Algorithmns and Datastructures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Bioinformatics Group / Department of Computer Science

Algorithmns and Datastructures, January 2017

Structure

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Balanced Trees
- Motivation
- AVL-Trees
- (a,b)-Trees
  - Introduction
  - Runtime Complexity
- Red-Black Trees

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Binary search tree:

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  - if the keys are inserted in ascending / descending order $(20, 19, 18, \ldots)$

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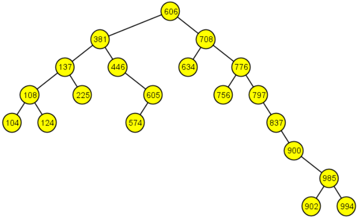

Figure: Binary search tree with random insert [Gna]

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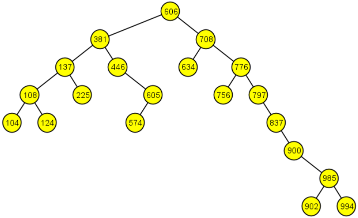

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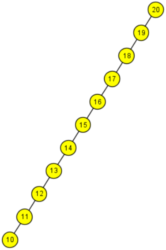

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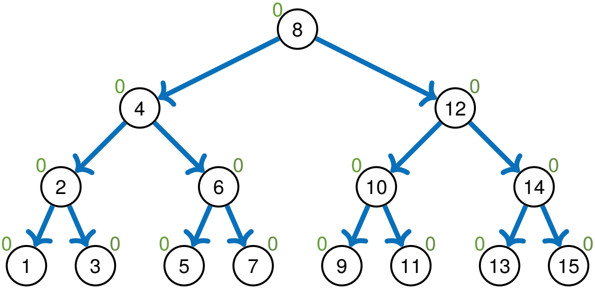


Figure: Example of an AVL-Tree

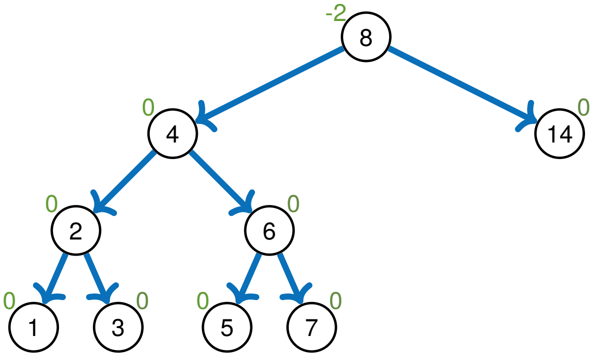


Figure: **Not** an AVL-Tree

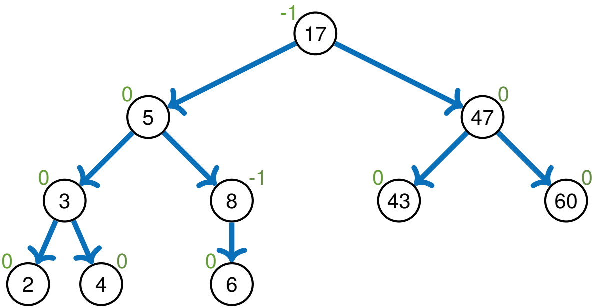


Figure: Another example of an AVL-Tree

Rotation:

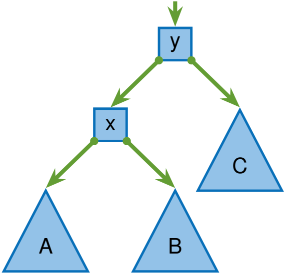
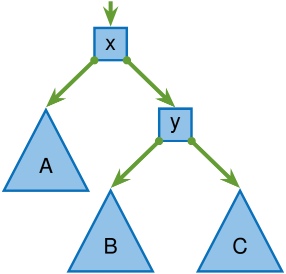


Figure: Before rotating

⇒



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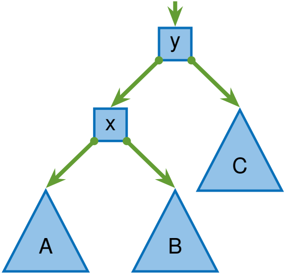

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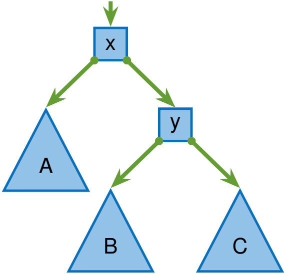

