

Algorithms and Datastructures

Levenshtein distance, Dynamic programming

Albert-Ludwigs-Universität Freiburg



**UNI
FREIBURG**

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Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, February 2018

Introduction

Edit distance

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Edit distance:

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- Measurement for similarity of two words / strings

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- Measurement for similarity of two words / strings
- Algorithm for efficient calculation
- General principle: dynamic programming

BioInfSearch

ejafjatljökuk
eyjafjallajökull
eyjafjallajökull movie
eyjafjallajölull trailer

Search!



Ulrich Latzenhofer; CC BY-SA 2.0

Wikipedia.org:

"Der Eyjafjallajökull ([ˈeɪjaˌfjatlaˌjœːkʏtʃ])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárpíng eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."



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eyjaföllajaküll

uniwersität verien 2017

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- Bioinformatics: Similarity of DNA-sequences

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- Cited 63437 times on Google Scholar (Sep. 2017)

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Definition of edit distance: (*Levenshtein-distance*)

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Edit distance

Example

1 2 3 4 5
DOOF

BLOED

Edit distance

Example



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DOOF



replace(1, B)

BOOF

BLOED

Edit distance

Example

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replace(2, L)

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Edit distance

Example

1 2 3 4 5

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replace(1, B)

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insert(4, E)

BLOEF

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Edit distance

Example

1 2 3 4 5

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replace(1, B)

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BLOF



insert(4, E)

BLOEF



replace(5, D)

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Edit distance

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BLOF



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BLOEF



replace(5, D)

BLOED

⏟
ED=4

Edit distance

Example

1 2 3 4 5

DOOF



BOOF



BLOF



BLOEF



BLOED

replace(1, B)

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insert(4, E)

replace(5, D)

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⏟
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Edit distance

Example

1 2 3 4 5

DOOF

↓

BOOF

↓

BLOF

↓

BLOEF

↓

BLOED

replace(1, B)

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replace(5, D)

⏟
ED=4

1 2 3 4 5

B LOED

DOOF

Edit distance

Example

1 2 3 4 5

DOOF



BOOF



BLOF



BLOEF



BLOED

replace(1, B)

replace(2, L)

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replace(5, D)

ED=4

1 2 3 4 5

B LOED



B LOEF

replace(5, F)

DOOF

Edit distance

Example

1 2 3 4 5

DOOF



BOOF



BLOF



BLOEF



BLOED

replace(1, B)

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1 2 3 4 5

B LOED



B LOEF



B LOF

replace(5, F)

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DOOF

Edit distance

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DOOF



BOOF



BLOF



BLOEF



BLOED

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B LOED



B LOEF



B LOF



BOOF

DOOF

replace(5, F)

delete(4)

replace(2, O)

Edit distance

Example

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DOOF



BOOF



BLOF



BLOEF



BLOED

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B LOED



B LOEF



B LOF



BOOF



DOOF

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DOOF



BOOF



BLOF



BLOEF



BLOED

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replace(5, D)

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B LOED



B LOEF



B LOF



BOOF



DOOF

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delete(4)

replace(2, \emptyset)

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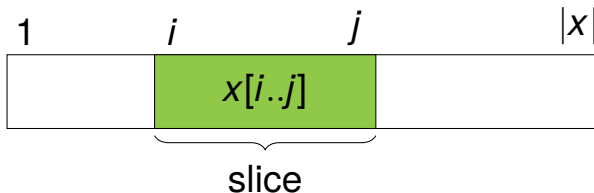
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$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

- $ED(x, y) \leq ED(x[1..n-1], y[1..m-1]) + 1 \quad n = |x|, m = |y|$



Solutions based on examples:

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Recursive approach:

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Recursive approach:

- Dividing in two halves? Not a good idea:

$$ED(\text{GRAU}, \text{RAUM}) = 2 \quad \text{but} \quad ED(\text{GR}, \text{RA}) + ED(\text{AU}, \text{UM}) = 4$$

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- Finding “smaller” sub problems?
Let's try it!

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- Let $\sigma_1, \dots, \sigma_k$ be a sequence of k operations where $k = \text{ED}(x, y)$ for $x \rightarrow y$ (transform x into y)
(We do not know this sequence but we assume it exists)



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The position of σ_{i+1} is \geq the position of σ_i where we only allow the positions to be equal on a delete operation

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1 2 3 4 5

DOOF



replace(1, B)

BOOF



replace(2, L)

BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

1 2 3 4 5 6 7

SAUDOOF



delete(1)

AUDOOF



delete(1)

UDOOF



delete(1)

DOOF



insert(4, O)

DOOOF

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- **Intuition:** The order of our sequence is not relevant (Therefore we can also sort them monotonously)

1	2	3	4	5
D	O	O	F	

B L O E D

1	2	3	4	5	6	7
S	A	U	D	O	O	F

D O O O F



Consider the last operation:

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- Solve **blue** part recursively

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DOOF

↓↓↓↓

BLOE

↓ insert

BLOED

Figure: Case 1a

DOOF

↓↓↓↓↓

BLOEDF

↓ delete

BLOED

Figure: Case 1b

DOOF

↓↓↓↓↓

BLOEF

↓ replace

BLOED

Figure: Case 1c



Consider the last operation:

