

Algorithms and Datastructures

O-Notation, L'Hopital

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\mathcal{O} -Notation

Motivation / Definition

Examples

Ω -Notation

Θ -Notation

Runtime

Summary

Limit / Convergence

L'Hôpital / l'Hospital

Practical use

We are interested in:

- Example: sorting
 - Runtime of Minsort “is growing as” n^2
 - Runtime of HeapSort “is growing as” $n \log n$
- Growth of a function in runtime $T(n)$
 - The role of constants (e.g. $1ns$) is minor
 - it is enough if relation holds for some $n \geq \dots$
- Describe the growth of the function **more formally**
 - By the means of Landau-Symbols [Wik]):
 - $\mathcal{O}(n)$ (Big O of n),
 - $\Omega(n)$ (Omega of n),
 - $\Theta(n)$ (Theta of n)

Big \mathcal{O} -Notation:

- Consider the function: $f: \mathbb{N} \rightarrow \mathbb{R}, n \mapsto f(n)$
 - \mathbb{N} : Natural numbers \rightarrow input size
 - \mathbb{R} : Real numbers \rightarrow runtime
- **Example:**
 - $f(n) = 3n$
 - $f(n) = 2n \log n$
 - $f(n) = \frac{1}{10}n^2$
 - $f(n) = n^2 + 3n \log n - 4n$

Big \mathcal{O} -Notation:

- Given two functions f and g :

$$f, g: \mathbb{N} \rightarrow \mathbb{R}$$

- **Intuitive:** f is Big-O of g (f is $\mathcal{O}(g)$)
 - ... if f relative to g does not grow faster than g
 - the growth rate matters, not the absolute values

Big \mathcal{O} -Notation:

■ Informal: $f = \mathcal{O}(g)$

- “=” corresponds to *is not isequal*
- ... if for some value n_0 for all $n \geq n_0$
- $f(n) \leq C \cdot g(n)$ for a constant C
- ($f = \mathcal{O}(g)$): From a value n_0 for all $n \geq n_0 \rightarrow f(n) \leq C \cdot g(n)$)

■ Formal: $f \in \mathcal{O}(g)$

Formal: $f \in \mathcal{O}(g)$

$$\mathcal{O}(g) = \{ f : \mathbb{N} \rightarrow \mathbb{R} \mid \exists n_0 \in \mathbb{N}, \exists C > 0, \forall n > n_0 : f(n) \leq C \cdot g(n) \}$$

“set of
all functions”

“for which”

“it exists”

“for all”

“such that”

Illustration of the Big O-Notation:



Figure: Runtime of two algorithms f_1, f_2

Example:

- $f(n) = 5n + 7, g(n) = n$
 $\Rightarrow 5n + 7 \in \mathcal{O}(g)$
 $\Rightarrow f \in \mathcal{O}(g)$
- **Intuitive:**
 $f(n) = 5n + 7 \rightarrow$ linear growth

Attention

$f(n) \leq g(n)$ is not guaranteed, better is $f(n) \leq C \cdot g(n) \quad \forall n > n_0$.

We have to proof: $\exists n_0, \exists C, \forall n \geq n_0: 5n+7 \leq C \cdot n$.

$$\begin{aligned} 5n+7 &\leq 5n+n \quad (\text{for } n \geq 7) \\ &= 6n \end{aligned}$$

$$\Rightarrow n_0 = 7, C = 6$$



Alternate proof:

$$\begin{aligned} 5n+7 &\leq 5n+7n \quad (\text{for } n \geq 1) \\ &= 12n \end{aligned}$$

$$\Rightarrow n_0 = 1, C = 12$$



Big O-Notation:

- We are only interested in the term with the highest-order, the fastest growing summand, the others will be ignored
- $f(n)$ is limited **from above** by $C \cdot g(n)$

Examples:

$$2n^2 + 7n - 20 \in \mathcal{O}(n^2)$$

$$2n^2 + 7n \log n - 20 \in$$

$$7n \log n - 20 \in$$

$$5 \in$$

$$2n^2 + 7n \log n + n^3 \in$$

Harder Example:

- Polynomes are simple
- More problematic: combination of complex functions

$$2\sqrt{x} + 3\ln x \in \mathcal{O}(??)$$

Omega-Notation:

■ Intuitive:

- $f \in \Omega(g)$, f is growing at least as fast as g
- So the same as Big-O but with *at-least* and not *at-most*

Formal: $f \in \Omega(g)$

$$\Omega(g) = \{f : \mathbb{N} \rightarrow \mathbb{R} \mid \exists n_0 \in \mathbb{N}, \exists C > 0, \forall n > n_0 : f(n) \geq C \cdot g(n)\}$$

“in $O(n)$
we had \leq ”



Example:

Proof of $f(n) = 5n + 7 \in \Omega(n)$:

$$\underbrace{5n+7}_{f(n)} \geq \underbrace{1 \cdot n}_{g(n)} \quad (\text{for } n \geq 1)$$

$$\Rightarrow n_0 = 1, C = 1$$



Illustration of the Omega-Notation:



Figure: Runtime of two algorithms f_1, f_2

Big Omega-Notation:

- We are only interested in the term with the highest-order, the fastest growing summand, the others will be ignored
- $f(n)$ is limited **from underneath** by $c \cdot g(n)$

Examples:

$$2n^2 + 7n - 20 \in \Omega(n^2)$$

$$2n^2 + 7n \log n - 20 \in$$

$$7n \log n - 20 \in$$

$$5 \in$$

$$2n^2 + 7n \log n + n^3 \in$$

Theta-Notation:

- **Intuitive:** f is Theta of g ...
 - ... if f is growing as much as g
 - $f \in \Theta(g)$, f is growing at the same speed as g

Formal: $f \in \Theta(g)$

$$\Theta(g) = \underbrace{\mathcal{O}(g) \cap \Omega(g)}_{\text{Intersection}}$$

Example:

$$\begin{aligned} f(n) &= 5n + 7, f(n) \in \mathcal{O}(n), f(n) \in \Omega(n) \\ \Rightarrow f(n) &\in \Theta(n) \end{aligned}$$

Proof for $\mathcal{O}(g)$ and $\Omega(g)$ look at slides 11 and 17



- f and g have the same “growth”

Big O-Notation $\mathcal{O}(n)$:

- f is growing **at most** as fast as g
- $C \cdot g(n)$ is the upper bound

Big Omega-Notation $\Omega(n)$:

- f is growing **at least** as fast as g
- $C \cdot g(n)$ is the lower bound

Big Theta-Notation $\Theta(n)$:

- f is growing at **the same** speed as g
 - $C_1 \cdot g(n)$ is the lower bound
 - $C_2 \cdot g(n)$ is the upper bound

Table: Common runtime types

Runtime	Growth
$f \in \Theta(1)$	constant time
$f \in \Theta(\log n) = \Theta(\log_k n)$	logarithmic time
$f \in \Theta(n)$	linear time
$f \in \Theta(n \log n)$	n-log-n time (nearly linear)
$f \in \Theta(n^2)$	squared time
$f \in \Theta(n^3)$	cubic time
$f \in \Theta(n^k)$	polynomial time
$f \in \Theta(k^n), f \in \Theta(2^n)$	exponential time

- So far discussed:
 - Membership in $O(\dots)$ proofed by hand:
Explicit calculation of n_0 and C
 - **However:** Both hint at **limits** in calculus

Definition of “Limit”

- The **limit** L exists for an infinite sequence f_1, f_2, f_3, \dots if for all $\varepsilon > 0$ one $n_0 \in \mathbb{N}$ exists, such that for all $n \geq n_0$ the following holds true: $|f_n - L| \leq \varepsilon$
- A function $f: \mathbb{N} \rightarrow \mathbb{R}$ can be written as a sequence
 $\Rightarrow \lim_{n \rightarrow \infty} f_n = L$

The limit is converging:

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0: |f_n - L| \leq \varepsilon$$

