

# Algorithms and Datastructures

Levenshtein distance, Dynamic programming

Albert-Ludwigs-Universität Freiburg



**UNI  
FREIBURG**

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Algorithms and Datastructures, February 2018

Introduction

Edit distance

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**Edit distance:**

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- Measurement for similarity of two words / strings
- Algorithm for efficient calculation
- General principle: dynamic programming

## BioInfSearch

ejafjatljökuk
eyjafjallajökull
eyjafjallajökull movie
eyjafjallajölull trailer

Search!



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### Wikipedia.org:

"Der Eyjafjallajökull ([ˈeɪjaˌfjatlaˌjœːkʏtʃ]](3), auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárping eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."





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- Bioinformatics: Similarity of DNA-sequences

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- Cited 63437 times on Google Scholar (Sep. 2017)

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**Definition of edit distance:** (*Levenshtein-distance*)

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# Edit distance

## Example

1 2 3 4 5  
DOOF

BLOED

# Edit distance

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1 2 3 4 5

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replace(1, B)

BOOF

BLOED

# Edit distance

## Example

1 2 3 4 5

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replace(2, L)

BLOF

BLOED

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## Example

1 2 3 4 5

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⏟  
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BLOF



BLOEF



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↓

BOOF

↓

BLOF

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BLOEF

↓

BLOED

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B LOED

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B LOEF

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DOOF

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BOOF



BLOF



BLOEF



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BOOF



BLOF



BLOEF



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BOOF

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BOOF



BLOF



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$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

■  $ED(x, y) \leq ED(x[1..n-1], y[1..m-1]) + 1 \quad n = |x|, m = |y|$





## Solutions based on examples:

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### Recursive approach:

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### Recursive approach:

- Dividing in two halves? Not a good idea:

$$ED(\text{GRAU}, \text{RAUM}) = 2 \quad \text{but} \quad ED(\text{GR}, \text{RA}) + ED(\text{AU}, \text{UM}) = 4$$

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- Finding “smaller” sub problems?  
Let's try it!





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- Let  $x, y$  be two strings
- Let  $\sigma_1, \dots, \sigma_k$  be a sequence of  $k$  operations where  $k = \text{ED}(x, y)$  for  $x \rightarrow y$  (transform  $x$  into  $y$ )  
(We do not know this sequence but we assume it exists)



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The position of  $\sigma_{i+1}$  is  $\geq$  the position of  $\sigma_i$  where we only allow the positions to be equal on a delete operation

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1 2 3 4 5

DOOF



replace(1, B)

BOOF



replace(2, L)

BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

1 2 3 4 5 6 7

SAUDOOF



delete(1)

AUDOOF



delete(1)

UDOOF



delete(1)

DOOF



insert(4, O)

DOOOF

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1	2	3	4	5
D	O	O	F	

B L O E D

1	2	3	4	5	6	7
S	A	U	D	O	O	F

D O O O F



**Consider the last operation:**

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- Solve **blue** part recursively

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DOOF

↓↓↓↓

BLOE

↓ insert

BLOED

Figure: Case 1a

DOOF

↓↓↓↓↓

BLOEDF

↓ delete

BLOED

Figure: Case 1b

DOOF

↓↓↓↓↓

BLOEF

↓ replace

BLOED

Figure: Case 1c



**Consider the last operation:**

### Consider the last operation:

- Solve **blue** part recursively



### Consider the last operation:

- Solve **blue** part recursively

W I N T E R



S O M M E R

↓ nothing

S O M M E R

### Display of solution:

- Alignment
- Example:

<u>S</u>	<u>A</u>	<u>U</u>	B	L	O	E	D
S	A	U	B	L	O	E	D

Figure: Case 2



## Dynamic programming:

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  - Edit distance: Each partial alignment has to be optimal

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**Figure:** Richard Bellman  
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- Optimal solutions consist of optimal partial solutions
  - Shortest paths: Each partial path has to be optimal
  - Edit distance: Each partial alignment has to be optimal
- Always solvable through dynamic programming  
(Caching of optimal partial solutions)



## Case analysis:

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  - $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$  and  $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

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Example:

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- Let  $n = |x|, m = |y|, m' = |z|$

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Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- Let  $n = |x|, m = |y|, m' = |z|$
- We note  $m' \in \{m-1, m, m+1\}$       why?





