Algorithmns and Datastructures

Hash Map, Universal Hashing

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
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Structure

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Feedback from the exercises

- Time effort from manageable to not manageable at all

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- For most using associative arrays (Python dictionary) tends to be a bit faster
  $\rightarrow$ Sorting the dictionary also takes time, depending on heterogeneity of the data (e.g. lots of locality names with

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Problem:

- Quickly find a element with a specific key
- Naive solution: Store pairs of key and value in a normal field
- For n keys searching requires $\Theta(n)$ time
- With a Hash Map this just requires $\Theta(1)$ in the best case, ... regardless how many elements are in the map!

Structure

Idea:

- Mapping the keys onto indices with a hash function
- Store the values at the calculated indices in a normal array

Example:

- Key set: $x = \{3904433, 312692, 5148949\}$

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- We need an array T with 5 elements.
  A "hashtable" with 5 "buckets"

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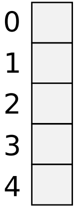
- Hash function: $h(x) = x \bmod 5$, in the range $[0, \ldots, 4]$
- We need an array T with 5 elements.
  A "hashtable" with 5 "buckets"
- The element with the key x is stored in $T[h(x)]$

Storage:

Figure: Hashtable T



Storage:

- insert(3904433,"A"): $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")

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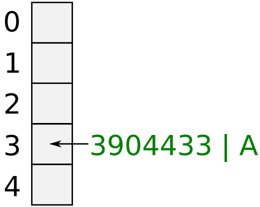

Storage:

- insert(3904433,"A"): h(3904433) = 3 $\Rightarrow$ T[3] = (3904433, "A")
- insert(312692, "B"): h(312692) = 2 $\Rightarrow$ T[2] = (312692, "B")

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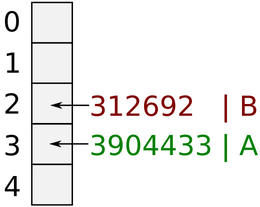

Storage:

- insert(3904433,"A"): $h(3904433) = 3 \Rightarrow T[3] =$ (3904433, "A")
- insert(312692, "B"): $h(312692) = 2 \Rightarrow T[2] =$ (312692, "B")
- insert(5148949, "C"): $h(5148949) = 4 \Rightarrow T[4] =$ (5148949, "C")

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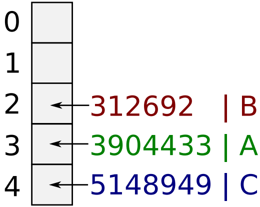

Searching:

- search(3904433): $h(3904433) = 3 \Rightarrow$ T[3] $\rightarrow$ (3904433, "A")

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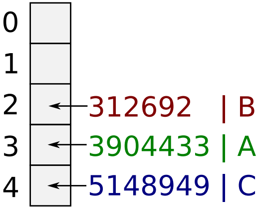

Searching:

- search(3904433): $h(3904433) = 3 \;\Rightarrow$ T[3] $\rightarrow$ (3904433, "A")
- search(123459): $h(123459) = 4 \;\Rightarrow$ T[4]

  $\Rightarrow$ Value with key 123459 does not exist

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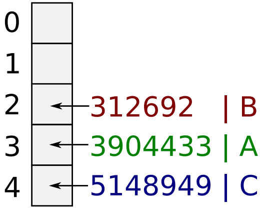


Associative Arrays

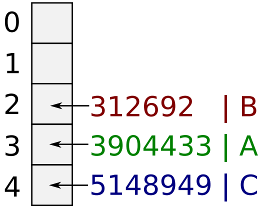
The Hash Map

Searching:

- search(3904433): $h(3904433) = 3 \;\Rightarrow$ T[3] $\rightarrow$ (3904433, "A")
- search(123459): $h(123459) = 4 \;\Rightarrow$ T[4]

  $\Rightarrow$ Value with key 123459 does not exist
- Search time for this example: $\mathcal{O}(1)$

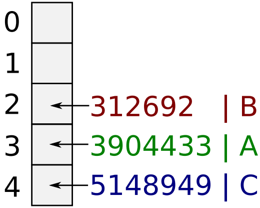
Figure: Hashtable T



Further inserting:

- insert(876543, "D"): $h(876543) = 3$

Figure: Hashtable T



Associative Arrays

Hash Collisions

Further inserting:

- insert(876543, "D"): $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") COLLISION!

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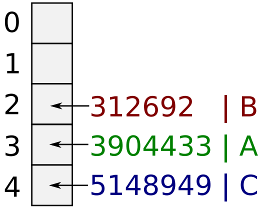

Further inserting:

- insert(876543, "D"): $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") COLLISION!
- This happens more often than expected
  - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

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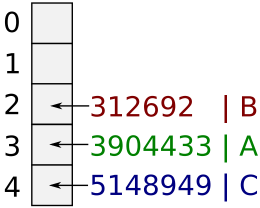

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Easiest Solution:

- Represent each bucket as list of key value pairs

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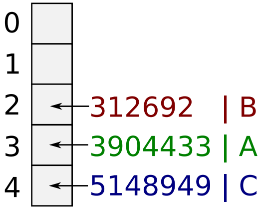

Problem:

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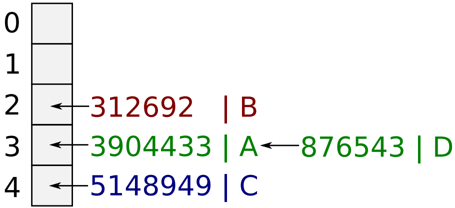
Easiest Solution:

