Algorithmns and Datastructures

Levenshtein distance, Dynamic programming

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Algorithmns and Datastructures, February 2017

Structure

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Introduction

Edit distance

Edit distance:

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- Measurement for similarity of two words / strings

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- Algorithm for efficient calculation

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Edit distance:

- Measurement for similarity of two words / strings
- Algorithm for efficient calculation
- General principle: dynamic programming

BioInfSearch

ejafjatlajökuk

eyjafjallajökull
eyjafjallajökull movie
eyjafjallajälull trailer

Search!

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Wikipedia.org:

"Der Eyjafjallajökull (['eɪjaˌfjatlaˌjœːkʏtl̥])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárþing eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."

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Hein Blöd    27568 Bremerhaven
Hein Bloed   27568 Bremerhafen
Hein Doof    27478 Cuxhaven
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eyjaföllajaküll
uniwersität verien 2017
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- Bioinformatics: Similarity of DNA-sequences

Search of similar proteins:

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- Changed life-science completely

Search of similar proteins:

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- Alignment $\hat{=}$ Edit distance
- Changed life-science completely
- Cited 63437 times on Google Scholar (Sep. 2017)

Structure

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Definition of edit distance: (*Levenshtein-distance*)

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- Let x, y be two strings
- Edit distance $\mathrm{ED}(x,y)$ of x and y:
  The minimal number of operations to transform x into y

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  The minimal number of operations to transform x into y
  - `Insert` a character

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  - `Insert` a character
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  - `Insert` a character
  - `Replace` a character with another
  - `Delete` a character

Edit distance
Example

1 2 3 4 5
DOOF

B L OED

Edit distance
Example

1 2 3 4 5
DOOF
$\downarrow$ `replace(1, B)`
BOOF

B L OED

Edit distance
Example

```
1 2 3 4 5
DOOF
   ↓       replace(1, B)
BOOF
   ↓       replace(2, L)
BLOF


BLOED
```

```
1 2 3 4 5
DOOF
   ↓      replace(1, B)
BOOF
   ↓      replace(2, L)
BLOF
   ↓      insert(4, E)
BLOEF

BLOED
```

```
1 2 3 4 5
DOOF
  ↓      replace(1, B)
BOOF
  ↓      replace(2, L)
BLOF
  ↓      insert(4, E)
BLOEF
  ↓      replace(5, D)
BLOED
```

Edit distance

Example

$$
\begin{array}{ll}
\text{1 2 3 4 5} & \\
\text{DOOF} & \\
\quad\downarrow & \texttt{replace(1, B)} \\
\text{BOOF} & \\
\quad\downarrow & \texttt{replace(2, L)} \\
\text{BLOF} & \\
\quad\downarrow & \texttt{insert(4, E)} \\
\text{BLOEF} & \\
\quad\downarrow & \texttt{replace(5, D)} \\
\text{BLOED} & \\
 & \underbrace{\hphantom{\texttt{replace(5, D)}}}_{\text{ED}=4}
\end{array}
$$

Edit distance
Example

1 2 3 4 5
DOOF

$\downarrow$ `replace(1, B)`

BOOF

$\downarrow$ `replace(2, L)`

BLOF

$\downarrow$ `insert(4, E)`

BLOEF

$\downarrow$ `replace(5, D)`

BLOED

$\underbrace{\texttt{replace(1, B)}\ \texttt{replace(2, L)}\ \texttt{insert(4, E)}\ \texttt{replace(5, D)}}_{\text{ED}=4}$

1 2 3 4 5
BLOED

Edit distance
Example

1 2 3 4 5 DOOF		
↓	`replace(1, B)`	1 2 3 4 5 B LOED
BOOF		
↓	`replace(2, L)`	
B LOF		
↓	`insert(4, E)`	
B LOEF		
↓	`replace(5, D)`	DOOF
B LOED		

$\underbrace{\hspace{6em}}_{\text{ED=4}}$

Edit distance
Example

1 2 3 4 5 DOOF		1 2 3 4 5 B LOED	
↓	`replace(1, B)`	↓	`replace(5, F)`
BOOF		B LOEF	
↓	`replace(2, L)`		
B LOF			
↓	`insert(4, E)`		
B LOEF			
↓	`replace(5, D)`		
B LOED		DOOF	

$\underbrace{\qquad\qquad\qquad}_{\text{ED}=4}$

Edit distance

Example

```
12345
DOOF
  ↓      replace(1, B)
BOOF
  ↓      replace(2, L)
BLOF
  ↓      insert(4, E)
BLOEF
  ↓      replace(5, D)
BLOED
```

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{ED=4}}$

```
12345
BLOED
  ↓      replace(5, F)
BLOEF
  ↓      delete(4)
BLOF

DOOF
```

Edit distance

Example

$$\begin{array}{ll}
\text{1 2 3 4 5} & \\
\text{DOOF} & \\
\downarrow & \texttt{replace(1, B)} \\
\text{BOOF} & \\
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\end{array}$$

$$\begin{array}{ll}
\text{1 2 3 4 5} & \\
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\downarrow & \texttt{replace(5, F)} \\
\text{BLOEF} & \\
\downarrow & \texttt{delete(4)} \\
\text{BLOF} & \\
\downarrow & \texttt{replace(2, O)} \\
\text{BOOF} & \\
 & \\
\text{DOOF} &
\end{array}$$

Edit distance
Example

```
12345
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  ↓     replace(1, B)
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  ↓     replace(2, L)
BLOF
  ↓     insert(4, E)
BLOEF
  ↓     replace(5, D)
BLOED
```

$\underbrace{\hspace{6em}}_{\text{ED=4}}$

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BLOED
  ↓     replace(5, F)
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  ↓     delete(4)
BLOF
  ↓     replace(2, O)
BOOF
  ↓     replace(1, D)
DOOF
```

Edit distance

Example

```
12345
DOOF
  ↓      replace(1, B)
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ED=4