Consider the last operation:

- Solve **blue** part recursively

Consider the last operation:

- Solve **blue** part recursively

W I N T E R



S O M M E R

↓ nothing

S O M M E R

Display of solution:

- Alignment

- Example:

—	—	—	B	L	O	E	D
S	A	U	B	L	O	E	D

Figure: Case 2



Dynamic programming:

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- Instances of Bellman's principle of optimality:

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Figure: Richard Bellman
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- Optimal solutions consist of optimal partial solutions
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Dynamic programming:

- Instances of Bellman's principle of optimality:
 - Shortest paths
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- Optimal solutions consist of optimal partial solutions
 - Shortest paths: Each partial path has to be optimal
 - Edit distance: Each partial alignment has to be optimal
- Always solvable through dynamic programming
(Caching of optimal partial solutions)



Figure: Richard Bellman
(1920 - 1984)

Case analysis:

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 - $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$ and $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

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- Let $n = |x|, m = |y|, m' = |z|$

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Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- Let $n = |x|$, $m = |y|$, $m' = |z|$
- We note $m' \in \{m-1, m, m+1\}$ why?



Case analysis:

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 - Case 1c: $\sigma_k = \text{replace}(m', y[m])$ [then $m' = m$]
- Case 2: σ_k does nothing at the outer end:
 - Then $z[m'] = y[m]$ and $x[n'] = z[m']$ and with that
 $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$ and $x[n] = y[m]$



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 - $ED(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$

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 - $ED(x[1..n-1], y) + 1$ and
 - $ED(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
 - $ED(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $ED(x, y) = |y|$

Case analysis:

- Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- Case 1b (delete): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
- Case 1c (replace): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (nothing): $\sigma_1, \dots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $ED(x, y)$ the minimum of
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- For $|y| = 0$ is $ED(x, y) = |x|$

```
def edit_distance(x, y):  
    if len(x) == 0:  
        return len(y)  
    if len(y) == 0:  
        return len(x)  
  
    ed1 = edit_distance(x, y[:-1]) + 1  
    ed2 = edit_distance(x[:-1], y) + 1  
    ed3 = edit_distance(x[:-1], y[:-1])  
    if x[-1] != y[-1]:  
        ed3 += 1  
  
    return min(ed1, ed2, ed3)
```



Recursive program:

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- The algorithm results in the following recursive formular:

$$\begin{aligned}T(n, m) &= T(n-1, m) + T(n, m-1) + T(n-1, m-1) + 1 \\&\geq T(n-1, m-1) + T(n-1, m-1) + T(n-1, m-1) \\&= 3 \cdot T(n-1, m-1)\end{aligned}$$

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⇒ The runtime is at least exponential



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Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

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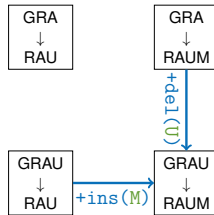
Visualization on the next slide:

- Operations always refer to the last position (indices are omitted)
- We also display the replaced character on a `replace` operation to visualize operations without costs
 $\Rightarrow \text{repl}(\text{A}, \text{A})$











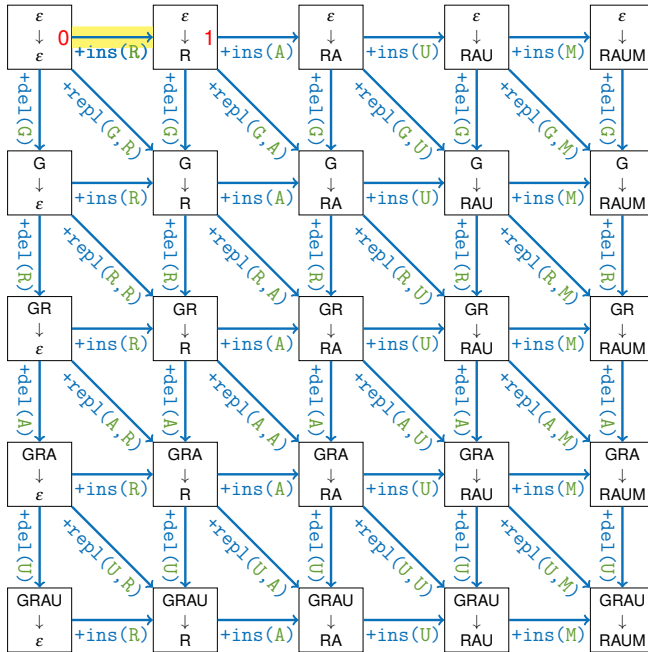




Fast algorithm:

We can determine the **edit distance** for all combination of partial strings from the top left to bottom right.











