

Unified Implementation of Finite Element Methods involving Jumps and Averages in Matlab

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Abstract

We provide unified implementations of the finite element methods involving jumps and averages in Matlab by combining the use of the software package varFEM, including the adaptive finite element methods for the Poisson equation and the C^0 interior penalty methods for the biharmonic equation. The design ideas can be extended to other Galerkin-based methods, for example, the discontinuous Galerkin methods and the virtual element methods.

1 Implementation of finite element methods in varFEM package

In this section we briefly review the finite element method and introduce the numerical implementation in varFEM, a Matlab software package for the finite element method. For simplicity, we consider the Poisson equation with the homogeneous Dirichlet boundary conditions.

1.1 The finite element method for the Poisson equation

Let Ω be a bounded Lipschitz domain in \mathbb{R}^2 with polygonal boundary $\partial\Omega$. Consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $f \in L^2(\Omega)$ is a given function. The continuous variational problem is to find $u \in V := H_0^1(\Omega)$ such that

$$a(u, v) = \ell(v), \quad v \in V,$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\sigma, \quad \ell(v) = \int_{\Omega} f v d\sigma.$$

For the finite element discretization, we discuss the conforming Lagrange elements. Let \mathcal{T}_h be a shape regular triangulation. The generic element will be denoted as K in the sequel. Define

$$V_h = \{v \in V : v|_K \in \mathbb{P}_k(K), \quad K \in \mathcal{T}_h\},$$

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where $k \leq 3$. The discrete problem is to seek $u_h \in V_h$ satisfying

$$a(u_h, v) = \ell(v), \quad v \in V_h. \quad (1.2)$$

1.2 Data structures for triangular meshes

We adopt the data structures given in *iFEM* [3]. All related data are stored in the Matlab structure `Th`, which is computed by using the subroutine `FeMesh2d.m` as

```
1 Th = FeMesh2d(node, elem, bdStr);
```

where the basic data structures `node` and `elem` are generated by

```
1 [node, elem] = squaremesh([x1 x2 y1 y2], h1, h2);
```

For clarity, we take a simple mesh shown in Fig. 1 as an example.

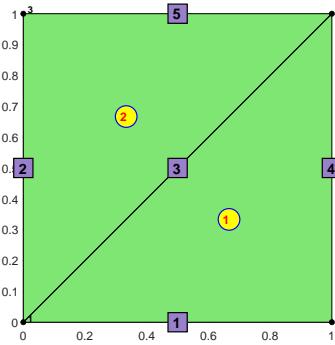


Fig. 1: Illustration of the data structures

The triangular meshes are represented by two basic data structures `node` and `elem`, where `node` is an $N \times 2$ matrix with the first and second columns contain x - and y -coordinates of the nodes in the mesh, and `elem` is an $NT \times 3$ matrix recording the vertex indices of each element in a counterclockwise order, where N and NT are the numbers of the vertices and triangular elements. For the mesh given in Fig. 1,

$$\text{elem} = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

In the current version, we only consider the \mathbb{P}_k -Lagrange finite element spaces with k up to 3. In this case, there are two important data structures `edge` and `elem2edge`. In the matrix `edge(1:NE, 1:2)`, the first and second rows contain indices of the starting and ending points. The column is sorted in the way that for the k -th edge, $\text{edge}(k, 1) < \text{edge}(k, 2)$ for $k = 1, 2, \dots, NE$. For the given triangulation,

$$\text{edge} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 4 \\ 3 & 4 \end{bmatrix}.$$

The matrix `elem2edge` establishes the map of local index of edges in each triangle to its global index in matrix `edge`. By convention, we label three edges of a triangle such that the i -th edge is opposite to the i -th vertex. For the given mesh,

$$\text{elem2edge} = \begin{bmatrix} 3 & 1 & 4 \\ 3 & 5 & 2 \end{bmatrix}.$$

In some cases, we may need to specify the left and right triangles sharing the same edge. For this purpose, we introduce another data structure `edge2elem`, which is an $\text{NE} \times 4$ matrix such that `edge2elem(k, 1)` and `edge2elem(k, 2)` are two triangles sharing the k -th edge for an interior edge. If the k -th edge is on the boundary, then we set `edge2elem(k, 1) = edge2elem(k, 2)`. For convenience, we also record the local index of the edge on the left and right triangles in `edge2elem(k, 3)` and `edge2elem(k, 4)`, respectively. For the given mesh,

$$\text{edge2elem} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

We refer the reader to <https://www.math.uci.edu/~chenlong/ifemdoc/mesh/auxstructuredoc.html> for some detailed information.

To deal with boundary integrals, we first extract the boundary edges from `edge` and store them in matrix `bdEdge`. In the input of `FeMesh2d.m`, the string `bdStr` is used to indicate the interested boundary part in `bdEdge`. For example, for the unit square $\Omega = (0, 1)^2$,

- `bdStr = 'x==1'` divides `bdEdge` into two parts: `bdEdgeType{1}` gives the boundary edges on $x = 1$, and `bdEdgeType{2}` stores the remaining part.
- `bdStr = {'x==1', 'y==0'}` separates the boundary data structure `bdEdge` into three parts: `bdEdgeType{1}` and `bdEdgeType{2}` give the boundary edges on $x = 1$ and $y = 0$, respectively, and `bdEdgeType{3}` stores the remaining part.
- `bdStr = []` implies that `bdEdgeType{1} = bdEdge`.

We also use `bdEdgeIdxType` to record the indices in matrix `edge`, and `bdNodeIdxType` to store the node indices for respective boundary parts. Note that we determine the boundary of interest by the coordinates of the midpoint of the edge, so '`x==1`' can also be replaced by a statement like '`x>0.99`'.

1.3 Implementation of the FEM in varFEM

FreeFEM is a popular 2D and 3D partial differential equations (PDE) solver based on finite element methods [4], which has been used by thousands of researchers across the world. The highlight is that the programming language is consistent with the variational formulation of the underlying PDEs, referred to as the variational formulation based programming in [5], where we

have developed an FEM package in a similar way of FreeFEM using the language of Matlab, named varFEM. The similarity here only refers to the programming style of the main or test script, not to the internal architecture of the software.

Consider the Poisson equation with the Dirichlet boundary condition on the unit square. The exact solution is given by

$$u(x, y) = xy(1 - x)(1 - y)\exp(-1000((x - 0.5)^2 + (y - 0.117)^2)).$$

The PDE data is generated by `pde = Poissonsdata_afem()`. The function file is simply given as follows.

```

1 function uh = varPoisson(Th,pde,Vh,quadOrder)
2
3 %% Assemble stiffness matrix
4 Coef = 1;
5 Test = 'v.grad';
6 Trial = 'u.grad';
7 kk = assem2d(Th,Coef,Test,Trial,Vh,quadOrder);
8
9 %% Assemble right hand side
10 Coef = pde.f; Test = 'v.val';
11 ff = assem2d(Th,Coef,Test,[],Vh,quadOrder);
12
13 %% Apply Dirichlet boundary conditions
14 g_D = pde.g_D;
15 on = 1;
16 uh = apply2d(on,Th,kk,ff,Vh,g_D);

```

In the above code, the structure `pde` stores the information of the PDE, including the exact solution `pde.uxexact`, the gradient `pde.Du`, etc. We set up the triple `(Coef, Test, Trial)` for the coefficients, test functions and trial functions in variational form, respectively. It is obvious that `v.grad` is for ∇v and `v.val` is for v itself. The routine `assem2d.m` computes the stiffness matrix corresponding to the bilinear form on the two-dimensional region, i.e.

$$A = (a_{ij}), \quad a_{ij} = a(\Phi_j, \Phi_i),$$

where Φ_i are the global shape functions of the finite element space `Vh`. The integral of the bilinear form, as $(\nabla\Phi_i, \nabla\Phi_j)_\Omega$, is approximated by using the Gaussian quadrature formula with `quadOrder` being the order of accuracy.

