

# Algorithms and Datastructures

## Open Addressing, Priority Queue

Albert-Ludwigs-Universität Freiburg



**UNI  
FREIBURG**

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science  
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## Hashing

- Recapitulation
- Treatment of hash collisions
- Open Addressing
- Summary

## Priority Queue

- Introduction

### Hashing:

- No hash function is good for all key sets!
  - This cannot work, because a big universe is mapped onto a small set:  $|\mathcal{U}| > m$
- For random key sets also simple hash function work, e.g.

$$\Rightarrow h(x) = x \bmod m$$

- Then the random keys make sure that it is distributed evenly
- To find a good hash function for every key set universal hashing is needed
  - Then however, for a fixed set of keys not every hash function is suitable, but only some

### Rehashing:

- It is possible to get bad hash functions with universal hashing, but it is unlikely
- This is determinable by monitoring the maximum bucket size
- If a pre-defined level is exceeded, then a **rehash** is performed

### How to rehash?

- New hash table with a new random hash function
- Copy elements into the new table
  - Expensive but happens not often
  - Therefore the average cost is low
  - Look at **amortized analysis** in the next lecture

### Buckets as linked list:

- Each bucket is a linked list
- Colliding keys are inserted into the linked list of a bucket, either sorted or appended at the end



- Operations in  $O(1)$  are possible if a suitable table size and hashfunction is selected
- Worst case  $O(n)$ , e.g. table size of 1
- Dynamic number of elements is possible

- For colliding keys we choose a new free entry
- Static, fixed number of elements
- The **probe sequence** determines for each key, in which sequence all hash table entries are searched for a free bucket
  - If an entry is already occupied, then iteratively the **following entry** can be checked. If a free entry is found the element is inserted
  - If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry have been found

### Definitions:

$h(s)$  Hash function for key  $s$

$g(s,j)$  Probing function for key  $s$  with overflow positions

$j \in \{0, \dots, m-1\}$  e.g.  $g(s,j)=j$

- The **probe sequence** is calculated by

$$h(s,j) = (h(s) - g(s,j)) \bmod m \in \{0, \dots, m-1\}$$



```
def insert(s, value):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
        j += 1  
  
    t[(h(s) - g(s, j)) mod m] \  
      = (s, value)
```



```
def lookup(s):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
  
        if t[(h(s) - g(s, j)) mod m][0] == s:  
            return t[(h(s) - g(s, j)) mod m]  
  
        j += 1  
  
    return None
```



Figure: Linear probe sequence

- Check the element with lower index:  $g(s, j) := j$   
⇒ Hash function:  $h(s, j) = (h(s) - j) \bmod m$
- This leads to the following probe sequence

$$h(s), h(s) - 1, h(s) - 2, \dots, \underbrace{0, m-1, m-2, \dots, h(s) + 1}_{\text{clipping}}$$



Figure: Linear probe sequence

- Can result in primary clustering
- Dealing with a hash collision will result in a higher probability of hash collisions in close entries

### Example:

- Keys: {12, 53, 5, 15, 2, 19}
- Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$
- $t.\text{insert}(12, \text{"A"}), h(12, 0) = 5$

0	1	2	3	4	5	6
					12, A	

- $t.\text{insert}(53, \text{"B"}), h(53, 0) = 4$

				53, B	12, A	
--	--	--	--	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

### Example:

- Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$
- t.insert (5, "C"),  $h(5, 0) = 5$ ,  $h(5, 1) = 4$ ,  $h(5, 2) = 3$

0	1	2	3	4	5	6
			5, C	53, B	12, A	

- t.insert (15, "D"),  $h(15, 0) = 1$

	15, D		5, C	53, B	12, A	
--	-------	--	------	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

### Example:

■ Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$

■ t.insert(2, "E"),  $h(2, 0) = 2$

0	1	2	3	4	5	6
	15, D	2, E	5, C	53, B	12, A	

■ t.insert(19, "F"),  $h(19, 0) = 5$ ,  $h(19, 1) = 4$ ,  
 $h(19, 2) = 3$ ,  $h(19, 3) = 2$ ,  $h(19, 4) = 1$ ,  $h(19, 5) = 0$

19, F	15, D	2, E	5, C	53, B	12, A	
-------	-------	------	------	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

