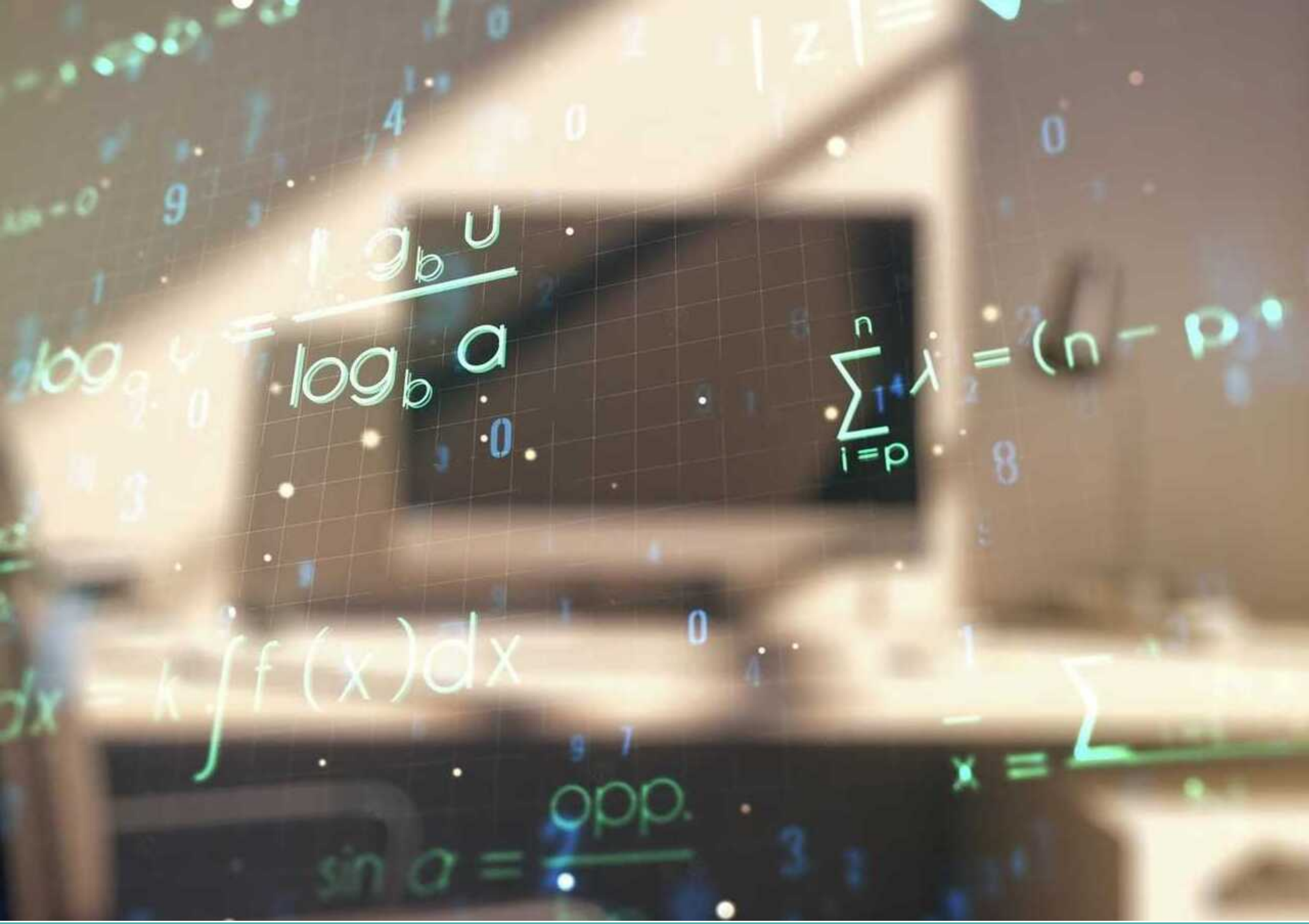

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Chapter 1 The Real and Complex Number Systems

Introduction

- *Ordered Set and Least-upper-bound*
- *Real Field and Properties*
- *The Complex Field*

1.1 Ordered Set

Definition 1.1 (Ordered Set)

*Suppose S be a set. An **order** on S is a relations, denoted by $<$, with the following properties:*

(1) Trichotomy: If $x, y \in S$, then exactly one of the following $x = y$, $x < y$, $y < x$ is true.

(2) Transitivity: If $x, y, z \in S$, $x < y$, and $y < z$, then $x < z$.

*The **ordered set** is a set S in which an order is defined.*

♣

Definition 1.2 (Supremum, Infimum)

Suppose S is a ordered set, $E \subset S$, and E is bounded above. Suppose there exists $\alpha \in S$ such that:

(1) α is an upper bound of E.

(2) If $\gamma < \alpha$, then γ is not an upper bound of E.

*Then $\alpha \in S$ is called the **least upper bound** of E (or the supremum of E) and is denoted by $\alpha = \sup E$.*

*The definition of **greatest lower bound** (infimum) is an analogous.*

♣

Remark The second statement is equivalent to: for all upper bounds γ, we have $\gamma \geq \alpha$.

Definition 1.3 (Least Upper Bound Property)

*An ordered set S has the **least-upper-bound property** if for all nonempty $E \subset S$ that is bounded above, then $\sup E$ exists in S.*

♣

Theorem 1.1

Suppose S is an ordered set the least-upper-bound property, then S has the greatest-lower-bound property, that is, for all nonempty $E \subset S$ that is bounded below, then $\inf E$ exists in S.

♡

Proof Let L be the set of all lower bounds of E, $L \neq \varnothing$. Since L is bounded above by elements in E, there exists

$\alpha = \sup L$ in S. It follows that $\alpha = \inf E$ because (1) for all $x \in E$, $\alpha \le x$ since x is an upper bound of L, it follows that $\alpha \in L$, and (2) $\gamma \le \alpha$ for all lower bounds $\gamma \in L$ by definition. This completes the proof.

Remark We construct the set of lower bounds to convert the l.u.b. property to g.l.b. The construction of L gives the following relationships: $L \le \alpha \le E$ and $\sup L = \alpha = \inf E$.

1.2 The Real Field

Definition 1.4 (Field, Ordered Field)

*A **field** $(F, +, \cdot)$ is a set F such that $(F, +)$ and $(F \setminus \{0\}, \cdot)$ are abelian group, and multiplicative is distributive to addition.*

*An **ordered field** is a field $(F, + \cdots)$ which is also an ordered set such that*

1. $x + y < x + z$ *if* $x, y, z \in F$ *and* $y < z$, *and*
2. $xy > 0$ *if* $x, y \in F$, $x, y > 0$.

Example 1.1 There exists no order that turns $\mathbb{C}$ into an ordered field.

Proposition 1.1 (Existence Theorem)

There exists an ordered field $\mathbb{R}$ which has the least upper bound property.

Remark Suppose $E \subset S$, $\alpha = \sup E$ if and only if for all $\varepsilon > 0$, there exists $x \in E$ such that $\alpha - \varepsilon < x \leq \alpha$.

Note *Well-ordering principle of $\mathbb{N}$: if E is a nonempty subset of $\mathbb{N}$, the E has a least element in it.*

Theorem 1.2

1. *Archimedean property: if $x, y \in \mathbb{R}$ and $x > 0$, there exists $N \in \mathbb{Z}_{>0}$ such that $nx > y$.*
2. *$\mathbb{Q}$ is dense in $\mathbb{R}$: if $x, y \in R$ and $x < y$, there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof (1) For the sake of contradiction, suppose there exists x, y such that $nx \leq y$ for all $n \in \mathbb{Z}_{>0}$. Let $E = \{nx \mid n \in \mathbb{Z}_{>0}\}$, clearly E is nonempty and bounded above by y, there exists $\alpha = \sup E$. There exists $nx \in E$ such that $\alpha - x < nx \leq \alpha$, it follows that $(n+1)x > \alpha$, contradicting the fact that $\alpha = \sup E$.

(2) There exists $n \in \mathbb{N}$ such that $n(y - x) > 1$, namely $ny - 1 > nx$. Apply the Archimedean property again, we obtain $m_1, m_2 \in \mathbb{Z}_{>0}$ such that $m_1 > nx$, $m_2 > -nx$, so $-m_2 < nx < m_1$. It follows that there exists m $(-m_2 \leq m \leq m_1)$ such that $m - 1 \leq nx < m$. Then $nx < m < ny$, so $x < m/n < y$ where $m/n \in \mathbb{Q}$. ∎

Remark Indeed, the set of all irrationals $\mathbb{Q}^c$ is also dense in $\mathbb{R}$.

Theorem 1.3

For every real $x > 0$ and every integer $n > 0$, there exists a unique real $y > 0$ such that $y^n = x$; in other words, $x^{1/n}$ exists and is unique.

Proof For the existence, let $E = \{t > 0 \mid t^n < x\}$. It is not hard to show E is nonempty (by choosing $t < \min(x, 1)$) and bounded above (by $1 + x$), so there exists $\alpha = \sup E$ by the least-upper-bound property.

We now prove $\alpha^n = x$ by contradiction. Notice that $b^n - a^n = (b-a)(b^{n-1} + ab^{n-1} + \cdots + a^{n-1}) < (b-a)nb^{n-1}$.

- Assume $\alpha^n > x$, put $h = (\alpha^n - x)/n\alpha^{n-1}$, then
$$\alpha^n - (\alpha - h)^n < h \cdot n\alpha^{n-1} \leq a^n - x.$$
That is, $x < (\alpha - h)^n < \alpha^n$, contradicting to the fact that $\alpha = \sup E$.
- Assume $\alpha^n < x$, put $h = \min\{1, (x - \alpha^n)/n(\alpha+1)^{n-1}\}$, then
$$(\alpha + h)^n - \alpha^n < h \cdot n(\alpha + h)^{n-1} \leq h \cdot n(\alpha + 1)^{n-1} \leq x - \alpha^n.$$
That is, $\alpha^n < (\alpha + h)^n < x$, contradicting to the fact that α is an upper bound.

Hence, $\alpha^n = x$. ■

Definition 1.5 (Extended Real Number System)

The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define $-\infty < x < +\infty$ for every $x \in R$. ♣

The extended real number system does not form a field.

1.3 The Complex Field and The Euclidean Spaces

Definition 1.6 (Complex Number)

*A **complex number** is an ordered pair (a, b) of real numbers. Let $x = (a, b)$ and $y = (c, d)$, we define the addition and multiplication by $x + y = (a + c, b + d)$ and $xy = (ac - bd, ad + bc)$.* ♣

Remark The complex number along with addition and multiplication forms a field $\mathbb{C}$, and it contains $\mathbb{R}$ as a subfield.

We define the *conjugate* of $x = (a, b)$ by $\bar{x} = (a, -b)$ and define the absolute value $|x| = (x\bar{x})^{1/2}$. The complex numbers have the following properties:

- $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$;
- $z + \bar{z} = 2\text{Re}(z)$, $z = \bar{z} = 2\text{Im}(z)$;
- $z\bar{z}$ is real and positive (exception when $z = 0$), so $|z| > 0$;
- $|\bar{z}| = |z|$;
- $|zw| = |z||w|$;
- $|\text{Re}(z)| \le |z|$;
- (triangle inequality) $|z + w| \le |z| + |w|$

  Proof:
$$
\begin{aligned}
|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
&= |z|^2 + 2\text{Re}(z\bar{w}) + |w|^2 \\
&\le |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2,
\end{aligned}
$$
  it follows that $|z + w| \le |z| + |w|$. ■

Proposition 1.2 (Schwarz Inequality)

If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are complex numbers, then
$$\left|\sum_{j=1}^{n} a_j \bar{b_j}\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$
♠

Proof Put $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, and $C = \sum a_j \bar{b_j}$. If $B = 0$, $b_1 = \cdots = b_n = 0$, so the conclusion is trivial. Suppose therefore $B > 0$, then
$$
\begin{aligned}
0 \le \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a_j} - \overline{Cb_j}) \\
&= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b_j} - BC \sum \bar{a_j} b_j + |C|^2 \sum |b_j|^2 \\
&= B^2 A - B\bar{C}C - BC\bar{C} + B|C|^2 \\
&= B(AB - |C|^2).
\end{aligned}
$$

Therefore, $AB - |C|^2 \geq 0$. ■

Proof (*Alternative*) We use the same notation of A, B, C as above. Notice that $AB = \sum_{i,j} a_i \bar{b_j} \bar{a}_i b_j$ and $|C|^2 = \sum_{i,j} a_i \bar{b_j} \bar{a}_j b_i$. Then

$$\begin{aligned}
AB &= \Big(\sum_i a_i \bar{a}_i\Big)\Big(\sum_j b_j \bar{b_j}\Big) \\
&= \Big(\sum_i a_i b_i\Big)\Big(\sum_j \bar{a}_j \bar{b_j}\Big) + \sum_{i,j} a_i \bar{b_j}(\bar{a}_i b_j - \bar{a}_j b_i) \\
&= \Big|\sum_i a_i b_j\Big|^2 + \sum_{i \leq j}(a_i \bar{b_j} - a_j \bar{b_i})(\bar{a}_i b_j - \bar{a}_j b_i) \\
&= |C|^2 + \sum_{i \leq j} |a_i \bar{b_j} - a_j \bar{b_i}|^2 \\
&\geq |C|^2.
\end{aligned}$$

Hence $AB \geq |C|^2$. ■

Definition 1.7 (Euclidean Space)

For $k \in \mathbb{Z}_{>0}$, $\mathbb{R}^k = \{\mathbf{x} : \mathbf{x} = (x_1, \cdots, x_k),\ x_i \in \mathbb{R}$ *for all* $i\}$ ♣

Let $\mathbf{x} = (x_1, \cdots, x_k)$ and $\mathbf{y} = (y_1, \cdots, y_k)$. The addition is defined by $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \cdots, x_k + y_k)$, the scalar multiplication is defined by $a\mathbf{x} = (ax_1, \cdots, ax_k)$, the inner product is defined by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i$, and the norm is defined by $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = (\sum_{i=1}^k x_i^2)^{1/2}$.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, $\alpha \in \mathbb{R}$, the following properties holds:

- $|\mathbf{x}| \geq 0$, and the equality holds if and only if $x = 0$;
- $|a\mathbf{x}| = |a||\mathbf{x}|$;
- (Cauchy-Schwartz) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$;
- (triangle inequality) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
- $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Chapter 2 Basic Topology

Introduction

- *Countable and Uncountable Sets*
- *Limit Point, Closed Set, Closure*
- *Open Relative (Subspace topology)*
- *Heine-Borel Theorem*
- *Neighborhoods, Open Sets*
- *Bounded Set, Dense Subset*
- *Open Cover, Compact Set*
- *Perfect Set, Connected Sets*

2.1 Countable Sets

Definition 2.1 (1-1 Correspondence)

Suppose A, B are sets, we say A and B are in 1-1 correspondence if there exists a bijection $f : A \to B$. We write $A \sim B$, and this relation is a equivalence relation.

