Algorithms and Datastructures

Linked Lists, Binary Search Trees

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Algorithms and Datastructures, January 2018

Structure

Sorted Sequences

Linked Lists

Binary Search Trees

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  - `next()/previous()`: Returns the element with the next bigger/smaller key. This enables iteration over all elements.

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- How could we implement this?

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Static array:

3	5	9	14	18	21	26	40	41	42	43	46

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- Lets have a closer look

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Binary Search Trees

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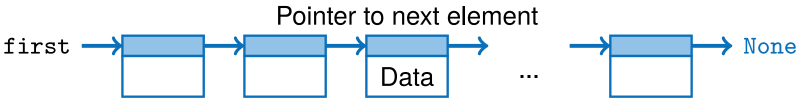

Abbildung: Linked list

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- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified
- No direct access of elements
  $\Rightarrow$ We have to iterate over the list

List with head / last element pointer:

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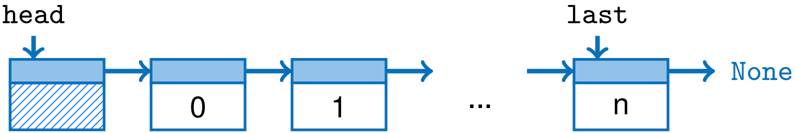

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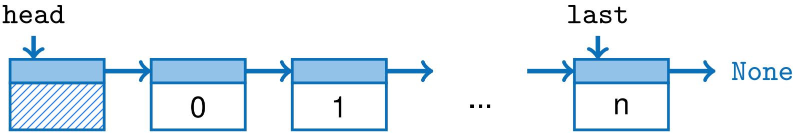

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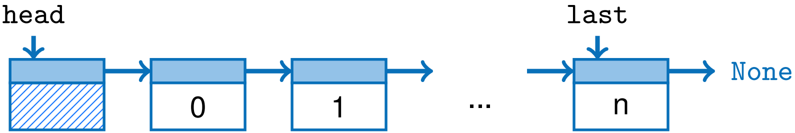

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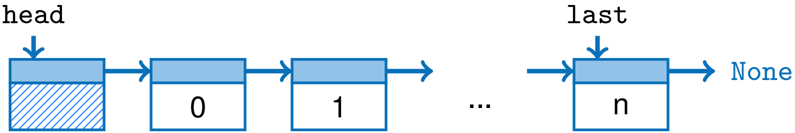

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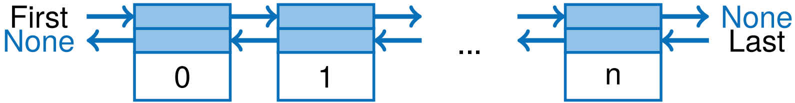

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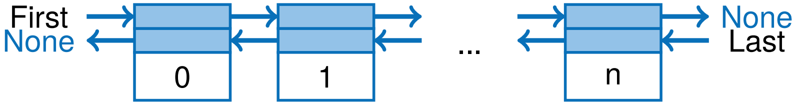

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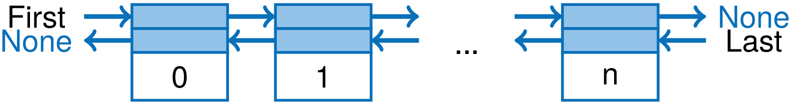

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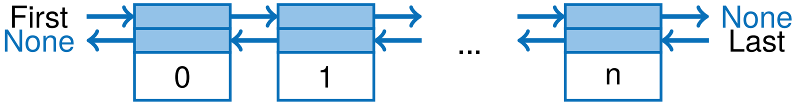

Abbildung: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

```
public class Listelem
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public class Listelem
{   //2 fields: integer and reference
```

Linked Lists

Implementation - Node/Element - Java

```
public class Listelem
{   //2 fields: integer and reference
    //private only available in class
    private int data;
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    public Listelem(int d)
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    //adopted from Mary K.Vernon
    //Introduction to Data Structures
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public int getData() {return data; }
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    //Integer represents possible data, e.g.
    //self defined refence datatypes
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class Listelem
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  Listelem* getNext() { return next; }
  void setNext(Listelem* n) { next = n; }
}
```