- Example for the proof of a limit
- Function $f(n) = 2 + \frac{1}{n}$ with limes $\lim_{n \rightarrow \infty} f(n) = 2$
- “Engineering” solution: use $n = \infty$

$$\frac{1}{\infty} = 0 \Rightarrow \lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} 2 + \frac{1}{n} = 2$$

- Now a more formal proof for $\lim_{n \rightarrow \infty} 2 + \frac{1}{n} = 2$
- We need to show: for all given ε there is an n_0 such that for all $n \geq n_0$

$$\left| 2 + \frac{1}{n} - 2 \right| = \left| \frac{1}{n} \right| \leq \varepsilon$$

- E.g.: for $\varepsilon = 0.01$ we get $\frac{1}{n} \leq \varepsilon$ for $n \geq 100$
- In general

$$n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$$

- Then we get:

$$\left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{n_0} = \frac{1}{\left\lceil \frac{1}{\varepsilon} \right\rceil} \leq \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \quad \square$$

Let $f, g: \mathbb{N} \rightarrow \mathbb{R}$ with an existing limit

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = L$$

Hence the following holds:

$$f \in \mathcal{O}(g) \quad \Leftrightarrow \quad \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \quad (1)$$

$$f \in \Omega(g) \quad \Leftrightarrow \quad \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0 \quad (2)$$

$$f \in \Theta(g) \quad \Leftrightarrow \quad 0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \quad (3)$$

$$f \in \mathcal{O}(g) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty$$

Forward proof (\Rightarrow):

$$f \in \mathcal{O}(g) \stackrel{\text{def. of } \mathcal{O}(n)}{\Rightarrow} \exists n_0, C \forall n \geq n_0 : f(n) \leq C \cdot g(n)$$

$$\Rightarrow \exists n_0, C \forall n \geq n_0 : \frac{f(n)}{g(n)} \leq C$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \leq C \quad \square$$

Backward proof (\Leftarrow):

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = C \quad \text{For some } C \in \mathbb{R} \text{ (Limit)}$$

$$\text{def. limes} \Rightarrow \exists n_0, \forall n \geq n_0 : \frac{f(n)}{g(n)} \leq C + \varepsilon \quad (\text{e.g. } \varepsilon = 1)$$

$$\Rightarrow \exists n_0, \forall n \geq n_0 : f(n) \leq \underbrace{(C+1)}_{O\text{-notation constant}} \cdot g(n)$$

$$\Rightarrow f \in \mathcal{O}(g) \quad \square$$

■ Intuitive:

$$\lim_{n \rightarrow \infty} 2 + \frac{1}{n} = 2 + \frac{1}{\infty} = 2$$

■ With L'Hôpital:

- Let $f, g : \mathbb{N} \rightarrow \mathbb{R}$
- If $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} g(n) = \infty/0$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

■ Holy inspiration

you need a doctoral degree for that

The limit can not be determined in the way of an Engineer:

$$\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = \frac{\lim_{n \rightarrow \infty} \ln(n)}{\lim_{n \rightarrow \infty} n} \quad \xrightarrow{\text{plugging in}} \quad \frac{\infty}{\infty}$$

Determine the limit with using L'Hôpital:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

Using L'Hôpital:

Numerator: **f(n)**: $n \mapsto \ln(n)$

Denominator: **g(n)**: $n \mapsto n$

$$\Rightarrow f'(n) = \frac{1}{n} \quad (\text{derivation from Numerator})$$

$$\Rightarrow g'(n) = 1 \quad (\text{derivation from Denominator})$$

$$\lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$$

What can we take for granted without proofing?

- Only things that are trivial
- It is always better to proof it

Examples:

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \text{is trivial}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0 \quad \text{is trivial}$$

$$\lim_{n \rightarrow \infty} \frac{\log(n)}{n} = 0 \quad \text{use L'Hopital}$$

Practical use:

- It is much easier to determine the runtime of an algorithm by using the \mathcal{O} -Notation
 - 1 Computing rules
 - 2 Practical use

