

Algorithms and Datastructures

Open Addressing, Priority Queue

Albert-Ludwigs-Universität Freiburg



**UNI
FREIBURG**

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, December 2016

Feedback

Exercises

Lecture

Hashing

Recapitulation

Treatment of hash collisions

Open Addressing

Summary

Priority Queue

Introduction

Feedback from the exercises



Feedback from the lecture



**UNI
FREIBURG**

Hashing:

- No hash function is good for all key sets!
 - This can cannot work, because a big universe is mapped onto a small set
- For random key sets also simple hash function work, e.g.

$$\Rightarrow h(x) = x \mod m$$

- Then the random keys make sure that it is distributed evenly
- To find a good hash function for every key set universal hashing is needed
 - Then however, for a fixed set of keys not every hash function is suitable, but only some

Rehashing:

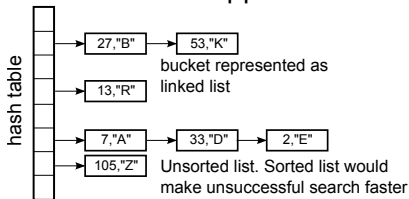
- It is possible to get bad hash functions with universal hashing, but it is unlikely
- This is determinable by monitoring the maximum bucket size
- If a pre-defined level is exceeded, then a **rehash** is performed

How to rehash?

- New hash table with a new random hash function
- Copy elements into the new table
 - Expensive but happens not often
 - Therefore the average cost is low
 - Look at **amortized analysis** in the next lecture

Buckets as linked list:

- Each bucket is a linked list
- Colliding keys are inserted into the linked list of a bucket, either sorted or appended at the end



- Operations in $O(1)$ are possible if a suitable table size and hash function is selected
- Worst case $O(n)$, e.g. table size of 1
- Dynamic number of elements is possible

- For colliding keys we choose a new free entry
- Static, fixed number of elements
- The **probe sequence** determines for each key, in which sequence all hash table entries are searched for a free bucket
 - If a Entry is already occupied, then iteratively the [following entry](#) can be checked. If a free entry is found the element is inserted.
 - If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry have been found.

Definitions:

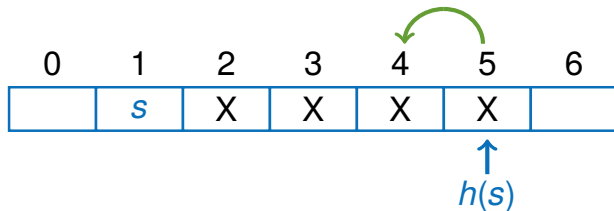
$h(s)$ Hash function for key s

$g(s, j)$ Probing function for key s with overflow positions

$j \in \{0, \dots, m-1\}$ e.g. $g(s, j) = j$

- The **probe sequence** is calculated by

$$h(s, j) = (h(s) - g(s, j)) \mod m \in \{0, \dots, m-1\}$$



```
def insert(s, value):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
        j += 1  
  
    t[(h(s) - g(s, j)) mod m] \  
      = (s, value)
```

```
def lookup(s):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
        is not None:  
  
        if t[(h(s) - g(s, j)) mod m][0] == s:  
            return t[(h(s) - g(s, j)) mod m]  
  
        j += 1  
  
    return None
```



Figure: Linear probe sequence

- Check the element with lower index: $g(s, j) := j$
 \Rightarrow Hash function: $h(s, j) = (h(s) - j) \bmod m$
- This leads to the following probe sequence

$$h(s), h(s) - 1, h(s) - 2, \dots, \underbrace{0, m-1, m-2, \dots, h(s) + 1}_{\text{clipping}}$$

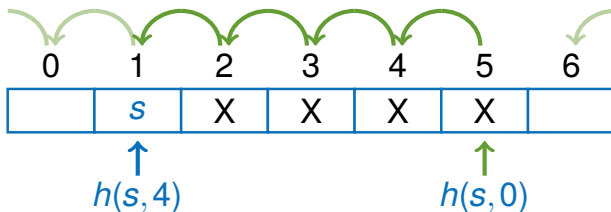


Figure: Linear probe sequence

- Can result in primary clustering
- Dealing with a hash collision will result in a higher probability of hash collisions in close entries

Example:

- Keys: {12, 53, 5, 15, 2, 19}
- Hash function: $h(s, j) = (s \bmod 7 - j) \bmod 7$
- $t.\text{insert}(12, \text{"A"}), h(12, 0) = 5$

