Algorithms and Datastructures

Shortest Path, Dijkstra Algorithm

Albert-Ludwigs-Universität Freiburg

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Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, March 2018

Structure

Structure

Graphs

Dijkstra Algorithm

For a graph $G = (V, E)$:

- A path of G is a sequence of edges $u_1, u_2, \dots, u_i \in V$ with
  - Undirected graph: $\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{i-1}, u_i\} \in E$
  - Directed graph: $(u_1, u_2), (u_2, u_3), \dots, (u_{i-1}, u_i) \in E$
- The length of a path is
  - Without weights: number of edges taken
  - With weights: sum of weigths of edges taken

For a graph $G = (V, E)$:

- The shortest path between two vertices u, v is the path $P = (u, \ldots, v)$ with the shortest length $d(u, v)$ or lowest costs
- The diameter of a graph is the longest shortest path

Structure

Graphs

Dijkstra Algorithm

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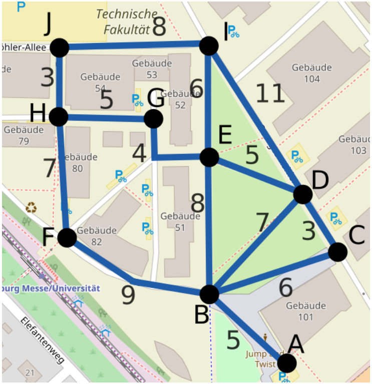
- Wanted: Shortest path from M to all other points
- Place pearls on crossings and clamp strings between them



Dijkstra Algorithm

Shortest Path without Computer

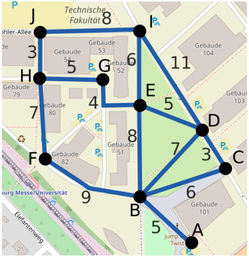
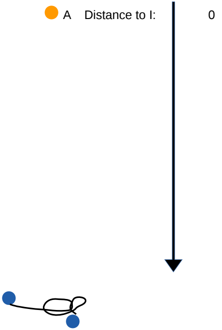
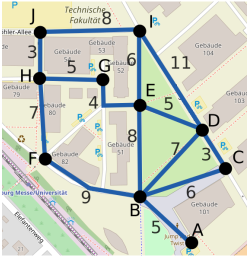


Figure: Based on OpenStreetMaps; CC BY-SA 2.0

- Take the net and pull it slowly upwards until fully lifted



- Each node (pearl) now has a specific height
- The distance to M is exactly the shortest path



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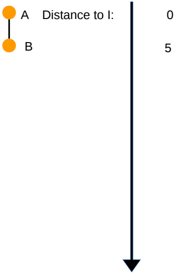

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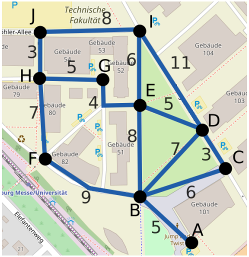
Dijkstra Algorithm

Shortest Path without Computer



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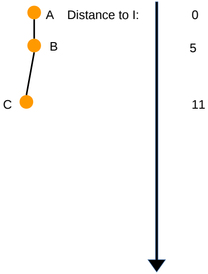

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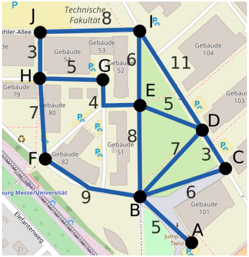

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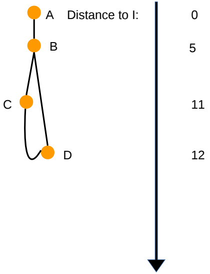

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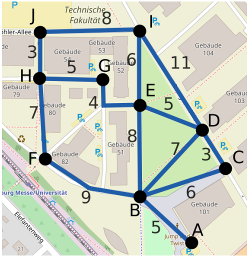

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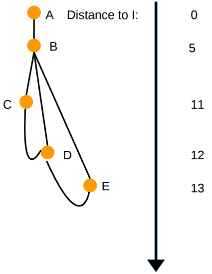

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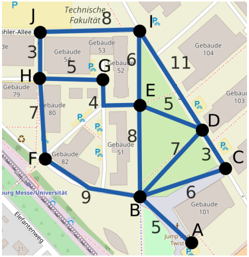

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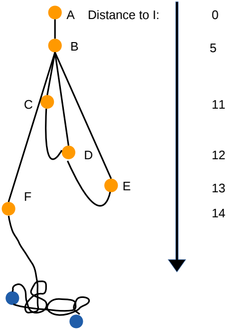

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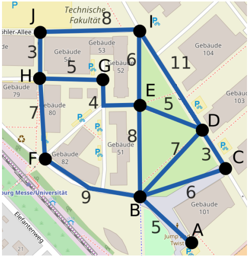
Dijkstra Algorithm

Shortest Path without Computer



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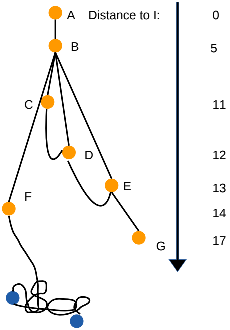

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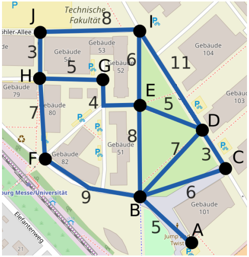

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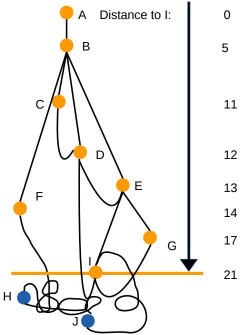

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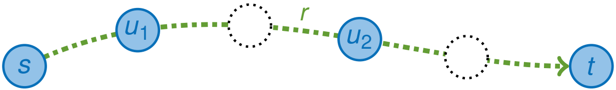


Figure: Shortest path from s to t

- Let r be the shortest path from s to t
- For each node u on path r the path from u to t is the shortest path

Proof:

- If there was a shorter path from s to u then we could choose this path to get faster to t
- Then r would not be the shortest path

Dijkstra Algorithm
Shortest Path

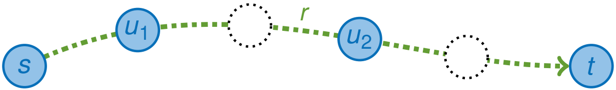


Figure: Shortest path from s to t

- This is also correct for all sub paths on r
- If the shortest path from s to t passes u_1 and u_2 then the sub path (u_1, u_2) is the shortest path from u_1 to u_2

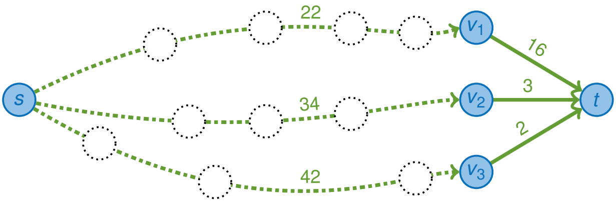


Figure: Shortest paths from s to t

- If we know the shortest path form s to the preceding nodes of t (v_1, v_2, v_3) we can determine the shortest path to t

Idea:

- Attach the cost of the shortest path to each node
- Let the information travel over the edges (message passing)
- In which order should we process the nodes?