### Squared probing:

- Motivation: Avoid local clustering

$$g(s, j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$

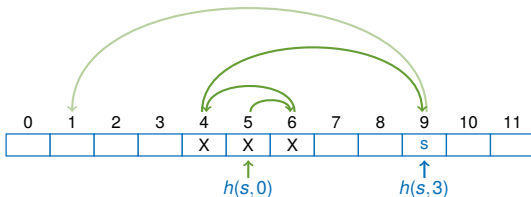


Figure: Squared probe sequence

- This leads to the following probe sequence

$$h(s), h(s) + 1, h(s) - 1, h(s) + 4, h(s) - 4, h(s) + 9, h(s) - 9, \dots$$

### Squared probing:

$$g(s, j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$

- If  $m$  is a prime number for which  $m = 4 \cdot k + 3$  then the probe sequence is a permutation of the indices of the hash tables
- Alternatively:  $h(s, j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \bmod m$
- Problem of secondary clustering  
No local clustering anymore, but keys with same hash value have similar probe sequence



### Uniform Probing:

- Motivation: So far uses function  $g(s, j)$  only the step counter  $j$  for linear and squared probing  
⇒ The probe sequence is independent of the key  $s$
- Uniform probing computes the sequence  $g(s, j)$  of permutations of all possible indices in dependency on key  $s$
- **Advantage:** Prevents clustering because different keys with the same hash value do not produce the same probe sequence
- **Disadvantage:** Hard to implement

### Double Hashing:



Figure: Double hashing probe sequence

- Motivation: Consider key  $s$  in probe sequence
- Use two independent hash functions  $h_1(s), h_2(s)$
- Hash function:  $h(s, j) = (h_1(s) + j \cdot h_2(s)) \bmod m$

### Double Hashing:

- Hash function:  $h(s, j) = (h_1(s) + j \cdot h_2(s)) \bmod m$
- probe sequence:

$$h_1(s), h_1(s) + h_2(s), h_1(s) + 2 \cdot h_2(s), h_1(s) + 3 \cdot h_2(s), \dots$$

- Works well in practical use
- This method is an approximation of uniform probing

### Example:

$$h_1(s) = s \mod 7$$

$$h_2(s) = (s \mod 5) + 1$$

$$h(s, j) = h_1(s) + j \cdot h_2(s) \mod 7$$

Table: Comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on  $h_1(s) \neq h_2(s)$

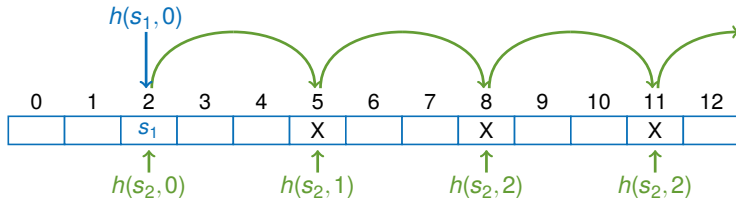


Figure: Double hashing

### Double hashing by Brent:

- Motivation:

Because different keys have different probe sequences, the sequence of the insertions has impact on efficiency of a successful search

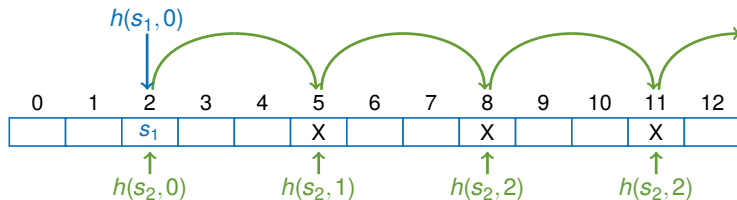


Figure: Double hashing

### Example:

- The key  $s_1$  is inserted at position  $p_1 = h(s_1, 0)$
- The hash function for  $s_2$  also results in  $p_2 = h(s_2, 0) = p_1$
- The locations  $h(s_2, j)$ ,  $j \in \{1, \dots, n\}$  are also occupied
- If we insert  $s_2$  at position  $h(s_2, n+1)$  the search will be inefficient

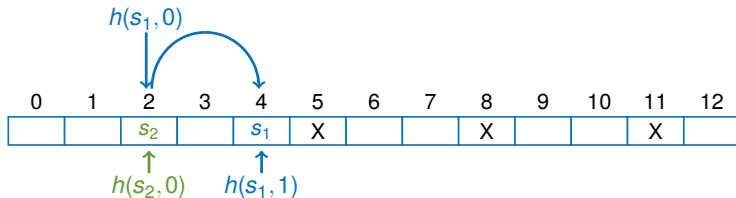


Figure: Double hashing by Brent

- Reversed sequence of keys would have been better
- **Brents Idea:**
  - Test if location  $h(s_1, 1)$  is free
  - If yes, move  $s_1$  from  $h(s_1, 0)$  to  $h(s_1, 1)$  and insert  $s_2$  at  $h(s_2, 0)$

