}_2)/4 + L_2^2M_3\ddot{\theta}_2 + (L_2M_2gsin(\theta_2))/2 + L_2M_3gsin(\theta_2) - \left(L_1L_2M_2\dot{\theta}_1^2sin(\theta_1-\theta_2)\right)/2 - L_1L_2M_3\dot{\theta}_1^2sin(\theta_1-\theta_2) + \left(L_2L_3M_3\dot{\theta}_3^2sin(\theta_2-\theta_3)\right)/2 + \left(L_1L_2M_2\ddot{\theta}_1cos(\theta_1-\theta_2)\right)/2 + L_1L_2M_3\ddot{\theta}_1cos(\theta_1-\theta_2) + \left(L_2L_3M_3\ddot{\theta}_3cos(\theta_2-\theta_3)\right)/2 = 0$$

- For $q_3 = \theta_3$, Euler-Lagrange Equation gives us:

$$\tau_3 = I_3\ddot{\theta}_3 + k_3\dot{\theta}_3 + (L_3^2M_3\ddot{\theta}_3)/4 + (L_3M_3gsin(\theta_3))/2 - \left(L_1L_3M_3\dot{\theta}_1^2sin(\theta_1-\theta_3)\right)/2 - \left(L_2L_3M_3\dot{\theta}_2^2sin(\theta_2-\theta_3)\right)/2 + \left(L_1L_3M_3\ddot{\theta}_1cos(\theta_1-\theta_3)\right)/2 + \left(L_2L_3M_3\ddot{\theta}_2cos(\theta_2-\theta_3)\right)/2 = 0$$

Note: MATLAB Symbolic Toolbox was used to derive the above equations

Solving EOM using ODE45

- State-space

$$\boldsymbol{s} = [\theta_1\ \dot{\theta}_1\ \theta_2\ \dot{\theta}_2\ \theta_3\ \dot{\theta}_3]'$$

- With the application of some torque control inputs at joint

$$\boldsymbol{\tau} = [\tau_1\ \tau_2\ \tau_3]'$$

- ODE45 solves using time derivative of state space in terms of state space. This requires determination expressions as follows:

$$\dot{\boldsymbol{s}} = \left[\dot{\theta}_1\ \ddot{\theta}_1\ \dot{\theta}_2\ \ddot{\theta}_2\ \dot{\theta}_3\ \ddot{\theta}_3\right]' = f(\boldsymbol{s})$$

- That is we need to determine $\ddot{\theta}_i's$ in terms or elements of $\boldsymbol{s}$. They are extremely difficult and long to derive as seen in next slide