We remark that the `Coef` has three forms:

1. A function handle or a constant.
2. The numerical degrees of freedom of a finite element function.
3. A coefficient matrix `CoefMat` resulting from the numerical integration.

In the computation, the first two forms in fact will be transformed to the third one. Given a

function $c(p)$, where $p = (x, y)$, the coefficient matrix is in the following form

$$\text{CoefMat} = \begin{bmatrix} c(p_1^1) & c(p_2^1) & \cdots & c(p_{n_g}^1) \\ c(p_1^2) & c(p_2^2) & \cdots & c(p_{n_g}^2) \\ \vdots & \vdots & \vdots & \vdots \\ c(p_1^{\text{NT}}) & c(p_2^{\text{NT}}) & \cdots & c(p_{n_g}^{\text{NT}}) \end{bmatrix}. \quad (1.3)$$

Here, $p_1^i, p_2^i, \dots, p_{n_g}^i$ are the quadrature points on the element K_i .

We display the numerical result in Fig. 2 for the uniform triangular mesh with $h_1 = h_2 = 1/50$ generated by

```
1 [node, elem] = squaremesh([0 1 0 1], 1/50, 1/50);
```

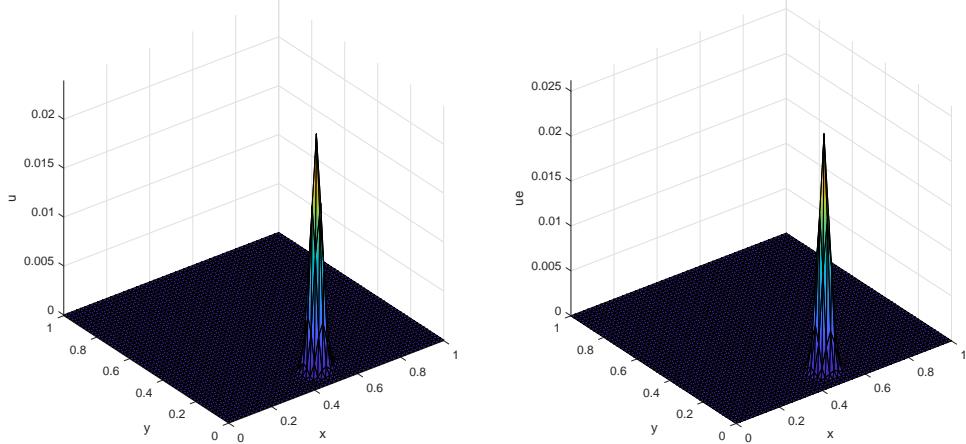


Fig. 2: Numerical and exact solutions with $h_1 = h_2 = 1/50$

1.4 Review on writing the subroutine `assem2d.m`

To make the discussions in the subsequent sections more clear, we now introduce the details of writing the subroutine `assem2d.m` to assemble a two-dimensional scalar bilinear form

$$a(v, u), \quad v = \varphi_i, \quad u = \phi_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n,$$

where the test function v and the trial function u are allowed to match different finite element spaces, which, for example, can be found in mixed finite element methods for Stokes problems. To handle different spaces, we write $\text{vh} = \{'\text{P1}', '\text{P2}'\}$ for the input of `assem2d.m`, where $\text{vh}\{1\}$ is for v and $\text{vh}\{2\}$ is for u . For simplicity, it is also allowed to write $\text{vh} = '\text{P1}'$ when v and u are in the same space.

Let us discuss the case where v and u are in the same space. Suppose that the bilinear form contains only first-order derivatives. Then the possible combinations are

$$\int_K avudx, \quad \int_K av_x u dx, \quad \int_K av_y u d\sigma,$$

$$\begin{aligned} & \int_K avu_x d\sigma, \quad \int_K av_x u_x d\sigma, \quad \int_K av_y u_x d\sigma, \\ & \int_K avu_y d\sigma, \quad \int_K av_x u_y d\sigma, \quad \int_K av_y u_y d\sigma. \end{aligned}$$

Of course, we often encounter the gradient form

$$\int_K a \nabla v \cdot \nabla u d\sigma = \int_K a(v_x u_x + v_y u_y) d\sigma.$$

We take the second bilinear form as an example. Let

$$a_K(v, u) = \int_K av_x u d\sigma,$$

and consider the \mathbb{P}_1 -Lagrange finite element. Denote the local basis functions to be ϕ_1, ϕ_2, ϕ_3 . Then the local stiffness matrix is

$$A_K = \int_K a \begin{bmatrix} \partial_x \phi_1 \\ \partial_x \phi_2 \\ \partial_x \phi_3 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix} d\sigma.$$

Let

$$v_1 = \partial_x \phi_1, \quad v_2 = \partial_x \phi_2, \quad v_3 = \partial_x \phi_3; \quad u_1 = \phi_1, \quad u_2 = \phi_2, \quad u_3 = \phi_3.$$

Then

$$A_K = (k_{ij})_{3 \times 3}, \quad k_{ij} = \int_K av_i u_j d\sigma.$$

The integral will be approximated by the Gaussian quadrature rule:

$$k_{ij} = \int_K av_i u_j d\sigma = |K| \sum_{p=1}^{n_g} w_p a(x_p, y_p) v_i(x_p, y_p) u_j(x_p, y_p),$$

where (x_p, y_p) is the p -th quadrature point. In the implementation, we in advance store the quadrature weights and the values of basis functions or their derivatives in the following form:

$$w_p, \quad \text{vi}(:, p) = \begin{bmatrix} v_i|_{(x_p^1, y_p^1)} \\ v_i|_{(x_p^2, y_p^2)} \\ \vdots \\ v_i|_{(x_p^{\text{NT}}, y_p^{\text{NT}})} \end{bmatrix}, \quad \text{uj}(:, p) = \begin{bmatrix} u_j|_{(x_p^1, y_p^1)} \\ u_j|_{(x_p^2, y_p^2)} \\ \vdots \\ u_j|_{(x_p^{\text{NT}}, y_p^{\text{NT}})} \end{bmatrix}, \quad p = 1, \dots, n_g,$$

where vi associated with v_i is of size $\text{NT} \times \text{ng}$ with the p -th column given by $\text{vi}(:, p)$. Let $\text{weight} = [w_1, w_2, \dots, w_{n_g}]$ and $\text{ww} = \text{repmat}(\text{weight}, \text{NT}, 1)$. Then k_{ij} for $a = 1$ can be computed as

```

1 k11 = sum(ww.*v1.*u1, 2);
2 k12 = sum(ww.*v1.*u2, 2);
3 k13 = sum(ww.*v1.*u3, 2);
4 k21 = sum(ww.*v2.*u1, 2);
5 k22 = sum(ww.*v2.*u2, 2);
6 k23 = sum(ww.*v2.*u3, 2);
7 k31 = sum(ww.*v3.*u1, 2);
8 k32 = sum(ww.*v3.*u2, 2);
9 k33 = sum(ww.*v3.*u3, 2);
10 K = [k11, k12, k13, k21, k22, k23, k31, k32, k33];

```

Here we have stored the local stiffness matrix A_K in the form of $[k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, k_{23}, k_{31}, k_{32}, k_{33}]$, and stacked the results of all cells together. By adding the contribution of the area, one has

```
1 Ndof = 3;
2 K = repmat(area,1,Ndof^2).*K;
```

For the variable coefficient case, such as $a(x, y) = x+y$, one can further introduce the coefficient matrix as

```
1 cf = @(pz) pz(:,1) + pz(:,2); % x+y;
2 cc = zeros(NT,ng);
3 for p = 1:ng
4     pz = lambda(p,1)*z1 + lambda(p,2)*z2 + lambda(p,3)*z3;
5     cc(:,p) = cf(pz);
6 end
```

where pz are the quadrature points on all elements. The above procedure can be implemented as follows.

```
1 K = zeros(NT,Ndof^2);
2 s = 1;
3 v = {v1,v2,v3}; u = {u1,u2,u3};
4 for i = 1:Ndof
5     for j = 1:Ndof
6         vi = v{i}; uj = u{j};
7         K(:,s) = area.*sum(ww.*cc.*vi.*uj,2);
8         s = s+1;
9     end
10 end
```

The bilinear form is assembled by using the built-in function `sparse.m` as in *iFEM*. In this case, the code is given as

```
1 ss = K(:);
2 kk = sparse(ii,jj,ss,NNdof,NNdof);
```

Here, (ii, jj) is called the sparse index.