0	1	2	3	4	5	6
					12, A	

- $t.\text{insert}(53, \text{"B"}), h(53, 0) = 4$

				53, B	12, A	
--	--	--	--	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

Example:

- Hash function: $h(s, j) = (s \bmod 7 - j) \bmod 7$
- t.insert (5, "C"), $h(5, 0) = 5$, $h(5, 1) = 4$, $h(5, 2) = 3$

0	1	2	3	4	5	6
			5, C	53, B	12, A	

- t.insert (15, "D"), $h(15, 0) = 1$

	15, D		5, C	53, B	12, A	
--	-------	--	------	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

Example:

■ Hash function: $h(s, j) = (s \bmod 7 - j) \bmod 7$

■ t.insert(2, "E"), $h(2, 0) = 2$

0	1	2	3	4	5	6
	15, D	2, E	5, C	53, B	12, A	

■ t.insert(19, "F"), $h(19, 0) = 5$, $h(19, 1) = 4$,
 $h(19, 2) = 3$, $h(19, 3) = 2$, $h(19, 4) = 1$, $h(19, 5) = 0$

19, F	15, D	2, E	5, C	53, B	12, A	
-------	-------	------	------	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

Squared probing:

- Motivation: Avoid local clustering

$$g(s, j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$

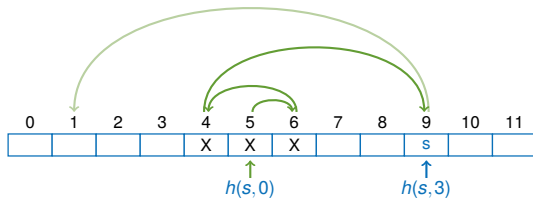


Figure: Squared probe sequence

- This leads to the following probe sequence

$$h(s), h(s) + 1, h(s) - 1, h(s) + 4, h(s) - 4, h(s) + 9, h(s) - 9, \dots$$

Squared probing:

$$g(s, j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$

- If m is a prime number for which $m = 4 \cdot k + 3$ then the probe sequence is a permutation of the indices of the hash tables.
- Alternatively: $h(s, j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \bmod m$
- Problem of secondary clustering
No local clustering anymore, but keys with same hash value have similar probe sequence

Uniform Probing:

- Motivation: So far uses function $g(s, j)$ only the step counter j for linear and squared probing
⇒ The probe sequence is independent of the key s
- Uniform probing computes the sequence $g(s, j)$ of permutations of all possible indices in dependency on key s
- **Advantage:** Prevents clustering because different keys with the same hash value do not produce the same probe sequence
- **Disadvantage:** Hard to implement

Double Hashing: $h_2(s)$ $h_2(s)$

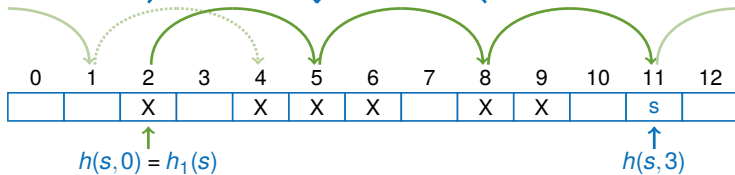


Figure: Double hashing probe sequence

- Motivation: Consider key s in probe sequence
- Use two independent hash functions $h_1(s), h_2(s)$
- Hash function: $h(s,j) = (h_1(s) + j \cdot h_2(s)) \bmod m$

Double Hashing:

- Hash function: $h(s, j) = (h_1(s) + j \cdot h_2(s)) \bmod m$
- probe sequence:

$$h_1(s), h_1(s) + h_2(s), h_1(s) + 2 \cdot h_2(s), h_1(s) + 3 \cdot h_2(s), \dots$$

- Works well in practical use
- This method is an approximation of uniform probing

Example:

$$h_1(s) = s \mod 7$$

$$h_2(s) = (s \mod 5) + 1$$

$$h(s, j) = h_1(s) + j \cdot h_2(s) \mod 7$$

Table: Comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on $h_1(s) \neq h_2(s)$

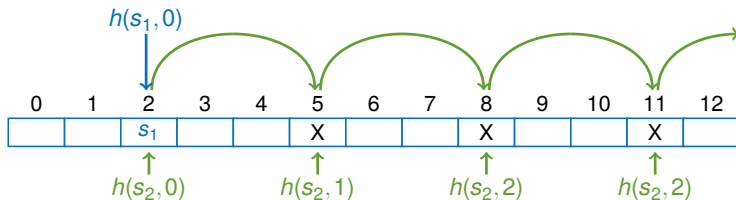


Figure: Double hashing

Double hashing by Brent:

- Motivation:

Because different keys have different probe sequences, the sequence of the insertions has impact on efficiency of a successful search



Figure: Double hashing

Example:

- The key s_1 is inserted at position $p_1 = h(s_1, 0)$
- The hash function for s_2 also results in $p_2 = h(s_2, 0) = p_1$
- The locations $h(s_2, j)$, $j \in \{1, \dots, n\}$ are also occupied
- If we insert s_2 at position $h(s_2, n+1)$ the search will be inefficient



Figure: Double hashing by Brent

- Reversed sequence of keys would have been better
- **Brents Idea:**
 - Test if location $h(s_1, 1)$ is free
 - If yes, move s_1 from $h(s_1, 0)$ to $h(s_1, 1)$ and insert s_2 at $h(s_2, 0)$

Idea:

- Motivation: Colliding elements are inserted in the hashtable sorted.
- Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is earlier possible because single probing steps have a fixed length

Implementation:

- Compare both keys if a collision occurs
- Insert the smaller key at p_1
- Search a position based on the diversion order for the bigger key

Example:

- The key 12 is saved at position $p_1 = h(12, 0)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $5 < 12$ we insert the key 5 at position p_1
- For the key 12 we iterate through the sequence

$h(12, 1), h(12, 2), h(12, 3), \dots$

Motivation:

- Having similar length of probe sequences for all elements. Total costs stay the same, but they are distributed evenly. Results in approximately similar search times for all elements.

Implementation:

- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)
- The key with the bigger search sequence is inserted at p_1 . The other key is assigned a new location based on the sequence

Example:

- The key 12 is saved at position $p_1 = h(12, 7)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $j_1 < j_2$ ($0 < 7$) the key 12 stays at position p_1
- For the key 5 we iterate through the sequence

$$h(5, 1), h(5, 2), h(5, 3), \dots$$

Problem:

- The key s_1 is inserted at position p_1
- The key s_2 returns the same hash value, but is inserted at position p_2 because of the probing order
- If s_1 is removed, it is impossible to find s_2

Solution:

- **Remove:** Elements are marked as removed, but not deleted
- **Inserting:** Elements marked as removed will be overwritten

Bucket as linked list: (dynamic, number of elements variable)

- Save colliding elements as linked list

Open hashing: (static, number of elements fixed)

- Determine a probe sequence, permutation of all hash values
- Linear, quadratic probing:
 - Easy to implement
 - Raise the probability of collisions because probing order does not depend on the key

Open hashing: (static, number of elements fixed)

- Uniform probing, double hashing:
 - Different probing orders for different keys
 - Avoids clustering of elements

Improving efficiency: (Brent, Ordered Hashing)

- Improve search efficiency by sorting colliding insertions
 - Abortion of unsuccessful search
 - Search sequence length balancing

Hashing:

- Efficient for dictionary operations:
 - Insert: $O(1) \dots O(n)$
 - Search: $O(1) \dots O(n)$
 - Remove: $O(1) \dots O(n)$
- Direct access of all elements in a hash table
- Using a hash function to find the position (hash value) in the hash table
- Hash function, size of the hash table and strategy to avoid hash collisions influence the efficiency of the datastructure

Definition:

- A priority queue saves a set of elements
- Each element contains a key and a value like a map
- There is a total order (like \leq) defined on the keys

Definition:

- The priority queue supports the following operations:

`insert(key, value)`: Inserts a new element into the queue

`getMin()`: Returns the element with the smallest key

`deleteMin()`: Removes the element with the smallest key

- Sometimes additional operations are defined:

`changeKey(item, key)`: Changes the key of the element

`remove(item)`: Removes the element from the queue

Special features:

- Multiple elements with the same key
 - No problem and for many applications necessary
 - If there is more than one element with the smallest key
 - `getMin()`: Returns just one of the possible elements
 - `deleteMin()`: Deletes the element returned by `getMin`
- Argument of `changeKey` and `remove` operations
 - There is no **quick-access** to a element in the queue
 - That's why `insert` and `getMin` return a reference (handle, accessor object)
 - `changeKey` and `remove` take this reference as argument
 - Therefore each element has to store its current position in the heap.

```
from queue import PriorityQueue

q = PriorityQueue()

e1 = (5, "A") # element with priority 5
q.put(e1); # insert element e1

# remove and return the lowest item
e2 = q.get()
```