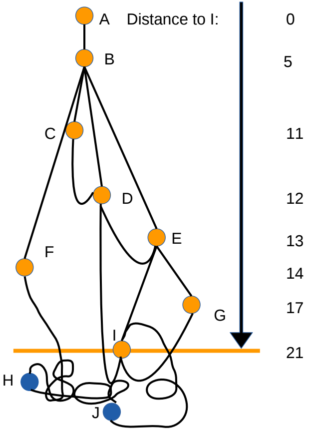
Inventor:

- Edsger Dijkstra (1930 - 2002)
- Computer scientist from Netherlands
- Won Turing-Award as one of few Europeans for his studies of structured programming
- Invented the Dijkstra-Algorithm in 1959

Figure: Portrait © Hamilton Richards - manuscripts of Edsger W. Dijkstra, University Texas at Austin

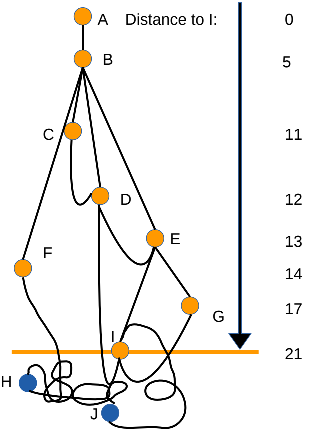
Example:

- Lift pearl *A* a little bit
- Connection to pearl *B* is hanging in the air
- Lift further until pearl *B* starts to lift at 5 m
- The shortest path to *B* is now known
- Lift further: The wires from *C*, *D*, *E* and *F* are now in the air
- One of the pearls *C*, *D*, *E* or *F* is the next one
  Which one?



Example:

- At 11 m pearl *C* gets lifted
- The wire to *D* is now in the air
- One of the pearls *D*, *E* and *F* is the next one
  Which one?
- At 12 m pearl *D* gets lifted
  ...
- How to translate this into an computer algorithm?



Dijkstra Algorithm

High level description: Three types of nodes

- Settled: For node u we know $\mathrm{dist}(s, u)$ 42
  (Pearl example: This pearl is hanging in the air)

- Active: For node u we know a tentative distance $\mathrm{td}(u) \geq \mathrm{dist}(s, u)$ (Can be optimal but doesn't have to) 37
  (Pearl example: This pearl is laying on the table but one connected wire is already in the air)

- Unreached: We have not reached the node yet
  (Pearl example: This preal is hanging in the air)

High level description:

- Each iteration take the active node u with the smallest $\mathrm{td}(u)$
  (The pearl getting lifted next)
- We update the state of the node u to settled
  (The pearl gets lifted)
- We check for each neighbor v of node u if we can reach v faster than currently possible
  (Check all outgoing wires from this pearl: Activate all connected pearls, update tentative distance if smaller)
- Iterate until no active nodes exist anymore

