Algorithms and Datastructures

Static Arrays, Dynamic Arrays, Amortized Analysis

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Structure

Static Arrays

- Static arrays exist in nearly every programming language
- They are initialized with a fixed size *n*
- **Problem:** The needed size is not always clear at compile time

Table: Static array with size $n = 5$

Index	0	1	2	3	4
Value	"a"	"b"	"c"	"d"	"e"

Static Arrays

Python

Python:

- We have dynamic sized lists
- Python does automatic resizing when needed

```
# Creates a list of "0"s with init. size 10
numbers = [0] * 10

# Prints number at index 7 ("0")
print("%d" % numbers[7])

# Saves number 42 at index 8
numbers[8] = 42

# Prints the number at index 8 ("42")
print("%d" % numbers[8])
```

Static Arrays

- The name “static array” has nothing to do with the keyword `static` from Java / C++
- Nor is the array allocated before the program starts
- The size of the array is static and can not be changed after creation
- The name “fixed-size array” would be more appropiate

Dynamic Arrays

Introduction

Dynamic arrays:

- The array is created with an initial size
- The size can be dynamically modified
- **Problem:** We need a dynamic structure to store the data

Dynamic Arrays

Python

Python:

```
greetings = ["Good morning", "ohai"]

greetings.append("Guten morgen")
greetings.append("bonjour")

# Prints text at index 2 ("Guten morgen")
print("%s" % greetings[2])

# Removes all elements
greetings.clear();
```

Dynamic Arrays

Implementation 1

- We store the data in a fixed-size array with the needed size
- **Append:**
    - Create fixed-size array with the needed size
    - Copy elements from the old to the new array
- **Remove:**
    - Create fixed-size array with the needed size
    - Copy elements from the old to the new array

Dynamic Arrays

Implementation 1

First implementation:

- We resize the array before each append
- We choose the size exactly as needed

Dynamic Arrays

Implementation 1 - Python

```
class DynamicArray:

    def __init__(self):
        self.size = 0
        self.elements = []

    def capacity(self):
        return len(self.elements)

    ...
```

Dynamic Arrays

Implementation 1 - Python

```
class DynamicArray:
    ...

    def append(self, item):
        newElements = [0] * (self.size + 1)

        for i in range(0, self.size):
            newElements[i] = self.elements[i]

        self.elements = newElements

        newElements[self.size] = item
        self.size += 1
```

Dynamic Arrays

Implementation 1

- Why is the runtime quadratic?

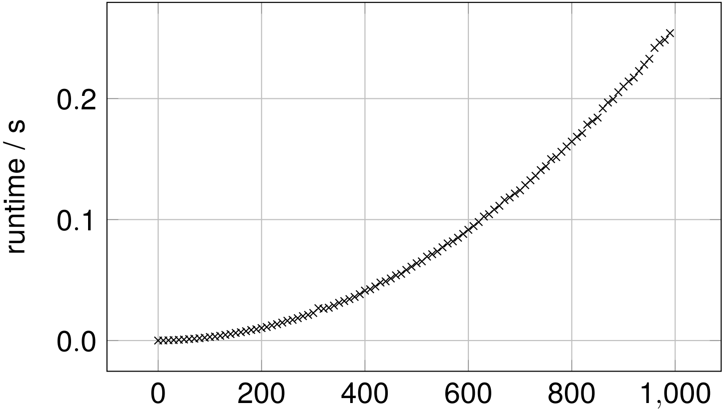
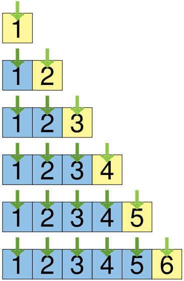


Figure: Runtime of *DynamicArray*

Dynamic Arrays

Implementation 1

Runtime:



$O(1)$	write 1 element
$O(1+1)$	write 1 element, copy 1 element
$O(1+2)$	write 1 element, copy 2 elements
$O(1+3)$	write 1 element, copy 3 elements
$O(1+4)$	write 1 element, copy 4 elements
$O(1+5)$	write 1 element, copy 5 elements
…	…

Dynamic Arrays

Implementation 1

Analysis:

- Let $T(n)$ be the runtime of n sequential append operations
- Let T_i be the runtime of each i-th operation
  - Then $T_i = A \cdot i$ for a constant A
  - We have to copy $i - 1$ elements

$$T(n) = \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} (A \cdot i) = A \cdot \sum_{i=1}^{n} i = A \cdot \frac{n^2 + n}{2}$$

$$= O(n^2)$$

Dynamic Arrays

Implementation 2

Idea:

- Better resize strategy
- We allocate more space than needed
- We over-allocate a constant amount of elements
  - Amount: $C = 3$ or $C = 100$

Dynamic Arrays

Implementation 2 - Python

```python
def append(self, item):
    if self.size >= len(self.elements):
        newElements = [0] * (self.size + 100)

        for i in range(0, self.size - 1):
            newElements[i] = self.elements[i]

        self.elements = newElements

    self.elements[self.size] = item
    self.size += 1
```

Dynamic Arrays

Implementation 2

- Why is the runtime still quadratic?

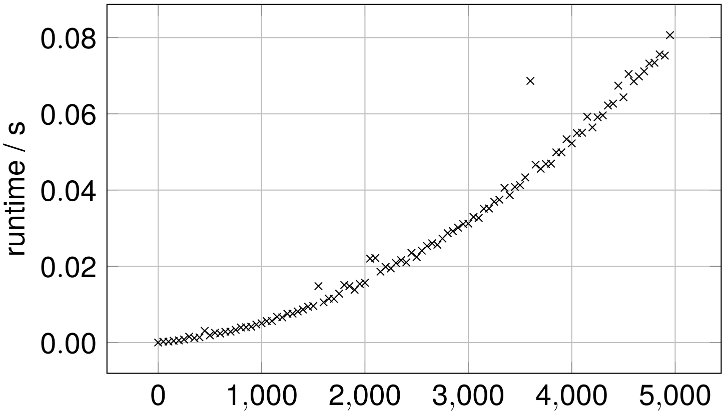
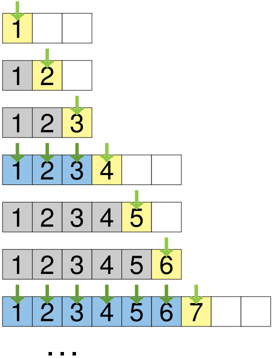


Figure: Runtime of *DynamicArray*

Dynamic Arrays

Implementation 2

Runtime for $C = 3$:



$O(1)$	write 1 element
$O(1)$	write 1 element
$O(1)$	write 1 element
$O(1+3)$	write 1 element, copy 3 elements
$O(1)$	write 1 element
$O(1)$	write 1 element
$O(1+6)$	write 1 element, copy 6 elements
…	…

Dynamic Arrays

Implementation 2

Analysis:

- Most of the append operations now just cost $O(1)$
- Every C steps the costs for copying are added: $C, 2 \cdot C, 3 \cdot C, \ldots$ this means:

$$\begin{aligned}
T(n) &= \sum_{i=1}^{n} A \cdot 1 + \sum_{i=1}^{n/C} A \cdot i \cdot C \\
&= A \cdot n + A \cdot C \cdot \sum_{i=1}^{n/C} i \\
&= A \cdot n + A \cdot C \cdot \frac{\frac{n^2}{C^2} + \frac{n}{C}}{2} \\
&= A \cdot n + \frac{A}{2 \cdot C} \cdot n^2 + \frac{A}{2} \cdot n = O(n^2)
\end{aligned}$$

- The factor of n^2 is getting smaller

Dynamic Arrays

Implementation 3

Idea:

- Double the size of the array

```
def append(self, item):
    if self.size >= len(self.elements):
        newElements = [0] \
            * max(1, 2 * self.size)

        for i in range(0, self.size):
            newElements[i] = self.elements[i]

        self.elements = newElements

    self.elements[self.size] = item
    self.size += 1
```

Dynamic Arrays

Implementation 3

- Now the runtime is linear with some bumps. Why?

