Recall 1d subdivision schemes

Fix $a_0, a_1, \ldots, a_m, b_0, \ldots, b_m$. Let

$$\begin{cases} f_{j+1}(q) = \sum_k a_k \cdot f_j(q - k2^{-j}) \\ f_{j+1}(q + \frac{1}{2^{j+1}}) = \sum_k b_k \cdot f_j(q - k2^{-j}) \end{cases}$$

$$f_0 : \mathbb{Z} \to \mathbb{R}, f_1 : \frac{\mathbb{Z}}{2} \to \mathbb{R}, f_2 : \frac{\mathbb{Z}}{4} \to \mathbb{R}, \ldots$$

If $f_j \to f_u, j \to \infty$, then $f_j(q - k2^{-j}) \approx f_u(q)$. So,

$$\sum_k a_k = 1, \sum_k b_k = 1$$

Recall 1d subdivision schemes

The subdivision operator $S\colon \ell_\infty \to \ell_\infty$ is linear and continuous. It is defined by the coefficients

$c_0, c_1, \ldots, c_N$, $c_{2k} = a_k$, $c_{2k+1} = b_k$.

$$\boxed{[Su](k) = \sum_{j\in\mathbb{Z}} c_{k-2j} \cdot u(j)}$$

For example, $[Su](0) = c_0 u(0) + c_2 u(-1) + c_4 u(-2) + \ldots$

$$\sum_k c_{2k} = \sum_k c_{2k+1} = 1$$

The convergence

How to construct the limit function f_u?

$f_0(k) = u(k) \quad k \in \mathbb{Z}, f_0 \colon \mathbb{Z} \to \mathbb{R}$

$\downarrow$

$f_1(\frac{k}{2}) = [Su](k) \quad k \in \mathbb{Z}, f_1 \colon \frac{1}{2}\mathbb{Z} \to \mathbb{R}$

$\downarrow$

$f_2(\frac{k}{4}) = [S^2 u](k) \quad k \in \mathbb{Z}, f_2 \colon \frac{1}{4}\mathbb{Z} \to \mathbb{R}$

$\ldots$

The scheme is **convergent**, if $\forall u \colon \mathbb{Z} \to \mathbb{R}\ \exists f_u \in C(\mathbb{R})$ such that $\lim_{j\to\infty} \|[S^j u] - f_u(\frac{\cdot}{2^j})\|_{\ell_\infty} = 0$

Refinement equations

It is sufficient to know the limit function only for the δ-sequence:

- let $\delta(k) = \delta_k^0$
- let the limit function for δ be f_δ.
- then $\forall u \in \ell_\infty \; f_u(x) = \sum_k f_\delta(x-k) \cdot u(k)$

Theorem 1: The limit function f_δ satisfies the following refinement equation:

$$\varphi(x) = \sum_k c_k \varphi(2x-k)$$

Refinable functions

The refinable function $\varphi(x)$ is a solution of the refinement equation

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k)$$

The mask of the equation is the characteristic trigonometric polynomial of the sequence c_k:

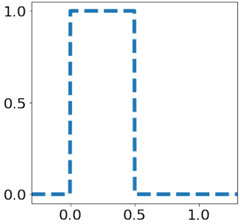
$$m(s) = \frac{1}{2} \sum_{k=0}^{N} c_k e^{-2\pi iks}.$$

$$\widehat{\varphi}(2s) = m(s)\widehat{\varphi}(s)$$

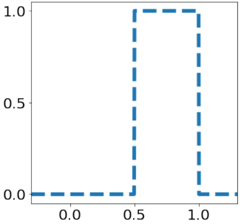
$$\widehat{\varphi}(0) = m(0)\widehat{\varphi}(0) \Rightarrow m(0) = 1, \sum_k c_k = 2$$

The simplest RE, $c_0 = c_1 = 1$

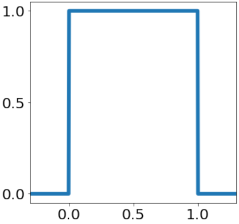
$$\varphi(2x) \quad + \quad \varphi(2x-1) \quad = \quad \varphi(x)$$



$+$

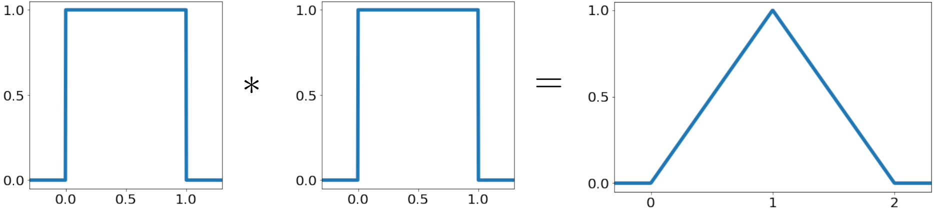


$=$



$$\widehat{\varphi}(2s) = m(s)\widehat{\varphi}(s) \Rightarrow \widehat{\varphi}(2s) = \frac{1+e^{-2\pi i s}}{2}\widehat{\varphi}(s)$$

1d B-splines

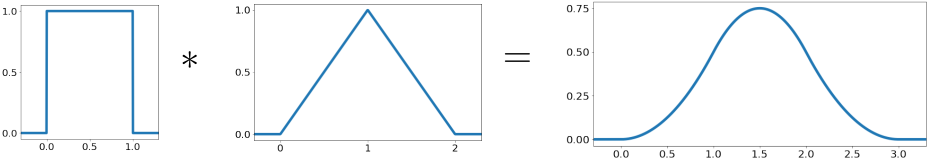


$$\widehat{\varphi}(2s) = m(s)\widehat{\varphi}(s) \Rightarrow \widehat{\varphi * \varphi}(2s) = m(s)^2\,\widehat{\varphi * \varphi}(s)$$

$$\widehat{\varphi_1}(2s) = \left(\frac{1 + e^{-2\pi i s}}{2}\right)^2 \widehat{\varphi_1}(s) = \frac{1 + 2e^{-2\pi i s} + e^{-4\pi i s}}{4}\widehat{\varphi_1}(s)$$

$$m_1(s) = \frac{1}{2}\left(\frac{1}{2} + e^{-2\pi i s} + \frac{1}{2}e^{-4\pi i s}\right)$$

1d B-splines

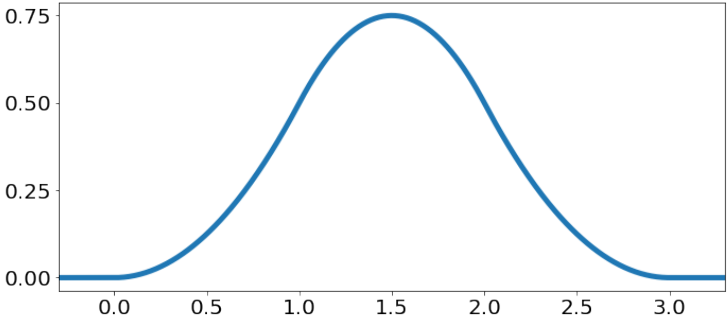


$$\widehat{\varphi_2}(2s) = m(s)^3\,\widehat{\varphi_2}(s)$$

$$m_2(s) = \frac{1}{2}\left(\frac{1}{4} + \frac{3}{4}e^{-2\pi i s} + \frac{3}{4}e^{-4\pi i s} + \frac{1}{4}e^{-6\pi i s}\right)$$

1d B-splines, B_3

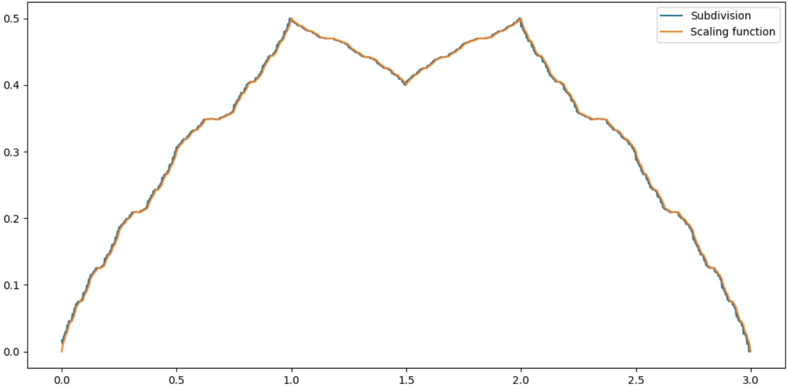
$c_0 = 0.25, c_1 = 0.75, c_2 = 0.75, c_3 = 0.25$
$Su[2k] = 0.25u[k] + 0.75u[k-1]$
$Su[2k+1] = 0.75u[k] + 0.25u[k-1]$



