Algorithms and Datastructures

Runtime analysis Minsort / Heapsort, Induction
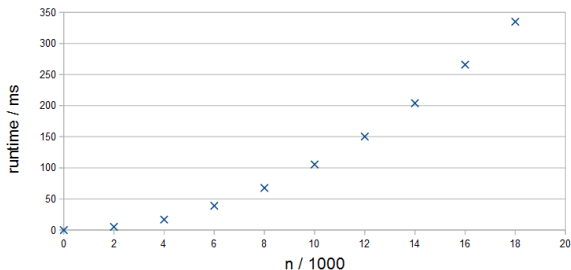
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Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, October 2017

Structure

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How long does the program run?

Runtime analysis - Minsort

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- **Observation:** It is going to be "disproportionaly" slower the more numbers are being sorted

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Runtime analysis - Minsort

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- **Problem:** The runtime is also depending on many other influences, especially:
  - Which kind of computer is the code executed on
  - What is running in the background
  - Which compiler is used to compile the code

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- **Problem:** The runtime is also depending on many other influences, especially:
    - Which kind of computer is the code executed on
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- **Abstraction 1:** Analyze the number of basic operations, rather than analyzing the runtime

Structure

Basic Operations

Incomplete list of basic operations:

- Arithmetic operation, for example: $a + b$
- Assignment of variables, for example: $x = y$
- Function call, for example: *minsort(lst)*

Basic Operations

Intuitive:

lines of code

Better:

lines of machine code

Best:

process cycles

Important:

The actual runtime has to be roughly proportional to the number of operations.

Structure

Runtime analysis - Minsort

How many operations does *Minsort* need?

- **Abstraction 2:** We calculate the upper (lower) bound, rather than counting the operations exactly

  Reason: Runtime is approximated by number of basic operations, but we can still infer:
  - Upper bound
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- **Basic Assmuption:**
  - n is size of the input data (i.e. array)
  - $T(n)$ number of operations for input n

Runtime analysis - Minsort

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- **Claim:** There are constants C_1 and C_2 such that:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

Runtime analysis - Minsort

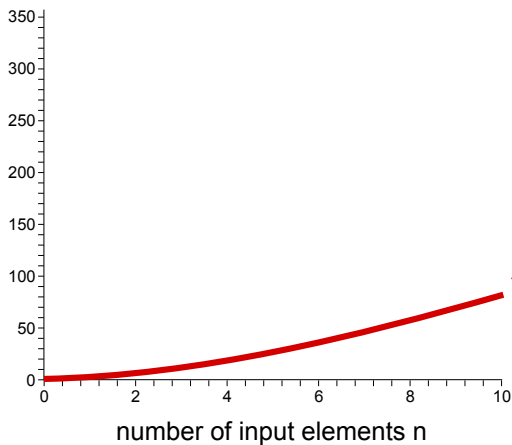How many operations does *Minsort* need?

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$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- This is called "quadratic runtime" (due to n^2)

Runtime Example

Number of Steps
(runtime [s])

$$T(n) = \frac{1}{2}n^2 + 3n$$
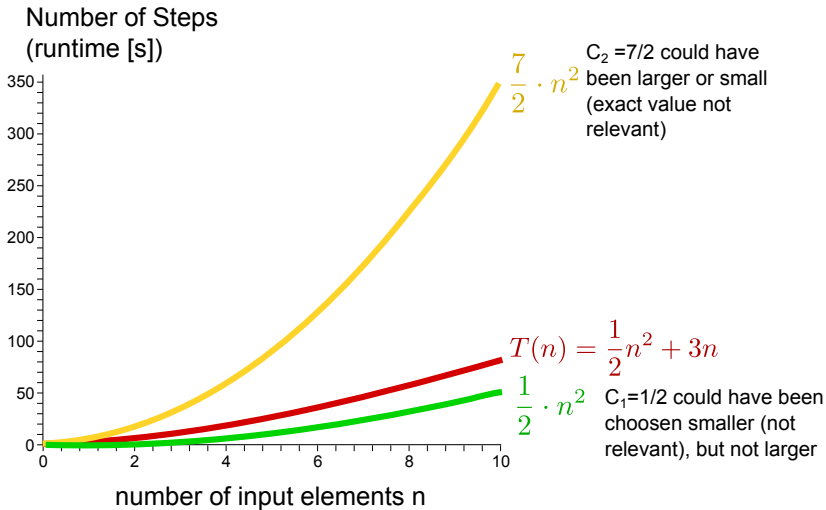
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$$\frac{7}{2} \cdot n^2$$

$C_2 = 7/2$ could have been larger or small (exact value not relevant)