■ Transitivity:

$$f \in \Theta(g) \wedge g \in \Theta(h) \rightarrow f \in \Theta(h)$$

$$f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h) \rightarrow f \in \mathcal{O}(h)$$

$$f \in \Omega(g) \wedge g \in \Omega(h) \rightarrow f \in \Omega(h)$$

■ Symmetry:

$$f \in \Theta(g) \leftrightarrow g \in \Theta(f)$$

$$f \in \mathcal{O}(g) \leftrightarrow g \in \Omega(f)$$

■ Reflexivity:

$$f \in \Theta(f) \quad f \in \Omega(f) \quad f \in \mathcal{O}(f)$$

■ Trivial:

$$\begin{aligned}f &\in \mathcal{O}(f) \\k \cdot \mathcal{O}(f) &= \mathcal{O}(f) \\ \mathcal{O}(f+k) &= \mathcal{O}(f)\end{aligned}$$

■ Addition:

$$\mathcal{O}(f) + \mathcal{O}(g) = \mathcal{O}(\max\{f, g\})$$

■ Multiplication:

$$\mathcal{O}(f) \cdot \mathcal{O}(g) = \mathcal{O}(f \cdot g)$$

- The input size for all examples is n
- Basic operations

$i1 = 0$	$\mathcal{O}(1)$
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- Sequences of basic operations

$i1 = 0$	$\mathcal{O}(1)$	}	$327 \cdot \mathcal{O}(1) = \mathcal{O}(1)$
$i2 = 0$	$\mathcal{O}(1)$		
...	...		
$i327 = 0$	$\mathcal{O}(1)$		

■ Loops

```
for i in range(0, n):  
    a[i] = 0
```

$$\left. \frac{\mathcal{O}(n)}{\mathcal{O}(1)} \right\} \mathcal{O}(1) \cdot \mathcal{O}(n) = \mathcal{O}(n)$$

```
for i in range(0, n):  
    a1[i] = 0  
    ...  
    a137[i] = 0
```

$$\left. \begin{array}{c} \frac{\mathcal{O}(n)}{\mathcal{O}(1)} \\ \dots \\ \mathcal{O}(1) \end{array} \right\} \left. \begin{array}{c} 137 \cdot \mathcal{O}(1) \\ = \mathcal{O}(1) \end{array} \right\} \mathcal{O}(1) \cdot \mathcal{O}(n) = \mathcal{O}(n)$$

■ Loops

```
for i in range(0, n):  
    for j in range(0, n-1):  
        a1[i][j] = 0  
        ...  
        a137[i][j] = 0
```

$$\left. \begin{array}{l} \frac{\mathcal{O}(n)}{\mathcal{O}(n-1)} \\ \frac{\mathcal{O}(1)}{\mathcal{O}(1)} \\ \dots \\ \frac{\mathcal{O}(1)}{\mathcal{O}(1)} \end{array} \right\} \left. \begin{array}{l} \mathcal{O}(n) \cdot \mathcal{O}(n) \\ = \mathcal{O}(n^2) \\ 137 \cdot \mathcal{O}(1) \\ = \mathcal{O}(1) \end{array} \right\} \left. \begin{array}{l} \mathcal{O}(1) \cdot \mathcal{O}(n^2) \\ = \mathcal{O}(n^2) \end{array} \right\}$$

■ Conditions

if $x < 100$:

$y = x$

else:

for i in range(0, n):

if $a[i] > y$:

$y = a[i]$

$$\left. \begin{array}{l} \frac{\mathcal{O}(1)}{\mathcal{O}(1)} \\ \frac{\mathcal{O}(n)}{\mathcal{O}(1)} \end{array} \right\} \left. \begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(n) \cdot \mathcal{O}(1) \\ = \mathcal{O}(n) \end{array} \right\} \begin{array}{l} \mathcal{O}(\max\{1, n\}) \\ = \mathcal{O}(n) \end{array}$$

- **Input:** List x with n numbers
- **Output:** $a[i]$ is the arithmetic mean of $x[0]$ to $x[i]$

```
def arithMean(x):  
    a = [0] * len(x)  
    for i in range(0, len(x)):  
        s = 0  
        for j in range(0, i+1):  
            s = s + x[j]  
  
        a[i] = s / (i+1)  
  
    return a
```

