

# Algorithmns and Datastructures

Runtime analysis Minsort / Heapsort, Induction

Albert-Ludwigs-Universität Freiburg



**UNI  
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## Runtime Example

Minsort

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# Feedback from the exercises

# Feedback from the lecture



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## Runtime Example

**Minsort**

## Basic Operations

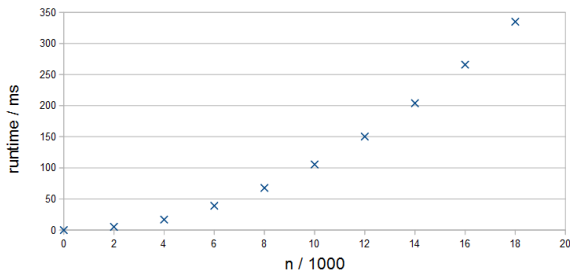
## Runtime analysis

Minsort

Heapsort

Introduction to Induction

## Logaritms



## How long does the program run?

- In the last lecture we had a schematic
- **Observation:** It is going to be “disproportional” slower the more numbers are being sorted
- How can we say more precisely what is happening?

## How can we analyse the runtime?

- Ideally we have a formula which provides the runtime of the program for an specific input



## How can we analyse the runtime?

- Ideally we have a formula which provides the runtime of the program for an specific input
- **Problem:** The runtime is also depending on many other influences, especially:
  - Which kind of computer is the code executed on
  - What is running in the background
  - Which compiler is used to compile the code

## How can we analyse the runtime?

- Ideally we have a formula which provides the runtime of the program for an specific input
- **Problem:** The runtime is also depending on many other influences, especially:
  - Which kind of computer is the code executed on
  - What is running in the background
  - Which compiler is used to compile the code
- **Abstraction 1:** Analyse the number of basic operations, rather than analysing the runtime

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## Logaritms

## Uncomplete list of basic operations:

- Arithmetic operation, for example:  $a + b$
- Allocation of variables, for example:  $x = y$
- Function call, for example: *Sorter.minSort(array)*

## Intuitive:

lines of code

## Better:

lines of machine  
code

## Best:

process cycles

## Important:

The actual runtime has to be roughly proportional to the number of operations.

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## Logaritms

How many operations does *Minsort* need?

- **Abstraction 2:** We calculate the upper (lower) bound, rather than counting the operations exactly

**Reason:** Runtime is unknown, but we know:

- Upper bound
  - Lower bound
- 
- **Basic Setting:**
    - $n$  is size of input (i.e. array)
    - $T(n)$  number of operations for input  $n$

How many operations does *Minsort* need?

- **Observation:** The number of operations needed is only depending on the size  $n$  of the array and not on the content!
- **Claim:** There are constants  $C_1$  and  $C_2$  such that:

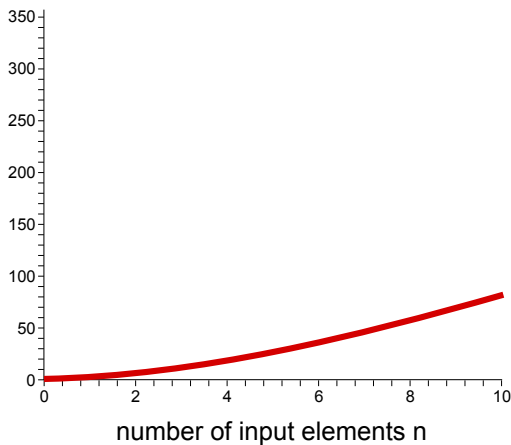
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- This is called “quadratic runtime” (due to  $n^2$ )



