

# Algorithms and Datastructures

## Cache Efficiency, Divide and Conquer

Albert-Ludwigs-Universität Freiburg



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Algorithms and Datastructures, March 2018

## Cache Efficiency

- Introduction

- Cache Organization

## Divide and Conquer

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- Up to now we always counted the **number of operations**
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- Up to now we always counted the **number of operations**
- Assuming this is a good measure for the runtime of a algorithm/tool
- Today we will see examples where this is not suitable

### Example:

- We sum up all elements of a field  $a$  of size  $n$  in ...
  - natural order:

$$\text{sum}(a) = a[1] + a[2] + \dots + a[n]$$

- random order:

$$\text{sum}(a) = a[21] + a[5] + \dots + a[8]$$

### Python:

```
def init(size):  
    """Creates the dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

### Python:

```
def run(param):  
    """Processes the dataset."""  
  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```





Figure: Summing elements in linear order

```
def init(size):  
    """Creates a randomly ordered dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
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    return (order, data)
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Figure: Summing elements in random order

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- Accessing elements in random order takes a lot longer (Factor 10) **Why?**
- The costs in terms of memory access are very different

## Cache Efficiency

Introduction

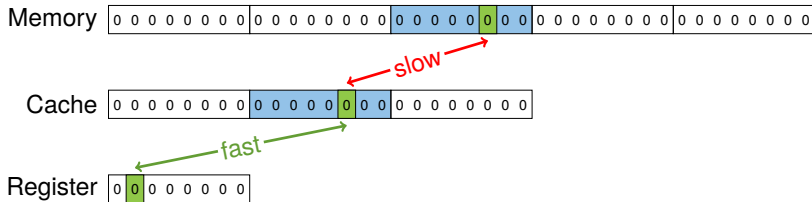
Cache Organization

## Divide and Conquer

Introduction

# Cache Efficiency

## CPU Cache



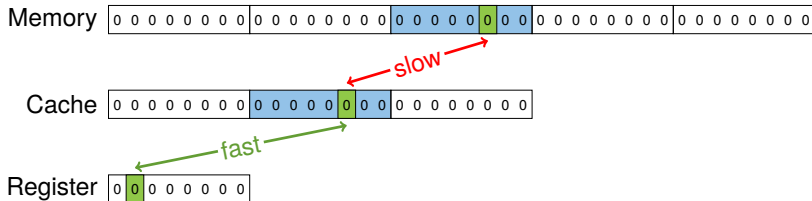


# Cache Efficiency

## CPU Cache



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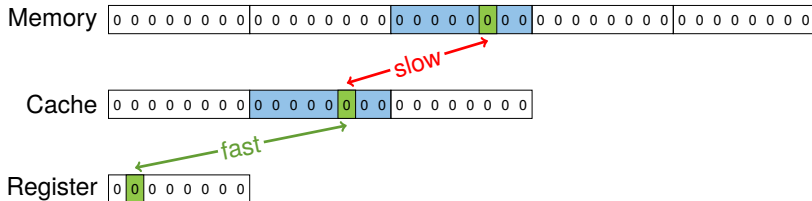
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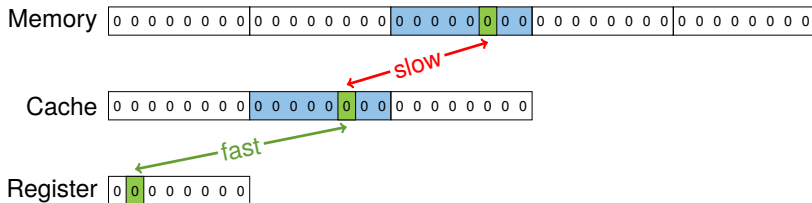


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- As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