Definition 2.2 (Countability)

For any $n \in \mathbb{Z}^+$, define $J_n := \{1, \cdots, n\}$, and let $J = \{1, \cdots\} = \mathbb{Z}_{>0}$. For any set A, we say:

- *A is* ***finite*** *if $A \sim J_n$ for some n (the empty set is considered finite).*
- *A is* ***countable*** *if $A \sim J$.*
- *A is* ***at most countable*** *if A is finite or countable, and A is uncountable if it is not at most countable.*

Example 2.1 $\mathbb{Z}$ is countable, because $f : \mathbb{N} \to \mathbb{Z}$, defined by $f(n) = n/2$ if n is even and $f(n) = -(n-1)/2$ if n is odd, is a bijection.

Definition 2.3 (Sequence)

A ***sequence*** *is a function defined on $\mathbb{N}$. If f is a sequence, we denote $x_n = f(n)$, and we write f as $\{x_n\}$.*

Proposition 2.1

Every infinite subset of a countable set A is countable.

Proof Let $E \subset A$ be an infinite subset. Since A is countable, there exists $\{x_n\} = A$. Let n_1 be the least positive integer such that $x_{n_1} \in E$, which exists by the well-ordering principle. Recursively, choose n_i from $E \setminus \{x_1, \cdots, x_{n-1}\}$, which is nonempty since E is infinite, such that n_i is the least positive integer such that $x_{n_i} \in E$. Putting $f(k) = x_{n_k}$ ($k \in \mathbb{Z}_+$), we obtain an 1-1 correspondence between E and J. ■

Note: That is, every subset of a countable set is at most countable.

Proposition 2.2

Let $\{E_n\}$, $n = 1, 2, \cdots$ be a sequence of countable sets, and put $S = \bigcup_{n=1}^{\infty} E_n$, then S is countable. In other words, the countable union of countable sets is countable.

Proof Let $E_n = \{x_{n,k}\}_{k=1}^{\infty}$ for all n, S can be enumerated as:

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

namely $S = \{x_{1,1}, x_{2,1}, x_{1,2}, x_{3,1}, \cdots\}$. Then S is at most countable. Since $E_1 \subset S$ is countable thus infinite, S is countable. ■

Corollary 2.1

The at most countable union of at most countable sets is at most countable.

Proposition 2.3

Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \cdots, a_n)$ where $a_k \in A$ $(k = 1, \cdots, n)$, and the element need not be distinct. Then B_n is countable. In other words, the finite cartesian product of countable sets is countable.

Proof We proceed by induction on n. If $n = 1$, the statement is trivial. For $n > 1$, suppose B_{n-1} is countable. Fix $b \in B_{n-1}$, let $E_b := \{(a, b) \mid a \in A\}$, which is countable since A is countable. Then $B_n = \bigcup_{b \in B_{n-1}} E_b$ is a countable union of countable sets, then B_n is countable by proposition 2.2. ■

Corollary 2.2

$\mathbb{Q}$ is countable.

Proof The set of $\mathbb{Z} \times \mathbb{Z}$ is countable by proposition 2.3, and $\mathbb{Q}$ can be view as the subset of $\mathbb{Z} \times \mathbb{Z}^* \subset \mathbb{Z} \times \mathbb{Z}$ by the map $f : (x, y) \mapsto x/y$, followed by $\mathbb{Q}$ is countable by 2.2. ■

Proposition 2.4 (Cantor)

$\mathbb{R}$ is uncountable.

Proof For the sake of contradiction, suppose $\mathbb{R}$ is countable, then so is $(0, 1) \subset \mathbb{R}$. Clearly, $(0, 1)$ is infinite. We can enumerate $(0, 1)$ as $\{x_n\}_{n=1}^{\infty}$, and let $x_n = 0.x_{n1}x_{n2}\cdots$ be the decimal representation. Choose $y = 0.b_1b_2\cdots$

for which $b_n \neq x_{nn}$ for all n. It follows that $y \neq x_n$ for all n since $b_n \neq x_{nn}$, so $y \notin \{x_n\}_{n=1}^{\infty}$, contradicting the fact that $y \in (0, 1)$. ■

2.2 Metric Spaces Topology

Definition 2.4 (Metric Spaces)

*A **metric space** is a set X with a distance function (**metric**) $d : X \to X \to \mathbb{R}$ such that:*

(a) $d(x, y) > 0$ if $p \neq q$, and $d(p, p) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

♣

Example 2.2 Let $X = \mathbb{R}^k$, defined $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ to be the usual Euclidean distance. d satisfies all the conditions in the above definition, so $(\mathbb{R}^k, d)$ is a metric space, and we called the Euclidean distance the "usual" distance in $\mathbb{R}^k$.

Definition 2.5 (Neighborhood, Open Set)

Let (X, d) be a metric space,

*(a) A **neighborhood** of p is a set $N_r(p)$ consisting of all $q \in X$ such that $d(p, q) < r$, for some $r > 0$. The number r is called the radius of $N_r(p)$.*

(b) A point E is an interior point of E if there is a neighborhood N of p such that $N \subset E$.

*(c) A set E is **open** if every point of E is an interior point.*

♣

Remark A set E is ***open*** if and only if for all $p \in E$, there exists $r > 0$ such that $N_r(p) \subset E$.

Proposition 2.5

Every neighborhood is an open set.

♠

Proof Consider the neighborhood $E = N_r(p)$. For all $p' \in E$, let $h = d(p, p') < r$, then $N_{r-h}(p') \subset E$, because for all $q \in E$, $d(p, q) \leq d(p, p') + d(p', q) < h + (r - h) = r$. Hence p' is an interior point for all $p' \in E$, thus E is open. ■

Definition 2.6 (Closed Set)

Let (X, d) be a metric space, suppose $E \subset X$,

*(a) A point p is a **limit point** of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$. If $p \in E$ and p is not a limit point of E, then p is called the **isolated point** of E.*

*(b) E is **closed** if every limit point of E is a point of E.*

♣

Remark Equivalently, p is a limit point if and only if $N_r^*(p) \cap E \neq \varnothing$, where we denote $N_r^*(p) := N_r(p) \setminus \{p\}$.

Proposition 2.6

If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Proof Proof by contradiction. Suppose p is a limit point of E, and there exists a neighborhood of p containing finitely many points $q_1, \cdots, q_n$. Put $r = \min_i d(p, q_i)$, then $r > 0$ since $\{q_i\}$ is finite. It follows that $N_r(p)$ contains no points of $E \setminus \{p\}$, contradicting that p is a limit point. ■

Remark Corollary: A finite point set has no limit points.

Definition 2.7 (Boundedness, Dense)

Let (X, d) be a metric space, suppose $E \subset X$,

*(a) The **complement** of E, denoted by E^c, is $E^c = \{p \in X \mid p \notin E\}$.*

*(b) E is **bounded** if there exists $M > 0$ and $p \in E$ such that $d(p, q) < M$ for all $q \in E$.*

*(c) E is **dense** in X if every point of X is a limit point of E or in E.*

DeMorgan's Law: Let $\{E_\alpha\}$ be a collection of sets E_α, then $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha^c)$.

Theorem 2.1

A set E is open if and only if its complement is closed.

Remark Corollary: A set F is closed if and only if its complement is open.

Proof ($\Rightarrow$) Suppose E is open, and let x be a limit point of E^c. If $x \in E$, there exists r' such that $N_{r'}(x) \subset E$, so $N_{r'}(x) \cap E^c = \varnothing$, contradicting that x is a limit point of E^c. Thus $x \in E^c$, implying that E^c is closed.

($\Leftarrow$) Suppose E^c is closed, and let $x \in E$. Since $x \notin E^c$, x is not a limit point of E^c, implying that there exists $r > 0$ such that $N_r^*(x) \cap E^c = N_r(x) \cap E^c = \varnothing$. It follows that $N_r(x) \subset E$, thus E is open. ■

Proposition 2.7

(a) Arbitrary unions and finite intersections of open sets are open.

(b) Arbitrary intersections and finite unions of closed sets are closed.

Proof (a) (i) Suppose $x \in G = \bigcup_\alpha G_\alpha$, x is a point G_β thus an interior point of G_β for some β. Then x is an interior point of G since $G_\beta \subset G$, so the arbitrary union of open sets is open. (ii) Suppose $x \in G = \bigcap_{i=1}^n G_i$, then for all i, there exists r_i such that $N_{r_i}(x) \subset G_i$. Put $r = \min\{r_i\}$, we have $N_r(x) \subset G_i$ for all i, so $N_r(x) \subset G$. Thus, x is an interior point of G, so G is open.

(b) By taking the complement and using DeMorgan's Law, we obtain (b) from (a).

Remark The infinite intersection of open sets is not necessarily open. For instance, $G_n = (-1/n, 1/n)$ for $n \in \mathbb{N}$, then $\bigcap G_n = \{0\}$ is not an open subset.

Definition 2.8 (Closure)

If X is a metric, $E \subset X$, and E' denotes the set of all limit points of E in X, the the ***closure*** *of E is the set $\bar{E} = E \cup E'$.*

Proposition 2.8

If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed,

(b) $E = \bar{E}$ if and only if E is closed, and

(c) $\bar{E} \subset F$ for every closed set F containing E.

Remark By (a) and (c), $\bar{E}$ is the smallest closed subset that contains E.

Proof (a) Let $p \in \bar{E}^c$, p is not in E nor a limit point of E, so there exists $r > 0$ such that $N_r(p) \cap E = \varnothing$. If $q \in N_r(p) \cap E'$, let $d = d(p, q)$, then there exists $q' \in E \cap N_{r-d}(q) \subset N_r(p) \cap E = \varnothing$, contradiction. Therefore, $N_r(p) \cap \bar{E} = N_r(p) \cap (E \cup E') = \varnothing$, then p is the interior point of $\bar{E}^c$, so $\bar{E}^c$ is open, followed by $\bar{E}$ is closed.

(b) Suppose $E = \bar{E}$, then E is closed by (a). Conversely, suppose E is closed, $E \subset \bar{E} = E \cap E' = E$, so $E = \bar{E}$.

(c) Since $E \subset F$, $E' \subset F$ because E' are limit points of F and thus in F because F is closed. Therefore, $\bar{E} = E \cup E' \subset F$.

Remark For (a), we show that for $p \in \bar{E}^c$, $N_r(p)$ contains no points of E, and it contains no limit points of E, otherwise it intersects E. Then we conclude p is an interior point.

Proposition 2.9

Let E be a nonempty set of $\mathbb{R}$ which is bounded above, and let $y = \sup E$. Then $y \in \bar{E}$, thus $y \in E$ if E is closed.

Proof Suppose $y \in E$, then obviously $y \in \bar{E}$. Suppose $y \notin E$, then for all $\varepsilon > 0$, there exists y' such that $y' \in N_\varepsilon(y) \cap E = N^*_\varepsilon(y) \cap E$. It implies that $y \in E'$, so $y \in \bar{E}$. ■

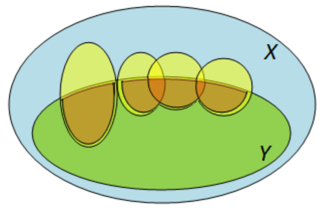
Definition 2.9 (Open Relative)

Let $Y \subset X$ be a non-empty subset. $E \subset Y$ is ***open relative*** *to Y if for each $p \in E$, there exists $r > 0$ such that $N_r(p) \cap Y \subset E$. Equivalently, there exists $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.*

Proposition 2.10

Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Remark E is open relative to $Y \subset$ means E is open in the ***subspace topology*** Y on X.