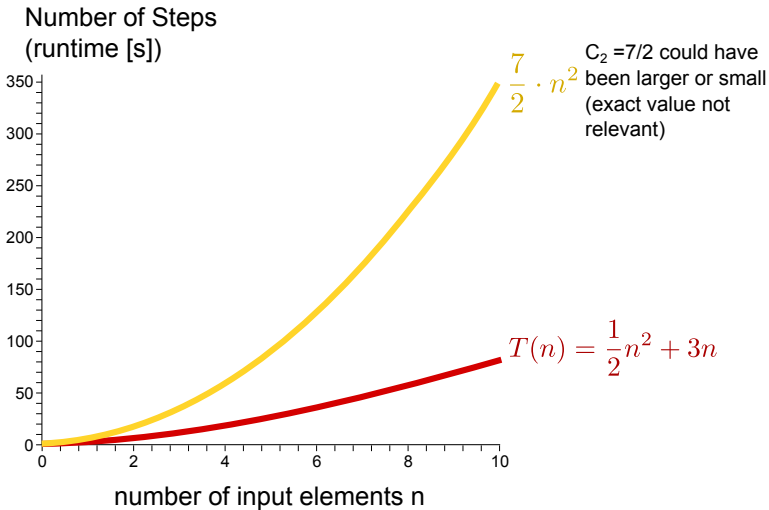
# Runtime Example

Number of Steps  
(runtime [s])

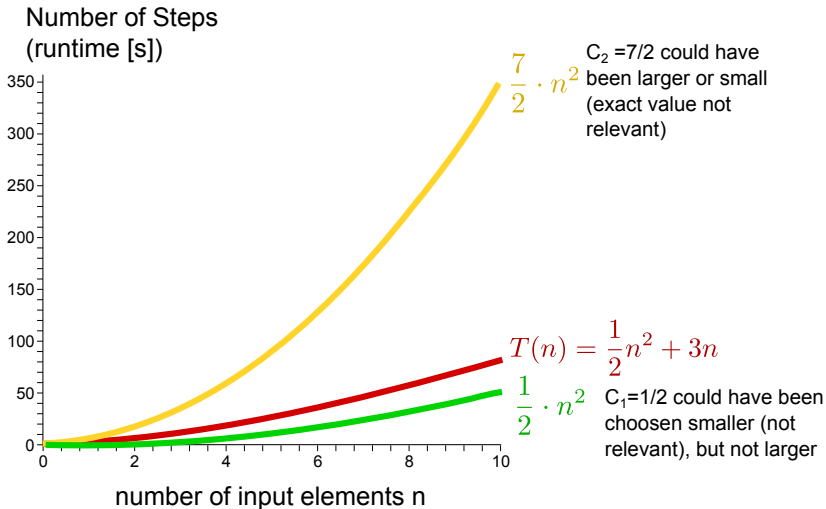


$$T(n) = \frac{1}{2}n^2 + 3n$$

# Runtime Example

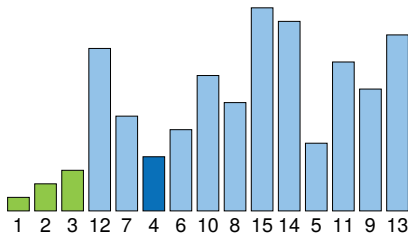


# Runtime Example



## We declare:

- Runtime of operations:  $T(n)$
- Number of Elements:  $n$
- Constants:  $C_1$  (lower bound),  $C_2$  (upper bound)  
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$
- Number of operations in round  $i$ :  $T_i$



**Figure:** *Minsort* at the iteration  $i = 4$ . We have to check  $n - 3$  elements

Compares at each  
iteration:

