# Algorithmns and Datastructures

## Hash Map, Universal Hashing

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Algorithmns and Datastructures, November 2016

# Structure

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- Feedback
  - Exercises
  - Lecture
- Associative Arrays
  - Introduction
  - Hash Map
- Universal Hashing
  - Introduction
  - Probability Calculation
  - Proof
  - Examples

# Feedback from the exercises

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- For most using associative arrays (Python dictionary) tends to be a bit faster
  $\rightarrow$ Sorting the dictionary also takes time, depending on heterogeneity of the data (e.g. lots of locality names with

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- For n keys searching requires $\Theta(n)$ time
- With a Hash Map this just requires $\Theta(1)$ in the best case, ... regardless how many elements are in the map!

# Structure

**Idea:**

- Mapping the keys onto indices with a hash function
- Store the values at the calculated indices in a normal array

**Example:**

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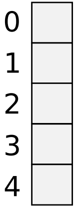
- Hash function: $h(x) = x \bmod 5$, in the range $[0, \ldots, 4]$
- We need an array T with 5 elements.
  A "hashtable" with 5 "buckets"
- The element with the key x is stored in $T[h(x)]$

**Storage:**

Figure: Hashtable T



**Storage:**

- insert(3904433,"A"): $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")

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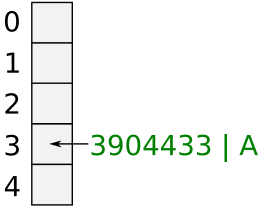

**Storage:**

- insert(3904433,"A"): $h$(3904433) = 3 ⇒ T[3] = (3904433, "A")
- insert(312692, "B"): $h$(312692) = 2 ⇒ T[2] = (312692, "B")

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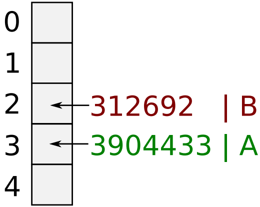

# Associative Arrays

## The Hash Map

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- insert(3904433,"A"): $h$(3904433) = 3 $\Rightarrow$ T[3] = (3904433, "A")
- insert(312692, "B"): $h$(312692) = 2 $\Rightarrow$ T[2] = (312692, "B")
- insert(5148949, "C"): $h$(5148949) = 4 $\Rightarrow$ T[4] = (5148949, "C")

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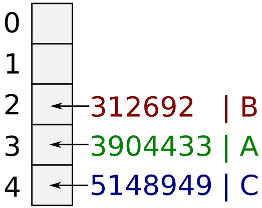

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**Searching:**

- search(3904433): $h(3904433) = 3 \Rightarrow$ T[3] $\rightarrow$ (3904433, "A")

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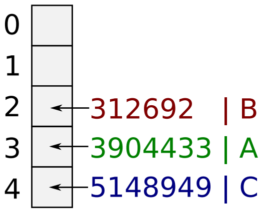


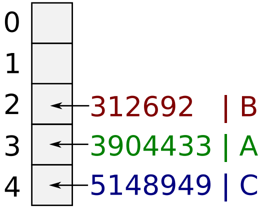
# Associative Arrays
## The Hash Map

**Searching:**

- search(3904433): $h$(3904433) = 3 $\Rightarrow$ T[3] $\rightarrow$ (3904433, "A")
- search(123459): $h$(123459) = 4 $\Rightarrow$ T[4]

  $\Rightarrow$ Value with key 123459 does not exist

Figure: Hashtable T



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- search(123459): $h(123459) = 4 \Rightarrow$ T[4]

  $\Rightarrow$ Value with key 123459 does not exist
- Search time for this example: $\mathcal{O}(1)$

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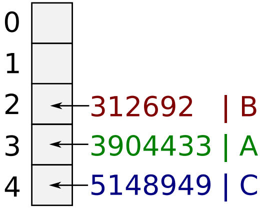
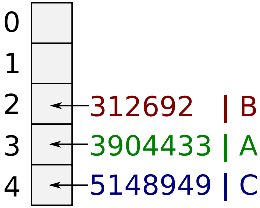


**Further inserting:**

- insert(876543, "D"): $h(876543) = 3$

Figure: Hashtable T



# Associative Arrays
## Hash Collisions

**Further inserting:**

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  $\Rightarrow$ T[3] = (876543, "D") COLLISION!