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$\frac{7}{2} \cdot n^2$

$C_2 = 7/2$ could have been larger or small (exact value not relevant)

$T(n) = \frac{1}{2}n^2 + 3n$

$\frac{1}{2} \cdot n^2$

$C_1 = 1/2$ could have been choosen smaller (not relevant), but not larger

number of input elements n

Runtime analysis - Minsort

We declare:

- Runtime of operations: $T(n)$
- Number of Elements: n
- Constants: C_1 (lower bound), C_2 (upper bound)

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- Number of operations in round i: T_i
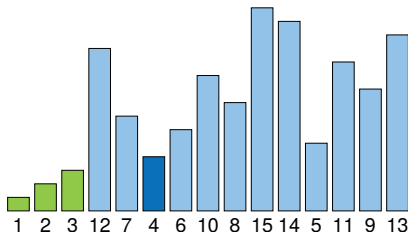

Figure: *Minsort* at the iteration $i = 4$. We have to check $n - 3$ elements

Runtime analysis - Minsort

Compares at each iteration:

Figure: *Minsort* with start data

Runtime analysis - Minsort

Figure: *Minsort* at iteration *i* = 1

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n - 0)$$

Runtime analysis - Minsort

$n-1$ elements left

1 9 5 12 7 4 6 10 8 15 14 3 11 2 13

Figure: *Minsort* at iteration $i = 2$

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$

Runtime analysis - Minsort

Figure: *Minsort* at iteration $i = 3$

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Runtime analysis - Minsort

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Figure: *Minsort* at iteration *i*

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$$T_{n-1} \le C_2' \cdot 2$$
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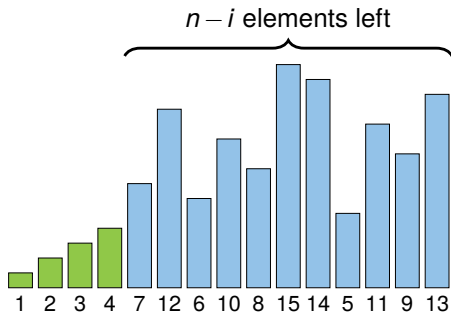
Runtime analysis - Minsort

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Figure: *Minsort* at iteration

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$$T(n) = C_2' \cdot (T_1 + \cdots + T_n) \le \sum_{i=1}^{n} \left(C_2' \cdot i \right)$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```python
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
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Remark: C_2' is cost of comparison $\Rightarrow$ assumed constant

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T(n) &\le \sum_{i=1}^{n} C_2' \cdot i \\
&= C_2' \cdot \sum_{i=1}^{n} i \\
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&= C_2' \cdot \frac{n(n+1)}{2}
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Runtime analysis - Minsort

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Excursion - Small Gauss Formula

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C_1' \cdot (n-i)$$

Runtime analysis - Minsort

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&\geq C_1' \cdot \frac{n \cdot n}{2 \cdot 2} = \frac{C_1'}{4} \cdot n^2
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Runtime analysis - Minsort

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Summarized:

$$\frac{C_1'}{4} \cdot n^2 \leq T(n) \leq C_2' \cdot n^2$$

Quadratic runtime proven:

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    - $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
        - $C \cdot n^2 = 10^{-9}\,\mathrm{s} \cdot 10^{12} = 10^3\,\mathrm{s} = 16.7\,\mathrm{min}$

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- **Quadratic runtime = "big" problems unsolvable**

Structure

Intuitive to extract minimum:

- **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.

Runtime - Heapsort

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Formal:

- Let $T(n)$ be the runtime for the *Heapsort* algorithm with n elements
- On the next pages we will proof $T(n) \leq C \cdot n \log_2 n$

Runtime - Heapsort

Depth of a binary tree:

- **Depth** d**:** longest path through the tree
- Complete binary tree has $n = 2^d - 1$ nodes
- Example: $d = 4$
  $\Rightarrow n = 2^4 - 1 = 15$

Figure: Binary tree with 15 nodes

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- If both has been proven, then $A(n)$ holds for all natural numbers n by **induction**

Induction - Example 1

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

Claim:

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- **Induction basis:** Assumption holds for $d = 1$

Root

$n(1) = 2^1 - 1 = 1$

Figure: Tree of depth 1 has 1 node

Induction - Example 1

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$n(1) = 2^1 - 1 = 1$
$\Rightarrow$ correct ✓

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$$n(d + 1) = 2 \cdot n(d) + 1$$