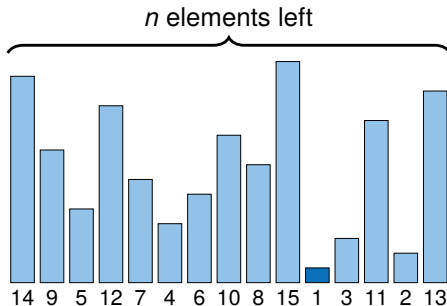


Figure: *Minsort* with start data

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

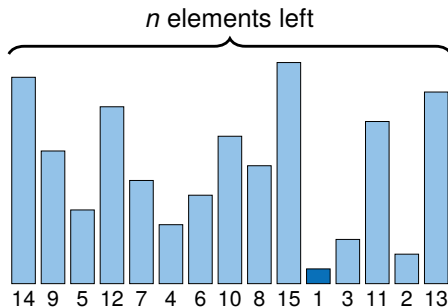
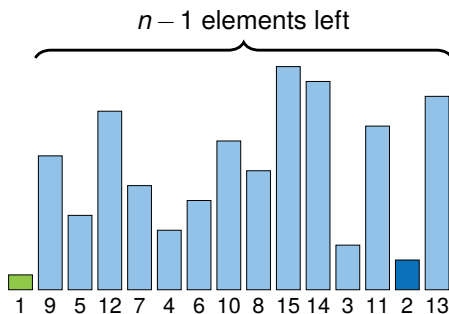


Figure: *Minsort* at iteration  $i = 1$

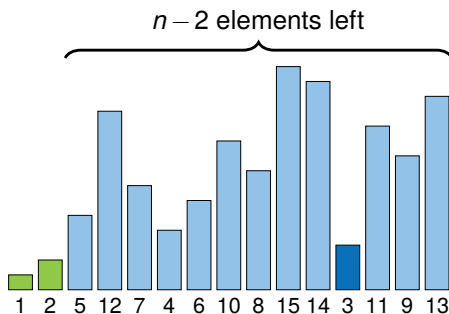


Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

Figure: *Minsort* at iteration  $i = 2$



Compares at each iteration:

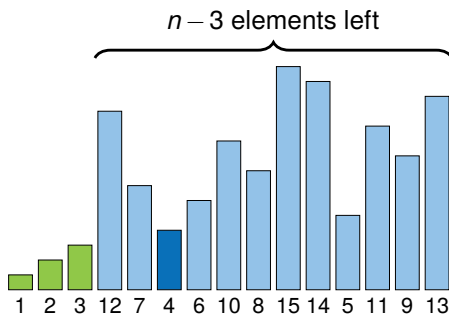
$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

Figure: *Minsort* at iteration  $i = 3$





Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

Figure: Minsort at iteration  $i = 4$

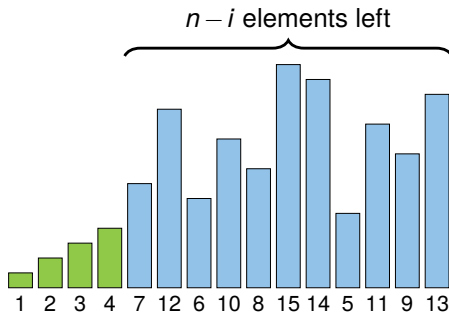


Figure: *Minsort* at iteration  $i$

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

$\vdots$

$$T_{n-1} \leq C'_2 \cdot 2$$

$$T_n \leq C'_2 \cdot 1$$

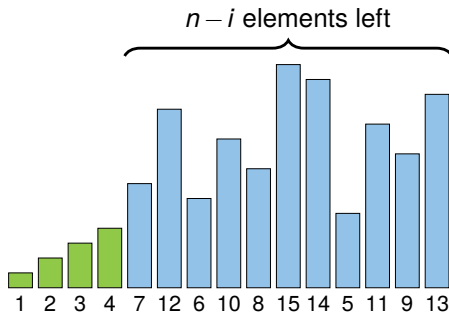


Figure: Minsort at iteration

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

$\vdots$

$$T_{n-1} \leq C'_2 \cdot 2$$

$$T_n \leq C'_2 \cdot 1$$

$$T(n) = C \cdot (T_1 + \dots + T_n) \leq C \cdot \sum_{i=1}^n i$$

## Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j  
  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
  
    return elements
```

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        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j  
        } const.  
        runtime  
  
    if minimum != i:  
        elements[i], elements[minimum] = \  
            elements[minimum], elements[i]  
  
    return elements
```

