

Algorithms and Datastructures

Runtime analysis Minsort / Heapsort, Induction

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Algorithms and Datastructures, October 2018

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

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Basic Operations

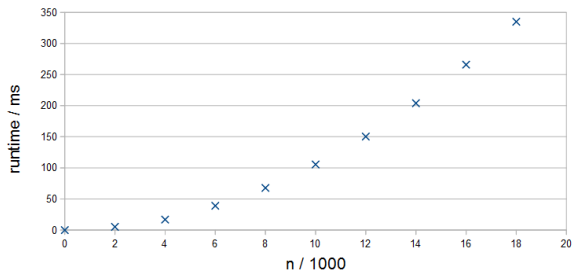
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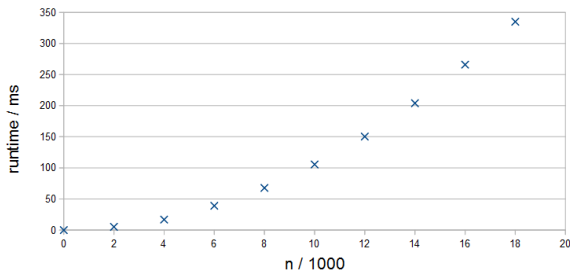


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 - What is running in the background
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- **Abstraction 1:** analyze the number of basic operations, rather than analyzing the runtime

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Incomplete list of basic operations:

- Arithmetic operation, for example: $a + b$
- Assignment of variables, for example: $x = y$
- Function call, for example: *minsort(lst)*

Intuitive:

lines of code

Better:

lines of machine
code

Best:

process cycles

Important:

The actual runtime has to be roughly proportional to the number of operations.

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- **Abstraction 2:** we calculate the upper (lower) bound, rather than exactly counting the number of operations

Reason: runtime is approximated by number of basic operations, but we can still infer:

- Upper bound
- Lower bound

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- **Abstraction 2:** we calculate the upper (lower) bound, rather than exactly counting the number of operations

Reason: runtime is approximated by number of basic operations, but we can still infer:

- Upper bound
 - Lower bound
- **Basic Assumption:**
 - n is size of the input data (i.e. array)
 - $T(n)$ number of operations for input n

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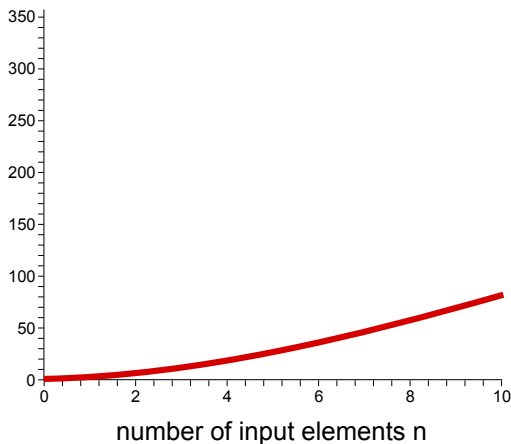
- **Observation:** the number of operations depends only on the size n of the array and not on the content!
- **Claim:** there are constants C_1 and C_2 such that:

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- This is called “quadratic runtime” (due to n^2)

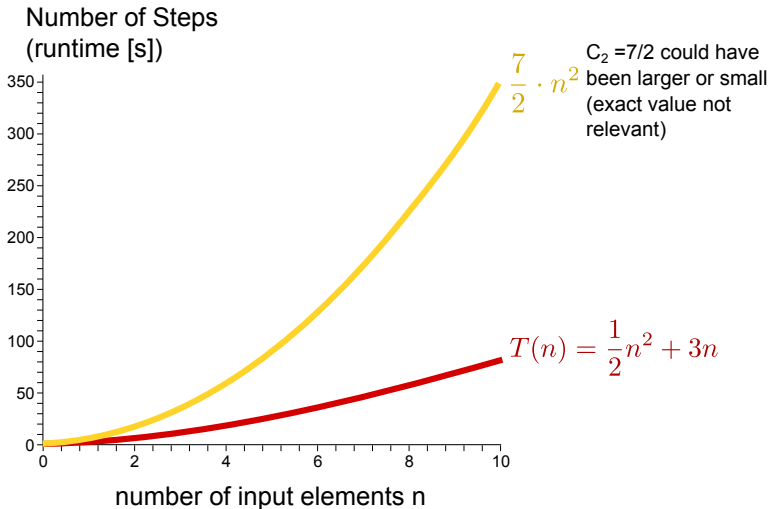
Runtime Example

Number of Steps
(runtime [s])