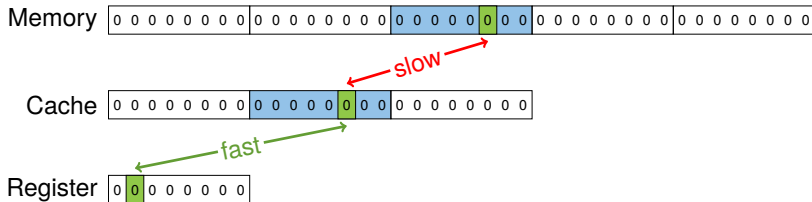
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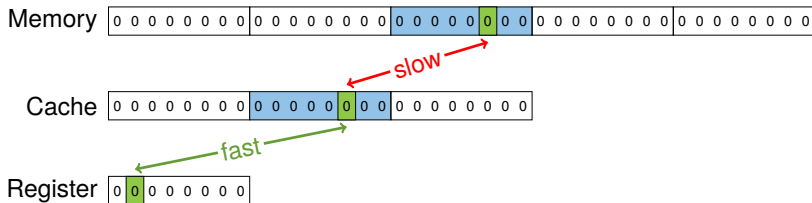


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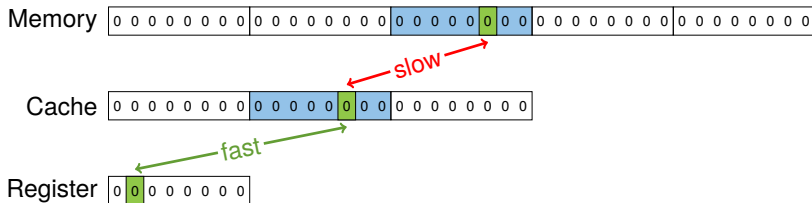
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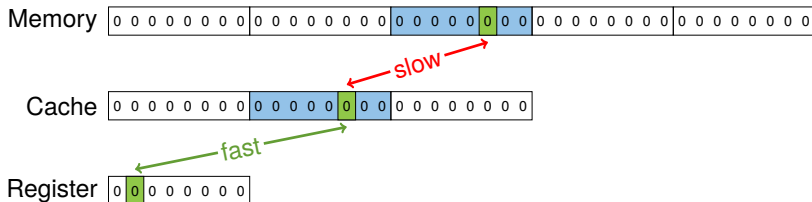
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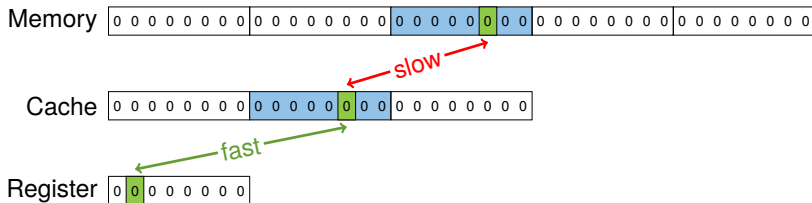
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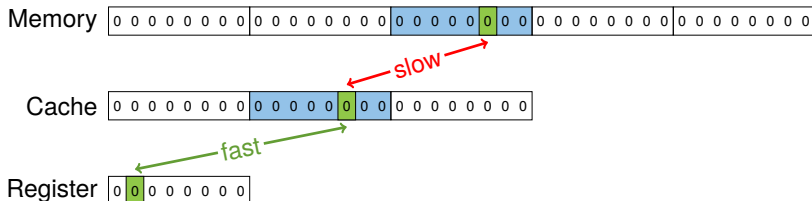
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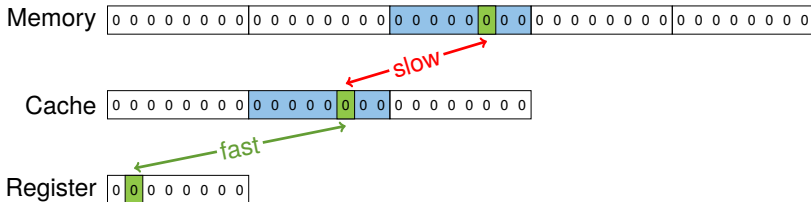
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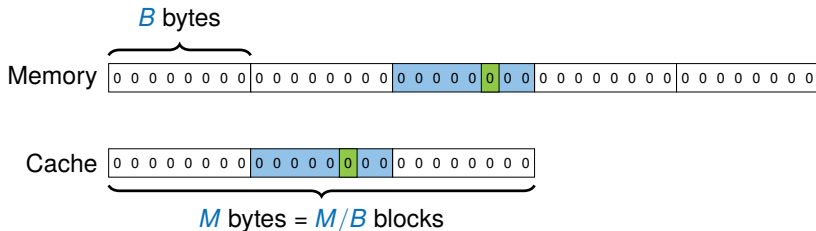


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- Details of discarding are not the topic for today

# Cache Efficiency

## Block Operations



## Terminology:



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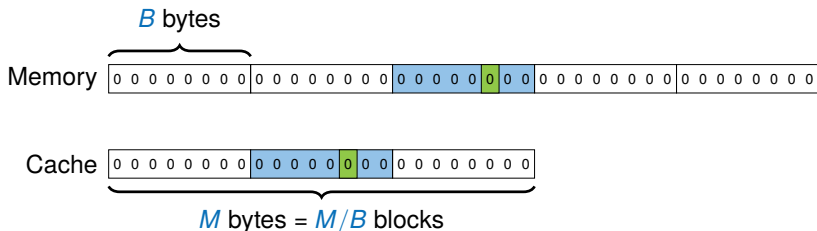
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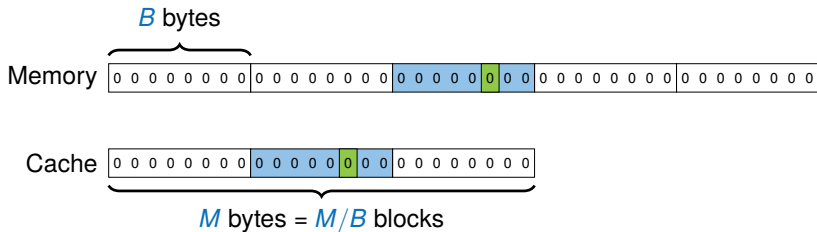


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- If data is not in fast memory, the corresponding block is loaded into the **cache**

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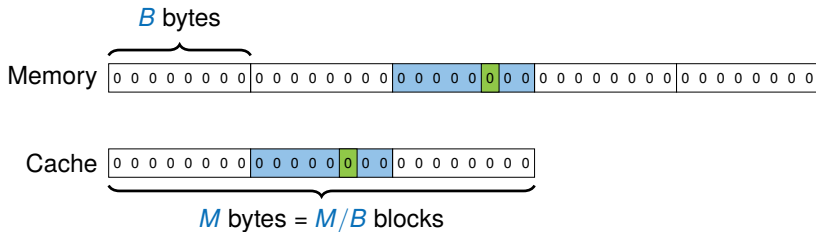
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- We use the number of **block operations** as runtime estimation
- We ignore runtime costs of cache accesses / management



**Figure:** Comparison good / bad locality

**Accessing the cache  $B$  times:**

- **Best case:** 1 block operation  $\rightarrow$  good locality
- **Worst case:**  $B$  block operations  $\rightarrow$  bad locality





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- Cache handling is only interesting when the input size is greater than  $M$

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  - Many operating systems use free system memory as disk cache



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- Block operations on disk-cache are called **IOs**  
(input / output operations)
- These also fall under the term **cache efficiency** or **IO efficiency**





