

# Algorithms and Datastructures

## Graphs, Depth-/Breadth-first Search, Graph-Connectivity

Albert-Ludwigs-Universität Freiburg



**UNI  
FREIBURG**

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science  
Algorithms and Datastructures, January 2017

## Graphs

- Introduction

- Implementation

- Application example



## Graphs - Overview:

### Graphs - Overview:

- Besides arrays, lists and trees the most common datastructure  
(Trees are a special type of graph)

### Graphs - Overview:

- Besides arrays, lists and trees the most common datastructure  
(Trees are a special type of graph)
- Representation of graphs in the computer

### Graphs - Overview:

- Besides arrays, lists and trees the most common datastructure  
(Trees are a special type of graph)
- Representation of graphs in the computer
- Breadth first search (BFS)

### Graphs - Overview:

- Besides arrays, lists and trees the most common datastructure  
(Trees are a special type of graph)
- Representation of graphs in the computer
- Breadth first search (BFS)
- Depth first search (DFS)

### Graphs - Overview:

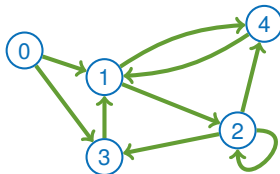
- Besides arrays, lists and trees the most common datastructure  
(Trees are a special type of graph)
- Representation of graphs in the computer
- Breadth first search (BFS)
- Depth first search (DFS)
- Connected components of a graph





## Terminology:

### Terminology:



### Terminology:



- Each Graph  $G = (V, E)$  consists of:

### Terminology:



- Each Graph  $G = (V, E)$  consists of:
  - A set of vertices (nodes)  $V = \{v_1, v_2, \dots\}$

## Terminology:



- Each Graph  $G = (V, E)$  consists of:
  - A set of vertices (nodes)  $V = \{v_1, v_2, \dots\}$
  - A set of edges (arcs)  $E = \{e_1, e_2, \dots\}$

## Terminology:



- Each Graph  $G = (V, E)$  consists of:
  - A set of vertices (nodes)  $V = \{v_1, v_2, \dots\}$
  - A set of edges (arcs)  $E = \{e_1, e_2, \dots\}$
- Each edge connects two vertices ( $u, v \in V$ )

### Terminology:



- Each Graph  $G = (V, E)$  consists of:
  - A set of vertices (nodes)  $V = \{v_1, v_2, \dots\}$
  - A set of edges (arcs)  $E = \{e_1, e_2, \dots\}$
- Each edge connects two vertices ( $u, v \in V$ )
  - Undirected edge:  $e = \{u, v\}$  (set)

## Terminology:



- Each Graph  $G = (V, E)$  consists of:
  - A set of vertices (nodes)  $V = \{v_1, v_2, \dots\}$
  - A set of edges (arcs)  $E = \{e_1, e_2, \dots\}$
- Each edge connects two vertices ( $u, v \in V$ )
  - Undirected edge:  $e = \{u, v\}$  (set)
  - Directed edge:  $e = (u, v)$  (tuple)



## Terminology:

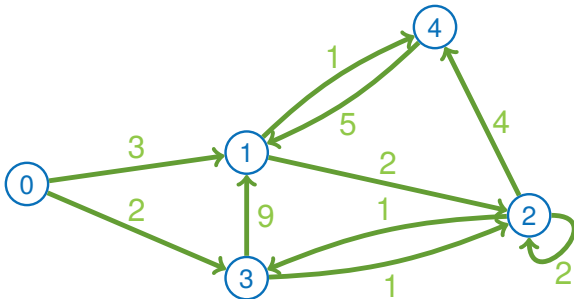


- Each Graph  $G = (V, E)$  consists of:
  - A set of vertices (nodes)  $V = \{v_1, v_2, \dots\}$
  - A set of edges (arcs)  $E = \{e_1, e_2, \dots\}$
- Each edge connects two vertices ( $u, v \in V$ )
  - Undirected edge:  $e = \{u, v\}$  (set)
  - Directed edge:  $e = (u, v)$  (tuple)
- Self-loops are also possible:  $e = (u, u)$  or  $e = \{u, u\}$



## Weighted graph:

### Weighted graph:



### Weighted graph:



- Each edge is marked with a real number named **weight**

### Weighted graph:



- Each edge is marked with a real number named **weight**
- The **weight** is also named **length** or **cost** of the edge depending on the application

