

# Algorithms and Datastructures

## Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Algorithms and Datastructures, January 2017

## Balanced Trees

- Motivation

- AVL-Trees

- (a,b)-Trees

  - Introduction

  - Runtime Complexity

- Red-Black Trees

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  - if the keys are inserted in ascending / descending order  
(20, 19, 18, ...)



## Gnarley trees:



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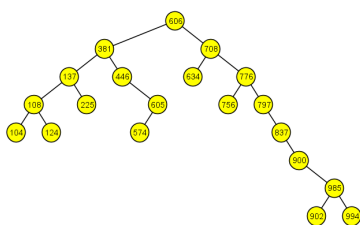


**Figure:** Binary search tree with random insert [Gna]

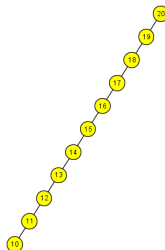


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**Figure:** Binary search tree with descending insert [Gna]



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- Can be interpreted as (2, 4)-tree
- Used in C++ `std::map`, Java `SortedMap`

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- With that the height of the search tree is always  $O(\log n)$
- We can perform all basic operations in  $O(\log n)$



Figure: Example of an AVL-Tree



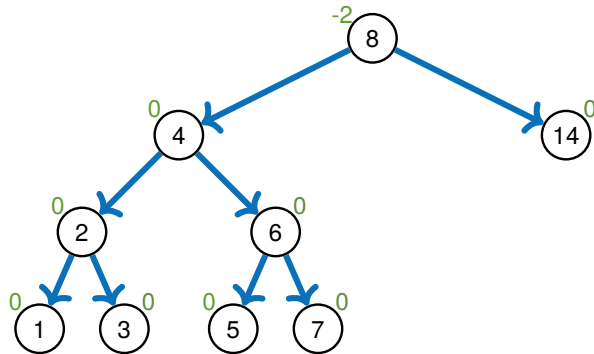


Figure: **Not** an AVL-Tree



Figure: Another example of an AVL-Tree

### Rotation:

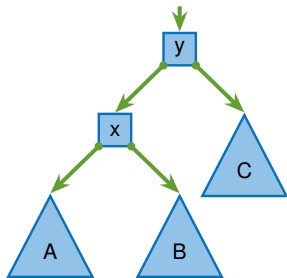


Figure: Before rotating

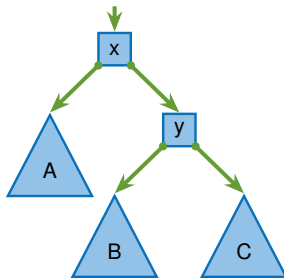


Figure: After rotating

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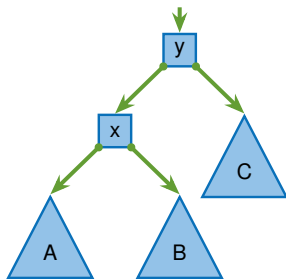


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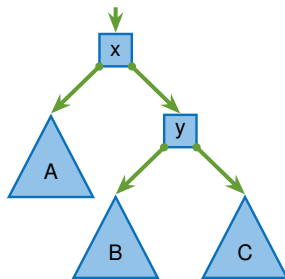


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- Central operation of **rebalancing**

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Figure: Inserting 1,..., 10 into an AVL-tree [Gna]



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- However not amortized update costs of  $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are  $(a,b)$ -trees

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### Idea:

- Save a varying number of elements per node
- So we have space for elements on an **insert** and balance operation



## $(a,b)$ -Tree:



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- We require:  $a \geq 2$  and  $b \geq 2a - 1$



### (2,4)-Tree:



Figure: Example of an (2,4)-tree

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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

### Not an (2,4)-Tree:



Figure: **Not** an (2,4)-tree

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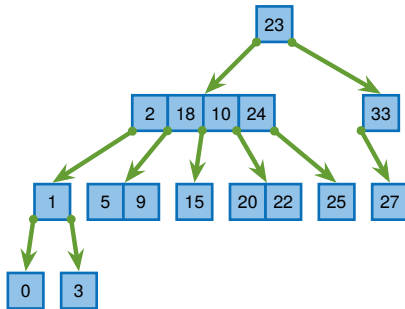