Proof ($\Rightarrow$) Suppose E is open relative to Y. To each $p \in E$ there is a positive number r_p such that $N_{r_p}(p) \cap Y \subset E$. Let $G = \bigcup_{p \in E} N_{r_p}(p)$, G is clearly open. Note that for all $p \in E$, $p \in N_{r_p}(p) \cap Y$, then $E = \bigcup_{p \in E}(N_{r_p}(p) \cap Y) = (\bigcup_{p \in E} N_{r_p}) \cap Y = G \cap Y$.

($\Leftarrow$) Suppose $E = Y \cap G$ for some open set G in X. For all $p \in E = G \cap Y$, there exists $r > 0$ such that $N_r(p) \subset G$ since G is open in X, then $N_r(p) \cap Y \subset G \cap Y = E$. Thus, E is open relative to Y. ∎

Example 2.3 Consider $E = (0, 1) \times \{0\}$. E is open (relative) to $Y = \mathbb{R} \times \{0\}$, considering E as a subset of Y. However, if we consider E as a subset of $X = \mathbb{R}^2$, E is not open.

2.3 Compact Space

Definition 2.10 (Open cover, Compactness)

*Suppose (X, d) is a metric space. An **open cover** of a set $E \subset X$ is a collection of open sets $\{G_\alpha \,|\, \alpha \in A\}$ such that $E \subset \bigcup_{\alpha \in A} G_\alpha$.*

*$K \subset X$ is **compact** if every open cover contains a finite subcover.* ♣

Proposition 2.11

Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y. ♠

Proof ($\Rightarrow$) Suppose K is compact relative to X, and assume $\{V_\alpha = G_\alpha \cap Y\}$ is an open cover open relative to Y. Then $\{G_\alpha\}$ is an open cover of K, so there is a finite subcover $\{G_i\}$ since K is compact relative to X. Thus, $K \subset (\bigcap_{i=1}^n G_i) \cap Y = \bigcap_{i=1}^n (G_i \cap Y) = \bigcap_{i=1}^n V_i$. It follows that there exists a finite subcover $\{V_i = G_i \cap Y\}$ of K open relative to Y, so K is open relative to Y.

($\Leftarrow$) The converse is an analogous. ■

Proposition 2.12

Compact subsets of a metric space are closed. ♠

Proof Suppose K is compact, we want to show K^c is open. Let $q \in K^c$ be given. For all $p \in K$, let $d = d(p, q)/2 > 0$, we define the neighborhoods $p \in U_p = N_d(p)$ and $q \in V_p = N_d(q)$. Note that $\{U_p \,|\, p \in K\}$ forms an open cover of K, so there exists a finite subcover $\{U_{p_i}\}$. Consider $V = \bigcap_{i=1}^n V_{p_i}$. Note that V is open, and $V \cap K = \varnothing$, since for all U_{p_i}, $K \cap U_{p_i} \subset V_{p_i} \cap U_{p_i} = \varnothing$. Therefore, p is an interior point, so K is closed since the choice of p is arbitrary. ■

Proposition 2.13

Closed subsets of compact sets are compact. ♠

Proof Suppose $F \subset K \subset X$ where F is closed relative to K and K is compact. Assume $\{U_\alpha\}$ is an open cover of F. Adding the open set F^c to $\{U_\alpha\}$ yields an open cover of K, so there exists a finite subcover $\{V_i\}$ of K since K is compact. Removing F^c from $\{V_i\}$ (if exists) gives a finite subcover of F. Hence F is compact. ■

Corollary 2.3

The intersection of a compact set and a closed set is compact. ♡

Proposition 2.14 (Finite intersection property)

If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Proof For the sake of contradiction, suppose $\bigcap K_\alpha = \varnothing$. Fix $K_1 \in \{K_\alpha\}$, then $K_1 \subset \bigcup K_\alpha^c$. By the compactness, there exists $\alpha_1, \cdots, \alpha_n$ such that $K_1 \subset \bigcup_{i=1}^n K_{\alpha_i}^c = (\bigcap_{i=1}^n K_{\alpha_i})^c$. Then $K_1 \cap \bigcap_{i=1}^n K_{\alpha_i} = \varnothing$, contradicting that finite intersections are nonempty. ∎

Note ***Corollary**: If $\{K_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^\infty K_n$ is nonempty.*

k-cell A **k-cell** is a set $I \subset \mathbb{R}^k$ of the fork $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ where $a_j < b_j$ for $j = 1, \cdots, k$.

Lemma 2.1

If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$ such that $I_n \supset I_{n+1}$, then $\bigcap_{n=1}^\infty I_n$ is nonempty.

Proof Let $I_n = [a_n, b_n]$ for all n, and put $E = \{a_n\}$. E is nonempty and bounded above by b_1, so there exists $x = \sup E$. For all m, notice that $a_1 \le a_2 \le \cdots a_m \le b_m \le \cdots \le b_2 \le b_1$, so $x \le b_m$. Also note that clearly $a_m \le x$ by the definition of supremum, thus $x \in I_m$. Hence $x \in \bigcap_{m=1}^\infty I_m$. ∎

Remark It is nor hard to show the intersection of a sequence of k-cells is nonempty.

Proposition 2.15

Every k-cell is compact.

Proof Proof by contradiction. Suppose $I \subset \mathbb{R}^k$ is a k-cell and is not compact. Put $\delta = \sqrt{\sum (a_i - b_i)^2}$. Let $c_j = (a_j + b_j)/2$, dividing $[a_j, b_j]$ into $[a_j, c_j] \cup [c_j, b_j]$ determines 2^k k-cell, and at least one of the k-cells, denoted by I_1, is not compact because I is not compact.

Continuing this process we obtain a sequence $\{I_n\}$ such that (a) $I_n \supset I_{n+1}$, (b) I_n cannot be covered by any finite subcollection of an open cover $\{G_\alpha\}$, and (c) $|x - y| \le 2^{-n}\delta$ if $x, y \in I_n$. There exists $x^* \in \bigcap I_n$ by Lemma 2.1 and $x^* \in G_\alpha$ for some α . Since G_α is open, there exists $r > 0$ such that $N_r(x^*) \subset G_\alpha$, and there exists $n \in \mathbb{Z}_{>0}$ such that $2^{-n} < r$ by the Archimedean property. This leads to a clear contradiction to (b). Hence I is compact. ∎

Lemma 2.2

Suppose K is compact and $E \subset K$ is an infinite subset. Then E has a limit point in K.

Proof Proof by contradiction. Suppose E has no limit point in K, then for all $q \in E$ there exists $\varepsilon_q > 0$ such that $N_{\varepsilon_q}^*(q) \cap E = \varnothing$. That is, $N_{\varepsilon_q}(q) \cap E = \{q\}$. The collection $\{N_{\varepsilon_q}(q) \,|\, q \in E\}$ forms an open cover, there exists a finite subcover by the compactness, contradicting to the fact that E is infinite. ∎

Theorem 2.2 (Heine-Borel Theorem)

Suppose E is a subset of $\mathbb{R}^k$ with Euclidean metric, then the following are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

♡

Remark In general, $(a) \not\Rightarrow (b)$ and $(a) \not\Rightarrow (c)$.

Proof $(a) \Rightarrow (b)$: E is bounded, so there is a k-cell containing E. Then E is a closed subset of compact set, so E is compact by Proposition 2.13.

$(b) \Rightarrow (c)$: Lemma 2.2.

$(c) \Rightarrow (a)$: Suppose E is not bounded, then E contains points $S = \{x_n\}_{n=1}^{\infty}$ such that $|x_n| > n$. S has no limit points since $N_{1/2}(p) \cap E$ contains at most two points, then (c) does not hold since S is infinite.

Now suppose E is not closed, then there exists a limit point x of E such that $x \notin E$. Construct $S = \{x_n\}_{n=1}^{\infty}$ such that $x_n \in N_{1/n} \cap E$. Assume y is another limit point of E, let $d = |x - y|/2 > 0$, and choose n_0 for which $1/n_0 \leq d$. Then $|x_n - y| \geq |x - y| - |x - x_n| \geq 2d - 1/n$, so $|x_n - y| \geq d$ for $n \geq n_0$. It implies that $N_d(y)$ is contains finitely many points in E, so y is not a limit point of E by Proposition (2.6). Then S is infinite and the only limit point is x but $x \notin E$. Therefore, E is closed and bounded by contrapositive. ■

Theorem 2.3 (Weierstrass)

Every bounded infinite subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

♡

Proof E is a subset of a k-cell $I \subset \mathbb{R}^k$ by the boundedness. Since I is compact, E has a limit point in $I \subset \mathbb{R}^k$ by Lemma (2.2). ■

2.4 Perfect Sets and Connected Sets

2.4.1 Perfect Sets

Definition 2.11 (Perfect Sets)

Suppose (X, d) is a metric space and $E \subset X$. E is ***perfect*** *if $E = E'$, equivalently, E is closed and has no isolated points. If $p \in E$ is not a limit point of E, p is called an* ***isolated point*** *of E.* ♣

Example 2.4 For fixed $a, b \in \mathbb{R}$, the closed interval $[a, b] \subset \mathbb{R}^1$ is perfect.

Proposition 2.16

Let P be a nonempty perfect set in $\mathbb{R}^k$, then P is uncountable. ♠

Proof P is infinite because it has a limit point. Assume P is countable and $P = \{x_i\}_{i=1}^{\infty}$. Fix $r_1 > 0$, let $V_1 = N_{r_1}(x_1)$. Since x_1 is a limit point, $V_1 \cap P \neq \varnothing$. We can construct recursively a sequence of neighborhoods $V_2, V_3, \cdots$ of points in E, for which (i) $\overline{V_{n+1}} \subset V_n$ and (ii) $x_n \notin \overline{V_{n+1}}$, and we know that $V_n \cap P \neq \varnothing$ since the center of V_n is a limit point of P.

Put $K_n = \overline{V_n} \cap P$. K_n is compact since $\overline{V_n}$ is compact and P is closed. Then $\bigcap_{i=1}^{\infty} K_n$ is nonempty by the Lemma (2.1). However, $x_n \neq K_{n+1}$ implies $\bigcap_{n=1}^{\infty} K_n = \varnothing$. By contradiction, P is uncountable. ■

Remark Key Claim: Given an open set U and $x \in X$, there exists an open subset $V \subsetneq U$ such that $x \notin V$, this holds by the Hausdorff axiom.

Key idea: We can construct a strictly decreasing sequence $\{V_n\}$ of neighborhoods of points of P, for which every V_n intersects P (by perfectness) but V_n converges to points outside of P (by excluding x_n in V_{n+1}). Then there is a contradiction regards to the intersection of $\{\overline{V_n} \cap P\}$.

Corollary 2.4

Every interval $[a, b]$ $(a < b)$ is uncountable. In particular, the set of all real numbers is uncountable. ♡

Example 2.5 *Cantor Set*: Let $E_0 = [0, 1]$. Recursively define E_n by removing the middle thirds of the intervals in E_{n-1}, e.g., $E_1 = [0, 1/3] \cup [2/3, 1]$. We obtain a sequence of compact sets E_n such that

(i) $E_1 \supset E_2 \supset \cdots$, and

(ii) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $P = \bigcap_{i=1}^{\infty} E_n$ is called the *Cantor Set*, and

- P is compact and P is nonempty by Lemma 2.1.
- P contains no segment. By the construction, the segment of the form $((3k + 1)/3^m, (3k + 2)/3^m)$ is not contained in P, but every segment (α, β) contains such segment, so P contains no segments.

- P is perfect. Let $x \in P$ and S be a segment containing x. Let I_n be the interval of E_n containing x, choose n large enough so that $I_n \subsetneq S$. Put x_n be the endpoint of of I_n such that $x_n \neq x$, it follows that $x_n \in P$ thus $x_n \in S \cap P$, so x is a limit point of P. Hence P is perfect.

The Cantor set is an example of totally disconnected, perfect, compact metric space.

2.4.2 Connected Sets

Definition 2.12 (Connectedness)

Suppose X is a metric space and $A, B \subset X$. A and B are said to be ***separated*** *if $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. A set $E \subset X$ is said to be* ***connected*** *if E is not a union of two nonempty separated sets.* ♣

Proposition 2.17

Suppose $E \subset \mathbb{R}$, E is connected if and only if it has the following property: if $x, y \in E$ and $x < z < y$, then $z \in E$. ♠

Proof ($\Rightarrow$) Proof by contrapositive. Assume there exists $z \in (x, y)$ such that $z \notin E$. Then $E = A_z \cup B_z$ where $A_z := E \cap (-\infty, z)$ and $B_z := E \cap (x, \infty)$. A_z, B_z are clearly nonempty and separated, then E is not connected.

($\Leftarrow$) Proof by contrapositive. Assume E is not connected and A, B is a separation. Choose $x \in A$ and $y \in B$, assume $x < y$ without loss of generality. Let $a = \sup(A \cup [x, y])$ and $b = \inf(B \cup [x, y])$. Clearly $a \leq b$. If $a < b$, choose $c \in (a, b)$, then $c \notin A \cup B = E$ but $x < c < y$, contradiction. Otherwise if $a = b$, $a \in \overline{A} \cap \overline{B}$ by Proposition 2.9, it means that $a \notin A \cup B = E$ since $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. Then $x < a < y$ and $a \notin E$. ■

Remark The following are criteria of connectedness:

(a) The subset set $E \subset X$ is connected if and only if there exists no disjoint nonempty open (relative to E) subsets A, B of E such that $E = A \cup B$.