Dijkstra Algorithm

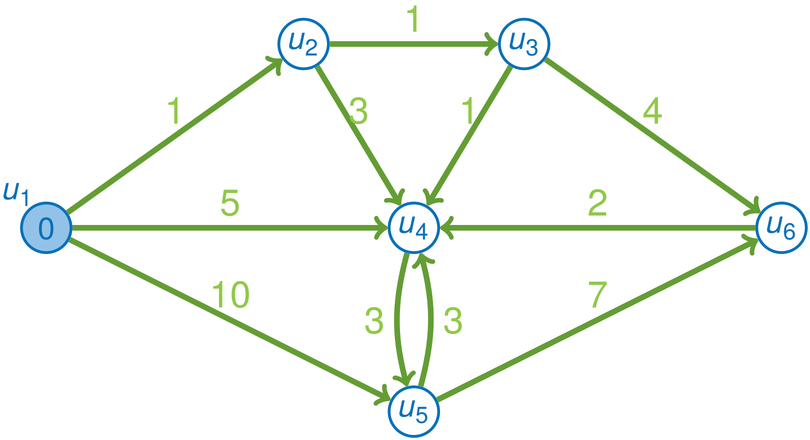


Figure: Start at u_1

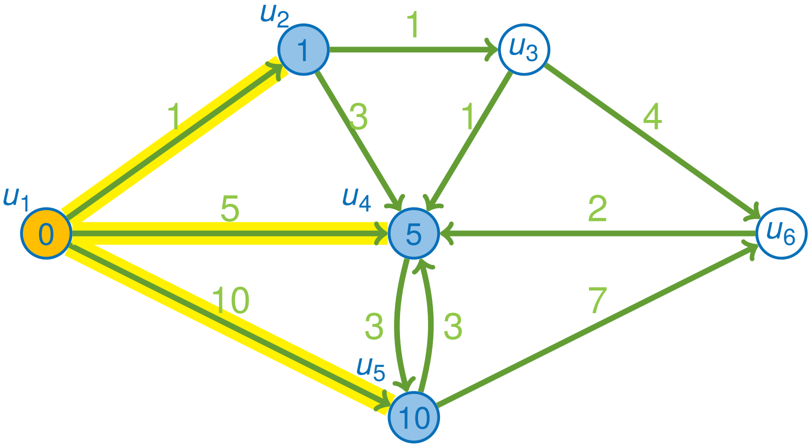


Figure: Iteration 1

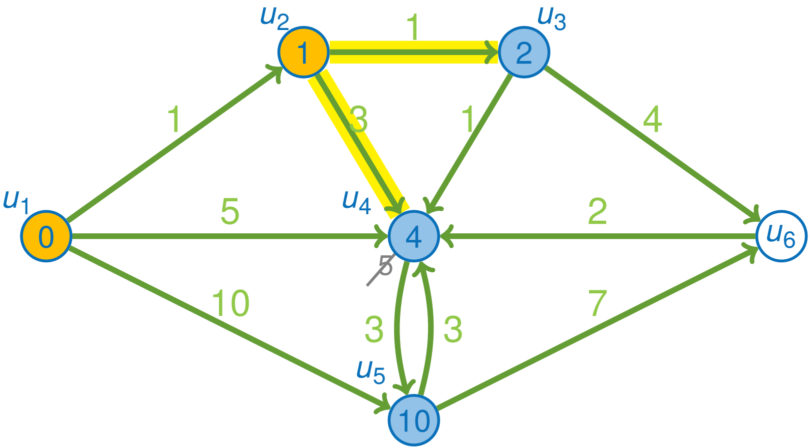


Figure: Iteration 2

Dijkstra Algorithm

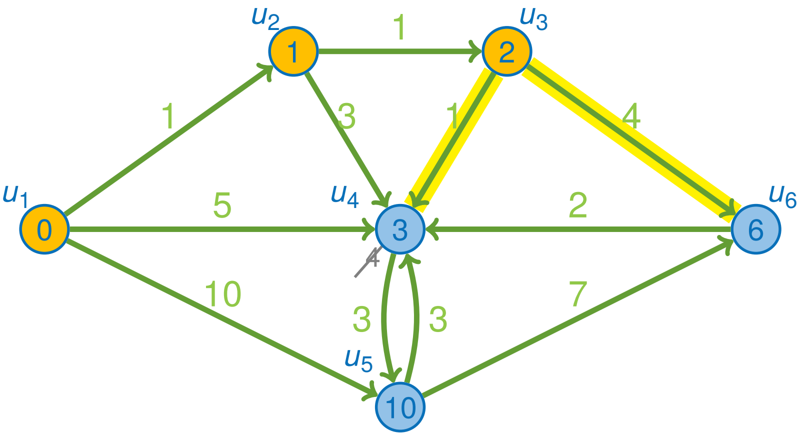


Figure: Iteration 3

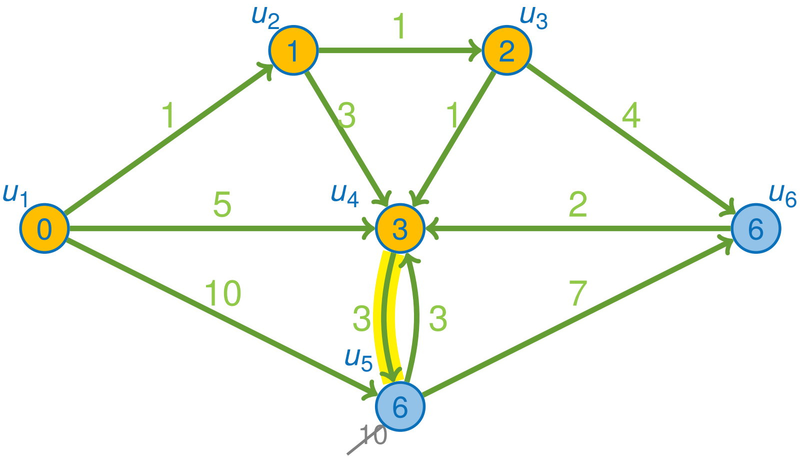


Figure: Iteration 4

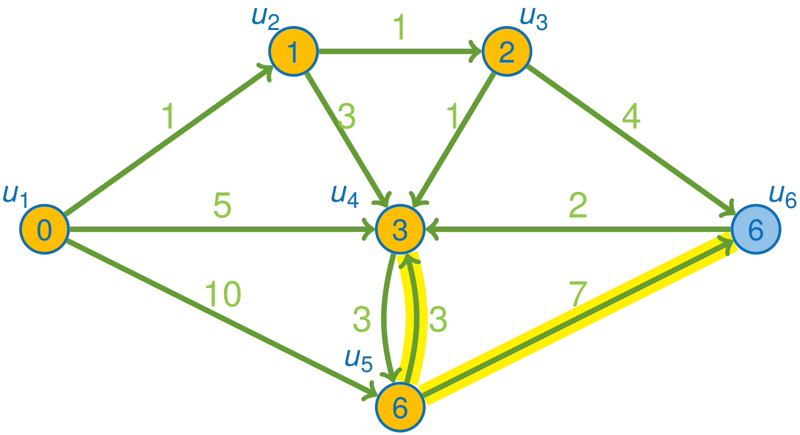


Figure: Iteration 5

Dijkstra Algorithm

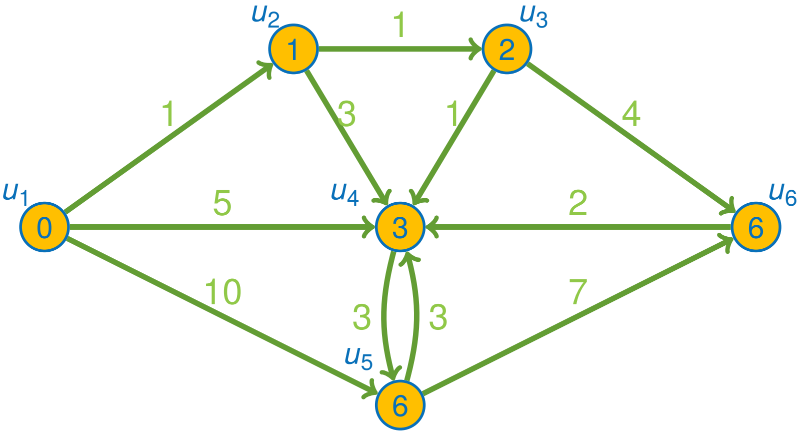


Figure: Iteration 6

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Proof:

- **Assumption 1:** All edges have a positive length
- **Assumption 2:** Each node has a unique distance $\text{dist}(s, u)$ to the start s
  (This was not the case on the previous slides)

  This results in an easy and intuitive proof.
  It is possible to show this without assumption 2. See references if interested

- With assumption 2 there exists a sorting u_1, u_2, ... with that:
  $$\text{dist}(s, u_1) < \text{dist}(s, u_2) < \text{dist}(s, u_3) < \ldots$$

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Proof:

- With **assumption 2** there exists a sorting u_1, u_2, $\dots$ with that:

$$\mathrm{dist}(s, u_1) < \mathrm{dist}(s, u_2) < \mathrm{dist}(s, u_3) < \dots$$

- We want to show that the *Dijkstra* algorithm finds the shortest path for each node u_i so that $\mathrm{td}(u_i) = \mathrm{dist}(s, u_i)$ holds
- Additionally we show that each node gets solved in order of the distance: Node u_i gets solved in iteration i

$$u_1, u_2, u_3, \dots$$

To show: Node u_i gets solved in round i

1. Node u_i contains the correct distance ($\mathrm{td}(u_i) = \mathrm{dist}(s, u_i)$) and is active
2. Node u_i has the smallest value for $\mathrm{td}(u_i)$ and gets selected by the algorithm