### **Example:** Road network

### **Example:** Road network

- Intersections:  
vertices

### **Example:** Road network

- Intersections:  
vertices
- Roads: edges



### **Example:** Road network

- Intersections:  
vertices
- Roads: edges
- Travel time:  
costs of the edges

### Example: Road network

- Intersections: **vertices**
- Roads: **edges**
- Travel time: **costs of the edges**

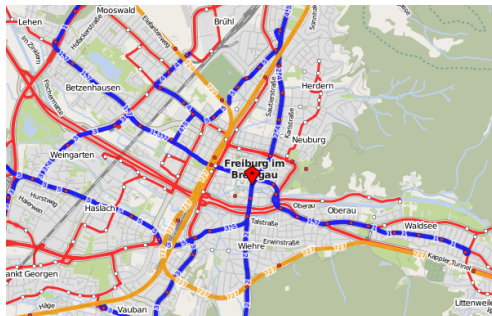


Figure: Map of Freiburg © OpenStreetMap

## Graphs

Introduction

**Implementation**

Application example



## How to represent this graph computationally?

### How to represent this graph computationally?

- 1 Adjacency matrix with space consumption  $\Theta(|V|^2)$

### How to represent this graph computationally?

- 1 Adjacency matrix with space consumption  $\Theta(|V|^2)$



Figure: Weighted graph with  
 $|V| = 4$ ,  $|E| = 6$

### How to represent this graph computationally?

- 1 Adjacency matrix with space consumption  $\Theta(|V|^2)$



Figure: Weighted graph with  
 $|V| = 4$ ,  $|E| = 6$

		end-vertex			
		0	1	2	3
start-vertex	0		2		3
	1			9	
	2				-1
	3		7	-2	

Figure: Adjacency matrix



## How to represent this graph computationally?





## How to represent this graph computationally?

2 Adjacency list / fields with space consumption  $\Theta(|V| + |E|)$

### How to represent this graph computationally?

- 2 Adjacency list / fields with space consumption  $\Theta(|V| + |E|)$   
Each list item stores the target vertice and the cost of the edge

### How to represent this graph computationally?

- 2 Adjacency list / fields with space consumption  $\Theta(|V| + |E|)$   
Each list item stores the **target vertex** and the **cost** of the edge



Figure: Weighted graph with  
 $|V| = 4, |E| = 6$

## How to represent this graph computationally?

- 2 Adjacency list / fields with space consumption  $\Theta(|V| + |E|)$   
 Each list item stores the **target vertex** and the **cost** of the edge



Figure: Weighted graph with  
 $|V| = 4, |E| = 6$

start-vertex	0	1, 2	3, 3
	1	2, 9	
	2	3, -1	
	3	1, 7	2, -2

Figure: Adjacency list



## Graph: Arrangement

### Graph: Arrangement

- Graph is fully defined through the [adjacency matrix / list](#)

### Graph: Arrangement

- Graph is fully defined through the [adjacency matrix / list](#)
- The arrangement is not relevant for visualisation of the graph

### Graph: Arrangement

- Graph is fully defined through the **adjacency matrix / list**
- The arrangement is not relevant for visualisation of the graph



Figure: Weighted graph with  
 $|V| = 4$ ,  $|E| = 6$



### Graph: Arrangement

- Graph is fully defined through the **adjacency matrix / list**
- The arrangement is not relevant for visualisation of the graph



Figure: Weighted graph with  
 $|V| = 4$ ,  $|E| = 6$



Figure: Same graph ordered by number - outer planar graph

```
class Graph:
    def __init__(self):
        self.vertices = []
        self.edges = []

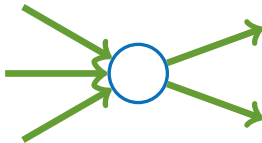
    def addVertice(self, vert):
        self.vertices.append(vert)

    def addEdge(self, fromVert, toVert, cost):
        self.edges.append( \
            (fromVert, toVert, cost))

    ...
```

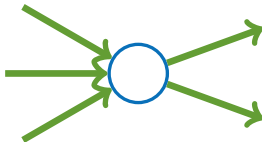
**Degree of a vertex:** Directed graph:  $G = (V, E)$

