

Algorithmns and Datastructures

Hash Map, Universal Hashing

Albert-Ludwigs-Universität Freiburg



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Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithmns and Datastructures, November 2016

Feedback

- Exercises
- Lecture

Associative Arrays

- Introduction
- Hash Map

Universal Hashing

- Introduction
- Probability Calculation
- Proof
- Examples

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- For most using associative arrays (Python dictionary) tends to be a bit faster
 - Sorting the dictionary also takes time, depending on heterogeneity of the data (e.g. lots of locality names with

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Problem:

- Quickly find a element with a specific key
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- For n keys searching requires $\Theta(n)$ time
- With a Hash Map this just requires $\Theta(1)$ in the best case, ... regardless how many elements are in the map!

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Idea:

- Mapping the keys onto indices with a [hash function](#)
- Store the values at the calculated indices in a normal array

Example:

- Key set: $x = \{3904433, 312692, 5148949\}$

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- We need an array **T** with **5** elements.
A "hashtable" with 5 "buckets"

Idea:

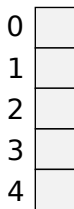
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Example:

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- Hash function: $h(x) = x \bmod 5$, in the range $[0, \dots, 4]$
- We need an array **T** with **5** elements.
A "hashtable" with 5 "buckets"
- The element with the key **x** is stored in **$T[h(x)]$**

Storage:

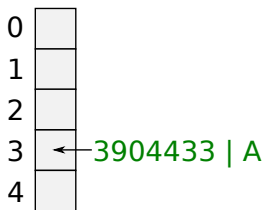
Figure: Hashtable T



Storage:

- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$

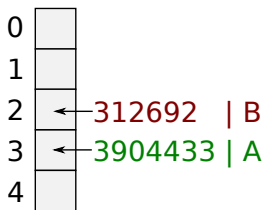
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Storage:

- $\text{insert}(3904433, "A")$: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- $\text{insert}(312692, "B")$: $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$

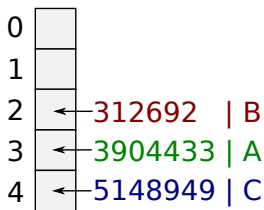
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- $\text{insert}(3904433, "A")$: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- $\text{insert}(312692, "B")$: $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$
- $\text{insert}(5148949, "C")$: $h(5148949) = 4 \Rightarrow T[4] = (5148949, "C")$

Figure: Hashtable T



Searching:

- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$

Figure: Hashtable T

0	
1	
2	← 312692 B
3	← 3904433 A
4	← 5148949 C

Searching:

- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$
- $\text{search}(123459): h(123459) = 4 \Rightarrow T[4]$
 \Rightarrow Value with key 123459 does not exist

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- $\text{search}(123459): h(123459) = 4 \Rightarrow T[4]$
 \Rightarrow Value with key 123459 does not exist
- Search time for this example: $\mathcal{O}(1)$

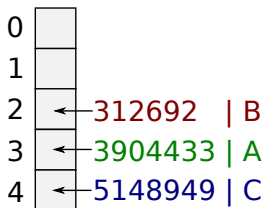
Figure: Hashtable T

0	
1	
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3	← 3904433 A
4	← 5148949 C

Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$

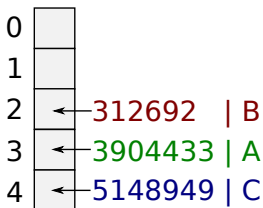
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Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$
 $\Rightarrow T[3] = (876543, "D")$ **COLLISION!**

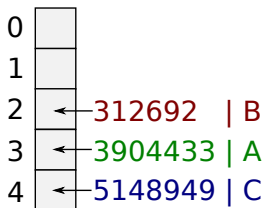
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Further inserting:

- $\text{insert}(876543, \text{"D"})$: $h(876543) = 3$
 $\Rightarrow T[3] = (876543, \text{"D"})$ **COLLISION!**
- This happens more often than expected
 - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

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Easiest Solution:

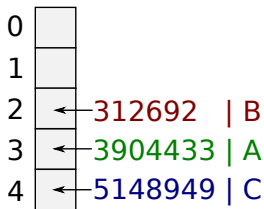
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Easiest Solution:

- Represent each bucket as list of key value pairs

Figure: Hashtable T



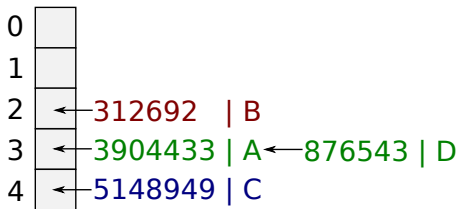
Problem:

- Two keys are equal $h(x) = h(y)$ but not the values $x \neq y$

Easiest Solution:

