

Algorithms and Datastructures

Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg



**UNI
FREIBURG**

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, March 2018

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Divide and Conquer

Introduction



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Concept:



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- **Recursive** application of the algorithm on smaller subproblems

Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
- **Recursive** application of the algorithm on smaller subproblems
- **Direct** solving of small subproblems

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

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Master theorem

Master theorem (Simple Form)

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Divide and Conquer

Maximum Subtotal



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Input:

Output:

Divide and Conquer

Maximum Subtotal

Input:

- Sequence X of n integers

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Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

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Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

Table: Input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Idea:



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- Combine both solutions into a overall solution
- The maximum is located in the **left half (A)** or the **right half (B)**
- The maximum interval can **overlap with the border (C)**

Principle:

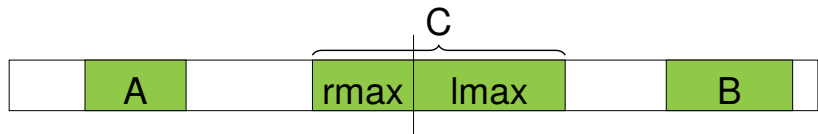


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- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$

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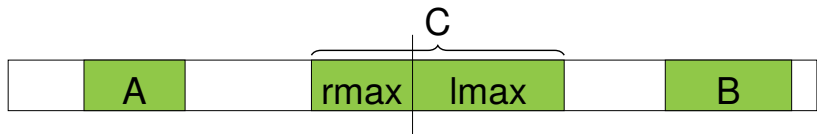
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- To solve *C* we have to calculate *rmax* and *lmax*

Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned
- To solve C we have to calculate rmax and lmax
- Overall solution is maximum of A B and C

Divide and Conquer

Maximum Subtotal - Python



```
def maxSubArray(X, i, j):
```


Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2
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def maxSubArray(X, i, j):  
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    A = maxSubArray(X, i, m)  
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    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])
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def maxSubArray(X, i, j):  
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    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

Divide and Conquer

Maximum Subtotal - Python



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#Alternative trivial case  
def maxSubArray(X, i, j):
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def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
```

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#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max  
def max(a, b, c):
```


Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
```

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

Divide and Conquer

Maximum Subtotal - Python



```
#Alternative implementation max
```

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```
def max(a, b):  
    if a > b:  
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#Alternative implementation max

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def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b  
  
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
<i>sum</i>	58	5	31	90	49	80
<i>lmax</i>	58	58	58	90	90	90

Table: *lmax* example

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- The *sum* and *lmax* are initialized with $X[i]$

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X	58	-53	26	59	-41	31
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<i>lmax</i>	58	58	58	90	90	90

- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update *sum*

Table: $lmax$ example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
$lmax$	58	58	58	90	90	90

- The sum and $lmax$ are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update sum
- If $sum > lmax$ then $lmax$ gets updated

Divide and Conquer

Maximum Subtotal



Call with array of four elements

`maxSubArray(-3,9,-4,7)`

in fact:

`maxSubArray(X,0,3)`
with `X=[-3,9,-4,7]`

Divide and Conquer

Maximum Subtotal



Call with array of four elements

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in fact:

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with $X = [-3, 9, -4, 7]$

Call with array of two elements

$\text{maxSubArray}(-3, 9)$

$\text{maxSubArray}(-4, 7)$

Divide and Conquer

Maximum Subtotal



Call with array of four elements

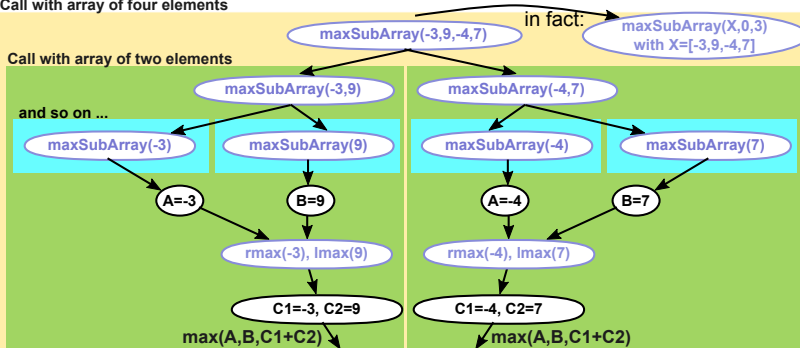


Divide and Conquer

Maximum Subtotal



Call with array of four elements



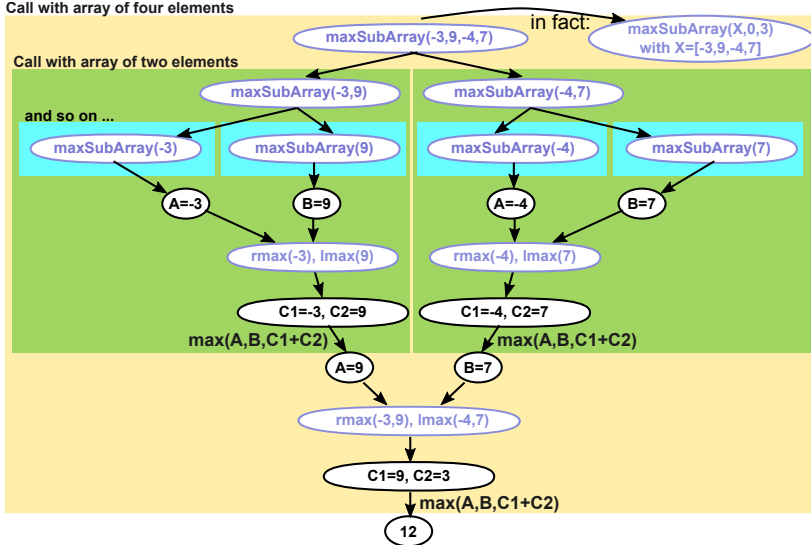
Divide and Conquer

Maximum Subtotal



Call with array of four elements

Call with array of two elements