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BLOED
  ↓      replace(5, F)
BLOEF
  ↓      delete(4)
BLOF
  ↓      replace(2, O)
BOOF
  ↓      replace(1, D)
DOOF
```

ED=4

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- $x[i..j]$ is the slice of x from i to j where $1 \leq i \leq j \leq |x|$

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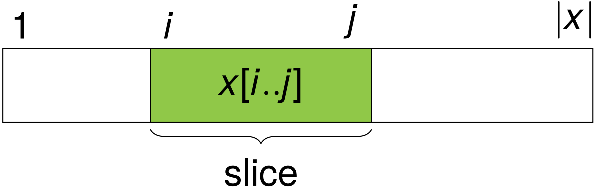

Edit distance

Trivial facts:

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- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$
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- $\mathrm{ED}(x,y) \geq \mathrm{abs}(|x| - |y|)$

$$\mathrm{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

Edit distance

Trivial facts:

- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$
- $\mathrm{ED}(x,\varepsilon) = |x|$
- $\mathrm{ED}(x,y) \geq \mathrm{abs}(|x| - |y|)$ $\qquad \mathrm{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$
- $\mathrm{ED}(x,y) \leq \mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ $\qquad n = |x|,\ m = |y|$

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Solutions based on examples:

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- From `VERIEN` to `FERIEN`?

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- From `AAEBEAABEAREEAEBA` to `RBEAAEEBAAAEBBAEAE`?

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- From `VERIEN` to `FERIEN`?
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- Searching biggest substrings can yield the solution but doesn't have to

Solutions based on examples:

- From VERIEN to FERIEN?
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- From AAEBEAABEAREEAEBA to RBEAAEEBAAAEBBAEAE?
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Recursive approach:

Solutions based on examples:

- From `VERIEN` to `FERIEN`?
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- Searching biggest substrings can yield the solution but doesn't have to

Recursive approach:

- Dividing in two halves? Not a good idea:

$$\mathrm{ED}(GRAU, RAUM) = 2 \quad \text{but} \quad \mathrm{ED}(GR, RA) + \mathrm{ED}(AU, UM) = 4$$

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- From `VERIEN` to `FERIEN`?
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Recursive approach:

- Dividing in two halves? Not a good idea:

  $\mathrm{ED}(GRAU, RAUM) = 2$ but $\mathrm{ED}(GR, RA) + \mathrm{ED}(AU, UM) = 4$

- Finding "smaller" sub problems?
  Let's try it!

Edit distance

Terminology:

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- Let x, y be two strings

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Terminology:

- Let x, y be two strings
- Let $\sigma_1, \ldots, \sigma_k$ be a sequence of k operations where $k = \mathrm{ED}(x, y)$ for $x \to y$ (transform x into y)
  (We do not know this sequence but we assume it exists)

Edit distance

Terminology:

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Terminology:

- We only consider monotonous sequences:
  The posititon of σ_{i+1} is $\geq$ the position of σ_i where we only allow the positions to be equal on a `delete` operation

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1 2 3 4 5		1 2 3 4 5 6 7	
DOOF		SAUDOOF	
$\downarrow$	`replace(1, B)`	$\downarrow$	`delete(1)`
BOOF		AUDOOF	
$\downarrow$	`replace(2, L)`	$\downarrow$	`delete(1)`
BLOF		UDOOF	
$\downarrow$	insert(4, E)	$\downarrow$	`delete(1)`
BLOEF		DOOF	
$\downarrow$	`replace(5, D)`	$\downarrow$	`insert(4, O)`
BLOED		DOOOF	

Edit distance

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- We only consider monotonous sequences:
  The positition of σ_{i+1} is $\geq$ the position of σ_i where we only allow the positions to be equal on a `delete` operation

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\begin{array}{lll}
\text{1 2 3 4 5} & & \\
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\quad\downarrow & \texttt{replace(1, B)} & \\
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\text{BLOF} & & \\
\quad\downarrow & \text{insert(4, E)} & \\
\text{BLOEF} & & \\
\quad\downarrow & \texttt{replace(5, D)} & \\
\text{BLOED} & &
\end{array}
\qquad
\begin{array}{ll}
\text{1 2 3 4 5 6 7} & \\
\text{SAUDOOF} & \\
\quad\downarrow & \texttt{delete(1)} \\
\text{AUDOOF} & \\
\quad\downarrow & \texttt{delete(1)} \\
\text{UDOOF} & \\
\quad\downarrow & \texttt{delete(1)} \\
\text{DOOF} & \\
\quad\downarrow & \texttt{insert(4, O)} \\
\text{DOOOF} &
\end{array}
$$

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- **Lemma:** For any x and y with $k = \mathrm{ED}(x, y)$ exists a monotonous sequence of k operations for $x \rightarrow y$

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Terminology:

- **Lemma:** For any x and y with $k = \mathrm{ED}(x, y)$ exists a monotonous sequence of k operations for $x \rightarrow y$
- **Intuition:** The order of our sequence is not relevant (Therefore we can also sort them monotonously)

1	2	3	4	5
D	O	O	F	
B	L	O	E	D

1	2	3	4	5	6	7
S	A	U	D	O	O	F
D	O	O	O	F		

Consider the last operation:

Consider the last operation:

- Solve blue part recursively

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- Solve blue part recursively

DOOF
↓↓↓↓
BLOE
↓insert
BLOED

Figure: Case 1a

DOOF
↓↓↓↓↓↓
BLOEDF
↓delete
BLOED

Figure: Case 1b

DOOF
↓↓↓↓↓
BLOEF
↓replace
BLOED

Figure: Case 1c

Consider the last operation:

Consider the last operation:

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Consider the last operation:

- Solve blue part recursively

W I N T E R
↓ ↓ ↓ ↓ ↓ ↓
S OMMER
↓nothing
S OMMER

Figure: Case 2

Display of solution:

- Alignment
- Example:

_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

Dynamic programming:

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Dynamic programming:

- Instances of Bellman's principle of optimality:

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    - Shortest paths

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Figure: Richard Bellman (1920 - 1984)

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Figure: Richard Bellman (1920 - 1984)

- Optimal solutions consist of optimal partial solutions
  - Shortest paths: Each partial path has to be optimal
  - Edit distance: Each partial alignment has to be optimal

Dynamic programming:

- Instances of Bellman's principle of optimality:
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Figure: Richard Bellman (1920 - 1984)

- Optimal solutions consist of optimal partial solutions
  - Shortest paths: Each partial path has to be optimal
  - Edit distance: Each partial alignment has to be optimal

- Always solvable through dynamic programming (Caching of optimal partial solutions)

Edit distance

Case analysis:

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Case analysis:

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- We consider the last operation σ_k
  - $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow z$ and σ_k: $z \rightarrow y$
    Example:

$$x = \text{DOOF},\ z = \text{SAUBLOEF},\ y = \text{SAUBLOED}$$

Edit distance

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- We consider the last operation σ_k
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    Example:

    $$x = \text{DOOF},\ z = \text{SAUBLOEF},\ y = \text{SAUBLOED}$$

  - Let $n = |x|$, $m = |y|$, $m' = |z|$

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  - $\sigma_1, \ldots, \sigma_{k-1}$: $x \to z$ and σ_k: $z \to y$
    Example:

$$x = \mathrm{DOOF},\ z = \mathrm{SAUBLOEF},\ y = \mathrm{SAUBLOED}$$

  - Let $n = |x|$, $m = |y|$, $m' = |z|$
  - We note $m' \in \{m-1, m, m+1\}$ why?

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  - Case 1a: $\sigma_k = \mathtt{insert}(m'+1, y[m])$ [then $m' = m-1$]

Edit distance

Case analysis:

- Case 1: σ_k does something at the outer end:
  - Case 1a: $\sigma_k = \texttt{insert}(m' + 1, y[m])$ [then $m' = m - 1$]
  - Case 1b: $\sigma_k = \texttt{delete}(m')$ [then $m' = m + 1$]

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Case analysis:

- Case 1: σ_k does something at the outer end:
  - Case 1a: $\sigma_k = \mathtt{insert}(m'+1, y[m])$ [then $m' = m-1$]
  - Case 1b: $\sigma_k = \mathtt{delete}(m')$ [then $m' = m+1$]
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- Case 2: σ_k does nothing at the outer end:

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Case analysis:

- Case 1: σ_k does something at the outer end:
  - Case 1a: $\sigma_k = \texttt{insert}(m'+1, y[m])$ [then $m' = m-1$]
  - Case 1b: $\sigma_k = \texttt{delete}(m')$ [then $m' = m+1$]
  - Case 1c: $\sigma_k = \texttt{replace}(m', y[m])$ [then $m' = m$]
- Case 2: σ_k does nothing at the outer end:
  - Then $z[m'] = y[m]$ and $x[n'] = z[m']$ and with that $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$ and $x[n] = y[m]$

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Case analysis:

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Case analysis:

- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: x $\rightarrow y[1..m-1]$

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- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y$

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Case analysis:

- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y$
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- Case 1c (`replace`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1, \ldots, \sigma_k$: $x[1..n-1] \rightarrow y[1..m-1]$

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1, \ldots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
- Case 1c (`replace`): $\sigma_1, \ldots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1, \ldots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1,\ldots,\sigma_{k-1}$: $x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1,\ldots,\sigma_{k-1}$: $x[1..n-1] \rightarrow y$
- Case 1c (`replace`): $\sigma_1,\ldots,\sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1,\ldots,\sigma_k$: $x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $\mathrm{ED}(x,y)$ the minimum of

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y$
- Case 1c (`replace`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1, \ldots, \sigma_k$: $x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $\mathrm{ED}(x,y)$ the minimum of
  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and

Edit distance

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This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $\mathrm{ED}(x, y)$ the minimum of
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Edit distance

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  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$

Edit distance

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  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
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  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1,\ldots,\sigma_{k-1}\colon x \rightarrow y[1..m-1]$
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  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $\mathrm{ED}(x,y) = |y|$

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Case analysis:

- Case 1a (`insert`): $\sigma_1,\ldots,\sigma_{k-1}$: $x \rightarrow y[1..m-1]$
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  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $\mathrm{ED}(x,y) = |y|$
- For $|y| = 0$ is $\mathrm{ED}(x,y) = |x|$

Edit distance
Implementation - Python

