Title

Contents

Abstract

Aąężźćńłóś

Chapter 1

Introduction

Predictions of the grawitational wave signal from early Universe cosmoliogical phase trnsition depend on the shape of effective potential of the theory. In this thesis we will investigate how different renormalisations schemes can change form of that potential.

Chapters 1-4 are mean as an introduction. New results are in 5 onward.

Chapter 2

Technical introduction

2.1 Models

2.1.1 Toy model

This model will be used throughout the whole thesis, unless stated otherwise.
For a toy model we choose theory of scalar electrodynamics, described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi D^\mu \Phi^\dagger - \lambda\Phi^4, \tag{2.1.1.0.1}$$

where Φ is a complex scalar field and the vector field present is $U(1)$ gauge boson.

Writing operator D more explicitly it reads:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\Phi + ieA_\mu\Phi)(\partial^\mu\Phi^\dagger - ieA^\mu\Phi^\dagger) - \lambda\Phi^4, \tag{2.1.1.0.2}$$

For the reasons that will be clear in reffinite momentum we will write Φ field as two real scalar fields φ_1 and φ_2, such that:

$$\Phi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \tag{2.1.1.0.3}$$

Then Lagrangian takes form:

$$\begin{aligned} \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)(\partial^\mu\varphi_1 - eA^\mu\varphi_2) \\ +\frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)(\partial^\mu\varphi_2 + eA^\mu\varphi_1) - \frac{1}{4}\lambda(\varphi_1^2 + \varphi_2^2)^2, \end{aligned} \tag{2.1.1.0.4}$$

which we will write for brevity as:

$$\begin{aligned} \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 \\ +\frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2 - \frac{1}{4}\lambda(\varphi_1^2 + \varphi_2^2)^2. \end{aligned} \tag{2.1.1.0.5}$$

For a better track of what is independent of numerical convention, we will also write:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 \\ +\frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2 - c_\lambda\lambda(\varphi_1^2 + \varphi_2^2)^2, \tag{2.1.1.0.6}$$

but $c_\lambda = \frac{1}{4}$ everywhere in the thesis if not stated otherwise.

2.1.2 Real model

$U(2) \times U(2)$ cośtam cośtam

2.2 Renormalisation schemes

2.2.1 $\overline{\text{MS}}$

The minimal-substraction scheme that favorises computation simplicity.
The standard structure on one loop is
constants are always the same.
It is bind to dimentional regularisation.

2.2.2 On-shell

Zero momentum limit version

2.2.3 Half $\overline{\text{MS}}$-Half On-shell

2.3 Effective potential

TO DO: some statements about effective potential in general

Chapter 3

Introduction – calculation of the unrenormalized effective potential

Tree level potential in our model is:

$$V_T = \frac{1}{4}\lambda(\varphi_1^2 + \varphi_2^2)^2. \tag{3.0.0.0.1}$$

The one loop correction to the effective potential is calculated as a sum of the following diagrams:

$$i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2n}\left(\frac{\lambda\frac{1}{2}\varphi_1^2}{k^2+i\varepsilon}\right)^n \tag{3.0.0.0.2}$$

$$i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2n}\left(\frac{1}{3}\frac{\lambda\frac{1}{2}\varphi_1^2}{k^2+i\varepsilon}\right)^n \tag{3.0.0.0.3}$$

$$i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2n}\left(\frac{2e^2\frac{1}{2}\varphi_1^2}{k^2+i\varepsilon}\right)^n (g^\mu{}_\mu - 1). \tag{3.0.0.0.4}$$

Summing all the diagrams in series it gives:

$$i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\lambda\frac{1}{2}\varphi_1^2}{k^2+i\varepsilon}\right)^n \tag{3.0.0.0.5}$$

$$i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{1}{3}\frac{\lambda\frac{1}{2}\varphi_1^2}{k^2+i\varepsilon}\right)^n \tag{3.0.0.0.6}$$

$$i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{2e^2\frac{1}{2}\varphi_1^2}{k^2+i\varepsilon}\right)^n (g^\mu{}_\mu - 1). \tag{3.0.0.0.7}$$

These integrals seems to be hideously infrared divergent. We can, however Wick rotate them to the Euklidean space and perform summation to get:

$$\mathit{logarithms} \tag{3.0.0.0.8}$$

Now, integrals have only singularity at $\varphi_1 = 0$.
There are now several ways to perform the integrals. We will present here briefly two

of them: dimentional regularisation (DR) and cut-off.
For all of our further calculations, we will use DR, but cut-off will apear in 4 as a method used in [1].

3.1 Cut-off

3.2 Dimentional regularisation

After passing to $D = 4 - 2\epsilon$ dimensions and using dimensional regularisation we have:

$$V_{1L} = \frac{1}{4}\frac{(\frac{1}{2}\lambda\varphi_1^2)^2}{(4\pi)^2}\left(-\frac{2}{\epsilon} + \gamma_E - \frac{3}{2} + \log\frac{(\frac{1}{2}\lambda\varphi_1^2)^2}{4\pi\mu^2}\right) \tag{3.2.0.0.1}$$

$$+\frac{1}{4}\frac{(\frac{1}{6}\lambda\varphi_1^2)^2}{(4\pi)^2}\left(-\frac{2}{\epsilon} + \gamma_E - \frac{3}{2} + \log\frac{(\frac{1}{6}\lambda\varphi_1^2)^2}{4\pi\mu^2}\right) \tag{3.2.0.0.2}$$

$$+\frac{1}{4}\frac{3(e^2\varphi_1^2)^2}{(4\pi)^2}\left(-\frac{2}{\epsilon} + \gamma_E - \frac{5}{6} + \log\frac{e^2\varphi^2}{4\pi\mu^2}\right) \tag{3.2.0.0.3}$$

Chapter 4

How did Coleman Weinberg do it?

4.1 $\lambda\varphi^4$ theory

Coleman and Weinberg in [1] start with "$\lambda\varphi^4$" model, namely, with Lagransian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}A(\partial_\mu\varphi)^2 - \frac{1}{2}B\varphi^2 - \frac{1}{4!}C\varphi^4, \tag{4.1.0.0.1}$$

where A, B, C are renormalisation constants.
They then proceeed to calculations of the renormalised effective potential in this theory.
Tree level potential is

$$V = \frac{\lambda}{4!}\varphi_c^4, \tag{4.1.0.0.2}$$

and up to the one loop level is:

$$V = \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}B\varphi_c^2 + \frac{1}{4!}C\varphi_c^4 + i\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\varphi_c^2}{k^2+i\epsilon}\right)^n. \tag{4.1.0.0.3}$$

As mentioned in [1] and 3, this expression seems "hideously infrared divergent". It is thus transformed into one in Euclidean space with apparent infrared divergence turned into logarithmic singularity at $\varphi_c = 0$:

$$V = \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}B\varphi_c^2 + \frac{1}{4!}C\varphi_c^4 + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(1 + \frac{\lambda\varphi_c^2}{2k^2}\right). \tag{4.1.0.0.4}$$

This is then calculated using cut-off method, with cut off ad $k^2 = \Lambda^2$. The result is:

$$V = \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}B\varphi_c^2 + \frac{1}{4!}C\varphi_c^4 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^2 + \frac{\lambda^2\varphi_c^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_c^2}{2\Lambda^2} - \frac{1}{2}\right). \tag{4.1.0.0.5}$$

4.1.1 Renormalisation

Then, they proceed, to renormalise the potential. First renormalisation condition is for the renormalised mass to vanish (eq. (3.5) in [1]).