(b) The subset set $E \subset X$ is connected if and only if the only subsets that are both open and closed (relative to E) are empty set and E itself.

Chapter 3 Numerical Sequences and Series

Introduction

- *Convergent Sequences*
- *Cauchy Sequences*
- *Subsequence and Subsequential Limits*

3.1 Convergent Sequences

3.1.1 Convergent Sequences

Definition 3.1 (Convergence)

A sequence $\{p_n\}$ in a metric space X is said to ***converge*** *if there is a point $p \in X$ such that for all $\varepsilon > 0$, there exists an integer N such that $d(p_n, p) < \varepsilon$ if $n \geq N$. We denote the convergence by $p_n \to p$ or $\lim_{n\to\infty} p_n = p$. If $\{p_n\}$ does not converge, it is said to* ***diverge****.*

Proposition 3.1

Let $\{p_n\}$ be a sequence in a metric space X,

(a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.

(b) If $p, p' \in X$ and $\{p_n\}$ converges to both p and p', then $p = p'$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

Proof (a) ($\Rightarrow$) The forward direction is trivial by definition, since for all $N_\varepsilon(p)$, we can choose N by definition such that $N_\varepsilon(p)$ contains all p_n for which $n \geq N$. ($\Leftarrow$) Conversely, let $\varepsilon > 0$ be given. Put $E = \{n \in \mathbb{Z}_{>0} : p_n \notin N_\varepsilon(p)\}$, E is finite. Let $N = \max E$, then $p_n \in N_\varepsilon(p)$ for all $n \geq N + 1$.

(b) Suppose $\{p_n\}$ converges to both p and p'. Assume $p \neq p'$, let $d = d(p, p')/2$. Then there exists N such that $p_n \in N_d(p) \cap N_d(p')$ for $n \geq N$, but $N_d(p) \cap N_d(p') = \varnothing$, contradiction.

(c) Suppose $p_n \to p$. There exists N such that $d(p_n, p) < 1$ for all $n \geq N$, then diameter is bounded by $M = \max\{d(p_1, p), \cdots, d(p_{N-1}, p), 1\}$.

(d) For all $n \in \mathbb{Z}_{>0}$, choose $p_n \in N_{1/n}(p)$, then the sequence $\{p_n\}$ converges to p. ■

Proposition 3.2

Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$;

(b) $\lim_{n\to\infty}(cs_n) = cs$, $\lim_{n\to\infty}(c+s_n) = c+s$, *for any number* c*;*

(c) $\lim_{n\to\infty} s_n t_n = st$*;*

(d) $\lim_{n\to\infty} 1/s_n = 1/s$, *given* $s_n \neq 0$ *and* $s \neq 0$.

Proof (d) Choose M such that $|s_n - s| < |s|/2$ if $n \geq M$, then we see $|s_n| > |s|/2$ $(n \geq m)$. Given $\varepsilon > 0$, there is an integer $N > M$ such that $n \geq N$ implies $|s_n - s| < |s|^2\varepsilon/2$, then

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| < \varepsilon.$$

■

Proposition 3.3

(a) *Suppose* $\boldsymbol{x}_n \in \mathbb{R}^k$ *and* $\boldsymbol{x_n} = (\alpha_{1,n}, \cdots, \alpha_{k,n})$*. Then* $\{\boldsymbol{x}_n\}$ *converges to* $\boldsymbol{x} = (\alpha_1, \cdots, \alpha_k)$ *if and only if* $\lim_{n\to\infty} a_{j,n} = a_j$ *for every* j.

1. *Suppose* $\{\boldsymbol{x}_n\}$ *and* $\{\boldsymbol{y}_n\}$ *are sequences in* $\mathbb{R}^k$, $\{\beta_n\}$ *is a sequence of real numbers, and* $\boldsymbol{x}_n \to \boldsymbol{x}$, $\boldsymbol{y}_n \to \boldsymbol{y}$, $\beta_n \to \beta$*. Then* $\lim_{n\to\infty}(\boldsymbol{x}_n + \boldsymbol{y}_n) = \boldsymbol{x} + \boldsymbol{y}$, $\lim_{n\to\infty}(\boldsymbol{x}_n \cdot \boldsymbol{y}_n) = \boldsymbol{x} \cdot \boldsymbol{y}$*, and* $\lim_{n\to\infty}(\beta_n \boldsymbol{x}_n) = \beta\boldsymbol{x}$*,*

3.1.2 Subsequence and Subsequential Limits

Definition 3.2 (Subsequence)

Given a sequence $\{p_n\}$*, consider a sequence* $\{n_k\}$ *of positive integers such that* $n_1 < n_2 < \cdots$*. The the sequence* $\{p_{n_i}\}$ *is called a* ***subsequence*** *of* $\{p_n\}$*. If* $\{p_{n_i}\}$ *converges, its limit is called a* ***subsequential limit*** *of* $\{p_n\}$.

Remark $\{p_n\}$ converges to p if and only if every subsequences of $\{p_n\}$ converge to p.

Proposition 3.4

(a) *If* $\{p_n\}$ *is a sequence in a compact metric space* X*, then some subsequence of* $\{p_n\}$ *converges to a point of* X.

(b) *Every bounded sequence in* $\mathbb{R}^k$ *contains a convergent subsequence.*

Proof (a) Let $E = \{p_n \mid n \in \mathbb{N}\}$. If E is finite, there is $p \in E$ appears infinitely many times, then the subsequence consisting only p converges to $p \in X$. If E is countable, E has a limit point p in X by Lemma 2.2. Choose n_i such that $d(p, p_{n_i}) < 1/i$ and $n_i > n_{i-1}$, which exists because $N_{1/i}(p) \cap E$ contains infinitely many points. Then $\{p_{n_i}\}$ converges to p.

(b) Follows directly from (a), since E bounded means it lies in some k-cell. ■

Proposition 3.5

The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Proof Let E^* be the set of all subsequential limits, and let q be a limit point of E^*. Choose $\{q_n\} \subset E^*$ such that $d(q_n, q) < 1/n$ for all n. For every $n \in \mathbb{N}$, there exists a subsequence $\{p_{n,i}\}_{i\in\mathbb{N}}$ converging to q_n, so there exists M such that $d(p_{n_i}, q_n) < 1/n$ for $i \geq M$, choose $m_n := n_i$ such that $i \geq M$ and $m_n > m_{n-1}$. Consider the subsequence $\{p_{m_i}\}$, for each $i \in \mathbb{N}$, $d(p_{m_i}, q) \leq d(p_{m_i}, q_i) + d(q_i, q) = 2/i$. Let $\varepsilon > 0$, there exists N such that $2/N < \varepsilon$, so $d(p_{m_i}, q) \leq 2/N < \varepsilon$ for $i \geq N$, hence $p_{m_i} \to q$. ■

Definition 3.3 (Upper and Lower Limits)

*Let $\{s_n\}$ be a sequence, let E be the set of subsequential limits (in the extended real number system), we define the **upper and lower limits** of $\{s_n\}$ to be $s^* = \sup E$ and $s_* = \inf E$, denoted by $s^* = \limsup_{n\to\infty} s_n$ and $s_* = \liminf_{n\to\infty} s_n$.*

Proposition 3.6

Let $\{s_n\}$ be a sequence of real number, let E and s^* be defined as above, then

(a) $s^* \in E$.

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the unique number with both properties. The result for s_* is analogous.

Proof (a) If $s^* = +\infty$, E is not bounded, so $s^* = +\infty \in E$. If $-\infty < s^* < +\infty$, since E is closed (Proposition 3.5), $s^* \in E$. If $s^* = -\infty$, $E = \{-\infty\}$, so $s^* = -\infty$.

(b) Assume there is $x > s^*$ such that $s_n \geq x$ for infinitely many values of n, then there is a subsequential limit y such that $y \geq x > s^*$, contradiction.

Uniqueness: Assume p, q satisfy both (a) and (b) and $p \neq q$. WLOG, let $p < q$, then there is x such that $p < x < q$. Since p satisfies (b), $s_n < x$ whenever $n \geq N$ for some N, so $q \notin E$, contradiction the fact that q satisfies (a). ■

Note *Suppose $s_n \leq t_n$ for $n \geq N$, where N is fixed, then $\liminf_{n\to\infty} s_n \leq \liminf_{n\to\inf} t_n$ and $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\sup} t_n$.*

3.1.3 Cauchy Sequence

Definition 3.4 (Cauchy Sequence)

*A sequence $\{p_n\}_{n\in\mathbb{N}}$ in a metric space (X, d) is a **Cauchy sequence** if for all $\varphi > 0$, there exists $N > 0$ such that $m, n \geq N$ implies $d(p_m, p_n) < \varepsilon$.*

Proposition 3.7

In a metric space, every convergent sequence is a Cauchy sequence.

Proof Suppose $p_n \to p$. Let $\varepsilon > 0$ be given, there exists $N > 0$ such that $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $n, m \geq N$, $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) = \varepsilon/2 + \varepsilon/2 = \varepsilon$, so $\{p_n\}$ is Cauchy.

Proposition 3.8

(a) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

(b) In particular, every Cauchy sequence converges in $\mathbb{R}^k$.

Proof (a) By Proposition 3.4, there exists a convergent subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ and denote by p the point it converges to. Let $\varepsilon > 0$ be given, There exists N such that $d(p_n, p_m) < \varepsilon/2$ for $n, m \geq N$ by Cauchy condition; and there exists $M > N$ and $d(p_{n_k}, p) < \varepsilon/2$ if $n_k \geq M$, by convergence of the subsequence. For $n \geq \max\{M, N\}$, choose p_{n_k} such that $n_k > M > N$, then $d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $\{p_n\}$ converges to p.

(b) Every Cauchy sequence is bounded in $\mathbb{R}^k$: diam $E_N < 1$ for some N, so the diameter of E is at most $\max\{x_1, \cdots, x_N, x_N + 1\}$. Hence E has a bounded closure in $\mathbb{R}^k$ and the proposition then follows from (a). ■

Remark The property that used in part (a) can be stated as: every Cauchy sequence with a convergent subsequence is convergent.

Definition 3.5 (Complete)

A metric space in which every Cauchy sequence converges is said to be ***complete****.*

Example 3.1 The set of all rational, denoted by $\mathbb{Q}$, is not complete. Consider the sequence "approaching" π.

Definition 3.6 (Monotonicity)

A sequence $\{s_n\}$ of real numbers is said to be ***monotonically increasing*** *if $s_n \leq s_{n+1}$ ($n = 1, \cdots$), and it is* ***monotonically decreasing*** *if $s_n \geq s_{n+1}$ ($n = 1, \cdots$).*

Proposition 3.9

Suppose $\{s_n\}$ is monotonic in $\mathbb{R}$. Then $\{s_n\}$ converges if and only if it is bounded.

Proof One direction follows directly from Proposition 3.1. For the other direction, without loss of generality, assume $\{s_n\}$ is monotonically increasing. Consider $E = \{s_n\}$, there exists $\alpha = \sup E$ by the l.u.b. property. Let $\varepsilon > 0$ be given, there exists $N > 0$ such that $\alpha - \varepsilon < s_N \leq \alpha$, then $\alpha - \varepsilon < s_n \leq \alpha$ for $n \geq N$ by the monotonicity. Hence $\{s_n\}$ converges to α. ■

Chapter 4 Continuity

Introduction

- *Limit of Functions*
- *Extreme Value Theorem*
- *Intermediate Value Theorem*
- *Normed Vector Space, Banach Space*
- *Continuity*
- *Uniform Continuity*
- *Discontinuity*

4.1 Limits of Functions

Definition 4.1 (Limit of Functions)

Let X, Y be metric spaces; suppose $E \subset X$, $f : E \to Y$, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$ or $\lim_{x\to p} f(x) = q$ if there is a point $q \in Y$ such that: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 < d_X(x, p) < \delta \Longrightarrow d_Y(f(x), q) < \varepsilon$$

♣

Proposition 4.1

Let X, Y, E, f, p be defined as above. Then $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.

♠

Proof ($\Rightarrow$) Suppose $\lim_{x\to p} f(x) = q$ and $\varepsilon > 0$, there exists $\delta > 0$ satisfying the definition above. For every sequence $\{p_n\}$ that satisfies the above properties, there exists N such that $0 < d_X(p_n, p) < \delta$ for $n \geq N$, in which $d_Y(p_n, p) < \varepsilon$. Hence $\lim_{n\to\infty} f(p_n) = q$.

($\Leftarrow$) Suppose $\lim_{x\to p} f(x) \neq q$, there exists $\varepsilon > 0$ such that for all $\delta > 0$, there is $x \in E$ such that $0 < d_X(p, x) < \delta$ but $d_Y(q, f(x)) \geq \varepsilon$. Construct a sequence $\{p_n\}$ by choosing $\delta_n = 1/n$, then it satisfies the desired properties but $d_Y(q, f(p_n)) \geq \varepsilon$, so $\lim_{n\to\infty} f(p_n) \neq q$. ■

Proposition 4.2

If f has a limit at p, the limit is unique.

