# Algorithms and Datastructures

## Hash Map, Universal Hashing

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# Structure


Associative Arrays
- Introduction
- Hash Map

Universal Hashing
- Introduction
- Probability Calculation
- Proof
- Examples

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# Associative Arrays

## How do we build a Map?

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- For n keys searching requires $\Theta(n)$ time
- With a hash map this just requires $\Theta(1)$ in the best case, ... regardless how many elements are in the map!

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- Mapping the keys onto indices with a hash function
- Store the values at the calculated indices in a normal array

**Example:**

- Key set: $x = \{3904433, 312692, 5148949\}$

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  A "hashtable" with 5 "buckets"

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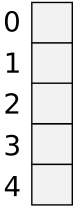
- Hash function: $h(x) = x \bmod 5$, in the range $[0, \ldots, 4]$
- We need an array T with 5 elements.
  A “hashtable” with 5 “buckets”
- The element with the key x is stored in $T[h(x)]$

**Storage:**

Figure: Hashtable T



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- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")

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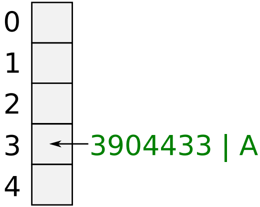

**Storage:**

- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")
- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow$ T[2] = (312692, "B")

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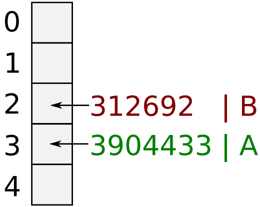

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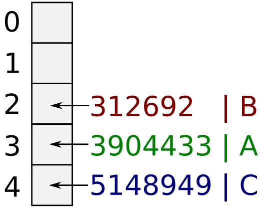

# Associative Arrays

## The Hash Map

**Searching:**

- `search(3904433)`: $h(3904433) = 3 \;\Rightarrow$ T[3] $\rightarrow$ (3904433, "A")

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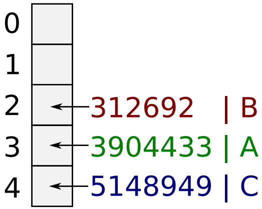
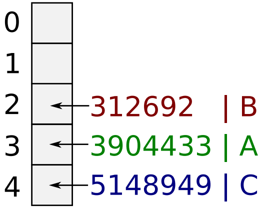


## Searching:

- `search(3904433)`: $h(3904433) = 3 \;\Rightarrow$ T[3] $\rightarrow$ (3904433, "A")
- `search(123459)`: $h(123459) = 4 \;\Rightarrow$ T[4]

  $\Rightarrow$ Value with key 123459 does not exist

Figure: Hashtable T



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- `search(3904433)`: $h(3904433) = 3 \;\Rightarrow$ T[3] $\rightarrow$ (3904433, "A")
- `search(123459)`: $h(123459) = 4 \;\Rightarrow$ T[4]

  $\Rightarrow$ Value with key 123459 does not exist
- Search time for this example: $\mathcal{O}(1)$

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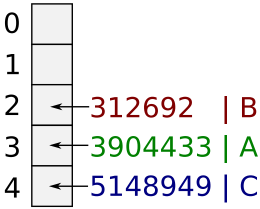
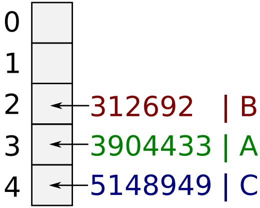


# Associative Arrays
## Hash Collisions

**Further inserting:**

- `insert(876543, "D")`: $h(876543) = 3$

Figure: Hashtable T



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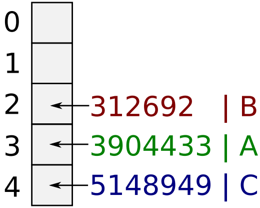

# Associative Arrays

## Hash Collisions

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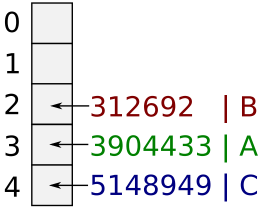
**Further inserting:**

- `insert(876543, "D")`: $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") $\Rightarrow$ Collision
- This happens more often than expected
  - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

Figure: Hashtable T



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- Represent each bucket as list of key value pairs

