AuE 8220: Autonomy: Mobility and Manipulation, Fall 2022

Homework #2: Rotation Matrices, Rotation Parameterizations, Homogenous Transformations

Assigned on: September 15th 2022 Due: September 22nd 2022, 1:00 PM

Problem 1

The relative orientation of two frames of reference are given as follows. Frame A forms the base/reference frame while Frame B is obtained by taking Frame A and rotating it by the following three **incremental relative rotations**: $R_{[z,pi/3]}$ followed by $R_{[x,pi/3]}$ and finally $R_{[z,pi/4]}$. **For the above sets of orientations determine:**

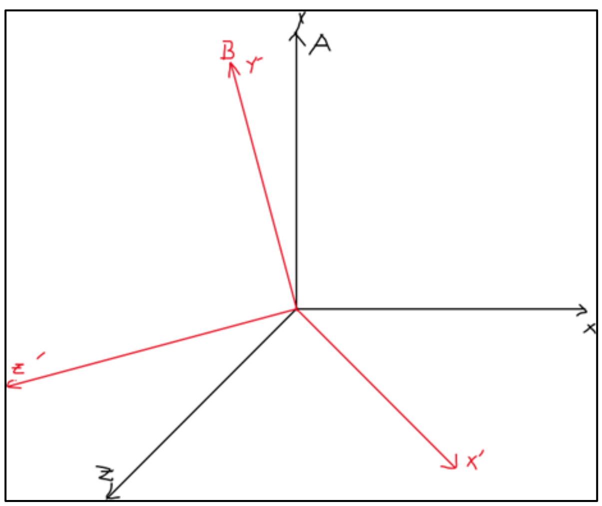
a) The alternate relative rotations representation called Z-Y-Z Euler Angles (obtained by three successive relative rotations first about z -axis by the angle phi, then around y-axis by theta, and finally rotated about z by psi – see section 2.5.1 in Spong, Hutchinson and Vidyasagar)

b) The Roll-Pitch-Yaw Angle representation (note these are absolute angles – see section 2.5.2 in Spong, Hutchinson and Vidyasagar)

c) The Axis/Angle representation (see section 2.5.3 in Spong, Hutchinson and Vidyasagar)

In particular, find the *singularities* and *multiple solutions* of these representations if any.

Solution



Step 1

As frame B is obtained through a series of relative transformations Z-X-Z, we find the equivalent rotation matrix on multiplying $R_z*R_x*R_z$ as shown:

$$R = R_{z,\frac{pi}{3}} R_{x,\frac{pi}{3}} R_{z,\frac{pi}{4}}$$

$$R = \begin{bmatrix} cos\pi/3 & -sin\pi/3 & 0 \\ sin\pi/3 & cos\pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\pi/3 & -sin\pi/3 \\ 0 & sin\pi/3 & cos\pi/3 \end{bmatrix} \begin{bmatrix} cos\pi/4 & -sin\pi/4 & 0 \\ sin\pi/4 & cos\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.04737 & -0.6797 & 0.75 \\ 0.6098 & 0.4356 & -0.433 \\ 0.6124 & 0.6124 & 0.5 \end{bmatrix}$$

Note: You can obtain the final matrix using Matlab coding as well.

a) **Z-Y-Z Euler Angles**

To obtain Z-Y-Z euler angles, we first need to obtain the final rotation matrix R_{ZYZ} as shown below:

$$R_{zyz}(\phi,\theta,\psi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{zyz}(\phi,\theta,\psi) = \begin{pmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{pmatrix}$$

We now equate the resultant matrix to the final rotation matrix obtained earlier:

$$\begin{pmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{pmatrix} = \begin{bmatrix} 0.04737 & -0.6797 & 0.75 \\ 0.6098 & -0.4356 & -0.433 \\ 0.6124 & 0.6124 & 0.5 \end{bmatrix}$$

Referring to equations 2.29 -2.34 derived by Vidyasagar, Spong, we will obtain two solutions for each angle:

$$\Theta = \text{atan2}(\pm\sqrt{1 - r_{33}^2}, r_{33})$$
$$= \pm 60\ degrees$$

Φ = atan2($\pm r_{23}, \pm r_{13}$)
= **120 or -60 degrees**

Ψ = atan2(($\pm r_{31}, \mp r_{32}$)
= **135 or -45 degrees**

b) The roll-pitch-yaw representation.