The rate of convergence of the subdivision schemes

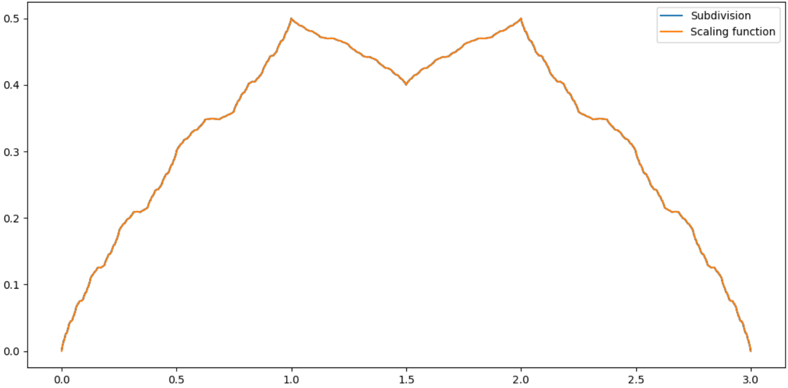
$c_0 = 1 - a,\ c_1 = a,\ c_2 = a,\ c_3 = 1 - a$

$$a = 0.4, 8 \text{ iterations}$$



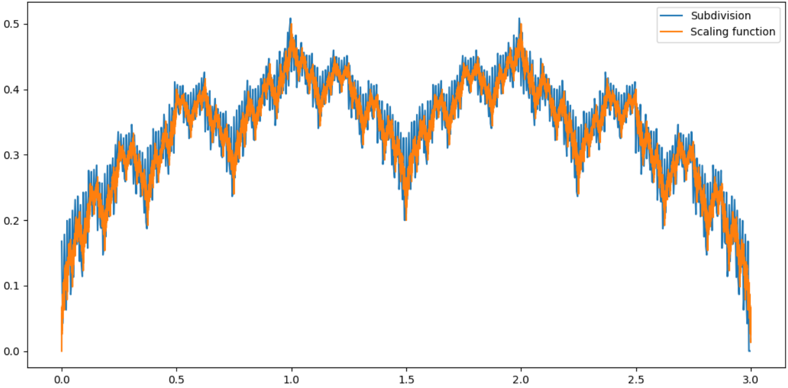
The rate of convergence of the subdivision schemes

$a = 0.4, 12$ iterations



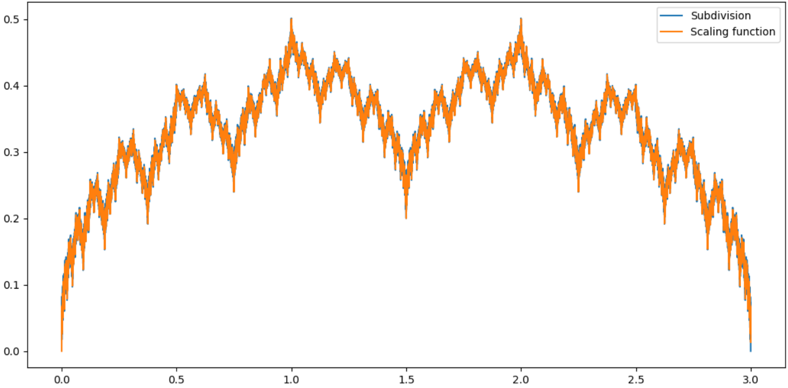
The rate of convergence of the subdivision schemes

$a = 0.2, 8$ iterations



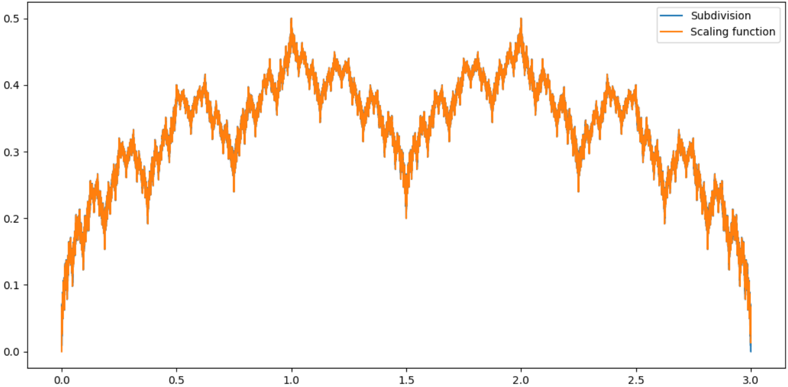
The rate of convergence of the subdivision schemes

$a = 0.2, 12$ iterations



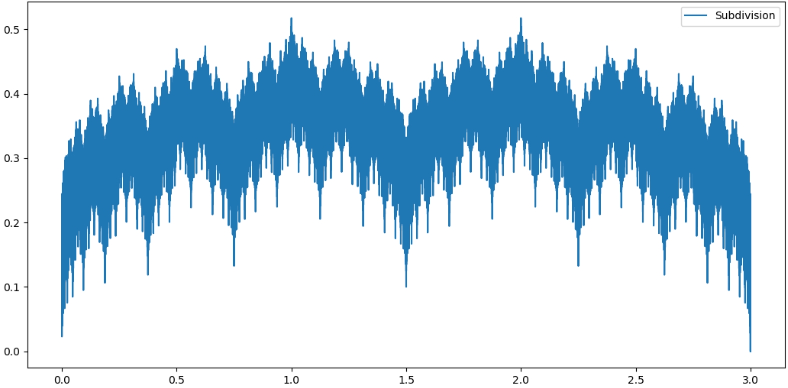
The rate of convergence of the subdivision schemes

$a = 0.2, 15$ iterations



The rate of convergence of the subdivision schemes

$a = 0.1, 15$ iterations



Regularity

It turns out that the rate of convergence depends on the regularity of function φ.

The Hölder exponent of regularity:
$\alpha_\varphi = \sup \{\alpha \geq 0 \ : \ \|\varphi(\cdot + h) - \varphi\|_C \leq Ch^\alpha, \forall h \in \mathbb{R}\}$

The Sobolev exponent of regularity (equivalent):
$s_\varphi = \sup \left\{ s > 0 \mid \int |\hat{\varphi}|^2 (|\xi|^2 + 1)^s d\xi < \infty) \right\}$

M. Charina and V. Yu. Protasov, *Smoothness of anisotropic wavelets, frames and subdivision schemes*, Applied and Computational Harmonic Analysis, 2017

The rate of convergence

Convergence: $\lim_{j\to\infty} \|S^j u - \varphi(\frac{\cdot}{2^j})\|_{\ell_\infty} = 0$

The rate of convergence: $\boxed{\|S^j u - \varphi(\frac{\cdot}{2^j})\|_{\ell_\infty} \le C\|u\|_{\ell_\infty}\tau^j.}$ If it holds that $\sum_k c_{2k} = \sum_k c_{2k+1} = 1$, then

$$\tau = \rho_C(T_0|_W, T_1|_W)$$

Let the support of φ be a subset of $[0, N]$.

$$(T_s)_{ij} = c_{2i-j+s}, \quad i,j = 0,\ldots,N-1; \quad s = 0,1$$

$$W = \left\{ x \in \mathbb{R}^N \mid \sum_k x_k = 0 \right\}$$

The rate of convergence and regularity

$$\rho_C(A_0, A_1) = \lim_{m\to\infty} \max_{\sigma} \|A_{\sigma(1)} \dots A_{\sigma(m)}\|^{1/m}, \ \sigma\colon \{1, \dots, m\} \to \{0, 1\}$$

In "most cases"

$$\alpha_\varphi = -\log_2 \rho_C(T_0|_W, T_1|_W)$$

(in 1d it is sufficient that integer shifts of φ are linearly independent).

$$\rho_2(A_0, A_1) = \lim_{m\to\infty} (\frac{1}{2^m} \sum_{\sigma} \|A_{\sigma(1)} \dots A_{\sigma(m)}\|^2)^{1/2m}$$

$$\alpha_{\varphi,2} = -\log_2 \rho_2(T_0|_W, T_1|_W)$$

The rate of convergence of cascade algorithm

Let T be an operator:

$$Tg(x) = \sum_k c_k g(2x - k)$$

The refinement equation: $T\varphi = \varphi$.