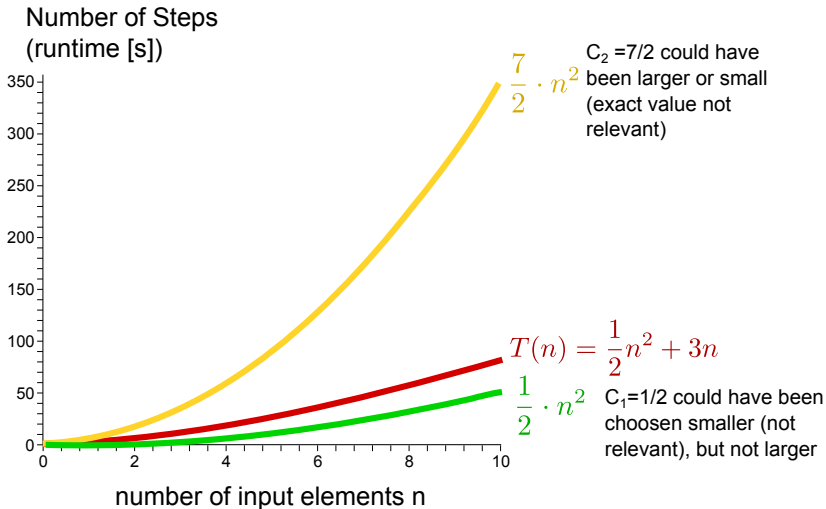


$$T(n) = \frac{1}{2}n^2 + 3n$$

Runtime Example



Runtime Example



We declare:

- Runtime of operations: $T(n)$
- Number of Elements: n
- Constants: C_1 (lower bound), C_2 (upper bound)
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$
- Number of operations in round i : T_i



Figure: *Minsort* at iteration $i = 4$. We have to check $n - 3$ elements

Runtime for each
iteration:

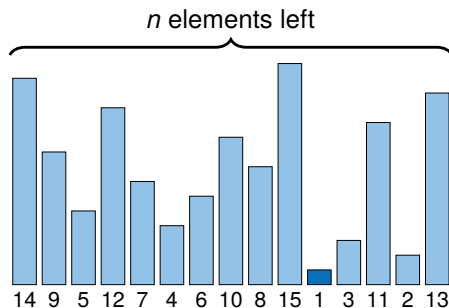


Figure: *Minsort* with start data

Runtime for each
iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

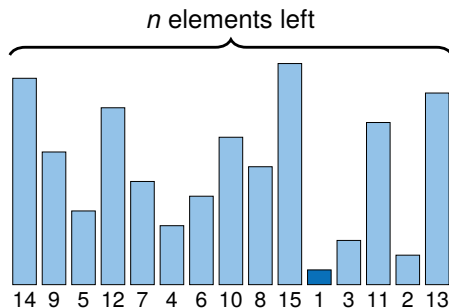


Figure: *Minsort* at iteration $i = 1$



Runtime for each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

Figure: *Minsort* at iteration $i = 2$



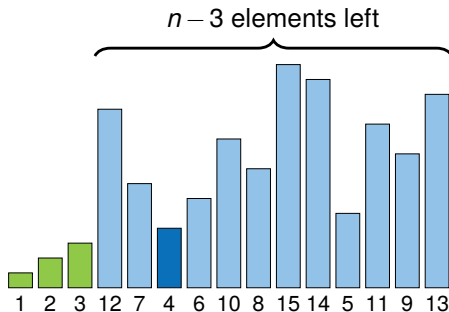
Runtime for each iteration:

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Figure: *Minsort* at iteration $i = 3$



Runtime for each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

Figure: *Minsort* at iteration $i = 4$



Figure: *Minsort* at iteration i

Runtime for each iteration:

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\vdots

$$T_{n-1} \leq C'_2 \cdot 2$$

$$T_n \leq C'_2 \cdot 1$$

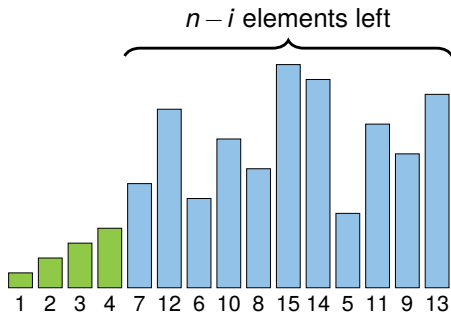


Figure: Minsort at iteration

Runtime for each iteration:

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$$T(n) = C'_2 \cdot (T_1 + \dots + T_n) \leq \sum_{i=1}^n (C'_2 \cdot i)$$

Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
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        if minimum != i:  
            elements[i], elements[minimum] = \  
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        runtime  
  
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Diagram illustrating the runtime analysis of the Minsort algorithm:

- The inner loop (for j in range(i+1, len(elements))) is highlighted in a light blue box.
- A bracket indicates that the inner loop runs $n-i-1$ times for each iteration of the outer loop.
- The inner loop body (if elements[j] < elements[minimum]: minimum = j) is highlighted in a darker blue box.
- A bracket indicates that the inner loop body runs const. runtime times for each iteration of the inner loop.
- A large bracket on the right indicates that the entire inner loop structure runs $n-1$ times.

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2$$

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Diagram illustrating the runtime analysis of the Minsort algorithm:

- The inner loop (for j in range(i+1, len(elements))) is annotated with "const. runtime".
- The inner loop is repeated $n-i-1$ times for each iteration of the outer loop.
- The outer loop (for i in range(0, len(elements)-1)) is repeated $n-1$ times.

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2$$

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Remark: C'_2 is cost of comparison \Rightarrow assumed constant

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Excursion - Small Gauss Formula

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper bound there exists a C_1 . Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i)$$

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How do we get to n^2 ?

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Summarized:

$$\frac{C'_1}{4} \cdot n^2 \leq T(n) \leq C'_2 \cdot n^2$$

Quadratic runtime proven:

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- **Quadratic runtime = “big” problems unsolvable**

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Formal:

- Let $T(n)$ be the runtime for the *Heapsort* algorithm with n elements
- On the next pages we will proof $T(n) \leq C \cdot n \log_2 n$

Depth of a binary tree:

- **Depth d :** longest path through the tree
- Complete binary tree has $n = 2^d - 1$ nodes
- Example: $d = 4$
 $\Rightarrow n = 2^4 - 1 = 15$

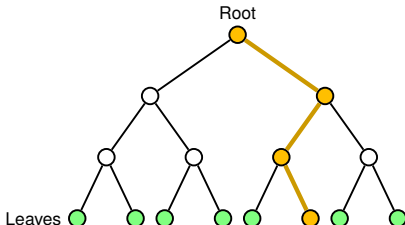


Figure: Binary tree with 15 nodes

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 - 2 **Induction step:** we show that the assumption is valid for all n (normally one step forward: $n = n + 1, A(1), \dots, A(n)$).

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- We show induction in two steps:
 - 1 **Induction basis:** we show that our assumption is valid for one value (for example: $n = 1, A(1)$).
 - 2 **Induction step:** we show that the assumption is valid for all n (normally one step forward: $n = n + 1, A(1), \dots, A(n)$).
- If both has been proven, then $A(n)$ holds for all natural numbers n by **induction**

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

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Figure: Tree of depth 1 has 1 node

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\Rightarrow correct ✓

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Induction - Example 1



Number of nodes $n(d)$ in a binary tree with depth d :

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Induction - Example 1



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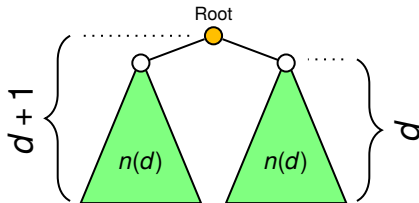


Figure: binary tree with subtrees

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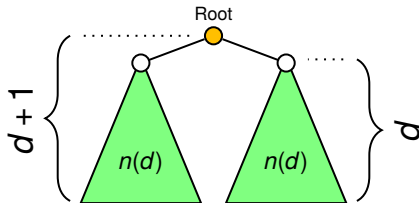


$$n(d+1) = 2 \cdot n(d) + 1$$

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$$\begin{aligned} n(d+1) &= 2 \cdot n(d) + 1 \\ &= 2 \cdot (2^d - 1) + 1 \end{aligned}$$