**Degree of a vertex:** Directed graph:  $G = (V, E)$



**Figure:** Vertex with in- / outdegree of 3 / 2

**Degree of a vertex:** Directed graph:  $G = (V, E)$



**Figure:** Vertex with in- / outdegree of 3 / 2

- **Indegree** of a vertex  $u$  is the number of **edge head ends** adjacent to the vertex

$$\deg^+(u) = |\{(v, u) : (v, u) \in E\}|$$

**Degree of a vertex:** Directed graph:  $G = (V, E)$

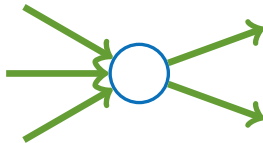


Figure: Vertex with in- / outdegree of 3 / 2

- **Indegree** of a vertex  $u$  is the number of **edge head ends** adjacent to the vertex

$$\deg^+(u) = |\{(v, u) : (v, u) \in E\}|$$

- **Outdegree** of a vertex  $u$  is the number of **edge tail ends** adjacent to the vertex

$$\deg^-(u) = |\{(u, v) : (u, v) \in E\}|$$

**Degree of a vertex:** Undirected graph:  $G = (V, E)$

**Degree of a vertex:** Undirected graph:  $G = (V, E)$



Figure: Vertex with degree of 4



**Degree of a vertex:** Undirected graph:  $G = (V, E)$



Figure: Vertex with degree of 4

- **Degree** of a vertex  $u$  is the number of **vertices** adjacent to the vertex

$$\deg(u) = |\{\{v, u\} : \{v, u\} \in E\}|$$



**Paths in a graph:**  $G = (V, E)$

**Paths in a graph:**  $G = (V, E)$

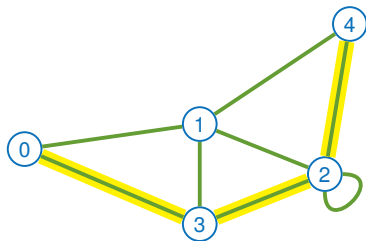


Figure: Undirected path of length 3  
 $P = (0, 3, 2, 4)$

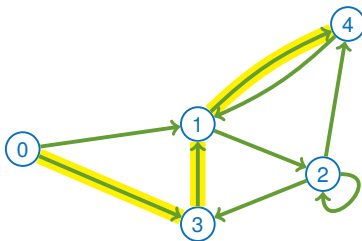


Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

**Paths in a graph:**  $G = (V, E)$



Figure: Undirected path of length 3  
 $P = (0, 3, 2, 4)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

- A path of  $G$  is a sequence of edges  $u_1, u_2, \dots, u_i \in V$  with

### Paths in a graph: $G = (V, E)$



Figure: Undirected path of length 3  
 $P = (0, 3, 2, 4)$

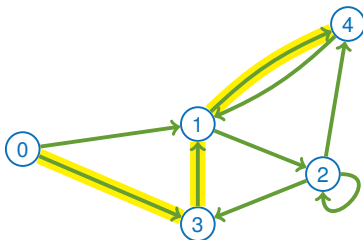


Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

- A path of  $G$  is a sequence of edges  $u_1, u_2, \dots, u_i \in V$  with
  - Undirected graph:  $\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{i-1}, u_i\} \in E$
  - Directed graph:  $(u_1, u_2), (u_2, u_3), \dots, (u_{i-1}, u_i) \in E$

**Paths in a graph:**  $G = (V, E)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

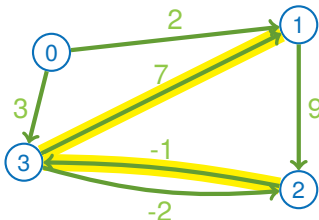


Figure: Weighted path with cost 6  
 $P = (2, 3, 1)$

**Paths in a graph:**  $G = (V, E)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

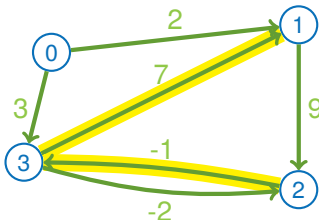


Figure: Weighted path with cost 6  
 $P = (2, 3, 1)$

**Paths in a graph:**  $G = (V, E)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

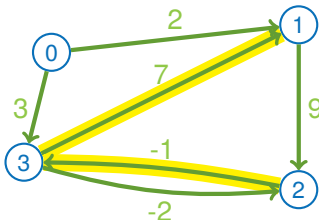


Figure: Weighted path with cost 6  
 $P = (2, 3, 1)$



**Paths in a graph:**  $G = (V, E)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$



Figure: Weighted path with cost 6  
 $P = (2, 3, 1)$

- The length of a path is: (also costs of a path)