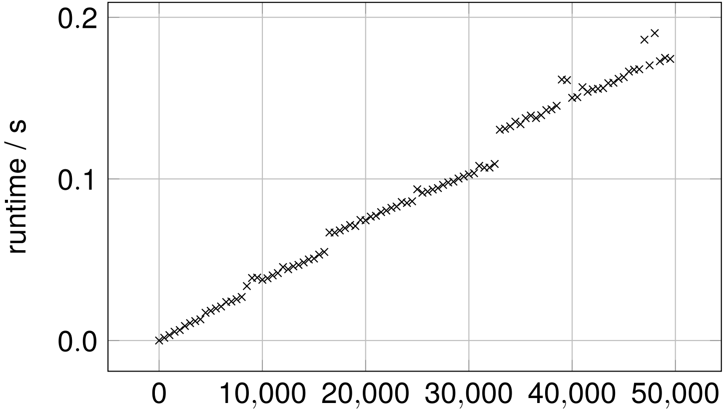
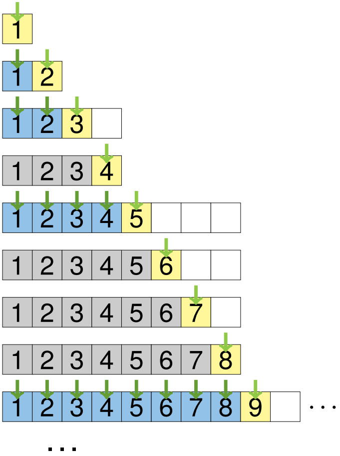


Figure: Runtime of *DynamicArray*

Dynamic Arrays

Implementation 2

Runtime for $C = 2$ (Double the size):



$O(1)$	write 1
$O(1+1)$	write 1, copy 1 element
$O(1+2)$	write 1, copy 2 elements
$O(1)$	write 1
$O(1+4)$	write 1, copy 4 elements
$O(1)$	write 1
$O(1)$	write 1
$O(1)$	write 1
$O(1+8)$	write 1, copy 8 elements
…	…

Dynamic Arrays

Implementation 3

Analysis:

- Now all appends cost $O(1)$
- Every 2^i steps we have to add the cost $A \cdot 2^i$ (for $i = 0, 1, 2, \ldots, k$ with $k = floor(log_2(n-1))$
- In total that accounts to:

$$\begin{aligned}
T(n) &= n \cdot A + A \cdot \sum_{i=0}^{k} 2^i = n \cdot A + A(2^{k+1} - 1) \\
&\leq n \cdot A + A \cdot 2^{(k+1)} \\
&= n \cdot A + 2 \cdot A \cdot 2^{(k)} \\
&\leq n \cdot A + 2 \cdot A \cdot n \\
&= 3 \cdot A \cdot n \\
&= O(n)
\end{aligned}$$

Dynamic Arrays

Shrinking

How do we shrink the array?

- If the array is half-full, we can shrink it by half, like for the append operation
- If we *append* directly after *shrinking* we have to extend the array again
    - We leave some space for following append operations

  ⇒ We only shrink the array to 75%

Dynamic Arrays

Shrinking

Analysis:

- **Difficult:** We have a random number of *append* / *remove* operations
- We can not exactly predict when resizing is happening

Dynamic Arrays

Amortized Analysis

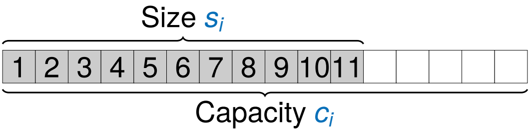


Figure: Static array with capacity c_i

Notation:

- We have n instructions $O = \{O_1, \dots, O_n\}$
- The size after operation i is s_i, with $s_0 := 0$
- The capacity after operation i is c_i, with $c_0 := 0$
- The cost of operation i is $\mathrm{cost}(O_i)$ (previously named T_i)

  Reallocation: $\mathrm{cost}(O_i) \leq A \cdot s_i$,
  Insert / Delete (Update): $\mathrm{cost}(O_i) \leq A$,