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### Not an (2,4)-Tree:

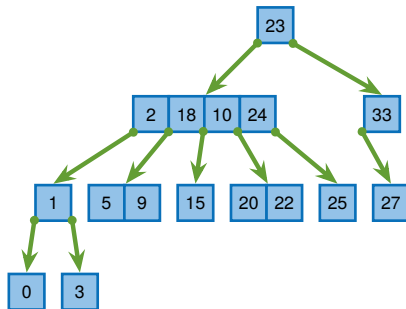


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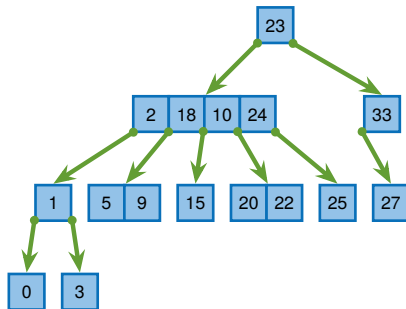


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- Degree of node too large / too small
- Leaves on different levels



**Searching an element:** (lookup)





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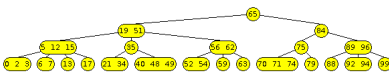
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BST AVL tree B tree Red-black tree AA tree Skiplist Max Heap Min Heap Treap Scapegoat tree Splay tree

Display



Control

50 Insert Find Delete Next

☐ Pause ☐ Small 4

#Nodes: 22 #Keys: 37 = 56% full Height: 3

Text

Search Found.



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- Then we **split** the node

Inserting an element: (`insert`)



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  - Thats why we have the limit  $b \geq 2a - 1$

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- If the node to split is the root we split it and create a new root node  
(The tree is now one level deeper)



**Removing an element: (remove)**



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Figure: Borrowing an element



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Figure: Combining two nodes

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- If the root has only one child left we take the child as new root  
(The tree shrinks one level)



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  - **insert** and **remove** often require only  $O(1)$  time

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 $\Rightarrow n \geq a^{d-1}$  and  $d \leq 1 + \log_a n = O(\log_a n)$
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  - **lookup** always takes  $\Theta(d)$
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  - Here is a counter-example for (2,3)-trees, analysis of (2,4)-trees

# $(a,b)$ -Trees

Runtime Complexity - Counter-example for  $(2,3)$ -Tree



## **$(2,3)$ -Tree:**



### (2,3)-Tree:

- Before executing `delete(11)`

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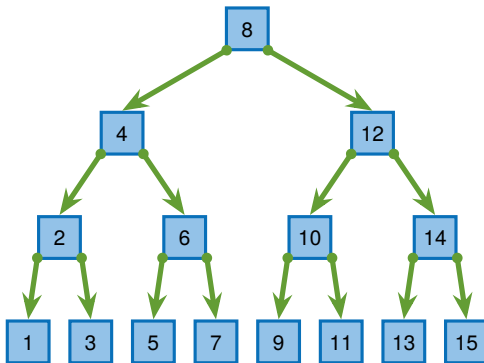


Figure: Normal (2,3)-Tree

### (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 1

### (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 2

### (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 3

### (2,3)-Tree:

- Executed `delete(11)`

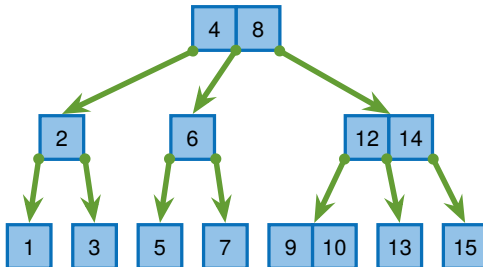


Figure: (2,3)-Tree - Delete step 4



# $(a,b)$ -Trees

Runtime Complexity - Counter example for  $(2,3)$ -Tree



## $(2,3)$ -Tree:



### (2,3)-Tree:

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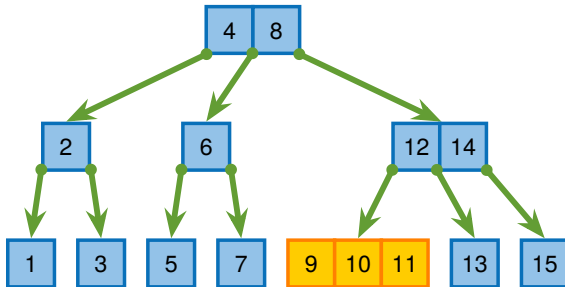