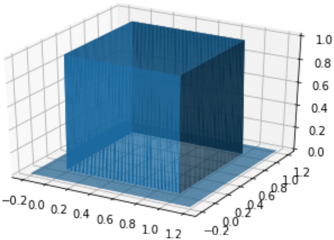
$\sum_{z \in \mathbb{Z}} f_0(x + z) \equiv 1.$
Convergence: $\| T^j f_0(x) - \varphi(x) \| \to 0, j \to \infty.$

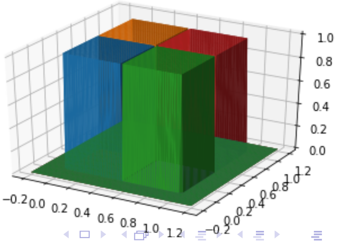
The rate of convergence: $\boxed{\| T^j f_0 - \varphi \|_C \leq C(\rho_C(T|_W))^j}$

Multivariate case

$$\varphi(x,y) = \chi_{[0,1]}(x)\chi_{[0,1]}(y)$$

$$\varphi(x,y) = \frac{1}{2}(\chi_{[0,1]}(2x)+\chi_{[0,1]}(2x-1))\frac{1}{2}(\chi_{[0,1]}(2y)+\chi_{[0,1]}(2y-1)) =$$
$$= \frac{1}{4}(\varphi(2x,2y)+\varphi(2x-1,y)+\varphi(2x,2y-1)+\varphi(2x-1,2y-1))$$





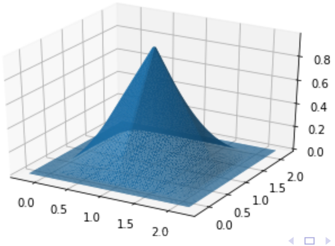
Multivariate case

The classical B-splines are the convolutions of the unit square.

$$\varphi(x,y) * \varphi(x,y) = (\chi_{[0,1]}(x) * \chi_{[0,1]}(x))(\chi_{[0,1]}(y) * \chi_{[0,1]}(y))$$

$$[Su](k) = \sum_{j\in\mathbb{Z}^2} c_{k_1-2j_1} c_{k_2-2j_2} u(j_1, j_2)$$

The number of coefficients in the subdivision scheme generated by B_k grows as k^d.



Subdivision based on the matrix dilations

The binary expansion $\to$ the multiplication by the powers of the expanding matrix $M \in \mathbb{Z}^{d \times d}$ (i.e. all eigenvalues $|\lambda_j| > 1$).

$$[Su](k) = \sum_{j \in \mathbb{Z}^d} c_{k-Mj} u(j)$$

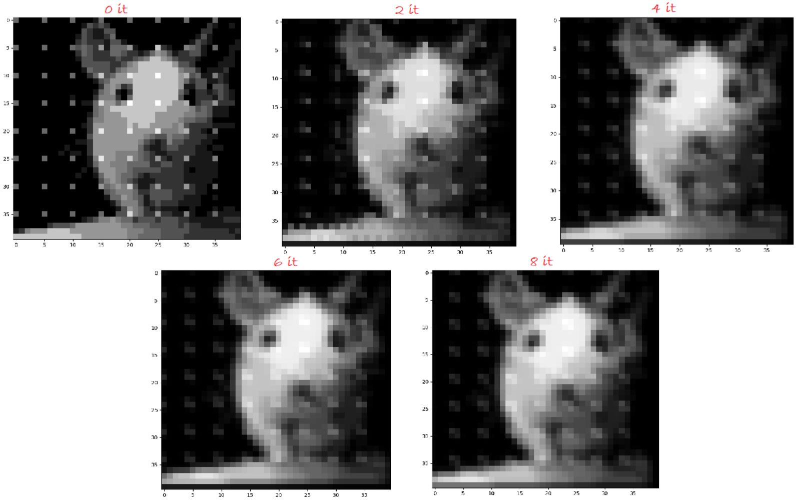
The direct product of 1d corresponds to the diagonal matrix:

$$[Su](k) = \sum_{j \in \mathbb{Z}^2} c_{k_1-2j_1} c_{k_2-2j_2} u(j_1, j_2)$$

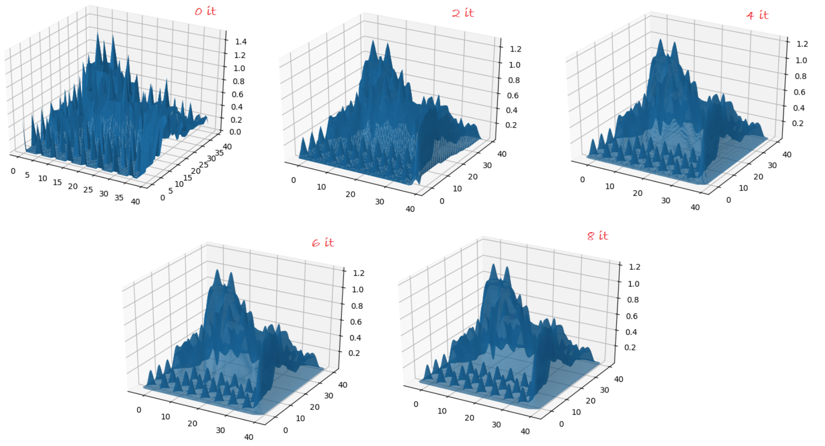
The refinement equation:

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} c_k \varphi(Mx - k)$$

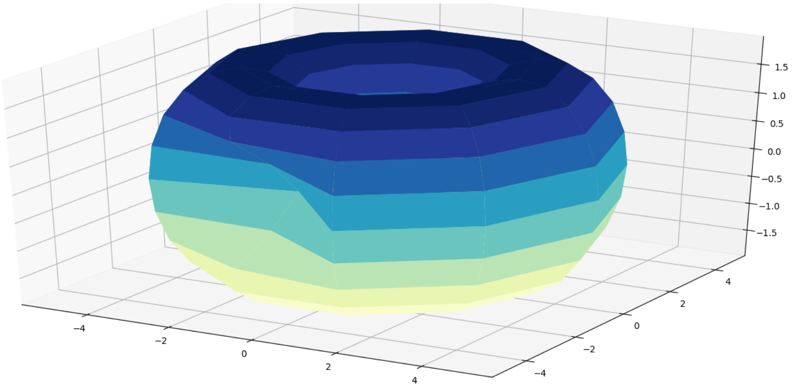
Smoothing with the subdivision based on Bear-4



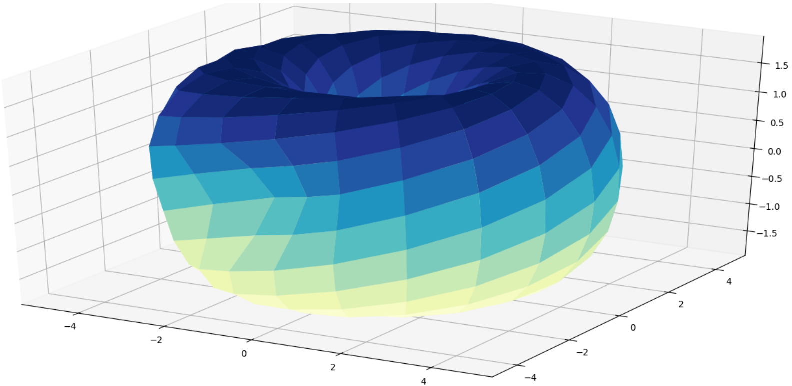
Smoothing with the subdivision based on Bear-4