They write is as:

$$\left.\frac{d^2V}{d\varphi_c^2}\right|_0 = 0, \tag{4.1.1.0.1}$$

which can be suspitious, as, according to discusion in [1] rigth after equation (2.12b), we have that mass is equal to $\left.\frac{d^2V}{d\varphi_c^2}\right|_{\langle\varphi\rangle}$, where $\langle\varphi\rangle$ is VEV of φ.
A priori, it is not guaranteed in our theory, that symmetry is unbroken and $\langle\varphi\rangle = 0$. However, it is also not explicitly broken yet, so we left it for now.
This condition give us, that

$$B = -\frac{\lambda\Lambda^2}{32\pi^2}, \tag{4.1.1.0.2}$$

as second derivative of the potential evaluated at zero is equal to:

$$\left.\frac{d^2V}{d\varphi_c^2}\right|_0 = B + \frac{\lambda\Lambda^2}{32\pi^2}. \tag{4.1.1.0.3}$$

Note that, if the derivative would be evaluated at some non zero $\langle\varphi\rangle$, this relation will vastly change.
After determining B, the following criterium for determining C is presented:

$$\left.\frac{d^4V}{d\varphi_c^4}\right|_M = \lambda. \tag{4.1.1.0.4}$$

We for now leave (although interesting) discusion of introducing the parameter M.
The resulting C is:

$$C = -\frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3}\right). \tag{4.1.1.0.5}$$

Note, that resulting C is completly independent of the value of B determined earlier, as B is not present in $\left.\frac{d^4V}{d\varphi_c^4}\right|_M$.
The resulting potential is:

$$V = \frac{\lambda}{4!}\varphi_c^4 + \frac{\lambda^2\varphi_c^4}{256\pi^2}\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right). \tag{4.1.1.0.6}$$

Then, the discussion proceeds, that, at $\varphi_c = 0$ we have now maximum, not minimum, and that the minimum of the potential occurs at the value of: φ_c determined by:

$$\lambda \ln\frac{\langle\varphi\rangle^2}{M^2} = -\frac{32}{3}\pi^2 + O(\lambda), \tag{4.1.1.0.7}$$

which is very far outside the expected range of validity of the one-loop approximation and must be rejected as superficial.
I am not sure what are the implications of this in physics.

4.2 Scalar electrodynamics

Now, we proceed, to present treatment of the theorie of our main interest conducted in [1].
They start with the theory with the lagransian [1](4.1):

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 + \frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2 - \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2 + \text{counterterms.} \tag{4.2.0.0.1}$$

Then, the resulting renormalised potential is presented [1](4.5):

$$V = \frac{\lambda}{4!}\varphi_c^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\varphi_c^4\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right), \tag{4.2.0.0.2}$$

whith only a brief note, that it is obtained after "straightforward computation" .
We will now investigate more carefully this omited step .
It is implied, that the procedure was the same as in the $\lambda\varphi^4$ case and we shall see, whether it was indeed the same, as well as, whether applied procedure is eligible for this theory.

Let us start with the effective potential with not yet calculated integrals and not yet evaluated renormalisation constants:

$$\begin{aligned} V =& \frac{\lambda}{4!}\varphi_c^4 - \frac{1}{2}B\varphi_c^2 - \frac{1}{4!}C\varphi_c^4 \\ &+ i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\varphi_c^2}{k^2+i\epsilon}\right)^n \\ &+ i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{6}\lambda\varphi_c^2}{k^2+i\epsilon}\right)^n \\ &+ i\int\frac{d^4k}{(2\pi)^4}3\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{e^2\varphi_c^2}{k^2+i\epsilon}\right)^n. \end{aligned} \tag{4.2.0.0.3}$$

This can be transformed as previously from this infrared divergent form, to the form with singularity only at $\varphi_c = 0$:

$$\begin{aligned} V =& \frac{\lambda}{4!}\varphi_c^4 - \frac{1}{2}B\varphi_c^2 - \frac{1}{4!}C\varphi_c^4 \\ &+ \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1 + \frac{\lambda\varphi_c^2}{2k^2}\right) \\ &+ \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1 + \frac{\lambda\varphi_c^2}{6k^2}\right) \\ &+ \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}3\ln\left(1 + \frac{e^2\varphi_c^2}{k^2}\right), \end{aligned} \tag{4.2.0.0.4}$$

and then calculated using cut-off method at $k^2 = \Lambda^2$:

$$
\begin{aligned}
V =& \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}B\varphi_c^2 + \frac{1}{4!}C\varphi_c^4 \\
&+ \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^2 + \frac{\lambda^2\varphi_c^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_c^2}{2\Lambda^2} - \frac{1}{2}\right) \\
&+ \frac{\lambda\Lambda^2}{3\cdot 64\pi^2}\varphi_c^2 + \frac{\lambda^2\varphi_c^4}{9\cdot 256\pi^2}\left(\ln\frac{\lambda\varphi_c^2}{6\Lambda^2} - \frac{1}{2}\right) \\
&+ \frac{3e^2\Lambda^2}{32\pi^2}\varphi_c^2 + \frac{3e^4\varphi_c^4}{64\pi^2}\left(\ln\frac{e^2\varphi_c^2}{\Lambda^2} - \frac{1}{2}\right).
\end{aligned} \tag{4.2.0.0.5}
$$

4.2.1 Renormalisation

Now, it starts the fun part.
The first imposed renormalisation condition in the previous ($\lambda\varphi^4$) case was:

$$
\left.\frac{d^2V}{d\varphi_c^2}\right|_0 = 0, \tag{4.2.1.0.1}
$$

where, it was stated, is equivalent, to renormalised mass being zero.
What is important, this "renormalized mass" is not meant to be a physicall mass, rather, the mass parameter, the constant that stand beside second power of the field, which can be defined as "$\left.\frac{d^2V}{d\varphi_c^2}\right|_0$" itself. Here it is not explicitly stated, whether this condition is being used in the case of scalar electrodynamics and we shall investigate that matter.
Let us start with the second renormalisation condition, namely:

$$
\left.\frac{d^4V}{d\varphi_c^4}\right|_M = \lambda, \tag{4.2.1.0.2}
$$

and reverse engeneer what happened in [1].
We can do this, because of the reason stated in 4.1.1 as B does not appear in $\left.\frac{d^4V}{d\varphi_c^4}\right|_M$.
We have that:

$$
\begin{aligned}
\left.\frac{d^4V}{d\varphi_c^4}\right|_M =& \lambda + C \\
&+ \frac{11\lambda^2}{32\pi^2} + \frac{3\lambda^2}{32\pi^2}\ln\frac{\lambda M^2}{2\Lambda^2} \\
&+ \frac{11\lambda^2}{288\pi^2} + \frac{\lambda^2}{96\pi^2}\ln\frac{\lambda M^2}{6\Lambda^2} \\
&+ \frac{(75-18\alpha)e^4}{16\pi^2} + \frac{9e^4}{8\pi^2}\ln\frac{e^2M^2}{\Lambda^2}.
\end{aligned} \tag{4.2.1.0.3}
$$

From this, for $\left.\frac{d^4V}{d\varphi_c^4}\right|_M = \lambda$, we conclude that:

$$\begin{aligned} C = -\Big(&\frac{11\lambda^2}{32\pi^2} + \frac{3\lambda^2}{32\pi^2}\ln\frac{\lambda M^2}{2\Lambda^2} \\ &+ \frac{11\lambda^2}{288\pi^2} + \frac{\lambda^2}{96\pi^2}\ln\frac{\lambda M^2}{6\Lambda^2} \\ &+ \frac{(75-18\alpha)e^4}{16\pi^2} + \frac{9e^4}{8\pi^2}\ln\frac{e^2M^2}{\Lambda^2}\Big). \end{aligned} \tag{4.2.1.0.4}$$

Subsituing this result to the potential result in:

$$\begin{aligned} V =& \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}B\varphi_c^2 \\ &- \frac{1}{4!}\Big(\frac{11\lambda^2}{32\pi^2} + \frac{3\lambda^2}{32\pi^2}\ln\frac{\lambda M^2}{2\Lambda^2} \\ &+ \frac{11\lambda^2}{288\pi^2} + \frac{\lambda^2}{96\pi^2}\ln\frac{\lambda M^2}{6\Lambda^2} \\ &+ \frac{(75-18\alpha)e^4}{16\pi^2} + \frac{9e^4}{8\pi^2}\ln\frac{e^2M^2}{\Lambda^2}\Big)\varphi_c^4 \\ &+ \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^2 + \frac{\lambda^2\varphi_c^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_c^2}{2\Lambda^2} - \frac{1}{2}\right) \\ &+ \frac{\lambda\Lambda^2}{3\cdot 64\pi^2}\varphi_c^2 + \frac{\lambda^2\varphi_c^4}{9\cdot 256\pi^2}\left(\ln\frac{\lambda\varphi_c^2}{6\Lambda^2} - \frac{1}{2}\right) \\ &+ \frac{3e^2\Lambda^2}{32\pi^2}\varphi_c^2 + \frac{3e^4\varphi_c^4}{64\pi^2}\left(\ln\frac{e^2\varphi_c^2}{\Lambda^2} - \frac{1}{2}\right), \end{aligned} \tag{4.2.1.0.5}$$

and after canceling:

$$\begin{aligned} V =& \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}B\varphi_c^2 \\ &+ \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^2 + \frac{\lambda\Lambda^2}{3\cdot 64\pi^2}\varphi_c^2 + \frac{3e^2\Lambda^2}{32\pi^2}\varphi_c^2 \\ &+ \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\varphi_c^4\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right). \end{aligned} \tag{4.2.1.0.6}$$

