

# Algorithms and Datastructures

## Linked Lists, Binary Search Trees

Albert-Ludwigs-Universität Freiburg



**UNI  
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Algorithms and Datastructures, January 2017

Feedback

Exercises

Lecture

Sorted Sequences

Linked Lists

Binary Search Trees

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# Feedback from the exercises



- The few people who gave feedback wrote that it was simple to doable.

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- Missing support in forum

# Feedback from the lecture



- Added german lecture recordings to current semester page

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- Lecture recordings are now password protected

Feedback

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  - **lookup(key)**: Find the element with the given **key**, if it is not available find the element with the smallest key  $>$ **key**
  - **next()/previous()**: Returns the element with the next bigger/smaller **key**. This enables iteration over all elements.



## Application examples:

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- We do not want to sort all elements every time on an **insert** operation
- How could we implement this?

### Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

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- `next` / `previous` in time  $O(1)$ 
  - They are next to each other
- `insert` and `remove` up to  $\Theta(n)$ 
  - We have to copy up to  $n$  elements



**Hash map:**



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- `next` / `previous` in time up to  $\Theta(n)$ 
  - The order of the elements is independent of the order of the keys

# Sorted Sequences

## Implementation 3 (good?) - Linked List

### **Linked list:**



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- Lets have a closer look

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## Linked list:



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- Elements are linked through references / pointer to the predecessor / successor
- Single / Doubly linked lists possible

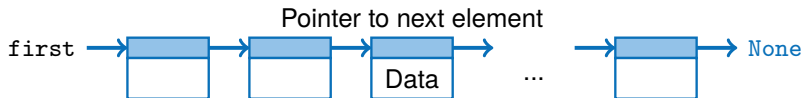


Figure: Linked list



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- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified
- No direct access of elements
  - ⇒ We have to iterate over the list



### List with head / last element pointer:

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Figure: Singly linked list



### List with head / last element pointer:

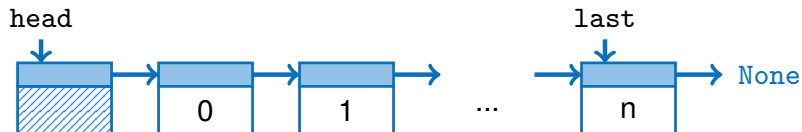


Figure: Singly linked list

- Head element has pointer to first list element

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- May also hold additional information:
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## Doubly linked list:

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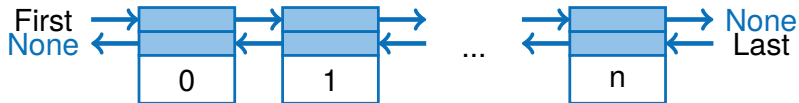


Figure: Doubly linked list

- Pointer to successor element

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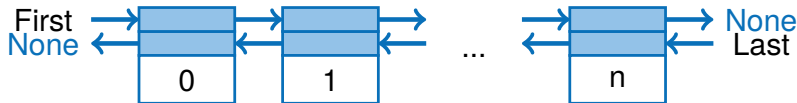


Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element

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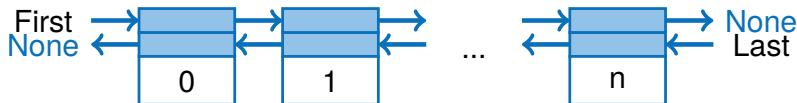


Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward





```
public class Listelem
```



```
public class Listelem  
{    //2 fields: integer and reference
```



```
public class Listelem
{
    //2 fields: integer and reference
    //private only available in class
    private int data;
    private Listelem next;
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    //2 constructors: for instance of class
    public Listelem(int d)
    { data = d; next = null; }
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    //adopted from Mary K.Vernon
    //Introduction to Data Structures
```

# Linked Lists

Implementation - Node/Element - Java



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```
//Function to read and write private fields  
public int getData() {return data; }  
public void setData(int d) { data = d; }
```



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public int getData() {return data; }  
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//Integer represents possible data, e.g.
//self defined reference datatypes
}
```

# Linked Lists

## Implementation - Node/Element - C++



```
class Listelem  
{
```



```
class Listelem
{
private:
    int data;
    Listelem* next;
```

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    int data;
    Listelem* next; //Pointer instead of reference
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# Linked Lists

Implementation - Node/Element - C++



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```
Listelem* getNext() { return next; }  
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}
```

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

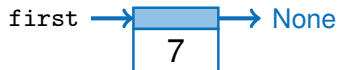
    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```



## Creating linked lists - Python:

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■ first = Node(7)
```



