

# Algorithms and Datastructures

## Hash Map, Universal Hashing

Albert-Ludwigs-Universität Freiburg



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## Associative Arrays

- Introduction

- Hash Map

## Universal Hashing

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- Probability Calculation

- Proof

- Examples

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- Naive solution: Store pairs of key and value in a normal field
- For  $n$  keys searching requires  $\Theta(n)$  time
- With a **hash map** this just requires  $\Theta(1)$  in the best case, ... regardless how many elements are in the map!



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- Mapping the keys onto indices with a [hash function](#)
- Store the values at the calculated indices in a normal array

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- Key set:  $x = \{3904433, 312692, 5148949\}$

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- We need an array **T** with **5** elements.  
A “hashtable” with 5 “buckets”

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- We need an array **T** with **5** elements.  
A “hashtable” with 5 “buckets”
- The element with the key **x** is stored in  **$T[h(x)]$**

### Storage:

Figure: Hashtable T



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■ `insert(3904433, "A")`:  $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$

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- `insert(3904433, "A")`:  $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- `insert(312692, "B")`:  $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$

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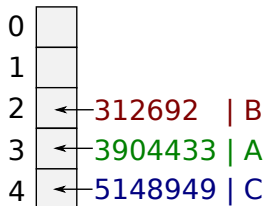
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- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$
- $\text{search}(123459): h(123459) = 4 \Rightarrow T[4]$   
 $\Rightarrow$  Value with key 123459 does not exist

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 $\Rightarrow$  Value with key 123459 does not exist
- Search time for this example:  $\mathcal{O}(1)$

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### Further inserting:

- `insert(876543, "D")`:  $h(876543) = 3$

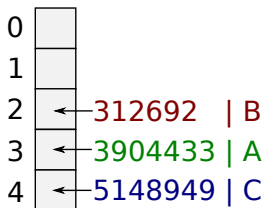
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### Further inserting:

- `insert(876543, "D")`:  $h(876543) = 3$   
 $\Rightarrow T[3] = (876543, "D") \Rightarrow$  **Collision**
- This happens more often than expected
  - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

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### Easiest Solution:

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

Figure: Hashtable T



### Best case:

- We have  $n$  keys which are equally distributed over  $m$  buckets
- We have  $\approx \frac{n}{m}$  pairs per bucket
- The runtime for searching is nearly  $\mathcal{O}(1)$  when **not**  $n \gg m$

**Best case**  
( $m = 5, n = 10$ )



### Worst case:

- All  $n$  keys are mapped onto the same bucket
- The runtime is  $\Theta(n)$  for searching

**Worst case**  
( $m = 5, n = 10$ )



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- Find a **set of keys** resulting in a degenerated **hash table**
  - *The **hash function** stays fixed*
  - *For table size of 100: Try  $100 \times (99 + 1)$  different numbers*
  - *Worst case: All 100 **key sets** map to one bucket*
- **Now:** Find a solution to avoid that problem

### Solution: universal hashing

- Out of a set of hash functions we randomly choose one



Figure: Hash func. 1



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### Solution: universal hashing

- Out of a set of hash functions we randomly choose one
  - The **expected result** of the hash function is an equal distribution over the buckets
  - This hash function stays fixed for the lifetime of table
- Optional: copy table with new hash when degenerated



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Figure: Hash func. 2



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- We call  $\mathcal{U}$  the set (universum) of possible keys

Key universe  $\mathcal{U}$



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$T$  (Hashtable)





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- Set of hash functions  $\mathbb{H} = \{h_1, h_2, \dots, h_n\}$  with  $h_i : \mathbb{U} \rightarrow \{0, \dots, m-1\}$