Remark 1.1. In varFEM, we use `Base2d.m` to load the information of vi and uj , for example, the following code gives the values of $\partial_x \phi$, where ϕ is a local basis function.

```
1 v = 'v.dx';
2 vbase = Base2d(v,node,elem,Vh{1},quadOrder);
```

Remark 1.2. The sparse index (ii, jj) can be simply given by the connectivity list `elem2dof` as

```
1 % assembly index
2 ii = reshape(repmat(elem2dof, Ndof,1), [], 1);
3 jj = repmat(elem2dof(:,1), Ndof, 1);
```

where `elem2dof` is an $NT \times Ndof$ matrix, with the i -th row representing the index vector of the degrees of freedom. For example, `elem2dof = elem` for the \mathbb{P}_1 -Lagrange element.

2 Adaptive finite element methods for the Poisson equation

In this section we briefly introduce the ingredients of the adaptive finite element method, and provide the overall structure of the implementation.

2.1 The ingredients of the adaptive FEM

For the conforming FEM, one can establish the following residual based a-posteriori error estimate

$$\|u - u_h\|_1 \lesssim \eta(u_h),$$

where $\eta = \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}$ and

$$\eta_K^2 = h_K^2 \|f + \Delta u_h\|_{0,K}^2 + \sum_{e \subset \partial K} h_e \|[\partial_{n_e} u_h]\|_{0,e}^2 \quad (2.1)$$

are referred to as the global and local error indicators, respectively. Here, $f + \Delta u_h$ is the interior residual, and

$$[\partial_{n_e} u_h] = \partial_{n_e} u_h|_{K^-} - \partial_{n_e} u_h|_{K^+}$$

is the jump term, where n_e is the unit outer normal vector of the target element $K = K^-$, and K^+ is the neighbouring triangle sharing e as an edge. Note that for an edge e on the domain boundary $\partial\Omega$, we assume $\partial_{n_e} u_h|_{K_2} = 0$. For some problems, we also need to introduce the average: $\{u\} = \frac{1}{2}(u|_{K^-} + u|_{K^+})$ for an interior edge and $\{u\} = u$ for a boundary edge.

Standard adaptive algorithms based on the local mesh refinement can be written as loops of the form

SOLVE → ESTIMATE → MARK → REFINE.

Given an initial subdivision \mathcal{T}_0 , to get \mathcal{T}_{k+1} from \mathcal{T}_k we first solve the FEM problem under consideration to get the numerical solution u_k on \mathcal{T}_k . The error is then estimated by using u_k , \mathcal{T}_k and the a posteriori error bound $\eta(u_h)$. The local error bound η_K is used to mark a subset \mathcal{M} of elements in \mathcal{T}_k for refinement. The marked triangles and possible more neighboring elements are refined in such a way that the subdivision meets certain conditions, for example, the resulting triangular mesh is still shape regular. The above procedures are included in the test script with an overview given as follows

```

1 for k = 1:maxIt
2   % Step 1: SOLVE
3   Th = FeMesh2d(node, elem, bdStr);
4   uh = varPoisson(Th, pde, Vh, quadOrder);
5   % Step 2: ESTIMATE
6   eta = Poisson_indicator(Th, uh, pde, Vh, quadOrder);
7   % Step 3: MARK
8   elemMarked = mark(elem, eta, theta);
9   % Step 4: REFINE
10  [node, elem] = bisect(node, elem, elemMarked);
11 end

```

Three important modules are involved: the local error indicator in Step 2, the marking algorithm in Step 3 and the local refinement algorithm in Step 4. We employ the Dörfler marking strategy to select the subset of elements and then use the newest vertex method to refine the mesh. Note that the subroutines `mark.m` and `bisect.m` are extracted from *iFEM* [3] with some minor modifications. In this article, we are only concerned with the computation of the error indicator. For the later two steps, we refer the reader to [3] for details on the implementation.

2.2 The unified implementation of the error indicator

2.2.1 The computation of the elementwise residuals

With the help of varFEM, the first term $h_K^2 \|f + \Delta u_h\|_{0,K}^2$ in (2.1) can be simply computed as

```

1 %% elementwise residuals
2 fc = interp2dMat(pde.f, '.val', Th, Vh, quadOrder);
3 uxxc = interp2dMat(uh, '.dxx', Th, Vh, quadOrder);
4 uyyc = interp2dMat(uh, '.ddy', Th, Vh, quadOrder);
5 Coef = (fc + uxxc + uyyc).^2;
6 [~, elemIh] = integral2d(Th, Coef, Vh, quadOrder);
7 elemRes = diameter.^2.*elemIh;

```

In the above code, `elemIh` stores all the local error indicators $[\eta_{K_1}, \dots, \eta_{K_{NT}}]^T$; The function `interp2dMat` is used to generate the coefficient matrix (see (1.3)). It is obvious that the coefficient matrix of $(f + \Delta u_h)^2$ is $(fc + uxxc + uyyc) .^2$, where `fc`, `uxxc` and `uyyc` are the coefficient matrices of f , $\partial_{xx} u_h$ and $\partial_{yy} u_h$, respectively.

In what follows, we focus on the unified implementation of the jump term or jump integral $\sum_{e \subset \partial K} h_e \|[\partial_{n_e} u_h]\|_{0,e}^2$. We remark that any other type of jump terms can be easily adapted or designed accordingly as will be seen.

2.2.2 The elementwise interior and exterior indices of the quadrature points

The integral over e is calculated by using the one-dimensional Gaussian numerical integration formula

$$\int_e f ds = |e|(w_1 f(p_1) + w_2 f(p_2) + \dots + w_{n_g} f(p_{n_g})),$$

where w_i and p_i are the quadrature weights and points on e , and n_g is the number of the quadrature points. Note that the endpoints of e are not included.

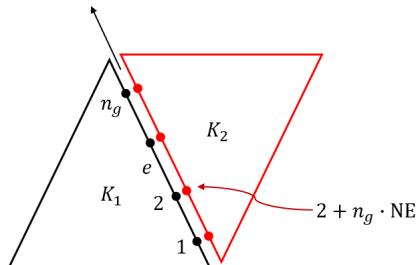


Fig. 3: Illustration of the quadrature points

- Indexing rule for the quadrature points. In Fig. 3 the direction of the edge e is specified by the arrow. In the implementation, the direction is determined by the data structure `edge` which satisfies `edge(k, 1) < edge(k, 2)`. For the first edge $e = e_1$ in `edge`, the quadrature points will be numbered by $1, 2, \dots, n_g$ or $1 : n_g$ for short when restricted to the left triangle. Similarly, the quadrature points on the i -th edge e_i will be numbered by

$$p_i^L = (1 : n_g) + (i - 1)n_g, \quad i = 1, 2, \dots, \text{NE}.$$

The right-hand restrictions are accordingly numbered as

$$p_i^R = p_i^L + n_g \cdot \text{NE}, \quad i = 1, 2, \dots, \text{NE}.$$

- Elementwise sign matrix. To characterize the direction of an edge on an element K , one can introduce the elementwise sign matrix

```
1 % sign of elementwise edges
2 sgnelem = sign([elem(:,3)-elem(:,2), elem(:,1)-elem(:,3), elem(:,2)-elem(:,1)]);
```

In some cases, it is better to restore the positive sign for edges on the boundary of the domain, or one can set the signs to be zero for later use:

```
1 E = false(NE,1); E(bdEdgeIdx) = 1; sgnbd = E(elem2edge);
2 sgnelem(sgnbd) = 0;
```

Here, the data structures `elem2edge` and `bdEdgeIdx` have been introduced in Subsect. 1.2.