Let us notice, that above potential differs from [1](4.5) only by the terms:

$$\frac{1}{2}B\varphi_c^2 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^2 + \frac{\lambda\Lambda^2}{3\cdot 64\pi^2}\varphi_c^2 + \frac{3e^2\Lambda^2}{32\pi^2}\varphi_c^2. \tag{4.2.1.0.7}$$

As such, in the renormalistion scheme used in [1], we must have that:

$$\frac{1}{2}B\varphi_c^2 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c^2 + \frac{\lambda\Lambda^2}{3\cdot 64\pi^2}\varphi_c^2 + \frac{3e^2\Lambda^2}{32\pi^2}\varphi_c^2 = 0, \tag{4.2.1.0.8}$$

thus:

$$B = -2\Big(\frac{\lambda\Lambda^2}{64\pi^2} + \frac{\lambda\Lambda^2}{3\cdot 64\pi^2} + \frac{3e^2\Lambda^2}{32\pi^2}\Big). \tag{4.2.1.0.9}$$

Then, the renormalised potential is indeed:

$$V = \frac{\lambda}{4!}\varphi_c^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\varphi_c^4\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right) \tag{4.2.1.0.10}$$

as in [1](4.5), but what is important is, then indeed

$$\left.\frac{d^2V}{d\varphi_c^2}\right|_0 = 0, \tag{4.2.1.0.11}$$

so this was preceisly the condition used to renormalised this potential.
This is precesuily the condition, that guarantees the φ^4 form of the potential.

TO DO: Napisać o stosunku mas

Chapter 5

$\overline{\text{MS}}$ renormalisation of the effective potential

Although the goal of our thesis is to renormalise effective potential on-shell $\overline{\text{MS}}$ renormalisation gives:

$$
\begin{aligned}
V_R = {} & c_\lambda \lambda \varphi_1^4 \\
& + \frac{1}{4} \frac{(\frac{1}{2}\lambda\varphi_1^2)^2}{(4\pi)^2} \left(-\frac{3}{2} + \log \frac{(\frac{1}{2}\lambda\varphi_1^2)^2}{\mu^2} \right) \\
& + \frac{1}{4} \frac{(\frac{1}{6}\lambda\varphi_1^2)^2}{(4\pi)^2} \left(-\frac{3}{2} + \log \frac{(\frac{1}{6}\lambda\varphi_1^2)^2}{\mu^2} \right) \\
& + \frac{1}{4} \frac{3(e^2\varphi_1^2)^2}{(4\pi)^2} \left(-\frac{5}{6} + \log \frac{e^2\varphi_1^2}{\mu^2} \right).
\end{aligned} \tag{5.0.0.0.1}
$$

Here, as will be apparent in the second, we are interested only in e^4 part, so from now on, we will write:

$$
V_R = c_\lambda \lambda \varphi_1^4 + \frac{3e^4}{64\pi^2} \left(-\frac{5}{6} + \log \frac{e^2\varphi_1^2}{\mu^2} \right) \varphi_1^4. \tag{5.0.0.0.2}
$$

We can bind e to λ and v at the loop level from the VEV definition:

$$
\left. \frac{\partial V_R}{\partial \varphi_1} \right|_v = 0. \tag{5.0.0.0.3}
$$

This gives the condition:

$$
4c_\lambda \lambda v^3 - \frac{e^4 v^3}{16\pi^2} - \frac{3e^4 v^3}{16\pi^2} \log \frac{e^2 v^2}{\mu^2} = 0. \tag{5.0.0.0.4}
$$

Setting scale parameter μ to the effective mass of the vector, namely ev, we have simpler form of:

$$
4c_\lambda \lambda v^3 - \frac{e^4 v^3}{16\pi^2} = 0. \tag{5.0.0.0.5}
$$

Which gives:

$$\lambda = \frac{e^4}{64 c_\lambda \pi^2}. \tag{5.0.0.0.6}$$

Writing potential with this substitutions yields:

$$V_R = \frac{3e^4}{64\pi^2}\Big(-\frac{1}{2} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{5.0.0.0.7}$$

This is exactly the same potential as obtained in [1] with different renormalisation. Equations 7.1.0.0.12 is very important in this discussion as it states, that λ is of order e^4 in our model. This post factum justifies our choice in taking only e^4 part, as other part was of order e^8.

5.1 Physical mass

We can now also bind square of the physical mass M_P^2 as the second derivative of the renormalised effective potential at VEV.

$$M_P^2 = \frac{\partial^2 V_R}{\partial\varphi_1^2}\Big|_v = \frac{9e^4}{16\pi^2}\Big(-\frac{1}{2}\Big)v^2 + \frac{3e^4v^2}{8\pi^2} + \frac{9e^4v^2}{32\pi^2}. \tag{5.1.0.0.1}$$

This gives that:

$$M_P^2 = \frac{3e^4v^2}{8\pi^2}. \tag{5.1.0.0.2}$$

From this we have that the ratio between scalar mass M_P^2 and vector mass $m(V)^2$ is:

$$\frac{M_P^2}{m(V)^2} = \frac{\frac{3e^4v^2}{8\pi^2}}{e^2v^2} = \frac{3e^2}{8\pi^2}. \tag{5.1.0.0.3}$$

We can now also express e^4 and λ in terms of M_P^2 and v:

$$e^4 = \frac{8M_P^2\pi^2}{3v^2} \tag{5.1.0.0.4}$$

$$\lambda = \frac{M_P^2}{24c_\lambda v^2}. \tag{5.1.0.0.5}$$

Writing potential in these terms gives:

$$V_R = \frac{M_P^2}{8v^2}\Big(-\frac{1}{2} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{5.1.0.0.6}$$

Chapter 6

On shell renormalisation of the effective potential – finite momentum approach

One of the goals, was to renormalsed potential in the classical on shell scheme. We will present this aprousch here. Following //cittation needed (Japończycy?)//. This approuch however, will have one big issue, described in 6.9

6.1 Second derivative condition

To calculate on-shell renormalisation we need to calculate self energy. However, it turns out, that simple calculation of self energy fails the test of comparison between the zero-momentum limit of the self energy and the second derivative of the effective potential.
Namely, it should be safisfied that:

$$\lim_{p^2 \to 0} \Sigma(p^2) = \frac{\partial^2 V_{eff}}{\partial \varphi_1^2}, \tag{6.1.0.0.1}$$

TO DO: napisać ile wychodzi

but it is not the case.
Let us show this.
We have that:

$$V_{eff} = \tag{6.1.0.0.2}$$

So:

$$\frac{\partial^2 V_{eff}}{\partial \varphi_1^2} = \tag{6.1.0.0.3}$$

On the other hand contribution to Σ constitued by following diagrams:

TO DO: diagrams

The values of these diagrams are as follows:

$$-i\frac{e^2}{M_V}\Big[M_V^2 a(M_2)+(-p^2-M_V^2+M_2^2)a(M_V)-(p^2+M_2^2)^2 b_0(p,0,M_2)$$
$$+(p^2+M_2^2-M_V^2)^2 b_0(p,M_V,M_2)\Big] \quad (6.1.0.0.4)$$

$$-i3e^2 a_b(M_V) \quad (6.1.0.0.5)$$
$$-i12c_\lambda \lambda a(M_1) \quad (6.1.0.0.6)$$
$$-i4c_\lambda \lambda a(M_2) \quad (6.1.0.0.7)$$

So:

$$\Sigma = -\frac{e^2}{M_V}\Big[M_V^2 a(M_2)+(-p^2-M_V^2+M_2^2)a(M_V)-(p^2+M_2^2)^2 b_0(p,0,M_2)$$
$$+(p^2+M_2^2-M_V^2)^2 b_0(p,M_V,M_2)\Big]$$
$$-3e^2 a_b(M_V)$$
$$-12c_\lambda \lambda a(M_1)$$
$$-4c_\lambda \lambda a(M_2) \quad (6.1.0.0.8)$$

Where:

$$a(M)= \quad (6.1.0.0.9)$$
$$b_0(p,M_1,M_2)= \quad (6.1.0.0.10)$$
$$a^b(M)= \quad (6.1.0.0.11)$$

and

$$b_0(0,M_1,M_2)= \quad (6.1.0.0.12)$$

Then, writing more explicitly we have that:

$$abcd \quad (6.1.0.0.13)$$

So, we see, that indeed:

$$\lim_{p^2\to 0}\Sigma(p^2)\neq \frac{\partial^2 V_{eff}}{\partial\varphi_1^2}. \quad (6.1.0.0.14)$$

6.2 Introducing explicit VEV

There is a solution to this:
From the $\overline{\text{MS}}$considerations, we know that Φ have non-zero VEV, let us call it v. Let us rotate Φ in such a way, that $\langle\varphi_1\rangle = v$ and $\langle\varphi_2\rangle = 0$, where now v is real.

