# Algorithms and Datastructures

## Runtime Complexity, Associative Arrays

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# Structure

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# Runtime Complexity

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- The runtime does not entirely depend on the size of the problem, but also on the type of input
- This results in:
  - Best runtime:
    Lowest possible runtime complexity for an input of size $n$
  - Worst runtime:
    Highest possible runtime complexity for an input of size $n$
  - Average / Expected runtime:
    The average of all runtime complexities for an input of size $n$

# Runtime Complexity

## Example 1 - Conditions

- Input: Field $a$ with $n$ elements
  $a[i] \in \mathbb{N},\ 0 \leq a[i] \leq n,\ 0 \leq i < n$
- Output: Field $a$ with $n$ elements $a[0] \neq 1$

```
if  a[0]  == 0:
    a[0] = 2
else:
    for i in range(0, n):
        a[i] = 2
```

$$\left.\begin{array}{l} \left.\begin{array}{l} \mathcal{O}(1) \\ \hline \mathcal{O}(1) \end{array}\right\} \mathcal{O}(1) \\ \left.\begin{array}{l} \mathcal{O}(n) \\ \hline \mathcal{O}(1) \end{array}\right\} \begin{array}{l} \mathcal{O}(n) \cdot \mathcal{O}(1) \\ = \mathcal{O}(n) \end{array} \end{array}\right\} \mathcal{O}(?)$$

- Best runtime: $\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1)$
- Worst runtime: $\mathcal{O}(1) + \mathcal{O}(n) = \mathcal{O}(n)$

# Runtime Complexity

## Example 1 - Average Runtime

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- The average runtime is determined by the average runtime for all input instances of size $n$
- Every element of $a$ can have $n$ different values
  $\Rightarrow n \cdot \ldots \cdot n = n^n$ different input instances of size $n$
  - a[ i ] == 1 in $n^{n-1}$ instances
  - a[ i ] != 1 in $n^n - n^{n-1} = n^{n-1} \cdot (n-1)$ instances
- Sum of all runtime complexities:

$$\underbrace{n^{n-1} \cdot \mathcal{O}(1)}_{\text{a[ i ] == 1}} + \underbrace{n^{n-1} \cdot (n-1) \cdot \mathcal{O}(n)}_{\text{a[ i ] != 1}}$$

- Average runtime:

$$\frac{n^{n-1} + n^{n-1} \cdot (n-1) \cdot n}{n^n} = \frac{1}{n} + n - 1 \in \mathcal{O}(n)$$

# Runtime Complexity

## Example 2 - Binary Addition

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- Input: $n$ digit dual number $a$
- Output: $n$ digit dual number $a+1$
- Runtime of the algorithm is determined by the number of bits getting changed (steps)
  1. "0" $\rightarrow$ "1"
  2. "1" $\rightarrow$ "0"
- Best runtime: 1 step = $\mathcal{O}(1)$
- Worst runtime: $n$ steps = $\mathcal{O}(n)$

Table: Binary addition

| Digits ($n$) | Input | Output | Steps |
|---|---|---|---|
| 10 | 1011100100 | 1011100101 | 1 |
| 4 | 1011 | 1100 | 3 |
| 8 | 11111111 | 00000000 | 8 |

# Runtime Complexity

## Example 2 - Average Steps

Table: Binary addition with $n = 1$

| Input | Output | Steps |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 1 |

Table: Binary addition with $n = 2$

| Input | Output | Steps |
|---|---|---|
| 00 | 01 | 1 |
| 01 | 10 | 2 |
| 10 | 11 | 1 |
| 11 | 00 | 2 |

$$\overline{\text{steps}} = \frac{1+1}{2} = 1$$

$$= 2 - \frac{1}{1} = 2 - \frac{1}{2^{n-1}}$$

$$\overline{\text{steps}} = \frac{1+2+1+2}{4} = \frac{3}{2}$$

$$= 2 - \frac{1}{2} = 2 - \frac{1}{2^{n-1}}$$



Table: Binary addition with $n = 3$

| Input | Output | Steps |
|---|---|---|
| 000 | 001 | 1 |
| 001 | 010 | 2 |
| 010 | 011 | 1 |
| 011 | 100 | 3 |
| 100 | 101 | 1 |
| 101 | 110 | 2 |
| 110 | 111 | 1 |
| 111 | 000 | 3 |

$$\overline{\text{steps}} = \frac{1+2+1+3+1+2+1+3}{8} = \frac{7}{4}$$

$$= 2 - \frac{1}{4} = 2 - \frac{1}{2^{n-1}}$$

$\Rightarrow$ Average runtime:

$$2 - \frac{1}{2^{n-1}} \in \mathcal{O}(1)$$



Table: Case analysis for instances of size $n$

| Input | Output | Instances | Steps |
|---|---|---|---|
| $\_\_\_\ldots\_\_\_0$ | $\_\_\_\ldots\_\_\_1$ | $2^{n-1}$ | 1 |
| $\_\_\_\ldots\_\_01$ | $\_\_\_\ldots\_\_10$ | $2^{n-2}$ | 2 |
| $\_\_\_\ldots\_011$ | $\_\_\_\ldots\_100$ | $2^{n-3}$ | 3 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\_01\ldots1111$ | $\_10\ldots0000$ | $2^1$ | n-1 |
| $011\ldots1111$ | $100\ldots0000$ | $2^0$ | n |
| $111\ldots1111$ | $000\ldots0000$ | 1 | n |

Average steps:

$$\frac{1\cdot 2^{n-1}+2\cdot 2^{n-2}+\cdots+(n-1)\cdot 2^1+n\cdot 2^0+n\cdot 1}{2^{n-1}+2^{n-2}+\cdots+2^1+2^0+1}=\frac{\sum_{i=1}^{n}\left(i\cdot 2^{n-i}\right)+n}{\sum_{i=0}^{n-1}2^i+1}$$



- Denominator:

$$\sum_{i=0}^{n-1} 2^i + 1 \overset{\text{geometric series}}{=} 2^n - 1 + 1 = 2^n$$

- Counter:

$$\sum_{i=1}^{n} \left( i \cdot 2^{n-i} \right) + n \overset{a=2a-a}{=} 2 \sum_{i=1}^{n} \left( i \cdot 2^{n-i} \right) - \sum_{i=1}^{n} \left( i \cdot 2^{n-i} \right) + n$$

$$= 1 \cdot 2^n + 2 \cdot 2^{n-1} + 3 \cdot n^{n-2} + \cdots + (n-1) \cdot 2^2 + n \cdot 2^1$$

$$- 1 \cdot 2^{n-1} - 2 \cdot 2^{n-2} - \cdots - (n-2) \cdot 2^2 - (n-1) \cdot 2^1 - n \cdot 2^0 + n$$

$$= \underbrace{2^n + 2^{n-1} + \cdots + 2^1 + 2^0}_{2^{n+1}-1} - 1 = 2^{n+1} - 2$$



Average steps:

$$\overline{steps} = \frac{\sum_{i=1}^{n} \left( i \cdot 2^{n-i} \right) + n}{\sum_{i=0}^{n-1} 2^i + 1} = \frac{2^{n+1} - 2}{2^n} = 2 - \frac{1}{2^{n-1}}$$

$$\lim_{n \to \infty} \left( 2 - \frac{1}{2^{n-1}} \right) = 2 \in \mathcal{O}(1)$$



**Normal array:**

$$A = [0 \Rightarrow A_0,\ 1 \Rightarrow A_1,\ 2 \Rightarrow A_2,\ 3 \Rightarrow A_3,\ \ldots]$$

- Searching elements by index
- Lookup of element with index ”3”:
  $\Rightarrow A[3] = A_3$



**Associative array:**

$$A = \begin{bmatrix} \text{"}Europa\text{"} \Rightarrow A_0, \text{"}Amerika\text{"} \Rightarrow A_1, \\ \text{"}Asien\text{"} \Rightarrow A_2, \text{"}Afrika\text{"} \Rightarrow A_3, \\ \dots \end{bmatrix}$$

- Searching elements by key
- The keys can be of any type with unique elements
- Lookup of element with key "*Afrika*":
  $\Rightarrow A[\text{"}Afrika\text{"}] = A_3$

# Associative Arrays

## Practical Example

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Table: Country data query from `http://geonames.org`

| ISO | ISO3 | Country | Continent | ... |
|---|---|---|---|---|
| AD | AND | Andorra | EU | ... |
| AE | ARE | United Arab Emirates | AS | ... |
| AF | AFG | Afghanistan | AS | ... |
| AG | ATG | Antigua and Barbuda | NA | ... |
| AI | AIA | Anguilla | NA | ... |
| AL | ALB | Albania | EU | ... |
| AM | ARM | Armenia | AS | ... |
| AO | AGO | Angola | AF | ... |
| AQ | ATA | Antarctica | AN | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |



**Task:** How many countries belong to each continent?

- We are interested in column 2 (country) and 3 (continent)
- There are two typical ways to solve this:
  - Using Sorting
  - Using an associative array



**Idea using sorting:**

- We sort the columns 2 and 3 by continent, so that all countries with the same continent are grouped in one block
- We count the size of the blocks

  **Disadvantage:**

  - Runtime of $\Theta(n \log n)$
  - We have to iterate the array twice (sort and then count)

  **Advantage:**

  - Easy to implement (even with simple linux / unix commands)



**Input:**

- The data is saved as tab seperated text (countryInfo.txt)
- Comments begin with a hash sign #

**Commands:**

- **grep**: Selects a specific set of lines (filter by ...)