- Elementwise interior indices. According to the indexing rule, one easily obtains the interior indices of the first sides of all triangles:

```
1 % first side
2 e1 = elem2edge(:,1); sgn1 = sgnelem(:,1);
3 id = repmat(1:ng, NT, 1);
4 id(sgn1<0,:) = repmat((ng:-1:1)+NE*ng, sum(sgn1<0), 1);
5 elemQuade1 = id + (e1-1)*ng;
```

Note that for the edges with positive and zero signs the natural indices are $1 : n_g$, while for those with negative signs the natural indices are reversed. One can similarly introduce the interior indices of other sides, and hence give the elementwise interior indices

```
1 elemQuadM = [elemQuade1, elemQuade2, elemQuade3];
```

where the letter `M` stands for “Minus”.

- Elementwise exterior indices. To compute the jump, we also need to introduce the elementwise exterior indices `elemQuadP`, where `P` is for “Plus”. Given an edge, assume that the interior indices of the quadrature points are i_1, \dots, i_{n_g} , and the exterior indices are i'_1, \dots, i'_{n_g} . Since $|i_k - i'_k| = n_g \cdot \text{NE}$, one just needs to subtract $n_g \cdot \text{NE}$ for those indices in `elemQuadM` greater than $n_g \cdot \text{NE}$, and add $n_g \cdot \text{NE}$ to those indices less than $n_g \cdot \text{NE}$.

```
1 index = ( elemQuadM > ng*NE ) ;
2 elemQuadP = elemQuadM + (-ng*NE)*index + ng*NE*(-index);
```

Obviously, for edges on the domain boundary, the exterior indices are greater than $n_g \cdot \text{NE}$.

2.2.3 The elementwise interior and exterior evaluations

Given u_h , we now determine the interior evaluations `elemuhM` and the exterior evaluations `elemuhP` of u_h corresponding to `elemQuadM` and `elemQuadP`. To this end, one can compute the evaluations of the basis functions at the quadrature points along the boundary ∂K . Given a triangle K , let λ_1, λ_2 and λ_3 be the barycentric coordinate functions. Since λ_i are usually used to construct the basis functions for the FEMs, one can first specify the evaluations of λ_i and the derivatives $\partial_x \lambda_i$ and $\partial_y \lambda_i$ at the quadrature points.

Given some points p_1, p_2, \dots, p_{n_G} on the triangle K , for example, the quadrature points for the integration of the bilinear forms. In varFEM, the evaluations of λ_i are stored in the following form

$$\begin{bmatrix} \lambda_1(p_1) & \lambda_2(p_1) & \lambda_3(p_1) \\ \lambda_1(p_2) & \lambda_2(p_2) & \lambda_3(p_2) \\ \vdots & \vdots & \vdots \\ \lambda_1(p_{n_G}) & \lambda_2(p_{n_G}) & \lambda_3(p_{n_G}) \end{bmatrix}.$$

At this time, one can choose p_j as the quadrature points along the boundary ∂K . Let $\partial K = e_1 \cup e_2 \cup e_3$. The quadrature points on e_i are denoted by $p_{1,e_i}, \dots, p_{n_g,e_i}$. In this case, $n_G = 3n_g$, and one can specify the values $\lambda_i(p_j)$ by using the 1-D quadrature points r_1, r_2, \dots, r_{n_g} . The Gaussian quadrature points and weights $r = [r_1, r_2, \dots, r_{n_g}]$ and $w = [w_1, w_2, \dots, w_{n_g}]$ on $[0, 1]$ are given by `quadpts1.m`:

```
1 [lambda1d, weight1d] = quadpts1(4); ng = length(weight1d);
2 [~, id] = sort(lambda1d(:, 1));
3 lambda1d = lambda1d(id, :); weight1d = weight1d(id);
```

Here we use `sort` to guarantee $r_1 < r_2 < \dots < r_{n_g}$. Note that

$$\text{lambda1d} = \begin{bmatrix} r_1 & r_{n_g} \\ r_2 & r_{n_g-1} \\ \vdots & \vdots \\ r_{n_g} & r_1 \end{bmatrix},$$

and all the row sums are 1, i.e., $r_1 + r_{n_g} = r_2 + r_{n_g-1} = \dots = 1$. By the definition of the barycentric coordinate functions, the coordinates $(\lambda_1, \lambda_2, \lambda_3 = 1 - \lambda_1 - \lambda_2)$ on the three sides of K are:

- the 1-th side: $(0, r_{n_g}, r_1), (0, r_{n_g-1}, r_2), \dots, (0, r_1, r_{n_g})$;
- the 2nd side: $(r_1, 0, r_{n_g}), (r_2, 0, r_{n_g-1}), \dots, (r_{n_g}, 0, r_1)$;
- the 3rd side: $(r_{n_g}, r_1, 0), (r_{n_g-1}, r_2, 0), \dots, (r_1, r_{n_g}, 0)$.

Therefore the $n_G = 3n_g$ points can be given as

```
1 function [lambdaBd, weightBd] = quadptsBd(order)
2 %%Gauss quadrature points along the boundary of triangles
3
4 % quadrature on [0,1]
5 [lambda1d, weight1d] = quadpts1(order); ng = length(weight1d);
6 [~, id] = sort(lambda1d(:, 1));
```

```

7 lambda1d = lambda1d(id,:); weight1d = weight1d(id);
8 % quadrature on each side
9 lambdae1 = [zeros(ng,1), lambda1d(:,2), lambda1d(:,1)];
10 lambdae2 = [lambda1d(:,1), zeros(ng,1), lambda1d(:,2)];
11 lambdae3 = 1 - lambdae1 - lambdae2;
12 % quadrature along the boundary of each element
13 lambdaBd = [lambdae1; lambdae2; lambdae3];
14 weightBd = repmat(weight1d,1,3);

```

With these quadrature points one can construct the evaluations of the basis functions. The details are omitted. Please refer to `Base2dBd.m` for implementation details. For the \mathbb{P}_1 element, the basis functions are $\phi_i = \lambda_i$, and the elementwise interior evaluations of u_h are given as follows

```

1 elemuhM = zeros(NT,3*ng);
2 for p = 1:3*ng
3     % interpolation at the p-th quadrature point
4     for i = 1:length(phi)
5         base = phi{i};
6         elemuhM(:,p) = elemuhM(:,p) + uh(elem2dof(:,i)).*base(:,p);
7     end
8 end

```

For other types of finite elements, one just needs to modify the basis functions `base`. For evaluations of the derivatives, for example, $\partial_x u_h$, one needs to change `base` to the partial derivative ∂_x of the basis functions. The elementwise exterior evaluations of u_h can be computed as

```

1 uhI = zeros(2*NE*ng,1);
2 uhI(elemQuadM) = elemuhM;
3 elemuhP = uhI(elemQuadP);

```

We remark that the exterior evaluations on the domain boundary are zero since the corresponding indices in `elemQuadM` are less than or equal to $n_g \cdot NE$.

The above discussions are summarized in a subroutine:

```

1 function [elemuhM,elemuhP] = elem2edgeInterp(wStr,Th,uh,Vh,quadOrder)

```

One can obtain the evaluations of u_h , $\partial_x u_h$ and $\partial_y u_h$ by setting `wStr` as `.val`, `.dx` and `.dy`, respectively. For convenience, we also return the elementwise unit normal vectors on the quadrature points, denoted by `elemQuadnx` and `elemQuadny`, where the first one is for n_x and the second one is for n_y . We remark that the normal vectors are simply the unit outer normal vector of the triangles.