$$
\begin{aligned}
\ddot{\theta}_1 = &-(2(32I_2I_3k_1\dot{\theta}_1 - 8I_3L_2^2m_2\tau_1 - 8I_2L_3^2m_3\tau_1 - 32I_3L_2^2m_3\tau_1 - 32I_2I_3\tau_1 - 4L_2^2L_3^2m_3^2 \\
&\tau_1 + 4L_2^2L_3^2m_3^2k_1\dot{\theta}_1 + 4L_2^2L_3^2m_3^2\tau_1\cos(2\theta_2 - 2\theta_3) + 8I_3L_2^2m_2k_1\dot{\theta}_1 + 8I_2L_3^2m_3k_1 \\
&\dot{\theta}_1 + 32I_3L_2^2m_3k_1\dot{\theta}_1 - 2L_2^2L_3^2m_2m_3\tau_1 + 4I_3L_1^2L_2^2m_2^2\dot{\theta}_1^2\sin(2\theta_1 - 2\theta_2) + 16I_3L_1^2L_2^2 \\
&m_3^2\dot{\theta}_1^2\sin(2\theta_1 - 2\theta_2) + 4I_2L_1^2L_3^2m_3^2\dot{\theta}_1^2\sin(2\theta_1 - 2\theta_3) + 16I_3L_1L_2m_2\tau_2\cos(\theta_1 - \\
&\theta_2) + 32I_3L_1L_2m_3\tau_2\cos(\theta_1 - \theta_2) + 16I_2L_1L_3m_3\tau_3\cos(\theta_1 - \theta_3) + 2L_2^2L_3^2m_2m_3k_1\dot{\theta}_1 + 4I_3L_1 \\
&L_2^3m_2^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 32I_3L_1L_2^3m_3^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 4I_2L_1L_3^3m_3^2\dot{\theta}_3^2 \\
&\sin(\theta_1 - \theta_3) - 4L_1L_2L_3^2m_3^2\tau_2\cos(\theta_1 + \theta_2 - 2\theta_3) - 8L_1L_2^2L_3m_3^2\tau_3\cos(\theta_1 - 2\theta_2 + \theta_3) + 4I_3 \\
&L_1L_2^2m_2^2g\sin(\theta_1) + 4I_2L_1L_3^2m_3^2g\sin(\theta_1) + 16I_3L_1L_2^2m_3^2g\sin(\theta_1) - 4L_2^2L_3^2m_3^2k_1\dot{\theta}_1\cos(2\theta_2 - \\
&2\theta_3) + 4I_3L_1L_2^2m_2^2g\sin(\theta_1 - 2\theta_2) + 16I_3L_1L_2^2m_3^2g\sin(\theta_1 - 2\theta_2) + 4I_2L_1L_3^2m_3^2g\sin(\theta_1 - 2\theta_3) \\
&+ 4L_1L_2L_3^2m_3^2\tau_2\cos(\theta_1 - \theta_2) + 8L_1L_2^2L_3m_3^2\tau_3\cos(\theta_1 - \theta_3) + 16I_2I_3L_1m_1g\sin(\theta_1) + 32I_2I_3L_1m_2g \\
&\sin(\theta_1) + 32I_2I_3L_1m_3g\sin(\theta_1) + 4I_2L_1L_2L_3^2m_3^2\dot{\theta}_2^2\sin(\theta_1 + \theta_2 - 2\theta_3) + 8I_3L_1L_2^2L_3m_3^2 \\
&\dot{\theta}_3^2\sin(\theta_1 - 2\theta_2 + \theta_3) + 24I_3L_1L_2^3m_2m_3\dot{\theta}_2^2\sin(\theta_1 - \theta_2) - 4L_1L_2^2L_3m_2m_3\tau_3\cos(\theta_1 - \\
&2\theta_2 + \theta_3) - 4L_1L_2L_3^2m_3^2k_2\dot{\theta}_2\cos(\theta_1 - \theta_2) - 8L_1L_2^2L_3m_3^2k_3\dot{\theta}_3\cos(\theta_1 - \theta_3) + 2L_1 \\
&L_2^2L_3^2m_1m_3^2g\sin(\theta_1) + 3L_1L_2^2L_3^2m_2m_3^2g\sin(\theta_1) + L_1L_2^2L_3^2m_2^2m_3g\sin(\theta_1) - L_1L_2^2L_3^2m_1m_3^2g\sin(\theta_1 - 2 \\
&\theta_2 + 2\theta_3) - L_1L_2^2L_3^2m_1m_3^2g\sin(\theta_1 + 2\theta_2 - 2\theta_3) - L_1L_2^2L_3^2m_2m_3^2g\sin(\theta_1 - 2\theta_2 + 2\theta_3) - L_1 \\
&L_2^2L_3^2m_2m_3^2g\sin(\theta_1 + 2\theta_2 - 2\theta_3) - L_1L_2^3L_3^2m_2m_3^2\dot{\theta}_2^2\sin(\theta_1 + \theta_2 - 2\theta_3) + L_1L_2^2L_3^3m_2m_3^2 \\
&\dot{\theta}_3^2\sin(\theta_1 - 2\theta_2 + \theta_3) - 16I_3L_1L_2m_2k_2\dot{\theta}_2\cos(\theta_1 - \theta_2) - 32I_3L_1L_2m_3k_2\dot{\theta}_2 \\
&\cos(\theta_1 - \theta_2) - 16I_2L_1L_3m_3k_3\dot{\theta}_3\cos(\theta_1 - \theta_3) + 2L_1L_2^2L_3^2m_2m_3^2g\sin(\theta_1 - 2\theta_2) + L_1L_2^2L_3^2 \\
&m_2^2m_3g\sin(\theta_1 - 2\theta_2) - L_1L_2^2L_3^2m_2m_3^2g\sin(\theta_1 - 2\theta_3) + 4I_3L_1L_2^2m_1m_2g\sin(\theta_1) + 4I_2L_1L_3^2m_1m_3g \\
&\sin(\theta_1) + 16I_3L_1L_2^2m_1m_3g\sin(\theta_1) + 8I_2L_1L_3^2m_2m_3g\sin(\theta_1) + 24I_3L_1L_2^2m_2m_3g\sin(\theta_1) + 4I_2L_1L_2L_3^2m_3^2 \\
&\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 8I_3L_1L_2^2L_3m_3^2\dot{\theta}_3^2\sin(\theta_1 - \theta_3) + 2L_1^2L_2^2L_3^2m_2m_3^2\dot{\theta}_1^2 \\
&\sin(2\theta_1 - 2\theta_2) + L_1^2L_2^2L_3^2m_2^2m_3\dot{\theta}_1^2\sin(2\theta_1 - 2\theta_2) - L_1^2L_2^2L_3^2m_2m_3^2\dot{\theta}_1^2\sin(2\theta_1 - 2 \\
&\theta_3) + 4L_1L_2L_3^2m_3^2k_2\dot{\theta}_2\cos(\theta_1 + \theta_2 - 2\theta_3) + 8L_1L_2^2L_3m_3^2k_3\dot{\theta}_3\cos(\theta_1 - 2\theta_2 + \theta_3) \\
&+ 16I_3L_1L_2^2m_2m_3g\sin(\theta_1 - 2\theta_2) + 4L_1L_2L_3^2m_2m_3\tau_2\cos(\theta_1 - \theta_2) + 16I_2I_3L_1L_2m_2\dot{\theta}_2^2\sin(\theta_1 \\
&- \theta_2) + 32I_2I_3L_1L_2m_3\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 16I_2I_3L_1L_3m_3\dot{\theta}_3^2\sin(\theta_1 - \theta_3) + 16I_3L_1^2L_2^2m_2 \\