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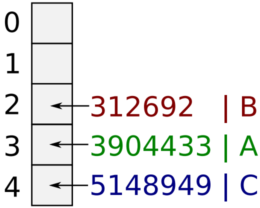
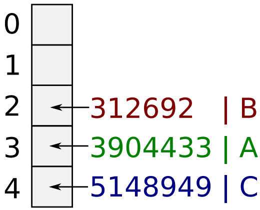


# Associative Arrays
## Hash Collisions

**Further inserting:**

- insert(876543, "D"):   $h$(876543) = 3
  $\Rightarrow$ T[3] = (876543, "D") COLLISION!
- This happens more often than expected
  - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

Figure: Hashtable T



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- Represent each bucket as list of key value pairs

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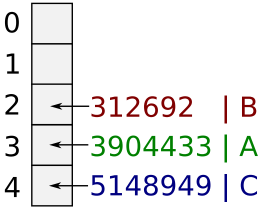

# Associative Arrays
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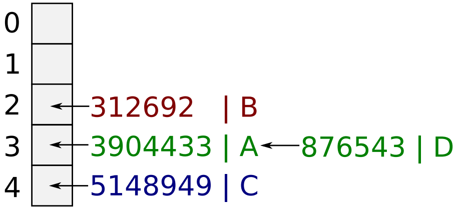
**Problem:**

- Two keys are equal $h(x) = h(y)$ but not the values $x \neq y$

**Easiest Solution:**

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

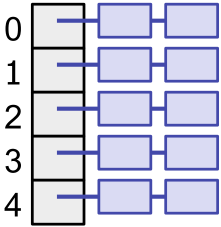
Figure: Hashtable T



Best case:

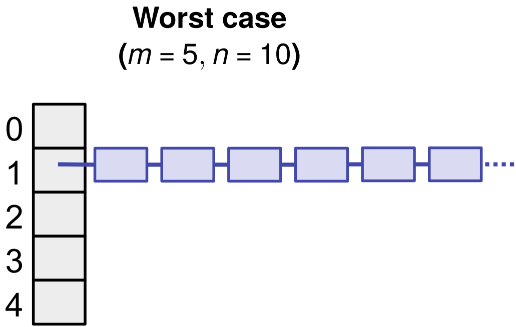
- We have $n$ keys which are equally distributed over $m$ buckets
- We have $\approx \frac{n}{m}$ pairs per bucket

**Best case**
**($m = 5$, $n = 10$)**



**Worst case:**

- All $n$ keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching



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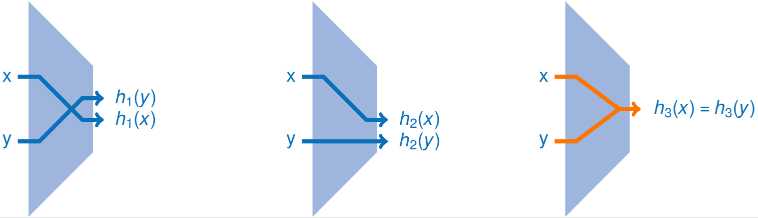
  - *for table size 100: try* $100 \times 99 + 1$ *different numbers*
  - *worst case: still 100 must have same hash bucket*
- **Now:** Find a solution to avoid that problem

**Solution:**

- We use a set of hash functions



**Solution:**

- We use a set of hash functions
- We choose a random hash function so that the expected result is an equal distribution over the buckets

  *this is fixed for the lifetime of table*

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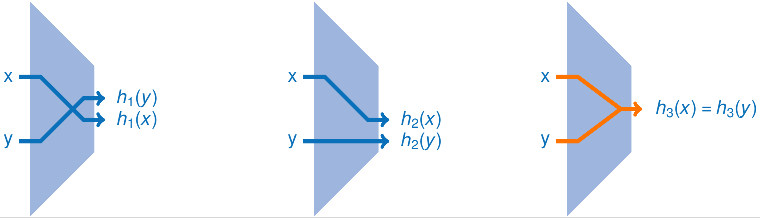