Keeping this in mind, we can rewrite Lagrangian in terms of shifted fields φ_1, φ_2 which have both zero VEV, now VEV is explicitly in the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu(\varphi_1 + v) - eA_\mu\varphi_2)^2 \tag{6.2.0.0.1}$$
$$+\frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu(\varphi_1 + v))^2 - c_\lambda\lambda((\varphi_1 + v)^2 + \varphi_2^2)^2.$$

This breaks the symmetry, but now there are more interaction terms in the Lagrangian and this leads to different self energy, now consistent with the second derivative of the effective potential, as will be shown in 6.4.

6.3 Contributions to Σ and T

Contributions to Σ and T constitutes now of the following diagrams:

TO DO: diagrams

The values of diagrams are as follows:
Contributing to Σ are:

$$- i\frac{e^2}{M_V}\Big[M_V^2 a(M_2) + (-p^2 - M_V^2 + M_2^2)a(M_V) - (p^2 + M_2^2)^2 b_0(p, 0, M_2)$$
$$+ (p^2 + M_2^2 - M_V^2)^2 b_0(p, M_V, M_2)\Big] \tag{6.3.0.0.1}$$

$$- i\frac{e^4v^2}{2M_V^4}\Big[2M_V^2 a(M_V) + p^4 b_0(p, 0, 0) - 2(p^2 - M_V^2)^2 b_0(p, M_V, 0)$$
$$+ 16M_V^4 b_0^b(p, M_V, M_V) + (p^4 - 4p^2M_V^2 - 4M_V^4)b_0(p, M_V, M_V)\Big]. \tag{6.3.0.0.2}$$

$$-i3e^2 a_b(M_V) = \tag{6.3.0.0.3}$$
$$-i12c_\lambda\lambda a(M_1) = \tag{6.3.0.0.4}$$
$$-i4c_\lambda\lambda a(M_2) = \tag{6.3.0.0.5}$$
$$-i288c_\lambda^2\lambda^2v^2 b_0(p, M_1, M_1) = \tag{6.3.0.0.6}$$
$$-i32c_\lambda^2\lambda^2v^2 b_0(p, M_2, M_2) = \tag{6.3.0.0.7}$$
$$\tag{6.3.0.0.8}$$

Contributing to T are:

$$-i3e^2 v a^b(M_V) = \tag{6.3.0.0.9}$$
$$-i12c_\lambda\lambda v a(M_1) = \tag{6.3.0.0.10}$$
$$-i4c_\lambda\lambda v a(M_2) = \tag{6.3.0.0.11}$$
$$-i4c_\lambda\lambda v^3 = . \tag{6.3.0.0.12}$$

Where

$$\begin{aligned} a(M) &= && (6.3.0.0.13)\\ b_0(p, M_1, M_2) &= && (6.3.0.0.14)\\ a^b(M) &= && (6.3.0.0.15)\\ b_0^b(p, M_1, M_2) &= . && (6.3.0.0.16) \end{aligned}$$

6.4 Second derivative condition – explicit VEV

Now, we have that:

$$\begin{aligned} \Sigma = &- \frac{e^2}{M_V}\Big[M_V^2 a(M_2) + (-p^2 - M_V^2 + M_2^2)a(M_V) - (p^2 + M_2^2)^2 b_0(p, 0, M_2)\\ &+ (p^2 + M_2^2 - M_V^2)^2 b_0(p, M_V, M_2)\Big]\\ &- \frac{e^4 v^2}{2M_V^4}\Big[2M_V^2 a(M_V) + p^4 b_0(p, 0, 0) - 2(p^2 - M_V^2)^2 b_0(p, M_V, 0)\\ &+ 16 M_V^4 b_0^b(p, M_V, M_V) + (p^4 - 4p^2 M_V^2 - 4M_V^4) b_0(p, M_V, M_V)\Big]\\ &- 3e^2 a_b(M_V)\\ &- 12 c_\lambda \lambda a(M_1)\\ &- 4 c_\lambda \lambda a(M_2)\\ &- 288 c_\lambda^2 \lambda^2 v^2 b_0(p, M_1, M_1)\\ &- 32 c_\lambda^2 \lambda^2 v^2 b_0(p, M_2, M_2), \qquad (6.4.0.0.1) \end{aligned}$$

where

$$\begin{aligned} a(M) &= && (6.4.0.0.2)\\ b_0(p, M_1, M_2) &= && (6.4.0.0.3)\\ a^b(M) &= && (6.4.0.0.4)\\ b_0^b(p, M_1, M_2) &= && (6.4.0.0.5) \end{aligned}$$

and

$$\begin{aligned} b_0(0, M_1, M_2) &= && (6.4.0.0.6)\\ b_0^b(0, M_1, M_2) &= && (6.4.0.0.7) \end{aligned}$$

So, we have that:

$$\Sigma = \qquad (6.4.0.0.8)$$

So indeed, we have now, that:

$$\lim_{p^2 \to 0} \Sigma(p^2) = \frac{\partial^2 V_{eff}}{\partial \varphi_1^2}, \qquad (6.4.0.0.9)$$

6.5 Counterterms

Following [1] we put the mass counterterm even though initialy the mass term was not present in the Lagrangian. We do so, because our theory has no a priory symmetry that would guarantee disapering mass term in renormalised Lagrangian. It will turn out to be crutial in our .
The Lagrangian with δZ, $\delta\lambda$ and δm counterterms looks like this:

$$\begin{aligned}\mathcal{L}_{\mathcal{R}} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} && (6.5.0.0.1)\\ &+ (1+\delta Z)(\frac{1}{2}(\partial_\mu(\varphi_1+v) - eA_\mu\varphi_2)^2 + \frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu(\varphi_1+v))^2)\\ &- (1+\delta Z)^2 c_\lambda(\lambda+\delta\lambda)((\varphi_1+v)^2+\varphi_2^2)^2\\ &- c_m\delta m((\varphi_1+v)^2+\varphi_2^2). && (6.5.0.0.2)\end{aligned}$$

Separating the terms with the first power of renormalisation constants and second power of φ_1, we obtain correction to the self energy equal to:

$$\delta\Sigma = -12c_\lambda v^2(2\lambda\delta Z_\varphi + \delta\lambda) - 2c_m\delta m - p^2\delta Z_\varphi, \quad (6.5.0.0.3)$$

where $p^2 = -\partial_\mu\varphi_1\partial^\mu\varphi_1$.
Separating the terms with the first power of renormalisation constants and first power of φ_1, we obtain correction to the tadpole equal to:

$$\delta T = -4c_\lambda v^3(2\lambda\delta Z_\varphi + \delta\lambda) - 2c_m\delta m v. \quad (6.5.0.0.4)$$

6.6 Renormalization conditions

First approach is to impose renormalisation conditions resembling classical on-shell. Here, Σ' stands for $\frac{\mathrm{d}\Sigma}{\mathrm{d}p^2}$ and, if not stated otherwise, Σ, $\delta\Sigma$ and Σ' are evaluated at $p^2 = M_P^2$, where M_P stands for physical mass. We denote real part as $\Re\,()$.

$$\begin{aligned} T + \delta T &= 0 && (6.6.0.0.1)\\ \Re(\Sigma) + \Re(\delta\Sigma) &= 0 && (6.6.0.0.2)\\ \Re(\Sigma') &= 0. && (6.6.0.0.3)\end{aligned}$$

This gives us:

$$\begin{aligned}\delta m &= \frac{-1}{4c_m}\left(\Re(\Sigma) - \frac{3}{v}T - M_P^2\Re(\Sigma')\right) && (6.6.0.0.4)\\ \delta\lambda &= \frac{1}{8c_\lambda v^2}\left(\Re(\Sigma) - \frac{1}{v}T - (16c_\lambda\lambda v^2 + M_P^2)\Re(\Sigma')\right) && (6.6.0.0.5)\\ \delta Z &= \Re(\Sigma'). && (6.6.0.0.6)\end{aligned}$$

We define M_P^2 as the second derivative of the tree potential evaluated at VEV, so:

$$M_P^2 = \frac{\partial^2}{\partial\varphi_1^2}c_\lambda\lambda\varphi_1^4\Big|_v = 12c_\lambda\lambda v^2 \quad (6.6.0.0.7)$$

and we impose, that it does not change after one loop contributions.