&m_3\dot{\theta}_1^2\sin(2\theta_1 - 2\theta_2) + 3L_1L_2^3L_3^2m_2m_3^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + L_1L_2^3L_3^2m_2^2m_3\dot{\theta}_2^2 \\
&\sin(\theta_1 - \theta_2) - 4L_1L_2L_3^2m_2m_3k_2\dot{\theta}_2\cos(\theta_1 - \theta_2) + L_1L_2^2L_3^2m_1m_2m_3g\sin(\theta_1) + 4I_2L_1L_2L_3^2m_2m_3 \\
&\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 4L_1L_2^2L_3m_2m_3k_3\dot{\theta}_3\cos(\theta_1 - 2\theta_2 + \theta_3) + 4I_3L_1L_2^2L_3m_2m_3\dot{\theta}_3^2 \\
&\sin(\theta_1 - 2\theta_2 + \theta_3)))/(64I_1I_2I_3 + 8I_3L_1^2L_2^2m_2^2 + 8I_1L_2^2L_3^2m_3^2 + 8I_2L_1^2L_3^2m_3^2 + 32I_3L_1^2L_2^2m_3^2 + 16I_2 \\
&I_3L_1^2m_1 + 16I_1I_3L_2^2m_2 + 64I_2I_3L_1^2m_2 + 16I_1I_2L_3^2m_3 + 64I_1I_3L_2^2m_3 + 64I_2I_3L_1^2m_3 - 8I_3L_1^2L_2^2m_2^2\cos(2 \\
&\theta_1 - 2\theta_2) - 32I_3L_1^2L_2^2m_3^2\cos(2\theta_1 - 2\theta_2) - 8I_2L_1^2L_3^2m_3^2\cos(2\theta_1 - 2\theta_3) - 8I_1L_2^2L_3^2m_3^2\cos(2\theta_2 \\
&- 2\theta_3) + 2L_1^2L_2^2L_3^2m_1m_3^2 + 6L_1^2L_2^2L_3^2m_2m_3^2 + 2L_1^2L_2^2L_3^2m_2^2m_3 + 4I_3L_1^2L_2^2m_1m_2 + 4I_2L_1^2L_3^2m_1m_3 + 16I_3 \\
&L_1^2L_2^2m_1m_3 + 4I_1L_2^2L_3^2m_2m_3 + 16I_2L_1^2L_3^2m_2m_3 + 48I_3L_1^2L_2^2m_2m_3 - 32I_3L_1^2L_2^2m_2m_3\cos(2\theta_1 - 2\theta_2) + L_1^2 \\
&L_2^2L_3^2m_1m_2m_3 - 4L_1^2L_2^2L_3^2m_2m_3^2\cos(2\theta_1 - 2\theta_2) - 2L_1^2L_2^2L_3^2m_2^2m_3\cos(2\theta_1 - 2\theta_2) - 2L_1^2L_2^2L_3^2m_1m_3^2 \\
&\cos(2\theta_2 - 2\theta_3) + 2L_1^2L_2^2L_3^2m_2m_3^2\cos(2\theta_1 - 2\theta_3) - 4L_1^2L_2^2L_3^2m_2m_3^2\cos(2\theta_2 - 2\theta_3))
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\theta}_2 = &-(2(32I_1I_3k_2\dot{\theta}_2 - 8I_3L_1^2m_1\tau_2 - 32I_3L_1^2m_2\tau_2 - 8I_1L_3^2m_3\tau_2 - 32I_3L_1^2m_3\tau_2 - 32I_1I_3\tau_2 \\
&- 4L_1^2L_3^2m_3^2\tau_2 + 4L_1^2L_3^2m_3^2k_2\dot{\theta}_2 + 4L_1^2L_3^2m_3^2\tau_2\cos(2\theta_1 - 2\theta_3) + 8I_3L_1^2m_1k_2\dot{\theta}_2 + 32I_3 \\
&L_1^2m_2k_2\dot{\theta}_2 + 8I_1L_3^2m_3k_2\dot{\theta}_2 + 32I_3L_1^2m_3k_2\dot{\theta}_2 - 2L_1^2L_3^2m_1m_3\tau_2 - 8L_1^2L_3^2m_2m_3\tau_2 - 4I_3 \\
&L_1^2L_2^2m_2^2\dot{\theta}_2^2\sin(2\theta_1 - 2\theta_2) - 16I_3L_1^2L_2^2m_3^2\dot{\theta}_2^2\sin(2\theta_1 - 2\theta_2) + 4I_1L_2^2L_3^2m_3^2 \\
&\dot{\theta}_2^2\sin(2\theta_2 - 2\theta_3) + 16I_3L_1L_2m_2\tau_1\cos(\theta_1 - \theta_2) + 32I_3L_1L_2m_3\tau_1\cos(\theta_1 - \theta_2) + 16I_1L_2L_3m_3 \\
&\tau_3\cos(\theta_2 - \theta_3) + 2L_1^2L_3^2m_1m_3k_2\dot{\theta}_2 + 8L_1^2L_3^2m_2m_3k_2\dot{\theta}_2 - 8I_3L_1^2L_2m_2^2g\sin(2\theta_1 - \theta_2) - 16I_3 \\
&L_1^2L_2m_3^2g\sin(2\theta_1 - \theta_2) - 16I_3L_1^3L_2m_2^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 32I_3L_1^3L_2m_3^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + 4I_1L_2L_3^3m_3^2\dot{\theta}_3^2 \\
&\sin(\theta_2 - \theta_3) - 4L_1L_2L_3^2m_3^2\tau_1\cos(\theta_1 + \theta_2 - 2\theta_3) - 8L_1^2L_2L_3m_3^2\tau_3\cos(\theta_2 - 2\theta_1 + \theta_3) + 8I_3L_1^2L_2m_2^2g \\
&\sin(\theta_2) + 4I_1L_2L_3^2m_3^2g\sin(\theta_2) + 16I_3L_1^2L_2m_3^2g\sin(\theta_2) - 4L_1^2L_3^2m_3^2k_2\dot{\theta}_2\cos(2\theta_1 - 2\theta_3) + 4 \\
&I_1L_2L_3^2m_3^2g\sin(\theta_2 - 2\theta_3) + 4L_1L_2L_3^2m_3^2\tau_1\cos(\theta_1 - \theta_2) + 8L_1^2L_2L_3m_3^2\tau_3\cos(\theta_2 - \theta_3) + 16 \\
&I_1I_3L_2m_2g\sin(\theta_2) + 32I_1I_3L_2m_3g\sin(\theta_2) + 4I_1L_1L_2L_3^2m_3^2\dot{\theta}_1^2\sin(\theta_1 + \theta_2 - 2\theta_3) + 8I_3L_1^2L_2 \\
&L_3m_3^2\dot{\theta}_3^2\sin(\theta_2 - 2\theta_1 + \theta_3) - 4I_3L_1^2L_2m_1m_2g\sin(2\theta_1 - \theta_2) - 8I_3L_1^2L_2m_1m_3g\sin(2\theta_1 - \\
&\theta_2) - 24I_3L_1^2L_2m_2m_3g\sin(2\theta_1 - \theta_2) - 4I_3L_1^3L_2m_1m_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 8I_3L_1^3L_2m_1m_3\dot{\theta}_1^2 \\
&\sin(\theta_1 - \theta_2) - 48I_3L_1^3L_2m_2m_3\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 4L_1^2L_2L_3m_2m_3\tau_3\cos(\theta_2 - 2\theta_1 + \theta_3) - 4L_1 \\
&L_2L_3^2m_3^2k_1\dot{\theta}_1\cos(\theta_1 - \theta_2) - 8L_1^2L_2L_3m_3^2k_3\dot{\theta}_3\cos(\theta_2 - \theta_3) + 3L_1^2L_2L_3^2m_2m_3^2g\sin(\theta_2) \\
