

# Algorithms and Datastructures

## Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg



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Algorithms and Datastructures, March 2018

## Divide and Conquer

- Concept

- Maximum Subtotal

## Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

  - Master theorem (Simple Form)

  - Master theorem (General Form)

# Divide and Conquer

## Introduction



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## Concept:



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If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
- **Recursive** application of the algorithm on smaller subproblems
- **Direct** solving of small subproblems



## Divide and Conquer

Concept

Maximum Subtotal

## Recursion Equations

Substitution Method

Recursion Tree Method

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# Divide and Conquer

Maximum Subtotal



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**Input:**

**Output:**

# Divide and Conquer

## Maximum Subtotal

### Input:

- Sequence  $X$  of  $n$  integers

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Table: Input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

**Output:** Sum: 187, Start: 2, End: 6

**Idea:**



### Idea:



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- Solve the left / right half of the problem **recursive**
- Combine both solutions into a overall solution
- The maximum is located in the **left half (A)** or the **right half (B)**
- The maximum interval can **overlap with the border (C)**

### Principle:

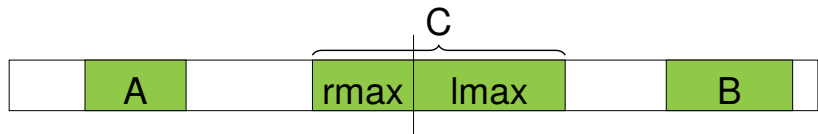


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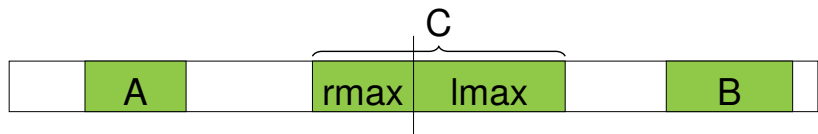
- Small problems are solved directly:  $n = 1 \Rightarrow \text{max} = X[0]$

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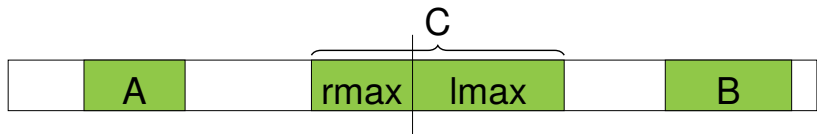
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- Big problems are decomposed into two subproblems and solved recursively. Subsolutions  $A$  and  $B$  are returned

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### Principle:



- Small problems are solved directly:  $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions  $A$  and  $B$  are returned
- To solve  $C$  we have to calculate  $\text{rmax}$  and  $\text{lmax}$
- Overall solution is maximum of  $A$ ,  $B$  and  $C$

# Divide and Conquer

## Maximum Subtotal - Python



```
def maxSubArray(X, i, j):
```



# Divide and Conquer

## Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2
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    A = maxSubArray(X, i, m)  
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    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])
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def maxSubArray(X, i, j):  
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    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

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#Alternative trivial case  
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## Maximum Subtotal - Python



```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

# Divide and Conquer

## Maximum Subtotal - Python



```
#Implementation max  
def max(a, b, c):
```



# Divide and Conquer

## Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
```

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

# Divide and Conquer

## Maximum Subtotal - Python



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#Alternative implementation max
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def max(a, b):  
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```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b  
  
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	$i$	$i + 1$	$\dots$	$\dots$	$j - 1$	$j$
$X$	58	-53	26	59	-41	31
<i>sum</i>	58	5	31	90	49	80
<i>lmax</i>	58	58	58	90	90	90



Table: *lmax* example

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- The *sum* and *lmax* are initialized with  $X[i]$

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- The *sum* and *lmax* are initialized with  $X[i]$
- We iterate over  $X$  from  $i + 1$  to  $j$  and update *sum*

Table:  $lmax$  example

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- The  $sum$  and  $lmax$  are initialized with  $X[i]$
- We iterate over  $X$  from  $i + 1$  to  $j$  and update  $sum$
- If  $sum > lmax$  then  $lmax$  gets updated

