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Structure



O-Notation

Motivation / Definition Examples

Ω-Notation

Θ-Notation

Runtime

Summary Limit / Convergence L'Hôpital / l'Hospital Practical use

- Example: sorting
 - Runtime of Minsort "is growing as" n^2
 - Runtime of HeapSort "is growing as" $n \log n$
- \blacksquare Growth of a function in runtime T(n)
 - The role of constants (e.g. 1ns) is minor
 - it is enough if relation holds for some $n \ge ...$
- Describe the growth of the function more formally
 - By the means of Landau-Symbols [Wik]):
 - \blacksquare $\mathcal{O}(n)$ (Big O of n),
 - \square $\Omega(n)$ (Omega of n),
 - \blacksquare $\Theta(n)$ (Theta of n)

Big *𝑉*-Notation:

- Consider the function: $f: \mathbb{N} \to \mathbb{R}, n \mapsto f(n)$
 - \blacksquare \mathbb{N} : Natural numbers \to input size
 - \blacksquare \mathbb{R} : Real numbers \rightarrow runtime

Example:

- $\blacksquare f(n) = 3n$
- $f(n) = 2n \log n$
- $f(n) = \frac{1}{10}n^2$
- $f(n) = n^2 + 3n \log n 4n$



Big *∅*-Notation:

 \blacksquare Given two functions f and g:

 $f,g:\mathbb{N}\to\mathbb{R}$

- Intuitive: f is Big-O of g (f is $\mathcal{O}(g)$)
 - \blacksquare ... if f relative to g does not grow faster than g
 - the growth rate matters, not the absolute values

Big *∅*-Notation:

- Informal: $f = \mathcal{O}(g)$
 - "=" corresponds to *is* not *isequal*
 - ... if for some value n_0 for all $n \ge n_0$
 - $f(n) \le C \cdot g(n)$ for a constant C
 - $(f = \mathcal{O}(g))$: From a value n_0 for all $n \ge n_0 \to f(n) \le C \cdot g(n)$
- Formal: $f \in \mathscr{O}(g)$

Formal: $f \in \mathcal{O}(g)$

$$\mathscr{O}(g) = \{ \ \mathsf{f} : \mathbb{N} \to \mathbb{R} \mid \exists n_0 \in \mathbb{N}, \ \exists C > 0, \ \forall n > n_0 \colon f(n) \leq C \cdot g(n) \}$$
 "set of "for which" "it exists" "for all" "such that" all functions"

Illustration of the Big O-Notation:

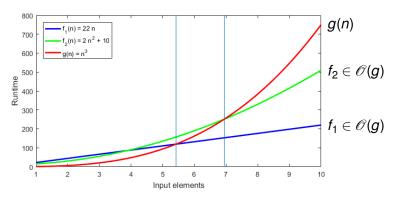


Abbildung: Runtime of two algorithms f_1, f_2

■
$$f(n) = 5n + 7$$
, $g(n) = n$
⇒ $5n + 7 \in \mathcal{O}(g)$
⇒ $f \in \mathcal{O}(g)$

Intuitive:

$$f(n) = 5n + 7 \rightarrow \text{linear growth}$$

Attention

 $f(n) \le g(n)$ is not guaranteed, better is $f(n) \le C \cdot g(n) \ \forall n > n_0$.

We have to proof: $\exists n_0, \exists C, \forall n \geq n_0$: $5n + 7 \leq C \cdot n$.

$$5n+7 \leq 5n+n \quad (\text{for } n \geq 7)$$

= $6n$

$$\Rightarrow$$
 $n_0 = 7$, $C = 6$

Alternate proof:

$$5n+7 \le 5n+7n \text{ (for } n \ge 1)$$

= 12n

$$\Rightarrow$$
 $n_0 = 1$, $C = 12$

Examples

Big O-Notation:

- We are only interested in the term with the highest-order, the fastest growing summand, the others will be ignored
- \blacksquare f(n) is limited from above by $C \cdot g(n)$

Examples:

$$2n^{2} + 7n - 20 \in \mathscr{O}(n^{2})$$

$$2n^{2} + 7n \log n - 20 \in$$

$$7n \log n - 20 \in$$

$$5 \in$$

$$2n^{2} + 7n \log n + n^{3} \in$$

Harder Example:

- Polynomes are simple
- More problematic: combination of complex functions

$$2\sqrt{x} + 3\ln x \in \mathcal{O}(\ref{eq:condition})$$

Omega-Notation:

- Intuitive:
 - $f \in \Omega(g)$, f is growing at least as fast as g
 - So the same as Big-O but with at-least and not at-most