&+ 2L_1^2L_2L_3^2m_2^2m_3g\sin(\theta_2) + L_1^2L_2L_3^2m_1m_3^2g\sin(2\theta_1 + \theta_2 - 2\theta_3) + L_1^2L_2L_3^2m_2m_3^2g\sin(2\theta_1 + \theta_2 \\
&- 2\theta_3) + L_1^3L_2L_3^2m_1m_3^2\dot{\theta}_1^2\sin(\theta_1 + \theta_2 - 2\theta_3) + 2L_1^3L_2L_3^2m_2m_3^2\dot{\theta}_1^2\sin(\theta_1 + \theta_2 - 2 \\
&\theta_3) + L_1^2L_2L_3^3m_2m_3^2\dot{\theta}_3^2\sin(\theta_2 - 2\theta_1 + \theta_3) - 16I_3L_1L_2m_2k_1\dot{\theta}_1\cos(\theta_1 - \theta_2) - 32I_3L_1 \\
&L_2m_3k_1\dot{\theta}_1\cos(\theta_1 - \theta_2) - 16I_1L_2L_3m_3k_3\dot{\theta}_3\cos(\theta_2 - \theta_3) + L_1^2L_2L_3^2m_2m_3^2g\sin(\theta_2 - 2 \\
&\theta_3) + 4I_1L_2L_3^2m_2m_3g\sin(\theta_2) + 24I_3L_1^2L_2m_2m_3g\sin(\theta_2) - 4I_1L_1L_2L_3^2m_3^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + 8 \\
&I_3L_1^2L_2L_3m_3^2\dot{\theta}_3^2\sin(\theta_2 - \theta_3) - 2L_1^2L_2^2L_3^2m_2m_3^2\dot{\theta}_2^2\sin(2\theta_1 - 2\theta_2) - L_1^2L_2^2L_3^2m_2^2 \\
&m_3\dot{\theta}_2^2\sin(2\theta_1 - 2\theta_2) + L_1^2L_2^2L_3^2m_1m_3^2\dot{\theta}_2^2\sin(2\theta_2 - 2\theta_3) + 2L_1^2L_2^2L_3^2m_2m_3^2 \\
&\dot{\theta}_2^2\sin(2\theta_2 - 2\theta_3) + 4L_1L_2L_3^2m_3^2k_1\dot{\theta}_1\cos(\theta_1 + \theta_2 - 2\theta_3) + 8L_1^2L_2L_3m_3^2k_3\dot{\theta}_3 \\
&\cos(\theta_2 - 2\theta_1 + \theta_3) + 4L_1L_2L_3^2m_2m_3\tau_1\cos(\theta_1 - \theta_2) + 4L_1^2L_2L_3m_1m_3\tau_3\cos(\theta_2 - \theta_3) + 12L_1^2L_2L_3m_2 \\
&m_3\tau_3\cos(\theta_2 - \theta_3) - 16I_1I_3L_1L_2m_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 32I_1I_3L_1L_2m_3\dot{\theta}_1^2\sin(\theta_1 - \\
&\theta_2) + 16I_1I_3L_2L_3m_3\dot{\theta}_3^2\sin(\theta_2 - \theta_3) - L_1^2L_2L_3^2m_1m_3^2g\sin(2\theta_1 - \theta_2) - 3L_1^2L_2L_3^2m_2m_3^2g\sin(2 \\
&\theta_1 - \theta_2) - 2L_1^2L_2L_3^2m_2^2m_3g\sin(2\theta_1 - \theta_2) - 16I_3L_1^2L_2^2m_2m_3\dot{\theta}_2^2\sin(2\theta_1 - 2\theta_2) - L_1^3L_2L_3^2m_1 \\
&m_3^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 6L_1^3L_2L_3^2m_2m_3^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - 4L_1^3L_2L_3^2m_2^2m_3\dot{\theta}_1^2 \\
&\sin(\theta_1 - \theta_2) + L_1^2L_2L_3^3m_1m_3^2\dot{\theta}_3^2\sin(\theta_2 - \theta_3) + 3L_1^2L_2L_3^3m_2m_3^2\dot{\theta}_3^2\sin(\theta_2 - \theta_3) - 4L_1 \\
&L_2L_3^2m_2m_3k_1\dot{\theta}_1\cos(\theta_1 - \theta_2) - 4L_1^2L_2L_3m_1m_3k_3\dot{\theta}_3\cos(\theta_2 - \theta_3) - 12L_1^2L_2L_3m_2m_3k_3 \\
&\dot{\theta}_3\cos(\theta_2 - \theta_3) - 4I_1L_1L_2L_3^2m_2m_3\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + 4I_3L_1^2L_2L_3m_1m_3\dot{\theta}_3^2\sin(\theta_2 - \\
&\theta_3) + 12I_3L_1^2L_2L_3m_2m_3\dot{\theta}_3^2\sin(\theta_2 - \theta_3) + 4L_1^2L_2L_3m_2m_3k_3\dot{\theta}_3\cos(\theta_2 - 2\theta_1 + \theta_3) - L_1^2 \\
&L_2L_3^2m_1m_2m_3g\sin(2\theta_1 - \theta_2) - L_1^3L_2L_3^2m_1m_2m_3\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + 4I_3L_1^2L_2L_3m_2m_3\dot{\theta}_3^2 \\
&\sin(\theta_2 - 2\theta_1 + \theta_3)))/(64I_1I_2I_3 + 8I_3L_1^2L_2^2m_2^2 + 8I_1L_2^2L_3^2m_3^2 + 8I_2L_1^2L_3^2m_3^2 + 32I_3L_1^2L_2^2m_3^2 + 16I_2 \\
&I_3L_1^2m_1 + 16I_1I_3L_2^2m_2 + 64I_2I_3L_1^2m_2 + 16I_1I_2L_3^2m_3 + 64I_1I_3L_2^2m_3 + 64I_2I_3L_1^2m_3 - 8I_3L_1^2L_2^2m_2^2\cos(2 \\
&\theta_1 - 2\theta_2) - 32I_3L_1^2L_2^2m_3^2\cos(2\theta_1 - 2\theta_2) - 8I_2L_1^2L_3^2m_3^2\cos(2\theta_1 - 2\theta_3) - 8I_1L_2^2L_3^2m_3^2\cos(2\theta_2 \\
&- 2\theta_3) + 2L_1^2L_2^2L_3^2m_1m_3^2 + 6L_1^2L_2^2L_3^2m_2m_3^2 + 2L_1^2L_2^2L_3^2m_2^2m_3 + 4I_3L_1^2L_2^2m_1m_2 + 4I_2L_1^2L_3^2m_1m_3 + 16I_3 \\
&L_1^2L_2^2m_1m_3 + 4I_1L_2^2L_3^2m_2m_3 + 16I_2L_1^2L_3^2m_2m_3 + 48I_3L_1^2L_2^2m_2m_3 - 32I_3L_1^2L_2^2m_2m_3\cos(2\theta_1 - 2\theta_2) + L_1^2 \\
&L_2^2L_3^2m_1m_2m_3 - 4L_1^2L_2^2L_3^2m_2m_3^2\cos(2\theta_1 - 2\theta_2) 3- 2L_1^2L_2^2L_3^2m_2^2m_3\cos(2\theta_1 - 2\theta_2) - 2L_1^2L_2^2L_3^2m_1m_3^2 \\
&\cos(2\theta_2 - 2\theta_3) + 2L_1^2L_2^2L_3^2m_2m_3^2\cos(2\theta_1 - 2\theta_3) - 4L_1^2L_2^2L_3^2m_2m_3^2\cos(2\theta_2 - 2\theta_3))
\end{aligned}
$$

