

# Algorithms and Datastructures

## Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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## Balanced Trees

- Motivation

- AVL-Trees

- (a,b)-Trees

  - Introduction

  - Runtime Complexity

- Red-Black Trees

### Binary search tree:

- With `BinarySearchTree` we could perform an `lookup` or `insert` in  $O(d)$ , with  $d$  being the `depth` of the tree
- Best case:  $d = O(\log n)$ 
  - If the keys are inserted randomly
- Worst case:  $d = O(n)$ 
  - if the keys are inserted in ascending / descending order  
(20, 19, 18, ...)

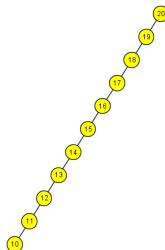


### Gnarley trees:

■ <http://people.ksp.sk/~kuko/bak>



**Figure:** Binary search tree with random insert [Gna]



**Figure:** Binary search tree with descending insert [Gna]

### Balanced trees:

- We do not want to rely on certain properties of our **key set**
- We explicitly want a depth of  $O(\log n)$
- We **rebalance** the tree from time to time

How do we get a depth of  $O(\log n)$ ?

- **AVL-Tree:**

- Binary tree with 2 children per node
- Balancing via “rotation”

- **(a,b)-Tree or B-Tree:**

- Node have between  $a$  and  $b$  children
- Balancing through **splitting** and **merging** nodes
- Used in data bases and file systems

- **Red-Black-Tree:**

- Binary tree with “black” and “red” nodes
- Balancing through “rotation” and “recoloring”
- Can be interpreted as (2, 4)-tree
- Used in C++ `std::map`, Java `SortedMap`

### AVL-Tree:

- Gregory Maximovich **Adelson-Velskii**, Yevgeniy Mikhailovlovich **Landis** (1963)
- Search tree with modified **insert** and **remove** operations while satisfying a **depth** condition
- Prevents degeneration of the search tree
- Height difference of left and right subtree is at maximum one
- With that the height of the search tree is always  $O(\log n)$
- We can perform all basic operations in  $O(\log n)$



Figure: Example of an AVL-Tree



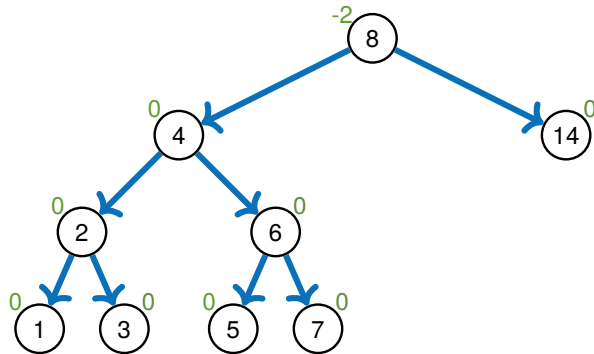


Figure: **Not** an AVL-Tree



Figure: Another example of an AVL-Tree

### Rotation:



Figure: Before rotating



Figure: After rotating

- Central operation of **rebalancing**
- After rotation to the right:
  - Subtree **A** is a layer higher and subtree **C** a layer lower
  - The parent child relations between nodes **x** and **y** have been swapped

### AVL-Tree:

- If a height difference of  $\pm 2$  occurs on an `insert` or `remove` operation the tree is rebalanced
- Many different cases of rebalancing
- **Example:** `insert` of 1,2,3,...
- <http://people.ksp.sk/~kuko/bak>



Figure: Inserting 1,..., 10 into an AVL-tree [Gna]

### Summary:

- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in  $O(\log n)$
- However not amortized update costs of  $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are  $(a,b)$ -trees

### (a,b)-Tree:

- Also known as **b-tree** (b for “balanced”)
- Used in data bases and file systems

### Idea:

- Save a varying number of elements per node
- So we have space for elements on an **insert** and balance operation

### (a,b)-Tree:

- All leaves have the same depth
- Each inner node has  $\geq a$  and  $\leq b$  nodes  
(Only the root node may have less nodes)



- Each node with  $n$  children is called “node of degree  $n$ ” and holds  $n - 1$  sorted elements
- Subtrees are located “between” the elements
- We require:  $a \geq 2$  and  $b \geq 2a - 1$

### (2,4)-Tree:



Figure: Example of an (2,4)-tree

- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)



### Not an (2,4)-Tree:



Figure: **Not** an (2,4)-tree

- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

### Searching an element: (lookup)

- The same algorithm as in [BinarySearchTree](#)
- Searching from the root downwards
- The keys at each node set the path

BST AVL tree B tree Red-black tree AA tree Skiplist Max Heap Min Heap Treap Scapegoat tree Splay tree

Display

Control

50 Insert Find Delete Next

☐ Pause ☐ Small 4

#Nodes: 77 #Keys: 77 = 56% full Height: 4

Text

Search Found.