Note: Difference in XYZ and ZYX convention (Source: Peter Corke Robotics Textbook):

The Cardanian angles are also known as roll, pitch and yaw angles. Confusingly there are two different versions in common use, the sequences XYZ and ZYX. Text books are not at all consistent on this matter. If there is any pattern to the inconsistency it is that the mobile robot community (drones, ground vehicles) uses ZYX while the robot manipulator community uses XYZ.

Why might that be? When describing the attitude of vehicles such as ships, aircraft and cars the convention is that the x-axis points in the forward direction and the z-axis points either up or down. That means the y-axis must point sideways according to the cross-product rule.

Imagine trying to describe the attitude of an aircraft. Our reference attitude is that the aircraft lies in the horizontal plane with its nose pointing in the direction of the world frame x-axis. The first thing we will do is to point the nose to the correct compass heading, that's a rotation within the xy-plane and about the world z-axis. Next, we are going to describe the pitch, the elevation of the front with respect to the horizontal plane, which is a rotation about the new y-axis. Finally, we describe the roll, the rotation about the forward axis of the vehicle, which is a rotation about the new x-axis. This leads to the ZYX angle sequence where the rotation matrix is given by

R(r,p,y)=Rz(y)Ry(p)Rx(r)

When describing the attitude of a robot gripper, as shown in Fig. 2.16, the convention is that the z-axis points forward and the x-axis is either up or down. This leads to the XYZ angle sequence

R(r,p,y)=Rx(y)Ry(p)Rz®

For this problem we use ZYX convention,

$$R_{zyx} = R$$

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} = \begin{bmatrix} 0.04737 & -0.6797 & 0.75 \\ 0.6098 & -0.4356 & -0.433 \\ 0.6124 & 0.6124 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\theta c\psi & s\phi s\psi + c\phi s\theta c\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & -c\phi s\psi + s\phi s\theta c\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix} = \begin{bmatrix} 0.04737 & -0.6797 & 0.75 \\ 0.6098 & -0.4356 & -0.433 \\ 0.6124 & 0.6124 & 0.5 \end{bmatrix}$$

Θ = atan2($-r_{31}$, $\pm\sqrt{1 - r_{31}^2}$)

= **-30 or -150 degrees**

Φ = atan2($\pm r_{21}, \pm r_{11}$)

= **85.56 or -93.44 degrees**

Ψ = atan2(($\pm r_{32}, \pm r_{33}$)

= **50.77 or -129.23 degrees**

c) The equivalent axis-angle transformation is obtained by equating the axis-angle representation matrix with the rotation matrix R.

$$R_{k,\theta} = R$$

$$\begin{bmatrix} k_x^2 v_\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y^2 v_\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z^2 v_\theta + c\theta \end{bmatrix}$$
$$= \begin{bmatrix} 0.04737 & -0.6797 & 0.75 \\ 0.6098 & 0.4356 & -0.433 \\ 0.6124 & 0.6124 & 0.5 \end{bmatrix}$$

Θ = $cos^{-1} \frac{r_{11} + r_{22} + r_{33} - 1}{2}$

Θ = 116.37 degrees

$$k = \frac{1}{2sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$K = \begin{bmatrix} 0.5834 \\ 0.0768 \\ 0.8086 \end{bmatrix}$$

Problem 2

The initial and final positions of corners of a unit cube are given in the inertial coordinate frame as:

The initial coordinates of each vertex is given by: $\begin{bmatrix} x_i^0 \\ y_i^0 \\ z_i^0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$.

The final coordinates of each vertex is:

$$\begin{bmatrix} x_i^f \\ y_i^f \\ z_i^f \end{bmatrix} = \begin{bmatrix} 0.5000 & 0.9698 & 1.0805 & 0.6107 & 1.3758 & 1.8456 & 1.9563 & 1.4865 \\ 0 & -0.8660 & -0.6160 & 0.2500 & 0.4330 & -0.4330 & -0.1830 & 0.6830 \\ 0 & -0.1710 & -1.1329 & -0.9619 & 0.2133 & 0.0423 & -0.9196 & -0.7486 \end{bmatrix}$$

Determine the homogenous transformation for the displacement.

Solution:

Let initial and final matrix be A and B respectively

As the initial and final coordinates of the cube vertices are given, to find out the equivalent homogenous matrix following steps should be exercised:

Convert initial and final matrices to homogenized form:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5 & 0.9698 & 1.0805 & 0.6107 & 1.3758 & 1.8456 & 1.9563 & 1.4865 \\ 0 & -0.8660 & -0.6160 & 0.2500 & 0.4330 & -0.4330 & -0.1830 & 0.6830 \\ 0 & -0.1710 & -1.1329 & -0.9619 & 0.2133 & 0.0423 & -0.9196 & -0.7486 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A homogenous transformation matrix can be writer as

$$H = \begin{bmatrix} r1 & r2 & r3 & d1 \\ r4 & r5 & r6 & d2 \\ r7 & r8 & r9 & d3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where rn represents rotation elements and dn as displacement.