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

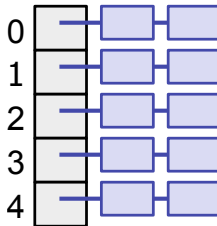
Figure: Hashtable T



Best case:

- We have n keys which are equally distributed over m buckets
- We have $\approx \frac{n}{m}$ pairs per bucket

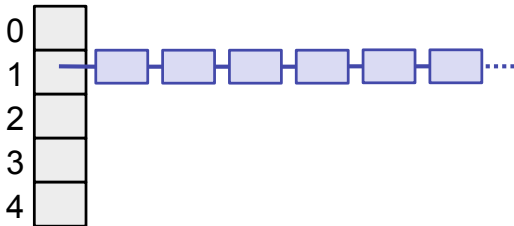
Best case ($m = 5, n = 10$)



Worst case:

- All n keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching

Worst case
($m = 5, n = 10$)



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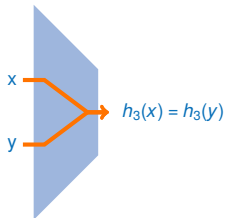
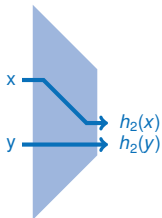
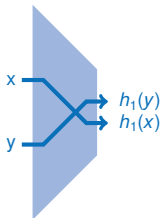
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- Find a set of keys so that it results in a degenerated hash table
 - *you may use the hash function*
 - *for table size 100: try $100 \times 99 + 1$ different numbers*
 - *worst case: still 100 must have same hash bucket*
- **Now:** Find a solution to avoid that problem

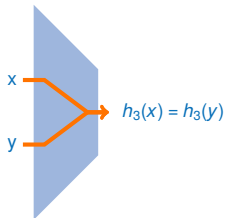
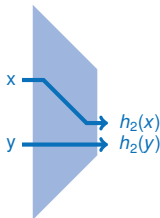
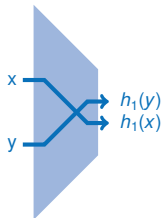
Solution:

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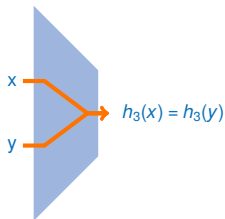
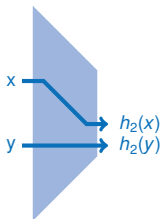
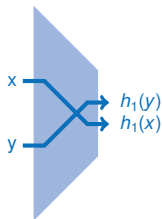
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- We choose a random hash function so that the **expected result** is an equal distribution over the buckets
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Solution:

- We use a set of hash functions
- We choose a random hash function so that the **expected result** is an equal distribution over the buckets
this is fixed for the lifetime of table
optional: copy table with new hash when degenerated
- This is called **universal hashing**



Definition:

- We call \mathcal{U} the set (universum) of possible keys, and $\mathcal{S} \subseteq \mathcal{U}$ the set of used keys

Key universe \mathcal{U}



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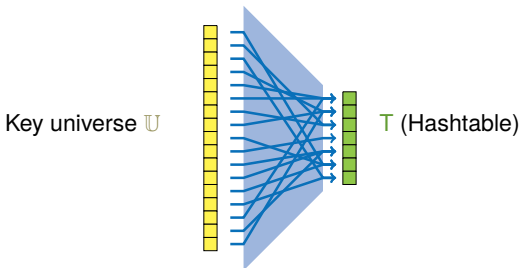


T (Hashtable)



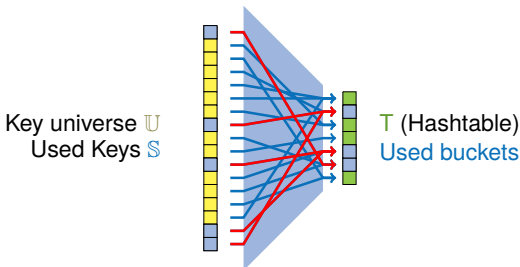
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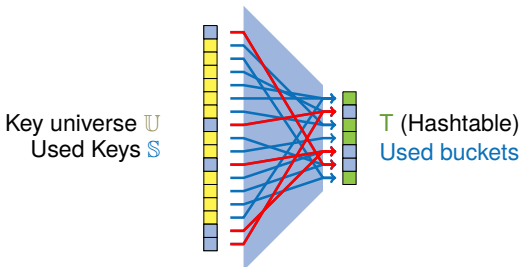
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- Idea: runtime should be $O(1 + \frac{|\mathcal{S}|}{m})$, where $\frac{|\mathcal{S}|}{m}$ is the table load



- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$

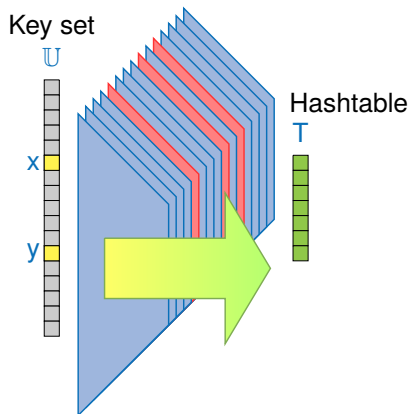


Figure: Set of hash functions \mathbb{H}

- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- An average of 3 out of 15 functions produce collisions

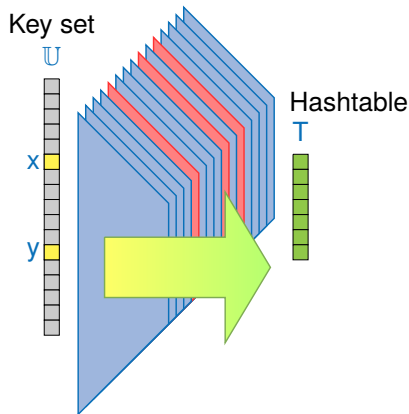


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Definition: \mathbb{H} is c -universal if $\forall x, y \in \mathbb{U} \mid x \neq y :$

Number of hash functions that create collisions

$$\frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}, \quad c \in \mathbb{R}$$

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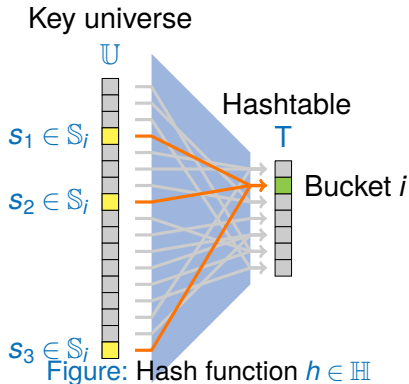
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Note: If the hash function assigns each key x and y randomly to buckets then:

$$\text{Prob}(\text{Collision}) = \frac{1}{m} \Leftrightarrow c = 1$$

- \mathbb{U} : Key universe
- \mathbb{S} : Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: Keys mapping to Bucket i (“synonyms”)
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$





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- Particularity: If $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

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Table: Throwing a dice

e	$P(e)$
1	$1/6$
2	$1/6$
3	$1/6$
4	$1/6$
5	$1/6$
6	$1/6$



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- Rolling a dice twice ($\Omega = \{1, \dots, 6\}^2$)
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- $E =$ if both eye numbers even, then $P(E) =$

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...
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(6, 6)	$1/36$	12

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 - Assigns a number to the result of an experiment
 - For example: X = Sum of eye numbers for rolling twice
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 - Example 1: $P(X = 2) =$
 - Example 2: $P(X = 4) =$

Table: Throwing a dice twice

e	$P(e)$	X
(1, 1)	$1/36$	2
(1, 2)	$1/36$	3
(1, 3)	$1/36$	4
...
(6, 5)	$1/36$	11
(6, 6)	$1/36$	12

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Sum of expected values: For independent (discrete) result variables X_1, \dots, X_n we can write:

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n)$$

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- X_2 : Expected number of eyes dice 2: $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$: Expected total number of eyes:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$



Corollary:

The probability of the event E is $p = P(E)$. Let X be the occurrences of the event E and n be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurrences of } 6) = \frac{1}{6} \cdot 60 = 10$$



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Def. \mathbb{E} -value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

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To proof:

$$\mathbb{E}[|S_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \quad \forall i$$



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$$\Rightarrow \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

Auxiliary calculation:

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- Which x, y lead to a relative collision count bigger than $\frac{c}{m}$?



Positive example:

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- Let \mathcal{H} be the set of all h for which:

$$h_{a,b}(x) = ((a \cdot x + b) \bmod p) \bmod m,$$

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- Exercise: show empirically that it is 2-universal



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- **Intuitive:** Scalar product with base m

$$a \bullet x = \sum_{0, \dots, k-1} a_i \cdot x_i$$

Example ($\mathbb{U} = \{0, \dots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned}h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10\end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned}h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7\end{aligned}$$

■ General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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[MS08] Kurt Mehlhorn and Peter Sanders.

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Hash Map - Theory

[Wik] [Hash table](#)

https://en.wikipedia.org/wiki/Hash_table

■ Hash Map - Implementations / API

[Cpp] [C++ - hash_map](#)

http://www.sgi.com/tech/stl/hash_map.html

[Jav] [Java - HashMap](#)

<https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html>

[Pyt] [Python - Dictionaries \(Hash table\)](#)

https://en.wikipedia.org/wiki/Hash_table