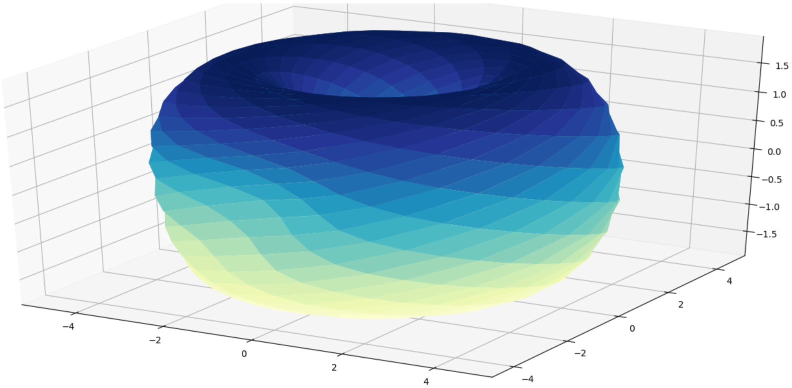
Torus example, iteration 0



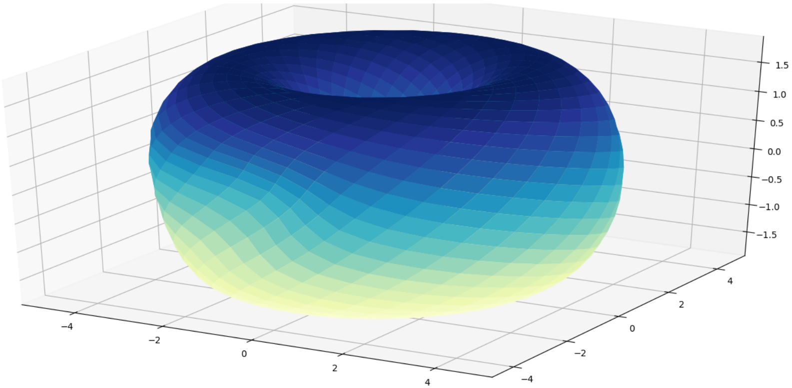
Torus example, iteration 1



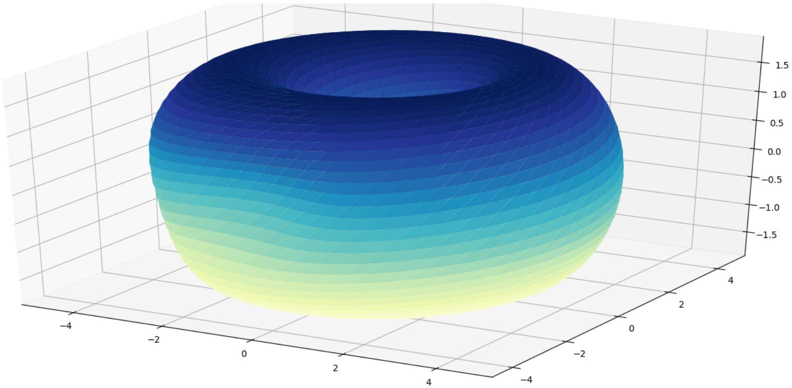
Torus example, iteration 2



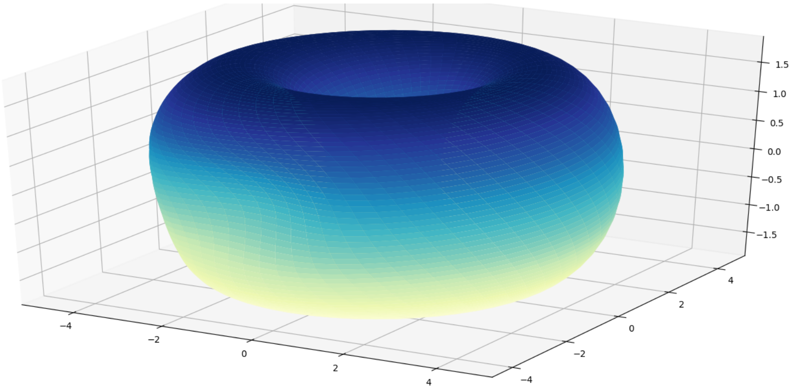
Torus example, iteration 3



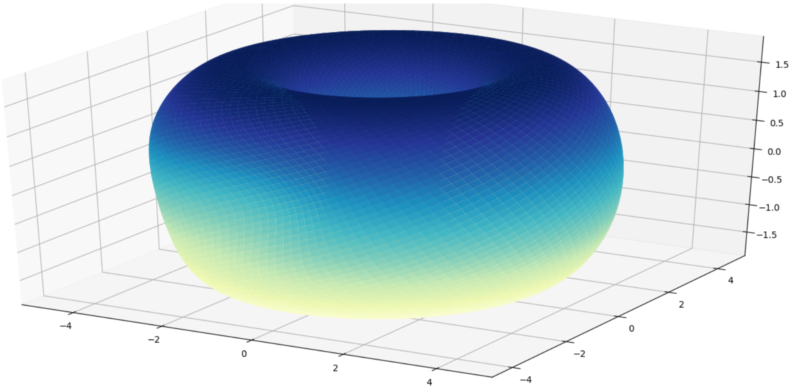
Torus example, iteration 4



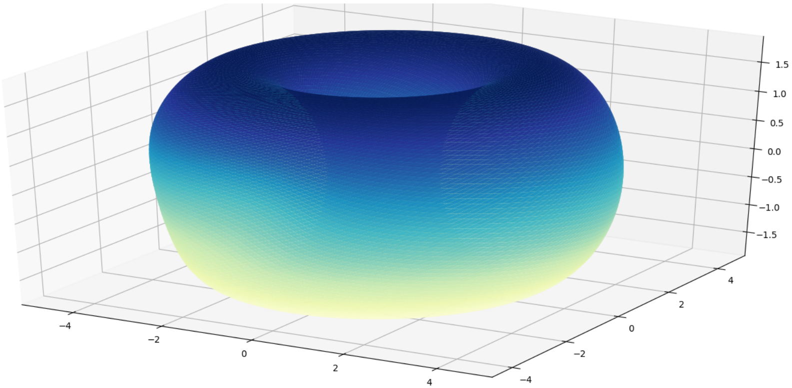
Torus example, iteration 5



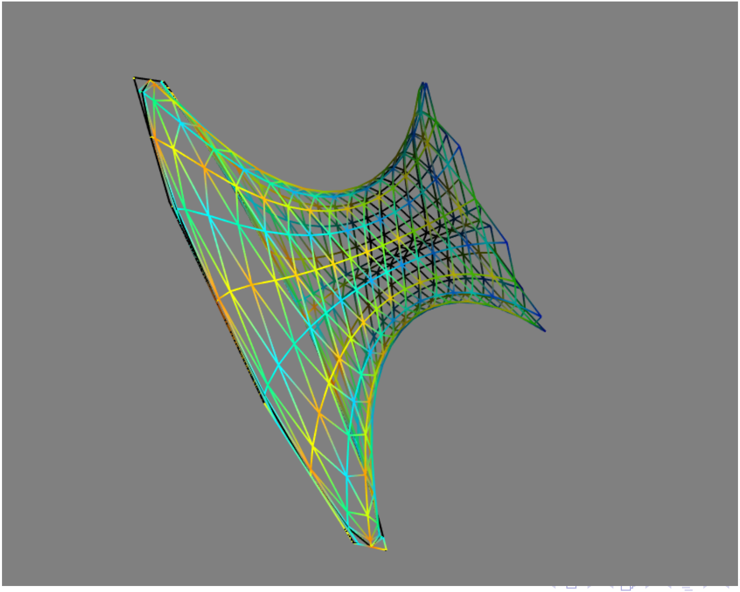
Torus example, iteration 6



Torus example, iteration 7



Catenoid



Multivariate B-splines

The classical B-splines start with the characteristic function of a square. Let us consider a more general approach: the B-spline is a convolution of characteristic functions of special compact sets.

For which sets their characteristic functions satisfy a refinement equation?

The convolution of the refinable functions is again a refinable function. Thus, we can consider subdivision schemes corresponding to such convolutions.

About number systems

Binary system

$$0.011010011001\ldots = \sum_{i=1}^{\infty} 2^{-i} s_i, \text{ where } s_i = 0 \text{ or } s_i = 1.$$

We obtain a segment

$$[0,1] = \left\{ \sum_{i=1}^{\infty} 2^{-i} s_i : s_i \in \{0,1\} \right\}.$$

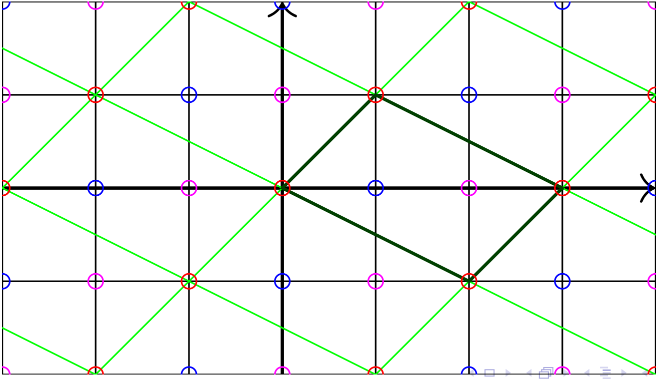
M-nary system

$$0.011010011001\ldots = \sum_{i=1}^{\infty} M^{-i} s_i, \text{ where } s_i \in D(M).$$

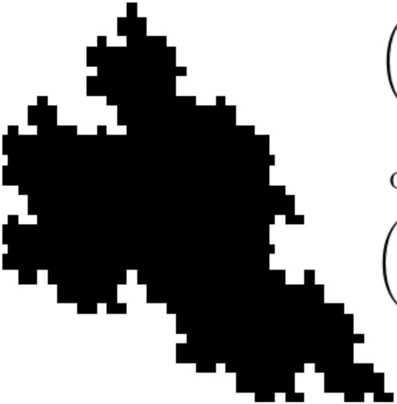
$$G = \left\{ \sum_{i=1}^{\infty} M^{-i} s_i : s_i \in D(M) \right\}.$$

What is $D(M)$?

The binary system: digits $\{0, 1\}$.
The M-nary system: digits $D(M)$ are integer vectors, the representatives from different quotient classes $\mathbb{Z}^d / M\mathbb{Z}^d$, i.e. if $d_1 \neq d_2 \in D(M)$, it holds $d_1 - d_2 \notin M\mathbb{Z}^d$. There are m digits, $m = |\det M|$.



Example 1: G with well-defined M, $D(M)$



$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

digits

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The properties of the set G

The set $G = \left\{ \sum_{i=1}^{\infty} M^{-i} s_i : s_i \in D(M) = \{d_0, \ldots, d_{m-1}\} \right\}$ can be split to m parts depending on s_1.