Formal: $f \in \Omega(g)$

$$\Omega(g) = \{f: \mathbb{N} \rightarrow \mathbb{R} \mid \exists n_0 \in \mathbb{N}, \exists C > 0, \forall n > n_0 : f(n) \geq C \cdot g(n)\}$$

Example:

Proof of
$$f(n) = 5n + 7 \in \Omega(n)$$
:

$$\underbrace{5n+7}_{f(n)} \geq \underbrace{1 \cdot n}_{g(n)} \quad (\text{for } n \geq 1)$$

$$\Rightarrow$$
 $n_0 = 1$, $C = 1$

Illustration of the Omega-Notation:

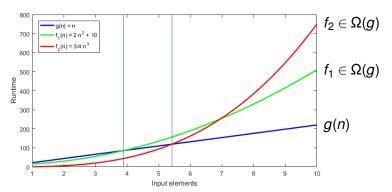


Abbildung: Runtime of two algorithms f_1, f_2

Big Omega-Notation:

- We are only interested in the term with the highest-order, the fastest growing summand, the others will be ignored
- \blacksquare f(n) is limited from underneath by $c \cdot g(n)$

Examples:

$$2n^{2}+7n-20 \in \Omega(n^{2})$$

$$2n^{2}+7n\log n-20 \in$$

$$7n\log n-20 \in$$

$$5 \in$$

$$2n^{2}+7n\log n+n^{3} \in$$

Theta-Notation:

- Intuitive: f is Theta of g ...
 - \blacksquare ... if f is growing as much as g
 - $f \in \Theta(g)$, f is growing at the same speed as g

Formal: $f \in \Theta(g)$

$$\Theta(g) = \underbrace{\mathscr{O}(g) \cap \Omega(g)}_{}$$

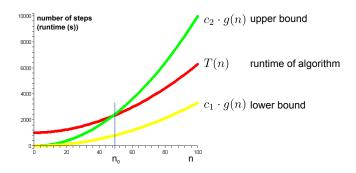
Intersection

Example:

$$f(n) = 5n + 7, f(n) \in \mathcal{O}(n), f(n) \in \Omega(n)$$

$$\Rightarrow f(n) \in \Theta(n)$$

Proof for $\mathcal{O}(g)$ and $\Omega(g)$ look at slides 11 and 17



■ f and g have the same "growth"

Big O-Notation $\mathcal{O}(n)$:

- \blacksquare *f* is growing at most as fast as *g*
- \blacksquare $C \cdot g(n)$ is the upper bound

Big Omega-Notation $\Omega(n)$:

- \blacksquare f is growing at least as fast as g
- $C \cdot g(n)$ is the lower bound

Big Theta-Notation $\Theta(n)$:

- \blacksquare *f* is growing at the same speed as *g*
 - $C_1 \cdot g(n)$ is the lower bound
 - $C_2 \cdot g(n)$ is the upper bound

Tabelle: Common runtime types

Runtime	Growth
$f \in \Theta(1)$	constant time
$f \in \Theta(\log n) = \Theta(\log_k n)$	logarithmic time
$f \in \Theta(n)$	linear time
$f \in \Theta(n \log n)$	n-log-n time (nearly linear)
$f \in \Theta(n^2)$	squared time
$f \in \Theta(n^3)$	cubic time
$f \in \Theta(n^k)$	polynomial time
$f \in \Theta(k^n), f \in \Theta(2^n)$	exponential time

- So far discussed:
 - Membership in O(...) proofed by hand: Explicit calculation of n_0 and C
 - However: Both hint at limits in calculus

Definition of "Limit"

- The limit L exists for an infinite sequence f_1, f_2, f_3, \ldots if for all $\varepsilon > 0$ one $n_0 \in \mathbb{N}$ exists, such that for all $n \ge n_0$ the following holds true: $|f_n L| \le \varepsilon$
- A function $f: \mathbb{N} \to \mathbb{R}$ can be written as a sequence $\Rightarrow \lim_{n \to \infty} f_n = L$

The limit is converging:

 $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \; \forall n \geq n_0 \colon \; |f_n - L| \leq \varepsilon$

- Example for the proof of a limit
- Function $f(n) = 2 + \frac{1}{n}$ with limes $\lim_{n\to\infty} f(n) = 2$
- "Engineering" solution: use $n = \infty$

$$\frac{1}{\infty} = 0 \Rightarrow \lim_{n \to \infty} f(n) = \lim_{n \to \infty} 2 + \frac{1}{n} = 2$$



