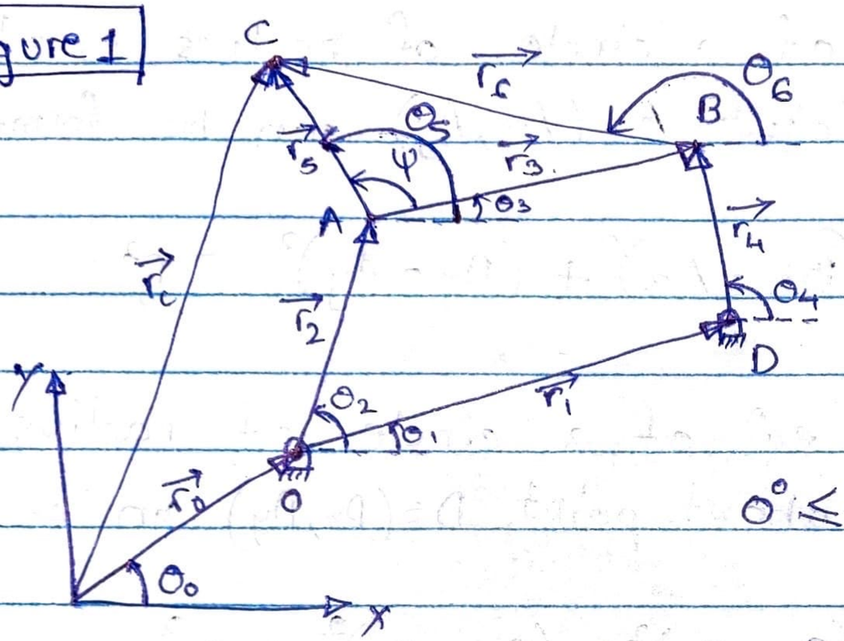
Figure 1



$r_0 = 0 \quad \theta_0 = 0°$
$r_1 = 4 \quad \theta_1 = 30°$
$r_2 = 2 \quad \theta_2 = \text{input}$
$r_3 = 3 \quad \theta_3 = \text{to find}$
$r_4 = 6 \quad \theta_4 = \text{to find}$
$r_5 = 4 \quad \psi = 30°$

$0° \le \theta_2 < 360°$
↑ input

* Problem 1A - Method of Circles

This method of analysis involves finding the expressions for two circles, calculating the intersection of which then provides solutions

Let points $O \equiv (0,0) \quad A \equiv (A_x, A_y) \quad B \equiv (B_x, B_y)$
$C \equiv (C_x, C_y) \quad D \equiv (D_x, D_y)$

Since we know $\vec{r_1} \equiv (r_1, \theta_1)$ & $\vec{r_2} \equiv (r_2, \theta_2)$, we can locate the positions of points A & D

$$A \equiv (A_x, A_y) = (r_2 \cos\theta_2 \, , \, r_2 \sin\theta_2)$$
$$D \equiv (D_x, D_y) = (r_1 \cos\theta_1 \, , \, r_1 \sin\theta_1)$$

The eqⁿ of a circle of radius $r_3 = |\vec{r_3}|$ centered about point $A \equiv (A_x, A_y)$ can be formulated as:

$$(B_x - A_x)^2 + (B_y - A_y)^2 = r_3^2 \qquad - (1)$$

Similarly, eqⁿ of a circle of radius $r_4 = |\vec{r_4}|$ centered about point $D \equiv (D_x, D_y)$ can be given by:

$$(B_x - D_x)^2 + (B_y - D_y)^2 = r_4^2 \qquad - (2)$$

Expanding eqⁿs (1) & (2), and subtracting (2) from (1)

$$\begin{aligned} &\cancel{B_x^2} + A_x^2 - 2A_xB_x + \cancel{B_y^2} + A_y^2 - 2A_yB_y = r_3^2 \\ -\ &\cancel{B_x^2} + D_x^2 - 2D_xB_x + \cancel{B_y^2} + D_y^2 - 2D_yB_y = r_4^2 \\ \hline &(A_x^2 + A_y^2) - (D_x^2 + D_y^2) + 2(D_x - A_x)B_x + \\ &\qquad 2(D_y - A_y)B_y = r_3^2 - r_4^2 \qquad - (3) \end{aligned}$$

we know that

$$(A_x^2 + A_y^2) = r_2^2 \quad \& \quad (D_x^2 + D_y^2) = r_1^2$$

$$\therefore \quad B_x = \underbrace{\frac{r_3^2 - r_4^2 + r_1^2 - r_2^2}{2(D_x - A_x)}}_{\text{Let} \rightarrow k_1} + \underbrace{\frac{2(A_y - D_y)}{2(D_x - A_x)}}_{\text{Let} \rightarrow k_2} B_y$$