Figure: (2,3)-Tree - Insert step 1

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- Executing `insert(11)`

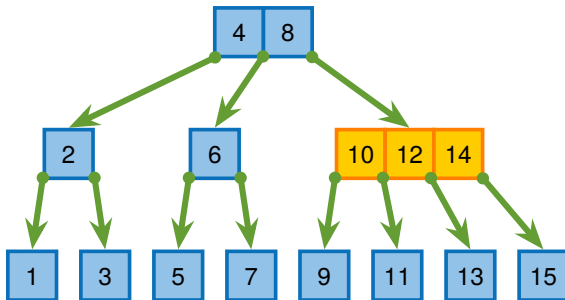


Figure: (2,3)-Tree - Insert step 2

### (2,3)-Tree:

- Executing `insert(11)`

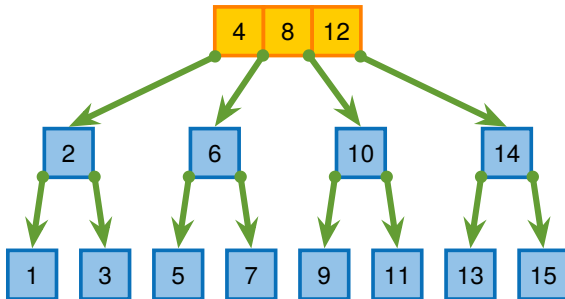


Figure: (2,3)-Tree - Insert step 3

### (2,3)-Tree:

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Figure: (2,3)-Tree - Insert step 4

### (2,3)-Tree:



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### (2,3)-Tree:

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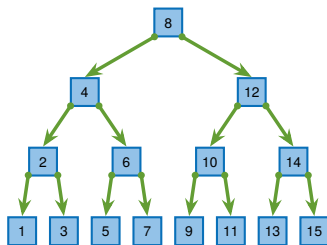


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### (2,3)-Tree:

- We are exactly where we started
- If  $b = 2a - 1$  then we can create a sequence of **insert** and **remove** operations where each operation costs  $O(\log n)$
- We need  $b \geq 2a$  instead of  $b \geq 2a - 1$

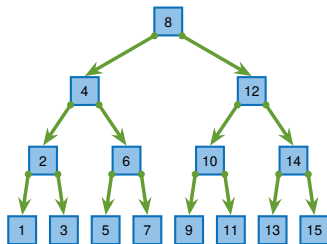


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### (2,4)-Tree:

- If all nodes have **2 children** we have to **combine** the nodes up to the root on a **remove** operation
- If all nodes have **4 children** we have to **split** the nodes up to the root on a **insert** operation
- If all nodes have **3 children** it takes some time to reach one of the previous two states

⇒ **Nodes of degree 3 are harmless**

Neither an **insert** nor a **remove** operation trigger rebalancing operations



### **$(2,4)$ -Tree:**





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### (2,4)-Tree:

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- Like with dynamic arrays:
  - **Reallocation** is expensive but it takes some time until the next expensive operation occurs
  - If we **overallocate** clever we have an amortized runtime of  $O(1)$



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### Example:



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Figure: Tree with potential  $\phi = 4$



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- The costs for operation  $i$  are coupled to the difference of the potential levels

$$c_i \leq A \cdot (\underbrace{\phi_i - \phi_{i-1}}_{\text{difference of potential levels}}) + B, \quad A > 0 \text{ and } B > A$$

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- With that each operation has an amortized cost of  $O(1)$

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Figure: Splitting a node on `insert`

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- Each splitted node creates a node of **degree 3**
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

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Costs:  $c_i \leq A \cdot m + B$

$$\begin{aligned}\Rightarrow c_i &\leq A \cdot (\phi_i - \phi_{i-1} + 1) + B \\ c_i &\leq A \cdot (\phi_i - \phi_{i-1}) + \underbrace{A + B}_{B'}\end{aligned}$$