Induction start:

1. - Only the start node $s = u_1$ is active and $\mathrm{td}(s) = 0$
   - Node u_1 gets solved and $\mathrm{td}(u_1) = \mathrm{dist}(s, u_1) = 0$
2. Only the start node u_1 is active

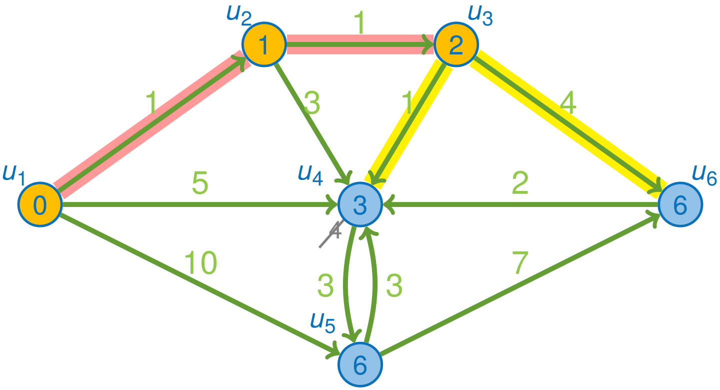
Induction step: $i = i + 1$

1. **To show:** Node u_{i+1} contains the correct distance ($\mathrm{td}(u_{i+1}) = \mathrm{dist}(s, u_{i+1})$) and is active
   - On the shortest path from s to u_{i+1} is a preceding node that:

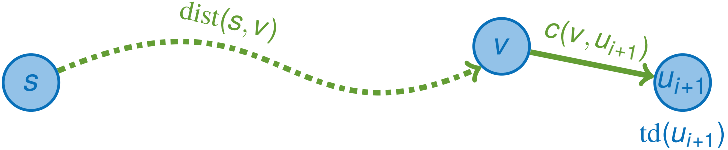
     $$\mathrm{dist}(s, u_{i+1}) = \mathrm{dist}(s, v) + \mathrm{c}(v, u_{i+1})$$

     ($\mathrm{c}(v, u_{i+1})$ are the costs of the edge)

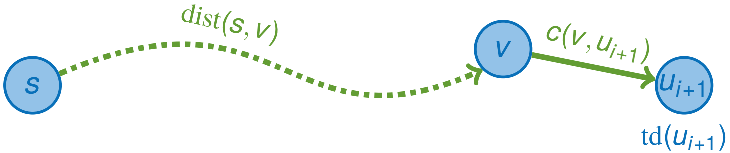
   - Hence $\mathrm{dist}(s, v) < \mathrm{dist}(s, u_{i+1})$ because $c > 0$ (c=cost of edge)
   - Because u_{i+1} is currently settled, the node v is one of the preceding nodes $u_1, \ldots, u_i$, hence $v = u_j$ with $0 \leq j \leq i$



- Preceding node of u_6 is $v = u_3$
- In round 3 $\mathrm{td}(u_6) = 2 + 4 = 6$ was already solved



1. **To show:** Node u_i contains the correct distance $\mathrm{td}(u_i) = \mathrm{dist}(s, u_i)$ and is active
   - With **induction assumption**: v already contains the correct distance which was evaluated in round j (edge from v to u_{i+1}) and is stored in $\mathrm{td}(u_{i+1})$
   - u_{i+1} is active because the preceding node was solved



2. **To show:** Node u_{i+1} has the smallest value for $\mathrm{td}(u_{i+1})$ and gets selected by the algorithm
   - All nodes with smaller dist are already solved
   - All other nodes u_k with $k > i+1$ have a greater $\mathrm{dist}(s, u_k)$ and with that the $\mathrm{td}(u_k)$ is greater or equal

   $\Rightarrow$ u_{i+1} is the node with the smallest td and gets selected by the algorithm

Implementation:

- We have to manage a set of active nodes
- We start with only the start node in our set
- At the start of each iteration we need the node u with the smallest $\mathrm{td}(u)$

  How to implement this?

Implementation:

- Using a priority queue with $\mathrm{td}(u)$ as keys
- The following problem occurs:
  - The tentative distance of an active node might change multiple times before it is settled
  - We have to change the key in our priority queue without removing the entry

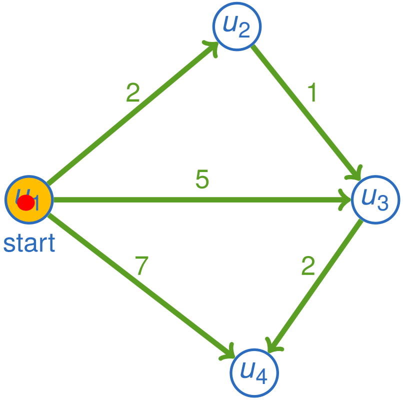
Limitations:

- Often only `insert`, `getMin` and `deleteMin` are implemented

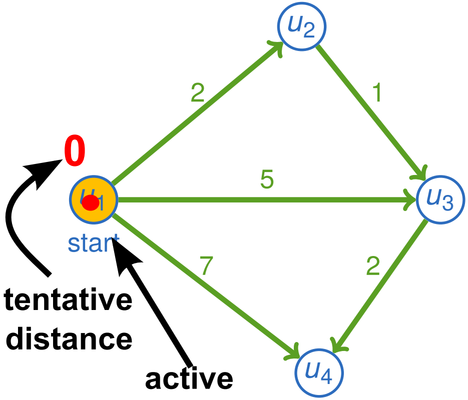
$\Rightarrow$ We only have access to the first element and not any desired one

Alternative:

- If a node reoccurs with a smaller dist we insert the element one more time into the priority queue
  (We do nothing if the distance is greater or equal)
- We do not remove the old entry
- The node always gets solved with the smallest distance because of the smaller key
- If a settled node reoccurs with a higher dist we remove it and do simply nothing



priority queue



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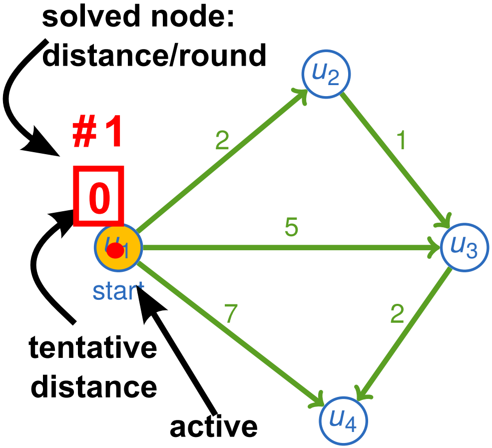

