

# Algorithmns and Datastructures

## Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg



**UNI  
FREIBURG**

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Algorithmns and Datastructures, March 2016

## Divide and Conquer

- Concept

- Maximum Subtotal

## Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

  - Master theorem (Simple Form)

  - Master theorem (General Form)

## Divide and Conquer

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# Divide and Conquer

## Introduction



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If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
  
- **Recursive** application of the algorithm on smaller subproblems
- **Direct** solving of small subproblems

## Divide and Conquer

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# Divide and Conquer

Maximum Subtotal



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### Input:

- Progression  $X$  of  $n$  integers

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Table: Input values

| Index | 0  | 1   | 2  | 3  | 4   | 5  | 6  | 7   | 8   | 9  |
|-------|----|-----|----|----|-----|----|----|-----|-----|----|
| Value | 31 | -41 | 59 | 26 | -53 | 58 | 97 | -93 | -23 | 84 |

**Output:** Sum: 187, Start: 2, End: 6

**Idea:**



### Idea:



- Solve the left / right half of the problem **recursive**

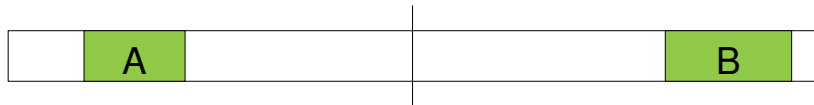


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- The maximum interval can **overlap with the border (C)**

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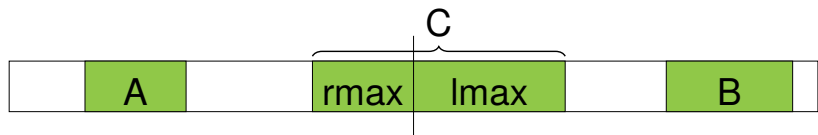
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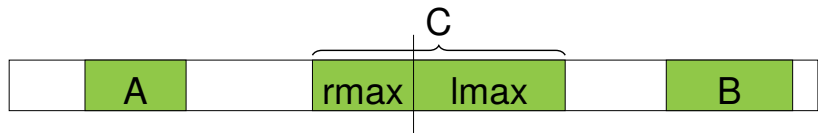
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- To solve  $C$  we have to calculate  $rmax$  and  $lmax$
- Overall solution is maximum of  $A B C$



# Divide and Conquer

## Maximum Subtotal - Python

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    #Solutions for A and B  
    A = maxSubArray(X, i, m)  
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    #rmax and lmax for bordercase C  
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    C = (C1[0] + C2[0], C1[1], C2[1])
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    #Solution is maximum of A,B,C  
    return max([A, B, C], \  
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    C = (C1[0] + C2[0], C1[1], C2[1])  
    #Solution is maximum of A,B,C  
    return max([A, B, C], \  
        key=lambda item: item[0])  
    #Simplification: only maximum
```

# Divide and Conquer

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#Alternative trivial case  
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def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
```



```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])
    ... # continue as before
```

# Divide and Conquer

## Maximum Subtotal - Python



```
#Implementation max  
def max(a, b, c):
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# Divide and Conquer

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#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
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def max(a, b, c):
    if a > b:
        if a > c:
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    else:
        if c > b:
            return c
        else:
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# Divide and Conquer

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def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b  
  
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```



```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

| index       | $i$ | $i + 1$ | ... | ... | $j - 1$ | $j$ |
|-------------|-----|---------|-----|-----|---------|-----|
| $X$         | 58  | -53     | 26  | 59  | -41     | 31  |
| <i>sum</i>  | 58  | 5       | 31  | 90  | 49      | 80  |
| <i>lmax</i> | 58  | 58      | 58  | 90  | 90      | 90  |

Table: *lmax* example

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- The *sum* and *lmax* are initialized with  $X[i]$

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- The *sum* and *lmax* are initialized with  $X[i]$
- We iterate over  $X$  from  $i + 1$  to  $j$  and update *sum*

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- The *sum* and *lmax* are initialized with  $X[i]$
- We iterate over  $X$  from  $i + 1$  to  $j$  and update *sum*
- If  $s > lmax$  then *lmax* gets updated