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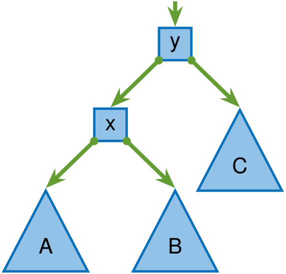

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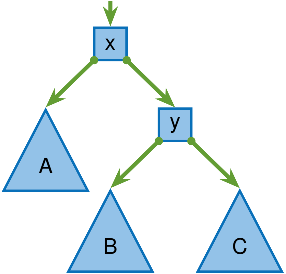

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- Central operation of rebalancing
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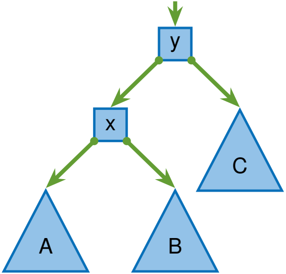

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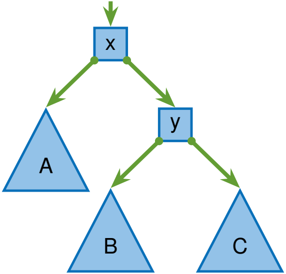

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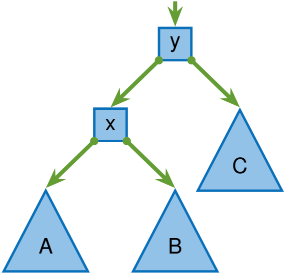

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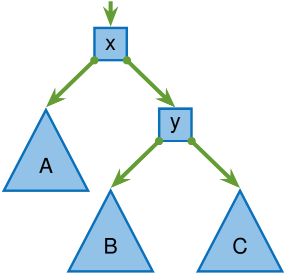

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- Central operation of rebalancing
- After rotation to the right:
  - Subtree A is a layer higher and subtree C a layer lower
  - The parent child relations between nodes x and y have been swapped

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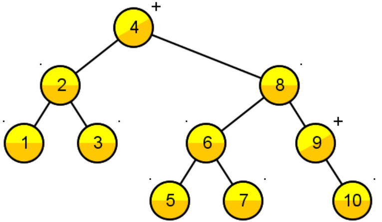

Figure: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

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- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b)-trees

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- So we have space for elements on an `insert` and balance operation

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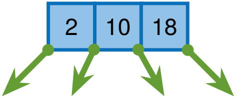

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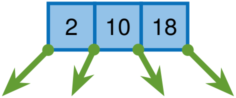

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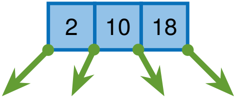

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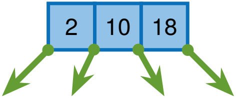

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- We require: $a \geq 2$ and $b \geq 2a-1$

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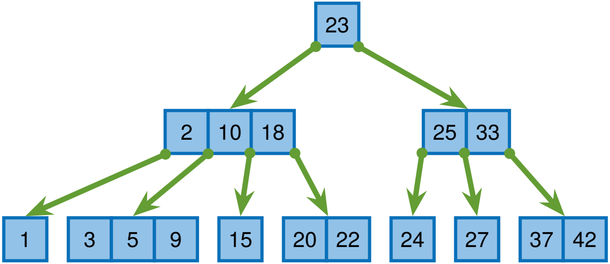

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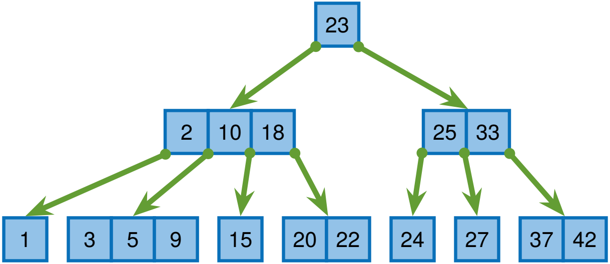

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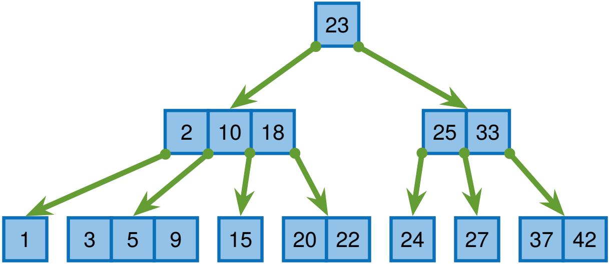

