

Algorithms and Datastructures

Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg



**UNI
FREIBURG**

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, March 2018

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Divide and Conquer

Introduction



UNI
FREIBURG

Concept:



Concept:

- **Divide** the problem into smaller subproblems

Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly

Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem

Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
- **Recursive** application of the algorithm on smaller subproblems

Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
- **Recursive** application of the algorithm on smaller subproblems
- **Direct** solving of small subproblems

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

Divide and Conquer

Maximum Subtotal



UNI
FREIBURG

Input:

Output:

Divide and Conquer

Maximum Subtotal

Input:

- Sequence X of n integers

Output:

Input:

- Sequence X of n integers

Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

Input:

- Sequence X of n integers

Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

Table: Input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Idea:



Idea:



- Solve the left / right half of the problem **recursive**

Idea:



- Solve the left / right half of the problem **recursive**
- Combine both solutions into a overall solution

Idea:



- Solve the left / right half of the problem **recursive**
- Combine both solutions into a overall solution
- The maximum is located in the **left half (A)** or the **right half (B)**

Idea:



- Solve the left / right half of the problem **recursive**
- Combine both solutions into a overall solution
- The maximum is located in the **left half (A)** or the **right half (B)**
- The maximum interval can **overlap with the border (C)**

Principle:

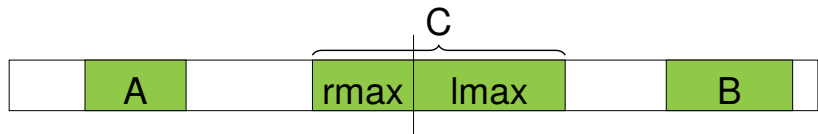


Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$

Principle:



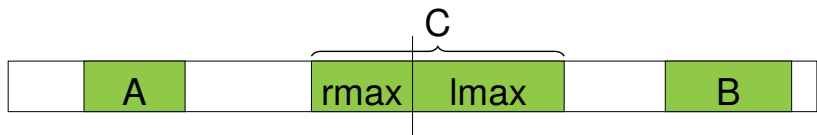
- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned

Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subolutions *A* and *B* are returned
- To solve *C* we have to calculate *rmax* and *lmax*

Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned
- To solve C we have to calculate rmax and lmax
- Overall solution is maximum of A , B and C

Divide and Conquer

Maximum Subtotal - Python



```
def maxSubArray(X, i, j):
```


Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2
```

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)
```

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])
```

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for cornercase C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

Divide and Conquer

Maximum Subtotal - Python



```
#Alternative trivial case  
def maxSubArray(X, i, j):
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

Divide and Conquer

Maximum Subtotal - Python



```
#Implementation max  
def max(a, b, c):
```


Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
```

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

Divide and Conquer

Maximum Subtotal - Python



```
#Alternative implementation max
```

#Alternative implementation max

```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b
```

#Alternative implementation max

```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b  
  
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
<i>sum</i>	58	5	31	90	49	80
<i>lmax</i>	58	58	58	90	90	90

Table: *lmax* example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
<i>sum</i>	58	5	31	90	49	80
<i>lmax</i>	58	58	58	90	90	90

- The *sum* and *lmax* are initialized with $X[i]$

Table: *lmax* example

index	i	$i + 1$	$j - 1$	j
X	58	-53	26	59	-41	31
<i>sum</i>	58	5	31	90	49	80
<i>lmax</i>	58	58	58	90	90	90

- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update *sum*

Table: $lmax$ example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
$lmax$	58	58	58	90	90	90

- The sum and $lmax$ are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update sum
- If $sum > lmax$ then $lmax$ gets updated

Divide and Conquer

Maximum Subtotal



Call with array of four elements

`maxSubArray(-3,9,-4,7)`

in fact:

`maxSubArray(X,0,3)`
with `X=[-3,9,-4,7]`

Divide and Conquer

Maximum Subtotal



Call with array of four elements

$\text{maxSubArray}(-3, 9, -4, 7)$

in fact:

$\text{maxSubArray}(X, 0, 3)$
with $X = [-3, 9, -4, 7]$

Call with array of two elements

$\text{maxSubArray}(-3, 9)$

$\text{maxSubArray}(-4, 7)$

Divide and Conquer

Maximum Subtotal



Call with array of four elements



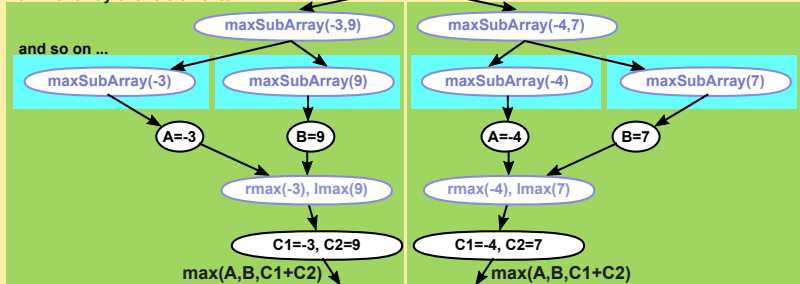
Divide and Conquer

Maximum Subtotal



Call with array of four elements

Call with array of two elements



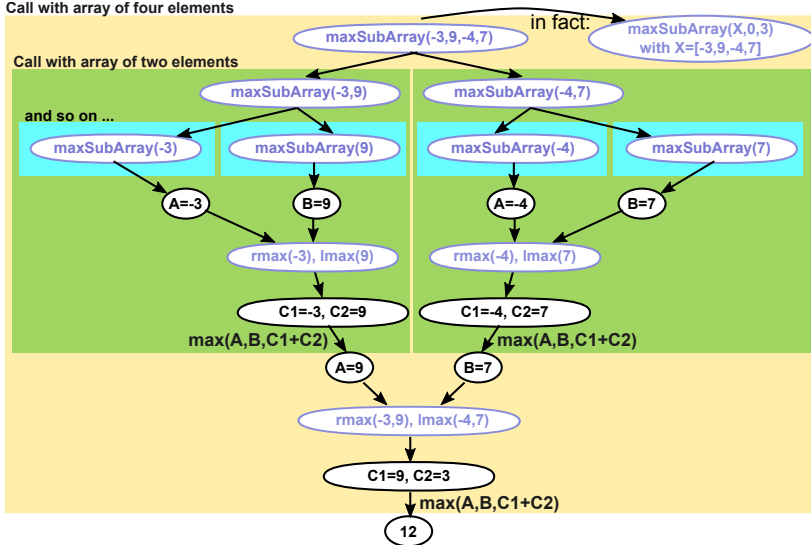
Divide and Conquer

Maximum Subtotal



Call with array of four elements

Call with array of two elements




```
def maxSubArray(X, i, j):  
    if i == j:  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
        return (X[i], i, i)                       # 0(1)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
        return (X[i], i, i)                       # 0(1)  
  
    m = (i + j) / 2                                # 0(1)  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)                             # O(n)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```



```
def maxSubArray(X, i, j):  
    if i == j:                                # O(1)  
        return (X[i], i, i)                 # O(1)  
  
    m = (i + j) / 2                           # O(1)  
    A = maxSubArray(X, i, m)                 # T(n/2)  
    B = maxSubArray(X, m + 1, j)             # T(n/2)  
  
    C1 = rmax(X, i, m)                       # O(n)  
    C2 = lmax(X, m + 1, j)                   # O(n)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
        return (X[i], i, i)                       # 0(1)  
  
    m = (i + j) / 2                                # 0(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)                             # 0(n)  
    C2 = lmax(X, m + 1, j)                         # 0(n)  
    C = (C1[0] + C2[0], C1[1], C2[1])              # 0(1)  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

```
def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)                             # O(n)  
    C2 = lmax(X, m + 1, j)                         # O(n)  
    C = (C1[0] + C2[0], C1[1], C2[1])              # O(1)  
  
    return max([A, B, C], \                       # O(1)  
               key=lambda item: item[0])
```