6.7 Partition by order in e

Here, we will be interested in only contributions up to order e^4. From our $\overline{\text{MS}}$considerations we can see, that λ should be of order e^4, that M_V^2 should be of order e^2 and that M_1, M_2 should be of order e^4. As that we are intersted only in following parts of contributions to Σ and T:

$$\begin{aligned}
\Sigma_{e^0} =& -\frac{e^2}{M_V^2}\Big[-p^4 b_0(p,0,M_2) + p^4 b_0(p,M_V,M_2)\Big] \\
& -\frac{e^4v^2}{M_V^4}\Big[p^4 b_0(p,0,0) - 2p^4 b_0(p,M_V,0) + p^4 b_0(p,M_V,M_V)\Big] \quad (6.7.0.0.1)\\
\Sigma_{e^2} =& -\frac{e^2}{M_V^2}\Big[-p^2 a(M_V) - 2p^2 M_V^2 b_0(p,M_V,M_2)\Big] \\
& -\frac{e^4v^2}{2M_V^4}\Big[4p^2 M_V^2 b_0(p,M_V,0) - 4p^2 M_V^2 b_0(p,M_V,M_V)\Big] \quad (6.7.0.0.2)\\
\Sigma_{e^4} =& -\frac{e^2}{M_V^2}a(M_V)\Big[-M_V^2 a(M_V) + M_V^4 b_0(p,M_V,M_2)\Big] \\
& -\frac{e^4v^2}{2M_V^4}\Big[2M_V^2 a(M_V) - 2M_V^4 b_0(p,M_V,0) \\
& + 16M_V^4 b_0^b(p,M_V,M_V) - 4M_V^4 b_0(p,M_V,M_V)\Big] \\
& - 3e^2 a^b(M_V) \\
& -\frac{e^2}{M_V^2}\Big[-2p^2 M_2^2 b_0(p,0,M_2) + 2p^2 M_2^2 b_0(p,M_V,M_2)\Big] \quad (6.7.0.0.3)\\
T_{e^4} =& - 3e^2 v a^b(M_V) - 4c_\lambda \lambda v^3. \quad (6.7.0.0.4)
\end{aligned}$$

6.8 Finitness of the renormalized potential

The divergent part of T, Σ and Σ' are:

$$\mathrm{div}T = -\frac{3e^4v^4}{16\pi^2}\Big(-\frac{2}{\epsilon}\Big) \quad (6.8.0.0.1)$$

$$\mathrm{div}\Sigma = \frac{6e^2(M_P^2 - 3e^2v^2)}{32\pi^2}\Big(-\frac{2}{\epsilon}\Big) \quad (6.8.0.0.2)$$

$$\mathrm{div}\Sigma' = \frac{3e^2}{16\pi^2}\Big(-\frac{2}{\epsilon}\Big). \quad (6.8.0.0.3)$$

After subsituing to $\delta\lambda$, δm, δZ and then to V_R we see that $\mathrm{div}V_R = 0$, thus renormalisation procedure succeeds in canceling divergences.

6.9 Resulting effective potential

Although full potential is to long to explicitly write in symbolic form, we will present it with some of the numerical values substituted. What we should note, is that in this case renormalized potential have both φ_1^2 term and φ_1^4 term, unlike in [1].

TO DO: diagramy, równania, takie, takie

6.10 Downsides of this approouch

The big downside of this method is that we incorporated the condition of not changing VEV between tree and one-loop level to our renormalization conditions. This results in inability to derive relationships at one-loop level between M_P^2, v and e from the condition that v is minimum of the first derivative of renormalised effective potential. Using this condition results in tautology, as we already used this condition for renormalisation.

Chapter 7

Zero momentum approach

Here, as well, we will consider potential up to order e^4.
We start with the 1-loop level potential with counterterms:

$$V_R^{\text{1-loop}} = \frac{3e^4}{64\pi^2}\Big(-\frac{2}{\epsilon} + \gamma_E - \frac{5}{6} + \log\frac{e^2\varphi_1^2}{4\pi\mu^2}\Big)\varphi_1^4 + c_\lambda\delta\lambda\varphi_1^4 + c_m\delta m\varphi_1^2. \quad (7.0.0.0.1)$$

Note, that when we will write v, meaning VEV, the vacuum expectation value, we will mean 1-loop level VEV.
Here we will present the approach more simmilar to one from [1]. The renormalisation conditions will be of the form:

$$\frac{d^k}{d\varphi_1^k}V\Big|_\alpha = \beta \quad (7.0.0.0.2)$$

where V is some part of the renormalized effective potential and α, β are some parameteres that will vary between the versions we will present here.
In this way renoramlisation conditions are stated only in terms of the effective potential itself without refering to self energy. As a on-shell condition we treat:

$$\frac{d^2}{d\varphi_1^2}V_R^{\text{1-loop}}\Big|_v = 0 \quad (7.0.0.0.3)$$

as it is, in a sense, a zero-momentum version of the condition of preservation of the tree level mass at the one-loop level.
In [1] the conditions used were (see 4):

$$\frac{d^2}{d\varphi_1^2}V_R\Big|_0 = 0 \quad (7.0.0.0.4)$$

$$\frac{d^4}{d\varphi_1^4}V_R\Big|_v = 24c_\lambda\lambda. \quad (7.0.0.0.5)$$

Note, that, in [1], as mentioned in 4, the convention is used, that $c_\lambda = 4!$.
We will present a few different versions of possible renormalisation conditions.

7.1 Analoguos to Coleman-Weinberg

As renormalisation conditions we impose that:

$$\frac{\partial^2}{\partial\varphi_1^2}V_R\Big|_0 = 0 \tag{7.1.0.0.1}$$

$$\frac{\partial^4}{\partial\varphi_1^4}V_R\Big|_v = 24c_\lambda\lambda. \tag{7.1.0.0.2}$$

These are exactly the same conditions as presented in [1]. We calculate it here once again for two reasons:
Firstly – we use different regularisation, so, although we expect for the resulting potential to be the same, the intermediate steps may (and will) differ. We think, that presenting them here can give a better inside what is going on and better reference to compare with other presented versions of the scheme.
Secondly – we will now write it with our convention of naming things, so it will be a better for reference to compare with other versions of the scheme for this reason too.
Codesponding derivatives are:

$$\frac{\partial^2}{\partial\varphi_1^2}V_R^{\text{1-loop}} = \frac{9e^4\varphi_1^2}{16\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2\varphi_1^2}{4\pi\mu^2}\Big)+\frac{21e^4}{32\pi^2}\varphi_1^2+12c_\lambda\delta\lambda\varphi_1^2+2c_m\delta m \tag{7.1.0.0.3}$$

$$\frac{\partial^4}{\partial\varphi_1^4}V_R^{\text{1-loop}} = \frac{9e^4\varphi_1^2}{8\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2\varphi_1^2}{4\pi\mu^2}\Big)+\frac{75e^4}{16\pi^2}+24c_\lambda\delta\lambda. \tag{7.1.0.0.4}$$

So the conditions take form:

$$2c_m\delta m = 0 \tag{7.1.0.0.5}$$

$$\frac{9e^4v^2}{8\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)+\frac{75e^4}{16\pi^2}+24c_\lambda\delta\lambda = 0. \tag{7.1.0.0.6}$$

Solving for $\delta\lambda$ and δm we have:

$$\delta m = 0 \tag{7.1.0.0.7}$$

$$\delta\lambda = -\frac{3e^4}{64\pi^2c_\lambda}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)-\frac{25e^4}{128\pi^2c_\lambda}. \tag{7.1.0.0.8}$$

Then, the renormalised potential is:

$$V_R = c_\lambda\lambda+\frac{3e^4\varphi_1^4}{64\pi^2}\Big(-\frac{25}{6}+\log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{7.1.0.0.9}$$