- Now a more formal proof for $\lim_{n\to\infty} 2 + \frac{1}{n} = 2$
- We need to show: for all given ε there is an n_0 such that for all $n \ge n_0$

$$\left|2+\frac{1}{n}-2\right|=\left|\frac{1}{n}\right|\leq\varepsilon$$

- E.g.: for ε = 0.01 we get $\frac{1}{n} \le \varepsilon$ for $n \ge 100$
- In general

$$n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$$

Then we get:

$$\left|\frac{1}{n}\right| = \frac{1}{n} \le \frac{1}{n_0} = \frac{1}{\left\lceil \frac{1}{\varepsilon} \right\rceil} \le \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \quad \Box$$

Let $f,g: \mathbb{N} \to \mathbb{R}$ with an existing limit

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=L$$

Hence the following holds:

$$f \in \mathscr{O}(g)$$
 \Leftrightarrow $\lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$ (1)

$$f \in \Omega(g)$$
 \Leftrightarrow $\lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$ (2)

$$f \in \Theta(g)$$
 \Leftrightarrow $0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$ (3)



$$f \in \mathscr{O}(g) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$$

Forward proof (\Rightarrow) :

$$f \in \mathscr{O}(g) \overset{\mathsf{def. of }\mathscr{O}(n)}{\Rightarrow} \exists n_0, C \ \forall n \geq n_0 : \ f(n) \leq C \cdot g(n)$$

$$\Rightarrow \exists n_0, C \ \forall n \geq n_0 : \frac{f(n)}{g(n)} \leq C$$

$$\Rightarrow \qquad \lim_{n \to \infty} \frac{f(n)}{g(n)} \leq C \quad \Box$$

Backward proof (⇐):

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = C \qquad \text{For some } C \in \mathbb{R} \text{ (Limit)}$$

$$\stackrel{\text{def. limes}}{\Rightarrow} \exists n_0, \forall n \geq n_0 : \qquad \frac{f(n)}{g(n)} \leq C + \varepsilon \quad (e.g. \ \varepsilon = 1)$$

$$\Rightarrow \exists n_0, \forall n \geq n_0 : \qquad f(n) \leq \underbrace{(C+1)}_{O-notation \ constant} \cdot g(n)$$

$$\Rightarrow \qquad f \in \mathscr{O}(g) \qquad \Box$$

Intuitive:

$$\lim_{n \to \infty} 2 + \frac{1}{n} = 2 + \frac{1}{\infty} = 2$$

■ With L'Hôpital:

Let
$$f, g : \mathbb{N} \to \mathbb{R}$$
If $\lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \infty/0$

$$\Rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}$$

Holy inspiration

you need a doctoral degree for that

The limit can not be determined in the way of an Engineer:

$$\lim_{n\to\infty}\frac{\ln(n)}{n}=\frac{\lim_{n\to\infty}\ln(n)}{\lim_{n\to\infty}n}\qquad \stackrel{\text{plugging in}}{\longrightarrow}\qquad \stackrel{\infty}{\longrightarrow}$$

Determine the limit with using L'Hôpital:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{f'(n)}{g'(n)}$$

Using L'Hôpital:

Numerator: $f(n): n \mapsto ln(n)$

Denominator: g(n): $n \mapsto n$

$$\Rightarrow f'(n) = \frac{1}{n}$$
 (derivation from Numerator)

 \Rightarrow g'(n) = 1 (derivation from Denominator)

$$\lim_{n\to\infty}\frac{f'(n)}{g'(n)}=\lim_{n\to\infty}\frac{1}{n}=0 \quad \Rightarrow \quad \lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{\ln(n)}{n}=0$$

Limits with L'Hôpital



What can we take for granted without proofing?

- Only things that are trivial
- It is always better to proof it

Examples:

$$\lim_{n \to \infty} \frac{1}{n} = 0 \qquad \text{is trivial}$$

$$\lim_{n \to \infty} \frac{1}{n^2} = 0 \qquad \text{is trivial}$$

$$\lim_{n \to \infty} \frac{\log(n)}{n} = 0 \qquad \text{use L'Hopital}$$



Practical use:

- It is much easier to determine the runtime of an algorithm by using the \(\mathcal{O}\)-Notation
 - Computing rules
 - Practical use