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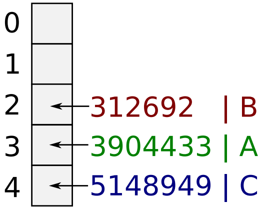

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## Hash Collisions

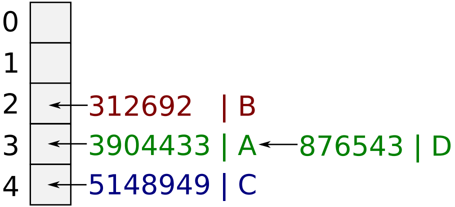
**Problem:**

- Two keys are equal $h(x) = h(y)$ but not the values $x \neq y$

**Easiest Solution:**

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

Figure: Hashtable T

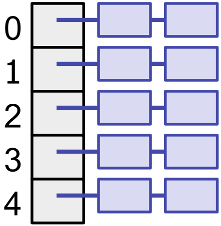


Best case:

- We have $n$ keys which are equally distributed over $m$ buckets
- We have $\approx \frac{n}{m}$ pairs per bucket
- The runtime for searching is nearly $\mathcal{O}(1)$ when **not** $n \gg m$

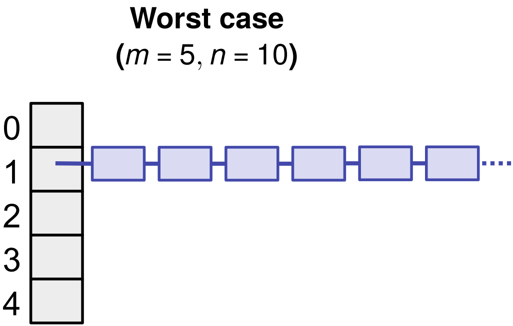
**Best case**

**($m = 5, n = 10$)**



Worst case:

- All $n$ keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching



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    - *The hash function stays fixed*
    - *For table size of 100: Try* $100 \times (99+1)$ *different numbers*
    - *Worst case: All 100 key sets map to one bucket*
- **Now:** Find a solution to avoid that problem

**Solution:** universal hashing

- Out of a set of hash functions we randomly choose one

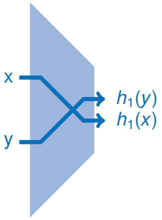



Figure: Hash func. 1

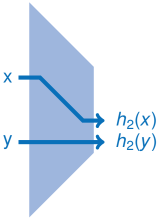
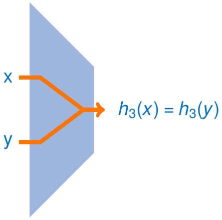



Figure: Hash func. 2

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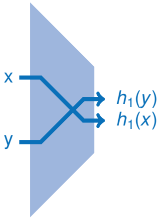


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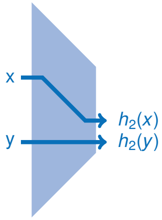
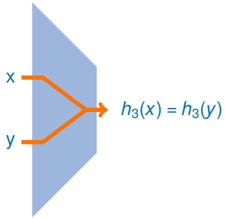



Figure: Hash func. 2

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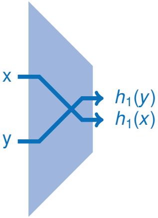
# Universal Hashing

## Idea

**Solution:** universal hashing

- Out of a set of hash functions we randomly choose one
- The expected result of the hash function is an equal distribution over the buckets
- This hash function stays fixed for the lifetime of table
  Optional: copy table with new hash when degenerated

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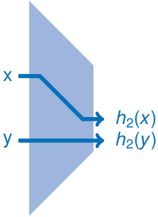


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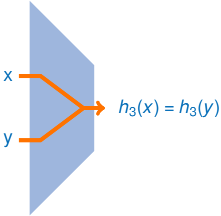


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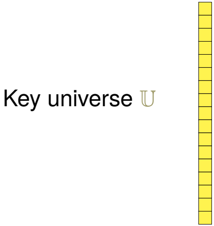
**Definition:**

- We call $\mathbb{U}$ the set (universum) of possible keys



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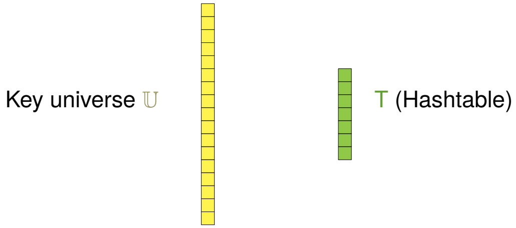

