Algorithms and Datastructures

Open Addressing, Priority Queue

Albert-Ludwigs-Universität Freiburg

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, November 2017

Structure

Hashing
- Recapitulation
- Treatment of hash collisions
- Open Addressing
- Summary

Priority Queue
- Introduction

Hashing:

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- To find a good hash function for every key set universal hashing is needed
  - Then however, for a fixed set of keys not every hash function is suitable, but only some

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  - Look at amortized analysis in the next lecture

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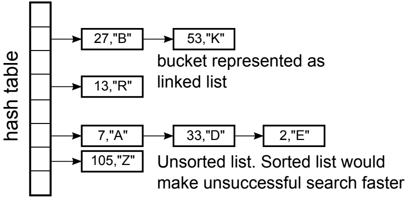

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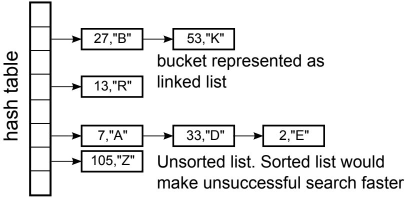

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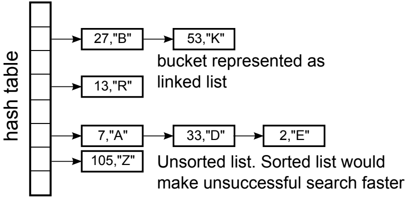

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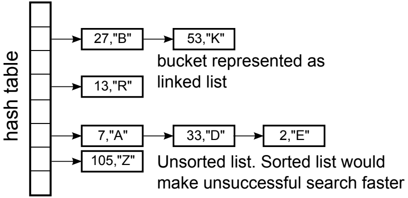

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- Dynamic number of elements is possible

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- The **probe sequence** determines for each key, in which sequence all hash table entries are searched for a free bucket
  - If a entry is already occupied, then iteratively the following entry can be checked. If a free entry is found the element is inserted
  - If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry have been found

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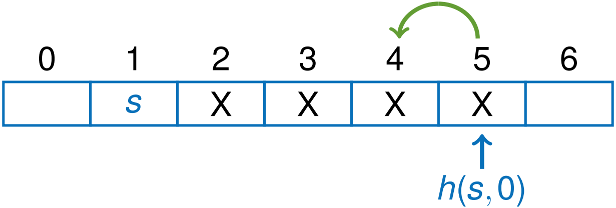
- The **probe sequence** is calculated by

$$h(s,j) = (h(s) - g(s,j)) \mod m \in \{0,\ldots,m-1\}$$



```
def insert(s, value):
    j = 0

    while t[(h(s) - g(s, j)) mod m] \
            is not None:
        j += 1

    t[(h(s) - g(s, j)) mod m] \
        = (s, value)
```

```
def lookup(s):
    j = 0

    while t[(h(s) - g(s, j)) mod m] \
            is not None:

        if t[(h(s) - g(s, j)) mod m][0] == s:
            return t[(h(s) - g(s, j)) mod m]

        j += 1

    return None
```

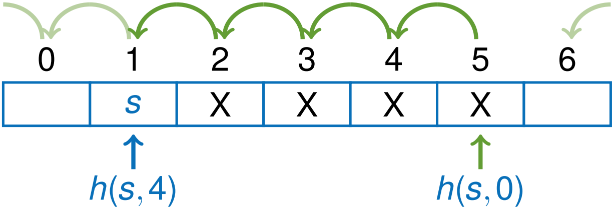
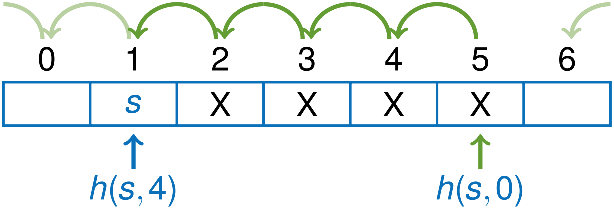


Abbildung: Linear probe sequence



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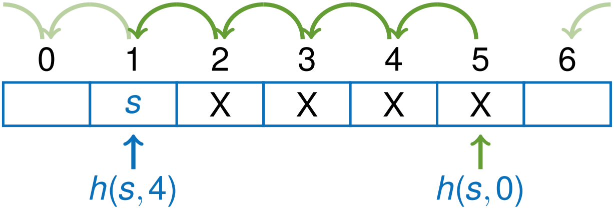
- Check the element with lower index: $g(s,j) := j$
  $\Rightarrow$ Hash function: $h(s,j) = (h(s) - j) \mod m$



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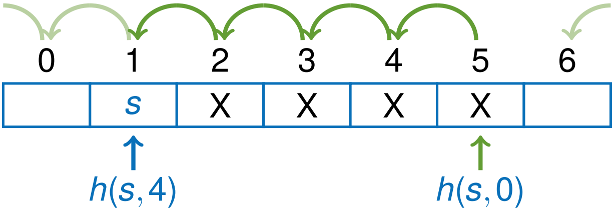
- Check the element with lower index: $g(s,j) := j$
  $\Rightarrow$ Hash function: $h(s,j) = (h(s) - j) \mod m$
- This leads to the following probe sequence

$$h(s), h(s)-1, h(s)-2, \ldots, \underbrace{0, m-1}_{\text{clipping}}, m-2, \ldots, h(s)+1$$



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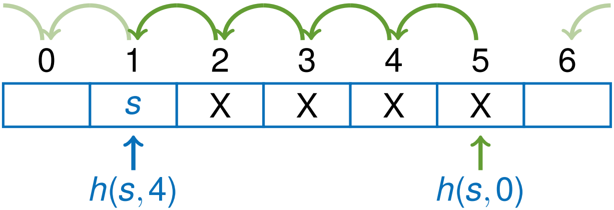
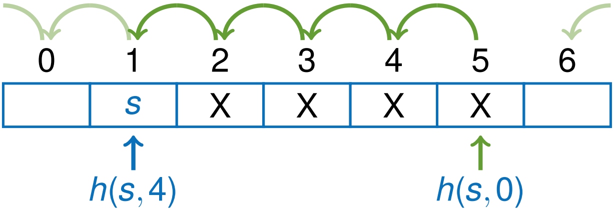