If we choose $s_1 = d_0$, we obtain the 1st part

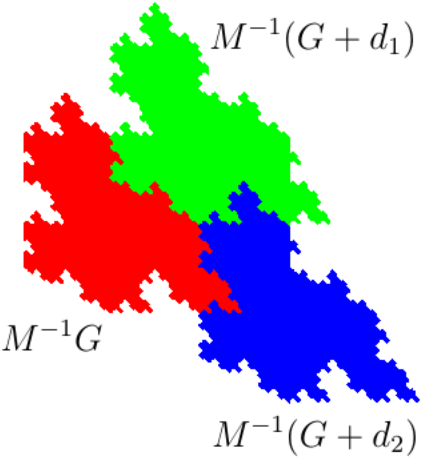
$$\left\{ M^{-1} d_0 + M^{-1} \sum_{i=2}^{\infty} M^{-(i-1)} s_i : s_i \in D(M) \right\} = M^{-1} d_0 + M^{-1} G.$$

...

If we choose $s_1 = d_{m-1}$, we obtain the m-th part

$$\left\{ M^{-1} d_{m-1} + M^{-1} \sum_{i=2}^{\infty} M^{-i+1} s_i : s_i \in D(M) \right\} = M^{-1} d_{m-1} + M^{-1} G.$$

The partition into 3 parts for Example 1



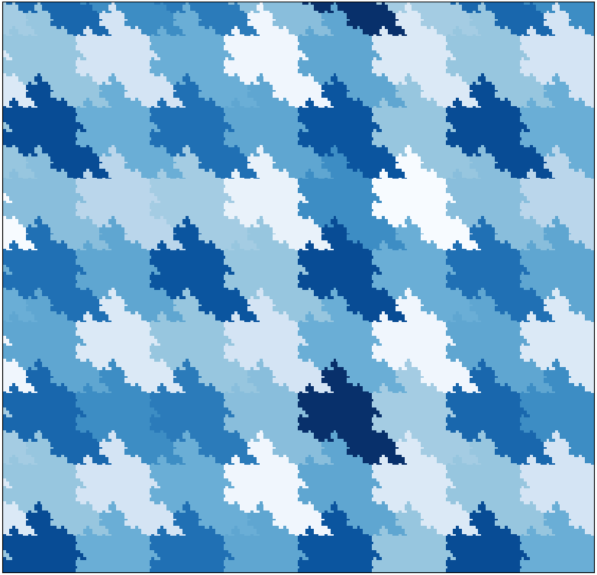
The properties of the set G

- Self-affinity: G is a disjunct, up to a null set, union of the sets $M^{-1}(G + d_i)$ similar to G.

$$\varphi(x) = \sum_{k \in D(M)} \varphi(Mx - k)$$

- Integer shifts of G cover the entire space in $|G|$ layers. If there is only one layer, then the set is called a **tile**.

Tiling of G from Example 1



Equivalent definition of a tile

The equivalent definition of a tile is based on properties.

G is a compact set in $\mathbb{R}^d$ with two properties:

- $\exists$ system of 'digits', integer vectors $d_0, d_1, \ldots d_{m-1}$, where $m = |\det M|$, such that $G = \bigcup_{i=1}^{m-1} M^{-1}(G + d_i)$
- $\bigcup_{k \in \mathbb{Z}^d} (G + k) = \mathbb{R}^d$ up to a null set

Both properties hold for 1d case

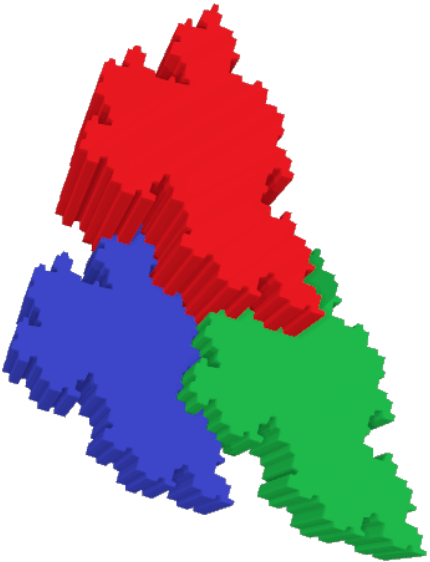
$$\mathbb{R}^1, M = 2,\ G = \left\{ \sum_{i=1}^{\infty} 2^{-i} d_i : d_i \in \{0, 1\} \right\}.$$

$G = [0, 1]$

1) $G = \frac{1}{2} G \cup \frac{1}{2}(G + 1)$

2) $\bigcup_{k \in \mathbb{Z}} (G + k) = \mathbb{R}$

Tiles are also a key ingredient for Haar systems in $\mathbb{R}^d$ (see J. Lagarias, Y. Wang, K. Grochenig, etc.).



Digits are significants

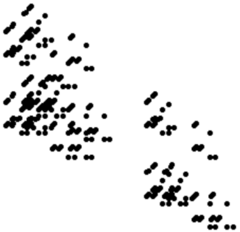
Matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

Digits $(0, 0)$, $(1, 0)$, $(0, 1)$

Digits are significant

Matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

Digits $(0,0)$, $(1,0)$, $(0,\mathbf{4})$



Digits are significant

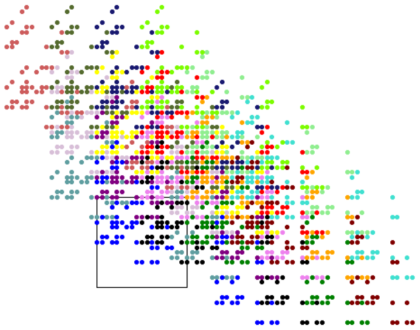
Matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

Digits $(0, 0)$, $(1, 0)$, $(0, 1)$

Digits are significant

Matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

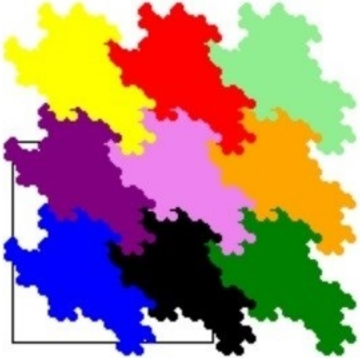
Digits $(0,0)$, $(1,0)$, $(0,4)$

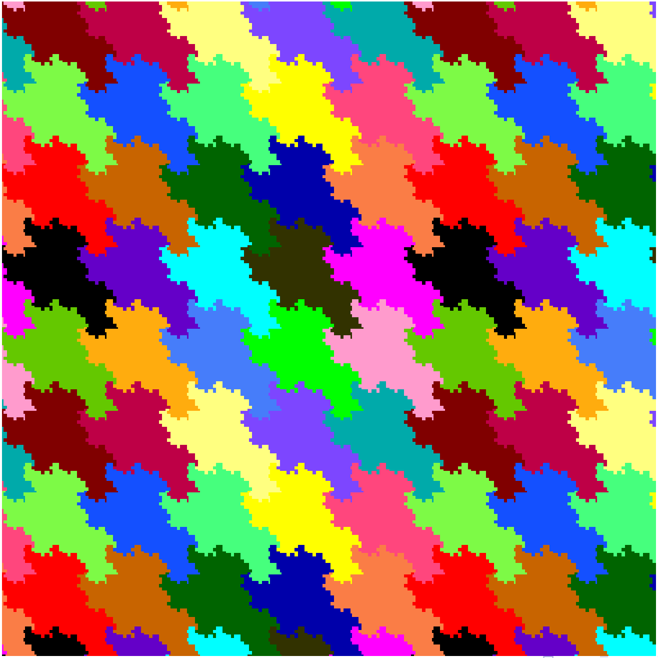


Examples

$$\begin{matrix} 1 & -1 \\ 1 & 2 \end{matrix}$$

(1, 0)
(0, 1)