```
def maxSubArray(X, i, j):  
    if i == j:  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
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def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
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    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
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    C1 = rmax(X, i, m)  
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Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

trivial case

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$\underbrace{\Theta(1)}_{\text{trivial case}}$

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

Recursion equation:

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- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

Divide and Conquer

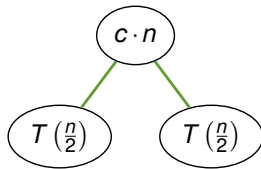
Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

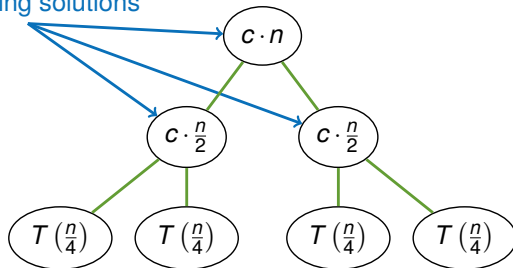
Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



combining solutions



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

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Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

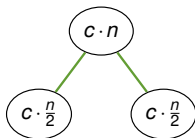
$$c \cdot n$$

1 node processing n elements
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



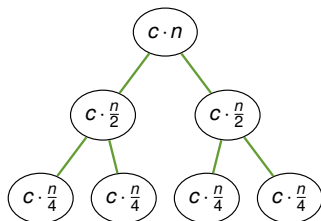
1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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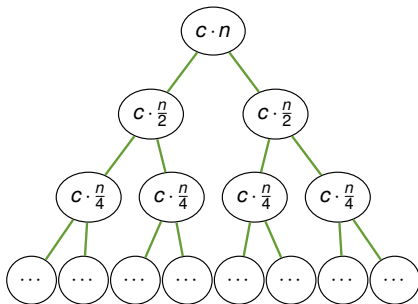
2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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 $\Rightarrow c \cdot n$

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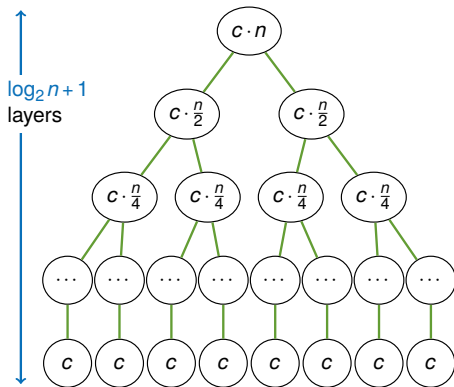
4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^j nodes processing $\frac{n}{2^j}$ elements
 $\Rightarrow 2^j c \cdot \frac{n}{2^j} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
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2^i nodes processing $\frac{n}{2^i}$ elements
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n nodes processing 1 element
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Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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Depth:

Divide and Conquer

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Depth:

- Top level with depth $i = 0$

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$
The costs of merging the solutions and solving of the trivial problems are the same here

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$

The costs of merging the solutions and solving of the trivial problems are the same here

$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Divide and Conquer

Maximum Subtotal - Summary



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- There is an approach running in $\mathcal{O}(n)$ if you assume that all subtotals are positive



Figure: Scanning the array in linear time

Divide and Conquer

Maximum Subtotal - Python



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#Implementation - linear runtime  
def maxSubArray(X):
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Divide and Conquer

Maximum Subtotal - Python



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        if rMax > tMax:  
            tMax, itMax = rMax, irMax  
  