# Divide and Conquer

## Maximum Subtotal



Call with array of four elements

`maxSubArray(-3,9,-4,7)`

in fact:

`maxSubArray(X,0,3)`  
with `X=[-3,9,-4,7]`

# Divide and Conquer

## Maximum Subtotal



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in fact:

$\text{maxSubArray}(X, 0, 3)$   
with  $X = [-3, 9, -4, 7]$

Call with array of two elements

$\text{maxSubArray}(-3, 9)$

$\text{maxSubArray}(-4, 7)$

# Divide and Conquer

## Maximum Subtotal



Call with array of four elements

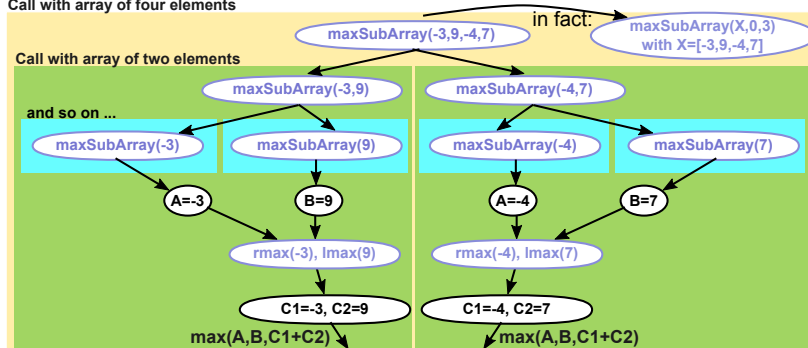


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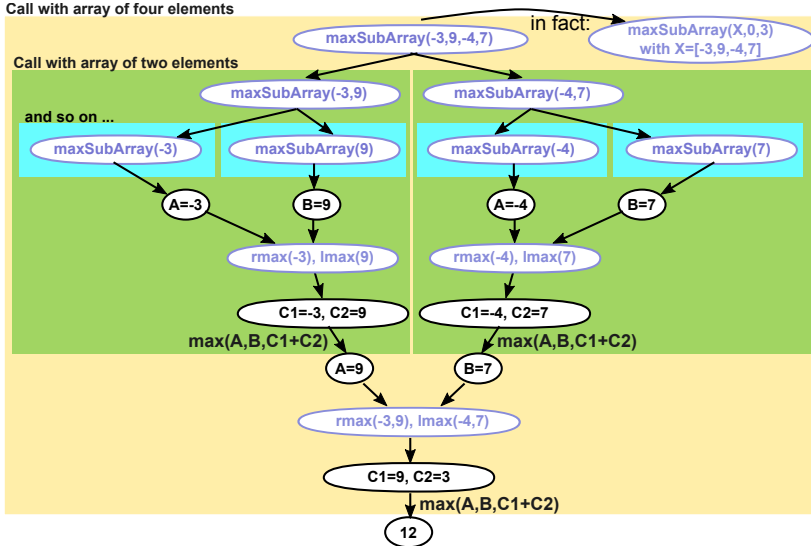
# Divide and Conquer

## Maximum Subtotal



Call with array of four elements

Call with array of two elements





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def maxSubArray(X, i, j):  
    if i == j:  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
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def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
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    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
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    A = maxSubArray(X, i, m)                       # T(n/2)  
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    C = (C1[0] + C2[0], C1[1], C2[1])              # O(1)  
  
    return max([A, B, C], \                       # O(1)  
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```

## Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

trivial case

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$\underbrace{\Theta(1)}_{\text{trivial case}}$

- There exist two constants  $a$  and  $b$  with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

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- There exist two constants  $a$  and  $b$  with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define  $c := \max(a, b)$ :

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

# Divide and Conquer

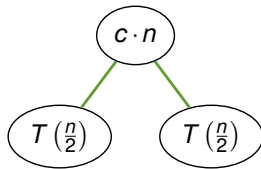
## Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime



# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

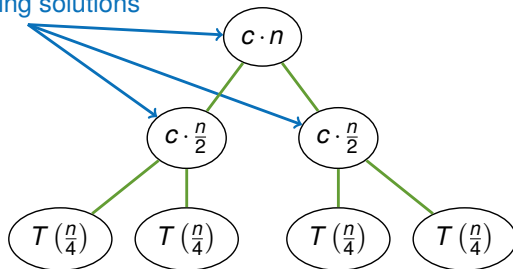
Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



combining solutions



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$

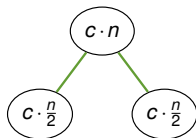
$$c \cdot n$$

1 node processing  $n$  elements  
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



1 node processing  $n$  elements  
 $\Rightarrow c \cdot n$

2 nodes processing  $\frac{n}{2}$  elements  
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



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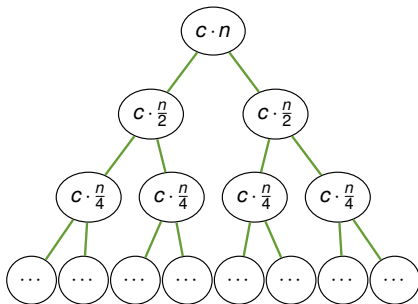
2 nodes processing  $\frac{n}{2}$  elements  
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing  $\frac{n}{4}$  elements  
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



1 node processing  $n$  elements  
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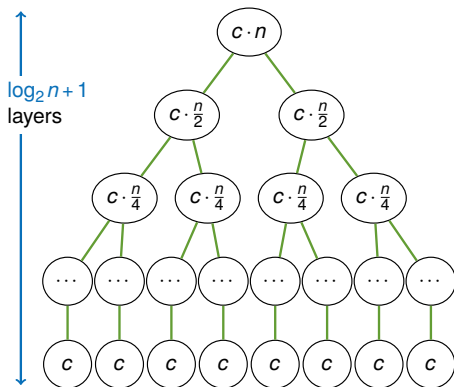
4 nodes processing  $\frac{n}{4}$  elements  
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$2^j$  nodes processing  $\frac{n}{2^j}$  elements  
 $\Rightarrow 2^j c \cdot \frac{n}{2^j} = c \cdot n$

Figure: Recursion tree method

# Divide and Conquer

## Maximum Subtotal - Illustration of $T(n)$



1 node processing  $n$  elements  
 $\Rightarrow c \cdot n$

2 nodes processing  $\frac{n}{2}$  elements  
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4 nodes processing  $\frac{n}{4}$  elements  
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

$2^i$  nodes processing  $\frac{n}{2^i}$  elements  
 $\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$

$n$  nodes processing 1 element  
 $\Rightarrow c \cdot n$

Figure: Recursion tree method



# Divide and Conquer

Maximum Subtotal - Illustration of  $T(n)$



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**Depth:**

# Divide and Conquer

Maximum Subtotal - Illustration of  $T(n)$



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## Maximum Subtotal - Summary



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- There is an approach running in  $\mathcal{O}(n)$  if you assume that all subtotals are positive



Figure: Scanning the array in linear time

# Divide and Conquer

## Maximum Subtotal - Python



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        if rMax > tMax:  
            tMax, itMax = rMax, irMax  
  
    return (tMax, itMax)
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## Divide and Conquer

Concept

Maximum Subtotal

## Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

### Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$



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- $T(n)$  is only defined for integers of  $\frac{n}{b}$  which is often ignored in benefit of a simpler solution



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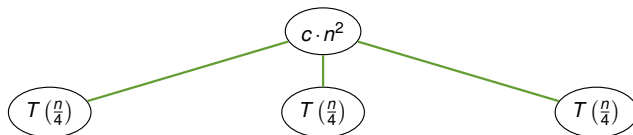
$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

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Figure: Recursion tree of example

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Figure: Recursion tree of example



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$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