Abbildung: Linear probe sequence

- Can result in primary clustering

Hashing

Open Addressing - Linear Probing



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- Can result in primary clustering
- Dealing with a hash collision will result in a higher probability of hash collisions in close entries

Example:

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- Keys: $\{12, 53, 5, 15, 2, 19\}$

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- `t.insert(12, "A")`, $h(12,0) = 5$

0	1	2	3	4	5	6
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- t.insert(53, "B"), $h(53,0) = 4$

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- t.insert(15, "D"), $h(15,0) = 1$

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- t.insert (2, "E"), $h(2,0) = 2$

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0	1	2	3	4	5	6
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- t.insert(19, "F"), $h(19,0) = 5$, $h(19,1) = 4$, $h(19,2) = 3$, $h(19,3) = 2$, $h(19,4) = 1$, $h(19,5) = 0$

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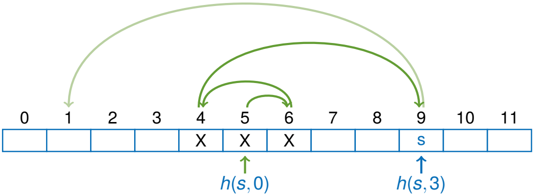

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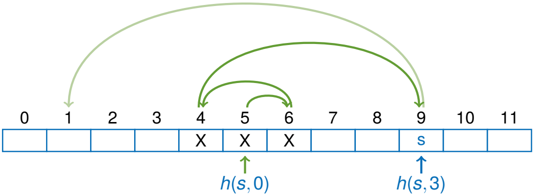

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- This leads to the following probe sequence

  $h(s),\ h(s)+1,\ h(s)-1,\ h(s)+4,\ h(s)-4,\ h(s)+9,\ h(s)-9,\ \ldots$

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- Alternatively: $h(s,j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \mod m$
- Problem of secondary clustering
  No local clustering anymore, but keys with same hash value have similar probe sequence

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- **Disadvantage:** Hard to implement

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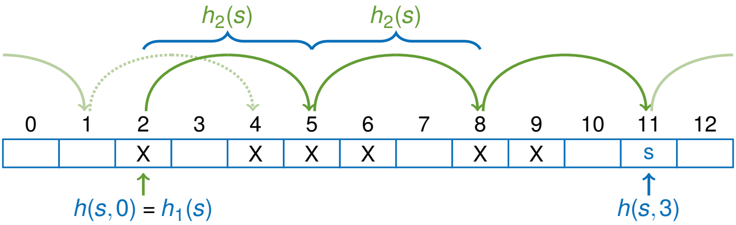

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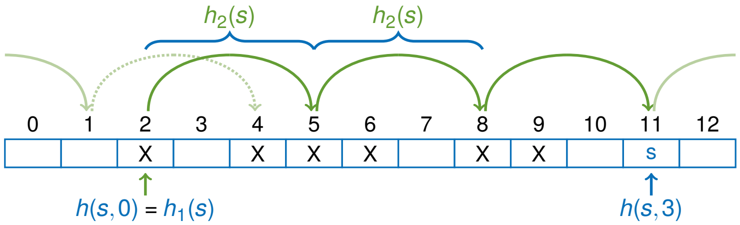

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Hashing

Open Addressing - Double Hashing

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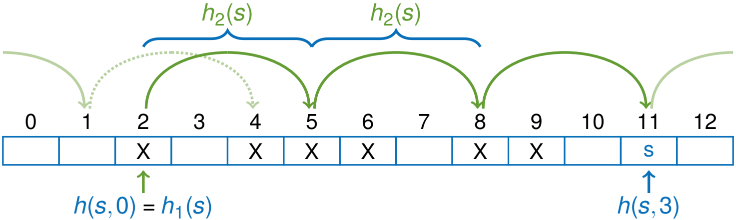

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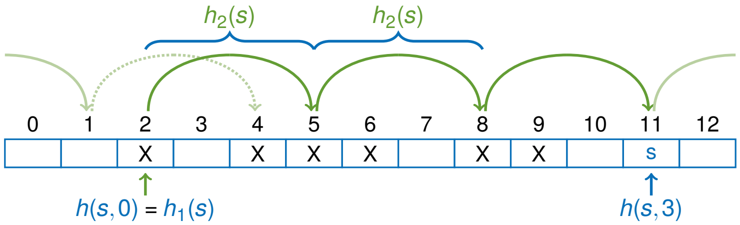

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- Works well in practical use
- This method is an approximation of uniform probing

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$$h(s,j) = h_1(s) + j \cdot h_2(s) \mod 7$$

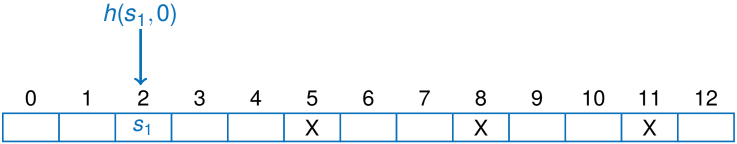
Tabelle: Comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on $h_1(s) \neq h_2(s)$

Hashing

Open Addressing - Double Hashing - Optimization



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Hashing

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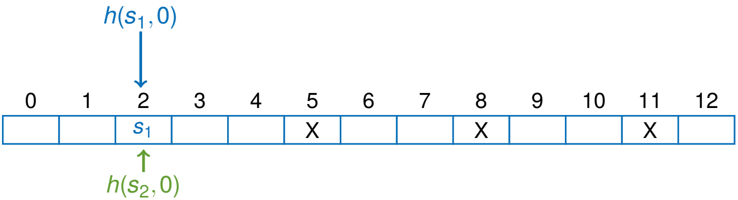