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            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
    return elements
```

Diagram illustrating the runtime analysis of the Minsort algorithm:

- The inner loop (for j in range(i+1, len(elements))) is labeled "const. runtime".
- The inner loop is repeated  $n-i-1$  times for each iteration of the outer loop.
- The outer loop (for i in range(0, len(elements)-1)) is repeated  $n-1$  times.

## Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
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```

const. runtime } n-i-1 times } n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2$$



## Alternative: Analyse the Code:

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def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
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            if elements[j] < elements[minimum]:  
                minimum = j  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
    return elements
```

Diagram illustrating the runtime analysis of the Minsort algorithm:

- The inner loop (for j in range(i+1, len(elements))) is highlighted in a light blue box.
- The inner loop is annotated with "const. runtime" and "n-i-1 times".
- The outer loop (for i in range(0, len(elements)-1)) is annotated with "n-1 times".

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2$$

## Alternative: Analyse the Code:

```
def minsort(elements):  
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                minimum = j  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
    return elements
```

Diagram illustrating the runtime analysis of the Minsort algorithm:

- The inner loop (for j in range(i+1, len(elements))) is highlighted in a light blue box.
- A bracket indicates that the inner loop runs  $n-i-1$  times for each iteration of the outer loop.
- The inner loop body (if elements[j] < elements[minimum]: minimum = j) is highlighted in a darker blue box.
- A bracket indicates that the inner loop body runs  $\text{const. runtime}$  times for each iteration of the inner loop.
- A large bracket on the right indicates that the entire inner loop structure runs  $n-1$  times (for each iteration of the outer loop).

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2 = \sum_{i=1}^{n-1} (n-i) \cdot C'_2 \leq \sum_{i=1}^n i \cdot C'_2$$