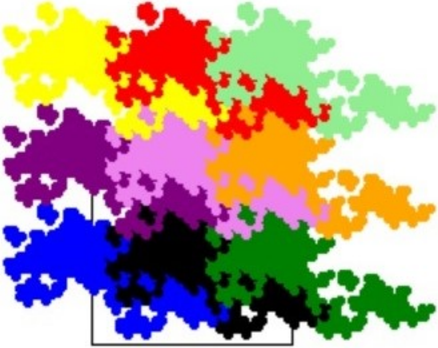


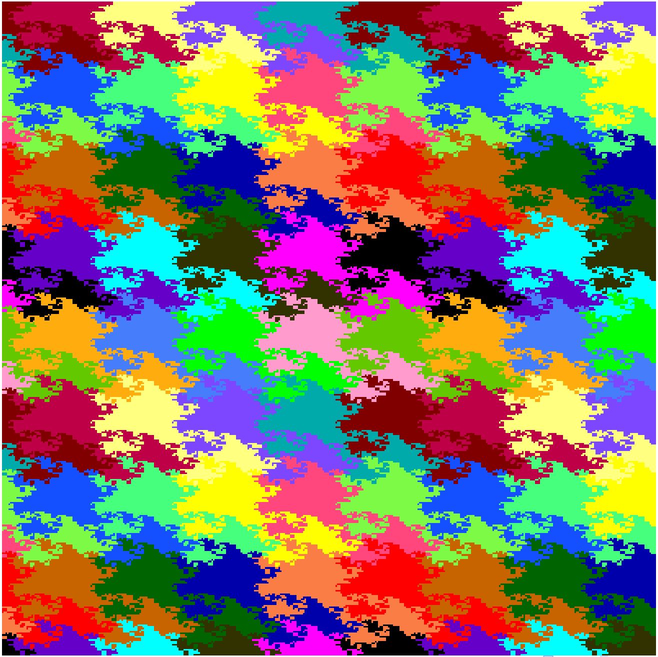


Examples

$$\begin{matrix} -1 & -3 \\ 1 & 0 \end{matrix}$$

(1, 0)
(1, 1)



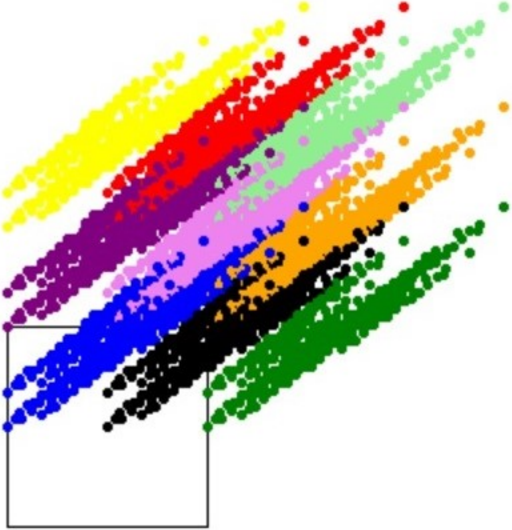


Examples

$$\begin{matrix} -1 & 3 \\ 1 & 0 \end{matrix}$$

(1, 0)

(1, 1)



Classification of tiles with 2 digits

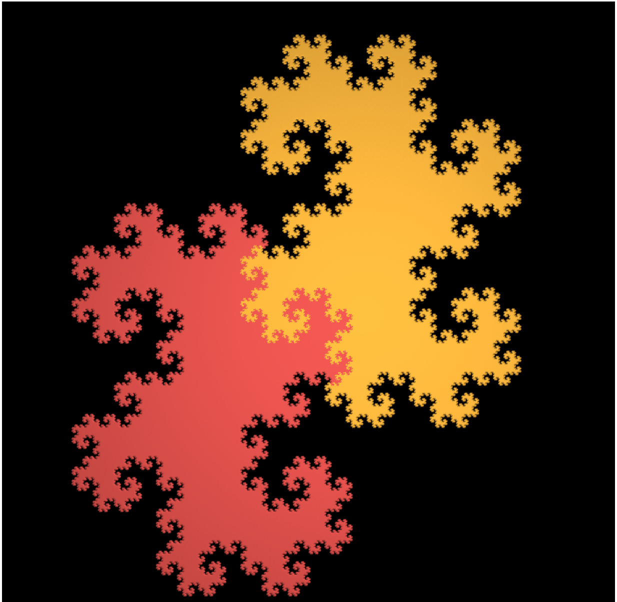
The less is m, the simpler is corresponding subdivision scheme. Thus, the case $d = 2$ is especially interesting.

Theorem

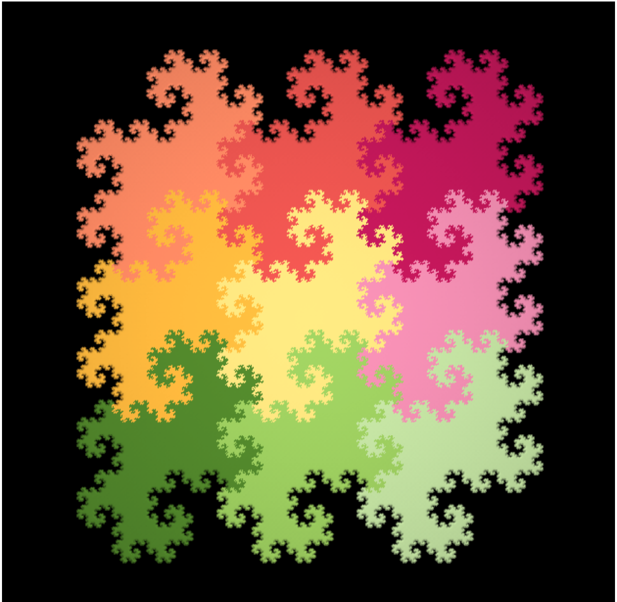
Up to an affine similarity, there are exactly three different tiles with 2 digits on the plane ($m = 2, d = 2$).

- *Dragon (twindragon)*
  regularity: $\approx$ **0.23819**
- *Bear (tame twindragon)*
  regularity: $\approx$ **0.39462**
- *Square*
  regularity: **0.5**