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```

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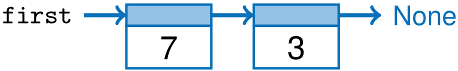
- `first.nextNode = Node(3)`



Linked Lists

Usage examples

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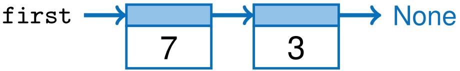
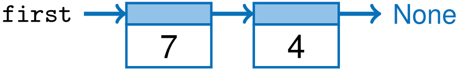


- `first.nextNode.value = 4`



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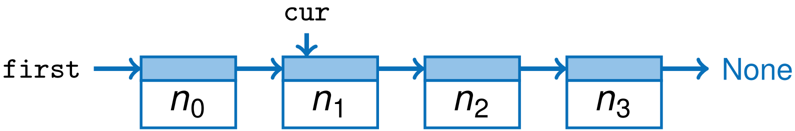

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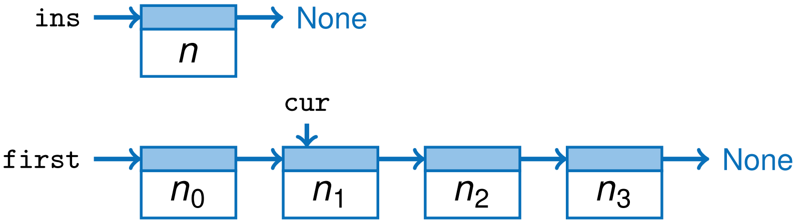

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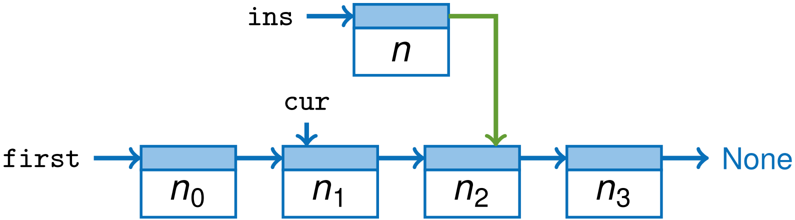

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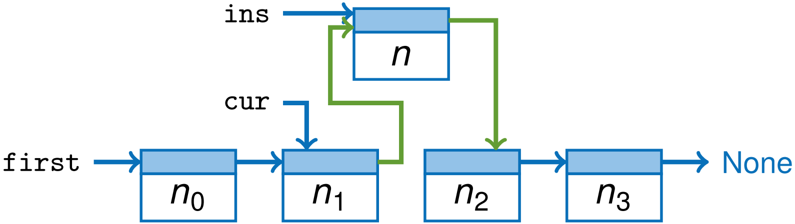

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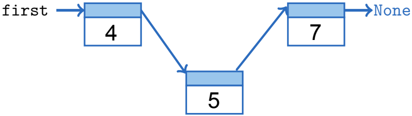

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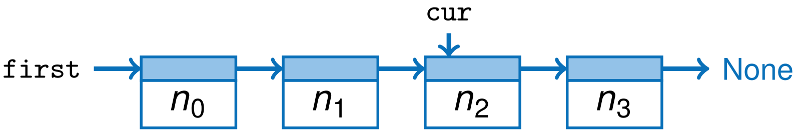

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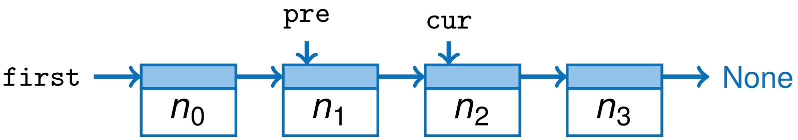

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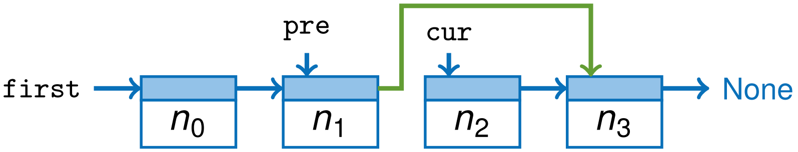