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

trivial case

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

$\underbrace{\Theta(1)}_{\text{trivial case}}$

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

$\underbrace{\Theta(1)}_{\text{trivial case}}$

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

Divide and Conquer

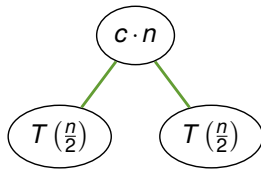
Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

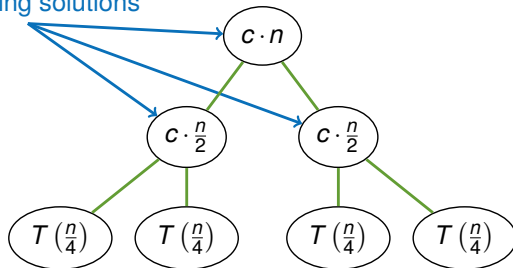
Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



combining solutions



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

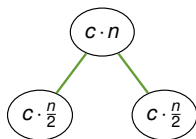
$$c \cdot n$$

1 node processing n elements
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

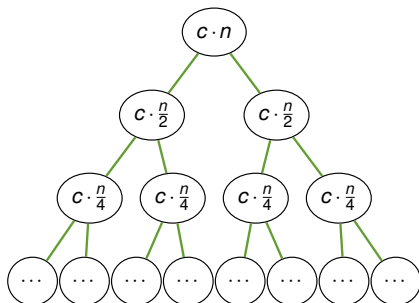
2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

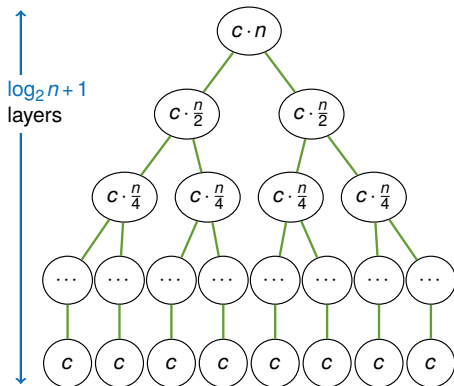
4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^j nodes processing $\frac{n}{2^j}$ elements
 $\Rightarrow 2^j c \cdot \frac{n}{2^j} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^i nodes processing $\frac{n}{2^i}$ elements
 $\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$

n nodes processing 1 element
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



UNI
FREIBURG

Depth:

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



Depth:

- Top level with depth $i = 0$

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$
The costs of merging the solutions and solving of the trivial problems are the same here

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$

The costs of merging the solutions and solving of the trivial problems are the same here

$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Divide and Conquer

Maximum Subtotal - Summary



Summary:

Summary:

- Direct solution is slow with $\mathcal{O}(n^3)$

Summary:

- Direct solution is slow with $\mathcal{O}(n^3)$
- Better solution with incremental update of sum was $\mathcal{O}(n^2)$

Summary:

- Direct solution is slow with $\mathcal{O}(n^3)$
- Better solution with incremental update of sum was $\mathcal{O}(n^2)$
- Divide and conquer approach results in $\mathcal{O}(n \log n)$

Summary:

- Direct solution is slow with $\mathcal{O}(n^3)$
- Better solution with incremental update of sum was $\mathcal{O}(n^2)$
- Divide and conquer approach results in $\mathcal{O}(n \log n)$
- There is an approach running in $\mathcal{O}(n)$ if you assume that all subtotals are positive

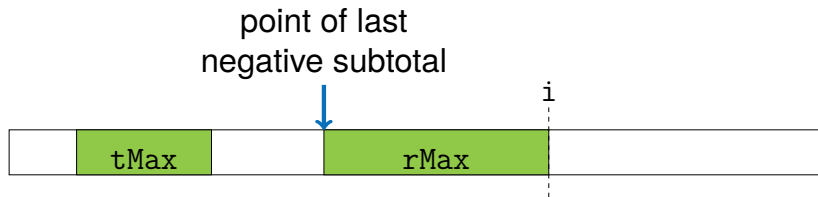


Figure: Scanning the array in linear time

Divide and Conquer

Maximum Subtotal - Python



```
#Implementation - linear runtime  
def maxSubArray(X):
```