Note: MATLAB Symbolic Toolbox was used to derive the above equations

Generalized Equations of Motion

- The above three equations can be combined into a generalized matrix-vector form as follows:

$$M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{F}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{\tau}$$

- Where,

$$\mathrm{M} = \begin{bmatrix} I_1 + (L_1^2 m_1)/4 + L_1^2 m_2 + L_1^2 m_3 & (L_1 L_2 m_2 \cos(\theta_1 - \theta_2))/2 + L_1 L_2 m_3 \cos(\theta_1 - \theta_2) & (L_1 L_3 m_3 \cos(\theta_1 - \theta_3))/2 \\ \begin{array}{c} (L_1 L_2 m_2 \cos(\theta_1 - \theta_2))/2 + \\ L_1 L_2 m_3 \cos(\theta_1 - \theta_2) \end{array} & I_2 + (L_2^2 m_2)/4 + L_2^2 m_3 & (L_2 L_3 m_3 \cos(\theta_2 - \theta_3))/2 \\ (L_1 L_3 m_3 \cos(\theta_1 - \theta_3))/2 & (L_2 L_3 m_3 \cos(\theta_2 - \theta_3))/2 & (m_3 L_3^2)/4 + I_3 \end{bmatrix}$$

$$\mathrm{F} = \begin{bmatrix} \begin{array}{c} k_1\dot{\theta}_1 + (L_1 m_1 g \sin(\theta_1))/2 + L_1 m_2 g \sin(\theta_1) + L_1 m_3 g \sin(\theta_1) + (L_1 L_2 m_2 \dot{\theta}_2^2 \\ \sin(\theta_1 - \theta_2))/2 + L_1 L_2 m_3 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (L_1 L_3 m_3 \dot{\theta}_3^2 \sin(\theta_1 - \theta_3))/2 \end{array} \\ \begin{array}{c} k_2\dot{\theta}_2 + (L_2 m_2 g \sin(\theta_2))/2 + L_2 m_3 g \sin(\theta_2) - (L_1 L_2 m_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2))/2 \\ - L_1 L_2 m_3 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + (L_2 L_3 m_3 \dot{\theta}_3^2 \sin(\theta_2 - \theta_3))/2 \end{array} \\ \begin{array}{c} k_3\dot{\theta}_3 + (L_3 m_3 g \sin(\theta_3))/2 - (L_1 L_3 m_3 \dot{\theta}_1^2 \sin(\theta_1 - \theta_3))/2 \\ -(L_2 L_3 m_3 \dot{\theta}_2^2 \sin(\theta_2 - \theta_3))/2 \end{array} \end{bmatrix}$$

$$\ddot{\boldsymbol{\theta}} = [\theta_1 \quad \theta_2 \quad \theta_3]'$$

Note: MATLAB Symbolic Toolbox was used to derive the above equations

Simulation Considerations

- Several Conditions are simulated like with or without damping and with or without torque controls at joint
- For torque control, PD feedback control used to reach desired points or follow desired trajectory
  - ❑ $\boldsymbol{\tau} = A\boldsymbol{\tau}' + B$ control was assumed with $A = M, and\ B = F$ and feedback control law $\boldsymbol{\tau}' = \ddot{\boldsymbol{\theta}}_d - K_v(\dot{\boldsymbol{\theta}} - \boldsymbol{\theta}_d) - K_p(\boldsymbol{\theta} - \boldsymbol{\theta}_d)$.
  - ❑ This reduces the system to: $\boldsymbol{\tau} = \ddot{\boldsymbol{e}} - K_v\dot{\boldsymbol{e}} + K_p\boldsymbol{e}, \quad \boldsymbol{e} = (\boldsymbol{\theta} - \boldsymbol{\theta}_d)$
  - ❑ K_v and K_p are differential and proportional gain matrices for the system.
  - ❑ For critically damped system we assume for desired control, $K_v = 2\sqrt{K_p}$. For this case, $K_p = 10I$ is assumed where I is a identity matrix.