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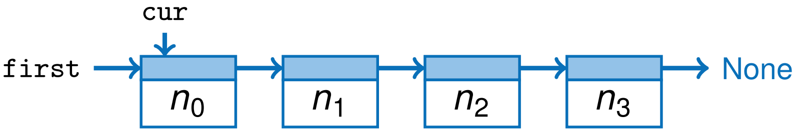

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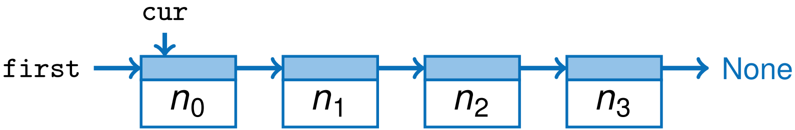

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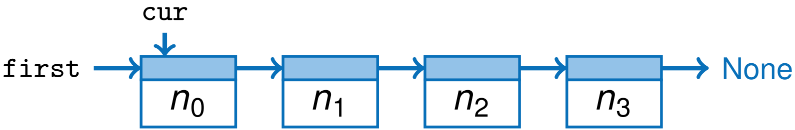

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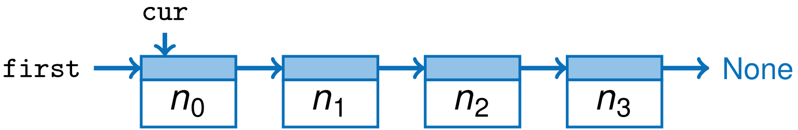

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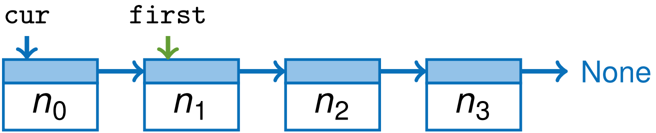

Removing a node `cur`: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```

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  - Iterating all nodes
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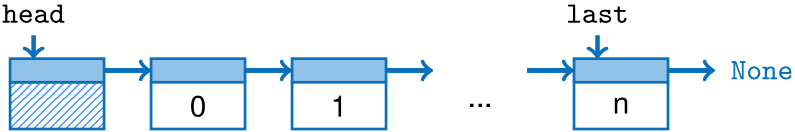
  - ...

Using a `head` node:

- Advantage:
  - Deleting the first node is no special case
- Disadvantage
  - We have to consider the first node at other operations
  - Iterating all nodes
  - Counting of all nodes
  - ...



```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```

```python
def append(self, value):
    ...

def insertAfter(self, cur, value):
    ...

def remove(self, cur):
    ...

def get(self, position):
    ...

def contains(self, value):
    ...
```

```java
/**
 * A singly linked list with data type int.
 */
public class LinkedList {

    private long itemCount;
    private Node head;
    private Node last;

    public LinkedList() {
        itemCount = 0;
        head = new Node();
        last = head;
    }
```