  `grep -v '^#' countryInfo.txt`

  `-v`: not

  `^#`: # at start of line

- **cut**: Selects specific columns of each line (tab separated)

  `cut -f5,9`

  `-f5,9`: columns 5 and 9 (columns 2, 3 of Table 6)

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**Commands:**

- **sort**: Sorts lines by a key

  `sort -t '	' -k2,2`

  `-t '	'`: Separator: Tab (Insert with `CTRL-V TAB`)

  `-k2,2`: Key from column 2 to 2

- **uniq**: Finds or counts unique keys

  `uniq -c`

  `-c`: count occurences of keys

- **head**: Displays a provided number of lines

  `head -n30`

  `-n30`: Displays the first 30 lines

- **less**: Displays the file page wise



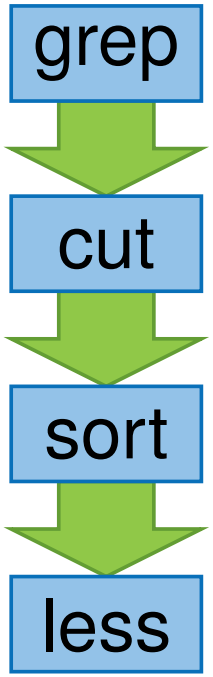
**Sort countries by continent:**

```
grep -v '^#' countryInfo.txt | cut -f5,9 \
   | sort -t '   ' -k2,2 | less
```

Table: Resulting data

| | |
|---|---|
| Algeria | AF |
| Angola | AF |
| Benin | AF |
| Botswana | AF |
| Burkina Faso | AF |
| Burundi | AF |
| Cameroon | AF |
| Cape Verde | AF |
| ⋮ | ⋮ |

Figure: Data pipeline





**Count countries per continent:**

```
grep -v '^#' countryInfo.txt | cut -f9 \
   | sort | uniq -c | sort -nr
```

Table: Resulting data

| | |
|---|---|
| 58 | AF |
| 54 | EU |
| 52 | AS |
| 42 | NA |
| 27 | OC |
| 14 | SA |
| 5 | AN |

Figure: Data pipeline



**Idea using associative arrays:**

- Take the continent as key
- Use a counter (occurences) or a list with all countries associated with this continent as value

**Advantage:**

- Runtime $\mathcal{O}(n)$, implied we can find an element in $\mathcal{O}(1)$ as in normal arrays



**Python:**

```
# creates a new map (called dictionary)
countries = {"DE" : "Deutschland", \
     "EN" : "England"}

# check if element exists
if "EN" in countries:
     print("Found %s!" % countries["EN"])

# map key "DE" to value 0
countries["DE"] = "Germany"

# delete key "DE"
del countries["DE"]
```



**Efficiency:**

- Depends on implementation
- Two typical implementations:
  - **Hashing:** Calculates a checksum of the key used as key of a normal array
    `search`: $\mathcal{O}(1) \ldots \mathcal{O}(n)$
    `insert`: $\mathcal{O}(1) \ldots \mathcal{O}(n)$
    `delete`: $\mathcal{O}(1) \ldots \mathcal{O}(n)$
  - **(Binary-)Tree:** Creates a sorted (binary) tree
    `search`: $\mathcal{O}(\log n) \ldots \mathcal{O}(n)$
    `insert`: $\mathcal{O}(\log n) \ldots \mathcal{O}(n)$
    `delete`: $\mathcal{O}(\log n) \ldots \mathcal{O}(n)$

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Table: Map implementions of programming languages

| | Hashing | (Binary-)Tree |
|---|---|---|
| Python | all dictionaries | |
| Java | `java.util.HashMap` | `java.util.TreeMap` |
| C++11/14 | `std::unordered_map` | `std::map` |
| C++98 | `__gnu_cxx::hash_map` | `std::map` |

# Further Literature

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- **General**


[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
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MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

# Further Literature

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- **Map - Implementations / API**

[Java] Java - HashMap
`https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html`

[Javb] Java - TreeMap
`https://docs.oracle.com/javase/7/docs/api/java/util/TreeMap.html`

[Pyt] Python - Dictionaries (Hash table)
`https://docs.python.org/3/tutorial/datastructures.html#dictionaries`

# Further Literature

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- **Map - Implementations / API**

[Cppa] C++ - hash_map
`http://www.sgi.com/tech/stl/hash_map.html`

[Cppb] C++ - map
`http://www.sgi.com/tech/stl/Map.html`