Divide and Conquer

Maximum Subtotal - Python



#Implementation - linear runtime

```
def maxSubArray(X):  
    # sum, start index  
    rMax, irMax = 0, 0 # current maximum  
    tMax, itMax = 0, 0 # total maximum
```

#Implementation - linear runtime

```
def maxSubArray(X):  
    # sum, start index  
    rMax, irMax = 0, 0 # current maximum  
    tMax, itMax = 0, 0 # total maximum  
  
    for i in range(len(X)):  
        if rMax == 0:  
            irMax = i  
        rMax = max(0, rMax + X[i])
```

#Implementation - linear runtime

```
def maxSubArray(X):  
    # sum, start index  
    rMax, irMax = 0, 0 # current maximum  
    tMax, itMax = 0, 0 # total maximum  
  
    for i in range(len(X)):  
        if rMax == 0:  
            irMax = i  
            rMax = max(0, rMax + X[i])  
  
        if rMax > tMax:  
            tMax, itMax = rMax, irMax  
  
    return (tMax, itMax)
```


Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$



Recursion equation:

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- n_0 is normally small, $f_0(n_0) \in \Theta(1)$

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- n_0 is normally small, $f_0(n_0) \in \Theta(1)$
- Normally $a > 1$ and $b > 1$

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- n_0 is normally small, $f_0(n_0) \in \Theta(1)$
- Normally $a > 1$ and $b > 1$
- Dependent on the strategy of solving $T(n)$ f_0 is ignored

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- n_0 is normally small, $f_0(n_0) \in \Theta(1)$
- Normally $a > 1$ and $b > 1$
- Dependent on the strategy of solving $T(n)$ f_0 is ignored
- $T(n)$ is only defined for integers of $\frac{n}{b}$ which is often ignored in benefit of a simpler solution

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



Substitution Method:

Substitution Method:

- Guess the solution and prove it with induction

Substitution Method:

- Guess the solution and prove it with induction
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

Substitution Method:

- Guess the solution and prove it with induction
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

- Assumption: $T(n) = n + n \cdot \log_2 n$



Induction:



Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &\stackrel{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \end{aligned}$$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &\stackrel{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \\ &= 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot (\log_2 n - 1)\right) + n \end{aligned}$$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned}T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\&\stackrel{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \\&= 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot (\log_2 n - 1)\right) + n \\&= n + n \log_2 n - n + n\end{aligned}$$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned}T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\&\stackrel{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \\&= 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot (\log_2 n - 1)\right) + n \\&= n + n \log_2 n - n + n \\&= n + n \log_2 n\end{aligned}$$



Substitution Method:



Substitution Method:

- Alternative assumption

Substitution Method:

- Alternative assumption
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 0 \end{cases}$$

Substitution Method:

- Alternative assumption
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 0 \end{cases}$$

- Assumption: $T(n) \in O(n \log n)$

Substitution Method:

- Alternative assumption
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 0 \end{cases}$$

- Assumption: $T(n) \in O(n \log n)$
- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$



Induction:

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \end{aligned}$$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \\ &= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \end{aligned}$$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned}T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\&\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \\&= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\&= c \cdot n \log_2 n - c \cdot n + n\end{aligned}$$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \\ &= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\ &= c \cdot n \log_2 n - c \cdot n + n \\ &\leq c \cdot n \log_2 n, \quad c \geq 1 \end{aligned}$$

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



Recursion tree method:

Recursion tree method:

- Can be used to make assumptions about the runtime

Recursion tree method:

- Can be used to make assumptions about the runtime
- Example:

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



Figure: Recursion tree of example

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

Figure: Recursion tree of example

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

Figure: Recursion tree of example



Figure: Levels of the recursion tree



Costs of connecting the partial solutions:
(excludes the last layer)

Costs of connecting the partial solutions:

(excludes the last layer)

- Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$

Costs of connecting the partial solutions:

(excludes the last layer)

- Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- Costs of partial problem on level i :

$$T_{ip}(n) = c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n \right)^2$$

Costs of connecting the partial solutions:

(excludes the last layer)

- Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- Costs of partial problem on level i :

$$T_{i_p}(n) = c \cdot \left(\left(\frac{1}{4} \right)^i \cdot n \right)^2$$

- Number of partial problems on level i : $n_i = 3^i$

Costs of connecting the partial solutions:

(excludes the last layer)

- Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- Costs of partial problem on level i :

$$T_{ip}(n) = c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n \right)^2$$

- Number of partial problems on level i : $n_i = 3^i$
- Costs on level i :

$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n \right)^2 = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$$

Costs of solving partial solutions: (only the last layer)



Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$
- Costs of partial problem on the **last level**: $T_{i+1_p}(n) = d$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$
- Costs of partial problem on the **last level**: $T_{i+1_p}(n) = d$
- With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \quad \Rightarrow n = 4^i \quad \Rightarrow i = \log_4 n$$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$
- Costs of partial problem on the **last level**: $T_{i+1_p}(n) = d$
- With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \quad \Rightarrow n = 4^i \quad \Rightarrow i = \log_4 n$$

- Number of partial problems on the **last level**:

$$n_{i+1} = 3^{\log_4 n}$$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$
- Costs of partial problem on the **last level**: $T_{i+1_p}(n) = d$
- With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \quad \Rightarrow n = 4^i \quad \Rightarrow i = \log_4 n$$

- Number of partial problems on the **last level**:

$$n_{i+1} = 3^{\log_4 n} = n^{\log_4 3} \quad \leftarrow \text{next slide}$$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$
- Costs of partial problem on the **last level**: $T_{i+1_p}(n) = d$
- With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \quad \Rightarrow n = 4^i \quad \Rightarrow i = \log_4 n$$

- Number of partial problems on the **last level**:

$$n_{i+1} = 3^{\log_4 n} = n^{\log_4 3} \quad \leftarrow \text{next slide}$$

- Costs on the **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right) \quad \text{uses } n = 3^{\log_3 n}$$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\begin{aligned}\log_4 n &= \log_4 \left(3^{\log_3 n} \right) \\ &= \log_3 n \cdot \log_4 3\end{aligned}$$

uses $n = 3^{\log_3 n}$

uses $\log a^b = b \cdot \log a$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right)$$

$$\text{uses } n = 3^{\log_3 n}$$

$$= \log_3 n \cdot \log_4 3$$

$$\text{uses } \log a^b = b \cdot \log a$$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right)$$

uses $n = 3^{\log_3 n}$

$$= \log_3 n \cdot \log_4 3$$

uses $\log a^b = b \cdot \log a$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3}$$

uses reformulation above

- Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right)$$

uses $n = 3^{\log_3 n}$

$$= \log_3 n \cdot \log_4 3$$

uses $\log a^b = b \cdot \log a$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3}$$

uses reformulation above

$$= \left(3^{\log_3 n} \right)^{\log_4 3}$$

uses $x^{a \cdot b} = (x^a)^b$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right)$$

$$\text{uses } n = 3^{\log_3 n}$$

$$= \log_3 n \cdot \log_4 3$$

$$\text{uses } \log a^b = b \cdot \log a$$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3}$$

uses reformulation above

$$= \left(3^{\log_3 n} \right)^{\log_4 3}$$

$$\text{uses } x^{a \cdot b} = (x^a)^b$$

$$= n^{\log_4 3}$$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\begin{aligned}\log_4 n &= \log_4 \left(3^{\log_3 n} \right) \\ &= \log_3 n \cdot \log_4 3\end{aligned}$$

uses $n = 3^{\log_3 n}$

uses $\log a^b = b \cdot \log a$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3}$$

uses reformulation above

$$\begin{aligned}&= \left(3^{\log_3 n} \right)^{\log_4 3} \\ &= n^{\log_4 3}\end{aligned}$$

uses $x^{a \cdot b} = (x^a)^b$

- This term will recur in the master theorem

Recursion Equations

Total costs



UNI
FREIBURG

Total costs:

Total costs:

- Costs of level i : $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$

Total costs:

- Costs of **level i**: $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- Costs of **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

Total costs:

- Costs of **level i**: $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- Costs of **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left(\begin{array}{c} \text{even with} \\ \text{infinite elements} \end{array} \right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