# Divide and Conquer

## Maximum Subtotal



Call with array of four elements

`maxSubArray(-3,9,-4,7)`

in fact:

`maxSubArray(X,0,3)`  
with `X=[-3,9,-4,7]`

# Divide and Conquer

## Maximum Subtotal



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in fact:

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with  $X = [-3, 9, -4, 7]$

Call with array of two elements

$\text{maxSubArray}(-3, 9)$

$\text{maxSubArray}(-4, 7)$

# Divide and Conquer

## Maximum Subtotal



Call with array of four elements



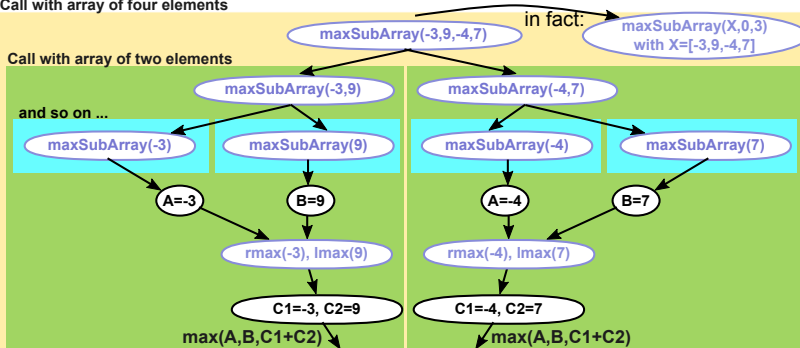


# Divide and Conquer

## Maximum Subtotal



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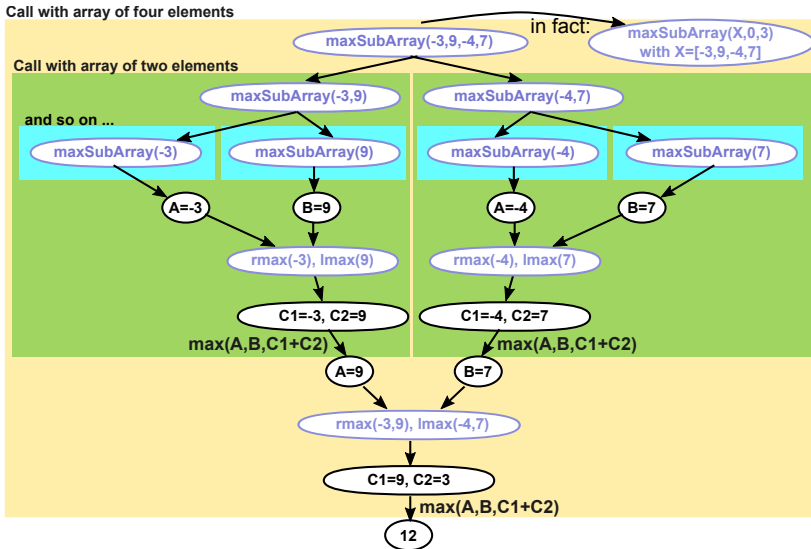
# Divide and Conquer

## Maximum Subtotal



Call with array of four elements

Call with array of two elements



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def maxSubArray(X, i, j):  
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    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
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def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
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    A = maxSubArray(X, i, m)                       # T(n/2)  
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## Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{cobination of solutions}} & n > 1 \end{cases}$$

$\underbrace{\Theta(1)}_{\text{trivial case}}$

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- There exist two constants  $a$  and  $b$  with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

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- We define  $c := \max(a, b)$ :

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$



# Divide and Conquer

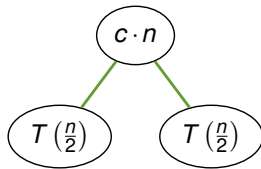
## Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



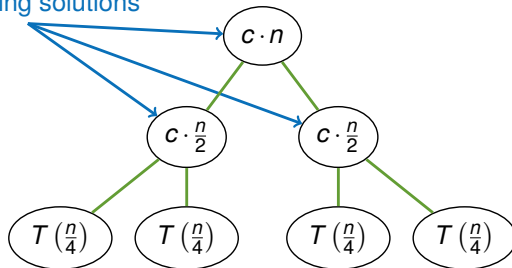
$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$

combining solutions



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$

$$c \cdot n$$

1 node processing  $n$  elements  
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