```python
def edit_distance(x, y):
    if len(x) == 0:
        return len(y)
    if len(y) == 0:
        return len(x)

    ed1 = edit_distance(x, y[:-1]) + 1
    ed2 = edit_distance(x[:-1], y) + 1
    ed3 = edit_distance(x[:-1], y[:-1])
    if x[-1] != y[-1]:
        ed3 += 1

    return min(ed1, ed2, ed3)
```

Recursive program:

Recursive program:

- The algorithm results in the following recursive formular:

$$\begin{aligned} T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\ &\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\ &= 3 \cdot T(n-1,m-1) \end{aligned}$$

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Recursive program:

- The algorithm results in the following recursive formular:

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- This results in $T(n,n) \geq 3^n$

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$$\begin{aligned} T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\ &\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\ &= 3 \cdot T(n-1,m-1) \end{aligned}$$

- This results in $T(n,n) \geq 3^n$

$\Rightarrow$ The runtime is at least exponential

Dynamic programming:

Edit distance

Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

Edit distance

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Visualization on the next slide:

Edit distance

Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

Visualization on the next slide:

- Operations always refer to the last position (indices are omitted)

Edit distance

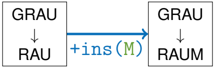
Dynamic programming:

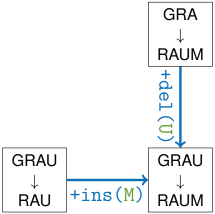
- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

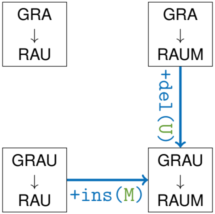
Visualization on the next slide:

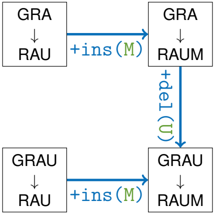