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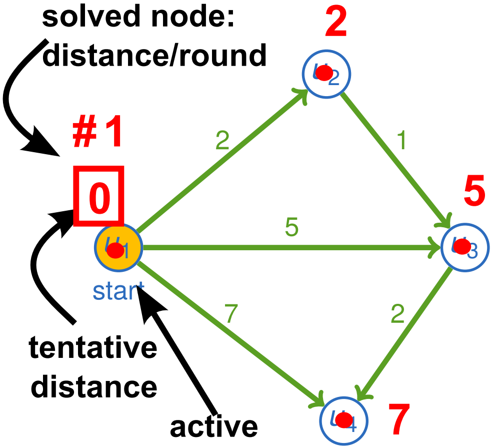
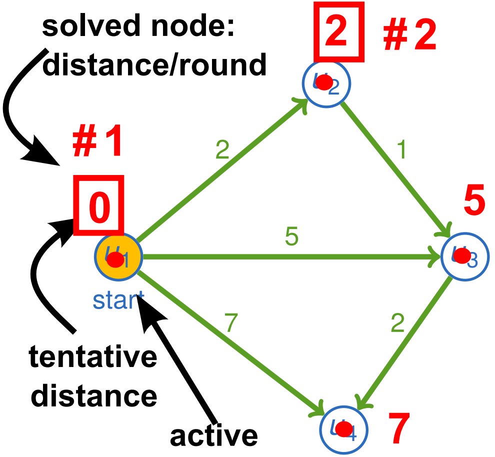


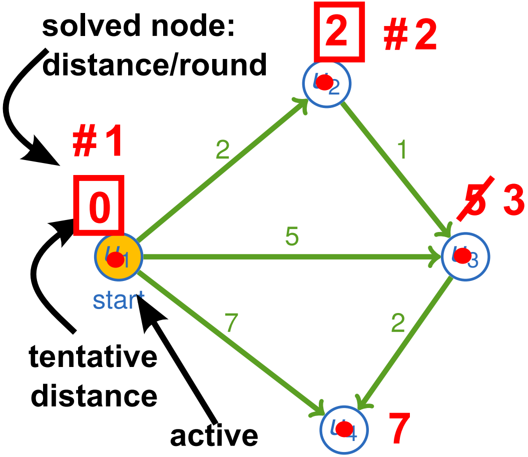
priority queue

(u2, 2)
(u3, 5)
(u4, 7)



priority queue

(u2, 2) → solved #2
(u3, 5)
(u4, 7)



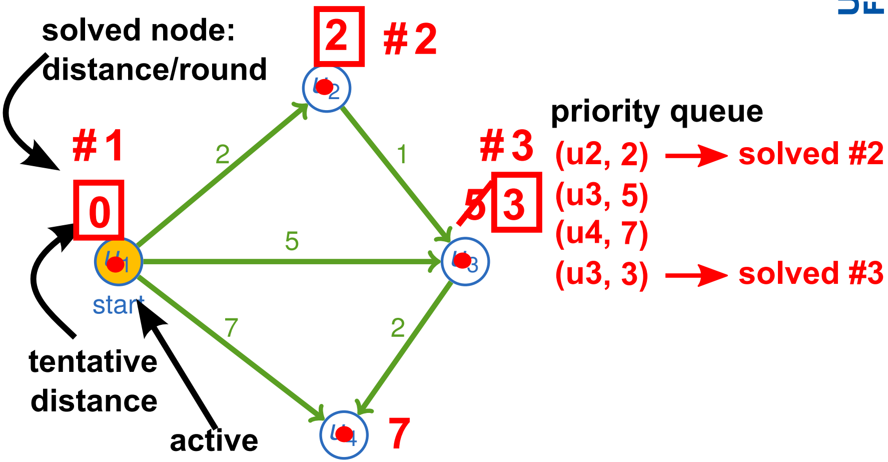
priority queue

(u2, 2) → solved #2

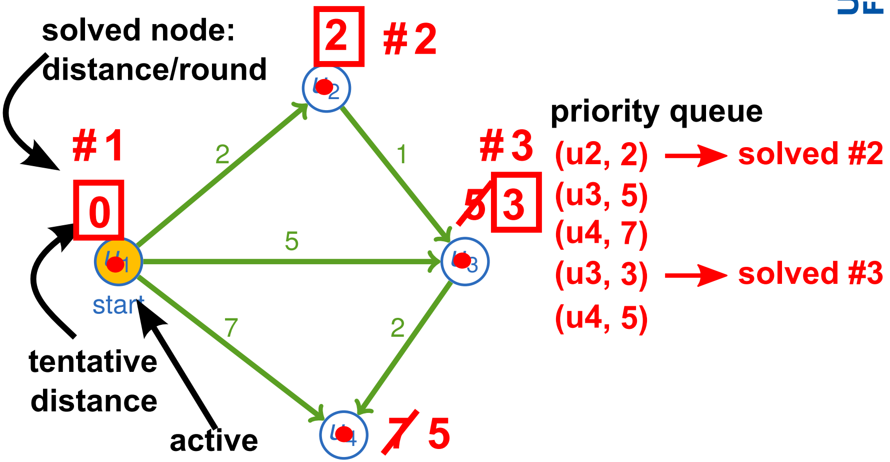
(u3, 5)

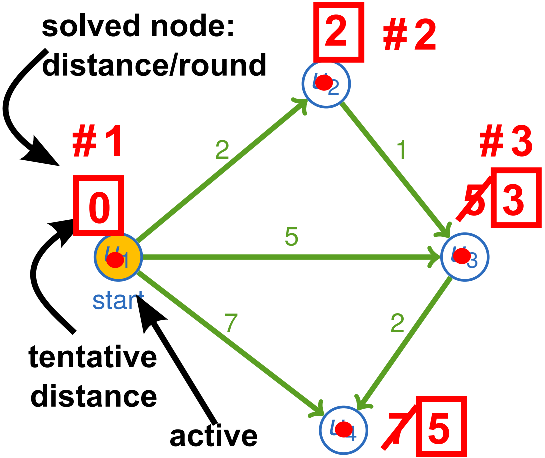
(u4, 7)

(u3, 3)