\mathcal{O} -Notation Runtime complexity

for i in range(0, len(x)):	$\frac{\mathcal{O}(n)}{\mathcal{O}(1)}$	}	$\mathcal{O}(n)$	}	$\mathcal{O}(n) \cdot \mathcal{O}(i)$ $= \mathcal{O}(n^2)$
s = 0	$\mathcal{O}(1)$				
for j in range(0, i+1):	$\frac{\mathcal{O}(i+1)}{\mathcal{O}(1)}$	}	$\mathcal{O}(i)$		
s = s + x[j]	$\mathcal{O}(1)$				
a[i] = s / (i+1)	$\mathcal{O}(1)$				

- How often will the instructions in the loop be executed, when the problem has size n ?

$$1 + 2 + \dots + n = \frac{n \cdot (n+1)}{2} \in \mathcal{O}(n^2)$$

Way of speaking:

- With the \mathcal{O} -Notation we look at the behavior of a function when $n \rightarrow \infty$
- We only analyze the runtime when $n \geq n_0$
- We talk about **asymptotic analysis**, when we discuss cost, runtime, etc. as $\mathcal{O}(\dots)$, $\Omega(\dots)$ or $\Theta(\dots)$

Attention:

- If you are using **asymptotic analysis**, you can not make any predictions about the runtime of smaller input sizes ($n < n_0$)
- For small input sizes (mostly $n < 10$), the runtime is predictably small
- n_0 does not necessarily have to be small

Examples:

- Let A and B be algorithms
 - A has the runtime $f(n) = 80n$
 - B has the runtime $f(n) = 2n \log_2 n$
- So $f = \mathcal{O}(g)$ but **not** $\Theta(g)$
 - \Rightarrow A is asymptotic faster than B
 - \Rightarrow There is a n_0 for that $n \geq n_0: f(n) \leq g(n)$

When is A faster than B?

We search the minimal n_0 :

$$f(n_0) = g(n_0)$$

$$80 n_0 = 2 n_0 \log_2 n_0$$

$$40 = \log_2 n_0$$

$$n_0 = 2^{40}$$

$$= (2^{10})^4 = (1024)^4$$

$$\approx (10^3)^4 = 10^{12}$$

$$\approx 1 \text{ trillion}$$

A is faster than B if n_0 has more than 1 trillion elements

- Logarithm of different bases differ only by a constant

$$\log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \cdot \log_b n$$

- Hence: $\log_a n \in \Theta(\log_b n)$
- For exponent this does not hold

$$3^n \notin \Theta(2^n)$$

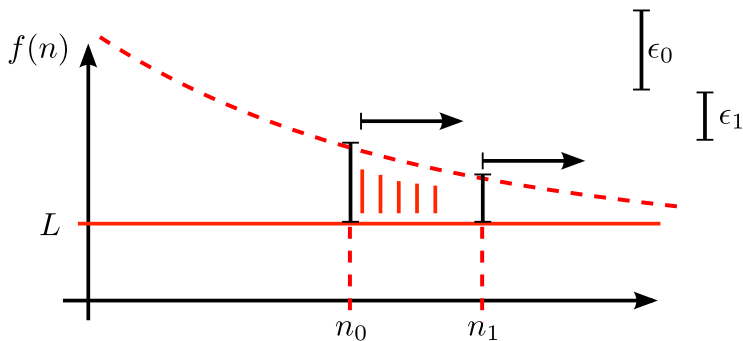
- Proof: Use equation (1) from Slide 31

$$3^n \in \mathcal{O}(2^n) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{3^n}{2^n} < \infty$$

- However:

$$\lim_{n \rightarrow \infty} \frac{3^n}{2^n} = \lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n = \infty$$

■ Figure for slide 28



■ General

- [MS08] [Kurt Mehlhorn and Peter Sanders.](#)
Algorithms and data structures, 2008.
[https://people.mpi-inf.mpg.de/~mehlhorn/
ftp/Mehlhorn-Sanders-Toolbox.pdf](https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf).

■ Big O notation

[Wik] [Big O notation](https://en.wikipedia.org/wiki/Big_O_notation)

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