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def minsort(elements):  
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            if elements[j] < elements[minimum]:  
                minimum = j  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
    return elements
```

Diagram illustrating the runtime analysis of the Minsort algorithm. The inner loop (for j) is highlighted in a light blue box, and the outer loop (for i) is highlighted in a light green box. Brackets on the right indicate the number of times each loop executes:

- The inner loop (for j) executes  $n-i-1$  times for each iteration of the outer loop.
- The outer loop (for i) executes  $n-1$  times.
- The total runtime is  $\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2$ .

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2 = \sum_{i=1}^{n-1} (n-i) \cdot C'_2 \leq \sum_{i=1}^n i \cdot C'_2$$

**Remark:**  $C'_2$  is cost of comparison  $\Rightarrow$  assumed constant

Finding an **upper bound**:  $T(n) \leq C_2 \cdot n^2$

$$T(n) \leq \sum_{i=1}^n C'_2 \cdot i$$

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$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \end{aligned}$$

Finding an **upper bound**:  $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \end{aligned}$$

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Finding an **upper bound**:  $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \\ &\leq C'_2 \cdot \frac{n(n+n)}{2}, \quad 1 \leq n \\ &= C'_2 \cdot \frac{2 \cdot n^2}{2} = C'_2 \cdot n^2 \end{aligned}$$

# Excursion - Small Gauss Formula

**Finding a lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a  $C_1$ . Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i)$$

**Finding a lower bound:**  $C_1 \cdot n^2 \leq T(n)$

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$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \end{aligned}$$

**Finding a lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a  $C_1$ . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \end{aligned}$$

How do we get to  $n^2$ ?

**Finding a lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a  $C_1$ . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \end{aligned}$$



**Finding a lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a  $C_1$ . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \\ &\geq C'_1 \cdot \frac{n \cdot n}{2 \cdot 2} \end{aligned}$$

**Finding a lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a  $C_1$ . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \\ &\geq C'_1 \cdot \frac{n \cdot n}{2 \cdot 2} = \frac{C'_1}{4} \cdot n^2 \end{aligned}$$

## Runtime Analysis:

- Upper bound:  $T(n) \leq C'_2 \cdot n^2$
- Lower bound:  $\frac{C'_1}{4} \cdot n^2 \leq T(n)$

## Summarized:

$$\frac{C'_1}{4} \cdot n^2 \leq T(n) \leq C'_2 \cdot n^2$$

## Quadratic runtime proven:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- The runtime is growing quadratic with the number of elements  $n$  in the list
- Let constants  $C_1$  and  $C_2$  for which  $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $2 \times$  elements  $\Rightarrow 4 \times$  runtime
  - $C = 1 \text{ ns}$  (1 simple instruction  $\approx 1 \text{ ns}$ )
  - $n = 10^6$  (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{12} = 10^3 \text{ s} = 16.7 \text{ min}$
  - $n = 10^9$  (1 billion numbers = 4 GB)
    - $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{18} = 10^9 \text{ s} = 31.7 \text{ years}$