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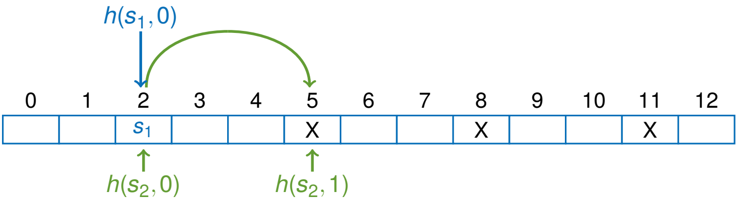
  Because different keys have different probe sequences, the sequence of the insertions has impact on efficiency of a sucessful search



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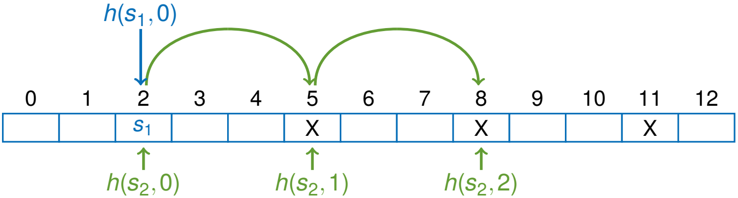

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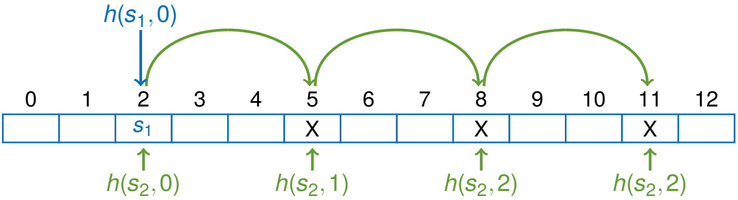

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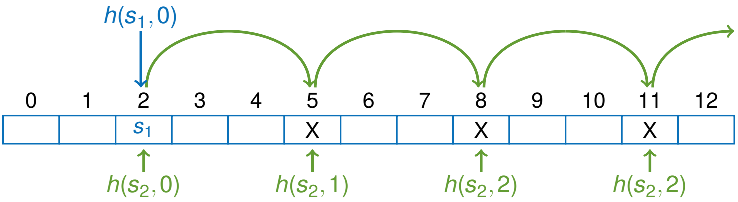

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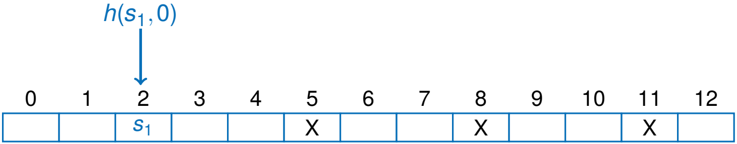

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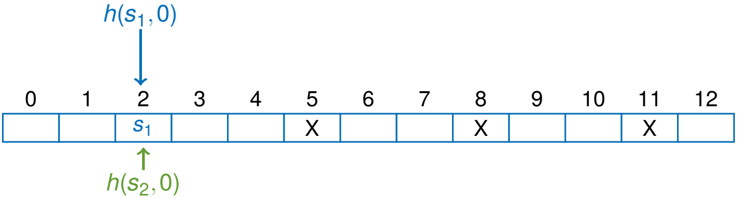

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- The hash function for s_2 also results in $p_2 = h(s_2, 0) = p_1$

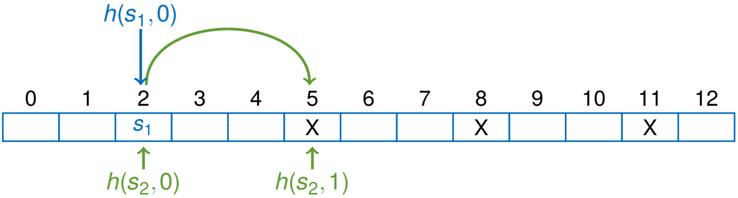

Abbildung: Double hashing

Example:

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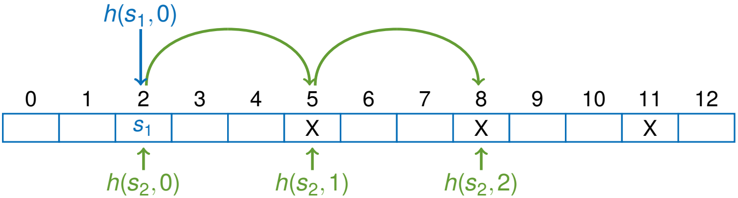
- The hash function for s_2 also results in $p_2 = h(s_2, 0) = p_1$
- The locations $h(s_2, j)$, $j \in \{1, \ldots, n\}$ are also occupied



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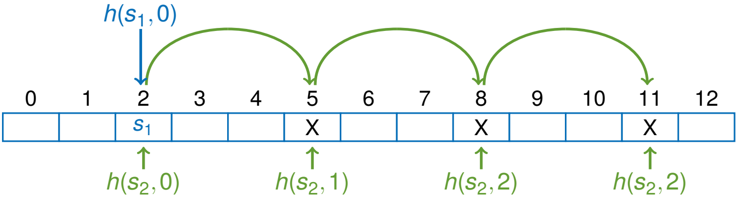

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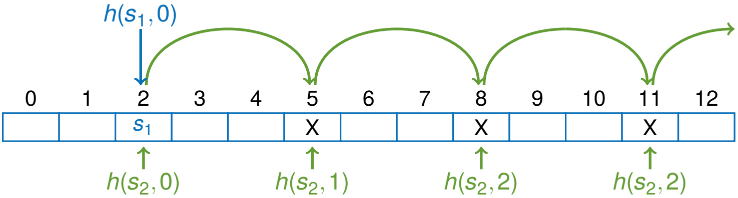

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- The locations $h(s_2, j)$, $j \in \{1, \ldots, n\}$ are also occupied
- If we insert s_2 at position $h(s_2, n+1)$ the search will be inefficient