Taking each column of B, A->B can be obtained using H as follows:

$$\begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix} = H * \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, each column of matrix B can be computed.

From the above equation, it is clear that the displacements d1, d2 and d3 are 0.5, 0 and 0 respectively.

To obtain rotation elements let's consider an example of second column of B and A

$$\begin{bmatrix} 0.9698 \\ -0.8660 \\ -0.1710 \\ 1 \end{bmatrix} = \begin{bmatrix} r1 & r2 & r3 & 0.5 \\ r4 & r5 & r6 & 0 \\ r7 & r8 & r9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

r1 + 0.5 = 0.9698

r1 = 0.4698

r4 = -0.8660

r7 = -0.1710

Similarly using the vertices (1,1,0) & (0,0,1) we get the values as follows,

r2 = 0.1107

r5 = 0.25

r8 = -0.9619

r3 = 0.8758

r6 = 0.433

r9 = 0.2133

Also, we can see that B = H*A, hence H = $A^{-1}B$ but since A is not a square matrix an inverse can't b obtained. In such cases a concept called Pseudo inverse is used that will be covered in the later part of the course.

Hence, Homogenous transformation matrix is

$$H = \begin{bmatrix} 0.4698 & 0.1107 & 0.8758 & 0.5 \\ -0.8660 & 0.25 & 0.433 & 0 \\ -0.1710 & -0.9619 & 0.2133 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3

Prove that the following matrix is a rotation matrix

$$\begin{bmatrix} C_\theta C_\psi & -C_\theta S_\psi & -S_\theta \\ S_\psi & C_\psi & 0 \\ -S_\theta C_\psi & -S_\theta S_\psi & C_\theta \end{bmatrix}$$

Solution:

Let the matrix above be R

Using symbolic toolbox in matlab, we try to verify properties of a Rotation Matrix.

Rotation matrices are square matrices, with real entries. More specifically, they can be characterized as orthogonal matrices with determinant 1; that is, a square matrix R is a rotation matrix if and only if $R^T = R^{-1}$ and det R = 1.

Testing if det {R} = 1

Matlab Code using symbolic toolbox

```
syms theta psi real
R = [cos(theta)*cos(psi) -cos(theta)*sin(psi) -sin(theta); -sin(psi) cos(psi) 0; -
sin(theta)*cos(psi) -sin(theta)*sin(psi) cos(theta)];
simplify(det(R))
```

Output:
2*sin(psi)^2*sin(theta)^2 - 2*sin(theta)^2 - 2*sin(psi)^2 + 1

As seen, since det{R} is not equal to 1, the matrix mentioned above can't be a rotation matrix.

Testing RR^T:

Output is:

```
ans =

[                          1, -2*cos(psi)*cos(theta)*sin(psi), -2*cos(psi)^2*cos(theta)*sin(theta)]
[    -2*cos(psi)*cos(theta)*sin(psi),                            1,                                  0]
[-2*cos(psi)^2*cos(theta)*sin(theta),                            0,                                  1]
```

As seen above, since the $R*R^T$ does not yield an identity matrix, the matrix shown above can't be a rotation matrix.

Problem 4

Find the values of the missing elements to complete the 3×3 rotation matrix representation of the location of a body fixed frame {M} with respect to an inertial frame {F}.

$$ {}^{F}R_{M} = \begin{bmatrix} ? & 0.707 & ? \\ 0.707 & ? & ? \\ ? & ? & 0 \end{bmatrix} $$

Solution:

Let unknown elements be as shown below:

$$ {}^{F}R_{M} = \begin{bmatrix} r_{11} & 0.707 & r_{13} \\ 0.707 & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

For {R} to be a rotation matrix, the following properties must be satisfied:
1) The column vectors (and hence the row vectors) must be unit vectors
2) The column vectors (and hence the row vectors) must be mutually orthogonal
3) det{R} = 1

4) $RR^T = 1$

Using matlab symbolic toolbox

```
syms r11 r13 r22 r23 r31 r32 real
eqns = [r11^2 + 0.707^2 + r31^2 == 1,
0.707^2 + r22^2 + r32^2 == 1,
r13^2 + r23^2 == 1,
r11*0.707 + 0.707*r22 + r31*r32 == 0,
0.707*r13 + r22*r23 == 0,
r11*r13 + 0.707*r23 == 0];
S = solve(eqns,[r11,r13,r22,r23,r31,r32]);
```