### Inserting an element: (`insert`)

- Search the position to insert the key into
- This position will always be an leaf
- Insert the element into the tree
- **Attention:** Nodes can have one element too many (Degree  $b + 1$ )
- Then we **split** the node

### Inserting an element: (`insert`)



Figure: Splitting a node

- If the degree is higher than  $b + 1$  we split the node
  - This results in a node with  $\text{ceil}\left(\frac{b-1}{2}\right)$  elements, a element for the parent node, and a node with  $\text{floor}\left(\frac{b-1}{2}\right)$  elements
  - That's why we have the limit  $b \geq 2a - 1$

### Inserting an element: (`insert`)

- If the degree is higher than  $b + 1$  we split the node
- Now the parent node can be of a higher degree than  $b + 1$
- We `split` the parent nodes the same way
- If the node to split is the root we split it and create a new root node  
(The tree is now one level deeper)

### Removing an element: (`remove`)

- Search the element in  $O(\log n)$  time
- **Case 1:** The element is contained by a leaf, remove it
- **Case 2:** The element is contained by an inner node
  - Search the `successor` in the right subtree
  - The `successor` is always contained by a leaf
  - Replace the element with its `successor` and delete the `successor` from the leaf
- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We `rebalance` the tree

### Removing an element: (`remove`)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree
- **Case a:** If the left or right neighbour node has a degree greater than  $a$  we **borrow** one element from this node



Figure: Borrowing an element

### Removing an element: (remove)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree
- **Case b:** We **combine** the node with its right or left neighbour



Figure: Combining two nodes



### Removing an element: (remove)

- Now the parent node can be of degree  $a - 1$
- We combine parent nodes the same way
- If the root has only one child left we take the child as new root  
(The tree shrinks one level)

### Runtime complexity of **lookup**, **insert** and **remove**:

- All operations in  $O(d)$  with  $d$  being the depth of the tree
- Each node (except the root) has more than  $a$  children  
 $\Rightarrow n \geq a^{d-1}$  and  $d \leq 1 + \log_a n = O(\log_a n)$
- If we look closer:
  - **lookup** always takes  $\Theta(d)$
  - **insert** and **remove** often require only  $O(1)$  time
  - Only in the **worst case** we have to **split** or **combine** all nodes on a path up to the root
  - We want to analyse in detail
  - Therefore instead of  $b \geq 2a - 1$  we need  $b \geq 2a$ .
  - Here is a counter-example for (2,3)-trees, analysis of (2,4)-trees