Example 1:

- Calculation of the sorted union of k sorted lists
(multi-way merge or k -way merge)

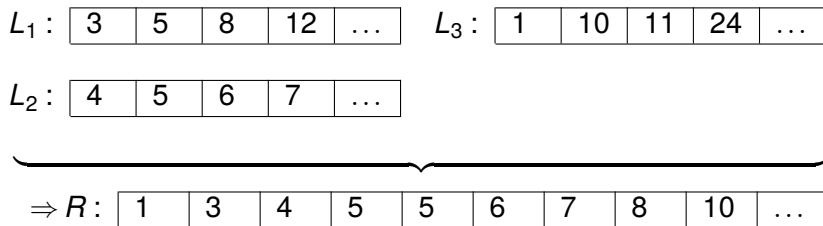


Figure: 3-way merge

Example 1:

- Calculation of the sorted union of k sorted lists (multi-way merge or k -way merge)
- Runtime: N = length of resulting list
 - Trivial: $\Theta(N \cdot k)$, minimum calculation $\Theta(k)$
 - Priority queue: $\Theta(N \cdot \log k)$, minimum calculation $\Theta(\log k)$

Example 2:

- For example Dijkstra's algorithm for computing the shortest path (\leftarrow following lecture)
- Among other applications it can be used for sorting

Idea:

- Save elements as tuples in a binary heap
- Summary from lecture 1 (*HeapSort*):
 - Nearly complete binary tree
 - **Heap condition:**
The key of each node \leq the keys of the children

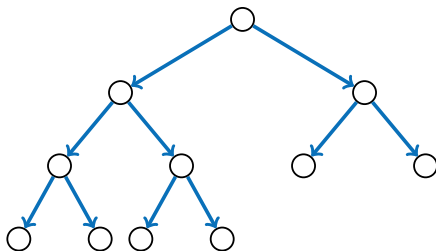


Figure: Heap with 11 nodes

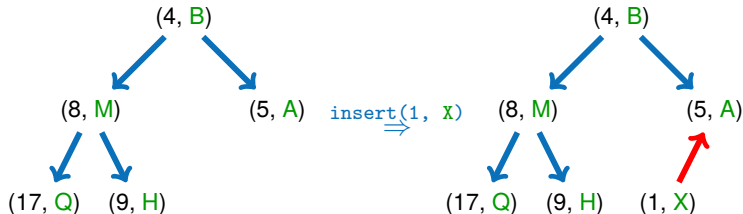


Figure: Min heap stored in array

Storing a binary heap:

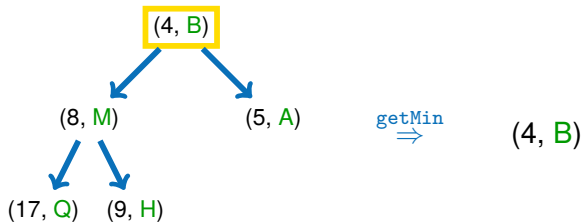
- Number nodes from top to bottom and left to right starting with 0 and store entries in array
- Children of node i are the nodes $2i+1$ and $2i+2$
- Parent node of node i is $\text{floor}((i-1)/2)$

Inserting an element: `insert(key, item)`



- Append the element at the end of the array
- The **heap condition** may be violated, but only at the last index
- Repair **heap condition** \Rightarrow We will see later how to do this

Returning the minimum: `getMin()`



- Else return the first element
- If the heap is empty return `None`

Removing the minimum: `deleteMin()`



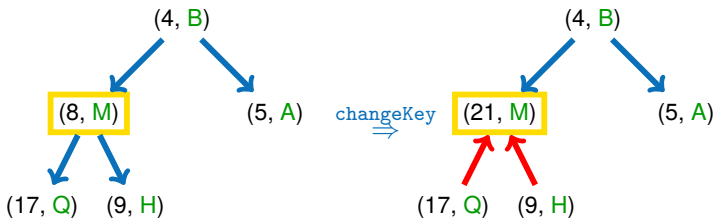
- Deleting the element with the lowest key
- Swap the last element with the first element and shrink the heap by one
- The **heap condition** may be violated, but only at the first index
- Repair **heap condition**

Changing the key (priority): `changeKey(item, key)`

- The element (queue item) is given as argument
- Replace the value of the key
- The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- Repair **heap condition**