### Idea:

- Motivation: Colliding elements are inserted in the hashtable sorted.
- Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is earlier possible because single probing steps have a fixed length

### Implementation:

- Compare both keys if a collision occurs
- Insert the smaller key at  $p_1$
- Search a position based on the diversion order for the bigger key



### Example:

- The key 12 is saved at position  $p_1 = h(12, 0)$
- We insert the key 5 into the hash map
- We assume  $h(5, 0)$  results in location  $p_1$
- Because  $5 < 12$  we insert the key 5 at position  $p_1$
- For the key 12 we iterate through the sequence

$h(12, 1), h(12, 2), h(12, 3), \dots$

### Motivation:

- Having similar length of probe sequences for all elements.  
Total costs stay the same, but they are distributed evenly.  
Results in approximately similar search times for all elements

### Implementation:

- If two keys  $s_1, s_2$  collide ( $p_1 = h(s_1, j_1) = h(s_2, j_2)$ ) we compare the length of the sequence ( $j_1$  or  $j_2$ )
- The key with the bigger search sequence is inserted at  $p_1$   
The other key is assigned a new location based on the sequence

### Example:

- The key 12 is saved at position  $p_1 = h(12, 7)$
- We insert the key 5 into the hash map
- We assume  $h(5, 0)$  results in location  $p_1$
- Because  $j_1 < j_2$  ( $0 < 7$ ) the key 12 stays at position  $p_1$
- For the key 5 we iterate through the sequence

$$h(5, 1), h(5, 2), h(5, 3), \dots$$

### Problem:

- The key  $s_1$  is inserted at position  $p_1$
- The key  $s_2$  returns the same hash value, but is inserted at position  $p_2$  because of the probing order
- If  $s_1$  is removed, it is impossible to find  $s_2$

### Solution:

- **Remove:** Elements are marked as removed, but not deleted
- **Inserting:** Elements marked as removed will be overwritten

### **Bucket as linked list:** (dynamic, number of elements variable)

- Save colliding elements as linked list

### **Open hashing:** (static, number of elements fixed)

- Determine a probe sequence, permutation of all hash values
- Linear, quadratic probing:
  - Easy to implement
  - Raise the probability of collisions because probing order does not depend on the key

### **Open hashing:** (static, number of elements fixed)

- Uniform probing, double hashing:
  - Different probing orders for different keys
  - Avoids clustering of elements

### **Improving efficiency:** (Brent, Ordered Hashing)

- Improve search efficiency by sorting colliding insertions
  - Abortion of unsuccessful search
  - Search sequence length balancing

### Hashing:

- Efficient for dictionary operations:
  - Insert:  $O(1) \dots O(n)$
  - Search:  $O(1) \dots O(n)$
  - Remove:  $O(1) \dots O(n)$
- Direct access of all elements in a hash table
- Using a hash function to find the position (hash value) in the hash table
- Hash function, size of the hash table and strategy to avoid hash collisions influence the efficiency of the datastructure

### Definition:

- A priority queue saves a set of elements
- Each element contains a key and a value like a map
- There is a total order (like  $\leq$ ) defined on the keys



### Definition:

- The priority queue supports the following operations:

`insert(key, value)`: Inserts a new element into the queue

`getMin()`: Returns the element with the smallest key

`deleteMin()`: Removes the element with the smallest key

- Sometimes additional operations are defined:

`changeKey(item, key)`: Changes the key of the element

`remove(item)`: Removes the element from the queue

### Special features:

- Multiple elements with the same key
  - No problem and for many applications necessary
  - If there is more than one element with the smallest key
    - `getMin()`: Returns just one of the possible elements
    - `deleteMin()`: Deletes the element returned by `getMin`
- Argument of `changeKey` and `remove` operations
  - There is no **quick-access** to a element in the queue
  - That's why `insert` and `getMin` return a reference (handle, accessor object)
  - `changeKey` and `remove` take this reference as argument
  - Therefore each element has to store its current position in the heap.

```
from queue import PriorityQueue

q = PriorityQueue()

e1 = (5, "A") # element with priority 5
q.put(e1); # insert element e1

# remove and return the lowest item
e2 = q.get()
```

### Example 1:

- Calculation of the sorted union of  $k$  sorted lists  
(multi-way merge or  $k$ -way merge)