- **Quadratic runtime = “big” problems unsolvable**

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**Heapsort**

Introduction to Induction

## Logaritms

## Intuitive:

- **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.
- **Heapsort:** The root-element is always the smallest (min-heap). We only need to repair a small part of the full tree after an delete operation.

## Formal:

- Let  $T(n)$  be the runtime for the *Heapsort* algorithm with  $n$  elements.
- On the next pages we will proof  $T(n) \leq C \cdot n \log_2 n$

## Depth of a binary tree:

- **Depth  $d$ :** longest path through the tree
- Complete binary tree has  $n = 2^d - 1$  nodes
- Example:  $d = 4$   
 $\Rightarrow n = 2^4 - 1 = 15$

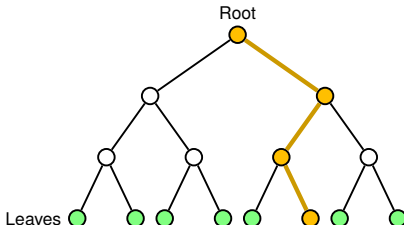


Figure: Binary tree with 15 nodes

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## Basics:

- You want to show assumption  $A(n)$  is valid  $\forall n \in \mathbb{N}$
- We show induction in two steps:
  - 1 **Induction basis:** we show that our assumption is valid at one point (for example:  $n = 1$ ).
  - 2 **Induction step:** we show that the assumption is valid for all  $n$  (normally one step forward:  $n = n + 1$ ).
- If both has been proven, then  $A(n)$  holds for all natural numbers  $n$  by **induction**

Claim:

A **complete** binary tree of depth  $d$  has  $n(d) = 2^d - 1$  nodes

- **Induction basis:** Formular holds for  $d = 1$

Root



$$n(1) = 2^1 - 1 = 1$$

$\Rightarrow$  correct ✓

Figure: Tree of depth 1 has 1 node

# Induction - Example 1



Number of nodes  $n(d)$  in a binary tree with depth  $d$ :

- **Induction assumption:**  $n(d) = 2^d - 1$

# Induction - Example 1



Number of nodes  $n(d)$  in a binary tree with depth  $d$ :

- **Induction assumption:**  $n(d) = 2^d - 1$
- **Induction basis:**  $n(1) = 2^1 - 1 = 2^1 - 1 = 1$  ✓

# Induction - Example 1



Number of nodes  $n(d)$  in a binary tree with depth  $d$ :

- **Induction assumption:**  $n(d) = 2^d - 1$
- **Induction basis:**  $n(1) = 2^1 - 1 = 2^0 - 1 = 1$  ✓
- **Induction step:** to show for  $d = d + 1$

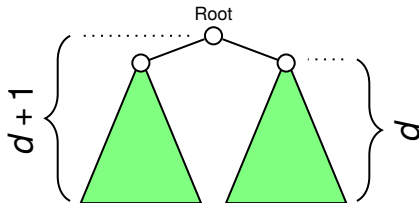
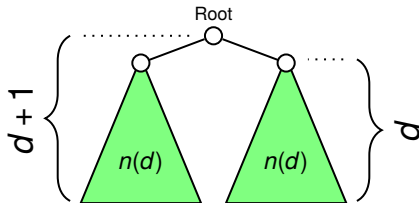


Figure: Binary tree with subtrees

# Induction - Example 1

Number of nodes  $n(d)$  in a binary tree with depth  $d$ :

- **Induction assumption:**  $n(d) = 2^d - 1$
- **Induction basis:**  $n(1) = 2^1 - 1 = 2^1 - 1 = 1$  ✓
- **Induction step:** to show for  $d = d + 1$



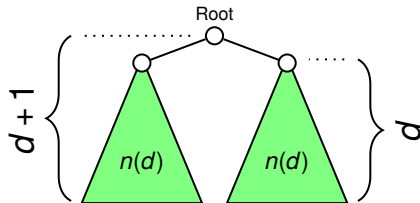
$$n(d+1) = 2 \cdot n(d) + 1$$

Figure: Binary tree with subtrees

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$$\begin{aligned} n(d+1) &= 2 \cdot n(d) + 1 \\ &= 2 \cdot (2^d - 1) + 1 \end{aligned}$$

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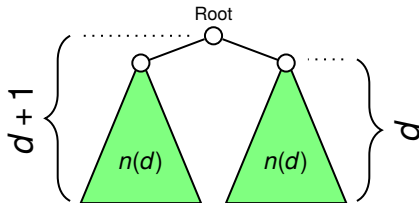


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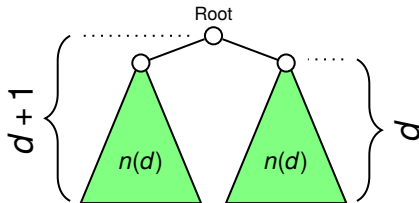


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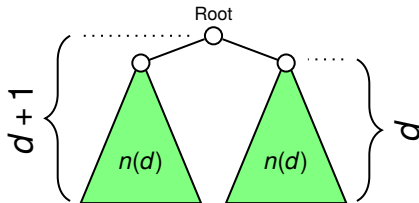


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⇒ **By induction:**  $n(d) = 2^d - 1 \quad \forall n \in \mathbb{N} \quad \square$

## Feedback

Exercises

Lecture

## Runtime Example

Minsort

## Basic Operations

## Runtime analysis

Minsort

**Heapsort**

Introduction to Induction

## Logaritms



## Heapsort has the following steps:

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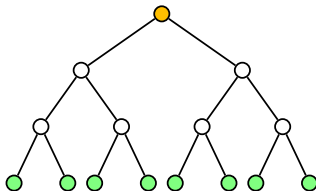
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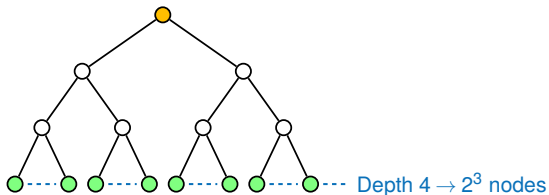