Total costs:

- Costs of **level i**: $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- Costs of **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \text{even with} \\ \text{infinite elements}}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

- Here: The costs of connecting the partial problems dominate

- **Geometric progression:**

Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

- **Geometric progression:**

Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

- **Geometric series:**

The series (cumulative sum) of a geometric sequence

- **Geometric progression:**

Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

- **Geometric series:**

The series (cumulative sum) of a geometric sequence

- For $|q| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \quad \Rightarrow \text{constant}$$

Recursion Equations

Proof of $O(n^2)$



UNI
FREIBURG

Proof of $\mathcal{O}(n^2)$:

Proof of $\mathcal{O}(n^2)$:

■ We know:

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{4}\right) + \Theta(n^2) \\ &\leq 3T\left(\frac{n}{4}\right) + c \cdot n^2 \end{aligned}$$

Proof of $\mathcal{O}(n^2)$:

- We know:

$$\begin{aligned}T(n) &= 3T\left(\frac{n}{4}\right) + \Theta(n^2) \\&\leq 3T\left(\frac{n}{4}\right) + c \cdot n^2\end{aligned}$$

- Assumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) \leq k \cdot n^2$$

Recursion Equations

Proof of $O(n^2)$



UNI
FREIBURG

Proof of $\mathcal{O}(n^2)$:



Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

$$T(n) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \end{aligned}$$

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16}k \cdot n^2 + c \cdot n^2 \end{aligned}$$

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16}k \cdot n^2 + c \cdot n^2 \\ &\leq k \cdot n^2 \qquad \text{for } k \geq \frac{16}{13}c \end{aligned}$$

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



Master theorem:

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...
 - ... which divides a problem of size n in a partial problems

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...
 - ... which divides a **problem of size n** in **a partial problems**
 - ... which solves each partial problem recursively with a **runtime of $T\left(\frac{n}{b}\right)$**

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...
 - ... which divides a **problem of size n** in **a partial problems**
 - ... which solves each partial problem recursively with a **runtime of $T\left(\frac{n}{b}\right)$**
 - ... which takes **$f(n)$** steps to merge all partial solutions



Master theorem:



Master theorem:

- In the examples we have seen that ...



Master theorem:

- In the examples we have seen that ...
 - Either the runtime of **connecting the solutions** dominates

Master theorem:

- In the examples we have seen that ...
 - Either the runtime of **connecting the solutions** dominates
 - Or the runtime of **solving the problems** dominates

Master theorem:

- In the examples we have seen that ...
 - Either the runtime of **connecting the solutions** dominates
 - Or the runtime of **solving the problems** dominates
 - Or both have **equal influence on runtime**

Master theorem:

- In the examples we have seen that ...
 - Either the runtime of **connecting the solutions** dominates
 - Or the runtime of **solving the problems** dominates
 - Or both have **equal influence on runtime**
- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

Recursion Equations

Master theorem (Simple Form)



Simple form:

Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n)}, \quad a \geq 1, b > 1, c > 0$$

in general form

Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n)}, \quad a \geq 1, b > 1, c > 0$$

in general form

- This yields a runtime of:

Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\substack{\text{Is any } f(n) \\ \text{in general form}}}, \quad a \geq 1, b > 1, c > 0$$

- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

Recursion Equations

Master theorem (Simple Form)

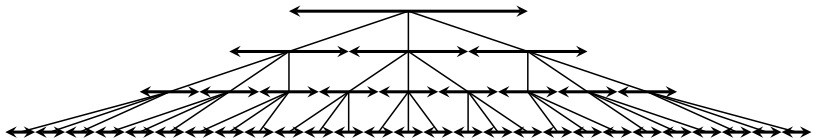


Figure: Simple recursion equation with $a = 3, b = 2$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

Recursion Equations

Master theorem (Simple Form)

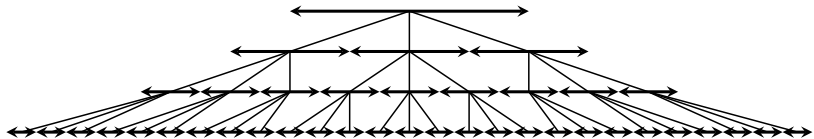


Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- Three partial problems with $\frac{1}{2}$ the size