### Creating linked lists - Python:

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■ `first.nextNode = Node(3)`



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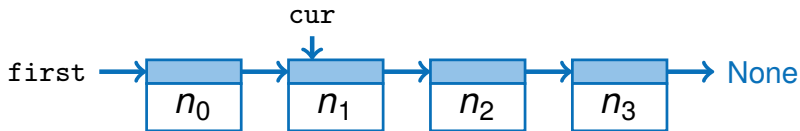
■ `first.nextNode = Node(3)`



■ `first.nextNode.value = 4`



Inserting a node after node `cur`:





**Inserting a node after node `cur`:**



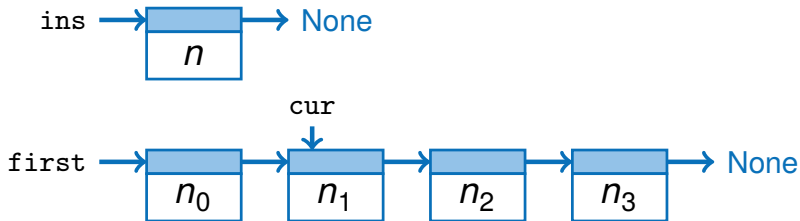


### Inserting a node after node `cur`:

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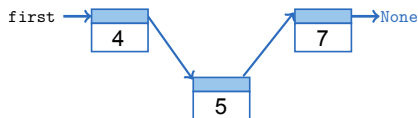


■ `cur.nextNode = Node (value ,cur.nextNode )`

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**Removing a node** `cur`:





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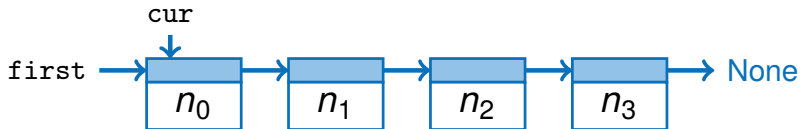
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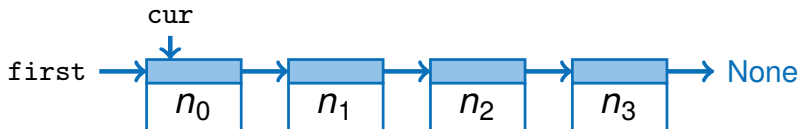
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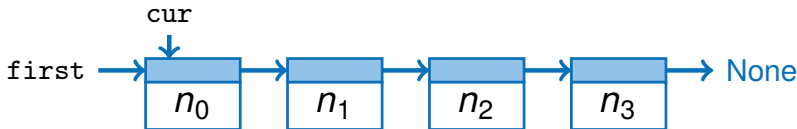
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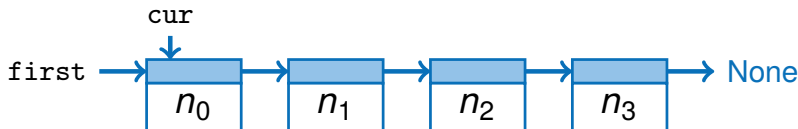
```
first = first.nextNode
```

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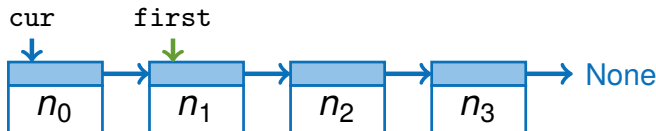


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### Removing a node `cur`: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```



### Using a head node:



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- Advantage:



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  - Iterating all nodes
  - Counting of all nodes

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```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```

```
def append(self, value):  
    ...  
  
def insertAfter(self, cur, value):  
    ...  
  
def remove(self, cur):  
    ...  
  
def get(self, position):  
    ...  
  
def contains(self, value):  
    ...
```

```
/**
 * A singly linked list with data type int.
 */
public class LinkedList {

    private long itemCount;
    private Node head;
    private Node last;

    public LinkedList() {
        itemCount = 0;
        head = new Node();
        last = head;
    }
}
```

```
    public int size() {  
        return itemCount;  
    }  
  
    public boolean isEmpty() {  
        return (itemCount == 0);  
    }  
  
    public void add (int data) { ... }  
    public void insertAfter(Node cur, int data)  
        { ... }  
    public void remove(Node cur) { ... }  
    public Node get(int position) { ... }  
    public boolean contains( int data) { ... }  
}
```