**Paths in a graph:**  $G = (V, E)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$



Figure: Weighted path with cost 6  
 $P = (2, 3, 1)$

- The **length of a path** is: (also costs of a path)
  - Without weights: **number of edges** taken

**Paths in a graph:**  $G = (V, E)$



Figure: Directed path of length 3  
 $P = (0, 3, 1, 4)$

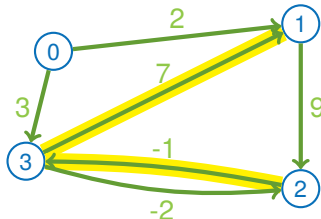


Figure: Weighted path with cost 6  
 $P = (2, 3, 1)$

- The **length of a path** is: (also costs of a path)
  - Without weights: **number of edges** taken
  - With weights: **sum of weights of edges** taken



**Shortest path in a graph:**  $G = (V, E)$

**Shortest path in a graph:**  $G = (V, E)$



Figure: Shortest path from 0 to 2 with cost / distance  $d(0,2) = ?$

### Shortest path in a graph: $G = (V, E)$



Figure: Shortest path from 0 to 2 with cost / distance  $d(0,2) = ?$

- The **shortest path** between two vertices  $u, v$  is the path  $P = (u, \dots, v)$  with the shortest length  $d(u, v)$  or lowest costs

### Shortest path in a graph: $G = (V, E)$



Figure: Shortest path from 0 to 2 with cost / distance  $d(0, 2) = 6$   
 $P = (0, 1, 4, 3, 2)$

- The shortest path between two vertices  $u, v$  is the path  $P = (u, \dots, v)$  with the shortest length  $d(u, v)$  or lowest costs



**Diameter of a graph:**  $G = (V, E)$



**Diameter of a graph:**  $G = (V, E)$

$$d = \max_{u,v \in V} d(u,v)$$



Figure: Diameter of graph is  $d = ?$

**Diameter of a graph:**  $G = (V, E)$

$$d = \max_{u,v \in V} d(u,v)$$



Figure: Diameter of graph is  $d = ?$

- The **diameter** of a graph is the length / the costs of the **longest shortest path**

**Diameter of a graph:**  $G = (V, E)$

$$d = \max_{u,v \in V} d(u,v)$$



Figure: Diameter of graph is  $d = 10$ ,  $P = (3, 2, 5)$

- The **diameter** of a graph is the length / the costs of the longest shortest path



**Connected components:**  $G = (V, E)$

**Connected components:**  $G = (V, E)$



Figure: Three connected components

- Undirected graph:

**Connected components:**  $G = (V, E)$



Figure: Three connected components

- Undirected graph:
  - All connected components are a partition of  $V$

$$V = V_1 \cup \dots \cup V_k$$

**Connected components:**  $G = (V, E)$



Figure: Three connected components

- Undirected graph:
  - All connected components are a partition of  $V$

$$V = V_1 \cup \dots \cup V_k$$

- Two vertices  $u, v$  are in the same connected component if a path between  $u$  and  $v$  exists

**Connected components:**  $G = (V, E)$



**Connected components:**  $G = (V, E)$

- Directed graph:

### Connected components: $G = (V, E)$

- Directed graph:
  - Named **strongly connected components**

### Connected components: $G = (V, E)$

- Directed graph:
  - Named **strongly connected components**
  - Direction of edge has to be regarded

### Connected components: $G = (V, E)$

- Directed graph:
  - Named **strongly connected components**
  - Direction of edge has to be regarded
  - Not part of this lecture

### **Graph Exploration:** (Informal definition)

### **Graph Exploration:** (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex

### **Graph Exploration:** (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex
- We visit each reachable vertex connected to  $s$

### Graph Exploration: (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex
- We visit each reachable vertex connected to  $s$
- **Breadth-first search**: in order of the smallest distance to  $s$



### Graph Exploration: (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex
- We visit each reachable vertex connected to  $s$
- **Breadth-first search**: in order of the smallest distance to  $s$
- **Depth-first search**: in order of the largest distance to  $s$