Parameters used for Simulation

- System Parameters

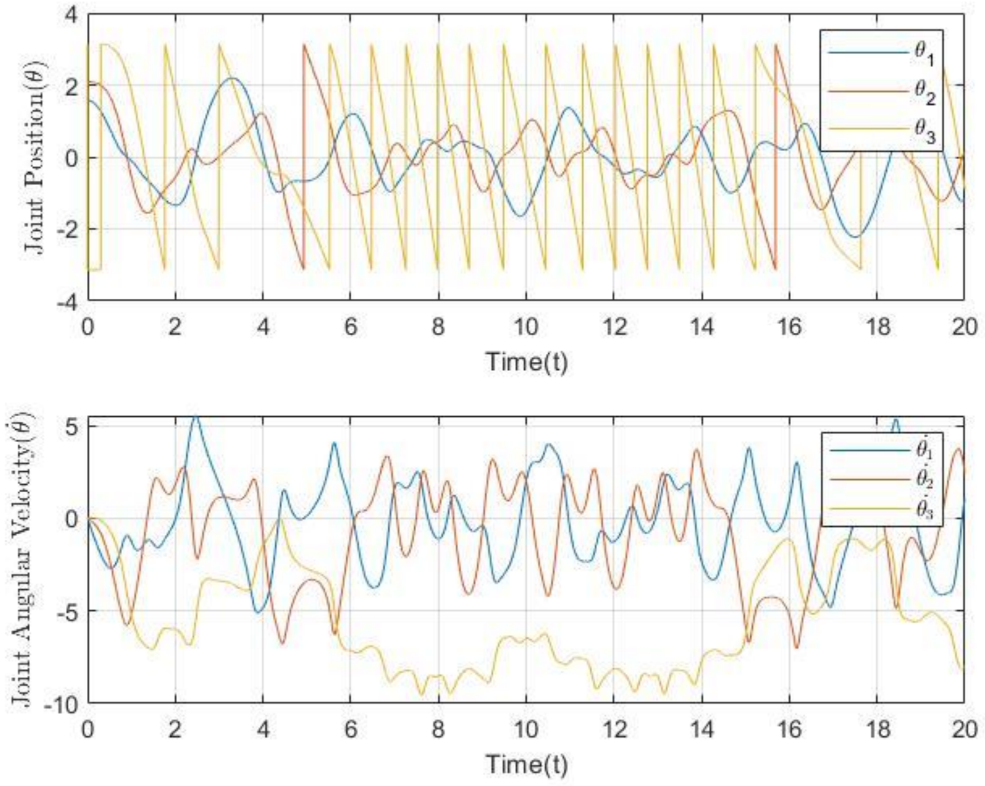
$$m_1 = m_2 = m_3 = 1\,kg$$
$$L_1 = L_2 = L_3 = 1\,m$$
$$I_1 = I_2 = I_3 = 1\,kg\,m^2$$

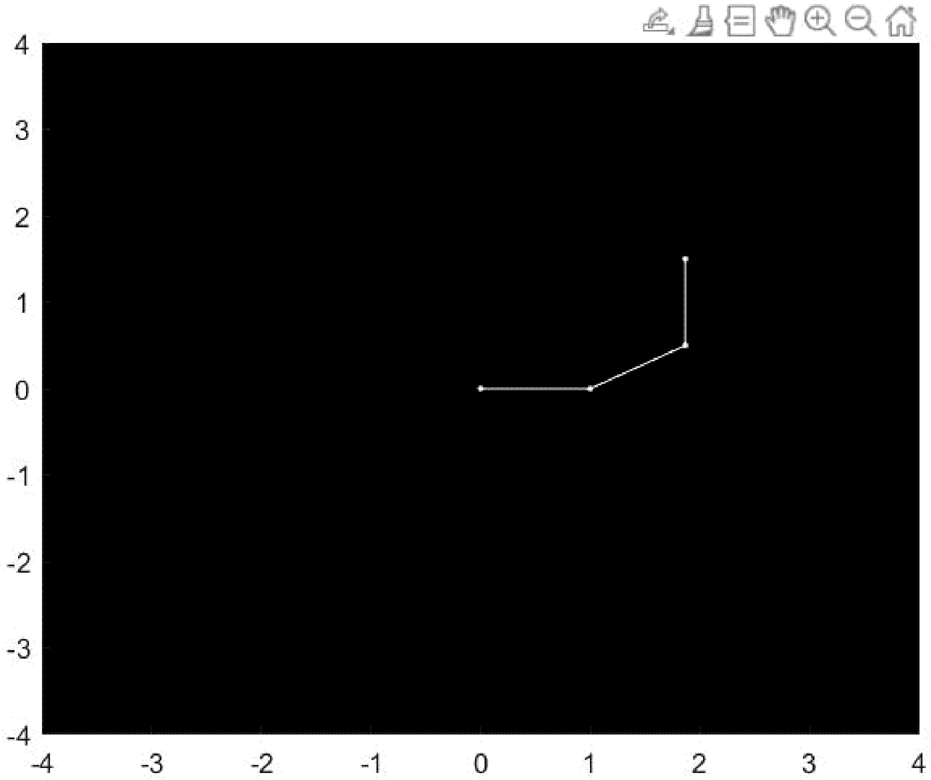
- Initial Conditions: $\boldsymbol{s} = \begin{bmatrix} \frac{\pi}{2} & 0 & \frac{2\pi}{3} & 0 & \pi & 0 \end{bmatrix}'$
- Simulation Time: 20 sec