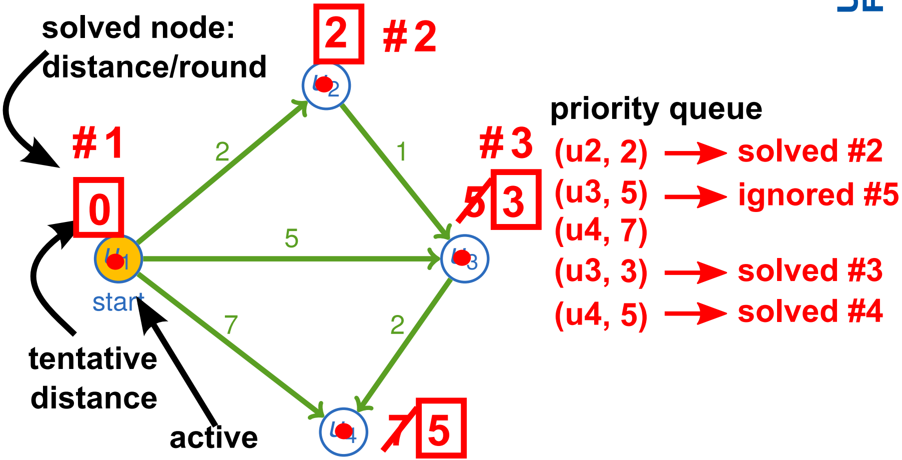
solved node:
distance/round
#1
0
#2
2
#3
~~5~~ 3
7
2
1
5
7
2
u_1
u_2
u_3
u_4
start
tentative
distance
active
priority queue
(u2, 2) → solved #2
(u3, 5)
(u4, 7)
(u3, 3) → solved #3

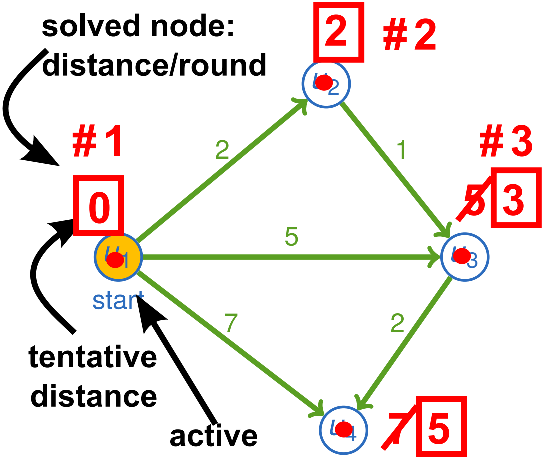

solved node:
distance/round
#1
0
#2
2
#3
~~5~~ 3
~~7~~ 5
2
1
5
7
2
u_1
u_2
u_3
u_4
start
tentative
distance
active
priority queue
(u2, 2) → solved #2
(u3, 5)
(u4, 7)
(u3, 3) → solved #3
(u4, 5)



priority queue

(u2, 2) → solved #2
(u3, 5)
(u4, 7)
(u3, 3) → solved #3
(u4, 5) → solved #4


solved node:
distance/round
#1
0
#2
2
#3
5
3
7
5
2
1
5
7
2
u_1
u_2
u_3
u_4
start
tentative
distance
active
priority queue
(u2, 2) → solved #2
(u3, 5) → ignored #5
(u4, 7)
(u3, 3) → solved #3
(u4, 5) → solved #4



priority queue

(u2, 2) ⟶ solved #2
(u3, 5) ⟶ ignored #5
(u4, 7) ⟶ ignored #6
(u3, 3) ⟶ solved #3
(u4, 5) ⟶ solved #4

Dijkstra Algorithm
Runtime analysis

Graph with n nodes and m edges: ($m \geq n$)

- Each node gets solved exactly one time
- When solving a node it's outgoing edges are taken into account
- Each edge triggers at maximum one `insert` operation
- The number of operations on the priority queue is at maximum $O(m)$
- This results in a runtime of $O(m \cdot \log m)$
  (log m because of at max. m elements in the priority queue)

Runtime of $O(m \cdot \log m)$:

- Because of $m \leq n^2$ we have a maximum runtime of $O(m \cdot \log n)$, because $\log n^2 = 2 \log n$
- With a complex priority queue the runtime can be reduced to $O(m + n \log n)$
  - For example with a **Fibonacci heap**
  - This results in a better runtime for complex graphs $m \sim n^2$
  - Complex heaps create a management overhead

  $\Rightarrow$ In practice $m \in O(n)$ with a **binary heap** being faster (See lecture 6)

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Termination criteria:

- Terminate as soon as the target node t is settled
  … never before because tentative distance might change:

$$\mathrm{td}(t) \geq \mathrm{dist}(s,t)$$

- Before the node t is solved all nodes u with
  $\mathrm{dist}(s,u) \leq \mathrm{dist}(s,t)$ are settled

Termination criteria:

- Not only the single source **single** target shortest path problem is solved by the Dijkstra algorithm but also the single source **all** targets problem
- This sounds wasteful but there is not a (much) better method for general graphs
  Intuitive: We only know that there is no shorter path if all nodes in the distance of $\mathrm{dist}(s, t)$ are evaluated

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Calculate the shortest path:

- With the current implementation of the Dijkstra algorithm we only get the length of the path
  How to get the path itself too?
- If we save the preceding node of the current shortest path on settling of each node we can reconstruct the path