### Graph Exploration: (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex
- We visit each reachable vertex connected to  $s$
- **Breadth-first search**: in order of the smallest distance to  $s$
- **Depth-first search**: in order of the largest distance to  $s$
- Not a problem on its own but is often used as subroutine of other algorithms

### Graph Exploration: (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex
- We visit each reachable vertex connected to  $s$
- **Breadth-first search**: in order of the smallest distance to  $s$
- **Depth-first search**: in order of the largest distance to  $s$
- Not a problem on its own but is often used as subroutine of other algorithms
  - Searching of connected components

### Graph Exploration: (Informal definition)

- Let  $G = (V, E)$  be a graph and  $s \in V$  a start vertex
- We visit each reachable vertex connected to  $s$
- **Breadth-first search**: in order of the smallest distance to  $s$
- **Depth-first search**: in order of the largest distance to  $s$
- Not a problem on its own but is often used as subroutine of other algorithms
  - Searching of connected components
  - Flood fill in drawing programmes

### Breadth-First Search:

- 1 We start with all vertices unmarked and mark visited vertices

### Breadth-First Search:

- 1 We start with all vertices unmarked and mark visited vertices
- 2 Mark the start vertex  $s$  (level 0)

### Breadth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s (level 0)**
- 3 Mark all unmarked **connected vertices (level 1)**

### Breadth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s** (**level 0**)
- 3 Mark all unmarked **connected vertices** (**level 1**)
- 4 Mark all unmarked **vertices connected** to a **level 1**-vertex (**level 2**)



### Breadth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s** (**level 0**)
- 3 Mark all unmarked **connected vertices** (**level 1**)
- 4 Mark all unmarked **vertices connected** to a **level 1**-vertex (**level 2**)
- 5 Iteratively mark reachable vertices for all levels

### Breadth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s (level 0)**
- 3 Mark all unmarked **connected vertices (level 1)**
- 4 Mark all unmarked **vertices connected** to a **level 1-vertex (level 2)**
- 5 Iteratively mark reachable vertices for all levels
- 6 All connected nodes are now marked and in the same **connected component** as the start vertex **s**