Dynamic Arrays

Amortized Analysis - Example

Operation			Size s_i	Capactity c_i	Costs $\text{cost}(O_i)$
O_1	append	realloc.	s_1	c_1	$A \cdot s_1$
O_2	append		s_2	$c_2 = c_1$	$A \cdot 1$
O_3	append		s_3	$c_3 = c_1$	$A \cdot 1$
O_4	remove		s_4	$c_4 = c_1$	$A \cdot 1$
O_5	remove	realloc.	s_5	c_5	$A \cdot s_5$
O_6	append		s_6	$c_6 = c_5$	$A \cdot 1$
O_7	remove		s_7	$c_7 = c_5$	$A \cdot 1$
O_8	append		s_8	$c_8 = c_5$	$A \cdot 1$
O_9	append	realloc.	s_9	c_9	$A \cdot s_9$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
O_n	append		s_n	c_n	$A \cdot 1$

Dynamic Arrays

Amortized Analysis - Example

Implementation:

- If O_i is an *append* operation and $s_{i-1} = c_{i-1}$:
  $\Rightarrow$ Resize array to $c_i = \left\lfloor \frac{3}{2} s_i \right\rfloor = \text{floor}\left(\frac{3}{2} s_i\right)$
  $\Rightarrow cost(O_i) = A \cdot s_i$

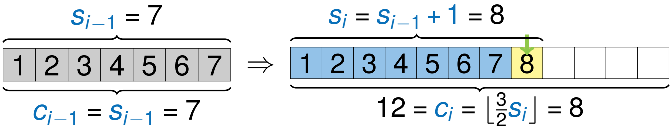


Figure: *Append* operation with reallocation

Result: after operation we have $c_i = \frac{3}{2} \cdot s_i$

Dynamic Arrays

Amortized Analysis - Example

Implementation:

- If O_i is an *remove* operation and $s_{i-1} \leq \frac{1}{3} c_{i-1}$:

  $\Rightarrow$ Resize array to $c_i = \left\lfloor \frac{3}{2} s_i \right\rfloor = \text{floor}\left(\frac{3}{2} s_i\right)$

  $\Rightarrow$ $cost(O_i) = A \cdot s_i$

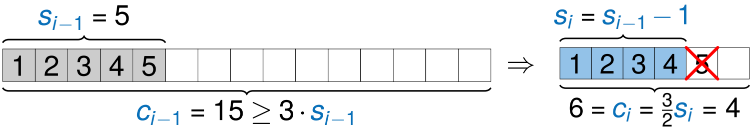


Figure: *Remove* operation with reallocation

Result: after operation we have again $c_i = \frac{3}{2} \cdot s_i$

Dynamic Arrays

Amortized Analysis - Proof

Idea for proof:

- Expensive are only operations where reallocations are necessary
- If we just reallocated, it takes some time until the next reallocation is required.
- **Assumption:** After a costly *reallocation* of size X we have at least X operations of runtime $O(1)$
- **Then:** Total cost of n operations is maximally $2 \cdot n$

Dynamic Arrays

Amortized Analysis - Proof

Table: Dynamic Array with $C_{\text{ext}} = \frac{3}{2}$

Operation			Size s_i	Capacity c_i	Costs $\mathrm{cost}(O_i)$
O_1	app.	realloc.	s_1	$c_1 = 4$	$C_1 \cdot s_1$
O_2	app.		s_2	$c_2 = c_1$	C_2
O_3	app.		s_3	$c_3 = c_1$	C_2
O_4	app.		s_4	$c_4 = c_1$	C_2
O_5	app.	realloc.	s_5	$c_5 = \lfloor \frac{3}{2} s_5 \rfloor = 7$	$C_1 \cdot s_5$
O_6	app.		s_6	$c_6 = c_5$	C_2
O_7	app.		s_7	$c_7 = c_5$	C_2
O_8	app.	realloc.	s_8	$c_8 = \frac{3}{2} s_8 = 12$	$C_1 \cdot s_8$
...	...	...	...	...	...