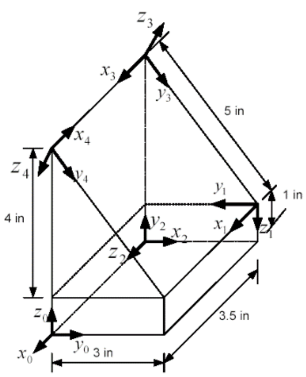
On solving the above equation, it yields no unique solution hence the matrix is not a rotation matrix.

Problem 5

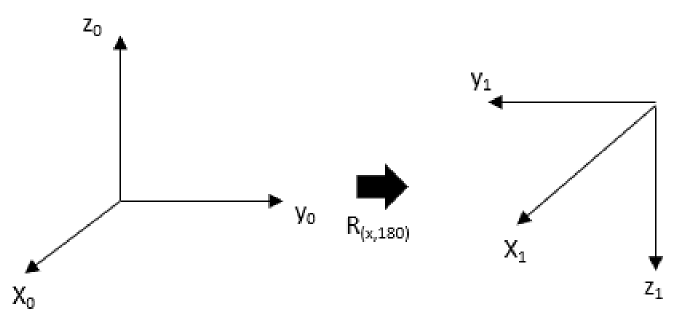
For the figure shown below, find:

(i) the 4×4 homogeneous transformation matrices, ${}^{i-1}A_i$ for i=1, 2, 3, 4 and

(ii) the 4×4 homogeneous transformation matrices ${}^{0}A_i$ for i=1, 2, 3, 4.



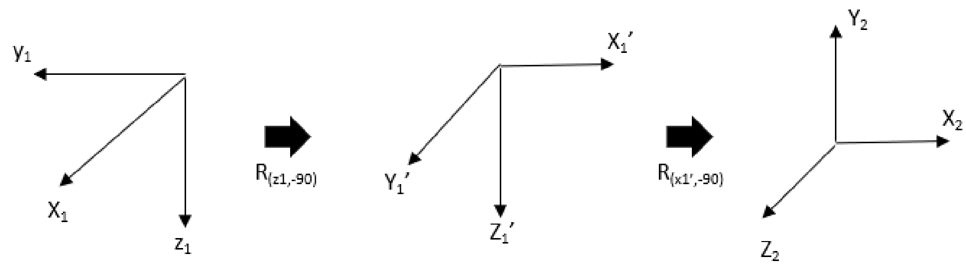
(i) can be obtained by carrying out a rotation about x-axis by 180deg, and translating by (-3.5,3,1) w.r.t frame 0 as shown below:



$$ {}^0A_1 = \begin{bmatrix} 1 & 0 & 0 & -3.5 \\ 0 & \cos(180^o) & -\sin(180^o) & 3 \\ 0 & \sin(180^o) & \cos(180^o) & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -3.5 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$\Rightarrow$ ${}^1A_2$ can be obtained by succesive rotation about z1 by -90deg & rotation about x1' by -90deg, and translating

by (0,3,1) w.r.t frame 1

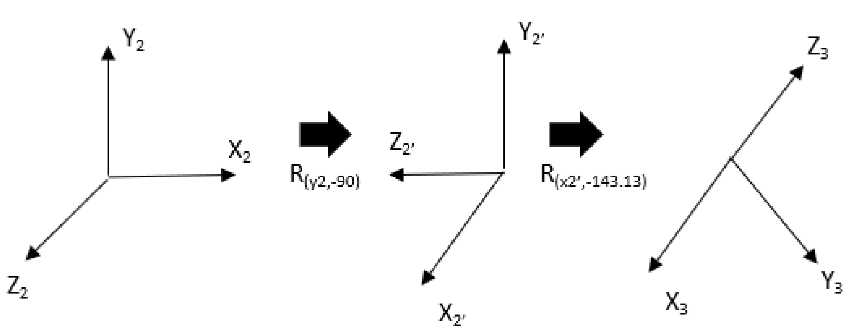


$$ R = R_{(z1,-90)}\,R_{(x1',-90)} = \begin{bmatrix} \cos(-90^o) & -\sin(-90^o) & 0 \\ \sin(-90^o) & \cos(-90^o) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-90^o) & -\sin(-90^o) \\ 0 & \sin(-90^o) & \cos(-90^o) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} $$

$$ {}^1A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$\Rightarrow$ ${}^2A_3$ can be obtained by succesive rotations - about y2 by -90deg -> about x2' by -143deg & translating by (0,5,0)