- The marked vertices create a “spanning tree” containing all reachable nodes

- The marked vertices create a “spanning tree” containing all reachable nodes

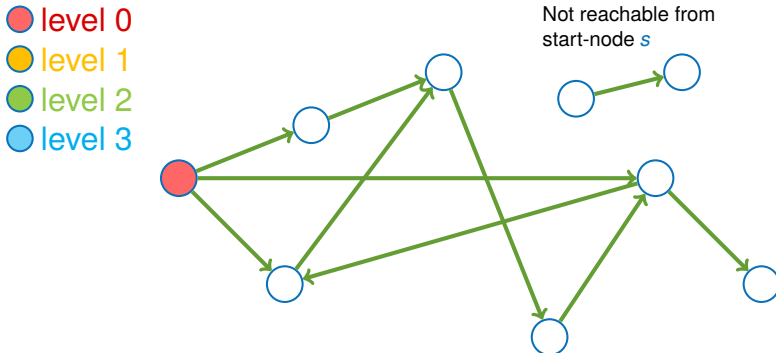


Figure: spanning tree of a breadth-first search

- The marked vertices create a “spanning tree” containing all reachable nodes

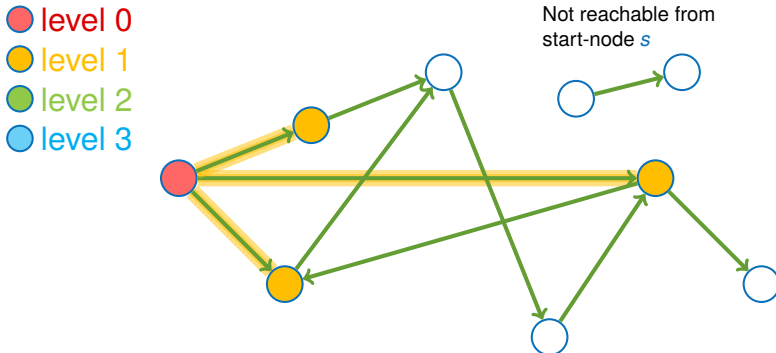


Figure: spanning tree of a breadth-first search

- The marked vertices create a “spanning tree” containing all reachable nodes



Figure: spanning tree of a breadth-first search

- UNIFREI



- The marked vertices create a “spanning tree” containing all reachable nodes

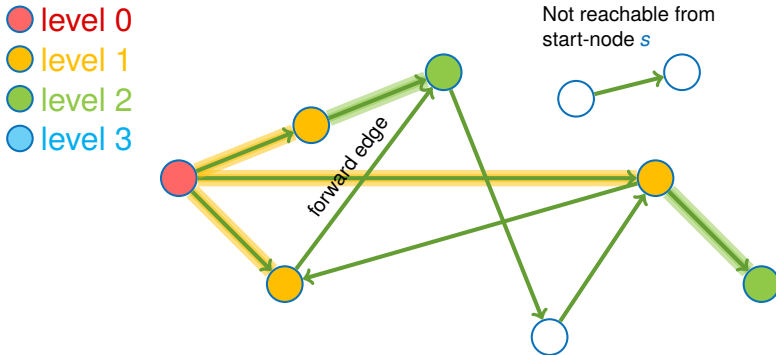


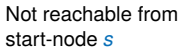
Figure: spanning tree of a breadth-first search



- UNIFREI



- UNIFREI



\_\_\_\_\_

- The marked vertices create a “spanning tree” containing all reachable nodes



Figure: spanning tree of a breadth-first search

### Depth-First Search:

### Depth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**

### Depth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s**

### Depth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s**
- 3 Pick an unmarked **connected vertex** and start a **recursive depth-first search** with the vertex as start vertex (continue on step 2)

### Depth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s**
- 3 Pick an unmarked **connected vertex** and start a **recursive depth-first search** with the vertex as start vertex (continue on step 2)
- 4 If no unmarked connected vertex exists go one vertex back and continue recursive search (reduce the recursion level by one)





### Depth-first search:

### Depth-first search:

- Search starts with **long paths** (searching with depth)

### Depth-first search:

- Search starts with **long paths** (searching with depth)
- Marks like **breadth-first search** all connected vertices

### Depth-first search:

- Search starts with **long paths** (searching with depth)
- Marks like **breadth-first search** all connected vertices
- If the graph is acyclic we get a **topological sorting**

### Depth-first search:

- Search starts with **long paths** (searching with depth)
- Marks like **breadth-first search** all connected vertices
- If the graph is acyclic we get a **topological sorting**
  - Each newly visited vertex gets marked by an increasing number

### Depth-first search:

- Search starts with **long paths** (searching with depth)
- Marks like **breadth-first search** all connected vertices
- If the graph is acyclic we get a **topological sorting**
  - Each newly visited vertex gets marked by an increasing number
  - The numbers increase with path length from the start vertex



- The marked vertices create a different spanning tree containing all reachable nodes