Which is identical to the one from [1], as described in 4.2.1 and all the results for expressing λ as e and v. As well as relations between λ, e and the pshysical mass would be the same as in [1].
We can bind λ to e and v at the loop level from the VEV definition:

$$\frac{\partial V_R}{\partial\varphi_1}\Big|_v = 0. \tag{7.1.0.0.10}$$

This gives the condition:

$$4c_\lambda \lambda v^3 - \frac{25e^4v^3}{32\pi^2} + \frac{3e^4v^3}{32\pi^2} = 0. \qquad (7.1.0.0.11)$$

Which gives:

$$\lambda = \frac{11e^4}{64c_\lambda \pi^2}. \qquad (7.1.0.0.12)$$

Writing potential with this substitution yields:

$$V_R = \frac{3e^4}{64\pi^2}\Big(-\frac{1}{2} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \qquad (7.1.0.0.13)$$

This is, as expected, exactly same potential as in [1].
We can now also bind square of the physical mass M_P^2 as the second derivative of the renormalised effective potential at VEV. The result for expressing M_P^2 in terms of e, as well as the ratio between scalar and vector mass will be the same as in the 5.1 and [1]as they depend only on the form of the potential as in 7.1.0.0.13.

$$M_P^2 = \frac{\partial^2 V_R}{\partial\varphi_1^2}\Big|_v = \frac{9e^4}{16\pi^2}\Big(-\frac{1}{2}\Big)v^2 + \frac{3e^4v^2}{8\pi^2} + \frac{9e^4v^2}{32\pi^2}. \qquad (7.1.0.0.14)$$

This gives that:

$$M_P^2 = \frac{3e^4v^2}{8\pi^2}. \qquad (7.1.0.0.15)$$

From this we have that the ratio between scalar mass M_P^2 and vector mass $m(V)^2$ is:

$$\frac{M_P^2}{m(V)^2} = \frac{\frac{3e^4v^2}{8\pi^2}}{e^2v^2} = \frac{3e^2}{8\pi^2}. \qquad (7.1.0.0.16)$$

We can now also express e^4 and λ in terms of M_P^2 and v (this part will be different than in 7.1.0.0.13 or [1]for λ):

$$e^4 = \frac{8M_P^2\pi^2}{3v^2} \qquad (7.1.0.0.17)$$

$$\lambda = \frac{11M_P^2}{24c_\lambda v^2}. \qquad (7.1.0.0.18)$$

Writing potential in these terms gives:

$$V_R = \frac{M_P^2}{8v^2}\Big(-\frac{1}{2} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \qquad (7.1.0.0.19)$$

7.2 Vanishing derivatives

As renormalisation conditions we impose that:

$$\frac{\partial^2}{\partial\varphi_1^2}V_R^{\text{1-loop}}\Big|_v = 0 \tag{7.2.0.0.1}$$

$$\frac{\partial^4}{\partial\varphi_1^4}V_R^{\text{1-loop}}\Big|_v = 0. \tag{7.2.0.0.2}$$

Codesponding derivatives are:

$$\frac{\partial^2}{\partial\varphi_1^2}V_R^{\text{1-loop}} =\frac{9e^4\varphi_1^2}{16\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2\varphi_1^2}{4\pi\mu^2}\Big)+\frac{21e^4}{32\pi^2}\varphi_1^2+12c_\lambda\delta\lambda\varphi_1^2+2c_m\delta m \tag{7.2.0.0.3}$$

$$\frac{\partial^4}{\partial\varphi_1^4}V_R^{\text{1-loop}} =\frac{9e^4\varphi_1^2}{8\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2\varphi_1^2}{4\pi\mu^2}\Big)+\frac{75e^4}{16\pi^2}+24c_\lambda\delta\lambda. \tag{7.2.0.0.4}$$

So the conditions take form:

$$\frac{9e^4v^2}{16\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)+\frac{21e^4}{32\pi^2}v^2+12c_\lambda\delta\lambda v^2+2c_m\delta m=0 \tag{7.2.0.0.5}$$

$$\frac{9e^4v^2}{8\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)+\frac{75e^4}{16\pi^2}+24c_\lambda\delta\lambda=0. \tag{7.2.0.0.6}$$

Solving for $\delta\lambda$ and δm we have:

$$\delta m =\frac{27e^4v^2}{32\pi^2c_m} \tag{7.2.0.0.7}$$

$$\delta\lambda =-\frac{3e^4}{64\pi^2c_\lambda}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)-\frac{75e^4}{128\pi^2c_\lambda}. \tag{7.2.0.0.8}$$

Then, the renormalised potential is:

$$V_R=c_\lambda\lambda\varphi_1^4+\frac{3e^4}{64\pi^2}\Big(-\frac{25}{6}+\log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4+\frac{27e^4v^2}{32\pi^2}\varphi_1^2. \tag{7.2.0.0.9}$$

We would like to write V_R in terms of M_P – the physical mass and v. First relation is $\lambda=\frac{M_P^2}{12c_\lambda v^2}$. Potential writen with this substitution becames:

$$V_R=\frac{M_P^2}{12v^2}\varphi_1^4+\frac{3e^4}{64\pi^2}\Big(-\frac{25}{6}+\log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4+\frac{27e^4v^2}{32\pi^2}\varphi_1^2. \tag{7.2.0.0.10}$$

To write e in terms of M_P and v we use the condition that

$$\frac{\partial}{\partial\varphi_1}V_R\Big|_v=0, \tag{7.2.0.0.11}$$

as v is by definition minimum of the potential.
It gives the relation:

$$\frac{M_P^2v}{3}+\frac{e^4v^3}{16\pi^2}\Big(-\frac{25}{2}\Big)+\frac{3e^4v^3}{32\pi^2}+\frac{27e^4v^3}{16\pi^2}=0. \tag{7.2.0.0.12}$$

Thus, we conclude that:

$$e^4 = \frac{-M_P^2\pi^2}{3v^2}. \tag{7.2.0.0.13}$$

This, unfortunatelly, is unacceptable, as then e is no longer a real number, which is unphysical. Thus, we conclude, that presented renormalisation method is not working and we need to search for another. One of possible ways is to expand the method to finite momentum.

TO DO: czy zostawić, to co poniżej

7.2.1 Comparison

We will now investigate, whether this has some chance of working by comparing above "zero momentum" method, only with first and second derivative, with "finite momentum" method and it's zero momentum limit.
With the conditions:

$$\frac{\partial}{\partial\varphi_1}V_R^{\text{1-loop}}\bigg|_v = 0 \tag{7.2.1.0.1}$$

$$\frac{\partial^2}{\partial\varphi_1^2}V_R^{\text{1-loop}}\bigg|_v = 0 \tag{7.2.1.0.2}$$

renormalisation constants $\delta\lambda$ and δm take form:

$$\delta\lambda = -\frac{e^4}{8\pi^2} - \frac{3e^4}{16\pi^2}\left(-\frac{2}{\epsilon} - \gamma_E + \log\frac{e^2v^2}{4\pi\mu^2}\right) \tag{7.2.1.0.3}$$

$$\delta m = \frac{3e^4v^2}{16\pi^2}. \tag{7.2.1.0.4}$$

And the renormalised potential is equal to:

$$V_R = \frac{M_P^2}{48c_\lambda v^2} + \frac{e^4\varphi_1^4}{128\pi^2}\left(6\log\frac{\varphi_1^2}{v^2} - 9\right) + \frac{3e^4v^2\varphi_1^2}{32\pi^2}. \tag{7.2.1.0.5}$$

Now, however, the condition for v is meaningless ass we already used it in renormalisation conditions. Nethertheless, we want to investigate how values of $\delta\lambda$ and δm are compared to other approaches.