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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

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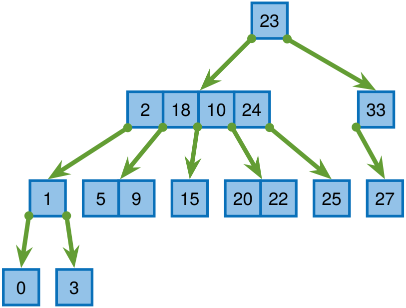

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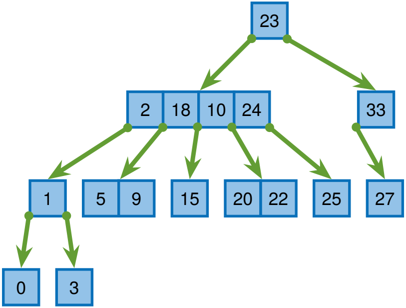

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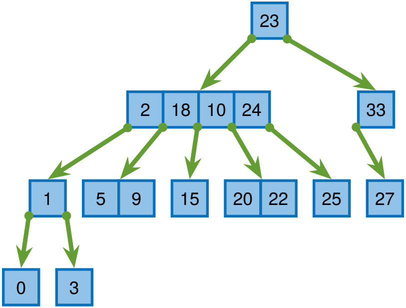

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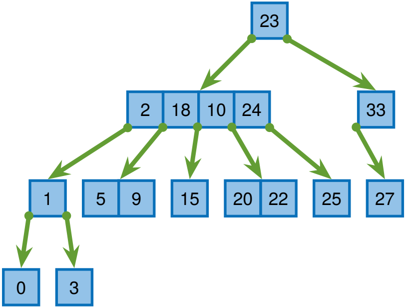

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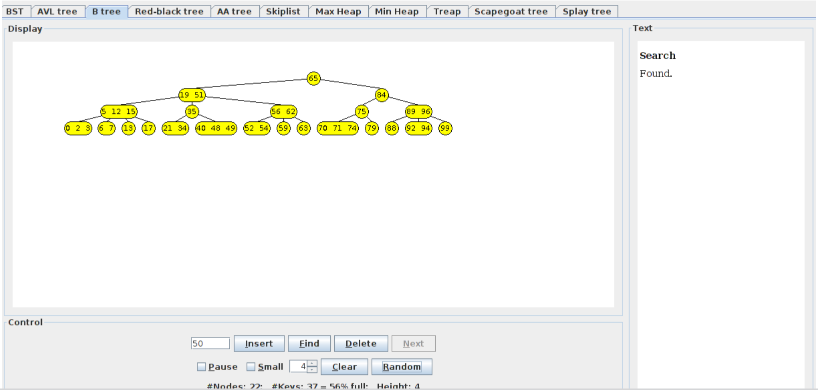

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- Then we **split** the node

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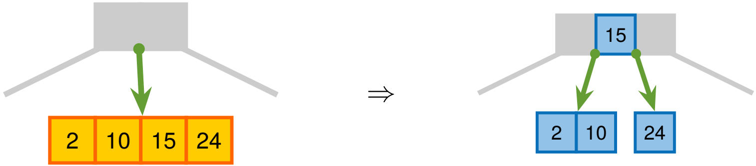

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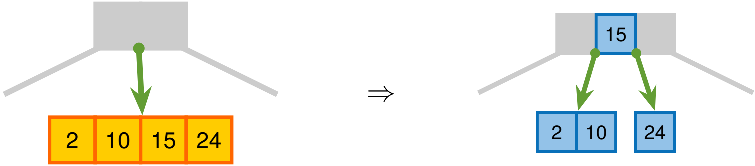

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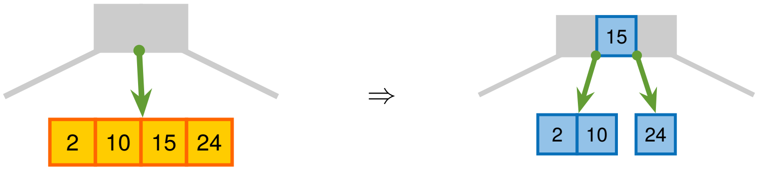