    return (tMax, itMax)
```


Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$



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Induction:



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$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

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Figure: Recursion tree of example

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Figure: Recursion tree of example

Recursion Equations

Recursion Tree Method



$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

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Figure: Levels of the recursion tree



Costs of connecting the partial solutions:
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- Transforming $3^{\log_4 n}$ uses general log rules

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uses $x^{a \cdot b} = (x^a)^b$

- This term will recur in the master theorem

Recursion Equations

Total costs



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$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left(\begin{array}{c} \text{even with} \\ \text{infinite elements} \end{array} \right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

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- Here: The costs of connecting the partial problems dominate

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Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

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- For $|q| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \quad \Rightarrow \text{constant}$$

Recursion Equations

Proof of $O(n^2)$



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Proof of $\mathcal{O}(n^2)$:

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■ We know:

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Proof of $\mathcal{O}(n^2)$:

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- Assumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

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Recursion Equations

Proof of $O(n^2)$



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- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

Recursion Equations

Master theorem (Simple Form)



Simple form:

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$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n)}, \quad a \geq 1, b > 1, c > 0$$

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- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 3, b = 2$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

Recursion Equations

Master theorem (Simple Form)

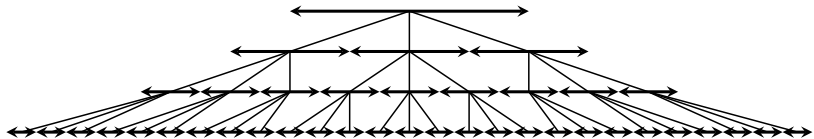


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- Three partial problems with $\frac{1}{2}$ the size

Recursion Equations

Master theorem (Simple Form)

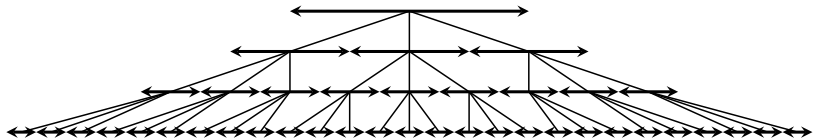


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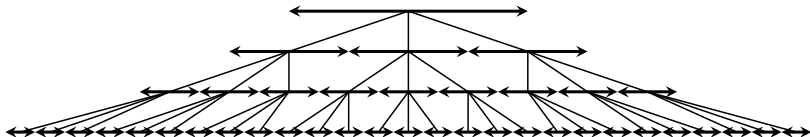


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Recursion Equations

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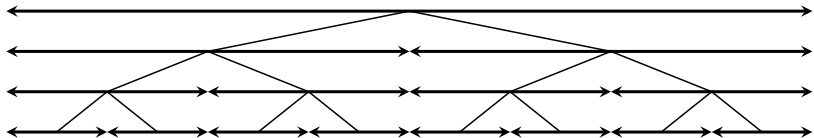


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Recursion Equations

Master theorem (Simple Form)

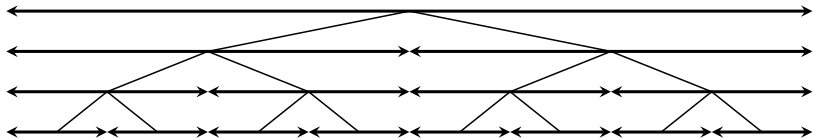


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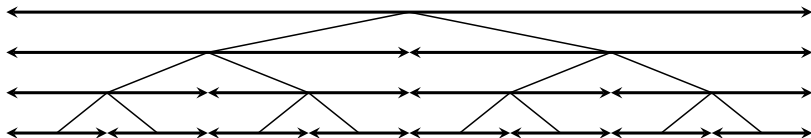


Figure: Simple recursion equation with $a = 2, b = 2$

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Recursion Equations

Master theorem (Simple Form)



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Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Recursion Equations

Master theorem (Simple Form)



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Case 3: $a < b$

Recursion Equations

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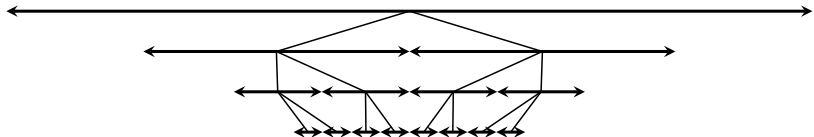


Figure: Simple recursion equation with $a = 2, b = 3$

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Recursion Equations

Master theorem (Simple Form)

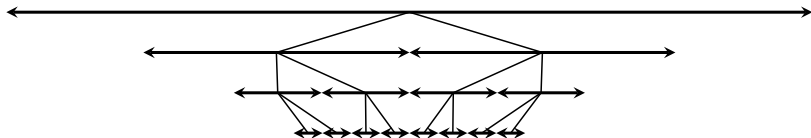


Figure: Simple recursion equation with $a = 2, b = 3$

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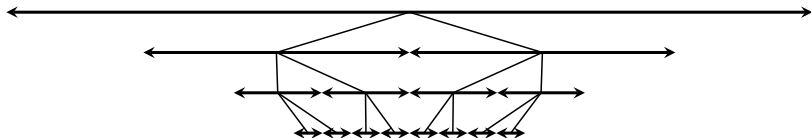


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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



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Solving the partial problems dominates
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- **Case 2:** $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log_b n$ layers

Master theorem (general form):

- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root)
dominates

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$



Case 1 - Example:

if $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

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$$\blacksquare T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

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■ $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, b = 3, f(n) = 17 \cdot n, \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$



Case 2: if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (General Form) - Case 2

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$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

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■ $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1, b = \frac{2}{3}, f(n) = 1, \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Case 3: if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$
Connecting all partial solutions in first layer (root) dominates

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Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if **regularity condition** also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$



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$n \log n$ is *asymptotically* larger than n ,
but not *polynomial* larger



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- **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems

$$T(n) \in \Theta(f(n))$$

■ General

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■ Master theorem

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