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  - Case 1c:  $\sigma_k = \text{replace}(m', y[m])$  [then  $m' = m$ ]
- Case 2:  $\sigma_k$  does nothing at the outer end:
  - Then  $z[m'] = y[m]$  and  $x[n'] = z[m']$  and with that  
 $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$  and  $x[n] = y[m]$



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- Case 1a (insert):  $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- Case 1b (delete):  $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
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```
def edit_distance(x, y):  
    if len(x) == 0:  
        return len(y)  
    if len(y) == 0:  
        return len(x)  
  
    ed1 = edit_distance(x, y[:-1]) + 1  
    ed2 = edit_distance(x[:-1], y) + 1  
    ed3 = edit_distance(x[:-1], y[:-1])  
    if x[-1] != y[-1]:  
        ed3 += 1  
  
    return min(ed1, ed2, ed3)
```



## Recursive program:

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- The algorithm results in the following recursive formular:

$$\begin{aligned}T(n, m) &= T(n-1, m) + T(n, m-1) + T(n-1, m-1) + 1 \\&\geq T(n-1, m-1) + T(n-1, m-1) + T(n-1, m-1) \\&= 3 \cdot T(n-1, m-1)\end{aligned}$$

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⇒ The runtime is at least exponential



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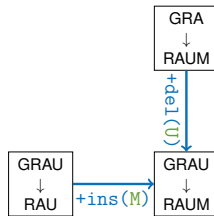
## Visualization on the next slide:

- Operations always refer to the last position (indices are omitted)
- We also display the replaced character on a `replace` operation to visualize operations without costs  
 $\Rightarrow \text{repl}(\text{A}, \text{A})$





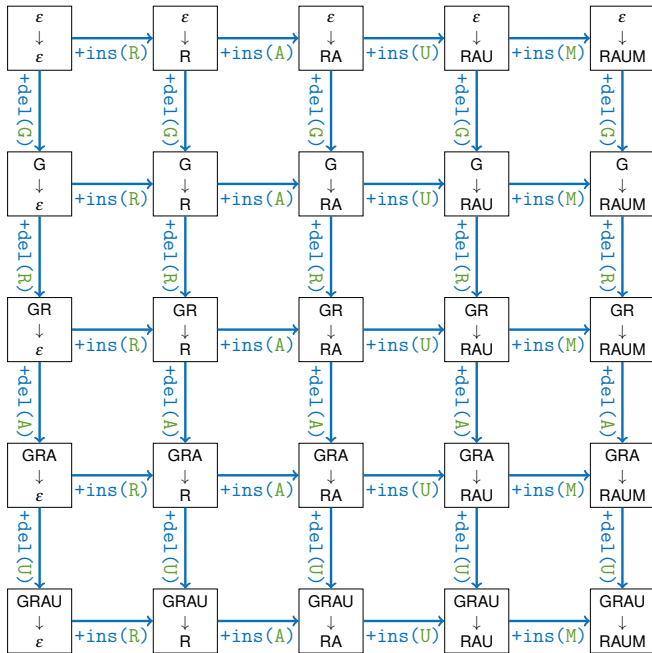












### Fast algorithm:

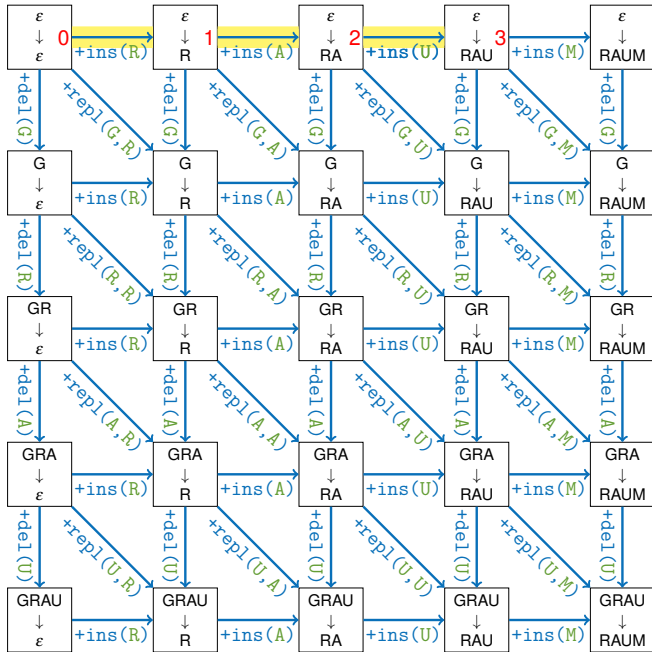
We can determine the **edit distance** for all combination of partial strings from the top left to bottom right.