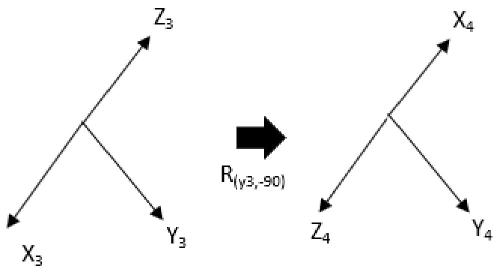
w.r.t to frame 2



$$ R = R_{(y2,-90)}\,R_{(x2',-143)} = \begin{bmatrix} \cos(-90^o) & 0 & \sin(-90^o) \\ 0 & 1 & 0 \\ -\sin(-90^o) & 0 & \cos(-90^0) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-143^o) & -\sin(-143^o) \\ 0 & \sin(-143^o) & \cos(-143^o) \end{bmatrix} = \begin{bmatrix} 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \\ 1 & 0 & 0 \end{bmatrix} $$

$$ {}^{2}A_3 = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 \\ 0 & -0.8 & 0.6 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$\Rightarrow$ ${}^{3}A_4$ can be obtained by rotating frame 3 about y3 by -180 and translating by (3.5,0,0) w.r.t frame 3



$$ {}^{3}A_4 = \begin{bmatrix} \cos(-180^o) & 0 & \sin(-180^o) & 3.5 \\ 0 & 1 & 0 & 0 \\ -\sin(-180^o) & 0 & \cos(-180^o) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 3.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(ii) $$ {}^{0}A_1 = \begin{bmatrix} 1 & 0 & 0 & -3.5 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}A_2 = {}^{0}A_1\,{}^{1}A_2 $$

$$ {}^{0}A_3 = {}^{0}A_2\,{}^{2}A_3 $$

$$ {}^{0}A_4 = {}^{0}A_3\,{}^{3}A_4 $$

```
clear all
% defining the homogenous transformation matrices calculated in (i)
A1_0 = [[1 0 0 -3.5];[0 -1 0 3];[0 0 -1 1];[0 0 0 1]];
A2_1 = [[0 0 1 0];[-1 0 0 3];[0 -1 0 1];[0 0 0 1]];
A3_2 = [[0 0.6 0.8 0];[0 -0.8 0.6 5];[1 0 0 0];[0 0 0 1]];
A4_3 = [[-1 0 0 3.5];[0 1 0 0];[0 0 -1 0];[0 0 0 1]];

%calculating the homogenous transformation matrices w.r.t frame zero
A2_0 = A1_0*A2_1;
A3_0 = A2_0*A3_2;
A4_0 = A3_0*A4_3;
```

Thus the homogenous matrices obtained are,

$$
{}^{0}A_2 = \begin{bmatrix} 0 & 0 & 1 & -3.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}A_3 = \begin{bmatrix} 1 & 0 & 0 & -3.5 \\ 0 & 0.6 & 0.8 & 0 \\ 0 & -0.8 & 0.6 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}A_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0.6 & -0.8 & 0 \\ 0 & -0.8 & -0.6 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Problem 6

Rodriques' formula for the rotation matrix during rotation of a rigid body about the unit vector u=[u_x, u_y, u_z]^T through an angle ϕ can be shown to be:

$$\mathbf{R} = \mathbf{I}\cos\phi + \mathbf{u}\mathbf{u}^T(1-\cos\phi) + \mathbf{U}\sin\phi$$

Using symbolic calculations only verify that it satisfies all the properties of a rotation matrix.

Solution:

As U is the cross-product matrix if the unit vectors:

$$U = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

As each of the u_x, u_y, u_z represent unit vectors, they need to be divided by their magnitude to convert them to unit vectors.

Matlab Code

```
syms ux uy uz phi real
mag = sqrt(ux^2 + uy^2 + uz^2);
ux = ux/mag;
uy = uy/mag;
uz = uz/mag;
u = [ux uy uz]';
I = eye(3);
U = [0 -uz uy;uz 0 -ux;-uy ux 0];
R = I*cos(phi) + u*u'*(1-cos(phi)) + U*sin(phi);
simplify(R*R')
simplify(det(R))
```

Output:

ans =

[1, 0, 0]

[0, 1, 0]

[0, 0, 1]

ans =

1

Hence, using symbolic toolbox we can verify that the solved matrix is a rotation matrix.