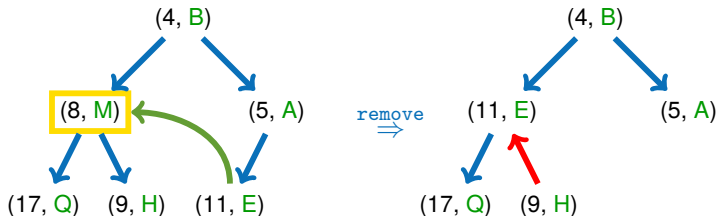


Changing the key (priority): `changeKey(item, key)`



- The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- Repair **heap condition**

Removing an element: `remove(item)`



- The element (queue item) is given as argument
- Replace the element with the last element and shrink the heap by one
- The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- Repair **heap condition**

Repairing after modifying operations:

- The heap condition can be violated after using `insert`, `deleteMin`, `changeKey`, `remove`, but only at one known position with index i
- Heap conditions can be violated in two directions:
 - Downwards: The key at index i is not \leq than the value of its children
 - Upwards: The key at index i is not \geq than the value of its parent
- We need two repair methods: `repairHeapUp`, `repairHeapDown`

repairHeapDown:

- Sift the element until the **heap condition** is valid
 - Change node with child, which has the lower key of both children
 - If the **heap condition** is violated repeat for the child node



Figure: Repairing the heap downwards

repairHeapDown:

- Sift the element until the **heap condition** is valid
 - Change node with child, which has the lower key of both children
 - If the **heap condition** is violated repeat for the child node

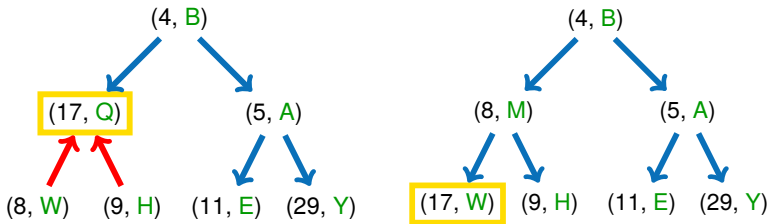


Figure: Repairing the heap downwards

repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node

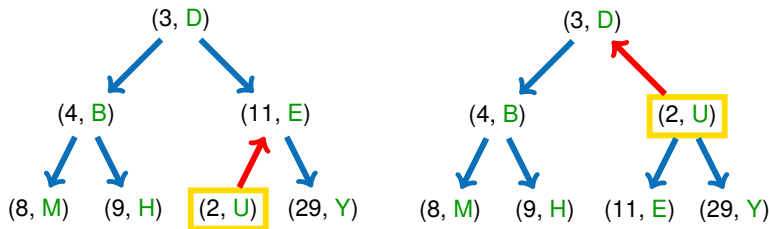


Figure: Repairing the heap upwards

repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node



Figure: Repairing the heap upwards

Index of a priority queue item:

- **Attention:** For `changeKey` and `remove` the item has to “know” where it is located in the heap
- Remember for `repairHeapUp` and `repairHeapDown`:
Update the index if moving an heap element

```
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```

Summary lecture 1:

- A full binary tree with n elements, has a **depth** of $O(\log n)$
- The maximum distance from the root to a leaf can be $O(\log n)$ elements
- Repairing the heap upwards and downwards:
We have only one path to traverse: $O(\log n)$

Runtime for methods

- **insert**, **deleteMin**, **changeKey**, **remove**:
We have to repair the heap: $O(\log n)$
- **getMin**: Return the element at index 0: $O(1)$

Improvements (Fibonacci heaps):

- `getMin`, `insert` and `decreaseKey` in amortized time of $O(1)$
- `deleteMin` in amortized time $O(\log n)$

Practical experience:

- The binary heap is simpler: Costs for managing the structure are low
- If the number of elements is relatively small so the difference is negligible
- Example:
 - For $n = 2^{10} \approx 1,000$ is the the `depth` $\log_2 n$ only 10
 - For $n = 2^{20} \approx 1,000,000$ is the `depth` $\log_2 n$ only 20

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Priority Queue - Implementations / API

[Cpp] [C++ - priority_queue](#)

`http:`

`//www.sgi.com/tech/stl/priority_queue.html`

[Jav] [Java - PriorityQueue](#)

`https://docs.oracle.com/javase/7/docs/api/
java/util/PriorityQueue.html`

[Pyt] [Python - PriorityQueue](#)

`https://docs.python.org/3/library/queue.
html#queue.PriorityQueue`