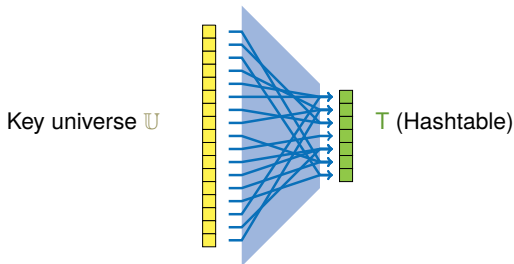


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- Idea: runtime should be  $O(1 + \frac{|\mathbb{S}|}{m})$ , where  $\frac{|\mathbb{S}|}{m}$  is the table load

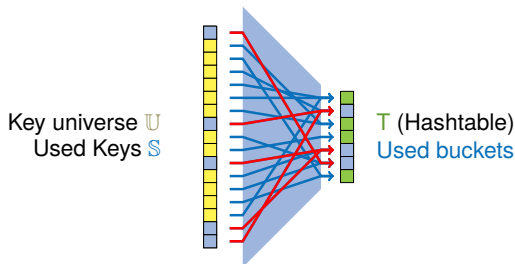


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- An average of 3 out of 15 functions produce collisions



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Number of hash functions that create collisions

$$\frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}, \quad c \in \mathbb{R}$$

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$$\text{Prob}(\text{Collision}) = \frac{1}{m} \Leftrightarrow c = 1$$

- $\mathbb{U}$ : Key universe
- $\mathbb{S}$ : Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$ : Keys mapping to Bucket  $i$  (“synonyms”)
- Ideal would be  $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$



Figure: Hash function  $h \in \mathbb{H}$

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- Particularity: If  $(m = \Omega(|\mathbb{S}|))$  then  $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$



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# Universal Hashing

## Probability Calculation



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**Sum of expected values:** For arbitrary discrete random variables  $X_1, \dots, X_n$  we can write:

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- $X_1$ : Expected result of dice 1:  $\mathbb{E}(X_1) = 3.5$

**Sum of expected values:** For arbitrary discrete random variables  $X_1, \dots, X_n$  we can write:

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- $X_2$ : Expected result of dice 2:  $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$ : Expected total number:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$



### Corollary:

The probability of the event  $E$  is  $p = P(E)$ . Let  $X$  be the occurrences of the event  $E$  and  $n$  be the number of executions of the experiment. Then  $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurrences of } 6) = \frac{1}{6} \cdot 60 = 10$$



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Def.  $\mathbb{E}$ -value:  $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

## Associative Arrays

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### Given:

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### To proof:

$$\mathbb{E}[|S_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \quad \forall i$$



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$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \quad y \in \mathbb{S} \setminus \{x\}$$

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$$\Rightarrow \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

**Auxiliary calculation:**

$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1) \\ &= P(h(y) = i) \\ &= P(h(y) = h(x)) \\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

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- Which  $x, y$  lead to a relative collision count bigger than  $\frac{c}{m}$ ?





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- Easy to implement but hard to proof
- Exercise: show empirically that it is 2-universal



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- The set of hash functions is  $c$ -universal:

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- We define:

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- **Intuitive:** Scalar product with base  $m$

$$a \bullet x = \sum_{0, \dots, k-1} a_i \cdot x_i$$

Example ( $\mathbb{U} = \{0, \dots, 999\}$ ,  $m = 10$ ,  $a = 348$ )

With  $a = 348$ :  $a_2 = 3$ ,  $a_1 = 4$ ,  $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With  $x = 127$ :  $x_2 = 1$ ,  $x_1 = 2$ ,  $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

## ■ General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Hash Map - Theory

[Wik] [Hash table](#)

[https://en.wikipedia.org/wiki/Hash\\_table](https://en.wikipedia.org/wiki/Hash_table)

## ■ Hash Map - Implementations / API

[Cpp] [C++ - hash\\_map](#)

[http://www.sgi.com/tech/stl/hash\\_map.html](http://www.sgi.com/tech/stl/hash_map.html)

[Jav] [Java - HashMap](#)

<https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html>

[Pyt] [Python - Dictionaries \(Hash table\)](#)

[https://en.wikipedia.org/wiki/Hash\\_table](https://en.wikipedia.org/wiki/Hash_table)