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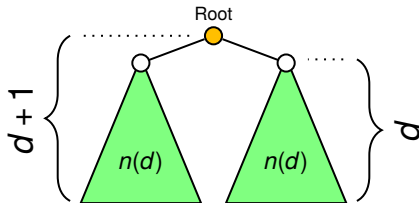- **Induction step:** to show for $d := d + 1$

Figure: Binary tree with subtrees

$$
\begin{aligned}
n(d+1) &= 2 \cdot n(d) + 1 \\
&= 2 \cdot \left(2^d - 1\right) + 1 \\
&= 2^{d+1} - 2 + 1 \\
&= 2^{d+1} - 1 \ \checkmark
\end{aligned}
$$

Induction - Example 1

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Figure: Binary tree with subtrees

$$n(d + 1) = 2 \cdot n(d) + 1$$
$$= 2 \cdot \left(2^d - 1\right) + 1$$
$$= 2^{d+1} - 2 + 1$$
$$= 2^{d+1} - 1 \checkmark$$

$\Rightarrow$ **By induction:** $n(d) = 2^d - 1 \ \forall n \in \mathbb{N} \ \square$

Structure

Heapsort has the following steps:

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Runtime - Heapsort

Heapsort has the following steps:

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    - Remove root as minimal element
    - Move last leaf to root position
    - Repair heap by `sifting`

Runtime of `heapify` depends on depth *d*:

Runtime - Heapsort
Heapify

Runtime of `heapify` depends on depth d:

Depth $4 \rightarrow 2^3$ nodes
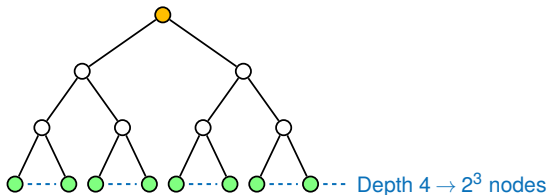
Runtime of `heapify` with depth of d:

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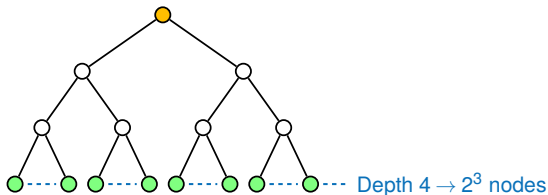
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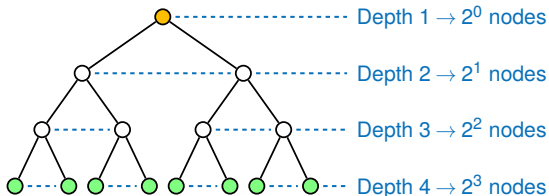
Depth $1 \rightarrow 2^0$ nodes

Depth $2 \rightarrow 2^1$ nodes

Depth $3 \rightarrow 2^2$ nodes

Depth $4 \rightarrow 2^3$ nodes

Runtime of heapify with depth of *d*:

- No costs at depth *d* with 2^{d-1} (or less) nodes
- The cost for sifting with depth 1 is at most $1C$ per node
- In general: Sifting costs are linear with path length **and** number of nodes

Runtime - Heapsort
Heapify

`Heapify` total runtime:

Depth $4 \to 2^3$ nodes

Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	

`Heapify` total runtime:

Depth 3 → 2^2 nodes
Depth 4 → 2^3 nodes
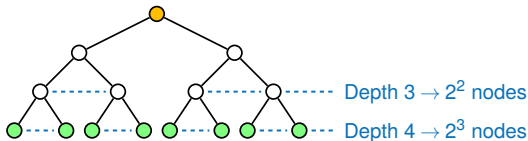
Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	

`Heapify` total runtime:

Depth 2 → 2^1 nodes
Depth 3 → 2^2 nodes
Depth 4 → 2^3 nodes

Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	

`Heapify` total runtime:

Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

`Heapify` total runtime:

Depth $1 \rightarrow 2^0$ nodes
Depth $2 \rightarrow 2^1$ nodes
Depth $3 \rightarrow 2^2$ nodes
Depth $4 \rightarrow 2^3$ nodes

Generally: Depth $d \rightarrow 2^{d-1}$ nodes
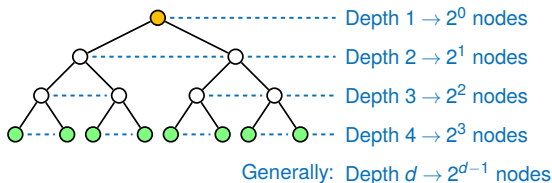
Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

In total: $$T(d) \leq \sum_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right)$$

`Heapify` total runtime:

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Depth $2 \to 2^1$ nodes

Depth $3 \to 2^2$ nodes

Depth $4 \to 2^3$ nodes

Generally: Depth $d \to 2^{d-1}$ nodes

Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	Standard
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	Equation
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