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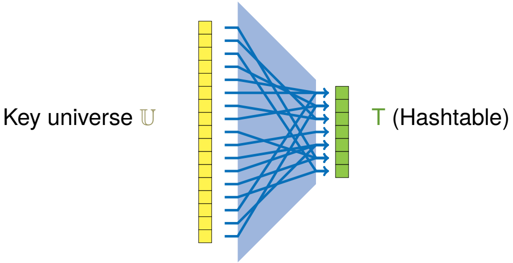


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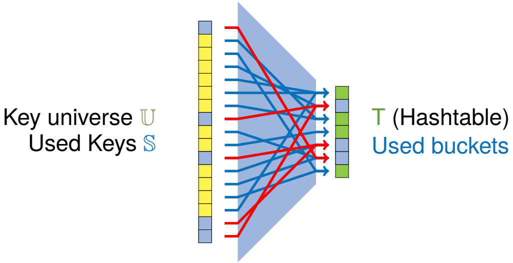


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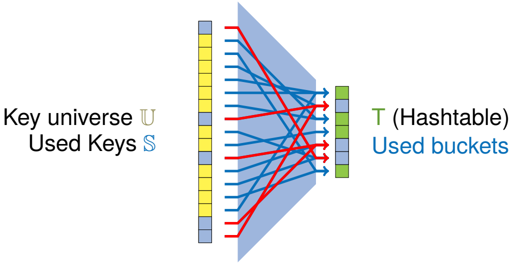
- Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load

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- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$

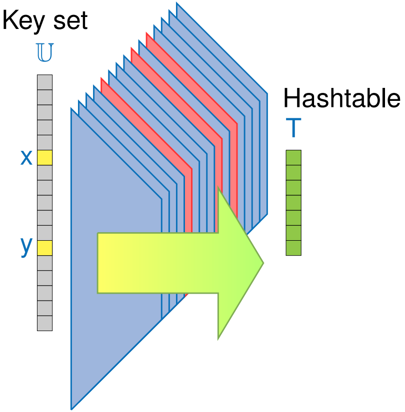


Figure: Set of hash functions $\mathbb{H}$

- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- An average of 3 out of 15 functions produce collisions

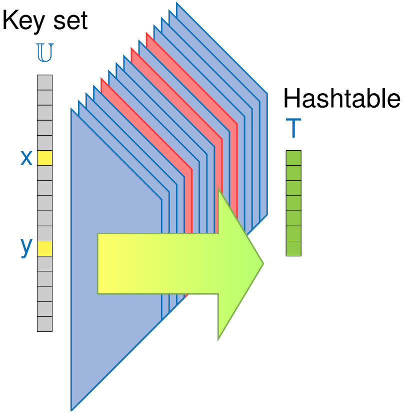


Figure: Set of hash functions $\mathbb{H}$

**Definition:** $\mathbb{H}$ is *c-universal* if $\forall x, y \in \mathbb{U} \mid x \neq y$ :

$$\frac{\overbrace{|\{h \in \mathbb{H} : h(x) = h(y)\}|}^{\text{Number of hash functions that create collisions}}}{\underbrace{|\mathbb{H}|}_{\text{Number of hash functions}}} \leq c \cdot \frac{1}{m}, \qquad c \in \mathbb{R}$$

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Note: If the hash function assigns each key $x$ and $y$ randomly to buckets then:

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Note: If the hash function assigns each key $x$ and $y$ randomly to buckets then:

$$Prob(Collision) = \frac{1}{m} \Leftrightarrow c = 1$$

- $\mathbb{U}$: Key universe
- $\mathbb{S}$: Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: Keys mapping to Bucket $i$ ("synonyms")
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$

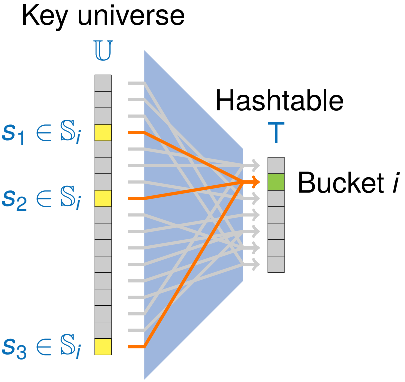


Figure: Hash function $h \in \mathbb{H}$

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- The expected average number of elements to search through per bucket is