2.2.4 The computation of the jump integral

With the above preparations, we are able to compute the jump integral

$$h_e \|[\partial_n u_h]\|_{0,e}^2 = h_e \int_e ([\partial_x u_h] n_x + [\partial_y u_h] n_y)^2 ds = h_e \cdot |e| \sum_{i=1}^{n_g} w_i ([\partial_x u_h] n_x + [\partial_y u_h] n_y)^2 (p_{i,e}).$$

- The first step is to compute the elementwise interior and exterior evaluations of $\partial_x u_h$ and $\partial_y u_h$:

```

1 %% elementwise interior and exterior evaluations at quadrature points
2 [elemuhxM,elemuhxP,elemnx,elemny] = elem2edgeInterp('.dx',Th,uh,Vh,quadOrder);
3 [elemuhyM,elemuhyP] = elem2edgeInterp('.dy',Th,uh,Vh,quadOrder);

```

- The elementwise jumps are

```

1 elem2Jumpx = elemuhxM - elemuhxP;
2 elem2Jumpy = elemuhyM - elemuhyP;

```

- The jump integral can be computed by looping of triangle sides:

```

1 elemJump = zeros(NT,1);
2 [~,weight1d] = quadpts1(quadOrder);
3 ng = length(weight1d);
4 for i = 1:3 % loop of triangle sides
5     hei = he(elem2edge(:,i));
6     id = (1:ng)+(i-1)*ng;
7     cei = hei;
8     neix = elemnx(:,id);
9     neiy = elemny(:,id);
10    Jumpnx = elem2Jumpx(:,id).*neix;
11    Jumpny = elem2Jumpy(:,id).*neiy;
12    Jumpn = (Jumpnx+Jumpny).^2;
13    elemJump = elemJump + cei.*hei.* (Jumpn*weight1d(:));
14 end

```

- We finally get the local error indicators

```

1 %% Local error indicator
2 eta = (abs(elemRes) + elemJump).^(1/2);

```

Here we add `abs` since the third-order quadrature rule has negative weights.

2.3 Numerical example

Consider the example given in Subsect. 1.3. We employ the Dörfler marking strategy with parameter $\theta = 0.4$ to select the subset of elements for refinement.

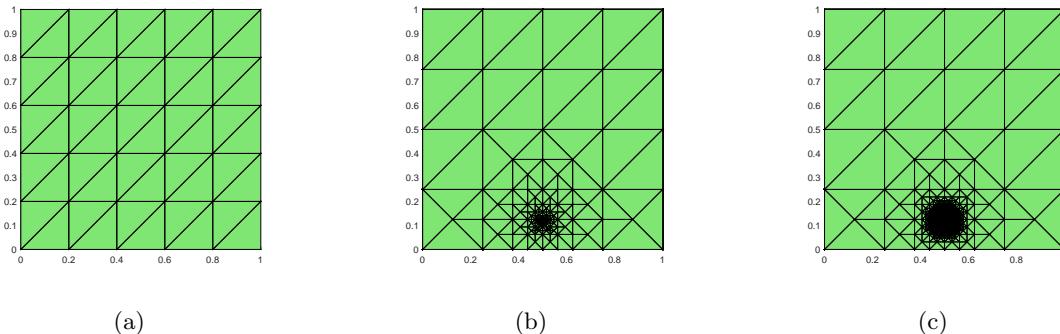


Fig. 4: The initial and the final adapted meshes. (a) The initial mesh; (b) After 20 refinement steps; (c) After 30 refinement steps

The initial mesh and the final adapted meshes after 20 and 30 refinement steps are presented in Fig. 4 (a-c), respectively, where the \mathbb{P}_1 element is used. We also plot the adaptive approximation in Fig. 5, which almost coincides with the exact solution. The convergence rates of the error estimators and the errors in H^1 norm are shown in Fig. 6, from which we observe the optimal

rates of convergence as predicted by the theory. The full code is available from varFEM package (<https://github.com/Terenceyuyue/varFEM>). The subroutine `PoissonVEM_indicator.m` is used to compute the local indicator and the test script is `main_Poisson_afem.m`.

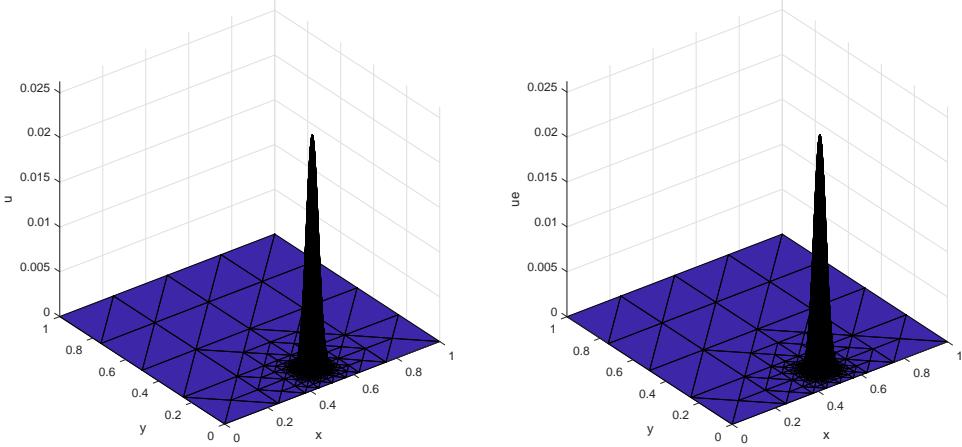


Fig. 5: The exact and numerical solutions

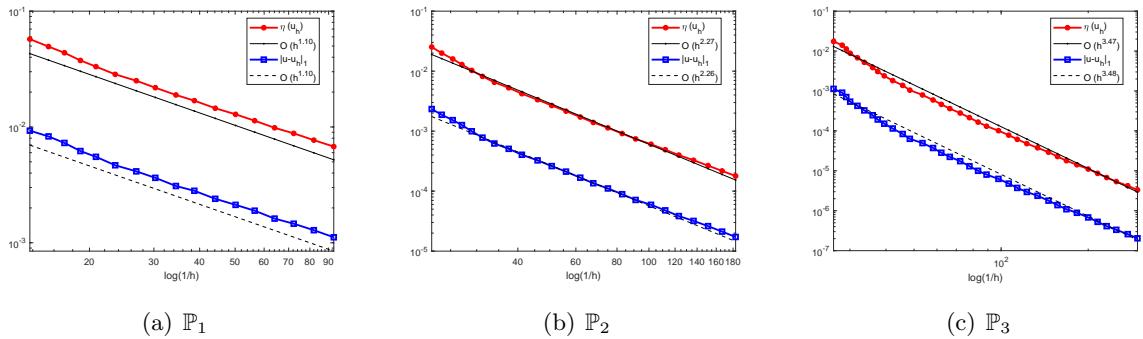


Fig. 6: The convergence rates of the error estimators and the errors in H^1 norm

3 C^0 interior penalty methods for the biharmonic equation

3.1 The introduction to the C^0 interior penalty method

Let $f \in L^2(\Omega)$ and $u \in V := H_0^2(\Omega)$ be the solution of the biharmonic equation $\Delta^2 u = f$. The variational problem is: Find $u \in V$ such that

$$a(u, v) = \ell(v), \quad v \in V,$$

where

$$a(u, v) = (\nabla^2 u, \nabla^2 v), \quad \ell(v) = (f, v).$$

Let $V_h := \mathbb{P}_2(\mathcal{T}_h) \cap H_0^1(\Omega)$ be the quadratic Lagrange finite element space. The symmetric C^0 interior penalty method is then to seek $u_h \in V_h$ such that (cf. [1, 2])

$$a_h(u_h, v_h) = \ell_h(v_h), \quad v_h \in V_h,$$

where $\ell_h(v_h) = (f, v_h)$ and the bilinear form is

$$a_h(u_h, v_h) := a_{\text{pw}}(u_h, v_h) - (\mathcal{J}(u_h, v_h) + \mathcal{J}(v_h, u_h)) + c_{\text{IP}}(u_h, v_h).$$