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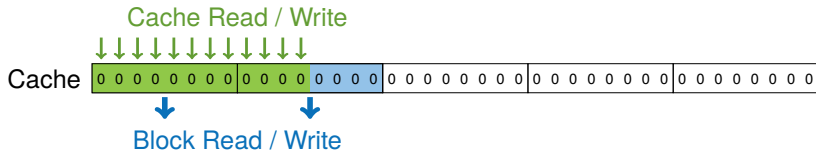


Figure: Good locality of sum operation

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Figure: Bad locality of sum operation

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- If **not**  $n \gg M$  this might occur with a high probability



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Figure: *QuickSort* with pivot-element

- **At start:** Pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- Do required changes *in place*



- **End point:**  $k$  is left to left-most element greater than pivot  
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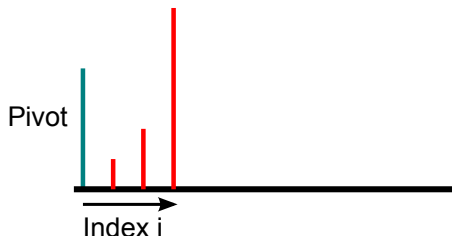
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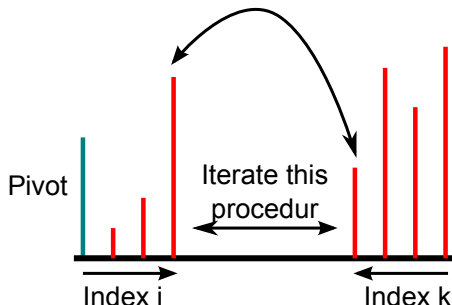
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### Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

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$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left( A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

# Cache Efficiency

## Block Operations - QuickSort



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- Let  $IO(n)$  be the number of **block operations** for input size  $n$
- Assumptions as before but recursion depth is  $k = \log_2 \frac{n}{B}$   
Why?

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in \mathcal{O}\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

## Cache Efficiency

Introduction

Cache Organization

## Divide and Conquer

Introduction



# Divide and Conquer

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- **Direct** solving of sufficiently small subproblems

# Divide and Conquer

Introduction - Python



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# Divide and Conquer

## Introduction - Python



- Function `solve` for solving a `problem` of size `n`



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```
def solve(problem):  
    if n < threshold:  
        return solution # solve directly  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
    return S1 + S2 + ... + Sk
```