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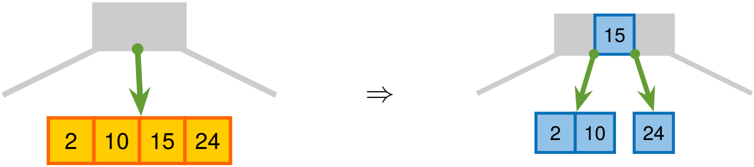

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  - This results in a node with $\mathrm{ceil}\left(\frac{b-1}{2}\right)$ elements, a element for the parent node, and a node with $\mathrm{floor}\left(\frac{b-1}{2}\right)$ elements
  - Thats why we have the limit $b \geq 2a-1$

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- If the node to split is the root we split it and create a new root node
  (The tree is now one level deeper)

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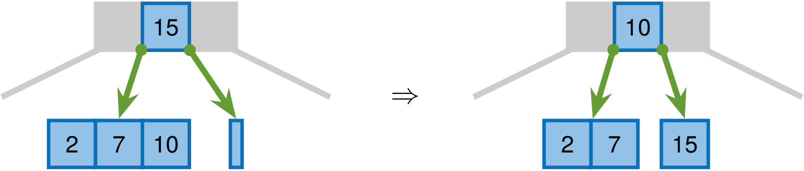

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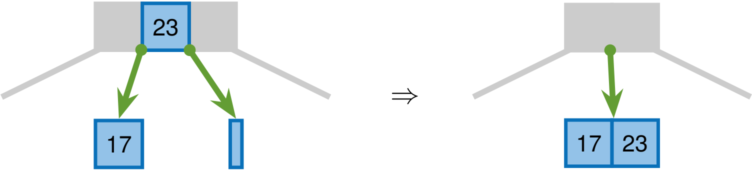

Figure: Combining two nodes

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- If the root has only one child left we take the child as new root
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  - Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$.

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  - Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$.
  - Here is a counter-example for (2,3)-trees, analysis of (2,4)-trees

(2,3)-Tree:

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- Before executing `delete(11)`

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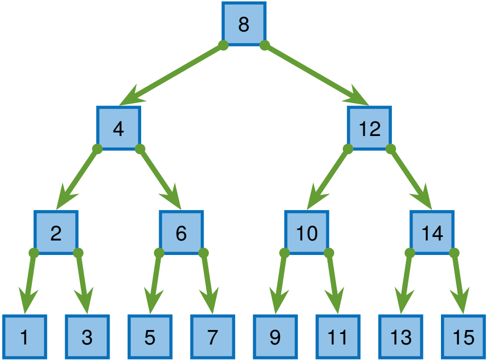

Figure: Normal (2,3)-Tree

(2,3)-Tree:

- Executing `delete(11)`

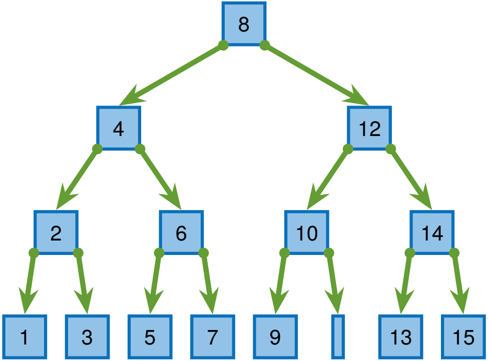

Figure: (2,3)-Tree - Delete step 1

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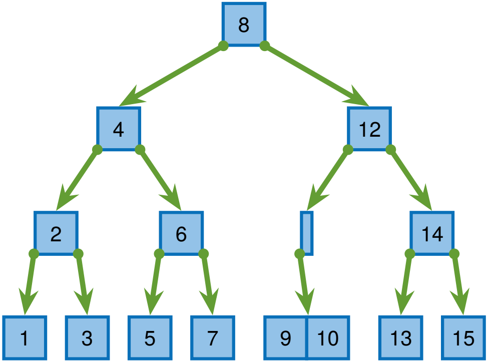

Figure: (2,3)-Tree - Delete step 2

(2,3)-Tree:

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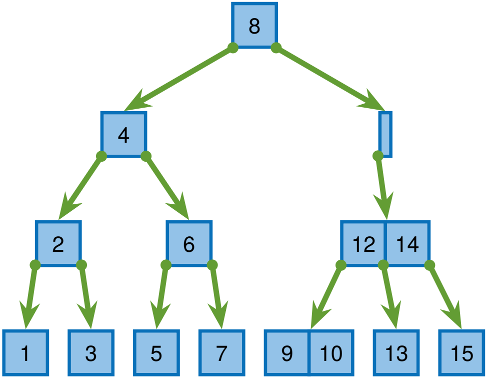