- **Initially:** heapify list of  $n$  elements
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  - Remove root as minimal element
  - Move last leaf to root position
  - Repair heap by sifting



Runtime of heapify depends on depth  $d$ :



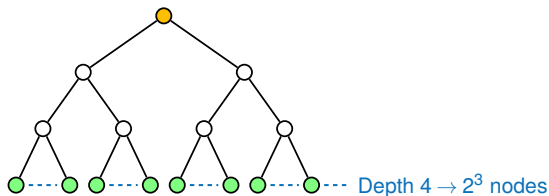
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Runtime of heapify with depth of  $d$ :

- No costs at depth  $d$  with  $2^{d-1}$  (or less) nodes

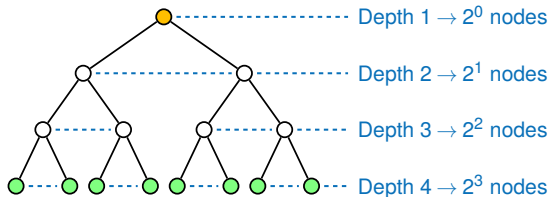
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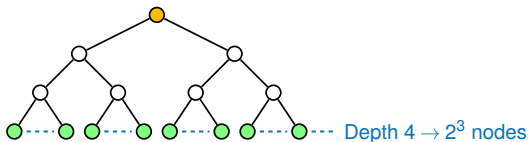
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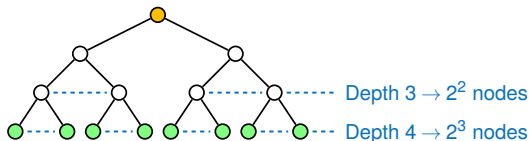
- No costs at depth  $d$  with  $2^{d-1}$  (or less) nodes
- The cost for sifting with depth 1 is at most  $1C$  per node
- In general: Sifting costs are linear with path length **and** number of nodes

Heapify total runtime:



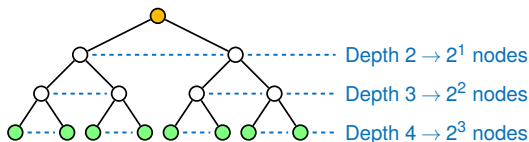
Depth	Nodes	Path length	Costs per node
$d$	$2^{d-1}$	0	$\leq C \cdot 0$

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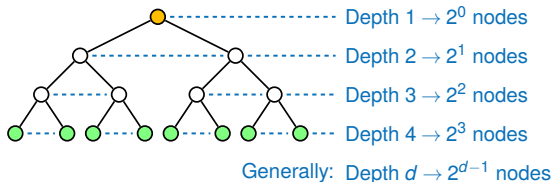
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Heapify total runtime:



Depth	Nodes	Path length	Costs per node
$d$	$2^{d-1}$	0	$\leq C \cdot 0$
$d-1$	$2^{d-2}$	1	$\leq C \cdot 1$
$d-2$	$2^{d-3}$	2	$\leq C \cdot 2$

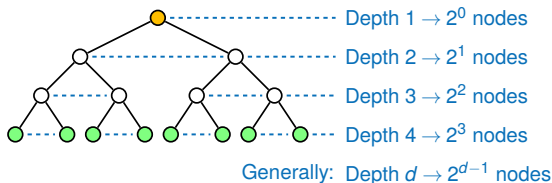
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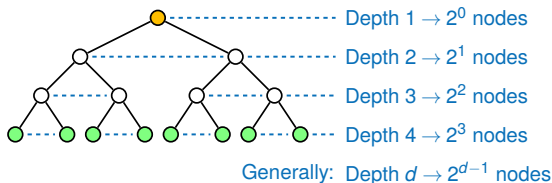
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**In total:** 
$$T(d) \leq \sum_{i=1}^d (C \cdot (i-1) \cdot 2^{d-i})$$

Heapify total runtime:



Depth	Nodes	Path length	Costs per node	Upper bound
$d$	$2^{d-1}$	0	$\leq C \cdot 0$	$\leq C \cdot 1$
$d-1$	$2^{d-2}$	1	$\leq C \cdot 1$	$\leq C \cdot 2$
$d-2$	$2^{d-3}$	2	$\leq C \cdot 2$	$\leq C \cdot 3$
$d-3$	$2^{d-4}$	3	$\leq C \cdot 3$	$\leq C \cdot 4$

**In total:** 
$$T(d) \leq \sum_{i=1}^d \left( C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^d \left( C \cdot i \cdot 2^{d-i} \right)$$

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- **Hence:** Resulting costs for heapify:

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- **But:** We want costs in relation to  $n$

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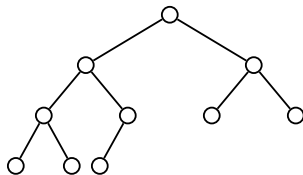


Figure: Partial binary tree

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth  $d$  has  $2^{d+1} - 1 \leq n$  nodes Why?
- $2^{d-1} - 1$  nodes in full tree till layer  $d - 1$

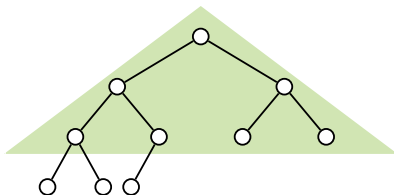


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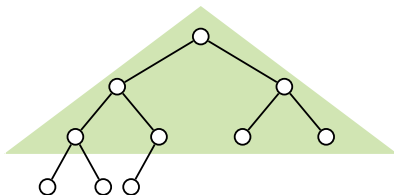


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- Equation times  $2^2$   
 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$

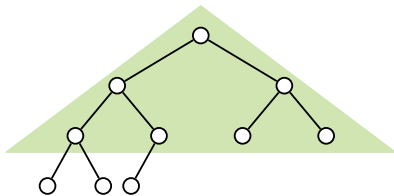


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■ At least 1 node in layer  $d$

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 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$

■ Cost for heapify:  
 $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

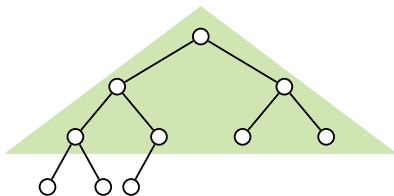


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Minsort

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Introduction to Induction

## Logaritms

- We want to proof (induction assumption):

$$\underbrace{\sum_{i=1}^d (i \cdot 2^{d-i})}_{A(d)} \leq \underbrace{2^{d+1}}_{B(d)}$$