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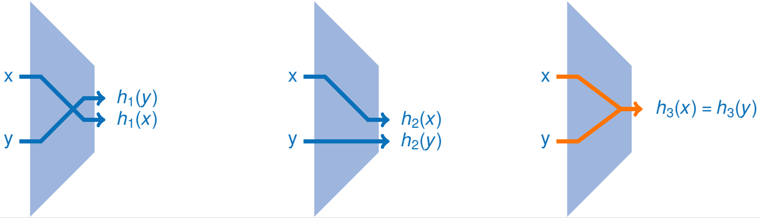
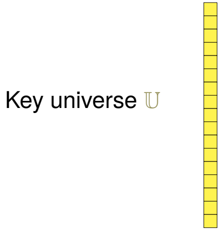
- This is called universal hashing



**Definition:**

- We call $\mathbb{U}$ the set (universum) of possible keys, and $\mathbb{S} \subseteq \mathbb{U}$ the set of used keys



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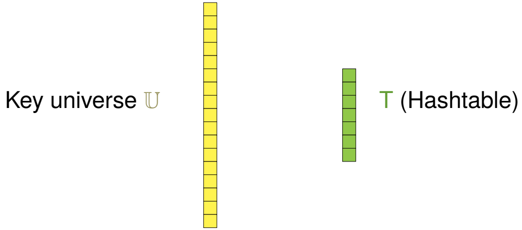

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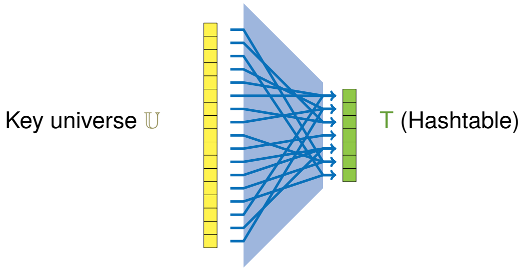

# Universal Hashing

## Definition

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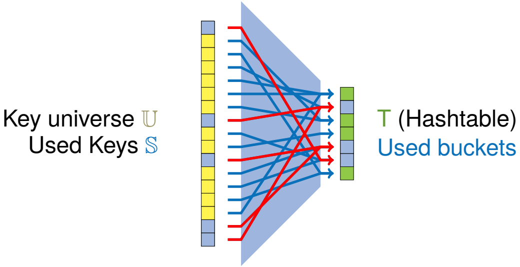

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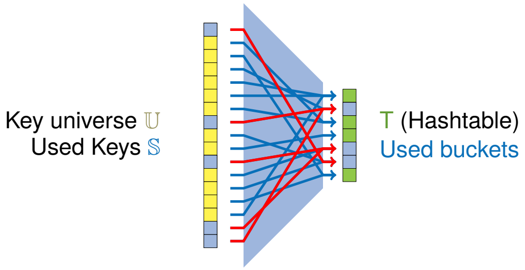
- Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load



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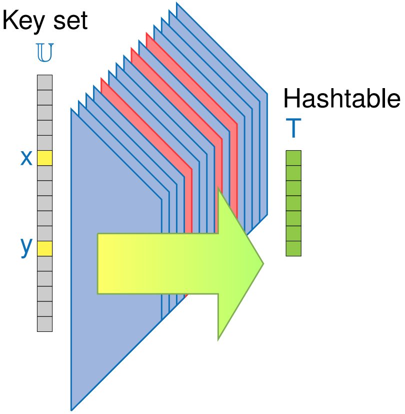
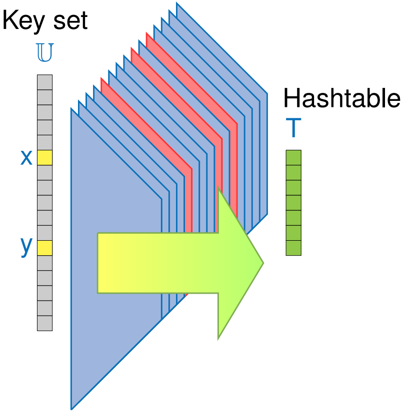



Figure: Set of hash functions $\mathbb{H}$

- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- An average of 3 out of 15 functions produce collisions