Figure: (2,3)-Tree - Delete step 3

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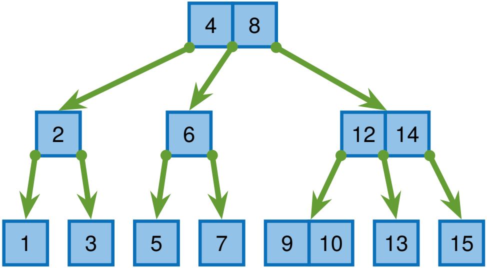

Figure: (2,3)-Tree - Delete step 4

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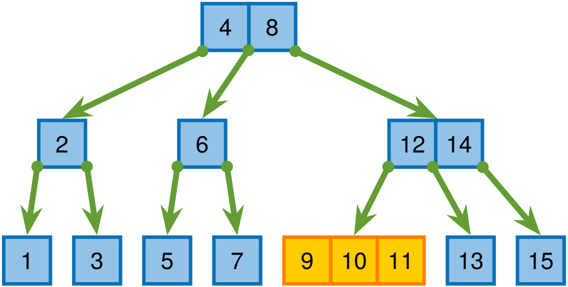

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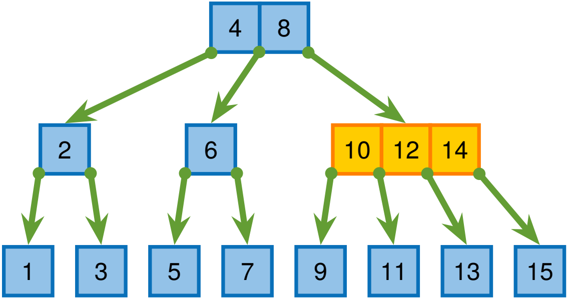

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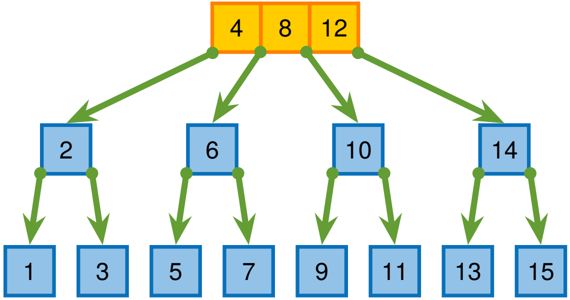

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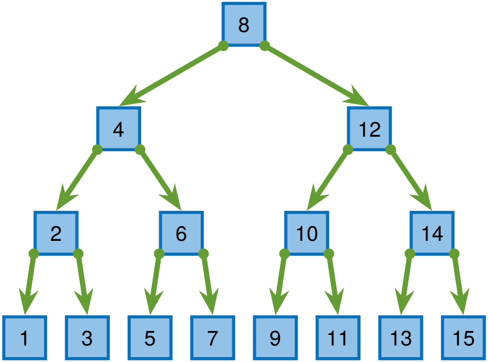

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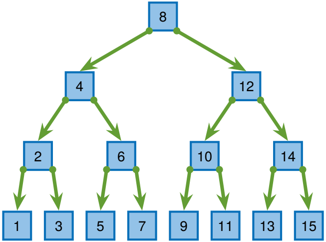

Figure: (2,3)-Tree

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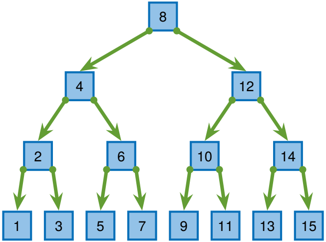
- We are exactly where we started



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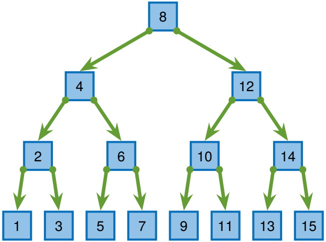

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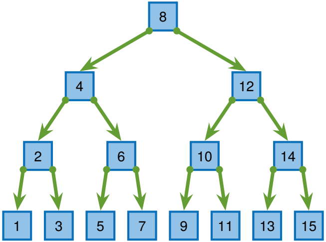
- We need $b \geq 2a$ instead of $b \geq 2a - 1$



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$\Rightarrow$ **Nodes of degree 3 are harmless**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