$$\therefore \quad B_x = k_1 + k_2 B_y \qquad - (4)$$

Sub. 4 in 2 ; we get

$$(k_1 + k_2 B_y - D_x)^2 + (B_y - D_y)^2 = r_4^2$$

Let $k_1 - D_x = k_3$

$$\therefore (k_2 B_y + k_3)^2 + (B_y - D_y)^2 = r_4^2$$

$$\therefore \underbrace{(k_2^2 + 1)}_{\text{Let} \rightarrow P} B_y^2 + \underbrace{(2k_2 k_3 - 2D_y)}_{\text{Let} \rightarrow Q} B_y + \underbrace{(k_3^2 + D_y^2 - r_4^2)}_{\text{Let} \rightarrow R} = 0$$

$$\therefore P B_y^2 + Q B_y + R = 0$$

$$\therefore (B_y)_{1,2} = \frac{-Q \pm \sqrt{Q^2 - 4PR}}{2P} \quad - (5)$$

$$\begin{cases} Q^2 - 4PR > 0 & \Rightarrow \text{2 real sol}^n\text{s.} \\ Q^2 - 4PR = 0 & \Rightarrow \text{repeated real sol}^n\text{s.} \\ Q^2 - 4PR < 0 & \Rightarrow \text{imaginary (complex conjugate)} \end{cases}$$

sol^{n}s. ← i.e. 4bar infeasible

Hence we proceed further only if $Q^2 - 4PR \geq 0$

Sub. B_{y_1} & B_{y_2} from 5 into 4, we get

$$(B_x)_{1,2} = k_1 + k_2 (B_{y_{1,2}}) \quad - (6)$$

Now that we know 2 possible positions of point $B = (B_x, B_y)$ in addition to previously known positions of points O, A and D, we can compute 2 possible values of θ_3 and θ_4

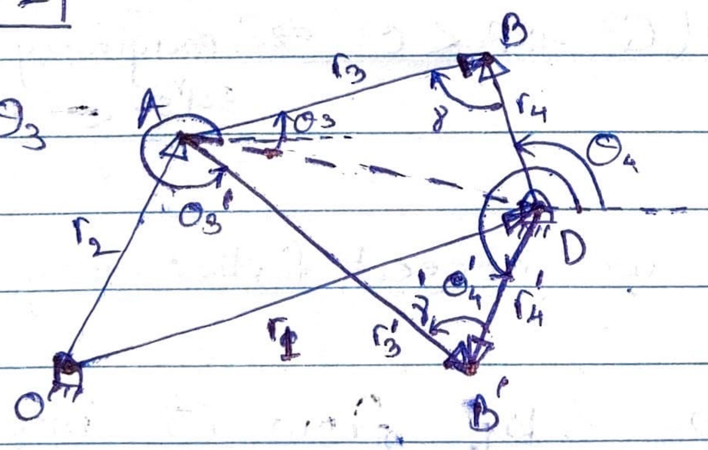
$$(\theta_3)_{1,2} = \text{atan2}\left(\left[(B_y)_{1,2} - A_y,\ (B_x)_{1,2} - A_x\right]\right) \qquad -(7)$$

$$(\theta_4)_{1,2} = \text{atan2}\left(\left[(B_y)_{1,2} - D_y,\ (B_x)_{1,2} - D_x\right]\right) \qquad -(8)$$

At this stage, there are 2 possible sets of $[B_x, B_y, \theta_3, \theta_4]$, i.e. $[B_{x_1}, B_{y_1}, \theta_{3_1}, \theta_{4_1}]$ and $[B_{x_2}, B_{y_2}, \theta_{3_2}, \theta_{4_2}]$, out of which one set corresponds to an uncrossed config. while the other corresponds to a crossed config.

$\boxed{\text{Figure 2}}$

$\gamma = \theta_4 - \theta_3$



OABD uncrossed config.

OAB'D crossed config.

We now apply the following check to determine which parameter set corresponds to which config.

Since $\{\vec{r_3}, \vec{r_4}\}$ and $\{r_3', r_4'\}$ are symmetrical about $\overleftrightarrow{AD}$, γ and γ' are equal in magnitude but opposite in direction (sign). This property can be exploited to differentiate between a crossed/uncrossed config.:

If $\gamma > 0 \Rightarrow$ uncrossed config.
If $\gamma < 0 \Rightarrow$ crossed config.

We choose values of θ_3 & θ_4 that satisfy the above criteria for the preferred config.