Dijkstra Algorithm

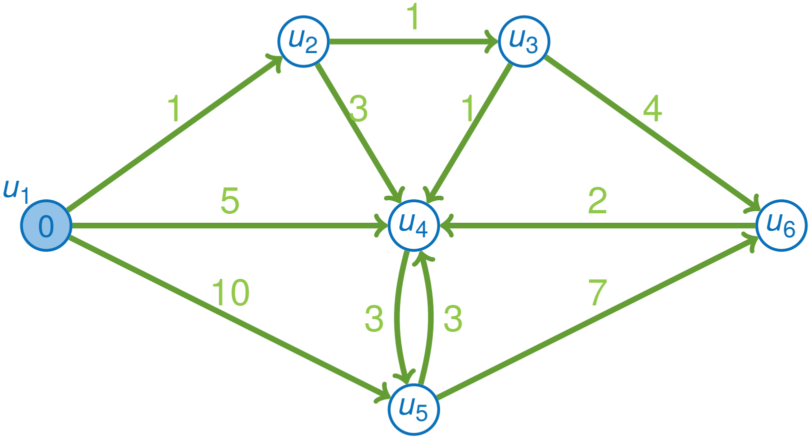


Figure: Start at u_1

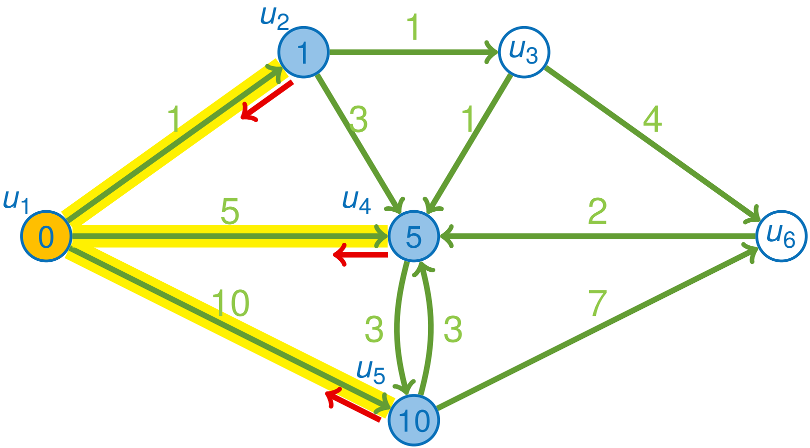


Figure: Iteration 1

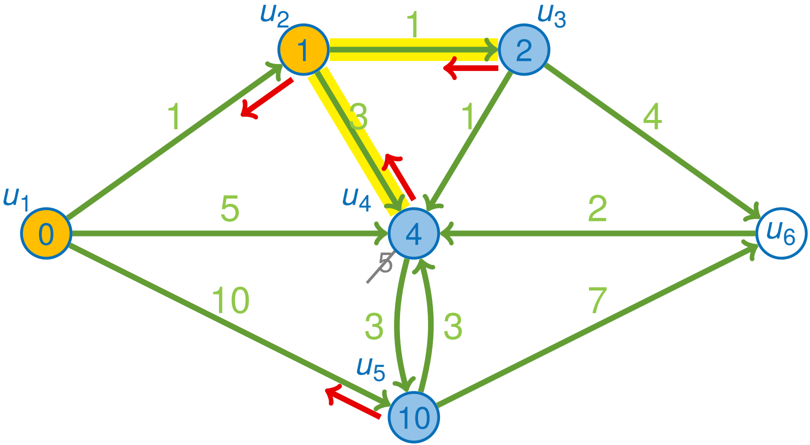


Figure: Iteration 2

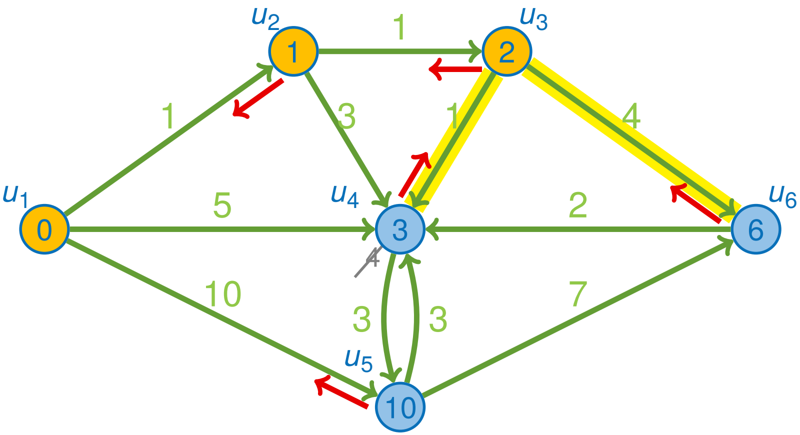


Figure: Iteration 3

Dijkstra Algorithm

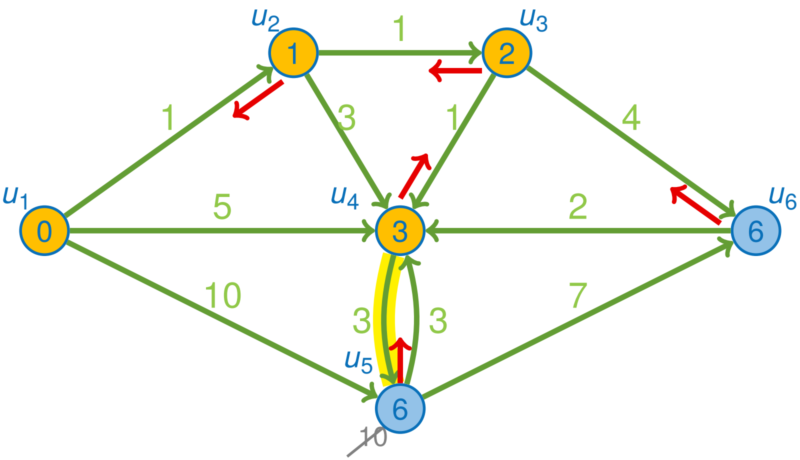


Figure: Iteration 4

Dijkstra Algorithm

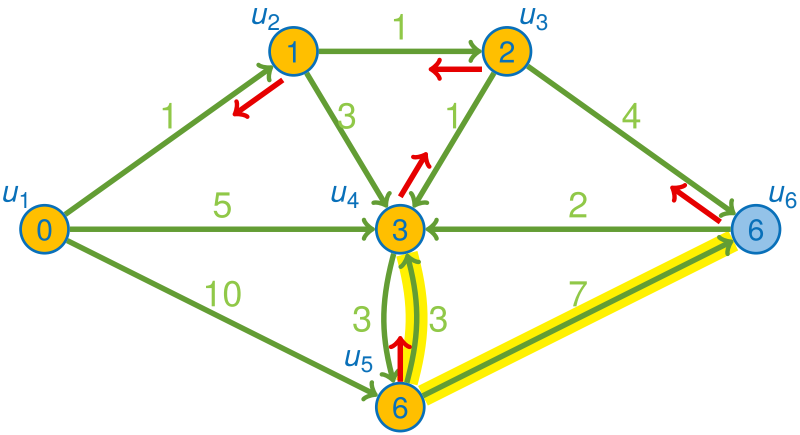
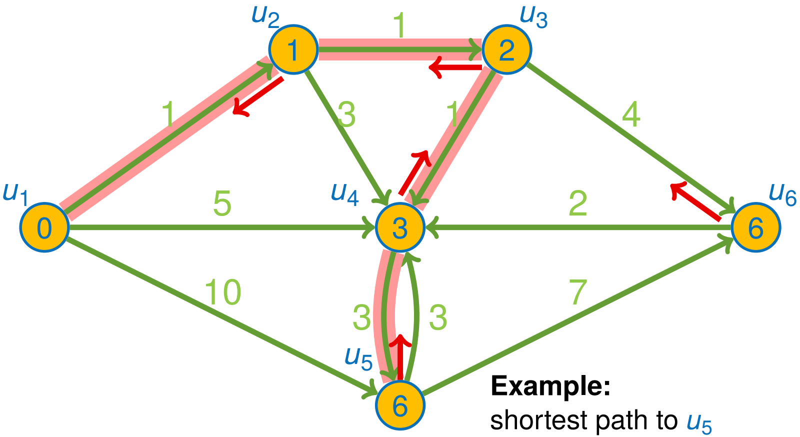


Figure: Iteration 5



Example:
shortest path to u_5

Figure: Iteration 6

Enhancement:

- In our proof we used the assumption that all costs are not negative (even > 0)
- With negative costs there might be negative cycles:

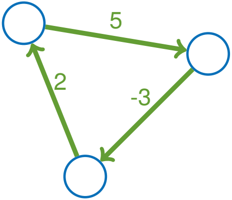


Figure: Here no problem ...