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Example:

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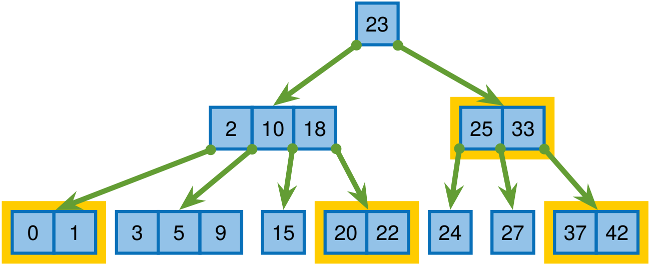

Figure: Tree with potential $\Phi = 4$

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- The costs for operation i are coupled to the difference of the potential levels

$$c_i \leq A \cdot \underbrace{(\Phi_i - \Phi_{i-1})} + B, \qquad A > 0 \text{ and } B > A$$

Number of harmless (degree 3) nodes at operation i. Can be -1, but not smaller than -1

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Case 1: *i-th* operation is an `insert` operation on a full node

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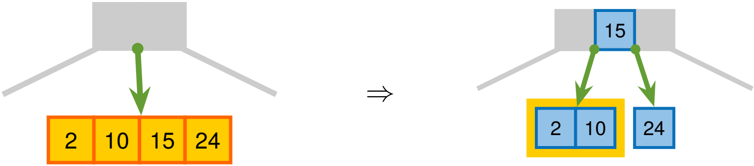

Figure: Splitting a node on `insert`

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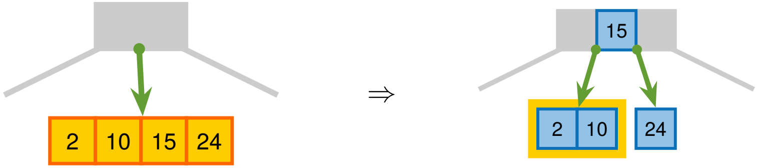

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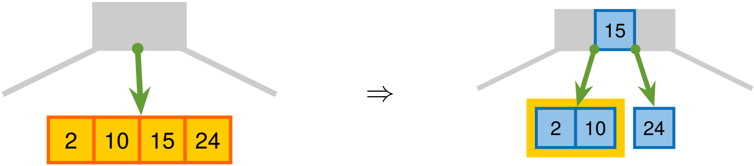

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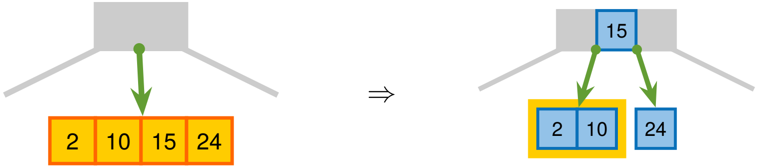

Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

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Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

Case 2: *i-th* operation is an `remove` operation

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- **Case 2.1:** Inner node

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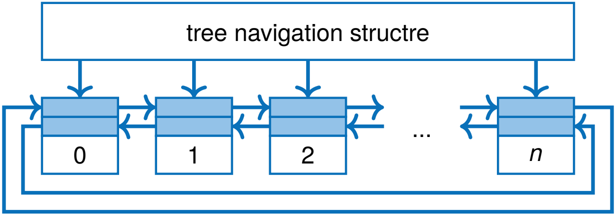

Figure: Tree with doubly linked list

Case 2: *i-th* operation is an `remove` operation

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrowing a node

Case 2: *i-th* operation is an `remove` operation

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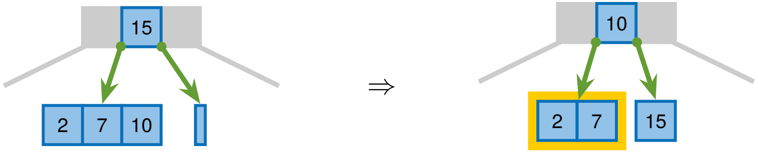

Figure: Borrowing an element case 2.1.1

Case 2: *i-th* operation is an `remove` operation

Case 2: *i-th* operation is an `remove` operation

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Case 2: *i-th* operation is an `remove` operation

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  - Case 2.1.2: Potential lowers by one