### (2,3)-Tree:

- Before executing `delete(11)`



Figure: Normal (2,3)-Tree

### (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 1

### (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 2

### (2,3)-Tree:

- Executing `delete(11)`

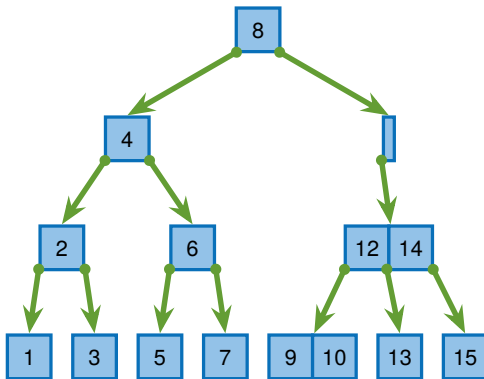


Figure: (2,3)-Tree - Delete step 3

### (2,3)-Tree:

- Executed `delete(11)`



Figure: (2,3)-Tree - Delete step 4

### (2,3)-Tree:

- Executing `insert(11)`

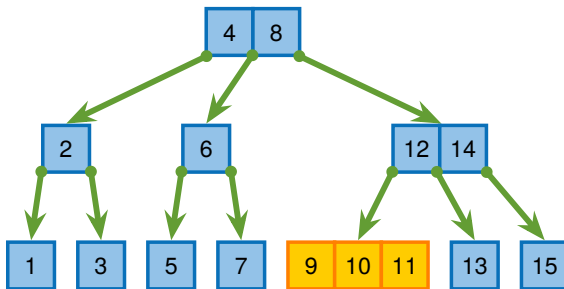


Figure: (2,3)-Tree - Insert step 1



### (2,3)-Tree:

- Executing `insert(11)`



Figure: (2,3)-Tree - Insert step 2

### (2,3)-Tree:

- Executing `insert(11)`

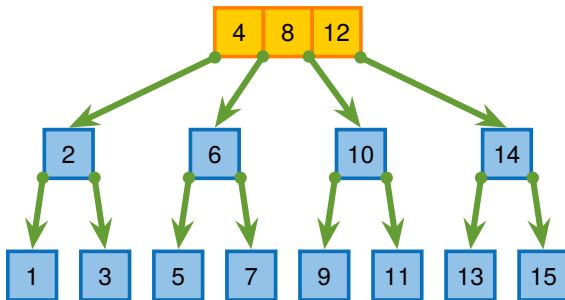


Figure: (2,3)-Tree - Insert step 3

### (2,3)-Tree:

- Executed `insert(11)`

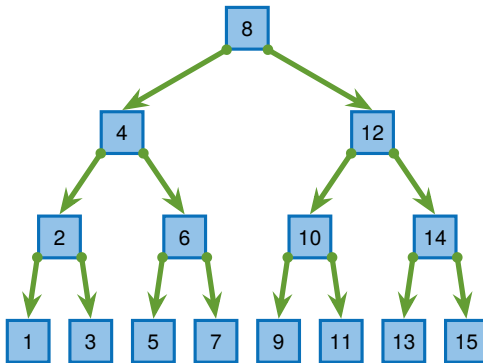


Figure: (2,3)-Tree - Insert step 4

### (2,3)-Tree:

- We are exactly where we started
- If  $b = 2a - 1$  then we can create a sequence of **insert** and **remove** operations where each operation costs  $O(\log n)$
- We need  $b \geq 2a$  instead of  $b \geq 2a - 1$

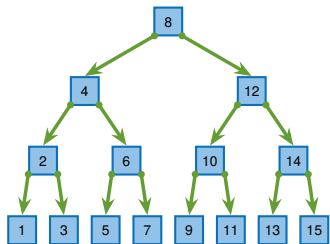


Figure: (2,3)-Tree

### (2,4)-Tree:

- If all nodes have **2 children** we have to **combine** the nodes up to the root on a **remove** operation
- If all nodes have **4 children** we have to **split** the nodes up to the root on a **insert** operation
- If all nodes have **3 children** it takes some time to reach one of the previous two states

⇒ **Nodes of degree 3 are harmless**

Neither an **insert** nor a **remove** operation trigger rebalancing operations

### (2,4)-Tree:

#### ■ Idea:

- After an expensive operation the tree is in a stable state
- It takes some time until the next expensive operation occurs
- Like with dynamic arrays:
  - **Reallocation** is expensive but it takes some time until the next expensive operation occurs
  - If we **overallocate** clever we have an amortized runtime of  $O(1)$

### Terminology:

- We analyze a sequence of  $n$  operations
- Let  $\phi_i$  be the potential of the tree after the  $i$ -th operation
- $n_3$  is the number of nodes with degree 3

### Example:

- Nodes of degree 3 are highlighted



Figure: Tree with potential  $\phi = 4$



### Terminology:

- Let  $c_i$  be the costs = runtime of the  $i$ -th operation
- We will show:
  - Each operation can maximally destroy one harmless node
  - For each further step, that incurs cost, the operation creates a further harmless node
- The costs for operation  $i$  are coupled to the difference of the potential levels

$$c_i \leq A \cdot \underbrace{(\phi_i - \phi_{i-1})} + B, \quad A > 0 \text{ and } B > A$$

Number of harmless (degree 3) nodes at operation  $i$ . Can be  $-1$ , but not smaller than  $-1$