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# Recursion Equations

## Recursion Tree Method



Figure: Levels of the recursion tree



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uses  $x^{a \cdot b} = (x^a)^b$

- This term will recur in the master theorem

# Recursion Equations

Total costs



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$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left( \begin{array}{c} \text{even with} \\ \text{infinite elements} \end{array} \right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

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- Here: The costs of connecting the partial problems dominate

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Quotient of two neighboring sequence parts is constant

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- For  $|q| < 1$ :

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \quad \Rightarrow \text{constant}$$

# Recursion Equations

Proof of  $O(n^2)$



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**Proof of  $\mathcal{O}(n^2)$ :**

### Proof of $\mathcal{O}(n^2)$ :

■ We know:

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{4}\right) + \Theta(n^2) \\ &\leq 3T\left(\frac{n}{4}\right) + c \cdot n^2 \end{aligned}$$

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- Assumption:  $T(n) \in \mathcal{O}(n^2)$ , so there exists a  $k > 0$  with

$$T(n) \leq k \cdot n^2$$



# Recursion Equations

Proof of  $O(n^2)$



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**Proof of  $\mathcal{O}(n^2)$ :**



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## Divide and Conquer

Concept

Maximum Subtotal

## Recursion Equations

Substitution Method

Recursion Tree Method

### Master theorem

Master theorem (Simple Form)

Master theorem (General Form)





## Master theorem:

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  - ... which takes  **$f(n)$**  steps to merge all partial solutions



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- **Simple form:** Special case with runtime of connecting the solutions  $f(n) \in O(n)$

# Recursion Equations

## Master theorem (Simple Form)



**Simple form:**

### Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n)}, \quad a \geq 1, b > 1, c > 0$$

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- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$



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Figure: Simple recursion equation with  $a = 3, b = 2$

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### Case 1: $a > b$

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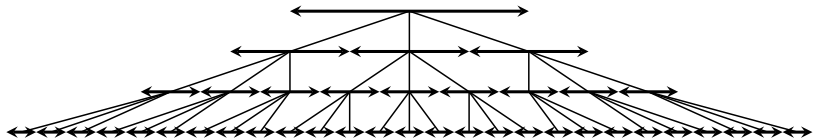


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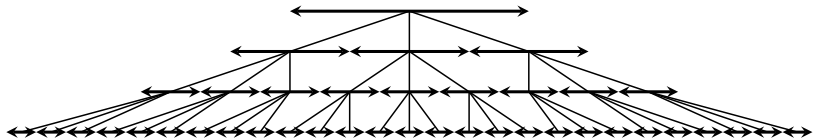


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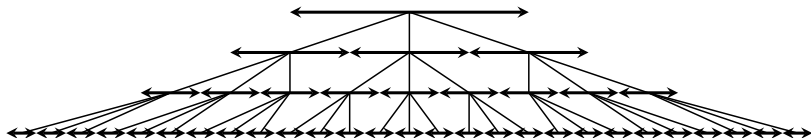


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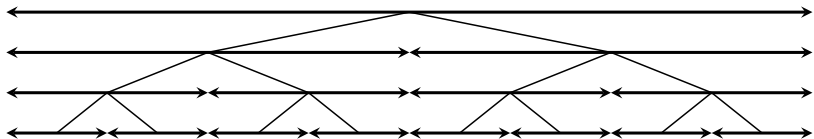


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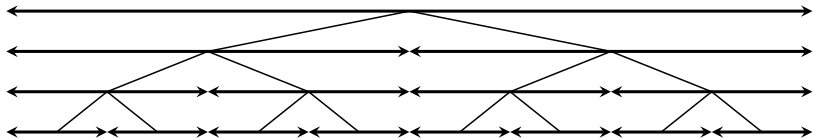


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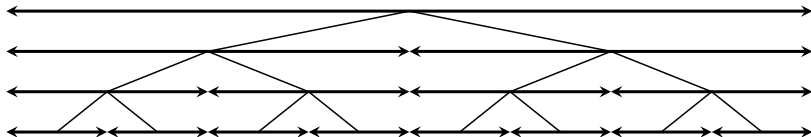