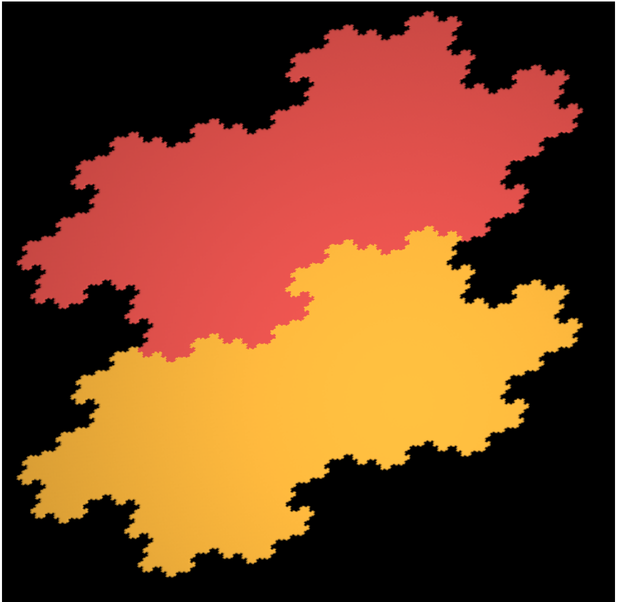
Dragon and its 2 parts



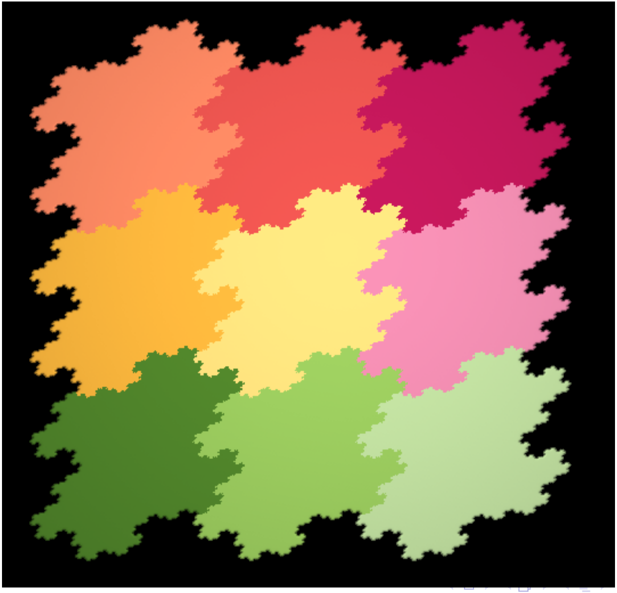
Tiling by Dragon



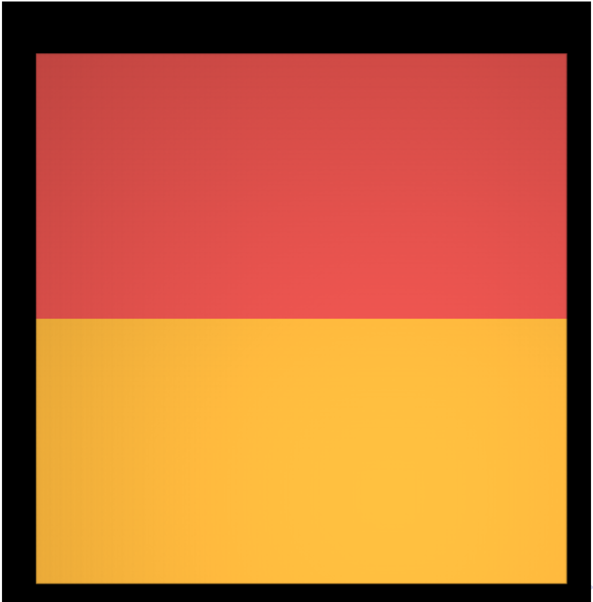
Bear and its 2 parts



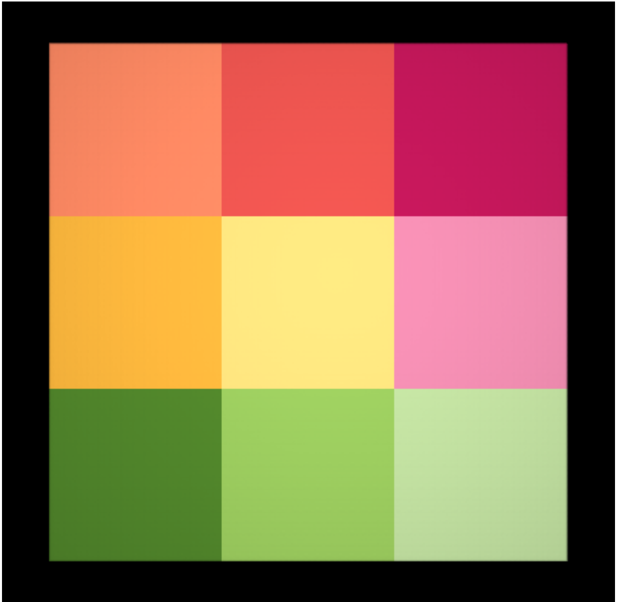
Tiling by Bear



Square and its 2 parts



Tiling by Square



Two types of Square

The square has the largest regularity $= \mathbf{0.5}$.
The direct product of one-dimensional schemes is the same square with the matrix

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

It has 4 coefficients in the scheme (in arbitrary dimension d it has 2^d coefficients). After convolution it has already 3^d coefficients and so on.
Our Square with the matrix

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

has 2 coefficients (there is such a tile in arbitrary dimension). After convolution we have $d+1$ coefficients.

2-digit case

A tile is **isotropic** if its matrix M is *isotropic* (i.e. it is similar to an orthonormal matrix multiplied by a number).

There are 3 types of 2-digit tiles (2-tiles) in $\mathbb{R}^2$ up to affine similarity. We call them Square, Dragon, Bear, all are isotropic. In $\mathbb{R}^3$ there are exactly 7 types of 2-tiles, there is only one isotropic among them, which is a cube.

Classification of isotropic 2-tiles

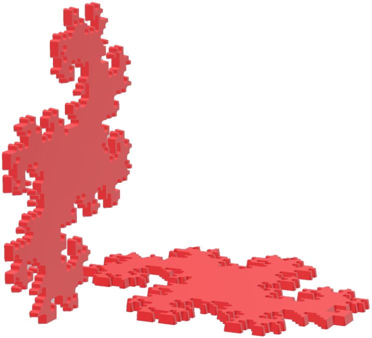
Theorem

If d is odd, then all isotropic 2-tiles in $\mathbb{R}^d$ are parallelepipeds.

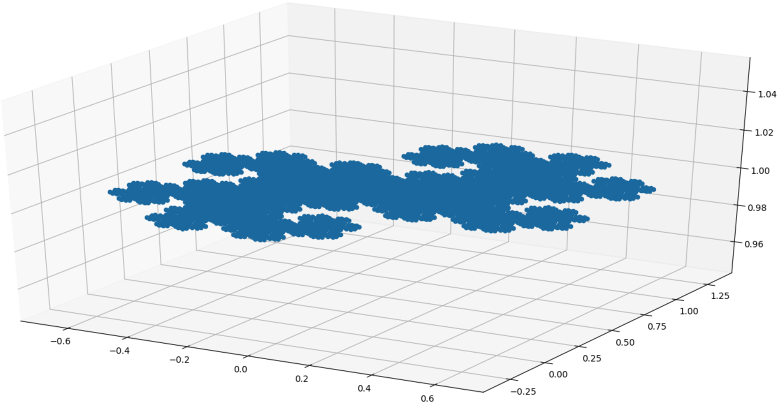
Classification of isotropic 2-tiles

Theorem

If $d = 2k$ is even, then in $\mathbb{R}^d$ there are, up to an affine similarity, exactly three isotropic 2-tiles: a parallelepiped, a direct product of k dragons, and a direct product of k bears.

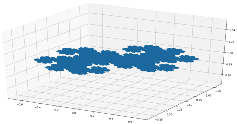


Dragon-1

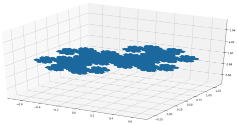


1.04
1.02
1.00
0.98
0.96
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
1.25
1.00
0.75
0.50
0.25
0.00
−0.25

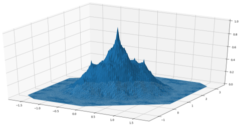
Multivariate B-splines



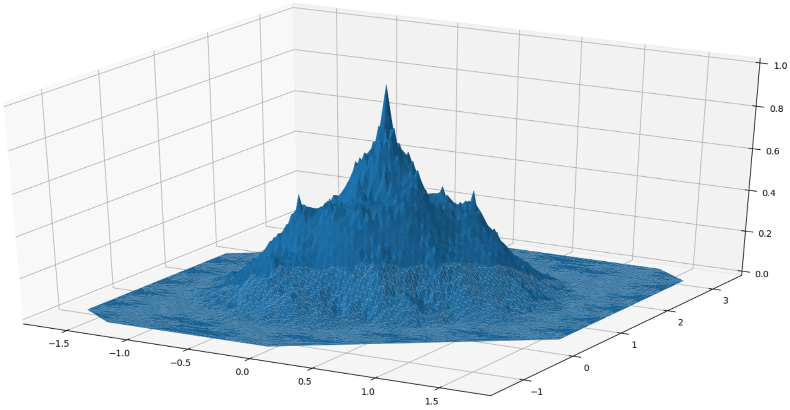
$*$



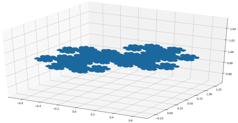
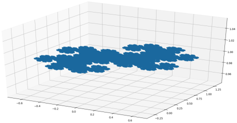
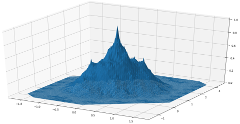
$=$

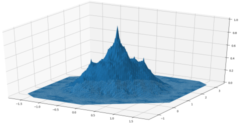
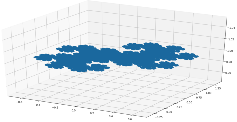
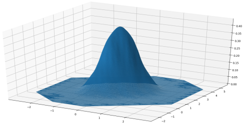