- With that each operation has an amortized cost of  $O(1)$

**Case 1:**  $i$ -th operation is an **insert** operation on a full node



Figure: Splitting a node on **insert**

- Each splitted node creates a node of **degree 3**
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

**Case 1:**  $i$ -th operation is an **insert** operation on a full node

- Let  $m$  be the number of nodes split
- The potential rises by  $m$
- If the “stop-node” is of **degree 3** then the potential goes down by one

$$\begin{aligned}\phi_i &\geq \phi_{i-1} + m - 1 \\ \Rightarrow m &\leq \phi_i - \phi_{i-1} + 1\end{aligned}$$

Costs:  $c_i \leq A \cdot m + B$

$$\begin{aligned}\Rightarrow c_i &\leq A \cdot (\phi_i - \phi_{i-1} + 1) + B \\ c_i &\leq A \cdot (\phi_i - \phi_{i-1}) + \underbrace{A + B}_{B'}\end{aligned}$$

**Case 2:** *i*-th operation is an **remove** operation

■ **Case 2.1:** Inner node

- Searching the successor in a tree is  $O(d) = O(\log n)$
- Normally the tree is coupled with a doubly linked list  
⇒ We can find the successor in  $O(1)$



Figure: Tree with doubly linked list

**Case 2:** *i*-th operation is an **remove** operation

■ **Case 2.1:** Borrowing a node

- Creates no additional operations
- Case 2.1.1: Potential rises by one



Figure: Borrowing an element case 2.1.1

**Case 2:** *i*-th operation is an **remove** operation

■ **Case 2.1:** Borrowing a node

- Creates no additional operations
- Case 2.1.2: Potential lowers by one



Figure: Borrowing an element case 2.1.2

**Case 2:**  $i$ -th operation is an **remove** operation

■ **Case 2.2:** Merging a node

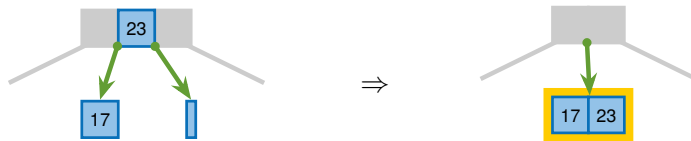


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree  $> 2$  or a degree 2 node, which can borrow from a neighbour
- The potential rises by  $m$
- If the “stop-node” is of **degree 2** then the potential eventually goes down by one
- Same costs as **insert**

### Lemma:

- We know:

$$c_i \leq A \cdot (\phi_i - \phi_{i-1}) + B, \quad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^n c_i = O(n)$$



### Proof:

$$\begin{aligned}\sum_{i=0}^n c_i &\leq \underbrace{A \cdot (\phi_1 - \phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\phi_2 - \phi_1) + B}_{\leq c_1} + \dots + \underbrace{A \cdot (\phi_n - \phi_{n-1}) + B}_{\leq c_n} \\ &= A \cdot (\phi_n - \phi_0) + B \cdot n && | \text{ telescope sum} \\ &= A \cdot \phi_n + B \cdot n && | \text{ we start with an empty tree} \\ &< A \cdot n + B \cdot n = O(n) && | \text{ number of degree 3 nodes} \\ &&& < \text{ number of nodes}\end{aligned}$$

### Red-Black Tree:

- Binary tree with **red** and **black** nodes
- Number of **black** nodes on path to leaves is equal
- Can be interpreted as **(2,4)-tree** (also named 2-3-4-tree)
- Each **(2,4)-tree**-node is a small red-black-tree with a **black** root node



Figure: Example of an red-black-tree [Gna]

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

### **Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Gnarley Trees

[Gna] **Gnarley Trees**

<https://people.ksp.sk/~kuko/gnarley-trees/>

## ■ AVL-Tree

[Wik] [AVL tree](#)

`https://en.wikipedia.org/wiki/AVL\_tree`

## ■ (a,b)-Tree

[Wika] [2-3-4 tree](#)

`https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934\_tree`

[Wikb] [\(a,b\)-tree](#)

`https://en.wikipedia.org/wiki/\(a,b\)-tree`

## ■ Red-Black-Tree

[Wik] [Red-black tree](https://en.wikipedia.org/wiki/Red%E2%80%93black_tree)

`https://en.wikipedia.org/wiki/Red%E2%80%93black\_tree`