- We denote the left side with  $A$ , the right side with  $B$

- **Induction basis:**  $d := 1$ :

$$A(d) \leq B(d)$$



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$$2^0 \leq 2^2 \quad \checkmark$$

**Induction step:** ( $d := d + 1$ ):

- **Idea:** Write down right formula and try to get  $A(d)$  and  $B(d)$  out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

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$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

$$\sum_{i=1}^{d+1} (i \cdot 2^{d+1-i}) \leq 2^{d+1+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$\vdots$

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$$2 \cdot A(n) + (d+1) \leq 2 \cdot B(n)$$

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$$2 \cdot A(n) + (d+1) \leq 2 \cdot B(n)$$

■ **Problem:** Does not work but claim still holds

## Working proof:

- Show a **little bit stronger** claim

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- **Advantage:** Results in a stronger induction assumption  
 $\Rightarrow$  **exercise**

## Feedback

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## Runtime Example

Minsort

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Introduction to Induction

## Logaritms

## **Runtime of the other operations:**

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$$2^{d-1} \leq n \Rightarrow d-1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

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- **Recall:** The number of elements is decreasing

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  - **Hence:**  $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$

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- **Recall:** The number of elements is decreasing
  - **Hence:**  $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
  - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \quad (\text{holds for } n > 2)$$

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- Heapify:  $T(n) \leq 4 \cdot n \cdot C$
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- Total runtime:  $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - Upper bound:  $C_2 \cdot n \log_2 n \geq T(n)$  (for  $n \geq 2$ )
  - Lower bound:  $C_1 \cdot n \log_2 n \leq T(n)$  (for  $n \geq 2$ )

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  - $\Rightarrow C_1$  and  $C_2$  are constant

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## Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient  $\frac{1}{\log_b a}$

### Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602 \dots \cdot 3.322 \dots = 2 \checkmark$
- $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3 \checkmark$

## Runtime of $n \log_2 n$ :

- Assume we have constants  $C_1$  and  $C_2$  with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- $2 \times$  elements  $\Rightarrow$  only slightly larger than  $2 \times$  runtime
  - $C = 1$  ns (1 simple instruction  $\approx 1$  ns)
  - $n = 2^{20}$  (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{20} \cdot 20 = 21.0 \text{ ms}$
  - $n = 2^{30}$  (1 billion numbers = 4 GB)
    - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{30} \cdot 30 = 32 \text{ s}$
- **Runtime  $n \log_2 n$  is nearly as good as linear!**

## ■ General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Mathematical Induction

[Wik] [Mathematical induction](https://en.wikipedia.org/wiki/Mathematical_induction)

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