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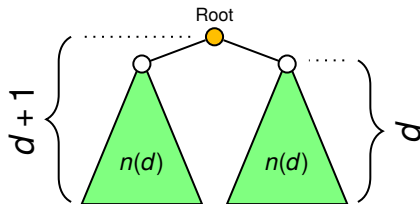


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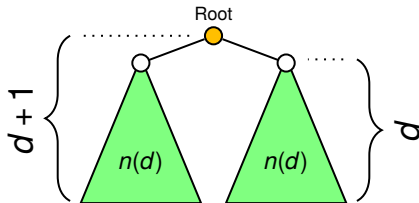


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⇒ **By induction:** $n(d) = 2^d - 1 \quad \forall n \in \mathbb{N} \quad \square$

Runtime Example

Minsort

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Heapsort

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Heapsort has the following steps:

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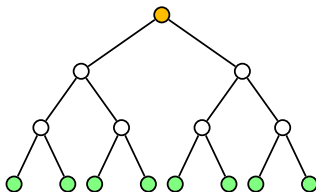
Heapsort has the following steps:

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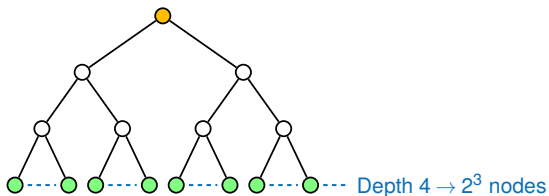
Heapsort has the following steps:

- **Initially:** heapify list of n elements
- **Then:** until all n elements are sorted
 - Remove root (=minimum element)
 - Move last leaf to root position
 - Repair heap by sifting

Runtime of heapify depends on depth d :



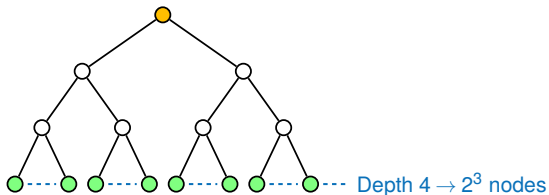
Runtime of heapify depends on depth d :



Runtime of heapify with depth of d :

- No costs at depth d with 2^{d-1} (or less) nodes

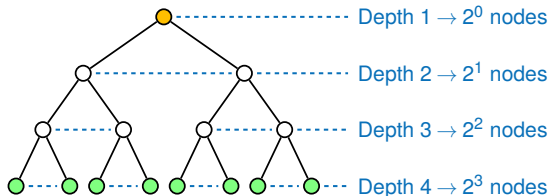
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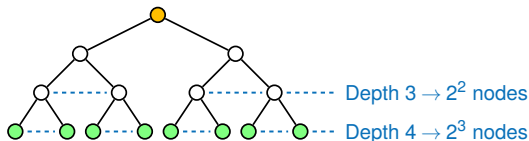
- No costs at depth d with 2^{d-1} (or less) nodes
- The cost for sifting with depth 1 is at most $1C$ per node
- In general: Sifting costs are linear with path length **and** number of nodes

Heapify total runtime:



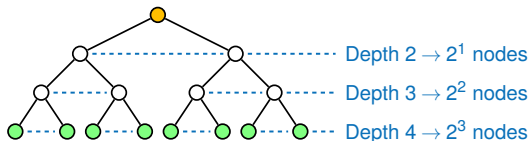
Depth	Nodes	Path length	Costs per node
d	2^{d-1}	0	$\leq C \cdot 0$

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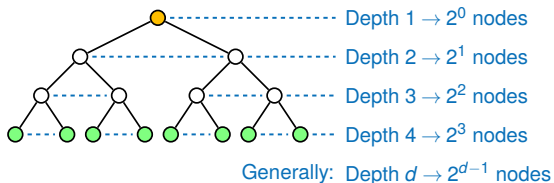
Depth	Nodes	Path length	Costs per node
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d	2^{d-1}	0	$\leq C \cdot 0$
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$d-2$	2^{d-3}	2	$\leq C \cdot 2$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$