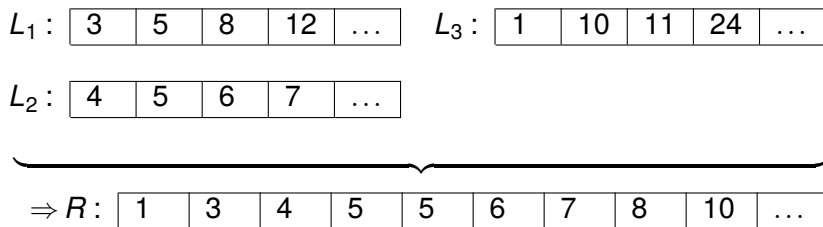


Figure: 3-way merge

### Example 1:

- Calculation of the sorted union of  $k$  sorted lists (multi-way merge or  $k$ -way merge)
- Runtime:  $N$  = length of resulting list
  - Trivial:  $\Theta(N \cdot k)$ , minimum calculation  $\Theta(k)$
  - Priority queue:  $\Theta(N \cdot \log k)$ , minimum calculation  $\Theta(\log k)$

### Example 2:

- For example Dijkstra's algorithm for computing the shortest path ( $\leftarrow$  following lecture)
- Among other applications it can be used for sorting

### Idea:

- Save elements as tuples in a binary heap
- Summary from lecture 1 (*HeapSort*):
  - Nearly complete binary tree
  - **Heap condition:**  
The key of each node  $\leq$  the keys of the children



Figure: Heap with 11 nodes



Figure: Min heap stored in array

### Storing a binary heap:

- Number nodes from top to bottom and left to right starting with 0 and store entries in array
- Children of node  $i$  are the nodes  $2i+1$  and  $2i+2$
- Parent node of node  $i$  is  $\text{floor}((i-1)/2)$

**Inserting an element:** `insert(key, item)`



- Append the element at the end of the array
- The **heap condition** may be violated, but only at the last index
- Repair **heap condition**  $\Rightarrow$  We will see later how to do this

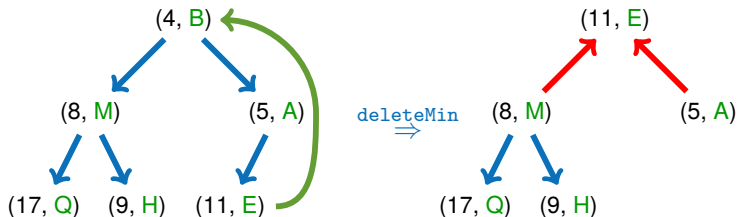


Returning the minimum: `getMin()`



- Else return the first element
- If the heap is empty return `None`

### Removing the minimum: `deleteMin()`



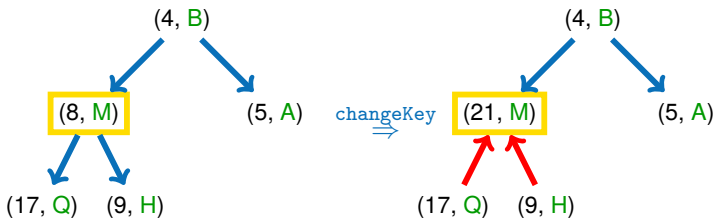
- Deleting the element with the lowest key
- Swap the last element with the first element and shrink the heap by one
- The **heap condition** may be violated, but only at the first index
- Repair **heap condition**

### Changing the key (priority): `changeKey(item, key)`

- The element (queue item) is given as argument
- Replace the key of the element
- The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- Repair **heap condition**

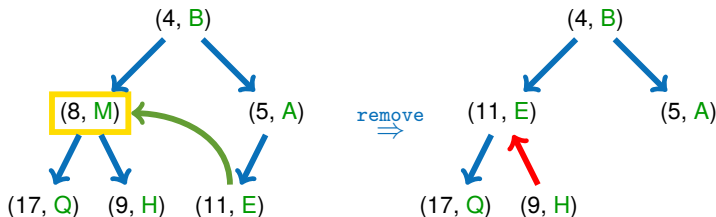


**Changing the key (priority):** `changeKey(item, key)`



- The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- Repair **heap condition**

### Removing an element: `remove(item)`



- The element (queue item) is given as argument
- Replace the element with the last element and shrink the heap by one
- The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- Repair **heap condition**

### Repairing after modifying operations:

- The heap condition can be violated after using `insert`, `deleteMin`, `changeKey`, `remove`, but only at one known position with index  $i$
- Heap conditions can be violated in two directions:
  - Downwards: The key at index  $i$  is not  $\leq$  than the value of its children
  - Upwards: The key at index  $i$  is not  $\geq$  than the value of its parent
- We need two repair methods: `repairHeapUp`, `repairHeapDown`

### repairHeapDown:

- Sift the element until the **heap condition** is valid
  - Change node with child, which has the lower key of both children
  - If the **heap condition** is violated repeat for the child node