♠

Proof Since the limit of a sequence $\{p_n\}$ is unique, the proposition follows directly from Proposition 4.1. ■

Binary Operations Suppose f, g are functions defined on E to $\mathbb{R}^k$, we define addition $f + g$ by $(f + g)(x) = f(x) + g(x)$ and multiplication fg by $(fg)(x) = f(x)g(x)$. Similarly, we define $f - g$ and f/g (defined only at points x such that $g(x) \neq 0$). The scalar multiplication λf is defined by $(\lambda f)(x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}$. The limit laws still holds.

Remark The change of variable in limits is stated as follows: If $x = g(t)$ is an invertible function with inverse g^{-1} in the deleted neighborhood of $t = b$, and $\lim_{t\to b} g(t) = a$, $\lim_{x\to a} g^{-1}(x) = b$, then either both the limits $\lim_{x\to a} f(x)$ and $\lim_{t\to b} f(g(t))$ exist and are equal or both of them don't exist.

4.2 Continuity

4.2.1 Continuous Functions

Definition 4.2 (Continuity)

*Suppose (X, d_X) and (Y, d_Y) are metric spaces. A function $f : X \to Y$ is **continuous at p** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for all x such that $d_X(x, p) < \delta$.*

*If f is continuous at every point of X, then f is **continuous on X**.*

♣

Proposition 4.3

Suppose $f : X \to Y$ and p is a limit point of E. Then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$.

♠

Remark If $p \in X$ is an isolated point, then f is continuous at $p \in X$.

Proposition 4.4 (Composition of Continuous Functions)

Suppose X, Y, Z are metric spaces, and $E \subset X$. If $f : E \to Y$ is continuous at $p \in E$, and $g : f(E) \to Z$ is continuous at $f(p)$, then $g \circ f : E \to Z$ is continuous at p.

♠

Proof Let $\varepsilon > 0$ be given. Since g is continuous, there exists $\delta > 0$ such that $d_Z(g(f(p)), g(f(q))) < \varepsilon$ if $d_Y(f(p), f(q)) < \delta$. Again since f is continuous, there exists $\lambda > 0$ such that $d_Y(f(p), f(q)) < \delta$ if $d_X(p, q) < \lambda$. Hence $d_Z((g \circ f)(p), (g \circ f)(q)) < \varepsilon$ if $d_X(p, q) < \lambda$, so $g \circ f$ is continuous by definition. ■

Proposition 4.5

A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

♠

Proof $(\Rightarrow)$: Suppose f is continuous and V is open in Y. For every $p \in f^{-1}(V)$, there exists $\varepsilon > 0$ such that $N_\varepsilon(f(p)) \subset V$, and by continuity of f there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ if $d_X(p, q) < \delta$ for all $q \in X$. It follows that $N_\delta(p) \subset f^{-1}(V)$, i.e., p is an interior point in $f^{-1}(V)$, thus $f^{-1}(V)$ is open.

$(\Leftarrow)$ Given $p \in X$ and $\varepsilon > 0$, let $V = N_\varepsilon(f(p))$ be the open neighborhood of $f(p)$. By the hypothesis $f^{-1}(V)$ is open, thus there exists $\delta > 0$ such that $N_\delta(p) \subset f^{-1}(V)$. In other words, $d_Y(f(p), f(q)) < \varepsilon$ if $d_X(p, q) < \delta$, so f is continuous at p. The choice of p is arbitrary implies that f is continuous on X. ■

Example 4.1 The converse does not necessarily hold. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/(x^2 + 1)$ is continuous on $\mathbb{R}$ (which is both open and closed). However, its image $f(\mathbb{R}) = (0, 1]$ is not open nor closed.

Corollary 4.1

A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(C)$ is open in X for every closed set C in Y. ♡

Example 4.2 Thomae's function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/n & \text{if } x = m/n \in \mathbb{Q}, \text{ where } m \in \mathbb{Z},\ n \in \mathbb{Z}_{>0},\ m, n \text{ coprime.} \end{cases}$$

This function is continuous at irrationals and discontinuous at rationals.

The function may also be continuous at finitely many points. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise is continuous only at $x = 0$.

4.2.2 Continuity and Compactness

Definition 4.3 (Bounded Function)

A mapping $f : E \to \mathbb{R}^k$ is said to be ***bounded*** *if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.* ♣

Proposition 4.6

Suppose $f : X \to Y$ is continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact. ♠

Proof Suppose $\{V_\alpha\}$ is an open cover of $f(X)$. Since $\{f^{-1}(V_\alpha)\}$ is an open cover of X because each $f^{-1}(V_\alpha)$ is open by Proposition 4.5, the compactness implies that there is a finite subcover $\{f^{-1}(V_i)\}_{i=1}^n$ of X. Note that hence $\{V_i\}_{i=1}^n$ is a finite subcover of $f(X)$ since $f(f^{-1})(E) \subset E$, it follows that $f(X)$ is compact. ■

Corollary 4.2

Suppose $f : X \to \mathbb{R}^k$ is continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $f(X)$ is closed and bounded, and f is thus bounded. ♡

Proposition 4.7 (Extreme Value Theorem)

Suppose f is a continuous real function on a compact metric space X, and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. Then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$. ♠

Proof Since $f(X)$ is closed and bounded, hence $f(X)$ contains M and m. ■

Proposition 4.8 (Inverse of Continuous Function)

Suppose f is a continuous bijective mapping of a compact metric space X into metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$ is a continuous mapping of Y onto X. ♠

Proof For every closed set $V \subset X$, V is compact, so $(f^{-1})^{-1}(V) = f(V)$ is compact and thus closed. Therefore, f^{-1} is continuous by Corollary (4.1). ■

Definition 4.4 (Uniform Continuity)

Suppose $f : X \to Y$ be a mapping of metric spaces, f is said to be ***uniformly continuous*** *on X if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ for all $p, q \in X$ such that $d_X(p, q) < \delta$.* ♣

Example 4.3 Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $\varepsilon > 0$, given $\delta > 0$, let $p = 1/\delta$ and $q = \delta/2 + 1/\delta$. Then $|p - q| = \delta/2 < \delta$, but

$$|f(p) - f(q)| = |1/\delta^2 - (\delta^2/4 + 1 + 1/\delta^2)| = 1 + \delta^2 > \varepsilon,$$

so f is not uniformly continuous on $\mathbb{R}$. Note that the issue is that $\mathbb{R}$ is not compact.

Proposition 4.9

Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X. ♠

Proof Let $\varepsilon > 0$ be given, choose $\delta_p > 0$ such that $d_X(p, q) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \varepsilon/2$. Since X is compact, there exists a finite cover of neighborhoods $\{N_{\delta_i/2}(p_i)\}$. Put $\delta = \min \delta_i/2$, then for all p, q such that $d_X(p, q) < \delta$, there exists p_i such that $p \in N_{\delta_i/2}(p_i)$, then $p, q \in N_{\delta_i}(p_i)$. Then $d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_i)) + d_Y(f(p_i), f(q)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence f is uniformly continuous. ■

Example 4.4 The compactness is essential. The continuous function f is not necessarily uniformly continuous even it is bounded. Consider $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$, and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sin(x^2)$. f and g are both bounded and continuous, yet they are not uniformly continuous.

4.2.3 Continuity and Connectedness

Proposition 4.10

Suppose $f : X \to Y$ where X, Y are metric spaces. If E is a connected subset of X, then $f(E)$ is connected. ♠

Proof Proof by contrapositive. Suppose $f(E)$ is not connected and A, B forms a separation of $f(E)$. Let $A' = f^{-1}(A) \cap E$ and $B' = f^{-1}(B) \cap E$, then $E = A' \cup B'$. A', B' are nonempty because $A, B \subset f(E)$ are nonempty, and $\overline{A'} \cap B' \subset f^{-1}(\overline{A} \cap B) = \varnothing$ (WLOG, $A \cap \overline{B'} = \varnothing$). Therefore, A', B' form a separation of E, so E

is not connected. ■

Proposition 4.11 (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ and $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$. ♠

Proof Since $[a, b]$ is connected, $f([a, b])$ is connected, so $c \in f([a, b])$ by connectedness. ■

4.3 Discontinuity, Monotonicity

Definition 4.5 (One-sided Limit)

Let f be defined on (a,b). Consider any point x such that $a \leq x < b$, we write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x,b) converging to x. The definition of $f(x-)$ is analogous. ♣

Remark The limit of f at x exists if and only if the one-sided limits coincide, namely $f(x+) = f(x-)$; in this case, $\lim_{t\to x} f(x) = f(x+) = f(x-)$.

Definition 4.6 (Discontinuity)

*Let f be defined on (a,b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the **first kind** (or a simple discontinuity). Otherwise the discontinuity is said to be of the **second kind**.* ♣

Remark There is two types of simple discontinuity: (a) $f(x+) \neq f(x-)$ (removable discontinuity), and (b) $f(x+) = f(x-) \neq f(x)$ (jump discontinuity).

Definition 4.7 (Monotonicity)

*Let $f : (a,b) \to \mathbb{R}$, then f is said to be **monotonically increasing** on (a,b) if $a < x < y < b$ implies $f(a) \leq f(b)$. The definition of monotonically decreasing function is analogous.* ♣

Proposition 4.12

Let f be monotonically increasing on (a,b). Then

(a) $f(x+)$ and $f(x-)$ exist at every point of $x \in (a,b)$.

(b) $\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$.

(c) If $a < x < y < b$, then $f(x+) \leq f(y-)$.

Analogous results hold for monotonically decreasing functions. ♠

Proof (a) Consider $S = \{f(t) \mid a < t < x\}$, there exists $A := \sup S$ since S is nonempty and bounded above by $f(x)$. Let $\varepsilon > 0$ be given, there exists $t_0 \in (a,x)$ such that $A - \varepsilon < f(t_0) \leq A$. Put $\delta = x - t$, then $|A - f(t)| < |A - f(t_0)| < \varepsilon$ if $|x - t| < \delta$. Thus, $f(x-) = A$ exists, and $f(x+)$ exists WLOG.

(b) By the definition of $f(x-)$ in Part (a), $\inf_{a<t<x} f(t) = f(x-)$, and $f(x-) \leq f(x)$ holds by monotonicity. The inequality for $f(x+)$ holds WLOG.

(c) This assertion follows directly from the inequality $f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = \sup_{x<t<y} f(t) = f(y-)$. ■

Note ***Corollary***: *Monotonic functions have no discontinuities of the second kind.*

Proposition 4.13

Let $f : (a, b) \to \mathbb{R}$ be monotonic real function, then the set of points at which f is discontinuous is at most countable.

Proof WLOG, assume f is monotonically increasing and $E = \{x \in (a, b) \mid f \text{ is discontinuous at } x\}$. Since f is increasing, $f(x-) < f(x+)$ if $x \in E$, then there exists $r_x \in \mathbb{Q}$ such that $f(x-) < r_x < f(x+)$. Define $\varphi : E \to \mathbb{Q}$ by $\varphi(x) = r_x$, then φ is clearly injective since $f(x+) \leq f(y-)$ if $x < y$. Therefore, $E \sim f(E) \subset \mathbb{Q}$, so E is at most countable. ■

4.4 Normed Vector Spaces

Definition 4.8 (Norm, Normed Vector Spaces)

*A **norm** on a vector space V is a function $\|\cdot\| : V \to [0, \infty)$ satisfying*

(i) *(positivity)* $0 \leq \|x\| < \infty$ *for all* $x \in V$

(ii) *(definiteness)* $\|x\| = 0$ *if and only if* $x = 0$

(iii) *(scalar multiplication)* $\|\alpha x\| = |\alpha| \|x\|$ *for all scalar* α *and* $x \in V$.

(iv) *(triangle inequality)* $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$.

The pair $(V, \|\cdot\|)$ *is called a **normed vector space**.* ♣

A function $\|\cdot\| : V \to [0, \infty)$ satisfying all properties above except (ii) is called a ***pseudonorm*** on V.

If $(V, \|\cdot\|)$ is a normed vector space, then the function $d : X \times X \to [0, \infty)$ is defined by $d(x, y) := \|x - y\|$ is a metric on V. This is called the usual metric or ***induced metric*** on V.

Definition 4.9 (Convergence)

Suppose $\{x_n\}$ *is a sequence in a normed vector space* $(V, \|\cdot\|)$. *The series* $\sum_{i=1}^{\infty} x_i$ *is said to **converge** if the sequence of partial sums* $\{s_n\}$, *where* $s_n = \sum_{i=1}^{n} x_i$, *converges to some* $x \in V$ *in the sense that* $\lim_{n\to\infty} \|x - \sum_{i=1}^{n} x_n\| = 0$. *In this case, we write* $\sum_{i=1}^{\infty} x_n = x$. ♣

Definition 4.10 (Banach Space)

*A **Banach space** is a normed vector space which is complete with respect to the induced metric.* ♣

Proposition 4.14

A normed vector space $(V, \|\cdot\|)$ *is Banach (namely complete) if and only if a series* $\sum_{i=1}^{\infty} x_i$ *converges whenever* $\sum_{i=1}^{\infty} \|x_i\|$ *converges.* ♠

Proof $(\Rightarrow)$: Let $S_n = \sum_{i=1}^{n} x_i$ and $T_n = \sum_{i=1}^{n} \|x_i\|$, suppose $\{T_n\}$ converges. Let $\varepsilon > 0$ be given. Since $\{T_n\}$ is Cauchy, so there exists $N > 0$ such that $n > m \geq N$ implies $|T_n - T_m| < \varepsilon$. Then

$$\|S_n - S_m\| = \left\| \sum_{i=n+1}^{m} x_i \right\| \leq \sum_{i=n+1}^{m} \|x_i\| = |T_n - T_m| < \varepsilon.$$

Hence $\{S_n\}$ is Cauchy and thus converges since X is complete.