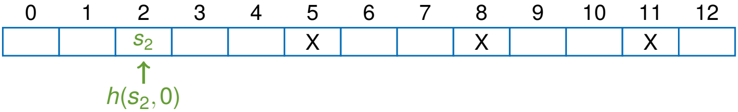
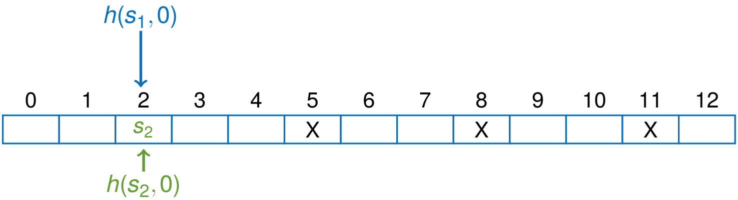


Abbildung: Double hashing by Brent

Hashing

Open Addressing - Double Hashing - Optimization



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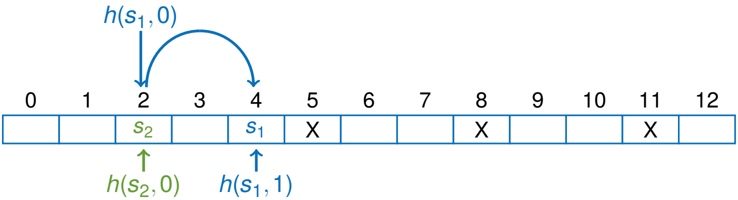

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Hashing

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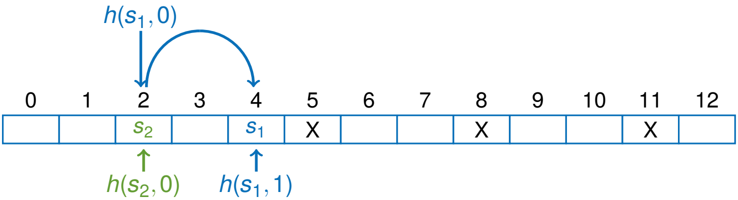
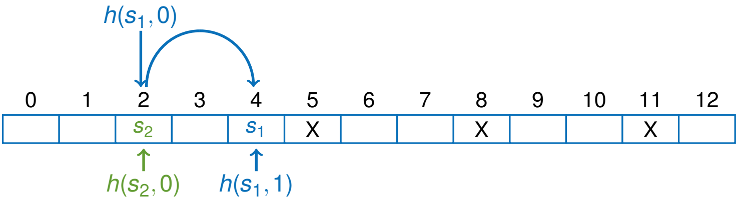


Abbildung: Double hashing by Brent

- Reversed sequence of keys would have been better



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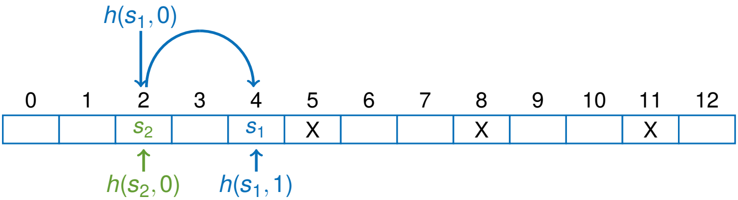
- Reversed sequence of keys would have been better
- **Brents Idea:**



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- Reversed sequence of keys would have been better
- **Brents Idea:**
  - Test if location $h(s_1, 1)$ is free

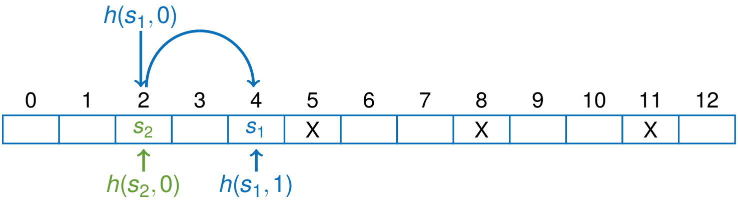

Abbildung: Double hashing by Brent

- Reversed sequence of keys would have been better
- **Brents Idea:**
  - Test if location $h(s_1, 1)$ is free
  - If yes, move s_1 from $h(s_1, 0)$ to $h(s_1, 1)$ and insert s_2 at $h(s_2, 0)$

Idea:

- Motivation: Colliding elements are inserted in the hashtable sorted.
- Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is earlier possible because single probing steps have a fixed length

Implementation:

- Compare both keys if a collision occurs
- Insert the smaller key at p_1
- Search a position based on the diversion order for the bigger key

Example:

- The key 12 is saved at position $p_1 = h(12, 0)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $5 < 12$ we insert the key 5 at position p_1
- For the key 12 we iterate through the sequence

$$h(12, 1),\ h(12, 2),\ h(12, 3),\ \ldots$$

Motivation:

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- Having similiar length of probe sequences for all elements. Total costs stay the same, but they are distributed evenly. Results in approximately similar search times for all elements

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- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)

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Implementation:

- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)
- The key with the bigger search sequence is inserted at p_1 The other key is assigned a new location based on the sequence

Example:

- The key 12 is saved at position $p_1 = h(12, 7)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $j_1 < j_2$ $(0 < 7)$ the key 12 stays at position p_1
- For the key 5 we iterate through the sequence

$$h(5, 1),\ h(5, 2),\ h(5, 3),\ \ldots$$

Problem:

- The key s_1 is inserted at position p_1
- The key s_2 returns the same hash value, but is inserted at position p_2 because of the probing order
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Solution:

- **Remove:** Elements are marked as removed, but not deleted
- **Inserting:** Elements marked as removed will we overwritten

Structure

Bucket as linked list: (dynamic, number of elements variable)

- Save colliding elements as linked list

Open hashing: (static, number of elements fixed)

- Determine a probe sequence, permutation of all hash values
- Linear, quadratic probing:
  - Easy to implement
  - Raise the probability of collisions because probing order does not depend on the key