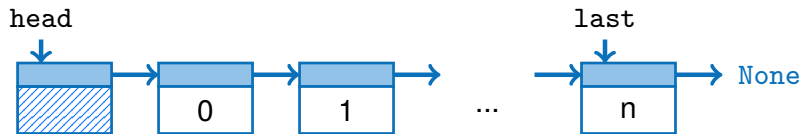
**Head, last:**



### Head, last:



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### Head, last:



- Head points to the first node, last to the last node
- We can append elements to the end of the list in  $O(1)$  through the last node
- We have to keep the pointer to last updated after all operations

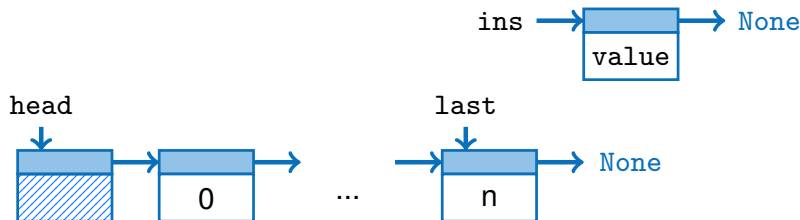


### Appending an element:

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```
■ def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

### Appending an element:



- ```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```
- The pointer to `last` avoids the iteration of the whole list



### Inserting after node `cur`:





**Inserting after node `cur`:**



### Inserting after node `cur`:

- The pointer to `head` is not modified

### Inserting after node cur:

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- ```
def insertAfter(self, cur, value):  
    if cur == last:  
        # also update last node  
        append(value)  
    else:  
        # last node is not modified  
        cur.nextNode = Node(value, \  
                             cur.nextNode)  
        itemCount += 1
```

### Remove node cur:





**Remove node** `cur:`



### **Remove node** `cur`:

- Searching the predecessor in  $O(n)$

### Remove node cur:

- Searching the predecessor in  $O(n)$

```
def remove(self, cur):  
    pre = first  
    while pre.nextNode != cur:  
        pre = pre.nextNode  
  
    pre.nextNode = cur.nextNode  
    itemCount -= 1  
  
    if pre.nextNode == None:  
        last = pre
```





**Getting a reference to node at `pos`:**



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### Getting a reference to node at `pos`:

- Iterate the entries of the list until at position ( $O(n)$ )

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```
def get(self, pos):  
    if pos < 0 or pos >= itemCount:  
        return None  
  
    cur = head  
    for i in range(0, pos):  
        cur = cur.nextNode  
  
    return cur
```



### Searching a value:



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- First element is head without an assigned value

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```
def contains(self, value):  
    cur = head  
  
    for i in range(0, itemCount):  
        cur = cur.nextNode  
        if cur.value == value:  
            return true  
  
    return false
```





**Runtime:**



### Runtime:

- Singly linked list:



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  - `next` in  $O(1)$



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- Better with `doubly linked lists`





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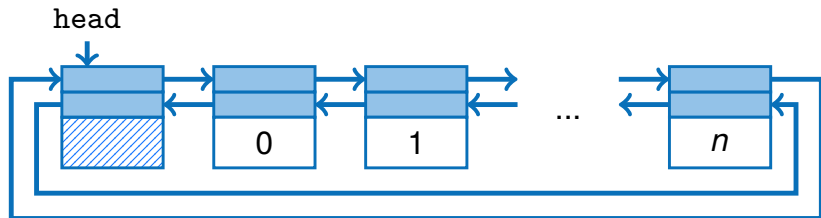
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    - a constant number of pointers needs to be modified
  - `lookup` in  $\Theta(n)$ 
    - Even if the elements are sorted we can only retrieve them in  $\Theta(n)$ .  
Why?



## Linked list in book:



# Linked Lists

List in real program



## Linked list in memory:



Feedback

Exercises

Lecture

Sorted Sequences

Linked Lists

Binary Search Trees



### **Runtime of a search tree:**



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  - The structure helps searching efficiently



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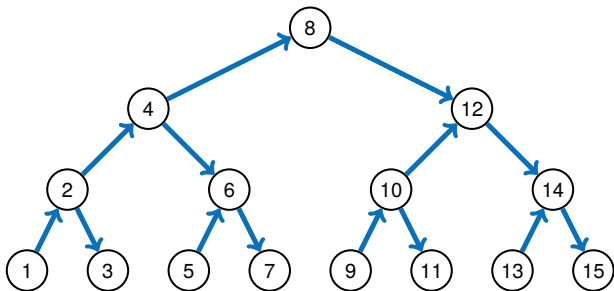