distance $4 \geq \left\lfloor \frac{s_1}{2} \right\rfloor$

distance $3 \geq \left\lfloor \frac{s_5}{2} \right\rfloor$

Dynamic Arrays

Amortized Analysis - Proof

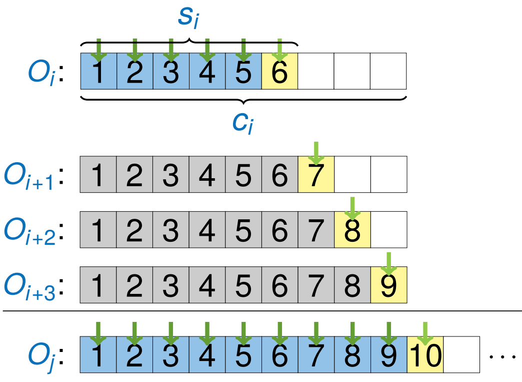
To show:

- **Lemma:** If a *reallocation* occurs at O_i the nearest *reallocation* is at O_j with $j - i > \frac{s_i}{2}$
- **Corollary:** $\mathrm{cost}(O_1) + \cdots + \mathrm{cost}(O_n) \leq 4A \cdot n$

Dynamic Arrays

Proof: Worst Case Same Operation

Table: Case 1: $\frac{1}{2}s_i$ *appends*



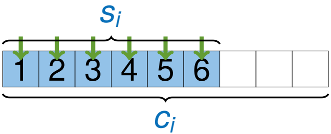
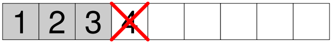
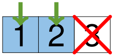
Array	Costs
O_i:	reallocation $A \cdot s_i$ (linear)
O_{i+1}:	A (constant)
O_{i+2}:	A (constant)
O_{i+3}:	A (constant)
O_j:	reallocation $A \cdot s_j$ (earliest realloc)

$\left.\begin{array}{l} O_{i+1} \\ O_{i+2} \\ O_{i+3} \end{array}\right\} \frac{s_i}{2}$ times

Dynamic Arrays

Amortized Analysis - Proof

Table: Case 2: $\frac{1}{2}s_i$ *removes*

	Array	Costs	
O_i:		reallocation $A \cdot s_i$ (linear)	
O_{i+1}:		A (constant)	$\frac{s_i}{2}$ times
O_{i+2}:		A (constant)	
O_{i+3}:		A (constant)	
O_j:		reallocation $A \cdot s_j$ (earliest reallocation)	

Dynamic Arrays
Amortized Analysis

Proof of lemma:

- If a reallocation happens at O_i and then again at O_j, then $j - i \geq s_i/2$
- After operation O_i the capacity is

$$c_i = \left\lfloor \frac{3}{2} \cdot s_i \right\rfloor$$

- Lets consider a operation O_i to O_k with $k - i \leq \frac{s_i}{2}$:
  - Case 1: Since the *reallocation* we have inserted at maximum $\text{floor}\left(\frac{1}{2} \cdot s_i\right)$ elementsation

$$s_k \leq s_i + \left\lfloor \frac{s_i}{2} \right\rfloor = \left\lfloor \frac{3}{2} s_i \right\rfloor = c_i$$

no reallocation needed

Proof of lemma - continued:

- Case 2: Since the *reallocation* we have removed at maximum $\left\lfloor \frac{1}{2} s_i \right\rfloor$ elements

$$s_k \geq s_i - \left\lfloor \frac{s_i}{2} \right\rfloor = \left\lceil \frac{1}{2} s_i \right\rceil$$

no reallocation needed

$$\Rightarrow 3 \cdot s_k \geq \left\lceil \frac{3}{2} s_i \right\rceil \geq \left\lfloor \frac{3}{2} s_i \right\rfloor = c_i$$

Dynamic Arrays

Amortized Analysis - Proof of Corollary

Corollary:

$$\mathrm{cost}(O_1) + \cdots + \mathrm{cost}(O_n) \leq 4A \cdot n$$

- Let the *reallocations* be at operations $\mathrm{cost}(O_{i_1}), \ldots, \mathrm{cost}(O_{i_m})$
- The cost of all *reallocations* are $A \cdot \left(s_{i_1} + \cdots + s_{i_m}\right)$
- With the lemma we know:

$$i_2 - i_1 > \frac{s_{i_1}}{2}, \quad i_3 - i_2 > \frac{s_{i_2}}{2}, \quad \ldots, \quad i_m - i_{m-1} > \frac{s_{i_{m-1}}}{2}$$