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Improving efficiency: (Brent, Ordered Hashing)

- Improve search efficiency by sorting colliding insertions
  - Abortion of unsuccessfull search
  - Search sequence length balancing

Hashing:

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- Efficient fo dictionary operations:

  `Insert`: $O(1) \ldots O(n)$
  `Search`: $O(1) \ldots O(n)$
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- Direct access of all elements in a hash table
- Using a hash function to find the position (hash value) in the hash table
- Hash function, size of the hash table and strategy to avoid hash collisions influence the efficiency of the datastructure

Structure

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    `getMin()`: Returns just one of the possible elements
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- Argument of `changeKey` and `remove` operations
  - There is no **quick-access** to a element in the queue
  - Thats why `insert` and `getMin` return a reference (handle, accessor object)
  - `changeKey` and `remove` take this reference as argument
  - Therefore each element has to store its current position in the heap.

```python
from queue import PriorityQueue

q = PriorityQueue()

e1 = (5, "A") # element with priority 5
q.put(e1); # insert element e1

# remove and return the lowest item
e2 = q.get()
```

Priority Queue

Application Example

Example 1:

- Calculation of the sorted union of k sorted lists (multi-way merge or k-way merge)

L_1 :

3	5	8	12	...

L_3 :

1	10	11	24	...

L_2 :

4	5	6	7	...

$\Rightarrow R$:

1	3	4	5	5	6	7	8	10	...

Abbildung: 3-way merge

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- For example Dijkstra's algorithm for computing the shortest path ($\leftarrow$ following lecture)

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Example 2:

- For example Dijkstra's algorithm for computing the shortest path ($\leftarrow$ following lecture)
- Among other applications it can be used for sorting

Idea:

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- Save elements as tuples in a binary heap

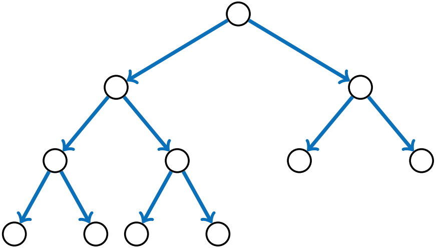


Abbildung: Heap with 11 nodes

Idea:

- Save elements as tuples in a binary heap
- Summary from lecture 1 (*HeapSort*):
  - Nearly complete binary tree
  - Heap condition:
    The key of each node $\leq$ the keys of the children

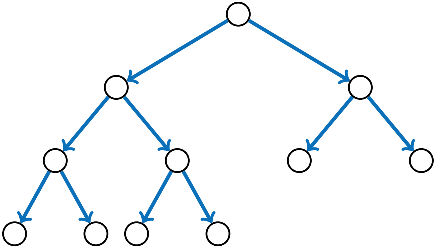
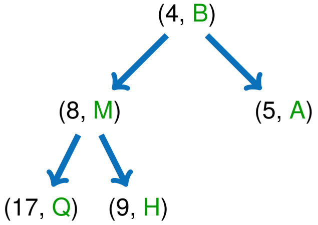


Abbildung: Heap with 11 nodes

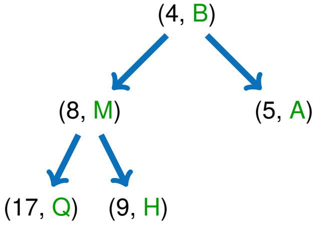
Priority Queue
Implementation



0	1	2	3	4
4, B	8, M	5, A	17, Q	9, H

Abbildung: Min heap stored in array

Priority Queue
Implementation



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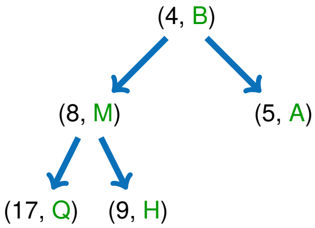

0	1	2	3	4
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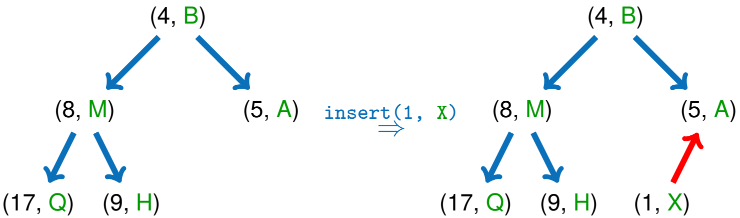
Abbildung: Min heap stored in array

Storing a binary heap:

- Number nodes from top to bottom and left to right starting with 0 and store entries in array
- Children of node i are the nodes $2i+1$ and $2i+2$
- Parent node of node i is $\text{floor}((i-1)/2)$

Inserting an element: `insert(key, item)`



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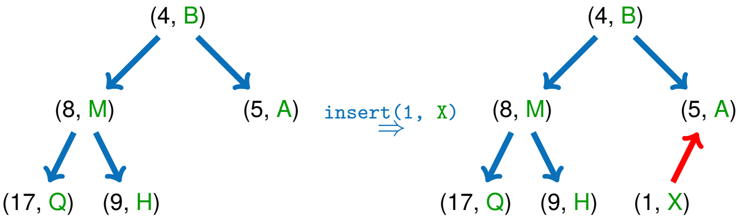

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Priority Queue

Implementation - Insertion

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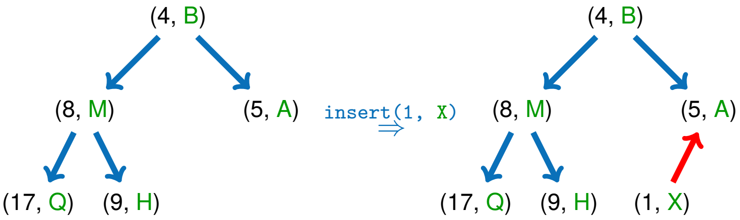

- Append the element at the end of the array
- The heap condition may be violated, but only at the last index