7.3 Half $\overline{\text{MS}}$-Half Onshell scheme

Due to (so far) lack of experimenttal data of coupling λ in considered theories, the on shell condition for that constatnt renders itself meaningless.
Thus, we propose mixed scheme, where we demand that the physical mass remain unchanged due to the one-loop corrections, but for the coupling case, we demand

only that the $\delta\lambda$ counterterm is such that the fourth derivative of the renormalzsed effective potential is finite – in the $\overline{\text{MS}}$manner.
Here we impose following renormalisation conditions:

$$\frac{\partial^2}{\partial\varphi_1^2}V_R^{\text{1-loop}}\Big|_v = 0 \tag{7.3.0.0.1}$$

$$\frac{\partial^4}{\partial\varphi_1^4}V_R^{\text{1-loop}}\Big|_v = \frac{9e^4}{8\pi^2}\Big(-\frac{5}{6}+\log\frac{e^2v^2}{\mu^2}\Big)+\frac{75e^4}{16\pi^2}. \tag{7.3.0.0.2}$$

Written in the full form these conditions take form:

$$\begin{aligned}&\frac{9e^4}{16\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)v^2\\ &+12c_\lambda\delta\lambda v^2+\frac{21e^4}{32\pi^2}v^2+2c_m\delta m=0\end{aligned} \tag{7.3.0.0.3}$$

$$\begin{aligned}&\frac{9e^4}{8\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\frac{5}{6}+\log\frac{e^2v^2}{4\pi\mu^2}\Big)+24c_\lambda\delta\lambda+\frac{75e^4}{16\pi^2}\\ &=\frac{9e^4}{8\pi^2}\Big(-\frac{5}{6}+\log\frac{e^2v^2}{\mu^2}\Big)+\frac{75e^4}{16\pi^2}.\end{aligned} \tag{7.3.0.0.4}$$

After solving equations for δm and $\delta\lambda$ we obtain:

$$\delta m = -\frac{3e^4v^2}{32c_m\pi^2}\Big(1+3\log\frac{e^2v^2}{\mu^2}\Big) \tag{7.3.0.0.5}$$

$$\delta\lambda = -\frac{3e^4}{64c_\lambda\pi^2}\Big(-\frac{2}{\epsilon}+\gamma_E-\log(4\pi)\Big). \tag{7.3.0.0.6}$$

The renormalized potential is then:

$$\begin{aligned}V_R =&c_\lambda\lambda\varphi_1^4+\\ &\frac{3e^4}{64\pi^2}\Big(-\frac{5}{6}+\log\frac{e^2\varphi_1^2}{\mu^2}\Big)\varphi_1^4\\ &+\frac{3e^4v^2}{32\pi^2}\Big(-1-3\log\frac{e^2v^2}{\mu^2}\Big)\varphi_1^2.\end{aligned} \tag{7.3.0.0.7}$$

From the tree level potential we have the relation $\lambda = \frac{M_P^2}{12c_\lambda v^2}$. Written in these terms we have:

$$\begin{aligned}V_R =&\frac{M_P^2}{12v^2}\varphi_1^4\\ &+\frac{3e^4}{64\pi^2}\Big(-\frac{5}{6}+\log\frac{e^2\varphi_1^2}{\mu^2}\Big)\varphi_1^4\\ &+\frac{3e^4v^2}{32\pi^2}\Big(-1-3\log\frac{e^2v^2}{\mu^2}\Big)\varphi_1^2.\end{aligned} \tag{7.3.0.0.8}$$

We can bind e to M_P and v at the loop level, from the definition of VEV:

$$\frac{\partial V_R}{\partial\varphi_1}\Big|_v = 0. \tag{7.3.0.0.9}$$

This gives the condition:

$$\frac{M_P^2 v}{3} - \frac{e^4 v^3}{4\pi^2} - \frac{3e^4 v^3}{8\pi^2} \log \frac{e^2 v^2}{\mu^2} = 0. \tag{7.3.0.0.10}$$

Setting scale parameter μ to the effective mass of the vector, namely ev, we have simpler form of:

$$-\frac{e^4 v^3}{4\pi^2} + \frac{M_P^2 v}{3} = 0. \tag{7.3.0.0.11}$$

Which gives:

$$e^4 = \frac{4M_P^2 \pi^2}{3v^2}. \tag{7.3.0.0.12}$$

Writing potential with this substitutions yields asdad:

$$V_R = \frac{M_P^2}{16v^2}\Big(\frac{1}{2} + \log \frac{\varphi_1^2}{v^2}\Big)\varphi_1^4 - \frac{M_P^2}{8}\varphi_1^2. \tag{7.3.0.0.13}$$

Potential can be written also in terms of v and e:

$$V_R = \frac{3e^4}{64\pi^2}\Big(\frac{1}{2} + \log \frac{\varphi_1^2}{v^2}\Big)\varphi_1^4 - \frac{3e^2 v^2}{32\pi^2}\varphi_1^2. \tag{7.3.0.0.14}$$

We can also derive direct relation between λ and e^4, namely:

$$\lambda = \frac{e^4}{16 c_\lambda \pi^2}, \tag{7.3.0.0.15}$$

and the ratio of masses of scalar and vector:

$$\frac{M_P^2}{m(V)^2} = \frac{\frac{3e^4 v^2}{4\pi^2}}{e^2 v^2} = \frac{3e^2}{4\pi^2}. \tag{7.3.0.0.16}$$

7.4 On shell with φ^4 potential

Although the previous renormalisation succeeded in making the potential finite and being in agreement with radiative symmetry breaking it has one last problem – the resulting potential have square term in the field.
We would like to investigate, whether one can renormalise theory such that tree level mass is the phsical mass, namely $\frac{\partial^2}{\partial \varphi_1^2} V_R\Big|_v$ (on-shell condition) and at the same time square term vanishes.
We will follow, what was done in [1], as described in 4.2.1 and by hand put the condition to square terms to vanish, which is equvalent to the condition for $\frac{\partial^2 V_R}{\partial \varphi_1^2}\Big|_0 = 0$.
We can see, that in the renormalised potential:

$$V_R = c_\lambda \lambda \varphi_1^4 + \frac{3e^4}{64\pi^2}\Big(-\frac{2}{\epsilon} + \gamma_E - \frac{5}{6} + \log \frac{e^2 \varphi_1^2}{4\pi \mu^2}\Big)\varphi_1^4 + c_\lambda \delta\lambda \varphi_1^4 + c_m \delta m \varphi_1^2 \tag{7.4.0.0.1}$$

only term square in the fields is $c_m \delta m \varphi_1^2$, therefore δm should be zero.
Note, that it is not the same as disregarding δm automaticly. As stated in [1], the theory has no a priori symmetry for δm to be 0 and we are respectful to that. It just so happens that in our regularisation scheme, if we want to have no square terms in the resulting potential (or to $\frac{\partial^2 V_R}{\partial \varphi_1^2}\Big|_0$ to vanish, which is equivalent), we need to put $\delta m = 0$. With different regularisation to satisfy this condition, we would have different δm, as seen in [1], written here at 4.2.1.0.9.
Therefore we impose following renormalising conditions:

$$\frac{\partial^2}{\partial \varphi_1^2} V_R^{\text{1-loop}}\Big|_v = 0 \tag{7.4.0.0.2}$$

$$\frac{\partial^2}{\partial \varphi_1^2} V_R\Big|_0 = 0. \tag{7.4.0.0.3}$$

Written in the full form these conditions take form:

$$\begin{aligned}&\frac{9e^4 v^2}{16\pi^2}\Big(-\frac{2}{\epsilon} + \gamma_E - \frac{5}{6} + \log\frac{e^2 v^2}{4\pi\mu^2}\Big)\\ &+ 12 c_\lambda \delta\lambda v^2 + \frac{21 e^4}{32\pi^2} v^2 + 2 c_m \delta m = 0 \end{aligned}\tag{7.4.0.0.4}$$

$$\frac{\partial^2}{\partial \varphi_1^2} V_R\Big|_0 = 0. \tag{7.4.0.0.5}$$

After solving equations for δm and $\delta\lambda$ we obtain:

$$\delta m = 0 \tag{7.4.0.0.6}$$

$$\delta\lambda = -\frac{3e^4}{64 c_\lambda \pi^2}\Big(-\frac{2}{\epsilon} + \gamma_E - \frac{5}{6} + \log\frac{e^2 v^2}{4\pi\mu^2}\Big) - \frac{7e^4}{128 c_\lambda \pi^2}. \tag{7.4.0.0.7}$$

The renormalized potential is then:

$$V_R = c_\lambda \lambda \varphi_1^4 + \frac{3e^4}{64\pi^2}\Big(-\frac{7}{6} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{7.4.0.0.8}$$

From the tree level potential we have the relation $\lambda = \frac{M_P^2}{12 c_\lambda v^2}$. Written in these terms we have:

$$V_R = \frac{M_P^2}{12 v^2}\varphi_1^4 + \frac{3e^4}{64\pi^2}\Big(-\frac{7}{6} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{7.4.0.0.9}$$