- Operations always refer to the last position (indices are omitted)
- We also display the replaced character on a `replace` operation to visualize operations without costs
  $\Rightarrow$ `repl(A, A)`

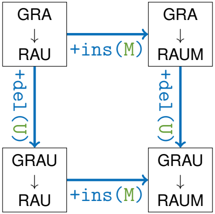
GRAU
$\downarrow$
RAUM

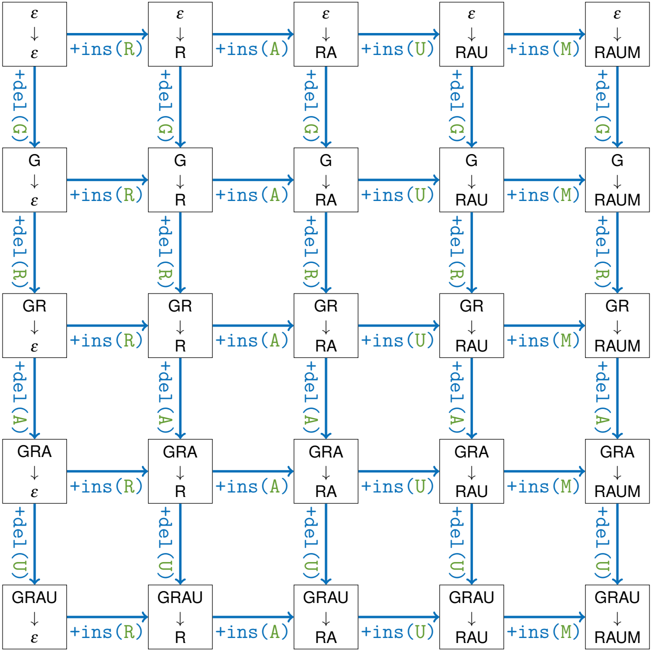

GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM


GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM

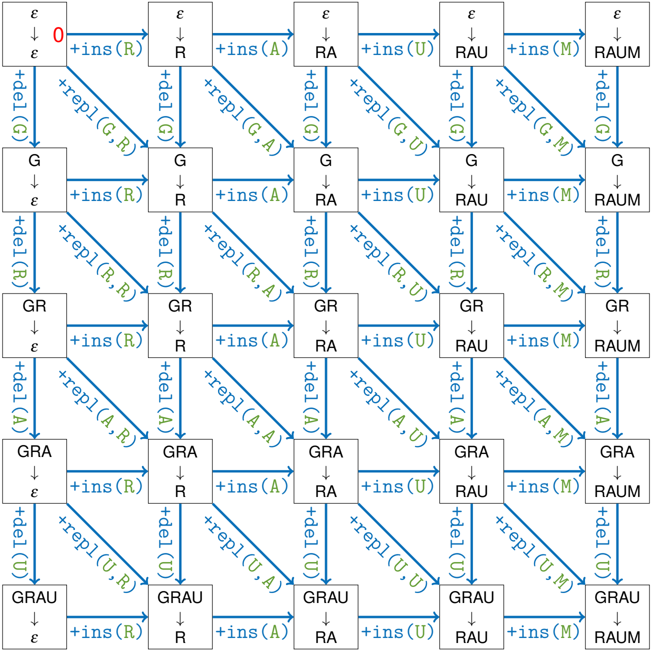

GRA
↓
RAU
GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM

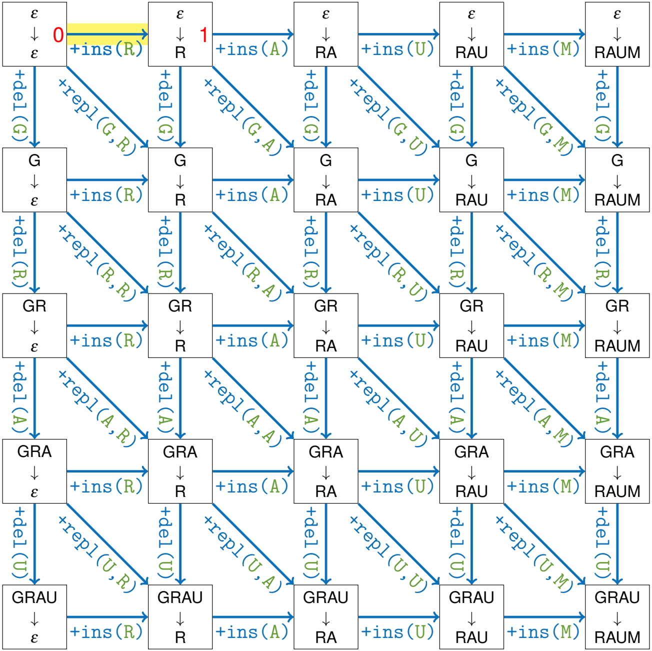

GRA
↓
RAU
+ins(M)
GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM


GRA ↓ RAU
+ins(M)
GRA ↓ RAUM
+del(U)
+del(U)
GRAU ↓ RAU
+ins(M)
GRAU ↓ RAUM

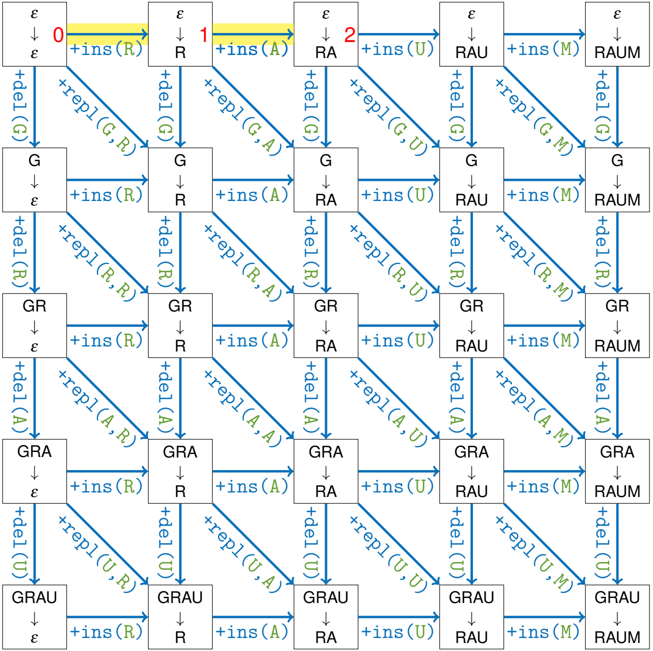

ε → ε
ε → R
ε → RA
ε → RAU
ε → RAUM
G → ε
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)

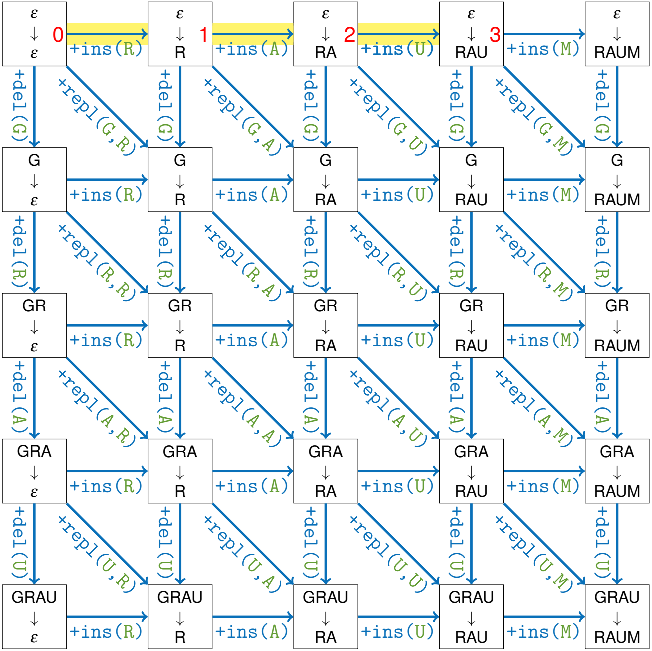
Fast algorithm:
We can determine the edit distance for all combination of partial strings from the top left to bottom right.

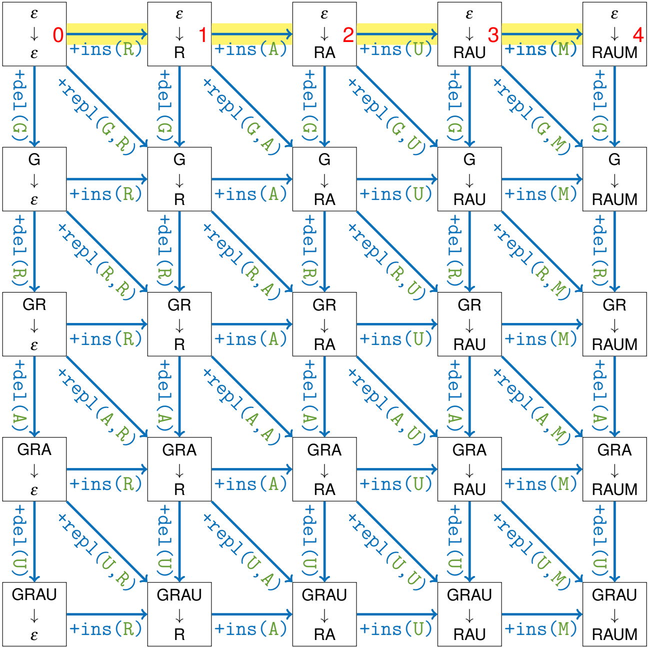

ε ↓ ε
0
+ins(R)
ε ↓ R
+ins(A)
ε ↓ RA
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

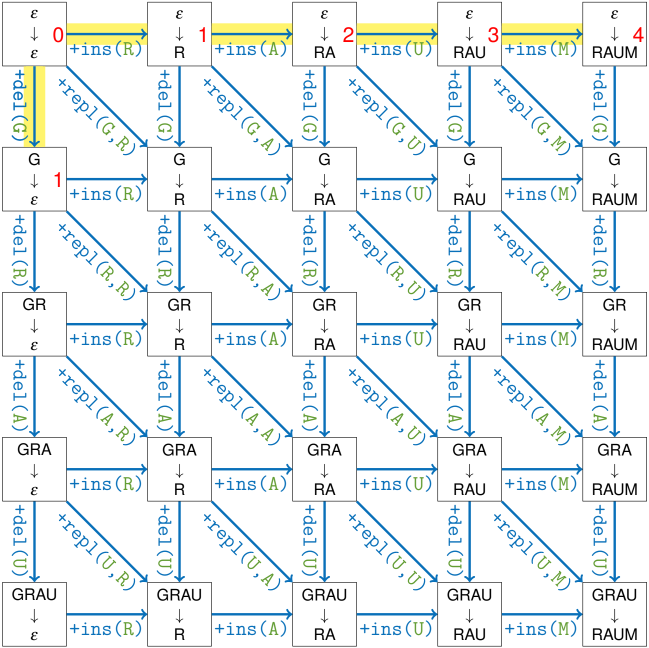


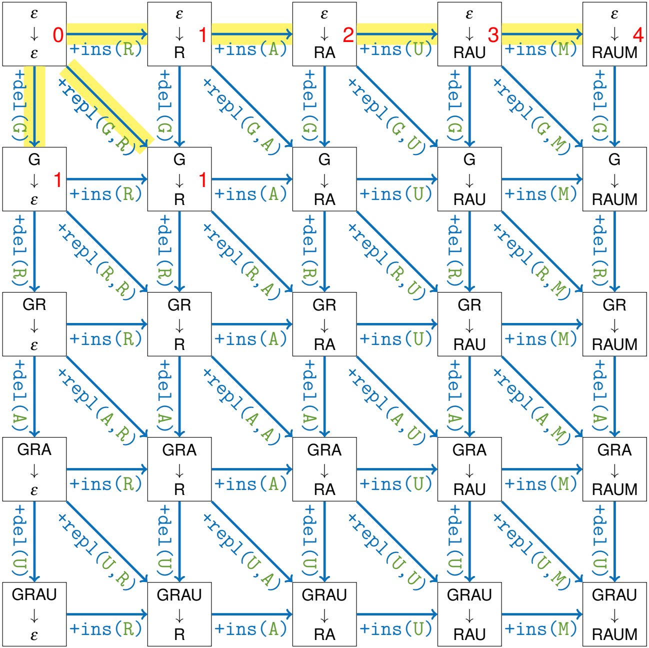
ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

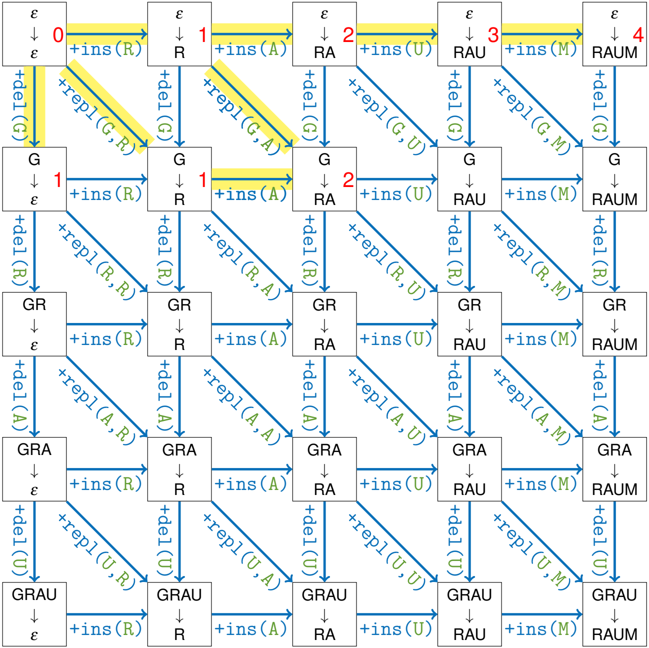

ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

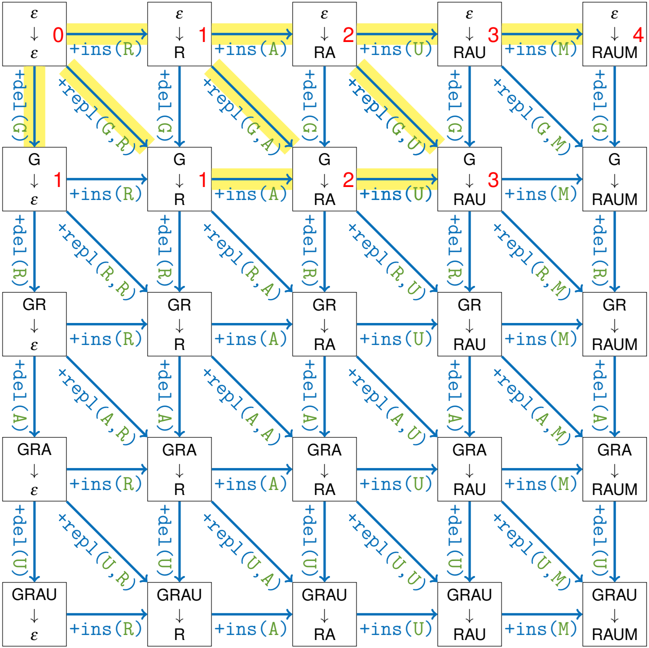

ε ↓ ε
ε ↓ R
ε ↓ RA
ε ↓ RAU
ε ↓ RAUM
0
1
2
3
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

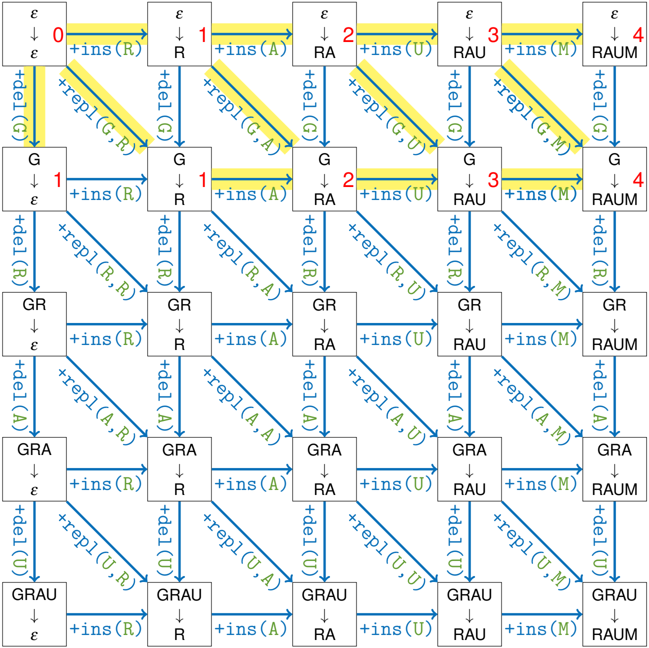

ε ↓ ε
0
ε ↓ R
1
ε ↓ RA
2
ε ↓ RAU
3
ε ↓ RAUM
4
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

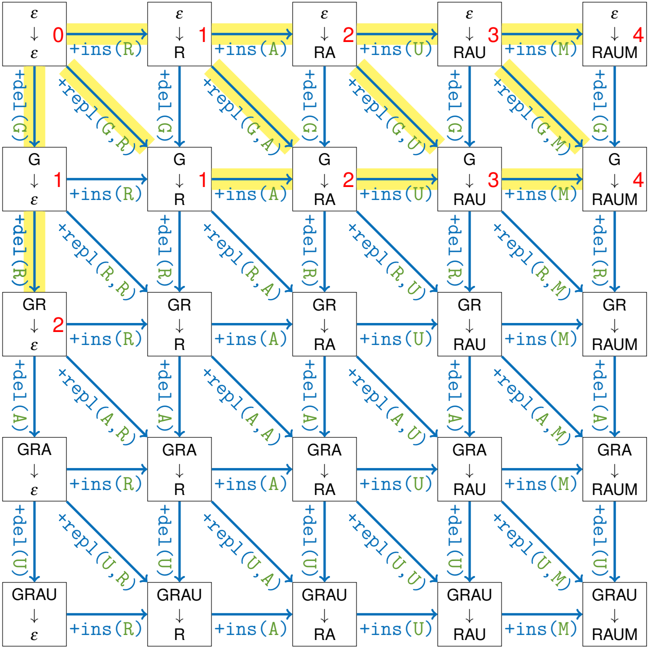

ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

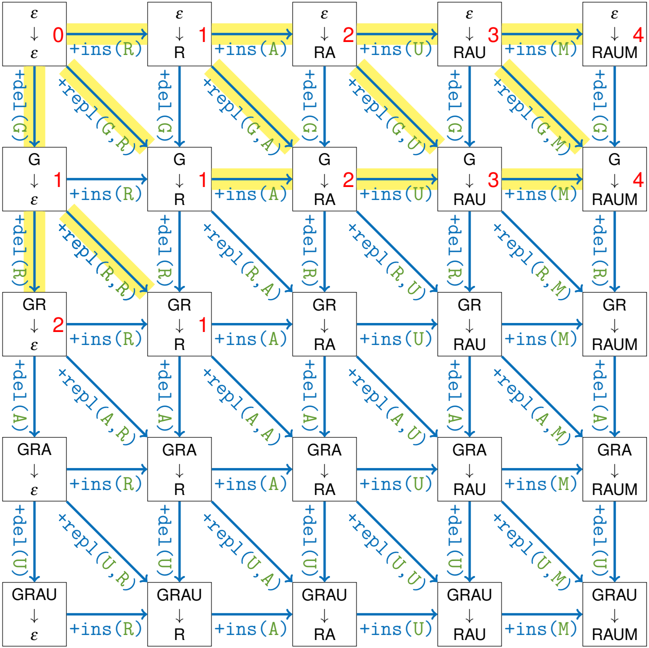

ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
3
+ins(M)
ε ↓ RAUM
4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
1
G ↓ R
1
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
3
+ins(M)
ε ↓ RAUM
4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
1
G ↓ R
1
G ↓ RA
2
G ↓ RAU
3
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
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+repl(A,A)
+repl(A,U)
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+repl(U,A)
+repl(U,U)
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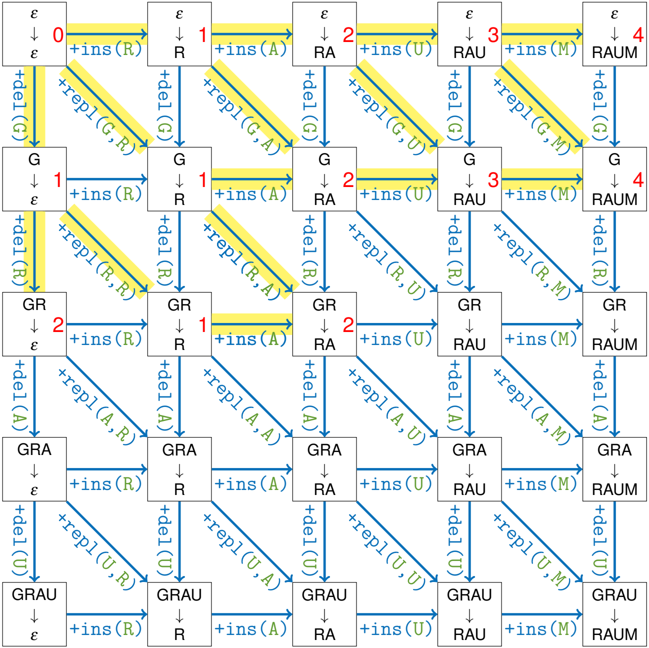
ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
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+repl(G,M)
+repl(R,R)
+repl(R,A)
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+repl(A,A)
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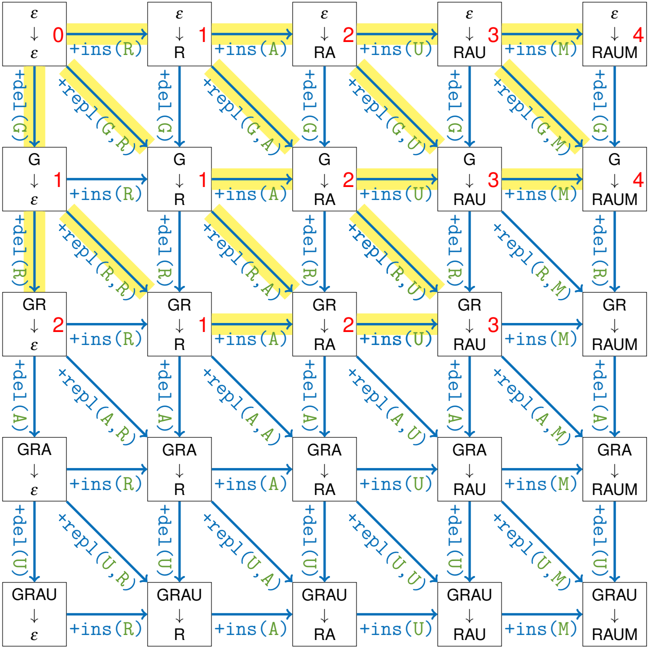
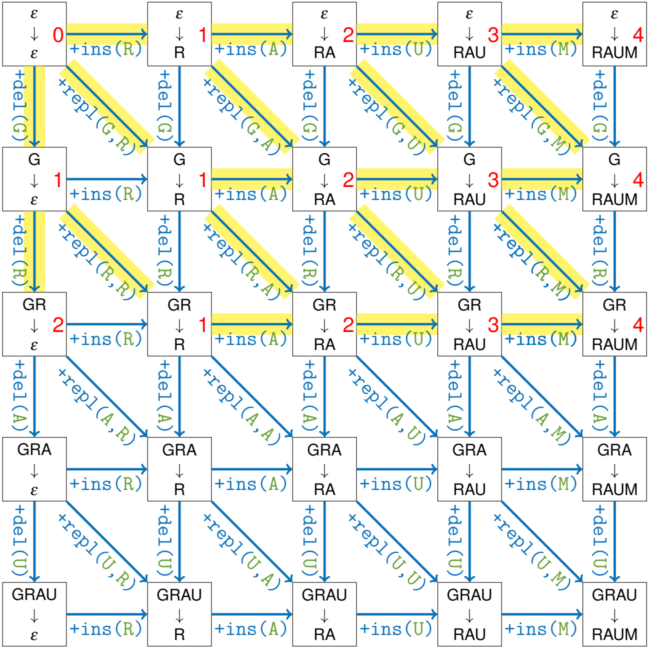
+repl(U,U)
+repl(U,M)





ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
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+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
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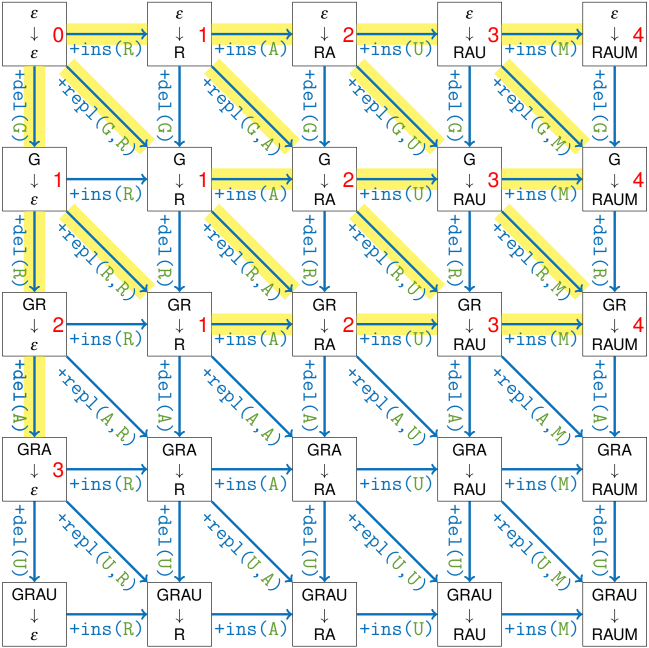
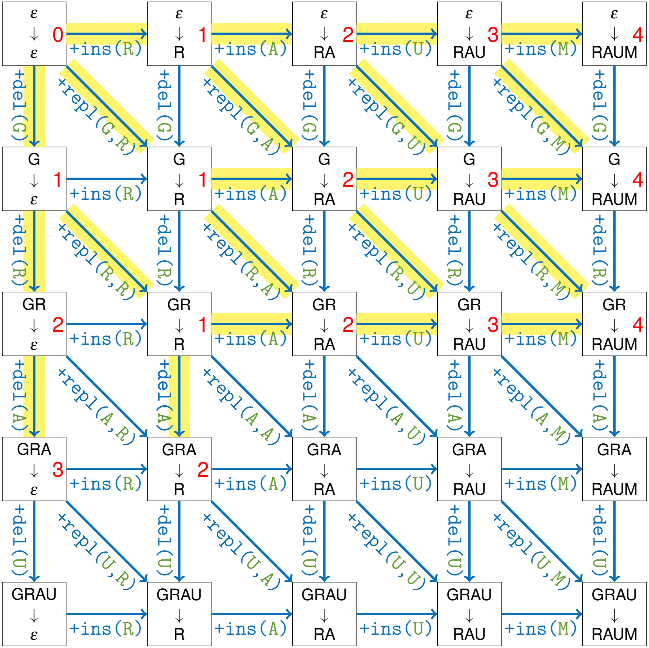
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)


ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
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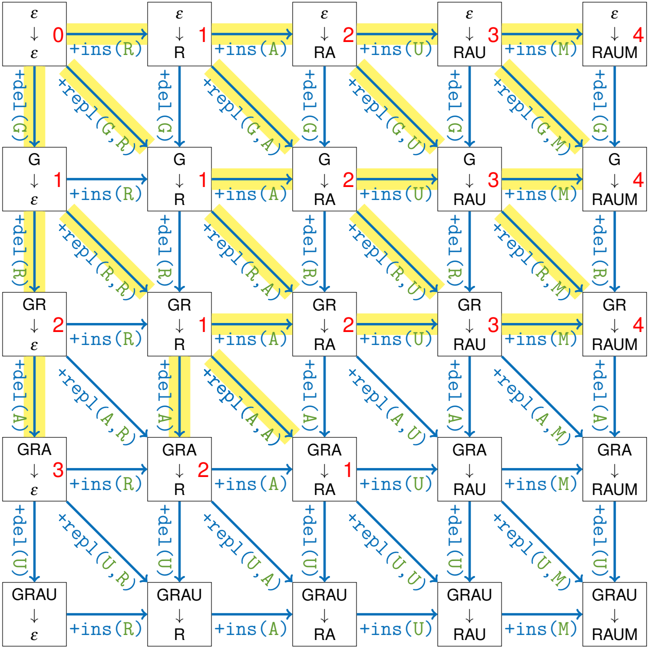
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
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GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
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+del(R)
+del(A)
+del(U)
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+repl(G,M)