Here, a_{pw} is the energy scalar product or the interior bilinear form,

$$a_{\text{pw}}(u_h, v_h) = \sum_{K \in \mathcal{T}} (\nabla^2 u_h, \nabla^2 v_h)_K.$$

\mathcal{J} is the jump term,

$$\mathcal{J}(u_h, v_h) := \sum_{e \in \mathcal{E}} \int_e \{\nabla^2 u_h n_e\} \cdot [\nabla v_h] ds, \quad (3.1)$$

where \mathcal{E} denotes the set of all edges of \mathcal{T}_h . The H^2 conformity is imposed weakly by the additional penalty term c_{IP} ,

$$c_{\text{IP}}(u_h, v_h) := \sum_{e \in \mathcal{E}} \frac{\sigma_e}{h_e} \int_e [\nabla u_h \cdot n_e] [\nabla v_h \cdot n_e] ds,$$

where $\sigma_e > 0$ is an edge-dependent parameter.

The parameter σ_e is used to guaranteed the coercivity or stability of the bilinear form. An automated mesh-dependent selection is proposed in [2], given by

$$\sigma_e := \begin{cases} \frac{3ah_e^2}{4} \left(\frac{1}{|K^+|} + \frac{1}{|K^-|} \right), & e \in \mathcal{E}^0, \\ \frac{3ah_e^2}{|K|} = 2 \cdot \frac{3ah_e^2}{4} \left(\frac{1}{|K^+|} + \frac{1}{|K^-|} \right), & e \in \mathcal{E}^\partial, \end{cases}$$

where we assume $K^+ = K^- = K$ when e is the boundary edge. By Theorem 3.1 there, every choice of $a > 1$ leads to the guaranteed stability.

Remark 3.1. The jump term given in [1] is

$$\sum_{e \in \mathcal{E}} \int_e \left\{ \frac{\partial^2 u_h}{\partial n_e^2} \right\} \left[\frac{\partial v_h}{\partial n_e} \right] ds,$$

which is equivalent to the one in (3.1). In fact, the C^0 continuity leads to $\partial_\tau v_h = 0$, and hence $\nabla v_h = \partial_n v_h n + \partial_\tau v_h \tau = \partial_n v_h n$, where $n = [n_1, n_2]^T$, $\tau = [t_1, t_2]^T$ and $n \cdot \tau = 0$. A direct manipulation yields

$$\begin{aligned} \left\{ \frac{\partial^2 u_h}{\partial n^2} \right\} \left[\frac{\partial v_h}{\partial n} \right] &= \{n^T \nabla^2 u_h n\} [\partial_n v_h] = ([\partial_n v_h] n)^T \{\nabla^2 u_h n\} \\ &= [\nabla v_h]^T \{\nabla^2 u_h n\} = \{\nabla^2 u_h n\} \cdot [\nabla v_h], \end{aligned}$$

as required.

3.2 The computation of the interior bilinear form

The shape function space is exactly the \mathbb{P}_2 -Lagrange finite element space. In contrast to the fully discontinuous Galerkin method, here the degrees of freedom (DoFs) are continuous, which makes the implementation very similar to that for the nonconforming FEMs.

The interior bilinear form

$$a_{\text{pw}}(u_h, v_h) = \sum_{K \in \mathcal{T}} (\nabla^2 u_h, \nabla^2 v_h)_K$$

can be easily computed and assembled in varFEM. As for the Poisson equation, we first use FeMesh2d.m to acquire the mesh-related data structures and geometric quantities.

```

1 %% Mesh
2 [node, elem] = squaremesh([0 1 0 1], 0.1); % h1 = h2 = h = 0.1
3 bdStr = [];
4 Th = FeMesh2d(node, elem, bdStr);

```

The PDE data is defined in `biharmonicdata_COIP.m`. For the current method, the finite element space is specified by `Vh = 'P2'` as follows.

```

1 %% PDE data
2 pde = biharmonicdata_COIP();
3 %% Finite element space
4 Vh = 'P2'; quadOrder = 5;

```

Noting that

$$(\nabla^2 u, \nabla^2 v) = v_{xx}u_{xx} + v_{xy}u_{xy} + v_{yx}u_{yx} + v_{yy}u_{yy},$$

we can compute the stiffness matrix with respect to a_{pw} in the following way:

```

1 %% Interior bilinear form
2 Coef = {1,1,1,1};
3 Test = {'v.dxx', 'v.dxy', 'v.dyx', 'v.dyy'};
4 Trial = {'u.dxx', 'u.dxy', 'u.dyx', 'u.dyy'};
5 A = assem2d(Th, Coef, Test, Trial, Vh, quadOrder);

```

3.3 The unified implementation of bilinear forms involving jump and average terms

Different from the jump integral in the a-posteriori error estimator, the jumps and averages of the interior penalty method appear in the bilinear forms. In the sequel we will present a unified design idea for computing and assembling the jump and penalty terms. For clarity, we shall take the jump term

$$\mathcal{J}(u_h, v_h) := \sum_{e \in \mathcal{E}} \int_e \{\nabla^2 u_h n_e\} \cdot [\nabla v_h] ds$$

as an example. It should be pointed out that the above bilinear form is not suitable for elementwise computation because the jump $[\Phi]$ of the global nodal basis function Φ involves all neighbouring cells. The most natural way is the edge-wise implementation since only the left and right triangles need to be considered.

3.3.1 The design ideas

Given an interior edge e , the left and right triangles sharing it as an edge will be denoted by K_1 and K_2 , respectively. The union $K_e = K_1 \cup K_2$ is referred to as the macro element of e in what follows. Denote the local basis functions (or their derivatives) on K_1 by $\phi_{i_1}, \dots, \phi_{i_6}$ and on K_2 by $\phi_{j_1}, \dots, \phi_{j_6}$, where the subscripts are the global numbers of the DoFs. After removing duplicates, the 12 basis functions in fact are associated with only 9 DoFs, which will be ordered from smallest to largest according to the subscripts. The corresponding “macro basis functions” are denoted by $\varphi_1, \varphi_2, \dots, \varphi_9$.

The design idea is based on a simple observation: If the macro element K_e is considered as an element K , and the basis function φ on K_e is treated as a basis function ϕ on K , then the jump terms can be computed and assembled in the way as for the interior bilinear form. For this purpose, we can establish the following correspondence:

$$\begin{aligned} \text{edge2dof} &\longrightarrow \text{elem2dof}, \\ \text{edgeBase} &\longrightarrow \text{Base2d}. \end{aligned}$$

Specifically, for the jumps and averages on e , only the 9 basis functions on the macro elements have contribution to the bilinear forms. For this reason, if e is considered as an element K for the interior bilinear form, and the local basis functions on the macro element are viewed as local basis functions on K , then the data structure `edge2dof` representing the connectivity on the macro element can be established as the data structure `elem2dof` for the bilinear form of the Poisson equation (see Remark 1.2). Accordingly, the evaluations of the basis functions can be stored as in `Base2d.m` (see Remark 1.1 and the introduction of `Base2dBd.m` in Subsect. 2.2.3), except that the quadrature points are replaced by the ones on e . In addition, for the sake of the vectorized implementation, we assume a virtual exterior triangle for a boundary edge e , with all the DoFs indexed by 1 and all the values of the basis functions set to zero, thus not affecting the assembly.