Finally, we compute position of point of interest 'C' on the coupler as follows:

w.r.t. $\theta_5 = \theta_3 + \psi$, — (9)

$$\vec{r_c} = \vec{r_2} + \vec{r_5}$$
$$= r_2\cos\theta_2\,\hat{i} + r_2\sin\theta_2\,\hat{j} + r_5\cos\theta_5\,\hat{i} + r_5\sin\theta_5\,\hat{j}$$
$$= (r_2\cos\theta_2 + r_5\cos\theta_5)\,\hat{i} + (r_2\sin\theta_2 + r_5\sin\theta_5)\,\hat{j}$$

$$\therefore \vec{r_5} = \underbrace{(A_x + r_5\cos\theta_5)}_{C_x}\,\hat{i} + \underbrace{(A_y + r_5\sin\theta_5)}_{C_y}\,\hat{j} \quad — (10)$$

* Problem 1B - Method of Loop Closure

Refering Fig. 1, the loop closure eqⁿ's can be written as

$$\vec{r_1} + \vec{r_4} = \vec{r_2} + \vec{r_3}$$

$$\Rightarrow \quad r_1 \cos\theta_1 + r_4 \cos\theta_4 = r_2 \cos\theta_2 + r_3 \cos\theta_3$$

and $\quad r_1 \sin\theta_1 + r_4 \sin\theta_4 = r_2 \sin\theta_2 + r_3 \sin\theta_3$

Let $\quad M = r_1 \cos\theta_1 - r_2 \cos\theta_2$

and $\quad N = r_1 \sin\theta_1 - r_2 \sin\theta_2$

$$\therefore \quad A + r_4 \cos\theta_4 = r_3 \cos\theta_3 \qquad -(1)$$

$$B + r_4 \sin\theta_4 = r_3 \sin\theta_3 \qquad -(2)$$

$$(1)^2 \Rightarrow A^2 + r_4^2 \cos^2\theta_4 + 2A r_4 \cos\theta_4 = r_3^2 \cos^2\theta_3 \qquad -(3)$$

$$(2)^2 \Rightarrow B^2 + r_4^2 \sin^2\theta_4 + 2A r_4 \sin\theta_4 = r_3^2 \sin\theta_3^2 \qquad -(4)$$

$$(3)+(4) \Rightarrow \underbrace{(A^2+B^2+r_4^2-r_3^2)}_{\text{Let} \rightarrow R} + \underbrace{(2 A r_4)}_{\text{Let} \rightarrow P} \cos\theta_4 + \underbrace{(2 B r_4)}_{\text{Let} \rightarrow Q} \sin\theta_4 = 0$$

$$\Rightarrow \quad \boxed{R + P\cos\theta_4 + Q\sin\theta_4 = 0} \qquad (5)$$

Freudenstein's Eqⁿ Form.

w.k.t. $\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}$ & $\cos\theta = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$

Let $t = \tan(\theta/2)$

$\Rightarrow \sin\theta_4 = \frac{2t}{1+t^2}$ & $\cos\theta_4 = \frac{1-t^2}{1+t^2}$

$$(5) \times 1+t^2 \Rightarrow R(1+t^2) + P(1-t^2) + Q(2t) = 0$$

$$\Rightarrow (R-P)t^2 + (2Q)t + (R+P) = 0$$

$$\Rightarrow t = \frac{-\cancel{2}Q \pm \sqrt{\cancel{4}Q^2 - \cancel{4}(R^2-P^2)}}{\cancel{2}(R-P)}$$

$$\Rightarrow t = \frac{-Q \pm \sqrt{Q^2-R^2+P^2}}{R-P} \quad \begin{matrix} \} \text{ Let} \rightarrow N \\ \} \text{ Let} \rightarrow D \end{matrix}$$

$$\begin{cases} Q^2-R^2+P^2 > 0 \Rightarrow 2 \text{ real sol}^n\text{s} \\ Q^2-R^2+P^2 = 0 \Rightarrow \text{repeated real sol}^n\text{s} \\ Q^2-R^2+P^2 < 0 \Rightarrow \text{imaginary (complex conjugate) sol}^n\text{s.} \end{cases}$$

We proceed further only if $Q^2-R^2+P^2 \geq 0$.

since $t = \tan(\theta_4/2)$

$$\Rightarrow \frac{\theta_4}{2} = \text{atan2}(N, D).$$

$$\Rightarrow (\theta_4)_{1,2} = 2 \cdot \text{atan2}\left(-Q \pm \sqrt{Q^2-R^2+P^2},\ R-P\right) \quad -(6)$$

we have

① ⇒ $A + r_4 \cos\theta_4 = r_3 \cos\theta_3$

② ⇒ $B + r_4 \sin\theta_4 = r_3 \sin\theta_3$

Sub. $(\theta_4)_{1,2}$ from ⑥ into ① & ②, we get

$$(\theta_3)_{1,2} = \operatorname{atan2}\left(B + r_4 \sin((\theta_4)_{1,2}),\ A + r_4 \cos((\theta_4)_{1,2})\right)$$

As described in Problem 1A, one of these sets corresponds to an uncrossed config. while the other corresponds to a crossed config. which can be determined by checking the sign (direction) of $\gamma = \theta_4 - \theta_3$.