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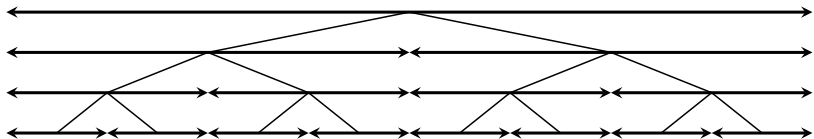


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Figure: Simple recursion equation with  $a = 2, b = 3$

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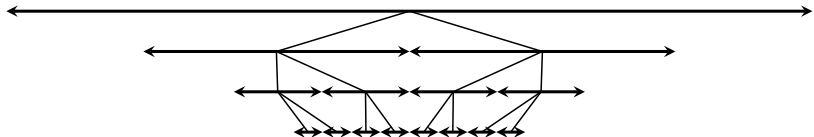


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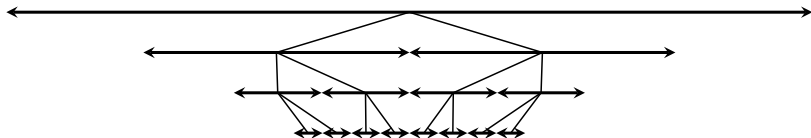


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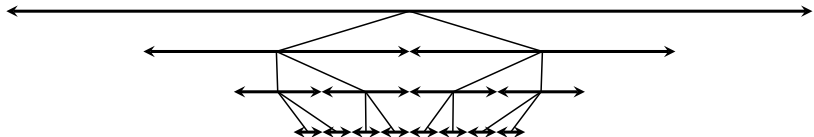


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- Runtime of  $\Theta(n)$

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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor  $\frac{a}{b}$

## Divide and Conquer

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**Master theorem**

Master theorem (Simple Form)

Master theorem (General Form)

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- **Case 1:**  $T(n) \in \Theta(n^{\log_b a})$  if  $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$ ,  $\varepsilon > 0$

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- **Case 2:**  $T(n) \in \Theta(n^{\log_b a} \log n)$  if  $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs,  $\log_b n$  layers



### Master theorem (general form):

- **Case 3:**  $T(n) \in \Theta(f(n))$  if  $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ ,  $\varepsilon > 0$

Connecting all partial solutions in first layer (root)  
dominates

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$



### Case 1 - Example:

if  $f(n) \in O(n^{\log_b a - \epsilon})$ ,  $\epsilon > 0$

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Solving the partial problems dominates (last layer, leaves)

$$\blacksquare T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

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■  $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, b = 3, f(n) = 17 \cdot n, \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$



**Case 2:** if  $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs,  $\log n$  layers

# Recursion Equations

## Master theorem (General Form) - Case 2



**Case 2:**  $T(n) \in \Theta(n^{\log_b a} \log n)$  if  $f(n) \in \Theta(n^{\log_b a})$

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■  $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 10 \cdot n$

$$a = 2, b = 2, f(n) = 10 \cdot n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$



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■  $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1, b = \frac{2}{3}, f(n) = 1, \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

**Case 3:** if  $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ ,  $\varepsilon > 0$   
Connecting all partial solutions in first layer (root) dominates

**Case 3:**  $T(n) \in \Theta(f(n))$  if  $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ ,  $\varepsilon > 0$

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$$a = 2, b = 2, f(n) = n^2, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Omega(n^{1+\varepsilon})$$

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Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if **regularity condition** also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$





### **Master theorem:**



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$n \log n$  is *asymptotically* larger than  $n$ ,  
but not *polynomial* larger



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- **Case 2:** Each layer has equal costs

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$$T(n) \in \Theta(n^{\log_b a} \log n), \quad \log n \text{ layers}$$

- **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems

$$T(n) \in \Theta(f(n))$$

## ■ General

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## ■ Master theorem

[Wik] [Master theorem](#)

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