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Figure: Tree with doubly linked list



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Figure: Merging two nodes

■ Potential rises by one

**Case 2:**  $i$ -th operation is an **remove** operation

■ **Case 2.2:** Merging a node

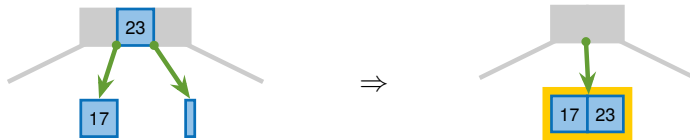


Figure: Merging two nodes

- Potential rises by one
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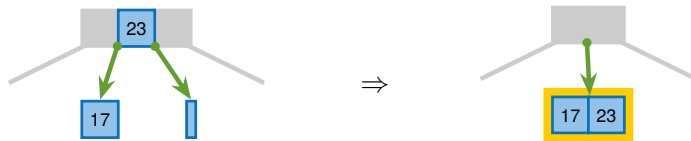


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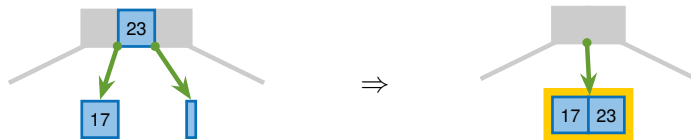


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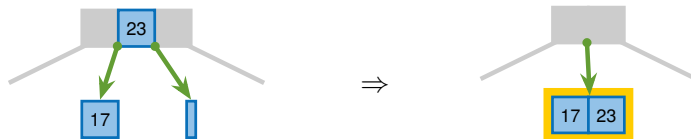


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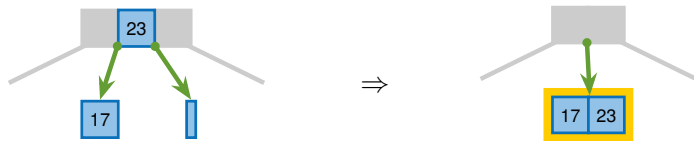


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- Same costs as **insert**



Lemma:

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- With that we can conclude:

$$\sum_{i=0}^n c_i = O(n)$$

### Proof:

$$\begin{aligned}\sum_{i=0}^n c_i &\leq \underbrace{A \cdot (\phi_1 - \phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\phi_2 - \phi_1) + B}_{\leq c_1} + \dots + \underbrace{A \cdot (\phi_n - \phi_{n-1}) + B}_{\leq c_n} \\ &= A \cdot (\phi_n - \phi_0) + B \cdot n && | \text{ telescope sum} \\ &= A \cdot \phi_n + B \cdot n && | \text{ we start with an empty tree} \\ &< A \cdot n + B \cdot n = O(n) && | \text{ number of degree 3 nodes} \\ &&& < \text{ number of nodes}\end{aligned}$$

## Balanced Trees

Motivation

AVL-Trees

(a,b)-Trees

Introduction

Runtime Complexity

**Red-Black Trees**



## Red-Black Tree:

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- Binary tree with red and black nodes

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- Binary tree with **red** and **black** nodes
- Number of **black** nodes on path to leaves is equal
- Can be interpreted as **(2,4)-tree** (also named 2-3-4-tree)
- Each **(2,4)-tree**-node is a small red-black-tree with a **black** root node





Figure: Example of an red-black-tree [Gna]

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

### **Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

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## ■ Gnarley Trees

[Gna] **Gnarley Trees**

<https://people.ksp.sk/~kuko/gnarley-trees/>

## ■ AVL-Tree

[Wik] [AVL tree](#)

`https://en.wikipedia.org/wiki/AVL\_tree`

## ■ (a,b)-Tree

[Wika] [2-3-4 tree](#)

`https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934\_tree`

[Wikb] [\(a,b\)-tree](#)

`https://en.wikipedia.org/wiki/\(a,b\)-tree`

## ■ Red-Black-Tree

[Wik] [Red-black tree](https://en.wikipedia.org/wiki/Red%E2%80%93black_tree)

`https://en.wikipedia.org/wiki/Red%E2%80%93black\_tree`