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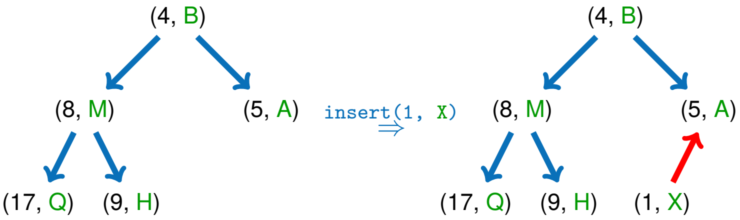
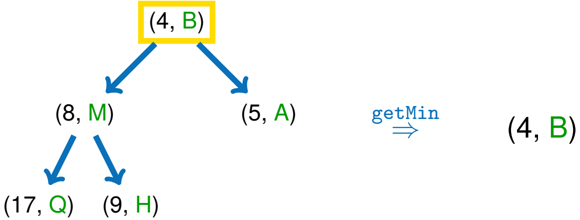


- Append the element at the end of the array
- The heap condition may be violated, but only at the last index
- Repair heap condition ⇒ We will see later how to do this

Returning the minimum: `getMin()`



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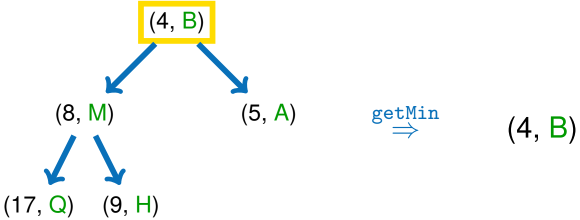

- Else return the first element

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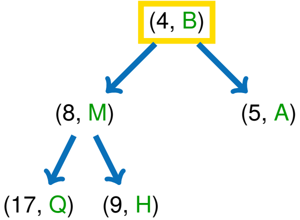
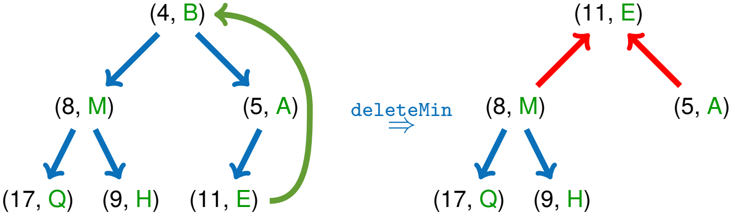


$\xrightarrow{\texttt{getMin}}$ (4, B)

- Else return the first element
- If the heap is empty return None

Removing the minimum: `deleteMin()`



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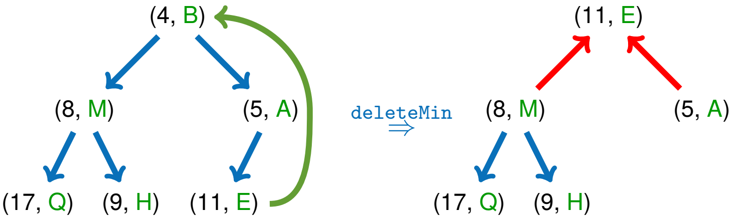

- Deleting the element with the lowest key

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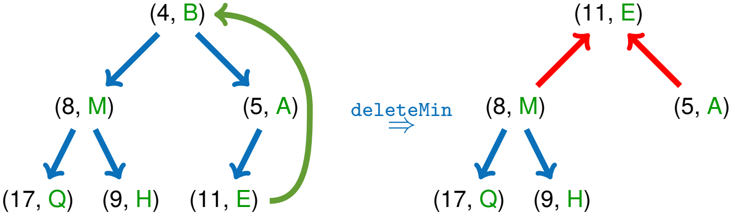

- Deleting the element with the lowest key
- Swap the last element with the first element and shrink the heap by one

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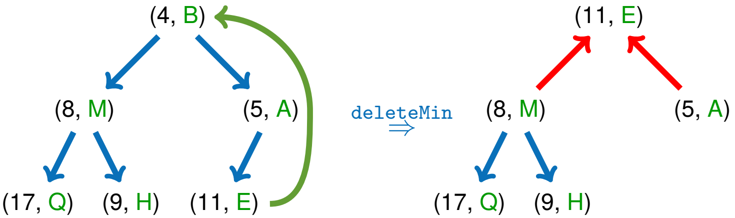

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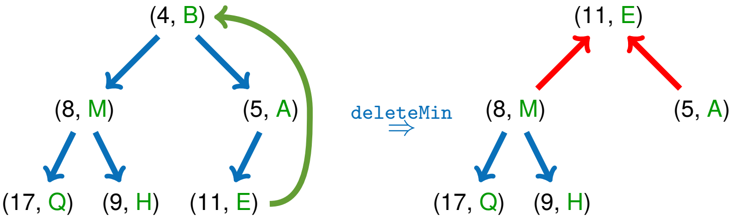

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- Swap the last element with the first element and shrink the heap by one
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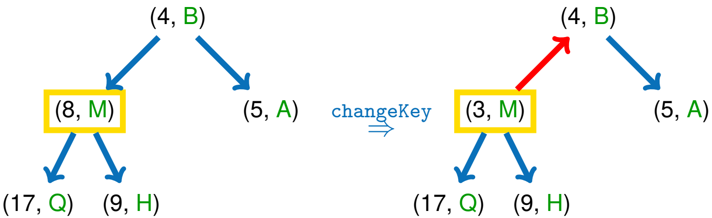
- Repair heap condition

Changing the key (priority): `changeKey(item, key)`

- The element (queue item) is given as argument
- Replace the key of the element
- The heap condition may be violated, but only at the element index and only in one direction (up / down)
- Repair heap condition