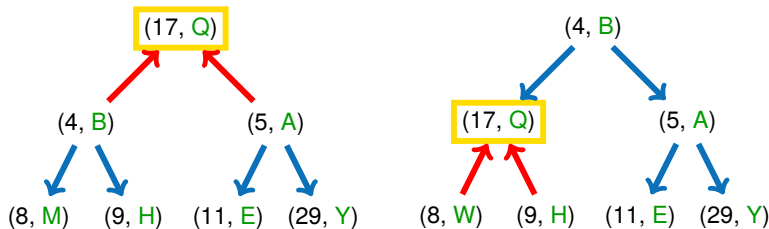


Figure: Repairing the heap downwards

### repairHeapDown:

- Sift the element until the **heap condition** is valid
  - Change node with child, which has the lower key of both children
  - If the **heap condition** is violated repeat for the child node

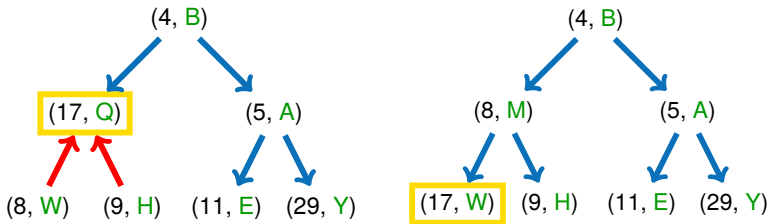


Figure: Repairing the heap downwards



### repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node

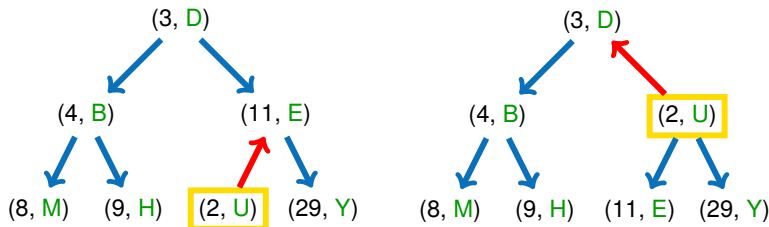


Figure: Repairing the heap upwards

### repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node

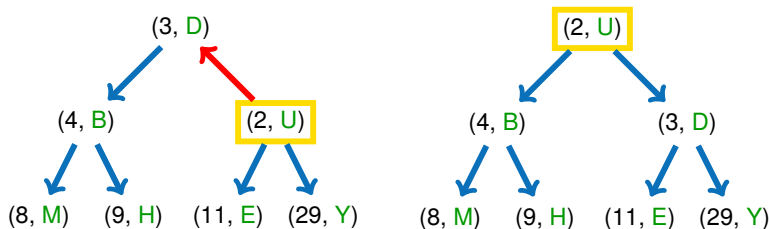


Figure: Repairing the heap upwards

### Index of a priority queue item:

- **Attention:** For `changeKey` and `remove` the item has to “know” where it is located in the heap
- Remember for `repairHeapUp` and `repairHeapDown`:  
Update the index if moving an heap element

```
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```

### Summary lecture 1:

- A full binary tree with  $n$  elements, has a **depth** of  $O(\log n)$
- The maximum distance from the root to a leaf can be  $O(\log n)$  elements
- Repairing the heap upwards and downwards:  
We have only one path to traverse:  $O(\log n)$

### Runtime for methods

- **insert**, **deleteMin**, **changeKey**, **remove**:  
We have to repair the heap:  $O(\log n)$
- **getMin**: Return the element at index 0:  $O(1)$

### Improvements (Fibonacci heaps):

- `getMin`, `insert` and `decreaseKey` in amortized time of  $O(1)$
- `deleteMin` in amortized time  $O(\log n)$

### Practical experience:

- The binary heap is simpler: Costs for managing the structure are low
- If the number of elements is relatively small so the difference is negligible
- Example:
  - For  $n = 2^{10} \approx 1,000$  is the the `depth`  $\log_2 n$  only 10
  - For  $n = 2^{20} \approx 1,000,000$  is the `depth`  $\log_2 n$  only 20

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Priority Queue - Implementations / API

[Cpp] [C++ - priority\\_queue](#)

`http:`

`//www.sgi.com/tech/stl/priority_queue.html`

[Jav] [Java - PriorityQueue](#)

`https://docs.oracle.com/javase/7/docs/api/  
java/util/PriorityQueue.html`

[Pyt] [Python - PriorityQueue](#)

`https://docs.python.org/3/library/queue.  
html#queue.PriorityQueue`