Dragon-2

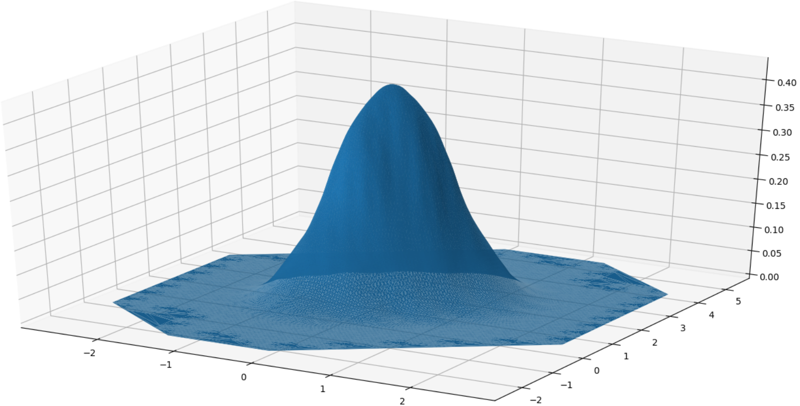


Multivariate B-splines

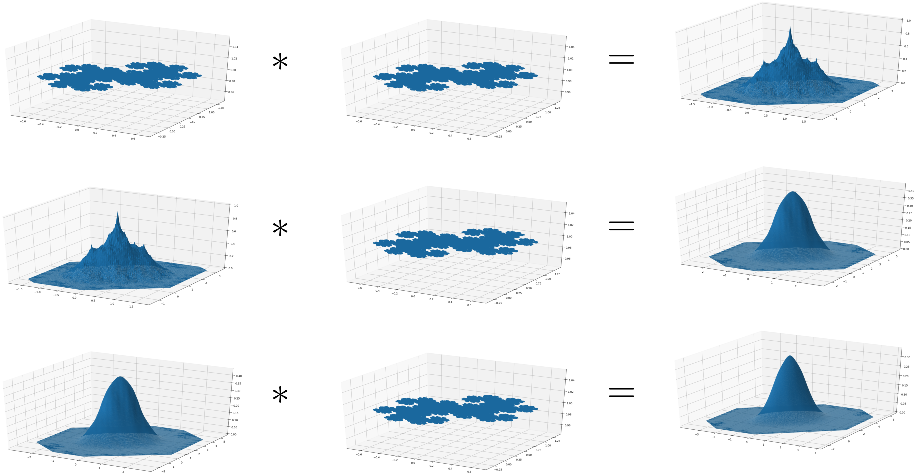
 $*$  $=$ 

 $*$  $=$ 

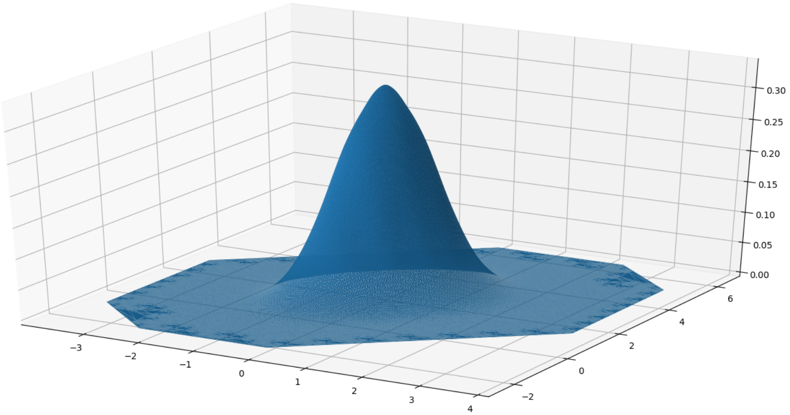
Dragon-3



Multivariate B-splines



Dragon-4



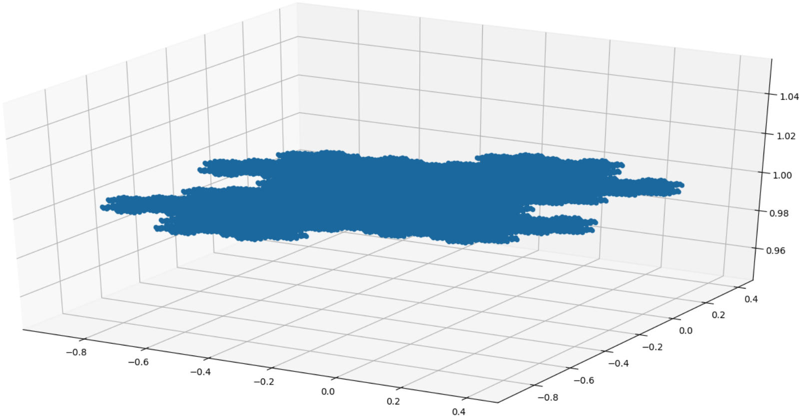
L_2-regularity

The order of B-spline	1st (B_1)	2nd (B_2)	3rd (B_3)	4th (B_4)	5th (B_5)
Square	0.5	1.5	2.5	3.5	4.5
Dragon	0.2382	1.0962	1.8039	2.4395	3.0557
Bear	0.3946	1.5372	2.6323	3.7092	4.7668

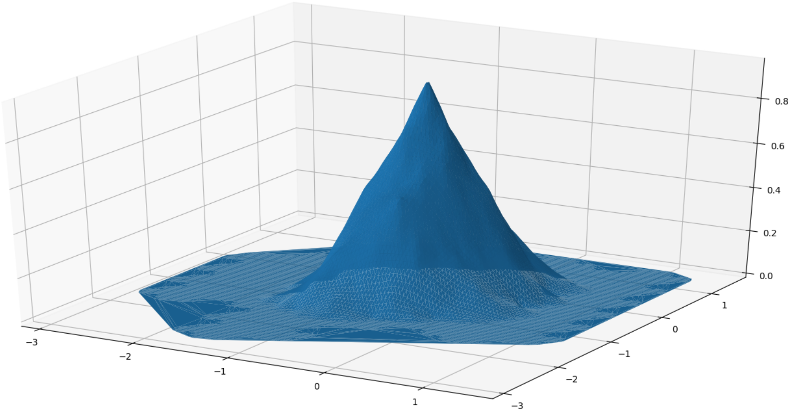
C-regularity

The order of B-spline	1st (B_1)	2nd (B_2)	3rt (B_3)	4th (B_4)
Square	0	1	2	3
Dragon	0	0.47637	1.5584	2.1924
Bear	0	0.7892	2.2349	3.0744

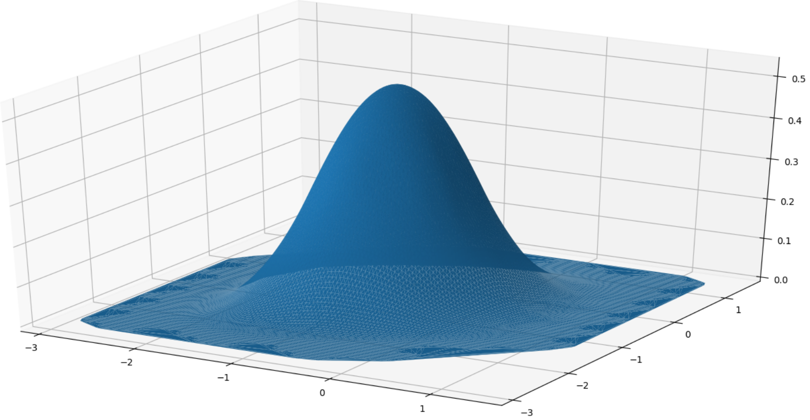
Bear-1



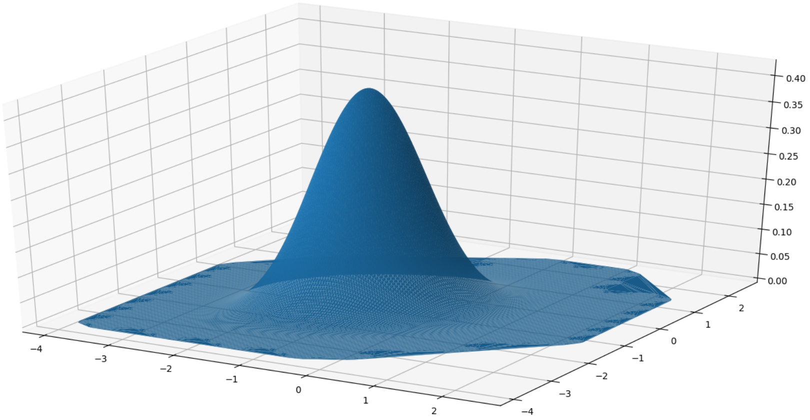
Bear-2



Bear-3



Bear-4



Bear-4, 1st order derivatives

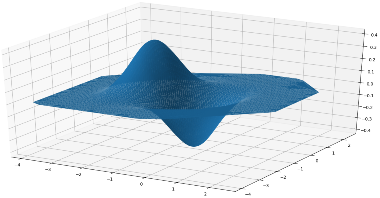


Figure: x

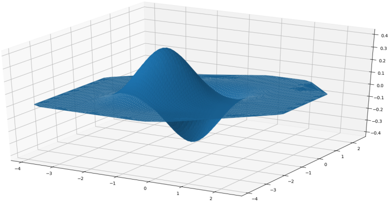
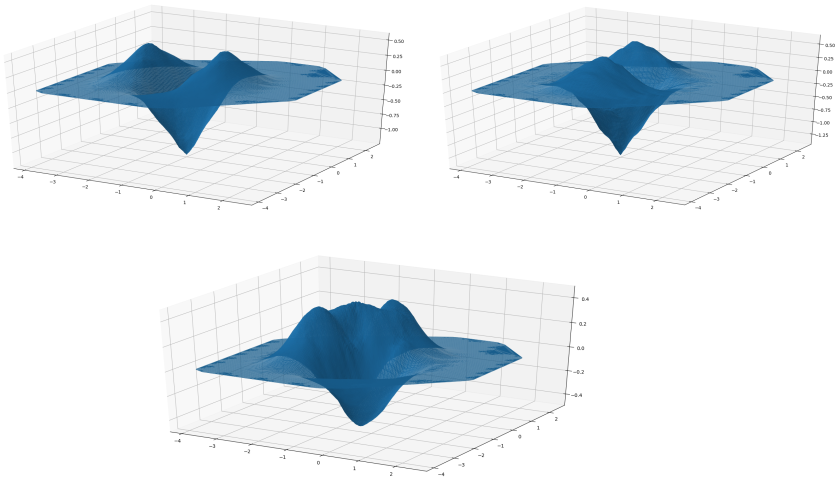


Figure: y

Bear-4, 2nd order derivatives



Bear-4, 3rd order derivatives