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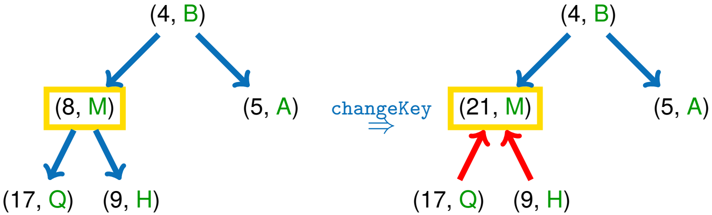

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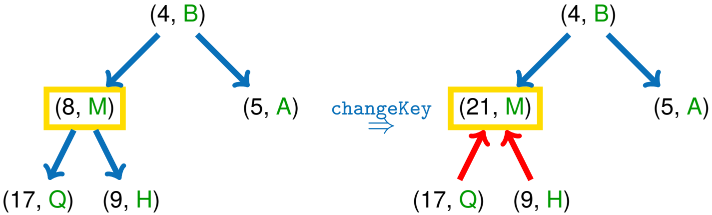

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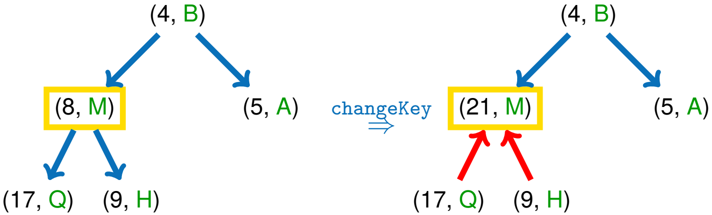
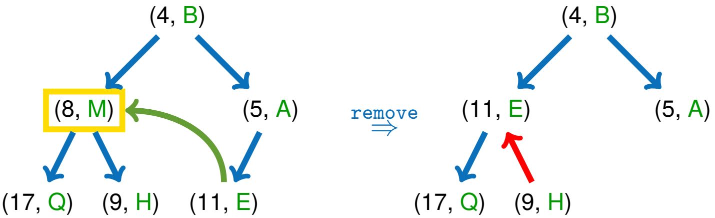


- The heap condition may be violated, but only at the element index and only in one direction (up / down)
- Repair heap condition

Removing an element: `remove(item)`



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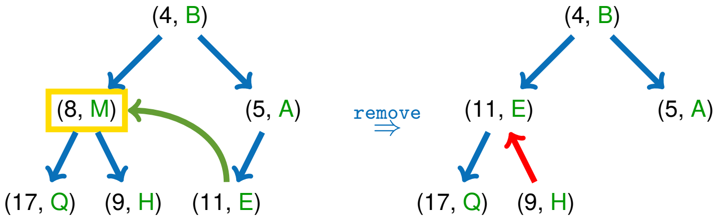

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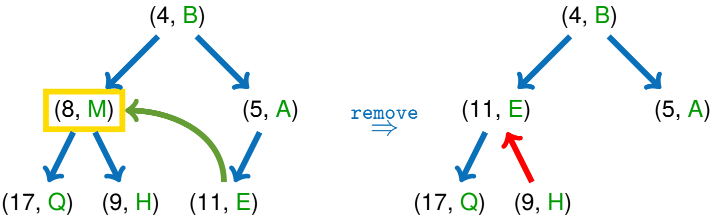

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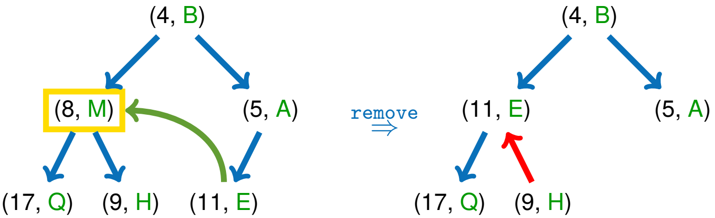

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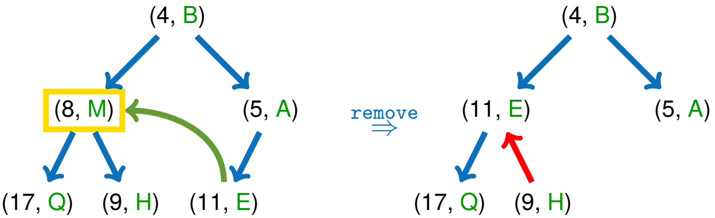

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  - Downwards: The key at index i is not $\leq$ than the value of its children
  - Upwards: The key at index i is not $\geq$ than the value of its parent
- We need two repair methods: `repairHeapUp`, `repairHeapDown`

Priority Queue

Implementation - Reparing the Heap

`repairHeapDown:`

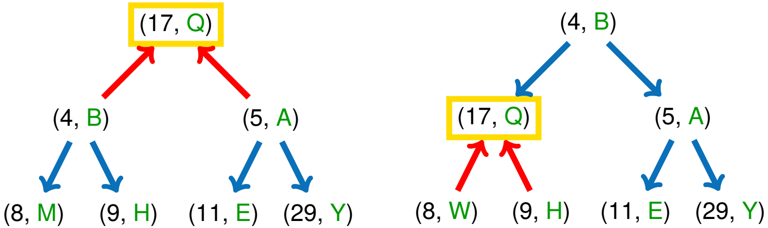

Abbildung: Repairing the heap downwards

`repairHeapDown:`

- Sift the element until the heap condition is valid

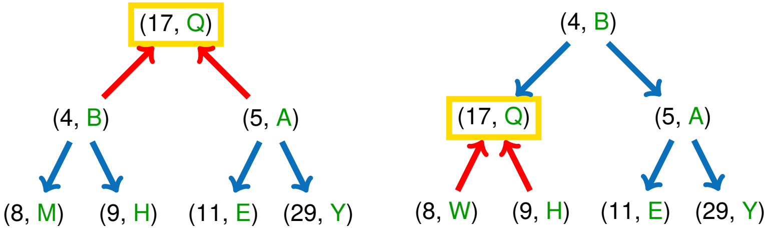

Abbildung: Repairing the heap downwards

`repairHeapDown:`

- Sift the element until the heap condition is valid
  - Change node with child, which has the lower key of both children