$k_1 = k_2 = k_3 = 0\ N\ (sec/rad)^2$

$\tau_1 = \tau_2 = \tau_3 = 0\ N\ m$

Case 1: W/O Damping and W/O Control

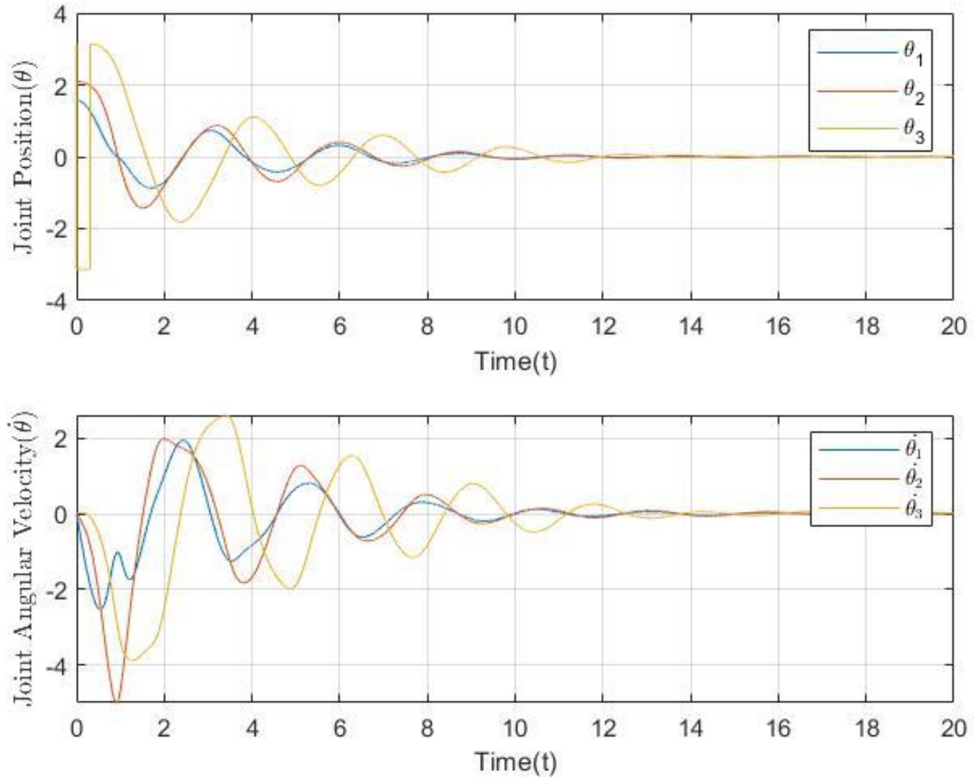


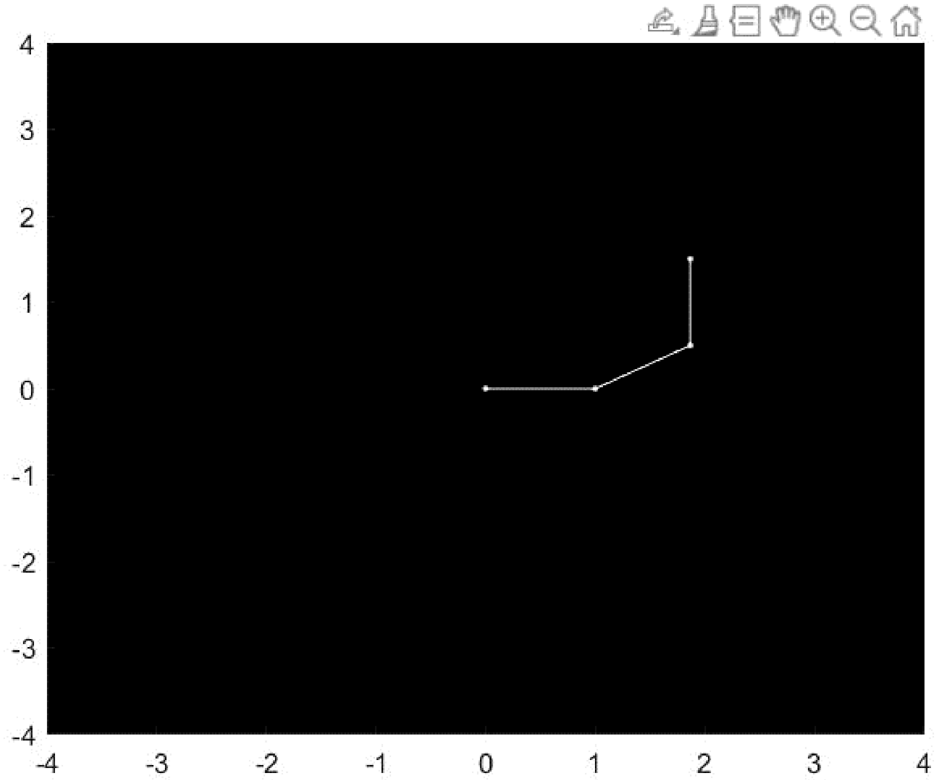


$$k_1 = k_2 = k_3 = 1\ N\ (sec/rad)^2$$

$$\tau_1 = \tau_2 = \tau_3 = 0\ N\ m$$

Case 2: W/ Damping and W/O Control

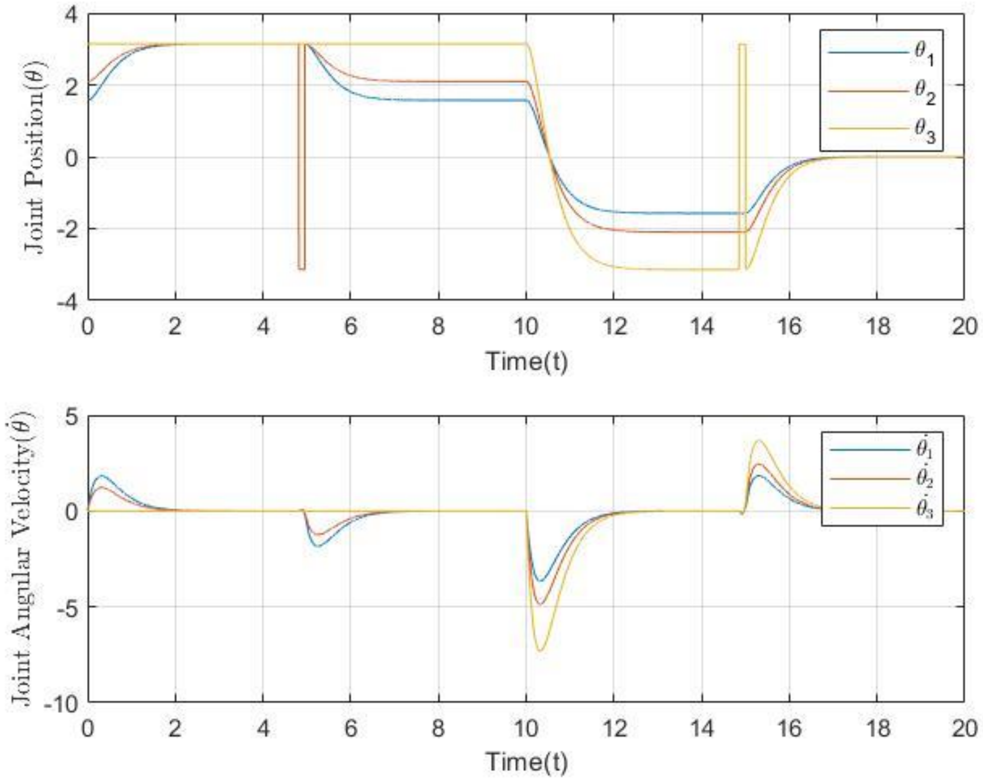


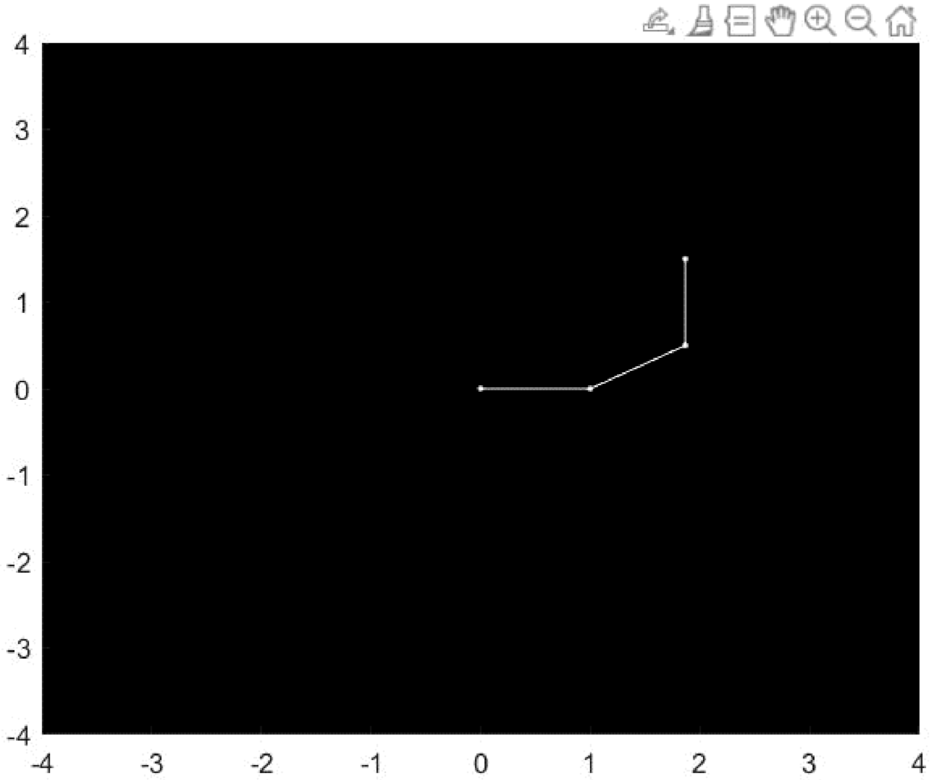


Case 3: W/ Damping and W/ Control Position Control

$$k_1 = k_2 = k_3 = 1\ N\ (sec/rad)^2$$

$$\theta_d = [\pi\ \pi\ \pi]',\ t = [0,5)$$
$$\theta_d = [\pi/2\ \pi/1.5\ \pi]',\ t = [5,10)$$
$$\theta_d = [-\pi/2\ \ -\pi/1.5 - \pi],t = [10,15)$$
$$\theta_d = [0\ 0\ 0]',\ t = [15,20)$$



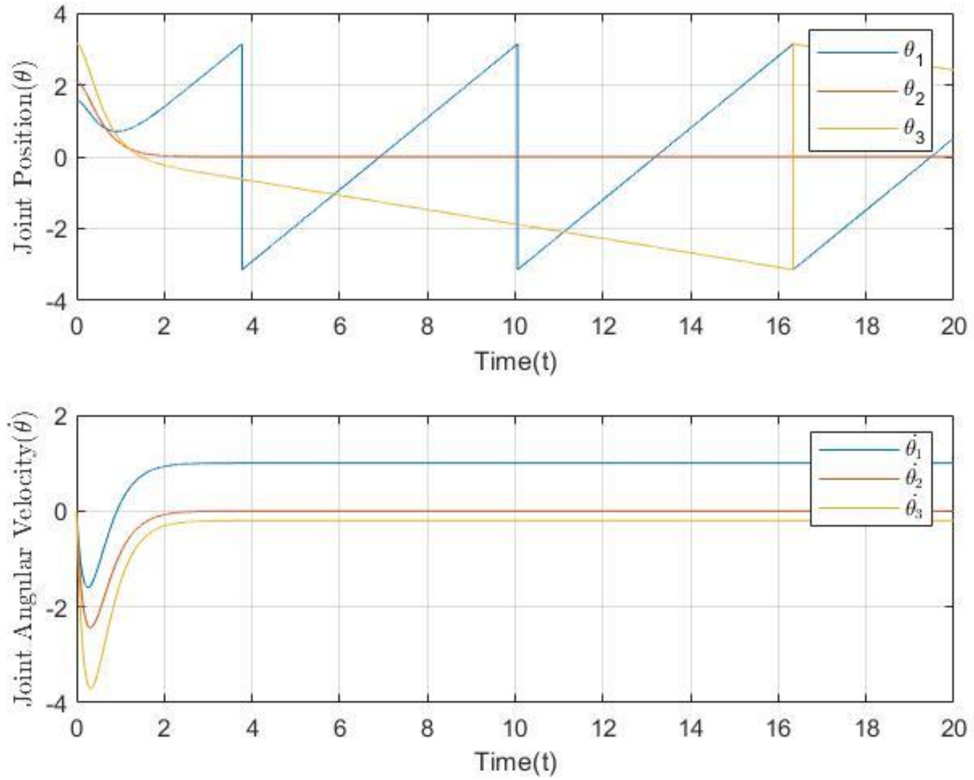


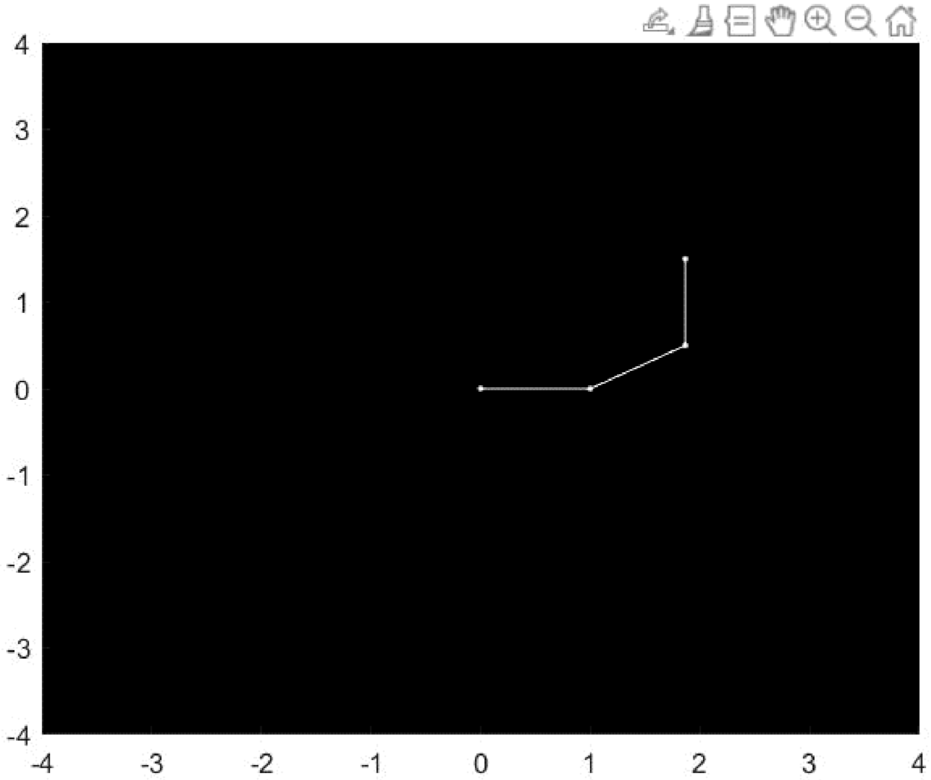
Case 4: W/ Damping and W/ Control Trajectory Following

$k_1 = k_2 = k_3 = 1\ N\ (sec/rad)^2$

Trajectory define by:

$\boldsymbol{\theta} = [t \quad 0 \quad -0.2t]'$



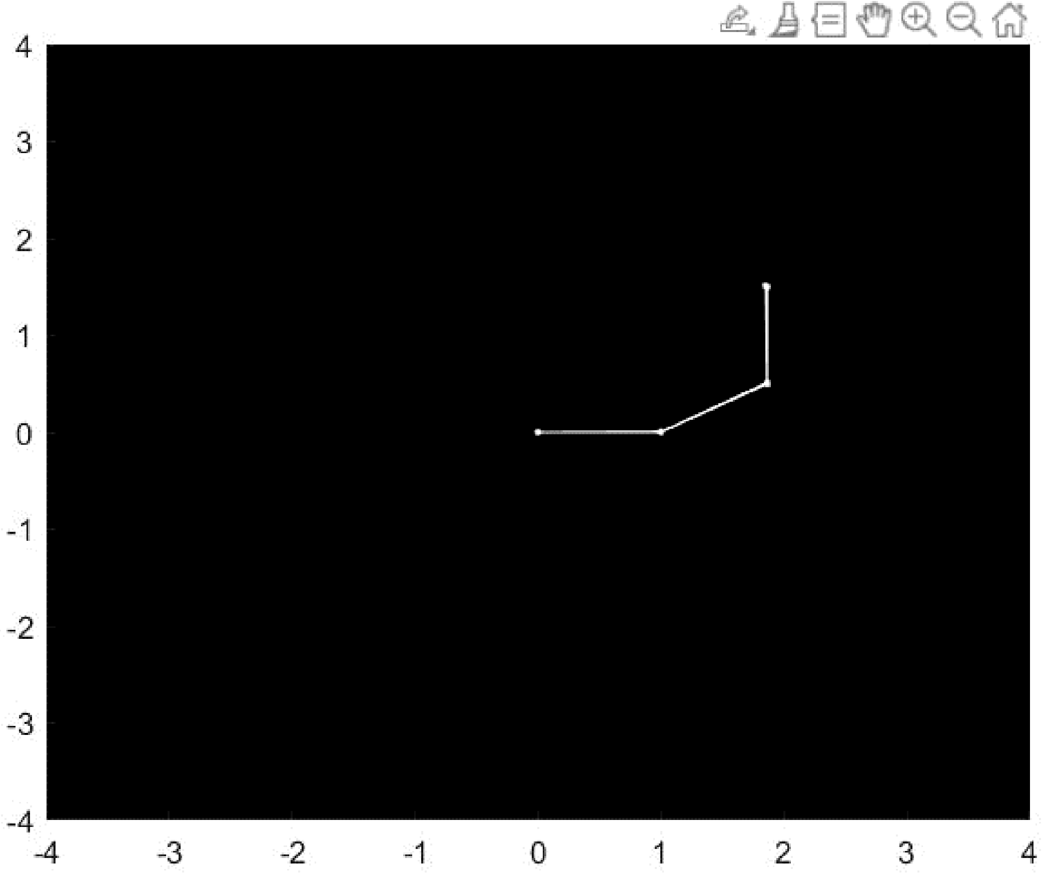


$k_1 = k_2 = k_3 = 0\ N\ (sec/rad)^2$

$\tau_1 = \tau_2 = \tau_3 = 0\ N\ m$

Case 5: W/O Damping and W/O Control

Chaos with 5 Pendulums

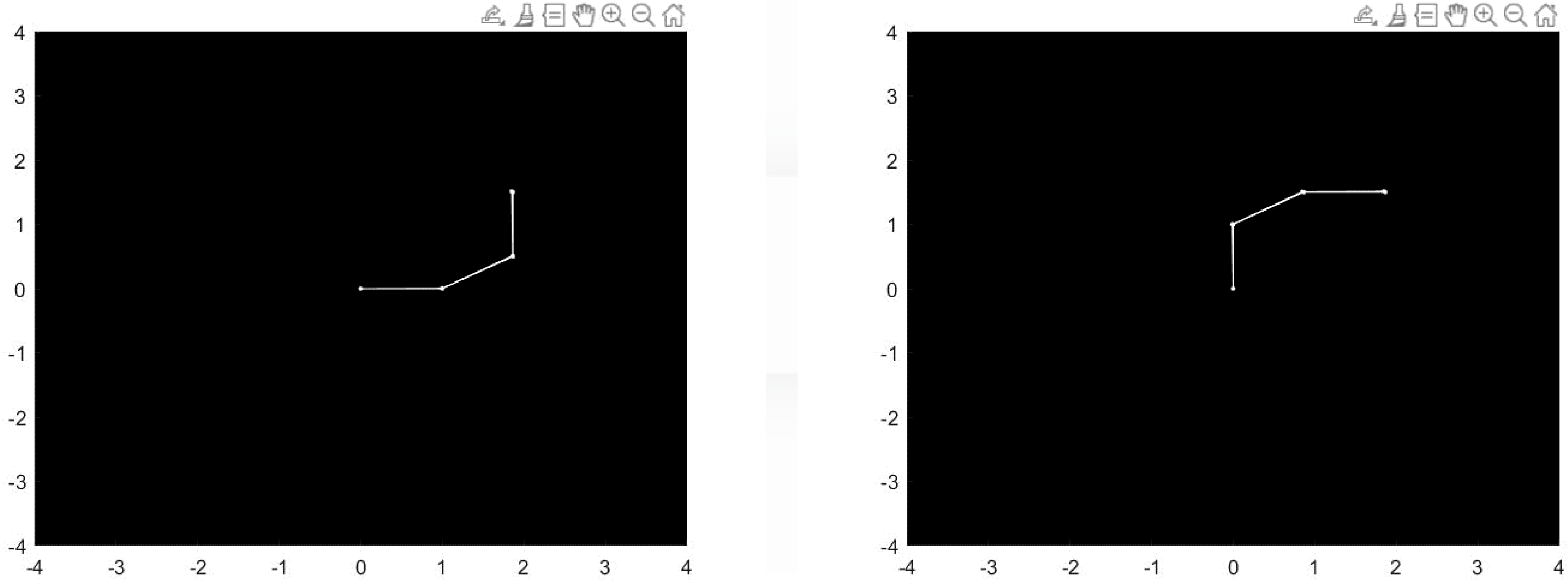


Note: Initial Conditions for each pendulum vary by increment of 0.1%

$k_1 = k_2 = k_3 = 0.5\ N\ (sec/rad)^2$

$\tau_1 = \tau_2 = \tau_3 = 0\ N\ m$

Case 6: W/ Damping and W/O Control Chaos with 5 Pendulums



Base IC: $\boldsymbol{s} = \begin{bmatrix} \frac{\pi}{2} & 0 & \frac{2\pi}{3} & 0 & \pi & 0 \end{bmatrix}'$

Base IC: $\boldsymbol{s} = \begin{bmatrix} \frac{\pi}{2} & 0 & \frac{2\pi}{3} & 0 & \pi & 0 \end{bmatrix}'$

Note: Initial Conditions for each pendulum vary by increment of 0.1%

Summary of Methods Applied

- Defined the designations to analyze Triple Pendulum system
- Identified the DOF and generalized coordinates of the system
- Determine the KE and PE of the system
- Used Euler-Lagrange Equation to determine the equations of motion of the system using MATLAB Symbolic Toolbox
- Used the EOMs to solve various case situations, and simulated and visualized the results; Use of ODE45 solver in MATLAB

Cases Analysis

- The system parameters are arbitrarily assumed and wouldn’t match the real-life situation.
- The control of the pendulum requires analysis of the desired system attributes to get appropriate gain matrices.
- The chaos of triple pendulum is apparent even with slight change in initial condition of the system.
- The chaos of triple pendulum is higher in an undamped system than in a damped system.
- The control of a triple pendulum is simply a control of a robotic arm where each of the joints are assumed to have control torques.

Future Work

- The system can be closely represented to real life conditions with actual parameters from a test rig.
- A test rig can be developed and the experimental and simulated results can be compared.
- To consider the unknown disturbances and reduce their effects PID control can be devised.
- Control at each joints can be replaced with an underactuated triple pendulum with control at its origin only. This system can be analyzed for various control possibilities like control inverted triple pendulum.

Link to Video Presentation:
https://ucincinnati.webex.com/ucincinnati/ldr.php?RCID=a32953848c338f3e088382cf16f5d8b3

Any Questions???

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THANK YOU!!!