```
        public int size() {
        return itemCount;
    }

    public boolean isEmpty() {
        return (itemCount == 0);
    }

public void add (int data) { ... }
    public void insertAfter(Node cur, int data)
        { ... }
    public void remove(Node cur) { ... }
    public Node get(int position) { ... }
    public boolean contains( int data) { ... }
}
```

Head, last:

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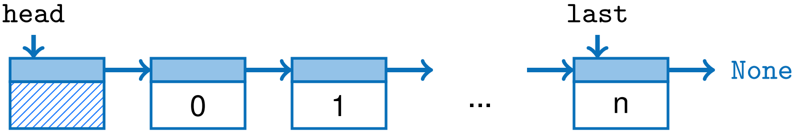

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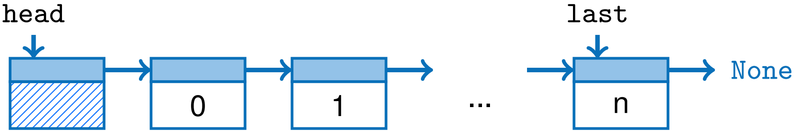

- `Head` points to the first node, `last` to the last node

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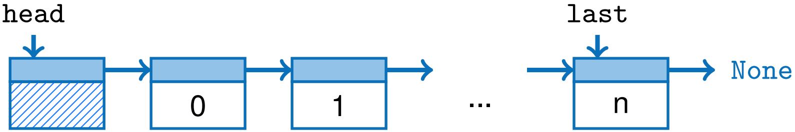

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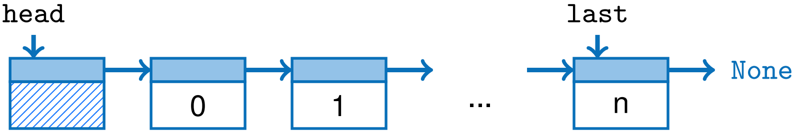

- `Head` points to the first node, `last` to the last node
- We can append elements to the end of the list in $O(1)$ through the `last` node
- We have to keep the pointer to `last` updated after all operations

Appending an element:

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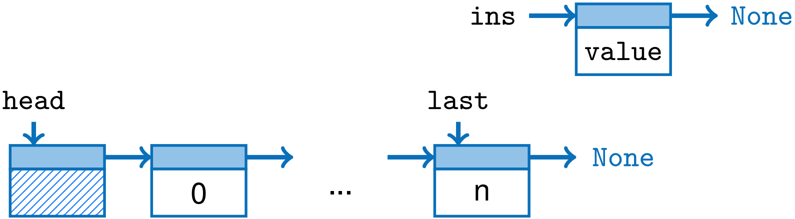

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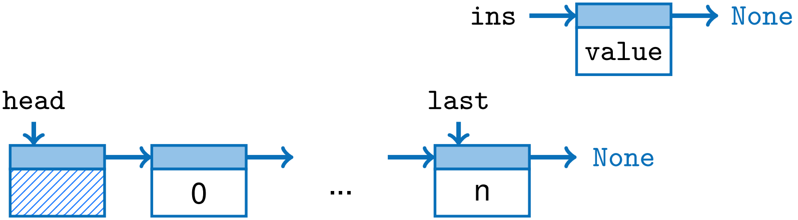


- ```
  def append(self, value):
      last.nextNode = Node(value)
      last = last.NextNode
      itemCount += 1
  ```
  ```

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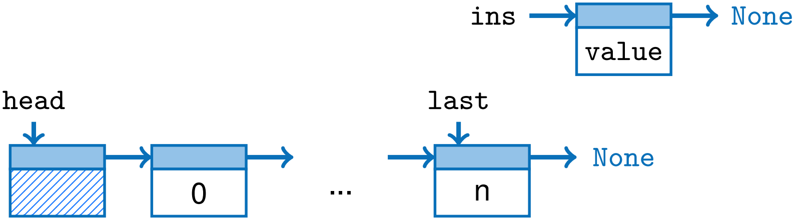


- ```
  def append(self, value):
      last.nextNode = Node(value)
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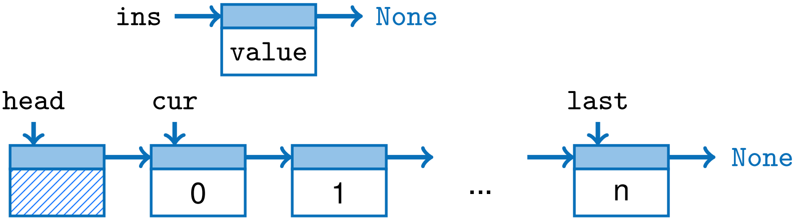
      itemCount += 1
  ```
- The pointer to `last` avoids the iteration of the whole list
  ```

Inserting after node `cur`:



Inserting after node `cur`:

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- The pointer to `head` is not modified

Inserting after node `cur`:

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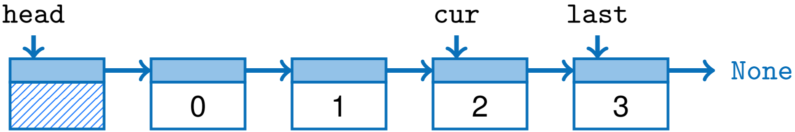
- ```
  def insertAfter(self, cur, value):
      if cur == last:
          # also update last node
          append(value)
      else:
          # last node is not modified
          cur.nextNode = Node(value, \
              cur.nextNode)
          itemCount += 1
  ```
  ```

Remove node `cur`:



Remove node `cur`:

Remove node `cur`:

- Searching the predecessor in $O(n)$

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- Searching the predecessor in $O(n)$

```
def remove(self, cur):
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
    itemCount -= 1

    if pre.nextNode == None:
        last = pre
```

Getting a reference to node at `pos`:

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- Iterate the entries of the list until at position ($O(n)$)

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- Iterate the entries of the list until at position ($O(n)$)

```
def get(self, pos):
    if pos < 0 or pos >= itemCount:
        return None

    cur = head
    for i in range(0, pos):
        cur = cur.nextNode

    return cur
```

Searching a `value`:

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- First element is head without an assigned value

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```
def contains(self, value):
    cur = head

    for i in range(0, itemCount):
        cur = cur.nextNode
        if cur.value == value:
            return true

    return false
```

Runtime:

Runtime:

- Singly linked list:

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  - `next` in $O(1)$

Runtime:

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  - `next` in $O(1)$
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  - `next` in $O(1)$
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  - `lookup` in $\Theta(n)$
- Better with doubly linked lists

Doubly linked list:

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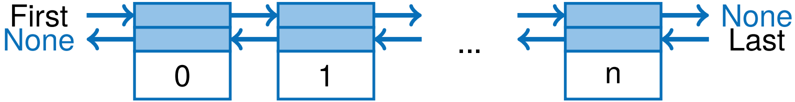

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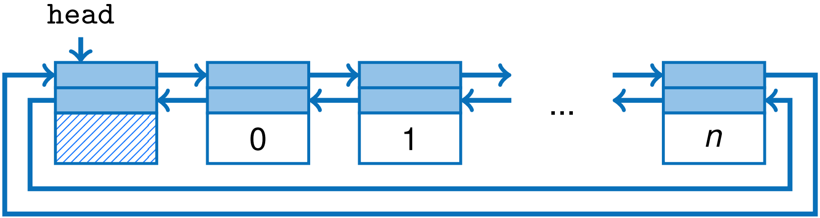

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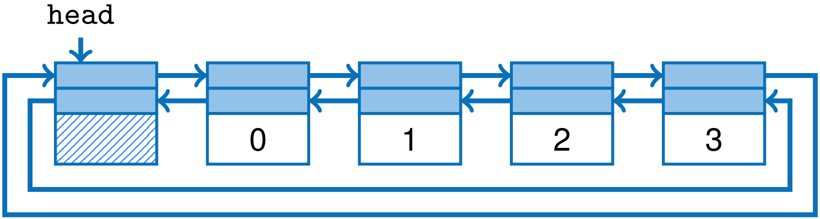
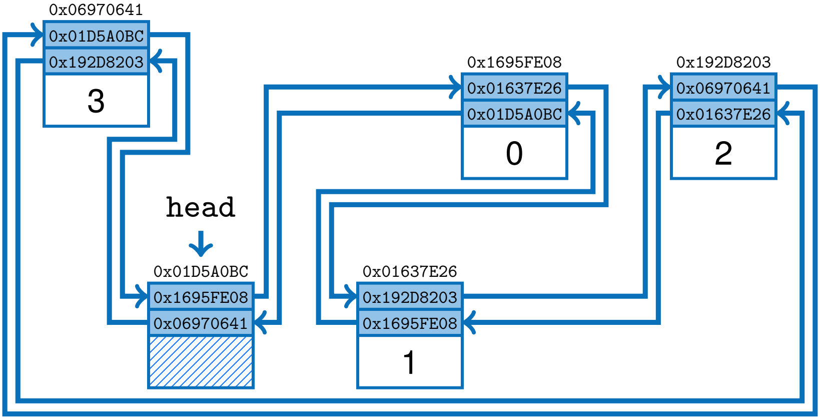
  - `insert` and `remove` in $O(1)$
    - a constant number of pointers needs to be modified
  - `lookup` in $\Theta(n)$