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m}$$

# Universal Hashing

## Definition

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- Particulary: If $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

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Table: Throwing a dice

| $e$ | $P(e)$ |
|---|---|
| 1 | $^1/_6$ |
| 2 | $^1/_6$ |
| 3 | $^1/_6$ |
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Table: Throwing a dice twice

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| $(1,2)$ | ${}^1/_{36}$ |
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| $\ldots$ | $\ldots$ |
| $(6,5)$ | ${}^1/_{36}$ |
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**Example:**

- Rolling a dice twice ($\Omega = \{1, \ldots, 6\}^2$)
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- $E$ = if both results are even, then $P(E) =$

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| $\dots$ | $\dots$ | $\dots$ |
| $(6,5)$ | $^1/_{36}$ | 11 |
| $(6,6)$ | $^1/_{36}$ | 12 |

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- Random variable
  - Assigns a number to the result of an experiment
  - For example: $X$ = Sum of results for rolling twice
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  - Example 1: $P(X = 2) =$
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Table: Throwing a dice twice

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**Sum of expected values:** For arbitrary discrete random variables $X_1, \ldots, X_n$ we can write:

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n)$$

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**Example:** Throwing two dice

- $X_1$: Expected result of dice 1: $\mathbb{E}(X_1) = 3.5$
- $X_2$: Expected result of dice 2: $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$: Expected total number:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

## Corollary:

The probability of the event $E$ is $p = P(E)$. Let $X$ be the occurences of the event $E$ and $n$ be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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### Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurences of } 6) = \frac{1}{6} \cdot 60 = 10$$

## Proof Corollary:

Indicator variable: $X_i$

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$$X_i = \begin{cases} 1, & \text{if event occurs} \\ 0, & \text{else} \end{cases}$$

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Def. $\mathbb{E}$-value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

# Structure


Associative Arrays

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**Given:**

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**To proof:**

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \forall i$$

**We know:**

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We define an indicator variable:

$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \qquad y \in \mathbb{S} \setminus \{x\}$$

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$$\Rightarrow \qquad |\mathbb{S}_i| = 1 + \sum_{y \in \mathbb{S} \setminus x} I_y$$

$$\Rightarrow \qquad \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

**Auxiliary calculation:**

$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1) \\ &= P(h(y) = i) \\ &= P(h(y) = h(x)) \\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

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$$
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**Hence:**

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- Which $x, y$ lead to a relative collision count bigger than $\frac{c}{m}$?

**Positive example:**

- Let $p$ be a big prime number, $p > m$ and $p \geq |\mathbb{U}|$

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**Positive example:**

- Let $p$ be a big prime number, $p > m$ and $p \geq |\mathbb{U}|$
- Let $\mathbb{H}$ be the set of all $h$ for which:

$$h_{a,b}(x) = ((a \cdot x + b) \mod p) \mod m,$$
$$\text{where } 1 \leq a < p,\ 0 \leq b < p$$

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- E.g.: $U = \{0, \ldots, 99\}, \quad p = 101, \quad a = 47, b = 5$

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- E.g.: $U = \{0, ..., 99\}, \quad p = 101, \quad a = 47, b = 5$
- Then $h(x) = ((47 \cdot x + 5) \mod 101) \mod m$
- Easy to implement but hard to proof
- Exercise: show empirically that it is 2-universal

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$$a = \sum_{0,\ldots,k-1} a_i \cdot m^i, \qquad k = \text{ceil}(\log_m |\mathbb{U}|)$$

$$x = \sum_{0,\ldots,k-1} x_i \cdot m^i$$

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- **Intuitive**: Scalar product with base $m$

$$a \bullet x = \sum_{0,\ldots,k-1} a_i \cdot x_i$$

## Example ($\mathbb{U} = \{0, \ldots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

# Further Literature

- **General for this Lecture**


[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
**Introduction to Algorithms.**
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

# Further Literature

- **Hash Map - Theory**

  [Wik] Hash table
  https://en.wikipedia.org/wiki/Hash_table

- **Hash Map - Implementations / API**

  [Cpp] C++ - hash_map
  http://www.sgi.com/tech/stl/hash_map.html

  [Jav] Java - HashMap
  https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html

  [Pyt] Python - Dictionaries (Hash table)
  https://en.wikipedia.org/wiki/Hash_table