Figure: A binary search tree





Figure: Another binary search tree

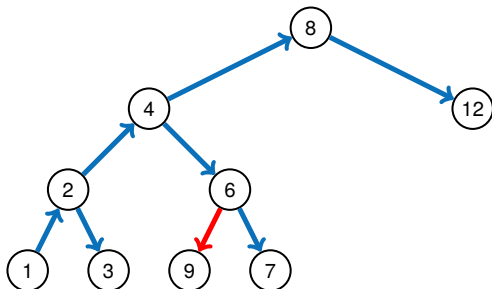


Figure: **Not** a binary search tree



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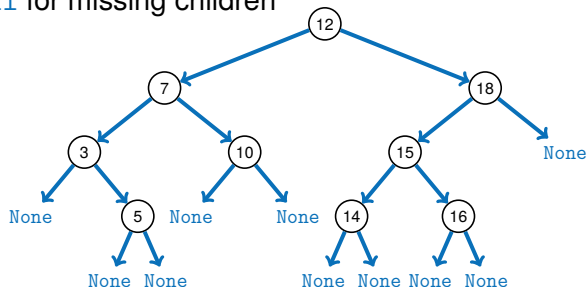


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Figure: Binary search tree with links



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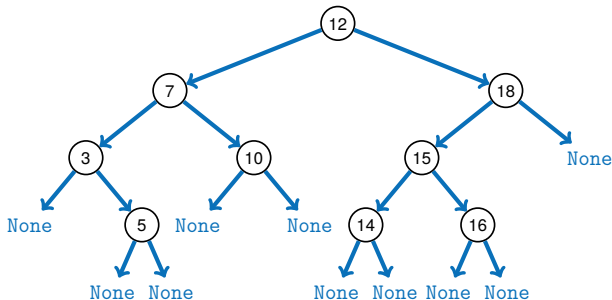
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■ `lookup(14)`

■ `lookup(6)`

■ `lookup(19)`

Figure: Binary search tree with total order “<”

# Binary Search Trees

## Implementation - Insert



### Insert:



### **Insert:**

- We search for the key in our search tree



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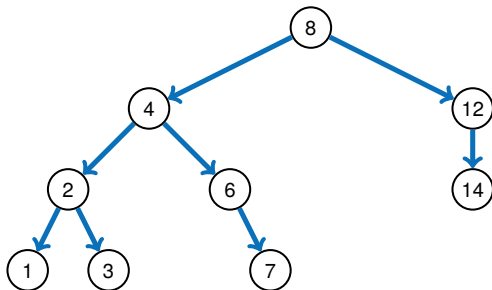
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Figure: Binary search tree with total order “<”

**Remove:** Case 1: The node “5” has no children

- Find **parent** of node “5” (“6”)
- Set left / right child of node “6” to **None** depending on position of node “5”



**Figure:** Binary search tree after deleting node “5”



**Remove:** Case 2: The node “12” has one child



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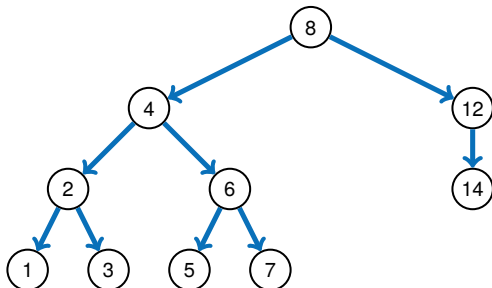


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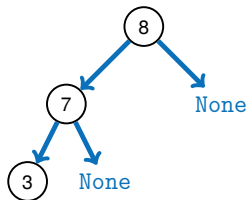
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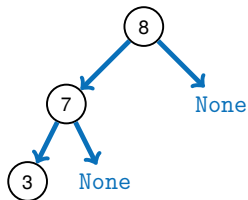


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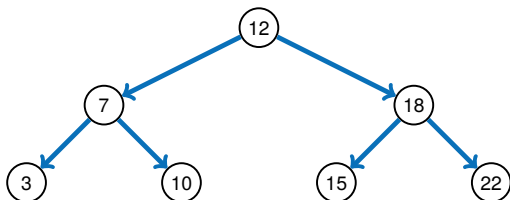


Figure: Complete binary tree  $d = \log n$

## ■ General

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## ■ **Linked List**

[Wik] [Linked list](#)

`https://en.wikipedia.org/wiki/Linked\_list`

## ■ **Binary Search Tree**

[Wik] [Binary search tree](#)

`https://en.wikipedia.org/wiki/Binary\_search\_tree`