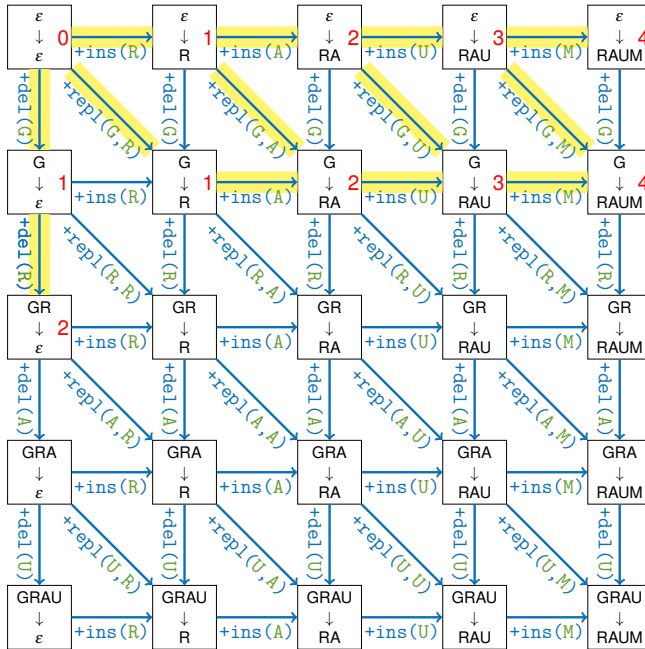










































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- We save at each recursion the most efficient previous entry (the **highlighted arrows** in our image)

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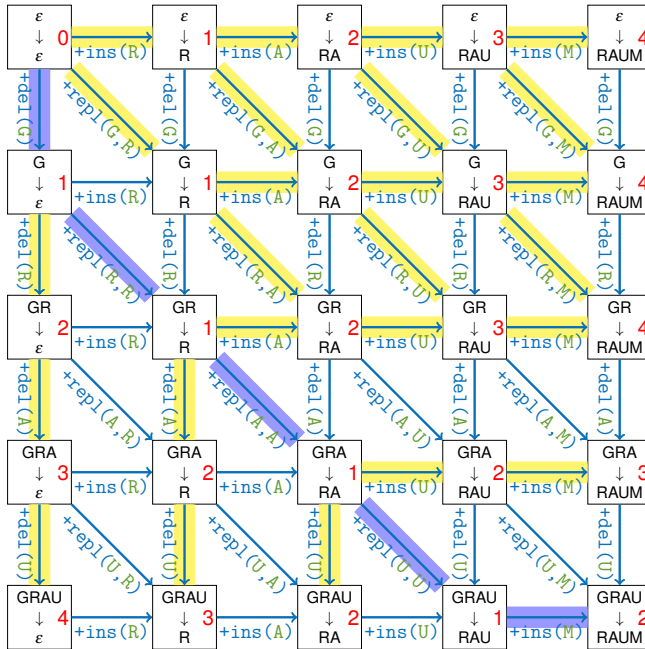
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- If we follow the highlighted path from (n, m) to $(1, 1)$ we get the optimum operations to transform x into y
 - If we can follow **more than one path** there exist more than one ideal **sequence**





General principle:

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- Recursive computation of ...
 - ... the same reoccurring partial problems
 - ... a limited number of partial problems

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- Recursive computation of ...
 - ... the same reoccurring partial problems
 - ... a limited number of partial problems
- Computation of the solutions for all partial problems
- In a order that unsolved partial problems consist of already solved partial problems
- The “path” to our solution normally gets computed while searching the best solution
- Dijkstra algorithm is basically dynamic programming!



Additional applications:

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- Solution in $O(n^3)$ time or $O(n^2)$ affine

$O(n^2)$ space consumption might be problematic:

Hirschberg algorithm:

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Hirschberg algorithm:

- Divide-and-conquer approach

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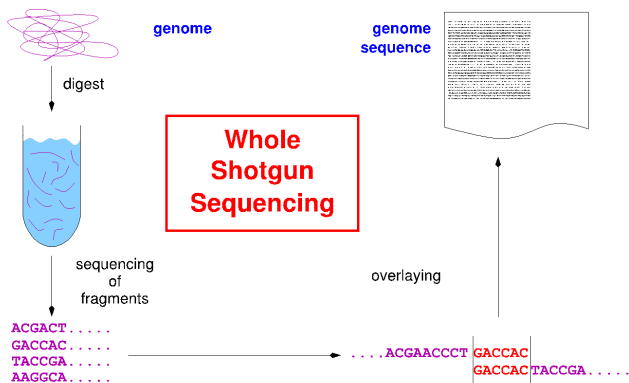
Hirschberg algorithm:

- Divide-and-conquer approach
- $O(n)$ space and $O(n^2)$ time consumption

Edit distance

Additional applications (III)





- Sequencing: $O(n^2)$ is too much



- Sequencing: $O(n^2)$ is too much
- Index: suffixtree, suffixarray, burrow-wheeler-transform

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ **Dynamic programming**

[Wik] [Dynamic programming](https://en.wikipedia.org/wiki/Dynamic_programming)

`https:`

`//en.wikipedia.org/wiki/Dynamic_programming`

■ **Edit distance**

[Wik] [Levenshtein distance](https://en.wikipedia.org/wiki/Levenshtein_distance)

`https:`

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