$(\Leftarrow)$: Let $\{x_n\}$ be Cauchy in X. For each $i \in \mathbb{Z}_{>0}$, choose N_i such that $N_i > N_{i-1}$ for which $n > m \geq N_i$ implies that $\|x_n - x_m\| \leq 1/2^i$. Define $y_i = x_{n_{i+1}} - x_{n_i}$, then $\|y_i\| = \|x_{n_{i+1}} - x_{n_i}\| \leq 1/2^i$, so $\sum_{i=1}^{k} \|y_i\|$ converges, followed by $\sum_{i=1}^{\infty} y_i$ converges. Note that $\sum_{i=1}^{n} y_i = x_{n_i+1} - x_{n_1}$, then $\{x_{n_i}\}$ is convergent. By the

Cauchy condition and the convergence of $\{x_{n_i}\}$, it is not hard to show that $\{x_n\}$ converges to $\lim_{i\to\infty} x_{n_i}$ using triangle inequality. ■

Definition 4.11 (Equivalent Norms)

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are called ***equivalent*** *if there exists $c_1, c_2 > 0$ such that $c_1\|x\|_2 \leq \|x\|_1 \leq c_2\|x\|_2$ for all $x \in X$.* ♣

Proposition 4.15

All norms on a finite dimensional vector space X are equivalent. ♠

Proof Suppose $\{e_1, \cdots, e_n\}$ is a basis of X, define $\|\sum_{i=1}^n a_i e_i\|_1 := \sum_{i=1}^n |a_i|$, and put $S := \{u \in X \mid \|u\|_1 = 1\}$. Given $\|\cdot\|_2$, we can define $f : (X, \|\cdot\|_1) \to \mathbb{R}$ by the equality $f(x) = \|x\|_2$. We now want to show f is continuous and S is compact, thus, it follows that im $f|_S = \{\|u\|_2 \mid u \in S\}$ has a maximum and a minimum by the extreme value theorem. Then for $x \in X$, we can put $u = x/\|x\|_1$. As shown above, using the above inequality, multiplying by $\|x\|_1$ yields the desired result $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_2$. ■

Chapter 5 Differentiation

5.1 Differentiation and Mean Value Theorems

Introduction

- *Differentiation*
- *Operations and Chain Rule*
- *Darboux*
- *Mean Value Theorems*
- *Taylor's Theorem*

5.1.1 Differentiation

Definition 5.1 (Differentiable)

For function $f : [a, b] \to \mathbb{R}$, we say f is differentiable at $x \in [a, b]$ if the limit of $\phi(t) := [f(t) - f(x)]/(t - x)$ exists when $t \to x$, i.e, the limit

$$\lim_{t\to x} \phi(t) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

exists. In this case, we denote by the limit $f'(x) := \lim_{t\to x} \phi(t)$. ♣

Proposition 5.1

If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$, then f is continuous at x. ♠

Proof Suppose f is differentiable at x, then $\lim_{t\to x}(f(t)-f(x)) = \lim_{t\to x} \phi(t)\cdot(t-x) = f'(x) \lim_{t\to x}(t-x) = 0$, so f is continuous.

Property *Suppose f, g are real-valued functions differentiable at x, then $f + g$, fg, and f/g are differentiable at x, and*

(a) $(f + g)'(x) = f'(x) + g'(x)$,

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$,

(c) $(f/g)'(x) = [f'(x)g(x) - f(x)g'(x)]/g(x)^2$ *if* $g(x) \neq 0$.

Theorem 5.1 (Chain Rule)

Suppose f is continuous on $[a, b]$ and differentiable at $x \in [a, b]$, and g is defined on $f([a, b])$ and differentiable at $f(x)$. If $h(t) = g(f(t))$, then h differentiable at x and

$$h'(x) = g'(f(x))f'(x).$$

♡

Proof Note that $f'(x)$ and $g'(f(x))$ exists by the differentiability, so

$$h'(x) = \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{t - x} = \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \frac{f(t) - f(x)}{t - x} = g'(f(x))f'(x).$$

Proposition 5.2 (Derivative of Inverse Function)

Let $f : X \to Y$ $(X, Y \subseteq \mathbb{R})$ be an invertible function that is differentiable at $p \in E$. Suppose that $f^{-1} : F \to E$ is continuous at $q := f(p)$ and that $f'(p) \neq 0$. Then f^{-1} is differentiable at $q = f(p)$, and we have $(f^{-1})'(q) = 1/f'(p)$.

5.1.2 Mean Value Theorems

Definition 5.2 (Local Extrema)

Let f be a real function on a metric space X. We say that f has a ***local maximum*** *at $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. Local minimums are defined likewise.*

Proposition 5.3 (Rolle's Theorem)

Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$. The analogous statement for local minima also holds.

Proof Since $f'(x)$ exists, $\lim_{t \to x} \phi(x)$ exists thus $\phi(x+)$ and $\phi(x-)$ exists. Note that $f(t) - f(x) \leq 0$ for all t, it follows that $\phi(x+) \leq 0$ and $\phi(x-) \geq 0$. Hence the existence of $f'(x)$ implies that $f'(x) = \lim_{t \to x} \phi = 0$. ■

Theorem 5.2 (Cauchy Mean Value Theorem)

If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Remark For non-degenerated cases, the condition is equivalent to: there exists x such that $g'(x)/f'(x) = [g(b) - g(a)]/[f(b) - f(a)]$.

Proof We may assume $f(b) - f(a) \neq 0$, otherwise the results follows directly from 5.3. Define $s(x) = f(x)/[f(b) - f(a)]$ and $t(x) = g(x)/[g(b) - g(a)]$, then $s(b) - s(a) = t(b) - t(a) = 1$. Notice that $(s - t)(b) = (s - t)(a)$, then by Rolle's Theorem, $(s - t)'(x) = 0$ for some $x \in (a, b)$, then $s'(x) = t'(x)$. Hence $g'(x)/f'(x) = [g(b) - g(a)]/[f(b) - f(a)]$. ■

Corollary 5.1 (Mean Value Theorem)

If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(x)$.

Proof Follows immediately from Cauchy MVT by taking $g(x) = x$. ■

Proposition 5.4

Suppose f is differentiable in (a, b). If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing; if $f'(x) = 0$, then f is constant; and if $f'(x) \leq 0$, then f is monotonically decreasing.

Proof Suppose $x_1 < x_2$, then $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$ for some $x \in (x_1, x_2)$ by MVT. The assertion follows immediately. ■

Proposition 5.5 (Darboux)

Suppose $f : [a, b] \to \mathbb{R}$ is differentiable, and $f'(a) < \lambda < f'(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = \lambda$.

Proof Let $g(x) = f(x) - \lambda t$. Note that $g'(a) < 0 < g'(b)$, there exists t_1, t_2 such that $g(t_1) < g(a)$ and $g(t_2) < g(b)$, so $g(a)$ and $g(b)$ are not the absolute minimum. Then minimum is attained at some $x \in [t_1, t_2] \subset (a, b)$, so $g'(x) = 0$ and thus $f'(x) = \lambda$. ■

Corollary 5.2

If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuity on $[a, b]$.

Example 5.1 The function f can be differentiable on $[a, b]$ but still have second kind of discontinuity. Suppose

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad \text{then } f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

f' is differentiable and has a second kind of discontinuity at $x = 0$ since $f'(0+)$ and $f'(0-)$ do not exist.

5.2 Derivative of Higher Order, Vector-Valued Functions

Theorem 5.3 (Taylor's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$, $n \in \mathbb{Z}_{>0}$, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

In general, the theorem shows that f can be approximated by a polynomial of degree $n - 1$, and it allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$. ♡

Proof Define $g(t) = f(t) - P(t) - M(t - \alpha)^n$, where M is defined for which $f(\beta) - P(\beta) + M(\beta - \alpha)$. Note that $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$ and $g(\beta) = 0$. Then there exists $x_1 \in (\alpha, \beta)$ such that $g'(x_1) = 0$ by MVT; continuing in this manner, we obtain $x_i \in (\alpha, x_{i-1})$ such that $g^{(i)}(x_i) = 0$. Therefore, $g^{(n)}(x_n) = 0$, thus $M = f'(x_n)/n!$.

Example 5.2 The Mean Value Theorem does not hold explicitly for vector-valued functions. Consider $F(t) = (\cos t, \sin t)$. $F(2\pi) - F(0) = (0, 0)$, but $2\pi F'(t) = 2\pi(-\sin t, \cos t) \neq (0, 0)$. It follows that $F'(t) \neq [F(2\pi) - F(0)]/(2\pi - 0)$ for all t. However, the following generalization holds.

Proposition 5.6

Suppose $\boldsymbol{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and $\boldsymbol{f}$ is differentiable in (a, b), then there exists $x \in (a, b)$ such that $|\boldsymbol{f}(b) - \boldsymbol{f}(b)| \leq (b - a)|\boldsymbol{f}'(x)|$. ♠

Proof If $f(b) - f(a) = 0$, the inequality holds immediately. Suppose $z = f(b) - f(a) \neq 0$, define $\varphi(t) = z \cdot f(t)$, then φ is a real-valued function differentiable on (a, b). By MVT, $\varphi(b) - \varphi(a) = (b - a)\varphi'(x)$ for some x, so $|z|^2 = z \cdot (f(b) - f(a)) = (b - a)z \cdot f'(x)$. Then

$$|z|^2 = |(b - a)z \cdot f'(x)| \leq (b - a)|z||f'(x)|,$$

where the inequality holds by Cauchy-Schwartz. Therefore $|z| \leq (b - a)|f'(x)|$. ■

Chapter 6 Sequences and Series of Functions

Introduction

- *Pointwise Convergence, Uniform Convergence*
- *Uniform Convergence Properties*
- *Criteria of Uniform Convergence*
- *Equicontinuous Family*

6.1 Uniform Convergence

Definition 6.1 (Convergence of Sequence of Functions)

*Suppose $E \subset X$ where X is a metric space and $\{f_n\}$ is a sequence of complex-valued functions defined on E. Define a function f by $f(x) = \lim_{n\to\infty} f_n(x)$, then f is the **limit function** of $\{f_n\}$, and $\{f_n\}$ is said to **converges to f pointwise**.*

Example 6.1 The double limit of a sequence of continuous function is not interchangeable, i.e., in general,

$$\lim_{t\to x}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\lim_{t\to x} f_n(x).$$

Suppose $m, n \in \mathbb{Z}_{>0}$, let $s_{m,n} = m/(m+n)$. Then for every fixed n, $\lim_{m\to\infty} s_{m,n} = 1$, so that $\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = 1$. On the other hand, for every fixed m, $\lim_{n\to\infty} s_{m,n} = 0$ so that $\lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = 0$.

Definition 6.2 (Uniform Convergence)

*A sequence of functions $\{f_n : E \to \mathbb{R}\}$ is said to **converge uniformly** to f on E if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all $x \in E$.*

Remark The difference between pointwise convergence and uniform convergence is that N depends only on $\varepsilon > 0$ in uniform convergence, and N depends on $\varepsilon > 0$ and $x \in E$ in pointwise convergence.

Proposition 6.1 (Uniformly Cauchy Criterion)

The sequence of functions $\{f_n\}$ converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m, n \geq N$ and $x \in E$ implies $|f_n(x) - f_m(x)| < \varepsilon$.

Proof ($\Rightarrow$) Suppose $\{f_n\}$ converges uniformly to f. Let $\varepsilon > 0$. There exists $N > 0$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in E$, so $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon$.

($\Leftarrow$) Suppose $\{f_n\}$ is uniformly Cauchy. The sequence converges pointwise to some f because $\{f_n(x)\}$ is Cauchy for all $x \in E$. Let $\varepsilon > 0$ be given, let N be chosen so that $|f_n(x) - f_m(x)| < \varepsilon/2$. Fix n and let $m \to \infty$, then $|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \varepsilon/2 < \varepsilon$ for all x, so $\{f_n\}$ converges uniformly. ■

Proposition 6.2

Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ *(*$x \in E$*), put* $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. *Then* $f_n \to f$ *uniformly on* E *if and only if* $M_n \to 0$ *as* $n \to 0$.