Dynamic Arrays

Amortized Analysis - Proof of Corollary

- We can conclude that:

$$\begin{aligned}
i_2 - i_1 &> \frac{s_{i_1}}{2} && \Rightarrow && s_{i_1} < 2(i_2 - i_1) \\
i_3 - i_2 &> \frac{s_{i_2}}{2} && \Rightarrow && s_{i-2} < 2(i_3 - i_2) \\
& && \vdots && \\
i_m - i_{m-1} &> \frac{s_{i_{m-1}}}{2} && \Rightarrow && s_{i_{m-1}} < 2(i_m - i_{m-1}) \\
& && && s_{i_m} \leq n \qquad \text{(trivial)}
\end{aligned}$$

Dynamic Arrays

Amortized Analysis - Proof of Corollary

- The costs of all reallocations are:

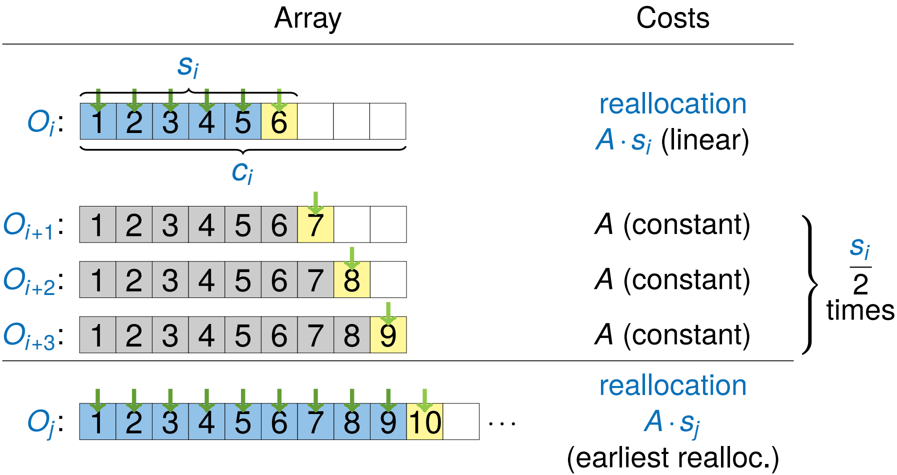
$$\begin{aligned} cost(realloc.) &= A \cdot \left(s_{i_1} + \cdots + s_{i_m}\right) \\ &< A \cdot (2(i_2 - i_1) + 2(i_3 - i_2) + \cdots + 2(i_m - i_{m-1}) + n) \\ &= A \cdot (2(i_m - i_1) + n) \\ &\leq A \cdot (2n + n) = 3A \cdot n \end{aligned}$$

- Additionally we have to consider the respective constant costs for a normal append or remove ($\leq A \cdot n$) therefore in total $\leq 4 \cdot A \cdot n$

Dynamic Arrays

Amortized Analysis - Alternate Proof of Corollary

Table: Case 1: $\frac{1}{2}s_i$ *appends*



Array	Costs
O_i: 1 2 3 4 5 6	reallocation $A \cdot s_i$ (linear)
O_{i+1}: 1 2 3 4 5 6 7	A (constant)
O_{i+2}: 1 2 3 4 5 6 7 8	A (constant)
O_{i+3}: 1 2 3 4 5 6 7 8 9	A (constant)
O_j: 1 2 3 4 5 6 7 8 9 10 …	reallocation $A \cdot s_j$ (earliest realloc.)

Dynamic Arrays

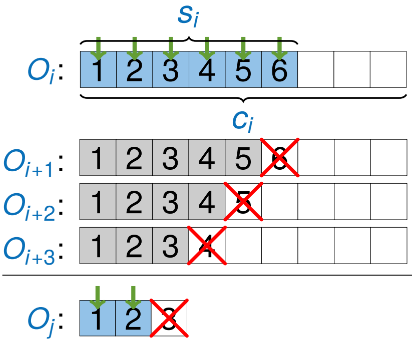
Amortized Analysis - Alternate Proof of Corollary