If $\gamma > 0$ ⇒ uncrossed config

If $\gamma < 0$ ⇒ crossed config.

Finally, we can compute position of point of interest 'c' on coupler as per eq^n ⑩ in sol^n of Problem 1A, i.e.

$\theta_5 = \theta_3 + \psi$ ... from ⑨ P-1A

$\vec{r}_5 = \underbrace{(A_x + r_5 \cos\theta_5)}_{Cx}\hat{i} + \underbrace{(A_y + r_5 \sin\theta_5)}_{Cy}\hat{j}$ ... from ⑩ P-1A

where, $A_x = r_2 \cos\theta_2$ & $A_y = r_2 \sin\theta_2$

* Problem 1c - Newton-Raphson Method

From loop closure, w.r.t.

$r_1\cos\theta_1 + r_4\cos\theta_4 = r_2\cos\theta_2 + r_3\cos\theta_3$

$r_1\sin\theta_1 + r_4\sin\theta_4 = r_2\sin\theta_2 + r_3\sin\theta_3$

$\Rightarrow$ $r_2\cos\theta_2 + r_3\cos\theta_3 - r_4\cos\theta_4 - r_1\cos\theta_1 = 0$

$r_2\sin\theta_2 + r_3\sin\theta_3 - r_4\sin\theta_4 - r_1\sin\theta_1 = 0$

We formulate the problem as follows:

$$\vec{F}(\theta) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} r_2\cos\theta_2 + r_3\cos\theta_3 - r_4\cos\theta_4 - r_1\cos\theta_1 \\ r_2\sin\theta_2 + r_3\sin\theta_3 - r_4\sin\theta_4 - r_1\sin\theta_1 \end{bmatrix} = \underset{\sim}{0} \quad - (1)$$

where, $\theta = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}$

Taking Taylor Series Expansion about $\theta = \theta_k + \Delta\theta$

$$\vec{F}(\theta_k + \Delta\theta) = \vec{F}(\theta_k) + \left[\frac{\partial \vec{F}}{\partial \theta}\right]_{\theta_k} \Delta\theta + \dots \text{ Higher Order Terms} \quad - (2)$$

where $\partial\vec{F}/\partial\theta$ is given by

$$\frac{\partial \vec{F}}{\partial \theta} = \begin{bmatrix} \partial f_1/\partial\theta_3 & \partial f_1/\partial\theta_4 \\ \partial f_2/\partial\theta_3 & \partial f_2/\partial\theta_4 \end{bmatrix} = \begin{bmatrix} -r_3\sin\theta_3 & r_4\sin\theta_4 \\ r_3\cos\theta_3 & -r_4\cos\theta_4 \end{bmatrix} \quad - (3)$$

We need to find roots of $\vec{F}$ (i.e. sol^{n}s of θ) for which $\vec{F}(\theta) = \underset{\sim}{0}$. Hence, let $\vec{F}(\theta_k + \Delta\theta) = \underset{\sim}{0}$. Ignoring higher order terms, eq^n (2) reduces to

$$\vec{F}(\theta_k) + \left[\frac{\partial \vec{F}}{\partial \theta}\right]_{\theta_k} \Delta\theta = \underset{\sim}{0}$$

$$\Rightarrow \Delta\theta = -\left[\frac{\partial \vec{F}}{\partial \theta}\right]_{\theta_k}^{-1} \vec{F}(\theta_k) \qquad - (4)$$

So, given an initial guess θ_k, we can update θ_k using Newton-Raphson update rule:

$$\theta_{k+1} = \theta_k + \Delta\theta \qquad - (5)$$

This iterative process of updating θ_k can be stopped by checking any of the following condition(s):

we use this criteria to stop

(i) $\vec{F}(\theta_k + \Delta\theta) \approx \underset{\sim}{0}$, i.e. $\vec{F}(\theta_k + \Delta\theta) \leq$ tolerance

(ii) abs $(\theta_{k+1} - \theta_k) <$ tolerance

(iii) no. of iterations $<$ max. no. of iterations