Characteristics

Transitivity:

$$f \in \Theta(g) \land g \in \Theta(h) \rightarrow f \in \Theta(h)$$

$$f \in \mathcal{O}(g) \land g \in \mathcal{O}(h) \rightarrow f \in \mathcal{O}(h)$$

$$f \in \Omega(g) \land g \in \Omega(h) \rightarrow f \in \Omega(h)$$

Symmetry:

$$f \in \Theta(g) \leftrightarrow g \in \Theta(f)$$

 $f \in \mathscr{O}(g) \leftrightarrow g \in \Omega(f)$

Reflexivity:

$$f \in \Theta(f)$$
 $f \in \Omega(f)$ $f \in \mathcal{O}(f)$

■ Trivial:

$$\begin{array}{rcl} f & \in & \mathcal{O}(f) \\ k \cdot \mathcal{O}(f) & = & \mathcal{O}(f) \\ \mathcal{O}(f+k) & = & \mathcal{O}(f) \end{array}$$

Addition:

$$\mathcal{O}(f) + \mathcal{O}(g) = \mathcal{O}(\max\{f, g\})$$

Multiplication:

$$\mathcal{O}(f)\cdot\mathcal{O}(g) = \mathcal{O}(f\cdot g)$$

Runtime Complexity

- The input size for all examples is *n*
- Basic operations

$$i1 = 0$$
 $\mathcal{O}(1)$

Sequences of basic operations

Loops

for i in range(0, n):

$$a[i] = 0$$

$$\mathcal{O}(n)$$

$$0(1) \cdot \mathcal{O}(n) = \mathcal{O}(n)$$
for i in range(0, n):

$$a1[i] = 0$$

$$0(1) \cdot \mathcal{O}(n) = \mathcal{O}(n)$$

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Loops

■ Conditions

- Input: List *x* with *n* numbers
- Output: a[i] is the arithmetic mean of x[0] to x[i]

```
def arithMean(x):
    a = [0] * len(x)
    for i in range(0, len(x)):
        s = 0
        for j in range(0, i+1):
            s = s + x[j]
        a[i] = s / (i+1)
```

for i in range(0, len(x)):
$$s = 0$$

$$for j in range(0, i+1):$$

$$s = s + x[j]$$

$$a[i] = s / (i+1)$$

$$O(n)$$

$$O(i)$$

$$O(i)$$

$$O(i)$$

$$O(i)$$

$$O(n)$$

$$O(n$$

■ How often will the instructions in the loop be executed, when the problem has size *n*?

$$1+2+\ldots+n=\frac{n\cdot(n+1)}{2}\in\mathscr{O}(n^2)$$

Discussion

Way of speaking:

- With the Ø-Notation we look at the behavior of a function when $n \to \infty$
- We only analyze the runtime when $n \ge n_0$
- We talk about asymptotic analysis, when we discuss cost, runtime, etc. as $\mathcal{O}(...)$, $\Omega(...)$ or $\Theta(...)$

- If you are using **asymptotic analysis**, you can not make any predictions about the runtime of smaller input sizes $(n < n_0)$
- For small input sizes (mostly n < 10), the runtime is predictably small
- \blacksquare n_0 does not necessarily have to be small

Discussion

Examples:

- Let A and B be algorithms
 - A has the runtime f(n) = 80n
 - B has the runtime $f(n) = 2n \log_2 n$
- So $f = \mathcal{O}(q)$ but **not** $\Theta(q)$
 - ⇒ A is asymptotic faster than B
 - ⇒ There is a n_0 for that $n \ge n_0$: $f(n) \le g(n)$

When is A faster then B?

We search the minimal n_0 :

$$f(n_0) = g(n_0)$$

$$80 n_0 = 2n_0 \log_2 n_0$$

$$40 = \log_2 n_0$$

$$n_0 = 2^{40}$$

$$= (2^{10})^4 = (1024)^4$$

$$\approx (10^3)^4 = 10^{12}$$

$$\approx 1 \text{ trillion}$$

A ist faster than B if n_0 has more than 1 trillion elements

Runtime Examples

Continued



Logarithm of different bases differ only by a constant

$$\log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \cdot \log_b n$$

- Hence: $\log_a n \in \Theta(\log_b n)$
- For exponent this does not hold

$$3^n\not\in\Theta(2^n)$$

Proof: Use equation (1) from Slide 31

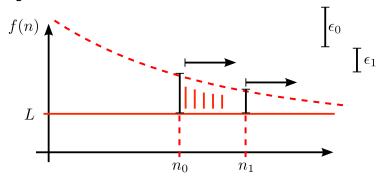
$$3^n \in \mathscr{O}(2^n) \Leftrightarrow \lim_{n \to \infty} \frac{3^n}{2^n} < \infty$$

However:

$$\lim_{n\to\infty}\frac{3^n}{2^n}=\lim_{n\to\infty}\left(\frac{3}{2}\right)^n=\infty$$



■ Figure for slide 28



■ General

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.

https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

■ Big O notation

[Wik] Big O notation

https://en.wikipedia.org/wiki/Big_O_notation