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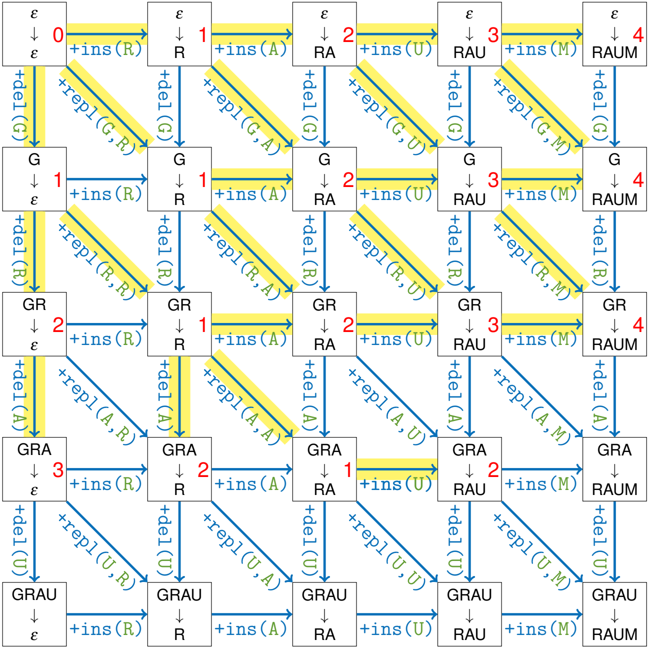
ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
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G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
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+repl(G,A)
+repl(G,U)
+repl(G,M)
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+repl(R,A)
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+repl(A,M)
+repl(U,R)
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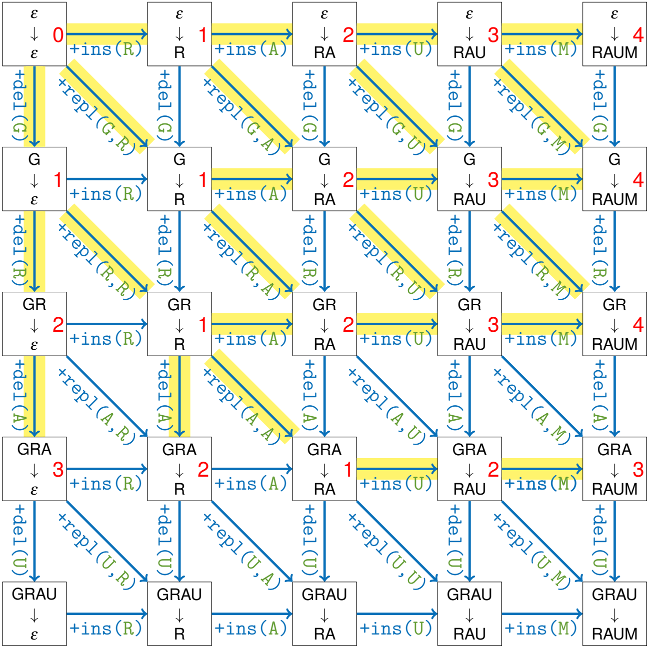
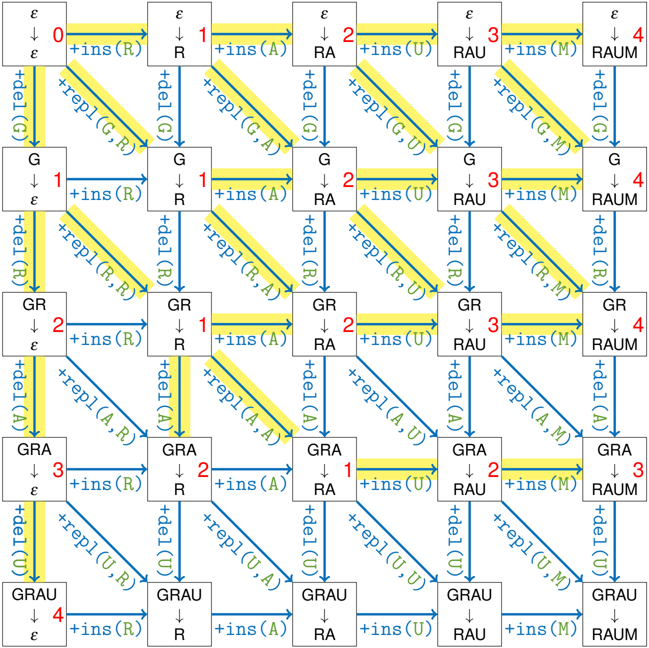
+repl(U,M)




ε ↓ ε 0
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ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
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+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
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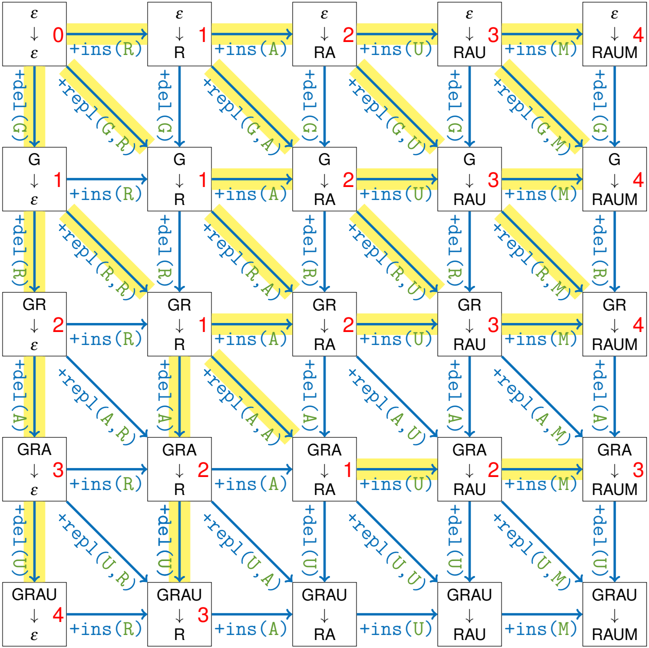

ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
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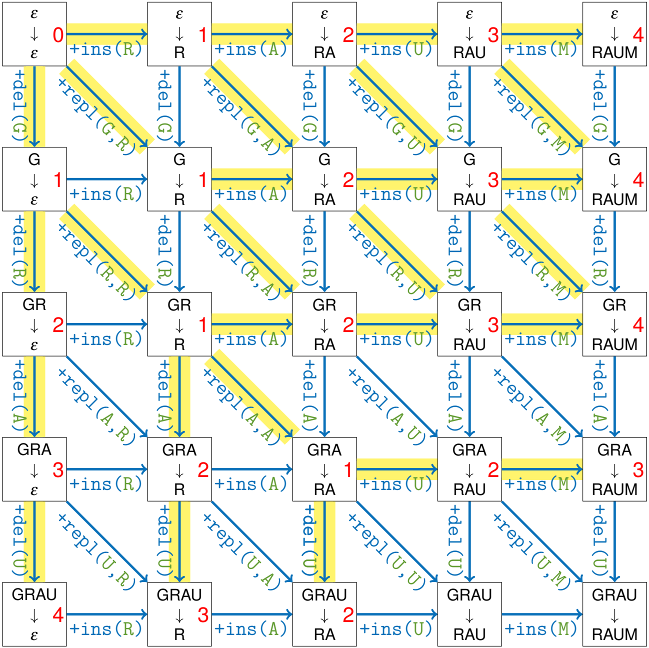
ε ↓ ε 0
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ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
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GRAU ↓ ε 4
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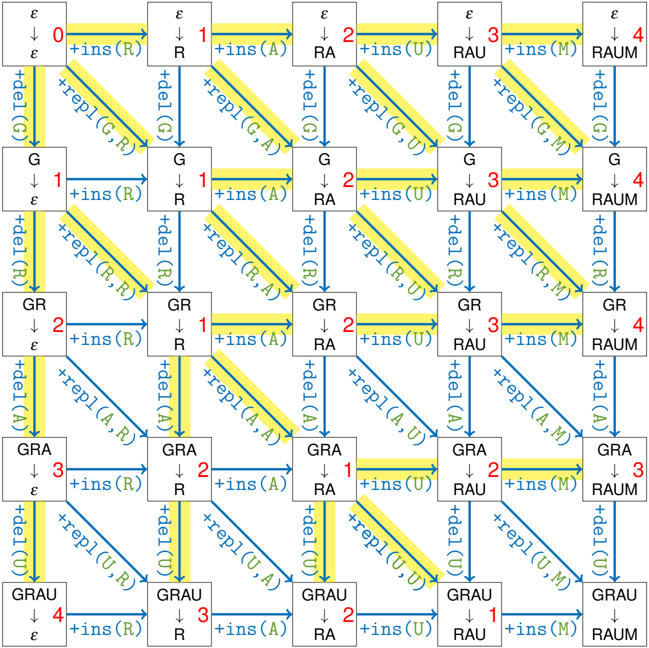
GRAU ↓ RA 2
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
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ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
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+repl(R,A)
+repl(R,U)
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+repl(A,A)
+repl(A,U)
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+repl(U,R)
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+repl(U,U)