    - Even if the elements are sorted we can only retrieve them in $\Theta(n)$.
      Why?

Linked list in book:



Linked list in memory:



Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Runtime of a search tree:

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- `next` and `previous` in $O(1)$

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- `next` and `previous` in $O(1)$
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- `insert` and `remove` in $O(log\ n)$
  - We will see why
- `lookup` in $O(log\ n)$
  - The structure helps searching efficiently

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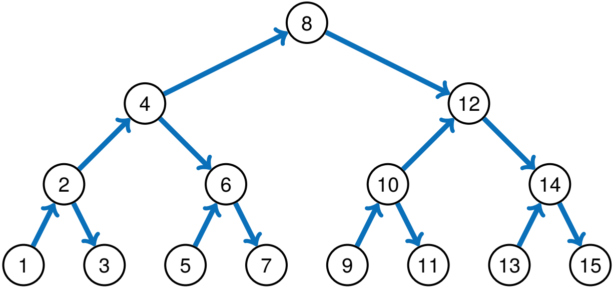

Abbildung: A binary search tree

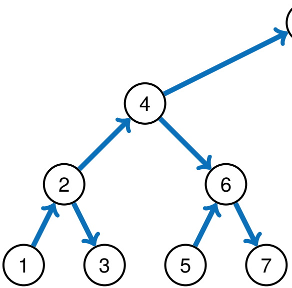

Abbildung: Another binary search tree

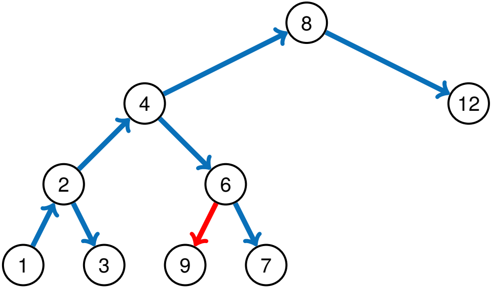

Abbildung: **Not** a binary search tree

Implementation:

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Implementation:

- For the heap we had all elements stored in an array
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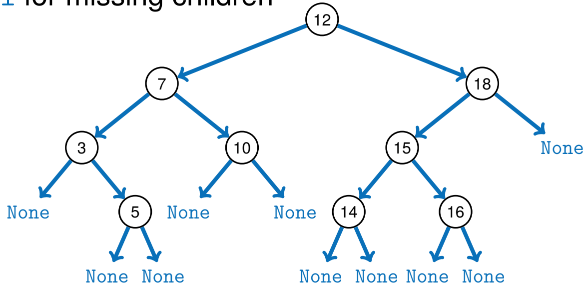

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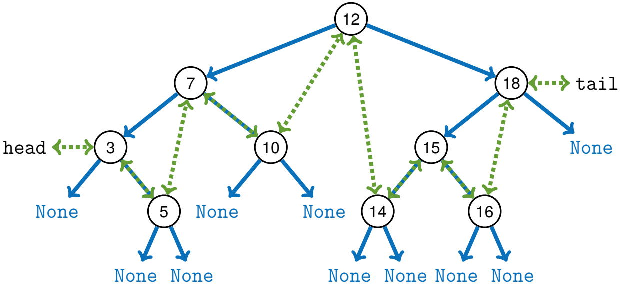

Abbildung: Binary search tree with links

Lookup:

Lookup:

- Definition:
  „ Search the element with the given key. If no element is found return the element with the next (bigger) key. “

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  - Go to the left / right until the child is `None` or the key is found

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  - If the key is not found return the next bigger one

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keys of left subtree < `node.key` < keys of right subtree

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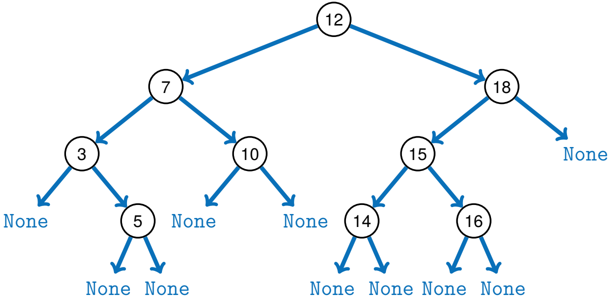

Examples:

Abbildung: Binary search tree with total order „<“

For each node applies the total order:

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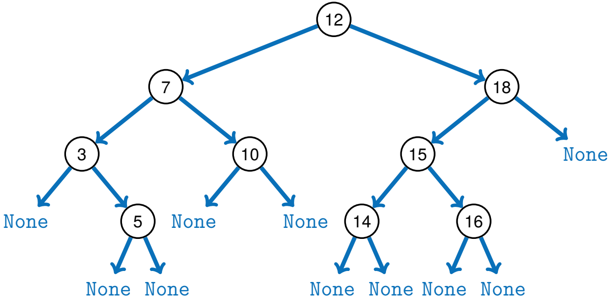