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search



- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

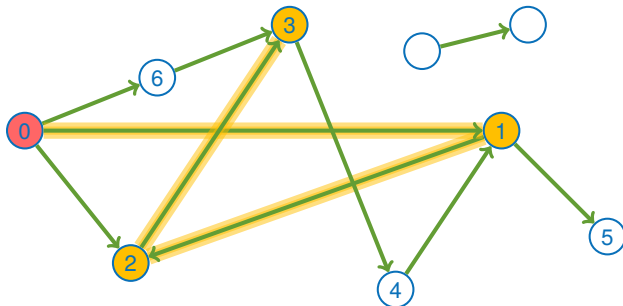


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

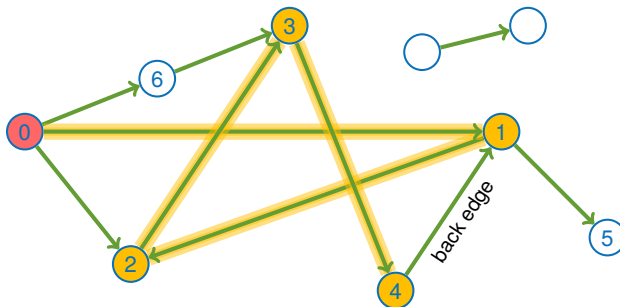


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

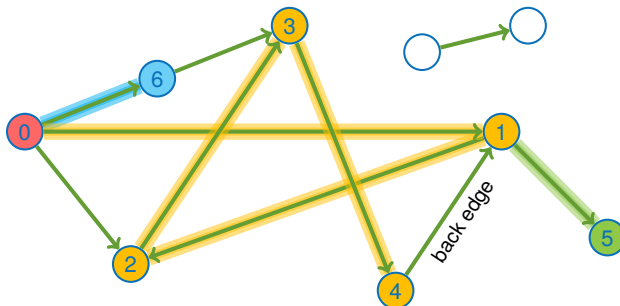


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

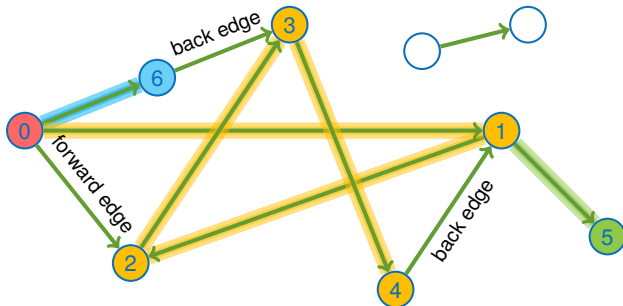


Figure: spanning tree of a depth-first search



- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

# Graphs

Why is this called Breadth - and Depth First Search?



**UNI  
FREIBURG**

### **Runtime complexity:**

- Constant costs for each visited vertex and edge

### Runtime complexity:

- Constant costs for each visited vertex and edge
- We get a runtime complexity of  $\Theta(|V'| + |E'|)$

### Runtime complexity:

- Constant costs for each visited vertex and edge
- We get a runtime complexity of  $\Theta(|V'| + |E'|)$
- Let  $V'$  and  $E'$  be the reachable vertices and edges

### Runtime complexity:

- Constant costs for each visited vertex and edge
- We get a runtime complexity of  $\Theta(|V'| + |E'|)$
- Let  $V'$  and  $E'$  be the reachable vertices and edges
- All vertices of  $V'$  are in the same connected component as our start vertex  $s$

### Runtime complexity:

- Constant costs for each visited vertex and edge
- We get a runtime complexity of  $\Theta(|V'| + |E'|)$
- Let  $V'$  and  $E'$  be the reachable vertices and edges
- All vertices of  $V'$  are in the same connected component as our start vertex  $s$
- This can only be improved by a constant factor

## Graphs

Introduction

Implementation

Application example



# Application example

## Image processing



**UNI  
FREIBURG**



- Connected component labeling

# Application example

## Image processing

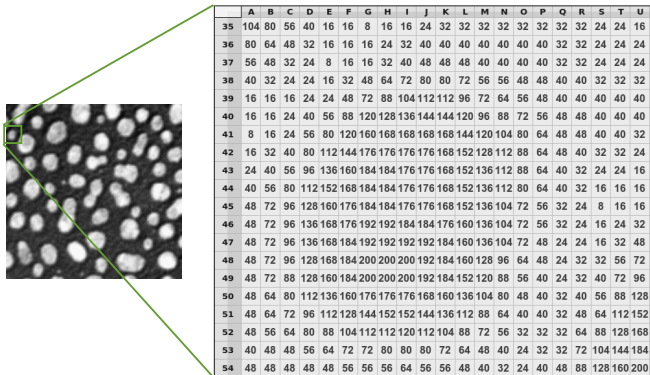


- Connected component labeling
- Counting of objects in an image

- Connected component labeling
- Counting of objects in an image



## What's object, what's background?



### Convert to black white using threshold:

value = 255 if value > 100 else 0





**Interpret image as graph:**

### **Interpret image as graph:**

- Each white pixel is a node



### **Interpret image as graph:**

- Each white pixel is a node
- Edges between adjacent pixels (normally 4 or 8 neighbors)

### **Interpret image as graph:**

- Each white pixel is a node
- Edges between adjacent pixels (normally 4 or 8 neighbors)
- Edges are not saved externally, algorithm works directly on array

### **Interpret image as graph:**

- Each white pixel is a node
- Edges between adjacent pixels (normally 4 or 8 neighbors)
- Edges are not saved externally, algorithm works directly on array
- Breadth- / depth-first search find all connected components (particles)