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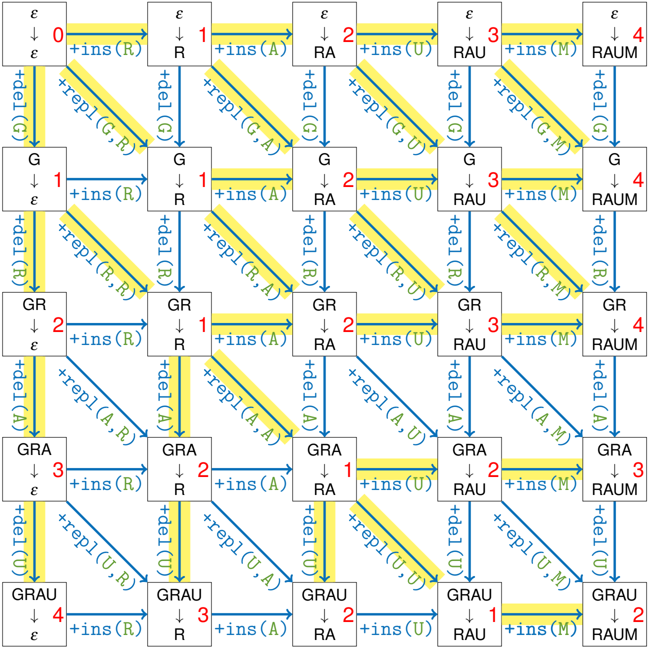

ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM 2
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
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+del(A)
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+repl(G,R)
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+repl(R,R)
+repl(R,A)
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+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

Edit distance

How to get the sequence of operations?

How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)

Edit distance

How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)
- There can be more than one arrows to the three previous entries

Edit distance

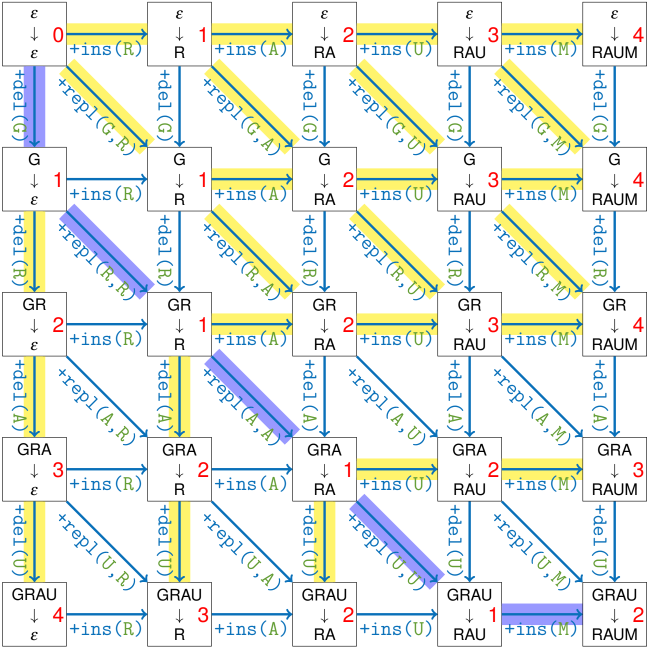
How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)
- There can be more than one arrows to the three previous entries
- If we follow the highlighted path from (n, m) to $(1, 1)$ we get the optimum operations to transform x into y

Edit distance

How to get the sequence of operations?

- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)
- There can be more than one arrows to the three previous entries
- If we follow the highlighted path from (n,m) to $(1,1)$ we get the optimum operations to transform x into y
  - If we can follow more than one path there exist more than one ideal sequence



ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM 2
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
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+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)

Edit distance

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- Recursive computation of …
  - … the same reoccuring partial problems