Proposition 6.3 (Weierstrass M-test)

Suppose $\{f_n\}$ *is a sequence of functions, and* $|f_n(x)| \le M_n$. *Then* $\sum f_n$ *converges uniformly on* E *if* $\sum M_n$ *converges.*

Proof $\{M_n\}$ is convergent and thus Cauchy. Let $\varepsilon > 0$. For n, m sufficiently large, $|\sum_{i=m}^n f_n(x)| \le \sum_{i=m}^n M_n < \varepsilon$ for all $x \in E$, so $\{\sum_{i=1}^n f_n\}$ is uniformly Cauchy. Then $\sum f_n$ is uniformly convergent. ■

6.2 Uniform Convergence, Continuity, and Differentiation

6.2.1 Uniform Convergence and Continuity

Theorem 6.1

Suppose $f_n \to f$ uniformly on E in a metric space. Let x be a limit point of E, and suppose that $A_n := \lim_{t\to x} f_n(t)$. Then $\{A_n\}$ converges, and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. That is,

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

♡

Proof (a) Let $\varepsilon > 0$. For sufficiently large N, $n \geq m \geq N$ implies $|f_n(t) - f_m(t)| < \varepsilon/2$ for all $t \in E$, so $|A_n - A_m| \leq \lim_{t\to x} |f_n(t) - f_m(t)| \leq \varepsilon/2 < \varepsilon$. This implies that $\{A_n\}$ is Cauchy, so it converges.

(b) Let $A := \lim_{n\to\infty} A_n$ and $f_n \to f$ uniformly. Let $\varepsilon > 0$ be given. Notice that

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

For all $t \in E$, for sufficiently large N, we have $|f(t) - f_n(t)| < \varepsilon/3$ by the uniform convergence given that $f_n \to f$ uniformly; $|f_n(t) - A_n| < \varepsilon/3$ by the definition; and $|A_n - A|$ by the definition of A. Therefore, $\lim_{t\to x} f(t) = A$, as desired. ■

Corollary 6.1

Suppose the sequence $\{f_n\}$ is continuous on E for each n, and $f_n \to f$ uniformly on E, then f is continuous on E.

♡

Remark The converse does not hold. Consider the below example where we let f_n be defined on $(0, 1)$. Then f is continuous but f_n does not converge uniformly.

Example 6.2 Suppose $f_n : [0, 1] \to \mathbb{R}$ is defined by $f_n(x) = x^n$. The sequence $\{f_n\}$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

We see that $\lim_{t\to 1}\lim_{n\to\infty} f_n(t) = \lim_{t\to 1} f(t) = 1$, whereas $\lim_{n\to\infty}\lim_{t\to 1} f_n(t) = \lim_{n\to\infty} 1 = 1$. Indeed, there exists x such that $x^n > 1/2$ for all n by intermediate value theorem, then $M_n = \sup_{x\in E} |f_n(x) - 0| > 1/2$. It implies that M_n does not converge to 0, so $\{f_n\}$ does not converge to f uniformly.

Proposition 6.4

Suppose K is compact, and

(i) $\{f_n\}$ is a sequence of continuous functions on K,

(ii) $\{f_n\}$ converges pointwise to a continuous function f on K,

(iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$.

Then $f_n \to f$ uniformly on K.

Proof Set $g_n = f_n - f$ for each $n \in \mathbb{Z}_{>0}$, then g_n are continuous and $g_n \to 0$ pointwise. Let $\varepsilon > 0$ be given. Let $K_n = \{x \in X \mid g_n(x) \geq \varepsilon\}$, K_n is closed because f is continuous, thus K_n is compact by Proposition 2.13. Note that $g_n(x) \geq g_{n+1}(x)$, so $K_n \supset K_{n+1}$. Since $g_n(x) \to 0$, $x \notin K_n$ for sufficiently large n, so $\bigcap K_n = \varnothing$. Then there exists $K_n = \varnothing$ by the finite intersection property, so $|g_n(x)| < \varepsilon$ for all $x \in X$. ∎

Example 6.3 Compactness is essential in the assumption of the above proposition. Consider $f_n = 1/(nx+1)$ defined on $(0,1)$, the sequence $\{f_n\}$ converges to 0. It satisfies all of the three conditions, but f_n does not converge to 0 uniformly.

Definition 6.3 ($\mathscr{C}(X)$)

If X is a metric space, then $\mathscr{C}(X)$ denotes the set of all complex-valued continuous bounded functions with domain X.

Note *$\mathscr{C}(X)$ is a normed vector space (over $\mathbb{C}$) by associating the supremum norm $\|f\| = \sup_{x\in X}|f(x)|$ to each function f.*

Proposition 6.5

The metric induced by the supremum norm makes $\mathscr{C}(X)$ into a complete metric space.

Proof Suppose $\{f_n\}$ is Cauchy in $\mathscr{C}(X)$. It is uniformly Cauchy, so it converges uniformly to a function f by Proposition 6.1. The continuity of f follows from Corollary 6.1. Since there exists N such that $\|f_n - f\| < 1$ if $n \geq N$, then f is bounded by $\|f_n\| + 1$. ∎

6.2.2 Uniform Convergence and Differentiation

The goal is to investigate the relationships between differentiability and uniform convergence. Suppose $\{f_n\}$ is a sequence of differentiable function on $[a,b] \subset \mathbb{R}$, and suppose $f_n \to f$ pointwise (or uniformly). The questions are

(a) Is the limit function f differentiable?

(b) If f differentiable on $[a,b]$, do we have $f_n'(x) \to f'(x)$ for $x \in [a,b]$?

Example 6.4 Define $f_n(x) = x^n$ for $x \in [0,1]$. The limit function is given by $f(x) = 1$ if $x = 1$ and $f(x) = 0$ otherwise. f is clearly not differentiable at $x = 0$ since it is not continuous, so (a) fails if the convergence is pointwise.

Define $f_n(x) = \sqrt{x^2 + 1/n}$ for $x \in [-1,1]$. The limit function is given by $f(x) = |x|$. Although $f_n \to f$ uniformly, f is still not differentiable at $x = 0$, so (a) fails even under uniform convergence.

Example 6.5 Define $f_n : [-1, 1] \to \mathbb{R}$ for each $n \in \mathbb{Z}_{>0}$ by $f_n(x) = x/[1 + (n-1)x^2]$. If $x = 0$, $f_n(0) = 0$ for all n; on the other hand, $\lim_{n\to\infty} f_n(x) = 0$ for each fixed x. The limit function is thus $f(x) = 0$, and it is differentiable. However,

$$f_n'(0) = \lim_{h\to 0} \frac{f_n(h) - f_n(0)}{h} = \lim_{h\to 0} \frac{h/[1+(n-1)h^2]}{h} = \lim_{h\to 0} \frac{1}{1+(n-1)h^2} = 1,$$

so f_n' does not converge to f'.

Proposition 6.6

There exists a real-valued continuous function which is nowhere differentiable.

Example 6.6 (Weierstrass) Define $\varphi(x) = |x|$ for $-1 \le x \le 1$ and let $\varphi(x+2) = \varphi(x)$, then $\varphi(x)$ is continuous on $\mathbb{R}$. Define the function (by a series of fractal sawtooth)

$$f = \sum_{n=0}^{\infty} f_n := \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x).$$

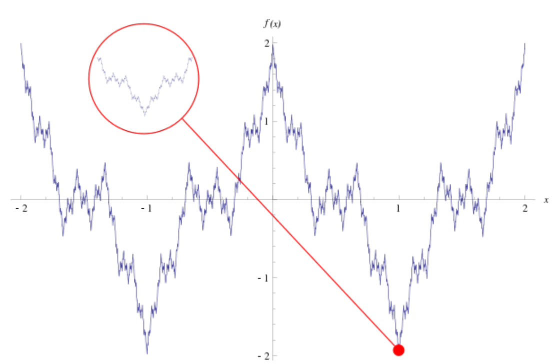
By Weierstrass M-test, the series $\sum f_n$ convergence uniformly to f, so f is continuous.

Fix $x \in \mathbb{R}$ and $m \in \mathbb{Z}_{>0}$, put $\delta_m = \pm 4^{-m}/2$, so there is no integer between $4^m x$ and $4^m(x+\delta_m)$. Define $\gamma_n = [\varphi(4^n(x+\delta_m)) - \varphi(4^n x)]/\delta_m$. Note that $|\gamma_n| \le 4^n$ because $|\varphi(s) - \varphi(t)| \le |s-t|$; in particular, if $n = m$, $|\gamma_n| = 4^n$, and if $n > m$, $\gamma_n = 0$ because $4^n\delta_m$ is even. Then

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n\right| \ge 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1).$$

Since $\delta_m \to 0$ when $m \to \infty$, the limit of the above expression does not exist, it follows that f is nowhere differentiable on $\mathbb{R}$.

The Weierstrass function is defined as Fourier series: $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$, where $0 < x < 1$.



Theorem 6.2

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$, such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f, and

$f'(x) = \lim_{n\to\infty} f_n'(x)$.

Proof (a) Let $\varepsilon > 0$ be given. Since $\{f_n(x_0)\}$ is convergent and thus Cauchy, there exists N_1 such that $n, m \geq N_1$ implies $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$. Since $\{f_n'\}$ converges uniformly and thus uniformly Cauchy, there exists N_2 such that $n, m \geq N_2$ implies $|f_n'(x) - f_m'(x)| < \varepsilon/2(b-a)$ for all x.

Let $N = \max\{N_1, N_2\}$. For $n, m \geq N$, by Mean Value Theorem, there exists $x \in (x, t)$ such that

$$|(f_n - f_m)(x) - (f_n - f_m)(t)| = |(f_n' - f_m'))(c)(x-t)| < \frac{\varepsilon}{2(b-a)}|x-t| \leq \frac{\varepsilon}{2}.$$

Then the triangle inequality implies that

$$|f_n(x) - f_m(x)| \leq |(f_n - f_m)(x) - (f_n - f_m)(t)| + |f_n(x_0) - f_m(x_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\{f_n\}$ is uniformly Cauchy and thus converge uniformly to some function f.

(b) Let f be the limit function of $\{f_n\}$, for fixed x, we define $\phi_n(t) := [f_n(t) - f_n(x)]/(t-x)$ for $t \neq x$, and define $\phi(t) = [f(t) - f(x)]/(t-x)$. As shown above, for $n, m \geq N$,

$$|\phi_n(t) - \phi_m(t)| < \frac{\varepsilon}{2(b-a)},$$

so $\{\phi_n\}$ converges uniformly, for $t \neq x$. Since $\{f_n\}$ converges to f, we conclude that $\lim_{n\to\infty} \phi_n(t) = \phi(t)$. Applying Theorem 6.1 yields $\lim_{n\to\infty} f_n'(x) = \lim_{t\to x} \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{t\to x} \phi(t) = f'(x)$. ■

6.3 Equicontinuous Families of Functions

Definition 6.4 (Pointwise bounded, Uniformly bounded)

*Let $\{f_n\}$ be a sequence of functions define on a set E, we say that $\{f_n\}$ is **pointwise bounded** on E if $\{f_n(x)\}$ is bounded for every $x \in E$, i.e., $|f_n(x)| < \phi(x)$ for all n and some finite-valued function ϕ.*

*We say that $\{f_n\}$ is **uniformly bounded** on E if there exists M such that $|f_n(x)| < M$ for all n and $x \in E$.* ♣

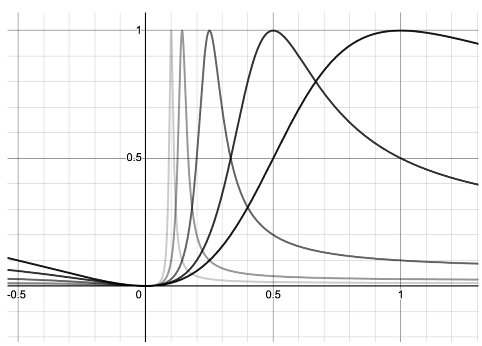
Remark In $\mathbb{C}$, every bounded sequence contains a convergent subsequence. However, the generalization fails to hold on the set of functions:

(i) It is not generally true that every sequence $\{f_n\}$ of bounded continuous functions (even if uniformly bounded on a compact set) contains a pointwise convergent subsequence. For instance, consider $f_n(x) = \sin nx$ on $[0, 2\pi]$.

However, a desired subsequence exists on a countable subset E_1 of E for the sequence of pointwise bounded functions. (See Proposition 6.7)

(ii) It is not generally true that every convergent sequence of functions $\{f_n\}$ (even if uniformly bounded on a compact set) contains a uniformly convergent subsequence? (See Example 6.7)

Example 6.7 Let $f_n(x) = x^2/[x^2 + (1 - nx)^2]$ for $x \in [0, 1]$. $\{f_n\}$ is uniformly bounded on $[0, 1]$ since $|f_n(x)| \le 1$ for all n and x, and $\lim_{n\to\infty} f_n(x) = 0$. However, $f_n(1/n) = 1$ for all n, so that no subsequence converge uniformly on $[0, 1]$.



The concept needed in this connection is "equicontinuity".

Proposition 6.7

If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges for every $x \in E$. ♠

Proof Suppose $E = \{x_i\}_{i\in\mathbb{Z}_{>0}}$. Note that $\{f_n(x_1)\}$ is bounded in $\mathbb{C}$, there is a subsequence $S_1 := \{f_{1,j}\}_j$ such that $\{f_{1,j}(x_1)\}_j$ is convergent. We define $S_i = \{f_{i,j}\}$ recursively as follows, for every $i > 1$, $S_{i-1}(x) = \{f_{i-1,j}(x_i)\}_j$ is bounded and infinite, so there is a subsequence $S_i := \{f_{i,j}\}_{j\in\mathbb{Z}_{>0}}$ of S_{i-1} for which converges at x_i.