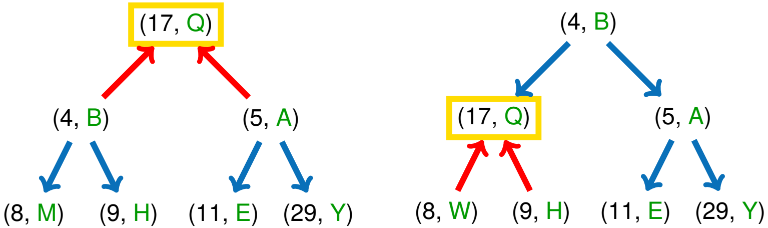

Abbildung: Repairing the heap downwards

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- Sift the element until the heap condition is valid
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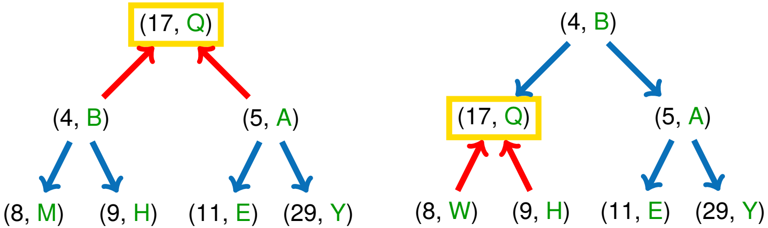

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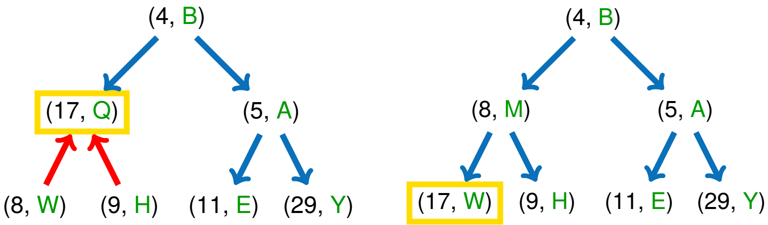

Abbildung: Repairing the heap downwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapUp:`

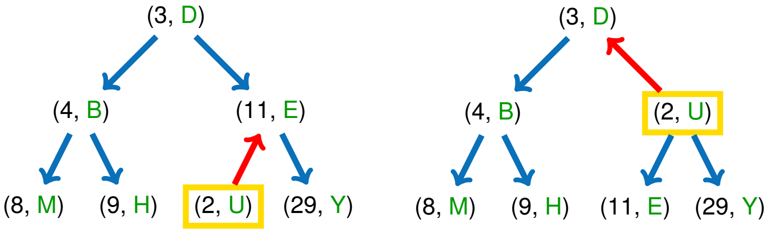

Abbildung: Repairing the heap upwards

`repairHeapUp:`

- Change node with parent

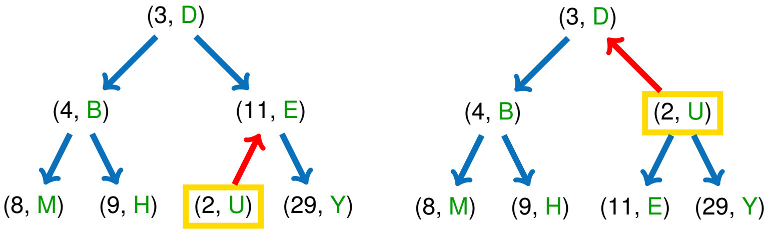

Abbildung: Repairing the heap upwards

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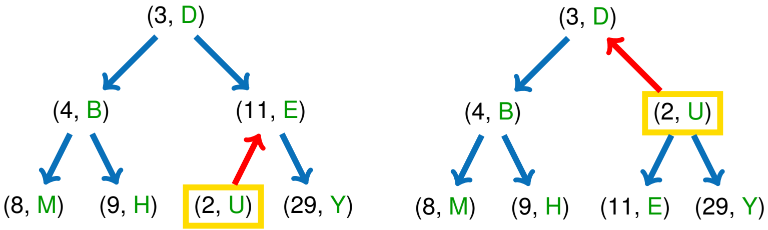

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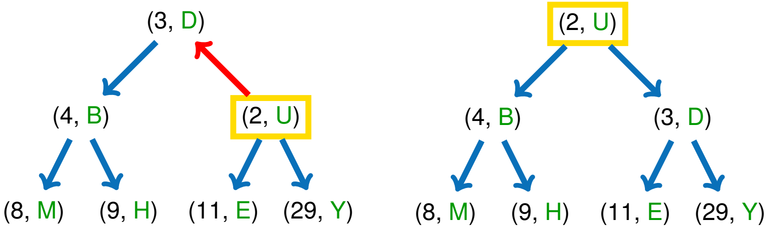

Abbildung: Repairing the heap upwards

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- Attention: For `changeKey` and `remove` the item has to „know" where it is located in the heap
- Remember for `repairHeapUp` and `repairHeapDown`: Update the index if moving an heap element

```python
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```

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- `insert`, `deleteMin`, `changeKey`, `remove`:
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- `getMin`: Return the element at index 0: $O(1)$

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Practical experience:

- The binary heap is simpler: Costs for managing the structure are low
- If the number of elements is relatively small so the difference is negligible
- Example:
  - For $n = 2^{10} \approx 1,000$ is the the depth $\log_2 n$ only 10
  - For $n = 2^{20} \approx 1,000,000$ is the depth $\log_2 n$ only 20

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson. **Introduction to Algorithms.** MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders. Algorithms and data structures, 2008. https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Priority Queue - Implementations / API**

  [Cpp] C++ - priority_queue
  http://www.sgi.com/tech/stl/priority_queue.html

  [Jav] Java - PriorityQueue
  https://docs.oracle.com/javase/7/docs/api/java/util/PriorityQueue.html

  [Pyt] Python - PriorityQueue
  https://docs.python.org/3/library/queue.html#queue.PriorityQueue