Recursion Equations

Master theorem (Simple Form)

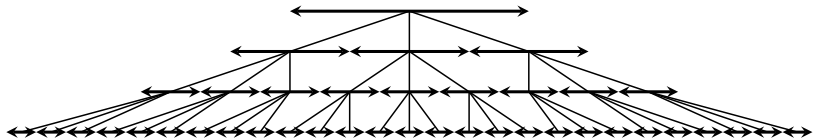


Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- Three partial problems with $\frac{1}{2}$ the size
- Solving the partial problems dominates (last layer, leaves)

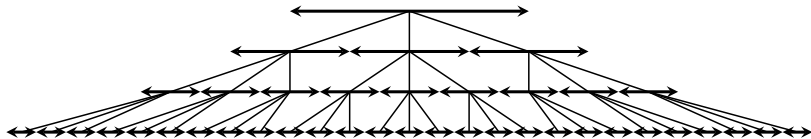


Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- Three partial problems with $\frac{1}{2}$ the size
- Solving the partial problems dominates (last layer, leaves)
- Runtime of $\Theta(n^{\log_b a})$

Recursion Equations

Master theorem (Simple Form)

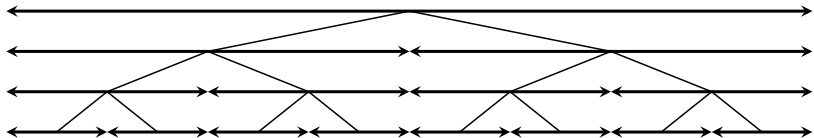


Figure: Simple recursion equation with $a = 2, b = 2$

Recursion Equations

Master theorem (Simple Form)

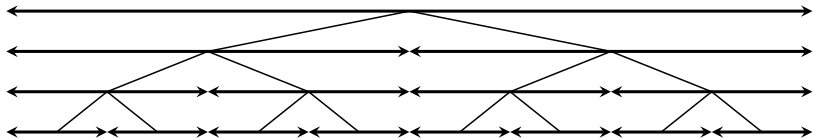


Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- Two partial problems with $\frac{1}{2}$ the size

Recursion Equations

Master theorem (Simple Form)

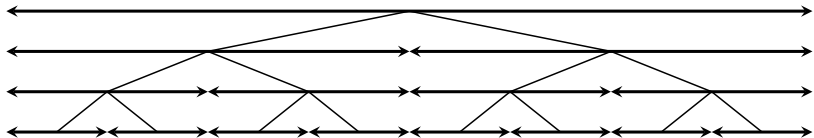


Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- Two partial problems with $\frac{1}{2}$ the size
- Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (Simple Form)

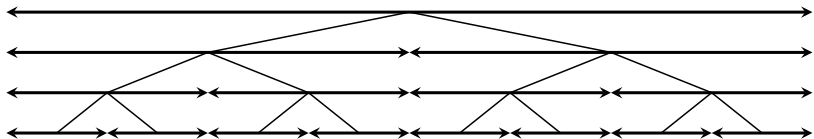


Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- Two partial problems with $\frac{1}{2}$ the size
- Each layer has equal costs, $\log n$ layers
- Runtime of $\Theta(n \log n)$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

Recursion Equations

Master theorem (Simple Form)

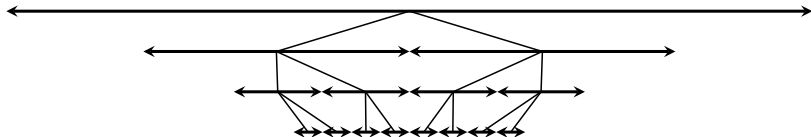


Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

- Two partial problems with $\frac{1}{3}$ the size

Recursion Equations

Master theorem (Simple Form)