The key point of the program is the construction of `edgeBase`, which comes down to determining the left and right evaluations of φ_i at the quadrature points z_1, z_2, \dots, z_{n_g} . To do so, let us introduce a matrix, referred to as the *macro basis matrix*, given by

$$P = \begin{bmatrix} \varphi_1(z_1^-) & \cdots & \varphi_1(z_{n_g}^-) & \varphi_1(z_1^+) & \cdots & \varphi_1(z_{n_g}^+) \\ \varphi_2(z_1^-) & \cdots & \varphi_2(z_{n_g}^-) & \varphi_2(z_1^+) & \cdots & \varphi_2(z_{n_g}^+) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_9(z_1^-) & \cdots & \varphi_9(z_{n_g}^-) & \varphi_9(z_1^+) & \cdots & \varphi_9(z_{n_g}^+) \end{bmatrix}_{9 \times 2n_g} =: [P^-, P^+],$$

where the first and the second n_g columns store the evaluations on the left and right triangles, respectively. The $\varphi(z^\pm)$ will be obtained from the evaluations of the basis functions $\phi_{i_1}, \dots, \phi_{i_6}$ and $\phi_{j_1}, \dots, \phi_{j_6}$. Note that the evaluations of the basis function ϕ on the element K are given in counterclockwise order along the boundary ∂K .

The idea of constructing the macro basis matrix P is shown in Algorithm 1. Let φ be a basis function on the macro element with the local index given by s . If φ “belongs to” the edge e , then

$$\varphi(\mathbf{z}^-) = P^-(s, :), \quad \varphi(\mathbf{z}^+) = P^+(r, :),$$

where $\mathbf{z} = [z_1, z_2, \dots, z_{n_g}]$. In contrast, if φ does not belong to e and suppose it is on the left triangle, then

$$\varphi(\mathbf{z}^-) = P^-(t, :), \quad \varphi(\mathbf{z}^+) = \mathbf{0}.$$

It is apparent that the initialization and the padding procedure correspond exactly to the above relations.

Algorithm 1 Construction of the macro basis matrix P

1. Initialization: P is initialized as a zero matrix.
2. Left and right padding: Let s_1, \dots, s_6 be the local indices of i_1, \dots, i_6 on the macro element. Then

$$P^-(s_k, :) = [\phi_{i_k}(z_1^-), \dots, \phi_{i_k}(z_{n_g}^-)], \quad k = 1, \dots, 6.$$

Similarly, let t_1, \dots, t_6 be the local indices of j_1, \dots, j_6 on the macro element. One has

$$P^+(t_k, :) = [\phi_{j_k}(z_1^+), \dots, \phi_{j_k}(z_{n_g}^+)], \quad k = 1, \dots, 6.$$

3. Reorder: On the left and right triangles, the evaluations of the basis functions are in counter-clockwise order. Hence, we need to modify the order of the columns of P^- and P^+ according to the orientation of e .
-

3.3.2 The vectorized implementation of the macro basis matrix

Following the previous idea in Algorithm 1, we are able to construct the macro basis matrix P s for any given edge. All these matrices will be collected in the matrix `edgePhi`, which takes the form of

$$\text{edgePhi} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{\text{NE}} \end{bmatrix}, \quad P_k \text{ is the macro basis matrix of } e_k.$$

It is relatively easy to obtain P_i edge by edge, but the large number of loops makes it inefficient. One way of the vectorized implementation is to deal with the i -th DoF of all the left or right cells simultaneously. The pseudo-code reads

```

1 % loop of basis functions on the left or right elements
2 for ib = 1:Ndof
3     % the current basis functions
4     base = phi{ib}; % (NT, 3*ng)
5     % indices in edgePhi
6     rowL = idMacro(:, ib) + rP; % rP = (0: 9: (NE-1)*9)'
7     rowR = idMacro(:, ib+Ndof) + rP;
8     % left padding
9     edgePhi(rowL, 1:ng) = subMat(base, Lrows, Lcols); % base(k1, (1:ng)+(e1-1)*ng)
10    % right padding: k1 ≠ k2
11    edgePhi(rowR, (1:ng)+ng) = subMat(base, Rrows, Rcols); % base(k2, (1:ng)+(e2-1)*ng)
12 end
13 % reorder w.r.t the orientation
14 edgePhi = edgePhi(ReverseRows, ReverseCols);

```

A brief explanation is given as follows:

- `idMacro` determines the local macro indices of the basis functions on the left or right triangles or the local row indices in each P_k . Note that $9(k - 1)$ should be added if these indices are associated with the k -th block of `edgePhi`, as indicated by the vector `rP`.

- `e1` records the local side indices of all edges on the left triangle. The matrix `base` has $3n_g$ columns with the first, the second and the last n_g columns corresponding to the first, the second and the third sides of the triangle, where each side records the values at the n_g quadrature points. The column indices corresponding to `e1` are obviously given by `(1:ng)+(e1-1)*ng`.
- `base(k1, (1:ng)+(e1-1)*ng)` means to extract the scattered entries in (i, j) locations, where i and j take the values in `k1` and `(1:ng)+(e1-1)*ng`, respectively. In Matlab, however, scripts like `A([1 2], [3 4])` are to extract the submatrix in the intersection of the indexed rows and columns, not the scattered entries. For this purpose, one can convert the (i, j) locations to the linear indices of the implicit column vector. For convenience, we have written a function `subMat.m` to realize the scattered extraction, where the built-in function `sub2ind.m` in Matlab is used.

With `edgePhi`, the base matrix given by φ_i can be obtained by extracting the i -th row of all P_k :

$$w_i = \begin{bmatrix} P_1(i,:) \\ P_2(i,:) \\ \vdots \\ P_{\text{NE}}(i,:) \end{bmatrix}, \quad i = 1, 2, \dots, 9.$$

They are further collected in cell array as

$$\text{edgeBaseM} = \{w_1^-, \dots, w_2^-\}, \quad \text{edgeBaseP} = \{w_1^+, \dots, w_2^+\}, \quad i = 1, 2, \dots, 9,$$

where

$$w_i^- = w_i(:, 1:n_g), \quad w_i^+ = w_i(:, (1:n_g) + n_g).$$

The above discussion will be summarized in the following subroutine

```
1 function [edgeBaseM, edgeBaseP, edge2dof] = edgeBase(wStr, Th, Vh, quadOrder)
```

To avoid repeated computations, we allow `wStr` to take more than one string as will be seen in the next subsection.

3.3.3 The computation and assembly of the jump and penalty terms

With the previous preparation, stiffness matrices for the jump and penalty terms can be computed and assembled as for the interior bilinear form. Let us take the jump term as an example. A simple calculation shows that

$$\begin{aligned} & [\nabla v_h] \cdot \{\nabla^2 u_h n_e\} \\ &= \begin{bmatrix} [v_{h,x}] \\ [v_{h,y}] \end{bmatrix} \cdot \begin{bmatrix} \{u_{h,xx} n_{e,x} + u_{h,xy} n_{e,y}\} \\ \{u_{h,yx} n_{e,x} + u_{h,yy} n_{e,y}\} \end{bmatrix} \\ &= [v_{h,x}] (\{u_{h,xx}\} n_{e,x} + \{u_{h,xy}\} n_{e,y}) + [v_{h,y}] (\{u_{h,yx}\} n_{e,x} + \{u_{h,yy}\} n_{e,y}). \end{aligned}$$

The first step is to obtain the basis matrix. Since the bilinear forms involve the first-order and second-order derivatives, we can compute the left and right evaluations as

```

1 %% edgeBase: left and right evaluations of basis functions on macro elements
2 wStr = {'.dx', '.dy', '.dxx', '.dxy', '.ddy'};
3 [edgeBaseM,edgeBaseP,edge2dof] = edgeBase(wStr,Th,Vh,quadOrder);
4 [wxM,wyM,wxxM,wxyM,wyyM] = deal(edgeBaseM{:});
5 [wxP,wyP,wxxP,wxyP,wyyP] = deal(edgeBaseP{:});

```

The sparse assembly index is similar to the one for the interior bilinear form. One just needs to replace `elem2dof` there by `edge2dof` (see Remark 1.2).

```

1 %% Sparse assembly index
2 ii = reshape(repmat(edge2dof, Ndofe,1), [], 1); % Ndofe = 9
3 jj = repmat(edge2dof(:,1), Ndofe, 1);

```

The `edge2dof` is of size $NE \times 9$, with each row representing a macro element. For the boundary edges, we have introduced virtual exterior triangles with the DoF indices set to 1. Since the realization is simple, we omit the details. Please refer to the subroutine `edgeBase.m`.