# Divide and Conquer

## Features



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- Realization of efficient solutions
  - If trivial solution is  $\in O(1)$
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  - And the number of subproblems is limited

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- Recursion depth should not get too big (stack / memory overhead)

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- Typically conceptual demanding

# Divide and Conquer

## Example - Maximum Subtotal



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Value	31	-41	59	26	-53	58	97	-93	-23	84

**Output:** Sum: 187, Start: 2, End: 6

### Application:

- Maximum profit of buying and selling shares



Figure: Stock value over time

### **Naive solution (brute force)**



### Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

Example - Maximum Subtotal - Python



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## Runtime - Upper bound

### Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

## Example - Maximum Subtotal



**Upper bound:**

# Divide and Conquer

## Example - Maximum Subtotal

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Table: Operations

$i$	Additions	$j$
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$



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- For each  $j$  we have at least  $\frac{n}{3}$  additions
- We need at least  $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$  steps

# Divide and Conquer

Example - Maximum Subtotal - Runtime



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**Runtime:**

### Runtime:

- With  $T(n) \in O(n^3)$  and  $T(n) \in \Omega(n^3)$  we know:

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- It is hard to solve the problem in a worse way ...

# Divide and Conquer

Example - Maximum Subtotal - Runtime



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**Current approach:**



### Current approach:

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$$S_{i,j} = X[i] + X[i+1] + \dots + X[j]$$

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### Better approach:

- Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i+1] + \dots + X[j] + X[j+1]$$

$$S_{i,j+1} = S_{i,j} + X[j+1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

# Divide and Conquer

Example - Maximum Subtotal - Python



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**Better solution:**

### Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
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■ Runtime  $\in O(n^2)$

# Divide and Conquer

## Example - Maximum Subtotal

### Divide and Conquer:



### Divide and Conquer Idea to solve:

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# Divide and Conquer

Example - Maximum Subtotal

## Divide and Conquer:



## Divide and Conquer Idea to solve:

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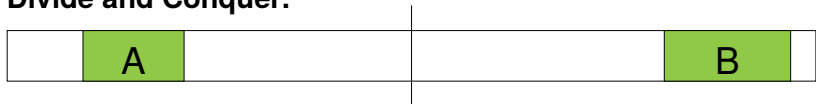
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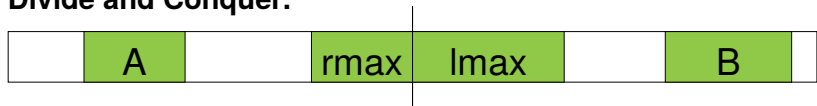
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# Divide and Conquer

## Example - Maximum Subtotal



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```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

## ■ General

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Caching

[Wik] [Cache](https://en.wikipedia.org/wiki/Cache)

`https://en.wikipedia.org/wiki/Cache`