Abbildung: Binary search tree with total order „<“

Examples:

- `lookup(14)`

For each node applies the total order:

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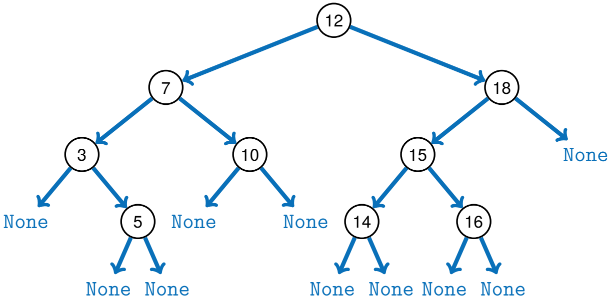

Abbildung: Binary search tree with total order „<“

Examples:

- `lookup(14)`
- `lookup(6)`

For each node applies the total order:

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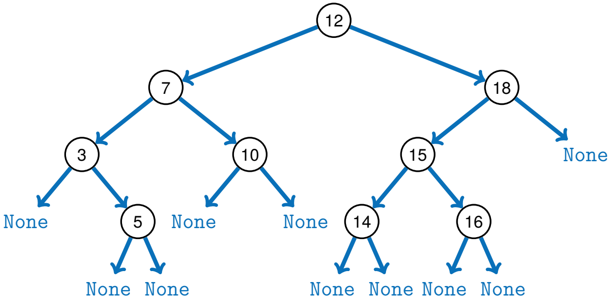

Abbildung: Binary search tree with total order „<“

Examples:

- `lookup(14)`
- `lookup(6)`
- `lookup(19)`

Insert:

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- We search for the key in our search tree

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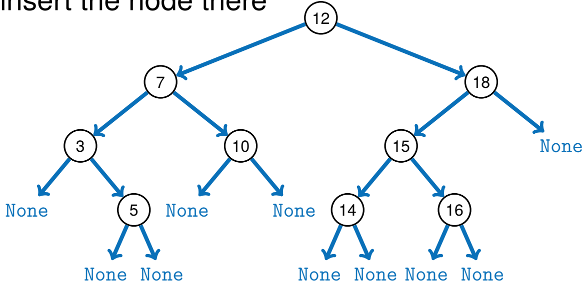

Abbildung: Binary search tree with total order „<“

Remove: Case 1: The node „5“ has no children

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- Find parent of node „5“ („6“)

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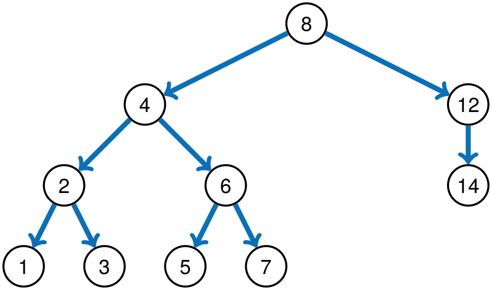

Abbildung: Binary search tree with total order „<“

Remove: Case 1: The node „5“ has no children

- Find parent of node „5“ („6“)
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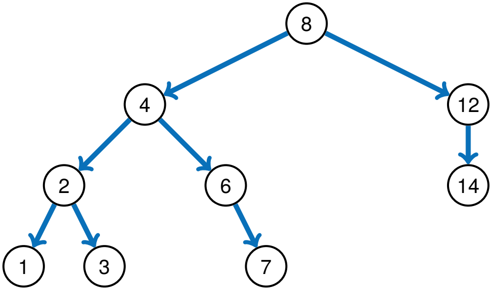

Abbildung: Binary search tree after deleting node „5“

Remove: Case 2: The node „12“ has one child

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- Find the child of node „12“ („14“)

Remove: Case 2: The node „12“ has one child

- Find the child of node „12“ („14“)
- Find the parent of node „12“ („8“)

Remove: Case 2: The node „12“ has one child

- Find the child of node „12“ („14“)
- Find the parent of node „12“ („8“)
- Set left / right child of node „8“ to „14“ depending on position of node „12“ (skip node „14“)

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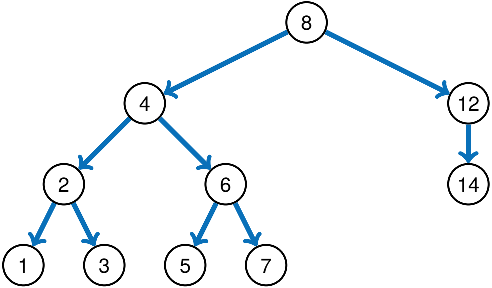

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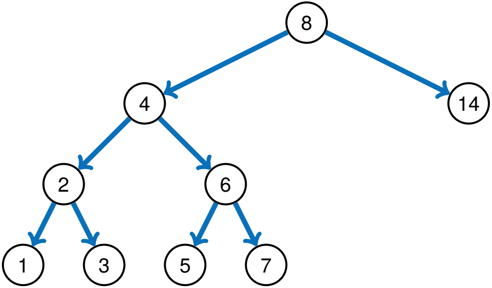