We can bind e to M_P and v at the loop level, from the definition of VEV:

$$\frac{\partial V_R}{\partial \varphi_1}\Big|_v = 0. \tag{7.4.0.0.10}$$

This gives the condition:

$$\frac{M_P^2 v}{3} + \frac{3e^4 v^3}{16\pi^2}\Big(-\frac{7}{6}\Big) + \frac{3e^4 v^3}{32\pi^2} = 0. \tag{7.4.0.0.11}$$

Which gives:

$$e^4 = \frac{8M_P^2\pi^2}{3v^2}. \tag{7.4.0.0.12}$$

Writing potential with this substitutions yields:

$$V_R = \frac{M_P^2}{8v^2}\Big(-\frac{1}{2} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{7.4.0.0.13}$$

Potential can be written also in terms of v and e:

$$V_R = \frac{3e^4}{64\pi^2}\Big(-\frac{1}{2} + \log\frac{\varphi_1^2}{v^2}\Big)\varphi_1^4. \tag{7.4.0.0.14}$$

Note, that, this is exactly same potential as in [1] and 5.0.0.0.7. We can now calculate following quantities: - relation between λ and e^4:

$$\lambda = \frac{e^4}{32c_\lambda\pi^2}, \tag{7.4.0.0.15}$$

- ratio between masses of scalar and vector:

$$\frac{M_P^2}{m(V)^2} = \frac{\frac{3e^4v^2}{8\pi^2}}{e^2v^2} = \frac{3e^2}{8\pi^2}. \tag{7.4.0.0.16}$$

Chapter 8

Summary

8.1 Summary table

Reg. – Regularisation scheme, Ren. – Renormalisation scheme, $k^2 = \Lambda^2$ – regularisation by cut-off at $k^2 = \Lambda^2$.

Reg.	Ren.					
$k^2 = \Lambda^2$	$\frac{d^2}{d\varphi_1^2} V_R \Big\vert_0 = 0$ $\frac{d^4}{d\varphi_1^4} V_R \Big\vert_v = 24 c_\lambda \lambda$	$\lambda = \frac{e^4}{\pi^2}$	$\lambda = \frac{M_P^2}{v^2}$	$e^4 = \frac{M_P^2}{\pi^2}$		
Dimentional regularisation	$\frac{d^2}{d\varphi_1^2} V_R \Big\vert_0 = 0$ $\frac{d^4}{d\varphi_1^4} V_R \Big\vert_v = 24 c_\lambda \lambda$					

Chapter 9

Conclusions

Appendix A

Introduction – mass term case

For comparison, we present here results for above methods used in the case with explicit mass term.

A.1 On-shell finite momentum approach

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For the comparison, we will present the same calculation, performed on the analoguos theory with explicit mass term. Similary as in the 6.2.0.0.1 we need to shift fields for 6.4.0.0.9 to be satisfied. The Lagrangian in this case is:

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu(\varphi_1+v) - eA_\mu\varphi_2)^2 \\ &+ \frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu(\varphi_1+v))^2 - c_m m^2((\varphi_1+v)^2+\varphi_2^2) - c_\lambda\lambda((\varphi_1+v)^2+\varphi_2^2)^2.\end{aligned} \tag{A.1.0.0.1}$$

With renormalisation constatnts:

$$\mathcal{L}_\mathcal{R} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{A.1.0.0.2}$$

$$\begin{aligned}&+ (1+\delta Z)(\frac{1}{2}(\partial_\mu(\varphi_1+v) - eA_\mu\varphi_2)^2 + \frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu(\varphi_1+v))^2) \\ &- (1+\delta Z)^2 c_\lambda(\lambda+\delta\lambda)((\varphi_1+v)^2+\varphi_2^2)^2 \\ &- (1+\delta Z)c_m(m+\delta m)((\varphi_1+v)^2+\varphi_2^2).\end{aligned} \tag{A.1.0.0.3}$$

Corrections then are:

$$\delta\Sigma = -12c_\lambda v^2(2\lambda\delta Z_\varphi + \delta\lambda) - 2c_m\delta m - 2c_m m^2\delta Z - p^2\delta Z \tag{A.1.0.0.4}$$

$$\delta T = -4c_\lambda v^3(2\lambda\delta Z_\varphi + \delta\lambda) - 2c_m\delta m v - 2c_m m^2 v\delta Z. \tag{A.1.0.0.5}$$

This changes the form of renormalisation constants to:

$$\delta m = \frac{-1}{4c_m}\left(\Re\left(\Sigma\right) - \frac{3}{v}T - (4c_m m^2 + M_P^2)\Re\left(\Sigma'\right)\right) \tag{A.1.0.0.6}$$

$$\delta \lambda = \frac{1}{8c_\lambda v^2}\left(\Re\left(\Sigma\right) - \frac{1}{v}T - (16c_\lambda \lambda v^2 + M_P^2)\Re\left(\Sigma'\right)\right) \tag{A.1.0.0.7}$$

$$\delta Z = \Re\left(\Sigma'\right). \tag{A.1.0.0.8}$$

The only difference is $4c_m m^2$ term in A.1.0.0.6.

A.2 "Zero momentum" approach

Here we will compare two kinds of "zero momentum" approach. First will be imposing renormalisation conditions in terms of only derivatives of effective potential. This is the zero momentum approach as first and second derivatives are limits of, respectively, taddpole and sef-energy in the zero momentum limit.
qSecond kind will be to calculate approach from reffinite momentum in the zero momentum limit.
Later we will discus some "potential only" version with different conditions and discuss whether adding finite momentum to it will produce satisfying results.

Potential only version

For comparison, we include also a version of this aprouch steming from refmass term. Inclusion of the mass term do not change the form of $\delta\lambda$ and δm. The frist difference occurs in the potential.
First we will describe the case with derivatives II and IV used in renormalisation conditions. Then the potential is equal:

$$V_R = c_\lambda \lambda \varphi_1^4 + c_m m^2 \varphi_1^2 + \frac{e^4\varphi_1^4}{64\pi^2}\left(3\log\frac{\varphi_1^2}{v^2} - \frac{25}{2}\right) + \frac{27e^4v^2\varphi_1^2}{32\pi^2}. \tag{A.2.0.0.1}$$

Now, we have that $M_P = 12c_\lambda\lambda v^2 + 2c_m m^2$. Now, we will use the condition, that:

$$\frac{\partial}{\partial\varphi_1}V^T\Big|_v = 0. \tag{A.2.0.0.2}$$

We have that $4c_\lambda\lambda v^3 + 2c_m m^2 v = 0$, so $\lambda = -\frac{c_m m^2}{2c_\lambda v^2}$, so

$$m^2 = \frac{-M_P^2}{4c_m} \quad \text{and} \tag{A.2.0.0.3}$$

$$\lambda = \frac{M_P^2}{8c_\lambda v^2}. \tag{A.2.0.0.4}$$

Writing V_R with respect to that gives:
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$$V_R = \frac{M_P^2\varphi_1^4}{8v^2} - \frac{M_P^2\varphi_1^2}{4} + \frac{e^4\varphi_1^4}{64\pi^2}\left(3\log\frac{\varphi_1^2}{v^2} - \frac{25}{2}\right) + \frac{27e^4v^2\varphi_1^2}{32\pi^2}. \tag{A.2.0.0.5}$$

From this we have, that:

$$\frac{e^4 v^3}{\pi^2} = 0, \tag{A.2.0.0.6}$$

which is also a not safisying result.
However, if we drop the condition, that $\frac{\partial}{\partial \varphi_1} V^T \Big|_v = 0$, we have potential in the form:

$$V_R = \frac{M_P^2 - 2c_m m^2}{12v^2}\varphi_1^4 + c_m m^2 \varphi_1^2 + \frac{e^4 \varphi_1^4}{64\pi^2}\left(3\log\frac{\varphi_1^2}{v^2} - \frac{25}{2}\right) + \frac{27e^4 v^2 \varphi_1^2}{32\pi^2}. \tag{A.2.0.0.7}$$

From thism, using the condition that $\frac{\partial}{\partial \varphi_1} V_R \Big|_v = 0$, we can derive the correspondence between e and M_P, v and m:

$$e^4 = -\frac{(M_P^2 + 4c_m m^2)\pi^2}{3v^2}, \tag{A.2.0.0.8}$$

which is finally a sensible result as it can be realised with real, positive e. However, then it must hold that $m^2 < -\frac{M_P^2}{4c_m}$.

Bibliography

[1] Sidney Coleman and Erick Weinberg, *Radiative corrections as the origin of spontaneous symmetry breaking*, Phys. Rev. D **7** (1973Mar), 1888–1910.