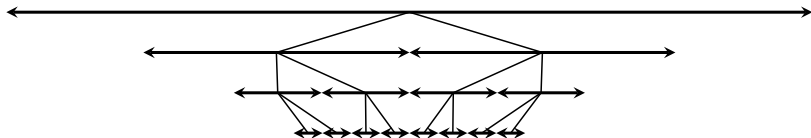


Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

- Two partial problems with $\frac{1}{3}$ the size
- Connecting all partial solutions dominates (first layer, root)



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

- Two partial problems with $\frac{1}{3}$ the size
- Connecting all partial solutions dominates (first layer, root)
- Runtime of $\Theta(n)$

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \quad a \geq 1, b > 1, c > 0$$

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \quad a \geq 1, b > 1, c > 0$$

■ ... yields to a runtime of:

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \quad a \geq 1, b > 1, c > 0$$

■ ... yields to a runtime of:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b \\ \Theta(n \log_b n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \quad a \geq 1, b > 1, c > 0$$

- ... yields to a runtime of:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b \\ \Theta(n \log_b n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



Master theorem (general form):

Master theorem (general form):

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

Master theorem (general form):

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates
(last layer, leaves)

Master theorem (general form):

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates
(last layer, leaves)

- **Case 2:** $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log_b n$ layers

Master theorem (general form):

- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$
Connecting all partial solutions in first layer (root)
dominates

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$



Case 1 - Example:

if $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates (last layer, leaves)



Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in O(n^{\log_b a - \epsilon})$, $\epsilon > 0$

Solving the partial problems dominates (last layer, leaves)

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates (last layer, leaves)

$$\blacksquare T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates (last layer, leaves)

■ $T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

■ $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, b = 3, f(n) = 17 \cdot n, \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$



Case 2: if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (General Form) - Case 2

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers



Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

■ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 10 \cdot n$

$$a = 2, b = 2, f(n) = 10 \cdot n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

■ $T(n) = 2 \cdot T(\frac{n}{2}) + 10 \cdot n$

$$a = 2, b = 2, f(n) = 10 \cdot n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

■ $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1, b = \frac{2}{3}, f(n) = 1, \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Case 3: if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$
Connecting all partial solutions in first layer (root) dominates



Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

■ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$

$$a = 2, b = 2, f(n) = n^2, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Omega(n^{1+\varepsilon})$$



Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

■ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

■ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$



Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T(\frac{n}{2}) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if **regularity condition** also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$



Master theorem:



Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$

Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$
- **Case 2:** $f(n) \notin \Theta(n^1)$

Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$
- **Case 2:** $f(n) \notin \Theta(n^1)$
- **Case 3:** $f(n) \notin \Omega(n^{1+\varepsilon})$

Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$
- **Case 2:** $f(n) \notin \Theta(n^1)$
- **Case 3:** $f(n) \notin \Omega(n^{1+\varepsilon})$

$n \log n$ is *asymptotically* larger than n ,
but not *polynomial* larger



Master theorem:



Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- Three cases depending on the dominance of the terms

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- Three cases depending on the dominance of the terms
- **Case 1:** Solving the partial problems is *polynomial* bigger than merging all solutions

$$T(n) \in \Theta(n^{\log_b a}), \quad T(n) \in \Theta(\text{number of leaves})$$

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- Three cases depending on the dominance of the terms
- **Case 1:** Solving the partial problems is *polynomial* bigger than merging all solutions

$$T(n) \in \Theta(n^{\log_b a}), \quad T(n) \in \Theta(\text{number of leaves})$$

- **Case 2:** Each layer has equal costs

$$T(n) \in \Theta(n^{\log_b a} \log n), \quad \log n \text{ layers}$$

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- Three cases depending on the dominance of the terms
- **Case 1:** Solving the partial problems is *polynomial* bigger than merging all solutions

$$T(n) \in \Theta(n^{\log_b a}), \quad T(n) \in \Theta(\text{number of leaves})$$

- **Case 2:** Each layer has equal costs

$$T(n) \in \Theta(n^{\log_b a} \log n), \quad \log n \text{ layers}$$

- **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems

$$T(n) \in \Theta(f(n))$$

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Master theorem

[Wik] [Master theorem](#)

https://en.wikipedia.org/wiki/Master_theorem