## Find connected components:

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	255

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	255

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels

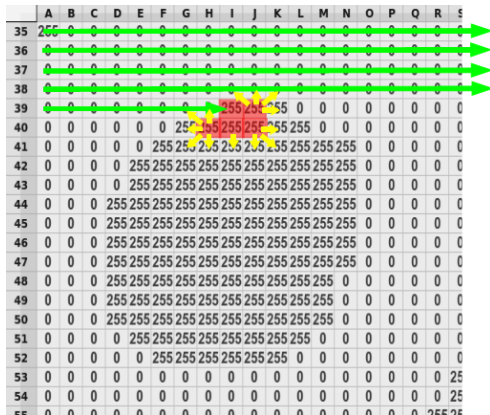


### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:



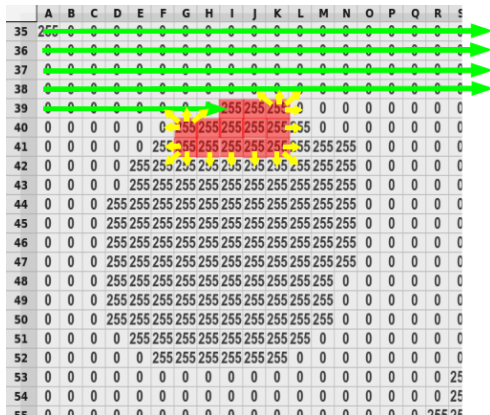
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:



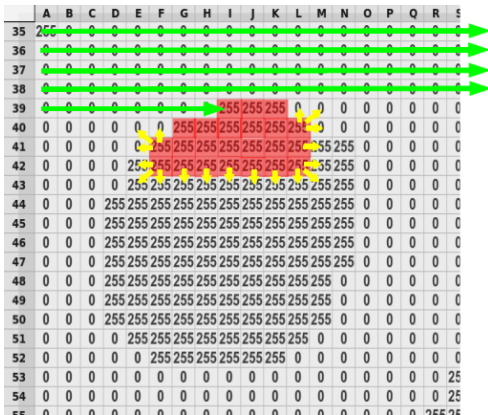
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:



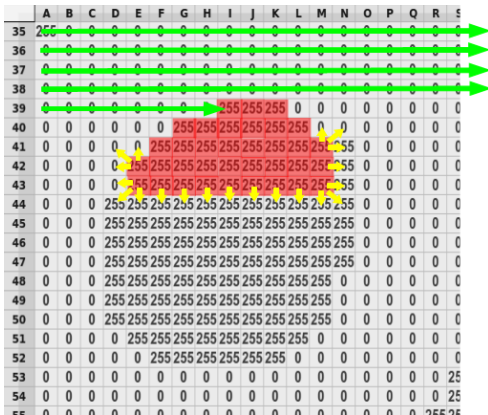
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:



- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1



### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

### Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	25	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as **component 2**
- ...

## Result of connected component labeling:

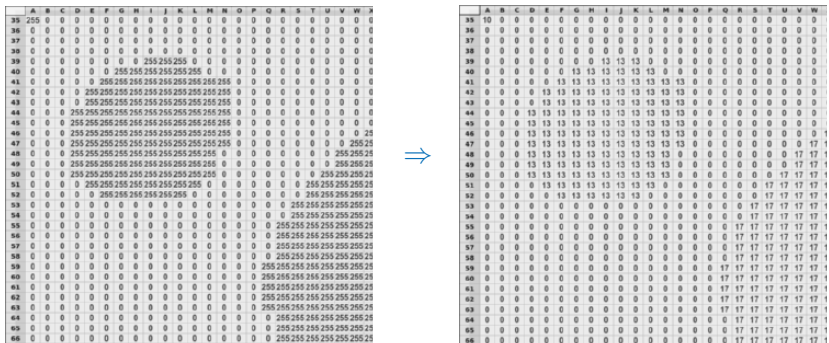


Figure: Result: particle indices instead of intensities

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Graph-Search

[Wika] [Breadth-first search](#)

`https://en.wikipedia.org/wiki/  
Breadth-first\_search`

[Wikb] [Depth-first search](#)

`https:  
//en.wikipedia.org/wiki/Depth-first\_search`

## ■ Graph-Connectivity

[Wik] [Connectivity \(graph theory\)](#)

`https://en.wikipedia.org/wiki/Connectivity\_  
\(graph\_theory\)`