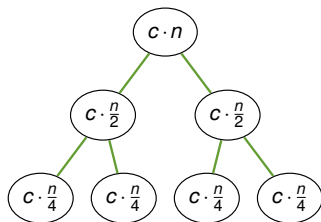
1 node processing  $n$  elements  
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2 nodes processing  $\frac{n}{2}$  elements  
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



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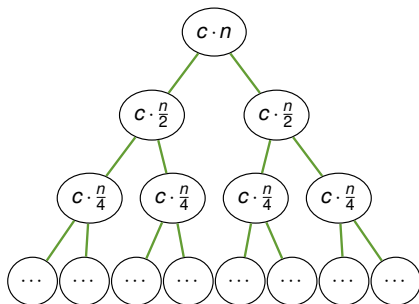
4 nodes processing  $\frac{n}{4}$  elements  
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

Figure: Recursion tree method



# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



1 node processing  $n$  elements  
 $\Rightarrow c \cdot n$

2 nodes processing  $\frac{n}{2}$  elements  
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing  $\frac{n}{4}$  elements  
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

$2^j$  nodes processing  $\frac{n}{2^j}$  elements  
 $\Rightarrow 2^j c \cdot \frac{n}{2^j} = c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



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$2^i$  nodes processing  $\frac{n}{2^i}$  elements  
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$n$  nodes processing 1 element  
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Figure: Recursion tree method

# Divide and Conquer

Maximum Subtotal - Illustration of  $T(n)$



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**Depth:**

# Divide and Conquer

Maximum Subtotal - Illustration of  $T(n)$



## Depth:

- Top level with depth  $i = 0$

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$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$



# Divide and Conquer

## Maximum Subtotal - Summary



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- There is an approach running in  $O(n)$  if you assume that all subtotals are positive

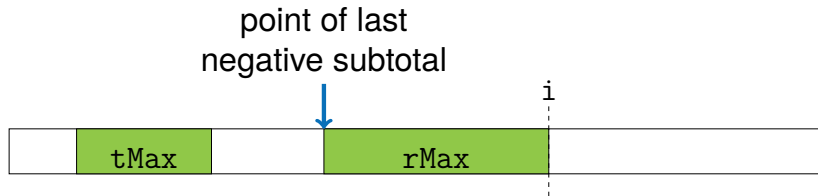


Figure: Scanning the array in linear time

# Divide and Conquer

## Maximum Subtotal - Python

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#Implementation - linear runtime  
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# Divide and Conquer

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    for i in range(len(X)):  
        if rMax == 0:  
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    for i in range(len(X)):  
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        if rMax > tMax:  
            tMax, itMax = rMax, irMax  
  