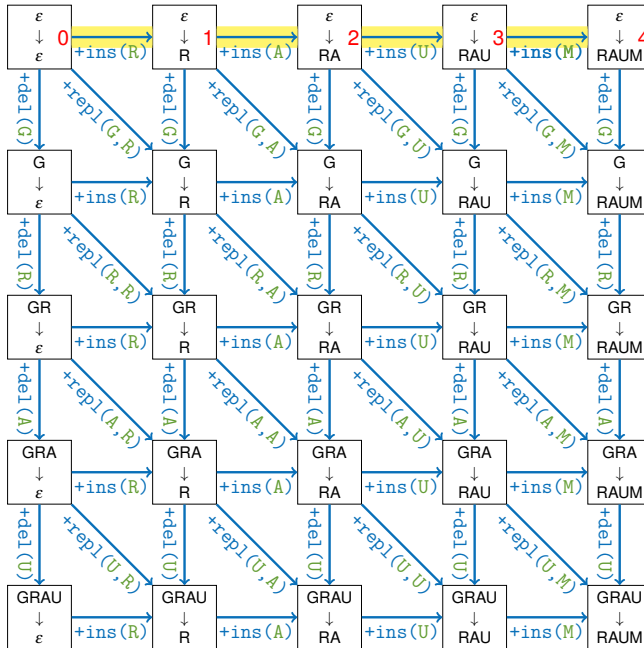














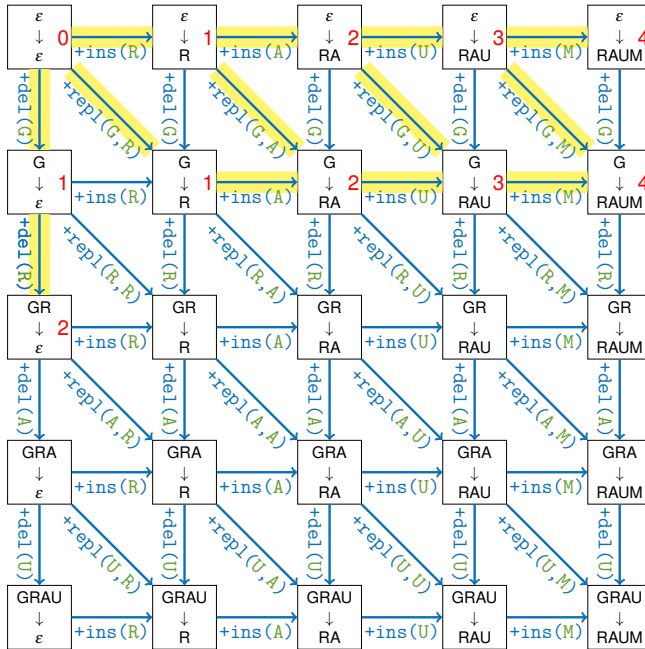












































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  - If we can follow **more than one path** there exist more than one ideal **sequence**





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  - ... the same reoccurring partial problems
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- Computation of the solutions for all partial problems
- In a order that unsolved partial problems consist of already solved partial problems
- The “path” to our solution normally gets computed while searching the best solution
- Dijkstra algorithm is basically dynamic programming!



## **Additional applications:**



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- Solution in  $O(n^3)$  time or  $O(n^2)$  affine



$O(n^2)$  space consumption might be problematic:

**Hirschberg algorithm:**

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### **Hirschberg algorithm:**

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### **Hirschberg algorithm:**

- Divide-and-conquer approach
- $O(n)$  space and  $O(n^2)$  time consumption

# Edit distance

## Additional applications (III)





- Sequencing:  $O(n^2)$  is too much



- Sequencing:  $O(n^2)$  is too much
- Index: suffixtree, suffixarray, burrow-wheeler-transform

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

### **Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ **Dynamic programming**

[Wik] [Dynamic programming](https://en.wikipedia.org/wiki/Dynamic_programming)

`https:`

`//en.wikipedia.org/wiki/Dynamic_programming`

## ■ **Edit distance**

[Wik] [Levenshtein distance](https://en.wikipedia.org/wiki/Levenshtein_distance)

`https:`

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