Consider the subsequence $S = \{f_{i,i}\}$ (diagonal process). Note that S is a subsequence of S_i except for the first $i - 1$

terms, so $S(x_i)$ converges for every $i = 1, \cdots$. ■

Definition 6.5 (Equicontinuity)

A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be ***equicontinuous*** *on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ for all $x, y \in E$ and $f \in \mathscr{F}$.* ♣

Remark The concept of equicontinuity is similar to uniform convergence. In the equicontinuous family, functions are uniformly continuous in the same extent, whereas in uniformly convergent sequence, functions converge in the same extent for every point.

Proposition 6.8

If K is a compact metric space, $f_n \in \mathscr{C}(K)$ for $n \in \mathbb{Z}_{>0}$, and $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K. ♠

Proof Let $\varepsilon > 0$ be given, and suppose the limit function is f. By uniform convergence, there exists N such that $|f_n(x) - f(x)| < \varepsilon/3$ for $n \geq N$ and $x \in K$. Since $\{f_n\}$ is continuous and K is compact, f is continuous by Corollary 6.1, and thus f is uniformly continuous by Proposition 4.9. Then there exists δ_0 such that $|f(x) - f(y)| < \varepsilon$ if $d(x, y) < \delta_0$. It follows that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| = \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for $d(x, y) < \delta_0$.

For $n < N$, f_n is uniformly continuous by Proposition 4.9 since f_n is continuous on a compact set. Then there exists $\delta_n > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ if $d(x, y) < \delta_n$.

Hence putting $\delta = \min\{\delta_0, \delta_1, \cdots, \delta_{N-1}\}$ suffices. ■

Remark We use the uniform convergence degenerate the case into finite case. For $n \geq N$, we use triangle inequality to convert f_n to f, which is uniformly continuous, and then bound $|f_n(x) - f_n(y)|$ for all $n \geq N$. For $n < N$, we can directly use uniform continuity for each individual f_n. Then taking the minimum of δ's suffices.

Theorem 6.3 (Arzelà–Ascoli)

If K is compact, $\{f_n\} \in \mathscr{C}(K)$ for $n \in \mathbb{Z}_{>0}$, and $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence. ♡

Proof (a) By equicontinuity, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < 1$ for $d(x, y) < \delta$. By compactness, $K = \{N_\delta(x_i)\}_{1 \leq i \leq N}$. For every $1 \leq i \leq N$, there exists M_i such that $|f_n(x_i)| < M_i$ by boundedness. Put $M = \max\{M_1, \cdots, M_n\}$. For every $x \in K$, there exists x_k such that $d(x, x_k) < \delta$, then for every $n \in \mathbb{Z}_{>0}$,

$$|f_n(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_k)| < 1 + M_k \leq 1 + M.$$

Hence $|f_n|$ is bounded uniformly by $1 + M$. ■

(b) For each n, K is covered by finite neighborhoods of the form $N_{1/n}(x_i)$, let K_n be the set of x_i's. Define $K' := \bigcup K_n$, then K' is countable, so there exists a subsequence $\{g_n\}_{n\in\mathbb{Z}_{>0}}$ of $\{f_n\}$ for which converges on K' by Proposition 6.7.

Let $\varepsilon > 0$ be given. By equicontinuity, there exists $\delta > 0$ such that $|g_n(x) - g_n(y)| < \varepsilon/3$ for $d(x, y) < \delta$ and all n. Choose $c > 0$ such that $1/c < \delta$, then for every $x \in K$, there exists $x_i \in K_c \subset K'$ such that $d(x, x_i) < \delta$ by the construction of K'. Also, by the convergence, for each $x_i \in K_n$, there exists N_i such that $|g_n(x_i) - g_m(x_i)|$ for all $n, m \geq N_i$. Put $N = \max\{N_i\}$. For $n, m \geq N$ and every $x \in E$, choose x_i as above, then

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)| = \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence $\{g_n\}$ converges uniformly on K. ■

Remark (a) We use compactness to degenerate the problem into finite points $\{x_i\}$. For every point x_i, $\{f_n(x_i)\}$ bounded by pointwise boundedness, then the equicontinuity allows us to bound the function on the neighborhood of x_i.

(b) For each δ, we can choose a finite subset by compactness so that their neighborhoods covers K. The equicontinuity implies that bounding $|f_n - f_m|$ on the finite subset allows us to bound $|f_n - f_m|$ on their neighborhoods. Then it suffices to prove a subsequence converges on the finite subset; this can be done because there exists a countable dense subset of K and thus a subsequence converging on it.

6.4 The Stone-Weierstrass Theorem

Property *The following equalities hold by considering the binomial distribution:*

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=1,$$
$$\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k}=\mathbb{E}[X]=nx,$$
$$\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k}=Var[X]=nx(1-x).$$

Proposition 6.9 (Weierstrass Approximation Theorem)

If f is a continuous complex function on $[a,b]$, there exists a sequence of polynomials P_n such that $\lim_{n\to\infty}P_n(x)=f(x)$ uniformly on $[a,b]$. If f is real, then P_n may be take real.

Proof Without loss of generality, we may assume $[a,b]=[0,1]$. Let $B_n(f)(x)=\sum_{k=0}^{n}f(k/n)\cdot b_{k,n}(x)$ where $b_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ is the Bernstein polynomial, we therefore want to show $B_n(f)\to f$ uniformly.

Since f is continuous on a compact set, f is uniformly continuous and bounded by some M. Let $\varepsilon>0$ be given. There exists $\delta>0$ such that $|f(x)<f(y)|<\varepsilon/2$ whenever $|x-y|<\delta$. Choose $N=M/\delta^2\varepsilon$. For $n\geq N$, since $f(x)=f(x)\cdot\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$,

$$\begin{aligned}|B_n(f)(x)-f(x)|&\leq\sum_{k=0}^{n}|f(k/n)-f(x)|\binom{n}{k}x^k(1-x)^{n-k}\\&=\underbrace{\sum_{|x-k/n|<\delta}|f(k/n)-f(x)|\,b_{k,n}(x)}_{A}+\underbrace{\sum_{|x-k/n|\geq\delta}|f(k/n)-f(x)|\,b_{k,n}(x)}_{B}.\end{aligned}\tag{6.4.1}$$

For $|x-k/n|<\delta$, $|f(k/n)-f(x)|<\varepsilon/2$, then

$$A\leq|f(k/n)-f(x)|\sum_{k=0}^{n}b_{k,n}(x)=|f(k/n)-f(x)|=\varepsilon/2.$$

For $|x-k/n|\geq\delta$, (Chebyshev's inequality)

$$\begin{aligned}B&\leq 2M\sum_{|x-k/n|\geq\delta}b_{k,n}(x)\leq\sum_{k=0}^{n}\frac{(x-k/n)^2}{\delta^2}b_{k,n}(x)=\frac{2M}{\delta^2n^2}\sum_{k=0}^{n}(nx-k)^2b_{k,n}(x)\\&=\frac{2M}{\delta^2n^2}\cdot nx(1-x)\leq\frac{2M}{\delta^2N}\cdot\frac{1}{4}=\frac{\varepsilon}{2}.\end{aligned}$$

Then $|B_n(f)(x)-f(x)|\leq A+B=\varepsilon$ for all $x\in[0,1]$, so $B_n(f)$ converges uniformly to f. ■

Remark The polynomial $B_n(f)(x)$ may be viewed as the weighted average of f on $[0,1]$ where the weight is given by the binomial distribution. For every x_0, when n approaches ∞, the binomial distribution is concentrated at x_0, so the term $b_{k,n}(x_0)$ vanishes when k/n is far from x, i.e, the polynomial $B_n(f)(x_0)$ converges to $f(x_0)$.

Corollary 6.2

For every interval $[-a,a]$, there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a,a]$. ♡

Definition 6.6 (Algebra of Functions, Uniform Closure)

*Let $\mathscr{A}$ be a family of of functions on a set E, then $\mathscr{A}$ is an **algebra** if $f+g, fg, cf \in \mathscr{A}$ for all $f, g \in \mathscr{A}$ and constant c.*

*If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \to f$ uniformly for $f_n \in \mathscr{A}$, then $\mathscr{A}$ is said to be **uniformly closed**. The **uniform closure** of $\mathscr{A}$ is the set of all limit functions of uniformly convergent sequences in $\mathscr{A}$.* ♣

Example 6.8 The set of polynomials on $\mathbb{R}$ is an algebra. $\mathbb{C}([a,b])$ is the uniform closure of the set of all polynomials on $[a,b]$, by the Weierstrass approximation problem.

Proposition 6.10

Suppose $\mathscr{B}$ is the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra. ♠

Proof Sketch: Suppose $f_n \to f$ uniformly and $g_n \to g$ uniformly. It is not hard to see that f, g are (uniformly) bounded on E, and $f_n + g_n \to f + g$, $f_n g_n \to fg$, and $cf_n \to cf$. Hence $f+g, fg, cf \in \mathscr{B}$, i.e., $\mathscr{B}$ is an algebra. By Proposition 2.9, the uniform closure $\mathscr{B}$ is (uniformly) closed. ■

Definition 6.7 (Separate Points, Vanish at No Points)

*Let $\mathscr{A}$ be a family of functions on a metric space E. $\mathscr{A}$ said to **separate points** on E if to every pair of distinct $x_1, x_2 \in E$, there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) = f(x_2)$.*

*If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ **vanishes at no point** of E.* ♣

Example 6.9 The algebra of all polynomials in one variables separates points and vanishes at no points. The algebra of all even polynomials on $[-1,1]$ does not separate points on $[-1,1]$ since $f(-1) = f(1)$ for all even polynomials f.

Proposition 6.11

Suppose $\mathscr{A}$ is an algebra of function on a set E, $\mathscr{A}$ separate points on E and vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains function f such that $f(x_1) = c_1$ and $f(x_2) = c_2$. ♠

Proof Since $\mathscr{A}$ separate points and vanishes at no point of E, there exists $g, h, k \in \mathscr{A}$ such that $g(x_1) \neq g(x_2)$, $g(x_1) \neq 0$, and $g(x_2) \neq 0$. Set $u = gk - g(x_1)k$ and $v = gh - g(x_2)h$. It is not hard to show $u(x_1) = v(x_2) = 0$ and $u(x_2), v(x_1) \neq 0$. Then the function $f := c_1 v/v(x_1) + c_2 u/u(x_2)$ is the desired function. ■

Theorem 6.4 (Stone-Weierstrass Theorem)

Let $\mathscr{A}$ be an algebra of real continuous functions of a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K. ♡

Proof **Step 1**: If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.

Proof: Let $a = \sup_{x \in K} |f(x)|$, and let $\varepsilon > 0$ be given. By Corollary 6.2, there exists $c_1, \cdots, c_n \in \mathbb{R}$ such that $|\sum_{i=1}^n c_i y^i - |y|| < \varepsilon$ for all $y \in [-a, a]$. Since $\mathscr{B}$ is an algebra, $g(x) = \sum_{i=1}^n c_i f(x)^i \in \mathscr{B}$, and $|g(x) - f(x)| < \varepsilon$ for all $x \in K$. Hence $|f|$ is an uniform limit of sequence in $\mathscr{B}$, so $|f| \in \mathscr{B}$ since $\mathscr{B}$ is uniformly closed.

Step 2: If $f, g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.

Proof: Notice that $\max(f, g) = ((f + g) + |f - g|)/2$, so $\max(f, g) \in \mathscr{B}$ follows immediately from the fact that $|f - g| \in \mathscr{B}$. The result holds for $\min(f, g)$, and the result may be extended to any finite set of functions.

Step 3: Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t - \varepsilon)$ for all $t \in K$.

Proof: For each $y \in K$, there exists $h_y \in \mathscr{B}$ such that $h_y(x) = f(x)$ and $h_y(y) = f(y)$ by Proposition 6.11. By the continuity of h_y there exists an open set J_y such that $h_y(t) > f(t) - \varepsilon$, and the compactness implies that $K \subset J_{y_1} \cup \cdots \cup J_{y_n}$ for some $y_1, \cdots, y_n$. Then setting $g_x := \max(h_{y_1}, \cdots, h_{y_n})$ suffices.

Step 4: Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathscr{B}$ such that $|h(x) - f(x)| < \varepsilon$ for $x \in K$.

Proof: Consider g_x for each $x \in K$. By the continuity of g_x, there exists open set V_x containing x such that $g_x(t) < f(t) + \varepsilon$. Since K is compact, $K \subset V_{x_1} \cup \cdots \cup V_{x_m}$ for some m. The setting $h := \min(g_{x_1}, \cdots, g_{x_m})$ suffices since $h(t) > f(t) - \varepsilon$ by Step 3 and the construction implies that $h(t) < f(t) + \varepsilon$. ■

Remark The Stone-Weierstrass Theorem does not hold for complex algebra.

Definition 6.8 (Self-Adjoint Algebra)

An algebra $\mathscr{A}$ of complex functions is said to be ***self-adjoint*** *if the complex conjugate $\overline{f} \in \mathscr{A}$ for all $f \in \mathscr{A}$.* ♣

Theorem 6.5

Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, and $\mathscr{A}$ separates points and vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$. ♡