    return (tMax, itMax)
```

## Divide and Conquer

Concept

Maximum Subtotal

## Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

### Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$



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- Describes the runtime for recursive functions:

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- $T(n)$  is only defined for integers of  $\frac{n}{b}$  which is often ignored in benefit of a simpler solution

## Divide and Conquer

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- Assumption:  $T(n) = n + n \cdot \log_2 n$



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## Divide and Conquer

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Maximum Subtotal

## Recursion Equations

Substitution Method

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$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

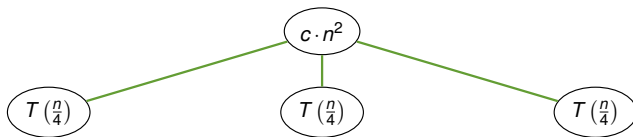


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Figure: Recursion tree of example

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Figure: Recursion tree of example

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

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Figure: Levels of the recursion tree



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- Costs on level  $i$ :

$$T_i(n) = 3^i \cdot c \cdot \left( \left( \frac{1}{4} \right)^i \cdot n \right)^2 = \left( \frac{3}{16} \right)^i \cdot c \cdot n^2$$



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- Costs on the **last level**:  $T_{i+1}(n) = d \cdot n^{\log_4 3}$



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uses  $x^{a \cdot b} = (x^a)^b$

- This term will recur in the master theorem

# Recursion Equations

Total costs



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**Total costs:**



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$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \text{even with} \\ \text{infinite elements}}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in O(n^2)$$

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- Here: The costs of connecting the partial problems dominate

# Recursion Equations

## Geometric Series



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Quotient of two neighbored progression parts is constant

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The series (cumulative sum) of a geometric progression

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The series (cumulative sum) of a geometric progression

- For  $|q| < 1$ :

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \Rightarrow \text{constant}$$



- **Geometric progression:**

Quotient of two neighbored progression parts is constant

- **Geometric series:**

The series (cumulative sum) of a geometric progression

- For  $|q| < 1$ :

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- Therefore constant

# Recursion Equations

Proof of  $O(n^2)$



UNI  
FREIBURG

**Proof of  $O(n^2)$ :**

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■ We know:

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## Divide and Conquer

Concept

Maximum Subtotal

## Recursion Equations

Substitution Method

Recursion Tree Method

### Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



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  - ... which takes  **$f(n)$**  steps to merge all partial solutions



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- **Simple form:** Special case with runtime of connecting the solutions  $f(n) \in O(n)$



# Recursion Equations

## Master theorem (Simple Form)



**Simple form:**

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$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{was any } f(n)}, \quad a \geq 1, b > 1, c > 0$$

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# Recursion Equations

## Master theorem (Simple Form)

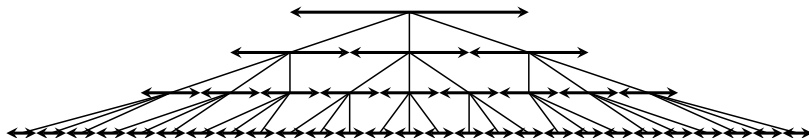


Figure: Simple recursion equation with  $a = 3, b = 2$

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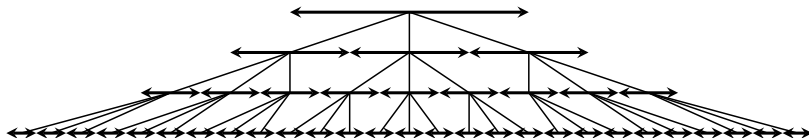