The stiffness matrices can be computed as follows.

```

1 %% Stiffness matrices J and C for jump and penalty terms
2 KJ = zeros(NE,Ndofe^2); KC = zeros(NE,Ndofe^2); % Ndofe = 9
3 s = 1;
4 for i = 1:Ndofe
5     for j = 1:Ndofe
6         % J
7         % J: first term
8         v1i = wxM{i}-wxP{i};
9         u1j = 0.5*Cavg.*((wxxM{j}+wxxP{j}).*nxg + 0.5*Cavg.*((wxyM{j}+wxyP{j}).*nyg;
10        % J: second term
11        v2i = wyM{i}-wyP{i};
12        u2j = 0.5*Cavg.*((wxyM{j}+wxyP{j}).*nxg + 0.5*Cavg.*((wyyM{j}+wyyP{j}).*nyg;
13        % J: combined
14        KJ(:,s) = he.*sum(ww.*(v1i.*u1j + v2i.*u2j), 2);
15
16        % C
17        vi = (wxM{i}-wxP{i}).*nxg + (wyM{i}-wyP{i}).*nyg;
18        uj = (wxM{j}-wxP{j}).*nxg + (wyM{j}-wyP{j}).*nyg;
19        KC(:,s) = se.*sum(ww.*vi.*uj, 2);
20
21        s = s+1;
22    end
23 end

```

Since the exterior values are set as zero and $\{u\} := u$ for the boundary edges, we need to multiply such averages by 2, which is realized by the factor `Cavg` in the code. For the penalty term, σ_e is given by `se`.

The final stiffness matrix can be assembled by the built-in function `sparse.m` as

```

1 %% Assemble the jump and penalty terms
2 J = sparse(ii,jj,KJ(:,NNdof,NNdof));
3 C = sparse(ii,jj,KC(:,NNdof,NNdof));
4 kk = A - J - J' + C;

```

The right-hand side and the boundary conditions can be implemented as for the Poisson equation.

```

1 %% Assemble the right-hand side
2 Coef = pde.f; Test = 'v.val';
3 ff = assem2d(Th,Coef,Test,[],Vh,quadOrder);
4
5 %% Apply Dirichlet boundary conditions
6 g_D = pde.g_D;
7 on = 1;
8 uh = apply2d(on,Th,kk,ff,Vh,g_D);

```

3.4 Numerical experiment

Let us consider the biharmonic equation with the exact solution given by $u = 10x^2y^2(1-x)^2(1-y)^2 \sin(\pi x)$. The numerical solutions for the mesh size $h = 0.1$ and $h = 0.01$ are displayed in Fig. 7, which are well matched with the exact solutions and are almost same with the results in [2].

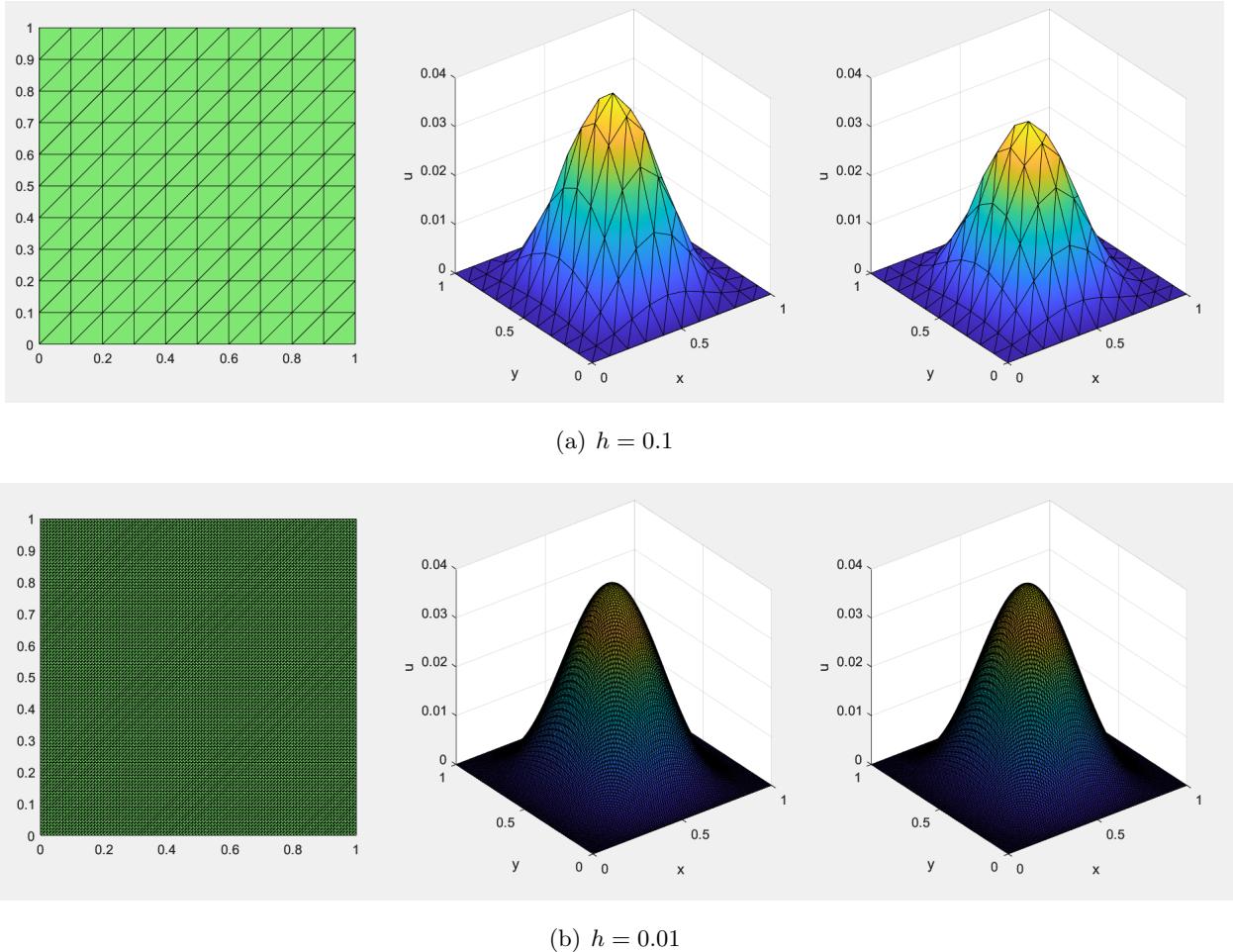


Fig. 7: Numerical and exact solutions of the C^0 interior penalty method for the biharmonic equation

We remark that the implementation in [2] involves some advanced data structures, which makes the code not easy to read and understand. The implementation in this article is more concise and unified for other jumps and averages. Due to the vectorization in constructing the macro basis matrices, our code is also of high efficiency. For the unit square, when $h = 0.01$, the computational

times of our implementation and the one in [2] are about 3.9 seconds and 2.6 seconds, respectively, where the slight overhead is due to the repeated computations in the several modules.

4 Concluding remarks

In this paper, unified implementations of the adaptive FEMs and C^0 interior penalty methods were presented. For the adaptive FEMs, the jumps and averages are associated with the known finite element functions, while the C^0 interior penalty method for the biharmonic equation is a typical example with jumps and averages in the bilinear forms. The design ideas can be extended to other types of Galerkin methods. The package is accessible on <https://github.com/Terenceyuyue/varFEM>.

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