- Total costs of $A \cdot \frac{3}{2} \cdot s_i$ for $\frac{s_i}{2} + 1$ operations
- Cost per operation:

$$\frac{\frac{3}{2}A \cdot s_i}{\frac{1}{2}s_i + 1} \leq \frac{\frac{3}{2}A \cdot s_i}{\frac{1}{2}s_i} = 3 \cdot A = \text{const.}$$

Dynamic Arrays

Amortized Analysis - Alternate Proof of Corollary



Costs

reallocation
$A \cdot s_i$ (linear)

A (constant)
A (constant)
A (constant)

$\frac{s_i}{2}$ times

reallocation
$A \cdot s_j$ (linear)

- Runtime analysis for local worst-case sequence
- Same total cost as previous slide

Dynamic Arrays

Amortized Analysis - Yet Another Proof of Corollary

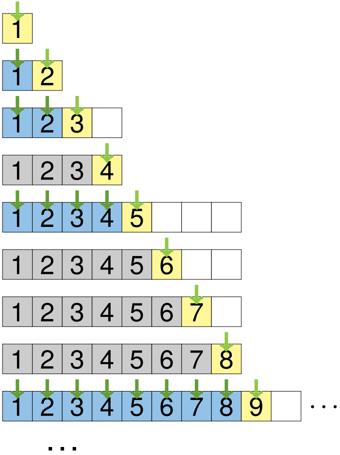
Bank account paradigm:

- **Idea:** “Save first, spend later”
- For each operation we deposit some coins on an “bank account”
  $\Rightarrow$ We still have constant costs
- When we have a linear operation (reallocation) we pay with the coins from our “bank account”
- For the “double the size” strategy we have to pay two coins per operation

Dynamic Arrays

Amortized Analysis - Yet Another Proof of Corollary

Double the size:



$\text{cost}(O_i)$	deposit / withdraw	account value
$O(1)$	+2	2
$O(1+1)$	+2 -1	3
$O(1+2)$	+2 -2	3
$O(1)$	+2	5
$O(1+4)$	+2 -4	3
$O(1)$	+2	5
$O(1)$	+2	7
$O(1)$	+2	9
$O(1+8)$	+2 -8	3
. . .	. . .	. . .

Dynamic Arrays

Amortized Analysis - Yet Another Proof of Corollary

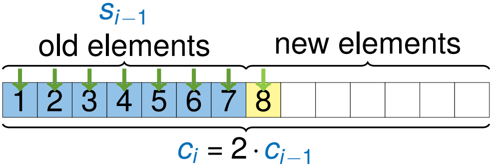


Figure: Array after realloc. (insert) operation

Why do we need to deposit 2 coints per operation?

1. Each newly inserted element has to be copied later (first coin)
2. Due to the factor of two there is for each new element also an old one in the array that also has to be copied (second coin)

Dynamic Arrays

Amortized Analysis - Yet Another Proof of Corollary

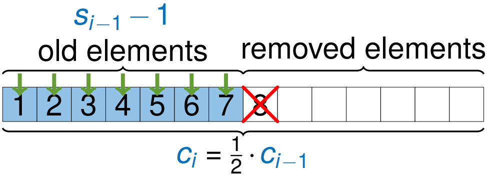


Figure: Array after realloc. (remove) operation

Shrinking strategy: If array 1/4 full shrink by half

- How many coins do we need per *remove* operation?
- **Worst case:** The previous remove operation triggered a *reallocation*

⇒ Array is half full

Dynamic Arrays

Amortized Analysis - Yet Another Proof of Corollary

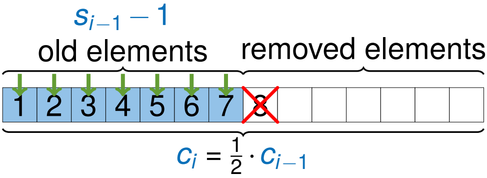


Figure: Array after realloc. (remove) operation

Shrinking strategy: If array 1/4 full shrink by half

- Array is half full
- The nearest *reallocation* is after removing $\frac{1}{4}c_i$ elements
- We have to copy $\frac{1}{4}c_i$ elements

⇒ 1 coin per operation is enough

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
`https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf`.

Further Literature

- **Amortized Analysis**

  [Wik] Amortized analysis
  `https://en.wikipedia.org/wiki/Amortized_analysis`