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$$\frac{\overbrace{|\{h \in \mathbb{H} : h(x) = h(y)\}|}^{\text{Number of hash functions that create collisions}}}{\underbrace{|\mathbb{H}|}_{\text{Number of hash functions}}} \leq c \cdot \frac{1}{m}, \qquad c \in \mathbb{R}$$

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$Prob(Collision) = \frac{1}{m} \Leftrightarrow c = 1$

- $\mathbb{U}$: Key universe
- $\mathbb{S}$: Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: Keys mapping to Bucket $i$ (“synonyms”)
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$

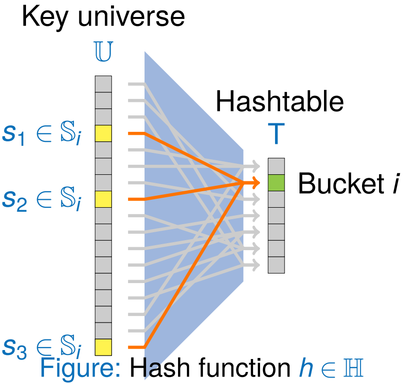


Figure: Hash function $h \in \mathbb{H}$

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- Particulary: If $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

# Structure

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Table: Throwing a dice

| $e$ | $P(e)$ |
|---|---|
| 1 | $1/6$ |
| 2 | $1/6$ |
| 3 | $1/6$ |
| 4 | $1/6$ |
| 5 | $1/6$ |
| 6 | $1/6$ |

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| $(1,2)$ | $^1/_{36}$ |
| $(1,3)$ | $^1/_{36}$ |
| $\ldots$ | $\ldots$ |
| $(6,5)$ | $^1/_{36}$ |
| $(6,6)$ | $^1/_{36}$ |

**Example:**

- Rolling a dice twice ($\Omega = \{1, \ldots, 6\}^2$)
- Each event $e \in \Omega$ has the probability $P(e) = {}^1/_{36}$
- $E$ = if both eye numbers even, then $P(E) =$

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Table: Throwing a dice twice

| $e$ | $P(e)$ | $X$ |
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| $(1,1)$ | $1/36$ | 2 |
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**Sum of expected values:** For independent (discrete) result variables $X_1, \ldots, X_n$ we can write:

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- $X = X_1 + X_2$: Expected total number of eyes:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

## Corollary:

The probability of the event $E$ is $p = P(E)$. Let $X$ be the occurences of the event $E$ and $n$ be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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## Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurences of } 6) = \frac{1}{6} \cdot 60 = 10$$

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Def. $\mathbb{E}$-value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

# Structure

**Given:**

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**To proof:**

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \forall i$$

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$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \qquad y \in \mathbb{S} \setminus \{x\}$$

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$$\Rightarrow \qquad \mathbb{E}\left(|\mathbb{S}_i|\right) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

**Auxiliary calculation:**

$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1)\\ &= P(h(y) = i)\\ &= P(h(y) = h(x))\\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

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$$\mathbb{E}\left[|\mathbb{S}_i|\right] = 1 + \sum_{y \in \mathbb{S} \setminus \{x\}} \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \square$$

# Structure

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- Which $x, y$ lead to a relative collision count bigger than $\frac{c}{m}$?

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- Then $h(x) = ((47 \cdot x + 5) \mod 101) \mod m$
- Easy to implement but hard to proof
- Exercise: show empirically that it is 2-universal

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- **Intuitive**: Scalar product with base $m$

$$a \bullet x = \sum_{0,\ldots,k-1} a_i \cdot x_i$$

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## Example ($\mathbb{U} = \{0, \ldots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

# Further Literature

- **General for this Lecture**


[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
**Introduction to Algorithms.**
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

# Further Literature

- **Hash Map - Theory**

  [Wik] Hash table
  https://en.wikipedia.org/wiki/Hash_table

- **Hash Map - Implementations / API**

  [Cpp] C++ - hash_map
  http://www.sgi.com/tech/stl/hash_map.html

  [Jav] Java - HashMap
  https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html

  [Pyt] Python - Dictionaries (Hash table)
  https://en.wikipedia.org/wiki/Hash_table