Abbildung: Binary search tree after delting node „12“

Remove: Case 3: The node „4“ has two children

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- Find the successor of node „4“ („5“)

Remove: Case 3: The node „4“ has two children

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- Replace the value of node „4“ with the value of node „5“

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- Find the successor of node „4“ („5“)
- Replace the value of node „4“ with the value of node „5“
- Delete node „5“ (the successor of node „4“) with remove-case 1 or 2

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Remove: Case 3: The node „4“ has two children

- Find the successor of node „4“ („5“)
- Replace the value of node „4“ with the value of node „5“
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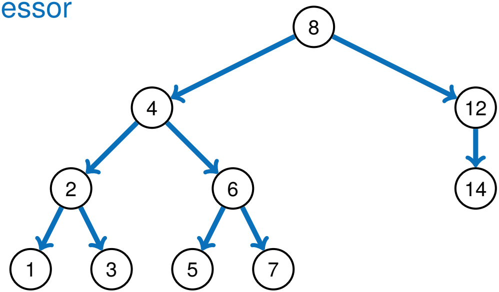
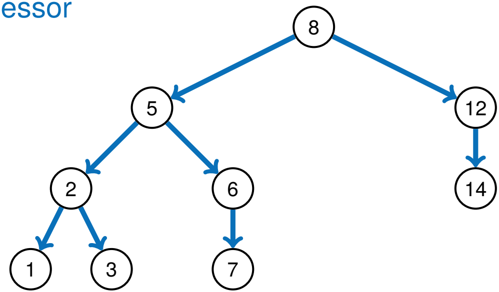


Remove: Case 3: The node „4“ has two children

- Find the successor of node „4“ („5“)
- Replace the value of node „4“ with the value of node „5“
- Delete node „5“ (the successor of node „4“) with remove-case 1 or 2
- There is no left node because we are deleting the predecessor



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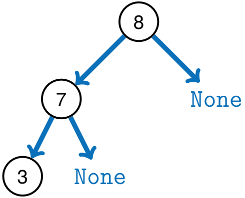

Abbildung: Degenerated binary tree $d = n$

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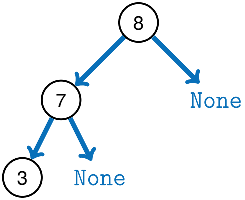

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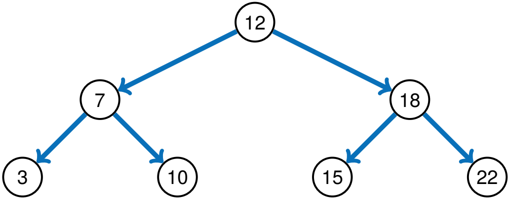

Abbildung: Complete binary tree $d = \log n$

- **General**

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Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
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- **Linked List**

  [Wik] Linked list
  https://en.wikipedia.org/wiki/Linked_list

- **Binary Search Tree**

  [Wik] Binary search tree
  https:
  //en.wikipedia.org/wiki/Binary_search_tree