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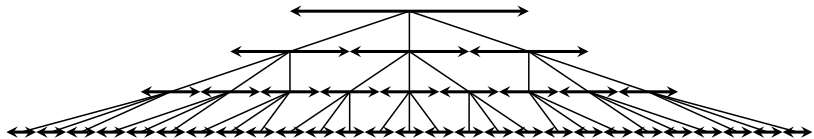


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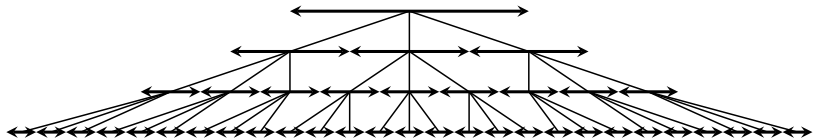


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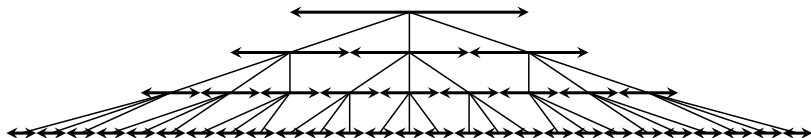


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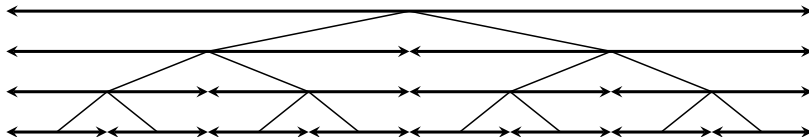


Figure: Simple recursion equation with  $a = 2, b = 2$

# Recursion Equations

## Master theorem (Simple Form)



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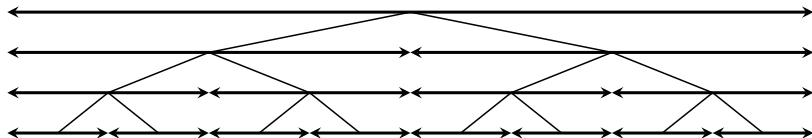


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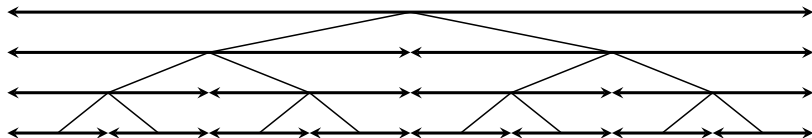


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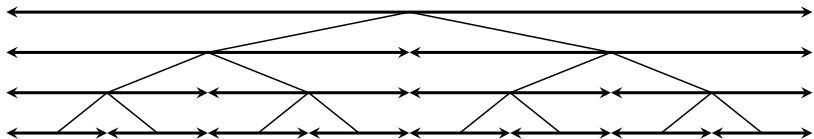


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# Recursion Equations

## Master theorem (Simple Form)



Figure: Simple recursion equation with  $a = 2, b = 3$

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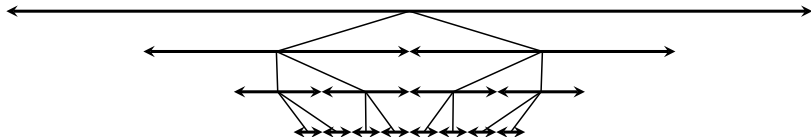


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**Case 3:**  $a < b$



# Recursion Equations

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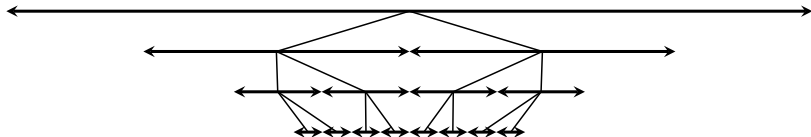


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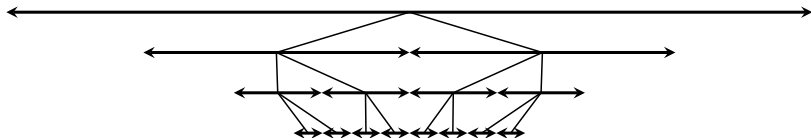


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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor  $\frac{a}{b}$

## Divide and Conquer

Concept

Maximum Subtotal

## Recursion Equations

Substitution Method

Recursion Tree Method

**Master theorem**

Master theorem (Simple Form)

Master theorem (General Form)



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- **Case 2:**  $T(n) \in \Theta(n^{\log_b a} \log n)$  if  $f(n) \in \Theta(n^{\log_b a})$

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Connecting all partial solutions dominates  
(first layer, root)

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$



### Case 1 - Example:

if  $f(n) \in O(n^{\log_b a - \epsilon})$ ,  $\epsilon > 0$

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$$\blacksquare T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

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■  $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, b = 3, f(n) = 17 \cdot n, \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$

# Recursion Equations

## Master theorem (General Form) - Case 2



**Case 2:** if  $f(n) \in \Theta(n^{\log_b a})$

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# Recursion Equations

## Master theorem (General Form) - Case 2

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■  $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 10 \cdot n$

$$a = 2, b = 2, f(n) = 10 \cdot n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

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■  $T(n) = T\left(\frac{2n}{3}\right) + 1$

$$a = 1, b = \frac{2}{3}, f(n) = 1, \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$



### Case 3:

if  $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ ,  $\varepsilon > 0$

Connecting all partial solutions dominates (first layer, root)



**Case 3:**  $T(n) \in \Theta(f(n))$  if  $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ ,  $\varepsilon > 0$

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$$a = 2, b = 2, f(n) = n^2, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Omega(n^{1+\varepsilon})$$

Check if **regularity condition** also holds:

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$



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$n \log n$  is *asymptotically* larger than  $n$ ,  
but not *polynomial* larger





### **Master theorem:**



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- **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems

$$T(n) \in \Theta(f(n))$$

## ■ General

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## ■ Master theorem

[Wik] [Master theorem](#)

[https://en.wikipedia.org/wiki/Master\\_theorem](https://en.wikipedia.org/wiki/Master_theorem)