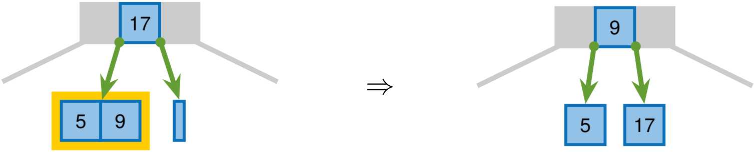

Figure: Borrowing an element case 2.1.2

Case 2: *i-th* operation is an `remove` operation

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- **Case 2.2:** Merging a node

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging a node

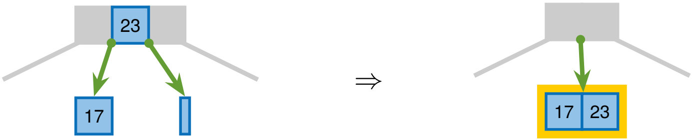

Figure: Merging two nodes

  - Potential rises by one

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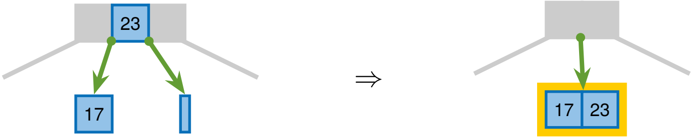

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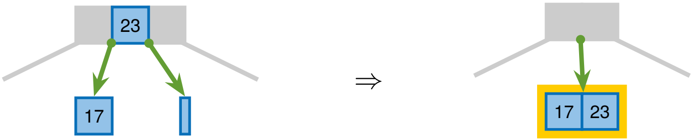

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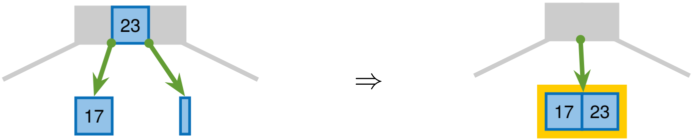

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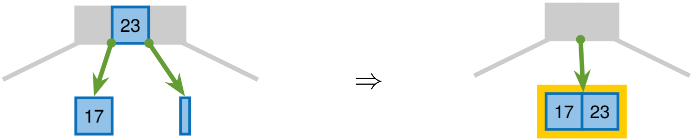

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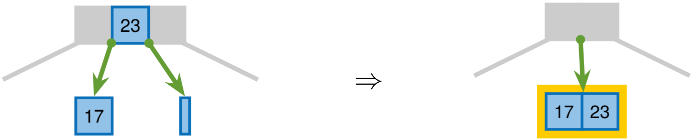

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- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree > 2 or a degree 2 node, which can borrow from a neighbour
- The potential rises by m
- If the "stop-node" is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

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- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

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- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^{n} c_i = O(n)$$

Proof:

$$\sum_{i=0}^{n} c_i \leq \underbrace{A\cdot(\Phi_1-\Phi_0)+B}_{\leq c_1}+\underbrace{A\cdot(\Phi_2-\Phi_1)+B}_{\leq c_1}+\cdots+\underbrace{A\cdot(\Phi_n-\Phi_{n-1})+B}_{\leq c_n}$$

$$= A\cdot(\Phi_n-\Phi_0)+B\cdot n \qquad | \text{ telescope sum}$$

$$= A\cdot\Phi_n+B\cdot n \qquad | \text{ we start with an empty tree}$$

$$< A\cdot n+B\cdot n = O(n) \qquad | \text{ number of degree 3 nodes } < \text{ number of nodes}$$

Structure

Red-Black Tree:

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- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

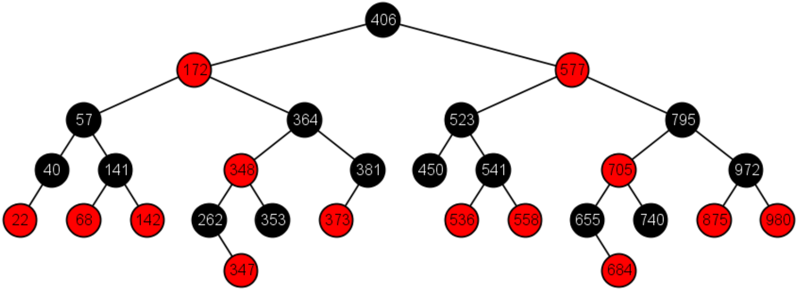

Figure: Example of an red-black-tree [Gna]

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- **Gnarley Trees**

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- **AVL-Tree**

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- **(a,b)-Tree**

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  [Wikb] (a,b)-tree
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- **Red-Black-Tree**

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