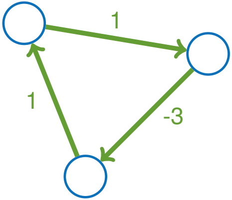
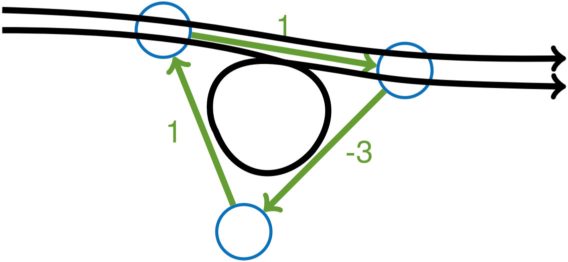


Figure: ... but here

Negative cycles:

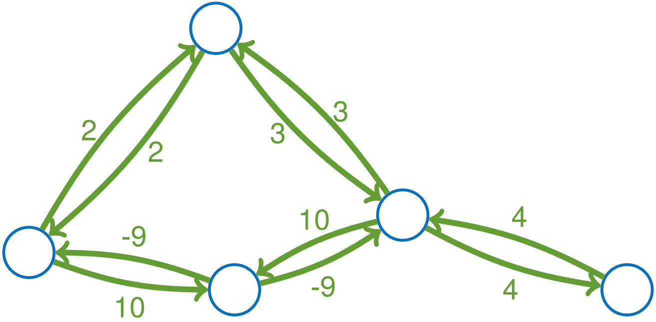


- No cycle:
  cost of 1
- 1 cycle:
  cost of 0
- 2 cycles:
  cost of -1
- 3 cycles:
  cost of -2
- . . .

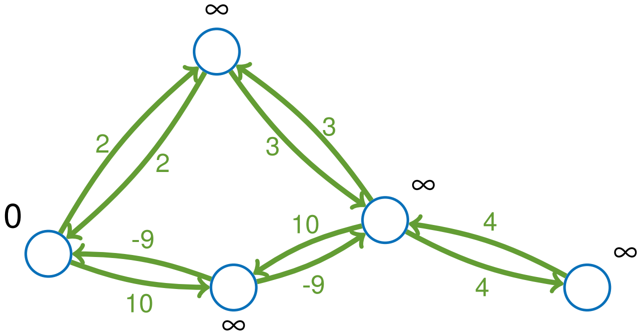
Enhancement:

- We need a different algorithm to deal with negative edges
  - For example the **Bellman-Ford** algorithm
  - If the graph is acyclic we can simply use a topological sorting (with DFS) and settling the nodes in order of this sorting
- Another (not only) in artificial intelligence used variant of the Dijkstra algorithm is the A* algorithm
  Additional information given:
  $h(u)$ = estimated value for $\mathrm{dist}(u, t)$

Dijkstra algorithm:
Message passing only from solved nodes



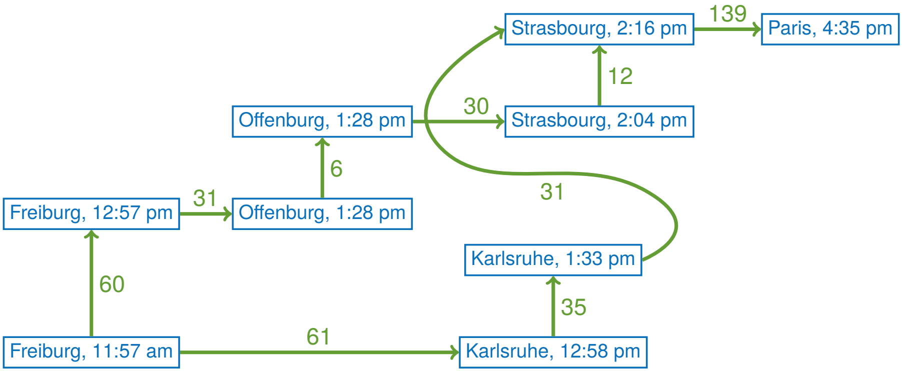
Bellman-Ford algorithm:
Message passing from all nodes until the path lengths are stable



Application example:

- Route planner for car trips (exercise sheet)
- Route planner for bus / train connections
  What could the graph look like?

Space-time graph:



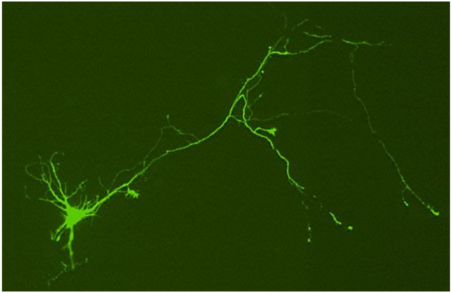
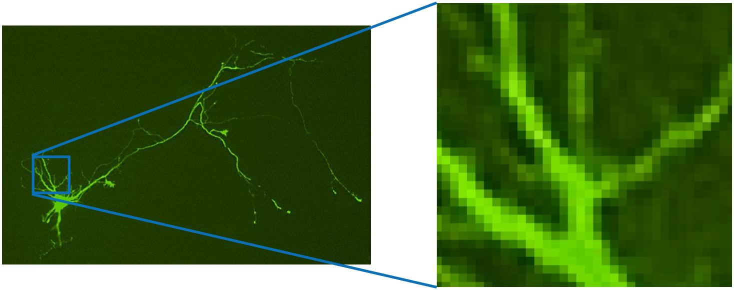


Figure: Neurons under fluorescence microscope

- **Task:** Measure length of axons (connections of neurons)
- Demo with ImageJ plugin NeuronJ
  `http://www.imagescience.org/meijering/software/neuronj/`

Dijkstra Algorithm

Application: Trace axons



- Image as graph: Each pixel is a node
- Implicit edges: Each pixel has an edge to it’s 8 neighbours (no need to save the edges)
- Costs for nodes (not edges): bright pixels are cheap, dark pixels are costly

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Dijkstra's algorithm**

  [Wik] Dijkstra's algorithm
  https://en.wikipedia.org/wiki/Dijkstra's_algorithm

- **Shortest path problem**

  [Wik] Shortest path problem
  https://en.wikipedia.org/wiki/Shortest_path_problem