In total: $\quad T(d) \leq \sum\limits_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum\limits_{i=1}^{d} \left(C \cdot i \cdot 2^{d-i} \right)$

Runtime - Heapsort
Heapify

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$d-1$	2^{d-2}	1	$\leq C \cdot 1$	$\leq C \cdot 2$
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	$\leq C \cdot 3$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	$\leq C \cdot 4$

In total: $T(d) \leq \sum_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^{d} \left(C \cdot i \cdot 2^{d-i} \right)$

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- **Hence:** Resulting costs for `heapify`:

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- **Hence:** Resulting costs for heapify:

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- **However:** We want costs in relation to n

`Heapify` total runtime:

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Figure: Partial binary tree

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- $2^{d-1} - 1$ nodes in full tree
  till layer $d - 1$

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Runtime - Heapsort
Heapify

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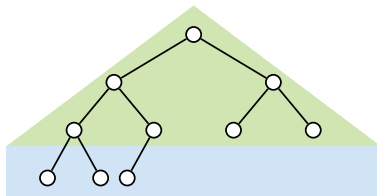 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$

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- Equation multiplied by 2^2
  $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$
- Cost for `heapify`:
  $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

Figure: Partial binary tree

Structure

- **We want to proof (induction assumption):**

$$\underbrace{\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right)}_{A(d) \leq B(d)} \leq \underbrace{2^{d+1}}$$

  - We denote the left side with *A*, the right side with *B*

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Induction - Example 2

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq 2^{d+1}$$

$$\sum_{i=1}^{1} \left(i \cdot 2^{1-i} \right) \leq 2^{1+1}$$

$$2^0 \leq 2^2 \; \checkmark$$

Induction step: $(d := d + 1)$:

- **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \qquad \Rightarrow \qquad A(d+1) \leq B(d+1)$$

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$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

$$\vdots$$

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$$2 \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) + 2 \cdot (d + 1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

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Induction - Example 2

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$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

- **Problem:** Does not work but claim still holds

Working proof:

- Show a little bit stronger claim

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- **Advantage:** Results in a stronger induction assumption

$$\Rightarrow \text{exercise}$$

Structure

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- **Recall**: The depth and number of elements is decreasing
  - **Hence**: $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$

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- **Recall**: The depth and number of elements is decreasing
  - **Hence**: $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
  - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \quad \text{(holds for } n > 2)$$

Runtime - Heapsort

Runtime costs:

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- Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - Upper bound: $C_2 \cdot n \log_2 n \geq T(n)$ (for $n \geq 2$)
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    - $\Rightarrow C_1$ and C_2 are constant

Structure

Base of Logarithms

Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient $\frac{1}{\log_b a}$

Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602\ldots \cdot 3.322\ldots = 2$ ✓
- $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3$ ✓

Runtime Example

Runtime of $n \log_2 n$**:**

- Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \le T(n) \le C_2 \cdot n \cdot log_2 n \quad \text{for } n \ge 2$$

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  - $C = 1\,\text{ns}$ (1 simple instruction $\approx 1\,\text{ns}$)
  - $n = 2^{20}$ (1 million numbers = $4\,\text{MB}$ with $4\,\text{B}$/number)
    - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{20} \cdot 20 = 21.0\,\text{ms}$

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        - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{20} \cdot 20 = 21.0\,\text{ms}$
    - $n = 2^{30}$ (1 billion numbers = 4 GB)
        - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{30} \cdot 30 = 32\,\text{s}$

Runtime Example

Runtime of $n \log_2 n$**:**

- Assume we have constants C_1 and C_2 with

  $$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot log_2 n \quad \text{for } n \geq 2$$

- $2\times$ elements $\Rightarrow$ only slightly larger than $2\times$ runtime

  - C = 1 ns (1 simple instruction $\approx$ 1 ns)
  - $n = 2^{20}$ (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{20} \cdot 20 = 21.0\,\text{ms}$
  - $n = 2^{30}$ (1 billion numbers = 4 GB)
    - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{30} \cdot 30 = 32\,\text{s}$

- **Runtime** $n \log_2 n$ **is nearly as good as linear!**

Further Literature

- **General for this Lecture**

  [CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
  Introduction to Algorithms.
  MIT Press, Cambridge, Mass, 2001.

  [MS08] Kurt Mehlhorn and Peter Sanders.
  Algorithms and data structures, 2008.
  https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Mathematical Induction**

  [Wik] Mathematical induction
      https://en.wikipedia.org/wiki/Mathematical_
      induction