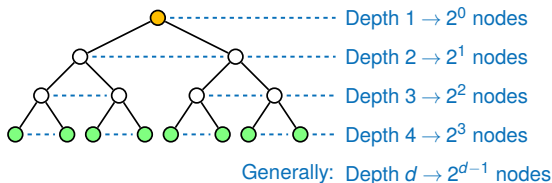
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d	2^{d-1}	0	$\leq C \cdot 0$
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$d-2$	2^{d-3}	2	$\leq C \cdot 2$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$

In total:
$$T(d) \leq \sum_{i=1}^d \left(C \cdot (i-1) \cdot 2^{d-i} \right)$$

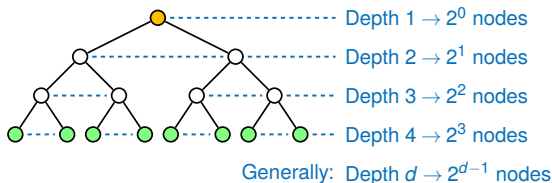
Heapify total runtime:



Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	Standard Equation
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

In total:
$$T(d) \leq \sum_{i=1}^d \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^d \left(C \cdot i \cdot 2^{d-i} \right)$$

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d	2^{d-1}	0	$\leq C \cdot 0$	$\leq C \cdot 1$
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- **However:** We want costs in relation to n

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Figure: Partial binary tree

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- Equation multiplied by 2^2
 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$

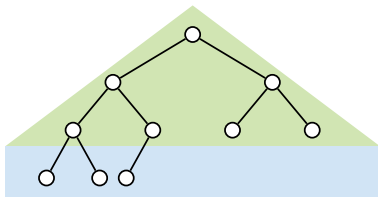


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- At least 1 node in layer d
- Equation multiplied by 2^2
 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$
- Cost for heapify:
 $\Rightarrow T(n) \leq C \cdot 4 \cdot n$



Figure: Partial binary tree

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Introduction to Induction

Logarithms

- We want to proof (induction assumption):

$$\underbrace{\sum_{i=1}^d (i \cdot 2^{d-i})}_{A(d)} \leq \underbrace{2^{d+1}}_{B(d)}$$

- We denote the left side with A , the right side with B

- **Induction basis:** $d := 1$:

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$$\sum_{i=1}^1 (i \cdot 2^{1-i}) \leq 2^{1+1}$$

$$2^0 \leq 2^2 \quad \checkmark$$

Induction step: ($d := d + 1$):

- **Idea:** Write down right-hand formula and try to get $A(d)$ and $B(d)$ out of it

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$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

\vdots

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$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

■ **Problem:** does not work but claim still holds

Working proof:

- Show a **little bit stronger** claim

$$\sum_{i=1}^d (i \cdot 2^{d-i}) \leq 2^{d+1} - d - 2 \leq 2^{d+1}$$

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- **Advantage:** results in a stronger induction assumption
 \Rightarrow **exercise**

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Runtime of the other operations:

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 - **Hence:** $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$

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- **Recall:** the depth and number of elements is decreasing
 - **Hence:** $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
 - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \quad (\text{holds for } n > 2)$$

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- Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
 - Upper bound: $C_2 \cdot n \log_2 n \geq T(n)$ (for $n \geq 2$)
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 - $\Rightarrow C_1$ and C_2 are constant

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Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient $\frac{1}{\log_b a}$

Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602 \dots \cdot 3.322 \dots = 2 \checkmark$
- $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3 \checkmark$

Runtime of $n \log_2 n$:

- Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

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 - $n = 2^{30}$ (1 billion numbers = 4 GB)
 - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{30} \cdot 30 = 32 \text{ s}$

Runtime of $n \log_2 n$:

- Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- $2 \times$ elements \Rightarrow only slightly larger than $2 \times$ runtime
 - $C = 1$ ns (1 simple instruction ≈ 1 ns)
 - $n = 2^{20}$ (1 million numbers = 4 MB with 4 B/number)
 - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{20} \cdot 20 = 21.0 \text{ ms}$
 - $n = 2^{30}$ (1 billion numbers = 4 GB)
 - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{30} \cdot 30 = 32 \text{ s}$
- **Runtime $n \log_2 n$ is nearly as good as linear!**

■ Course literature

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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■ Mathematical Induction

[Wik] [Mathematical induction](https://en.wikipedia.org/wiki/Mathematical_induction)

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