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  - … a limited number of partial problems
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- In a order that unsolved partial problems consist of already solved partial problems
- The “path” to our solution normally gets computed while searching the best solution
- Dijkstra algorithm is basically dynamic programming!

Additional applications:

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- *Edit distance*: global alignment with $O(n^2)$ space and time consumption

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_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

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_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

  - Solution in $O(n^3)$ time or $O(n^2)$ affine

Edit distance
Additional applications (II)

$O(n^2)$ space consumption might be problematic:

Hirschberg algorithm:

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- Divide-and-conquer approach

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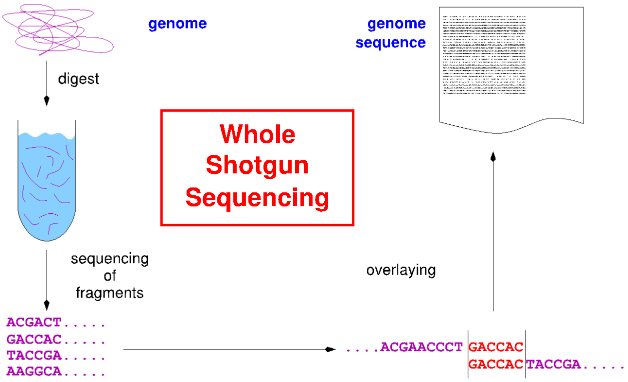
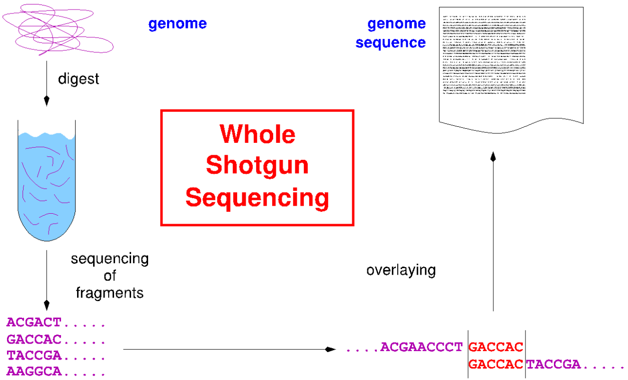
Hirschberg algorithm:

- Divide-and-conquer approach
- $O(n)$ space and $O(n^2)$ time consumption

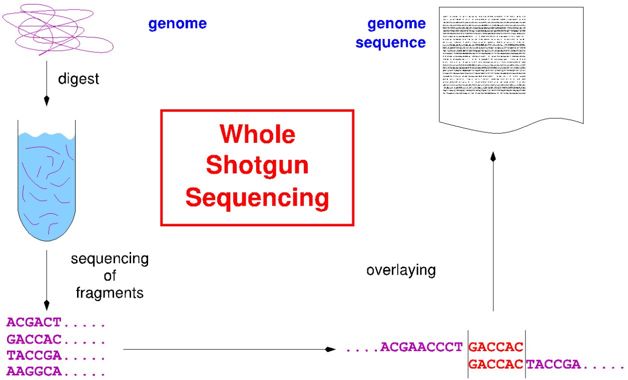
Additional applications (III)



Edit distance
Additional applications (III)



- Sequencing: $O(n^2)$ is too much



- Sequencing: $O(n^2)$ is too much
- Index: suffixtree, suffixarray, burrow-wheeler-transform

Further Literature

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- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Dynamic programming**

[Wik] Dynamic programming
https://en.wikipedia.org/wiki/Dynamic_programming

- **Edit distance**

[Wik] Levenshtein distance
https://en.wikipedia.org/wiki/Levenshtein_distance