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

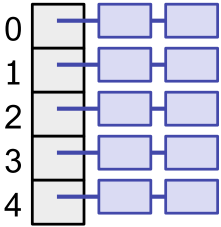
Figure: Hashtable T



Best case:

- We have n keys which are equally distributed over m buckets
- We have $\approx \frac{n}{m}$ pairs per bucket

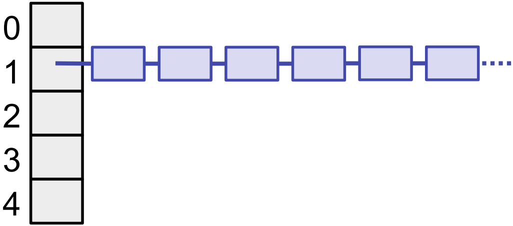
Best case
(m = 5, n = 10)



Worst case:

- All n keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching

Worst case
($m = 5$, $n = 10$)



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  - *for table size 100: try* $100 \times 99 + 1$ *different numbers*

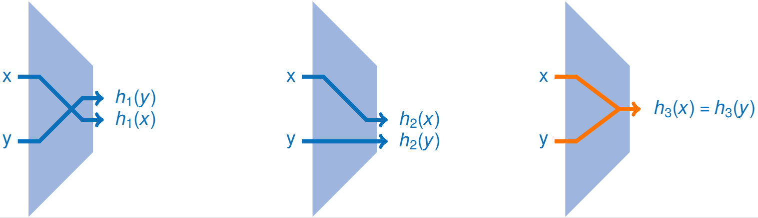
Universal Hashing

Thought Experiment

- I create a hash function
- Find a set of keys so that it results in a degenerated hash table
  - *you may use the hash function*
  - *for table size 100: try* $100 \times 99 + 1$ *different numbers*
  - *worst case: still 100 must have same hash bucket*
- **Now:** Find a solution to avoid that problem

Solution:

- We use a set of hash functions

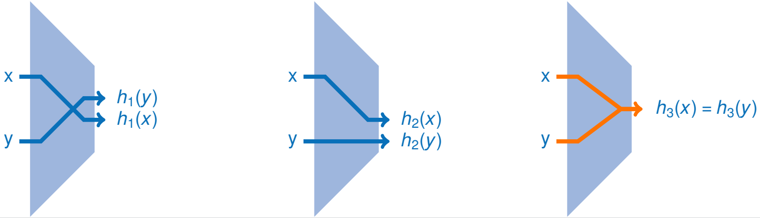


Solution:

- We use a set of hash functions
- We choose a random hash function so that the expected result is an equal distribution over the buckets

this is fixed for the lifetime of table

optional: copy table with new hash when degenerated



Solution:

- We use a set of hash functions
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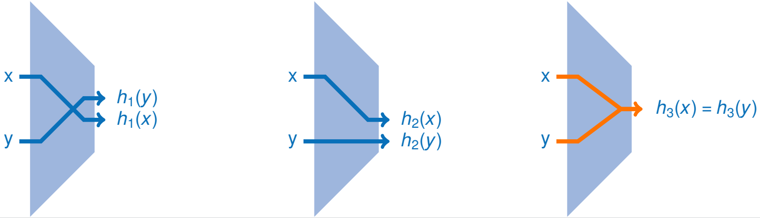
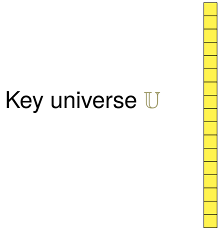
  optional: copy table with new hash when degenerated
- This is called universal hashing



Definition:

- We call $\mathbb{U}$ the set (universum) of possible keys, and $\mathbb{S} \subseteq \mathbb{U}$ the set of used keys



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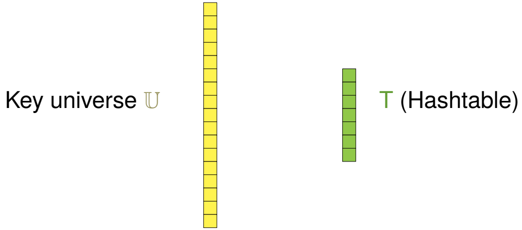

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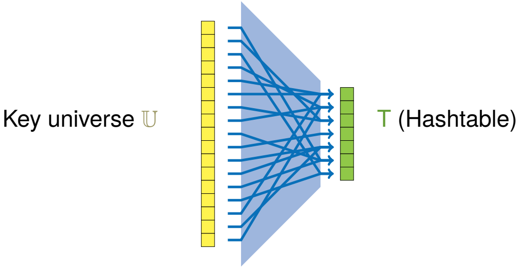

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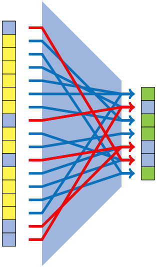
Key universe $\mathbb{U}$
Used Keys $\mathbb{S}$



T (Hashtable)
Used buckets

Universal Hashing

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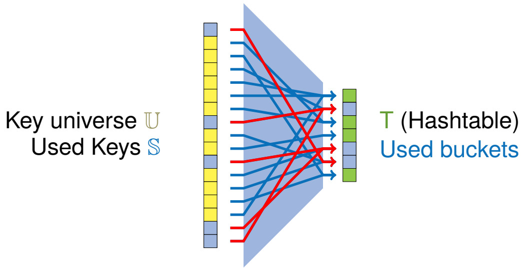
- The size m of the hash table T
- Set of hash functions $\mathbb{H} = \{h_1, h_2, \ldots, h_n\}$ with $h_i : \mathbb{U} \rightarrow \{0, \ldots, m-1\}$
- Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load



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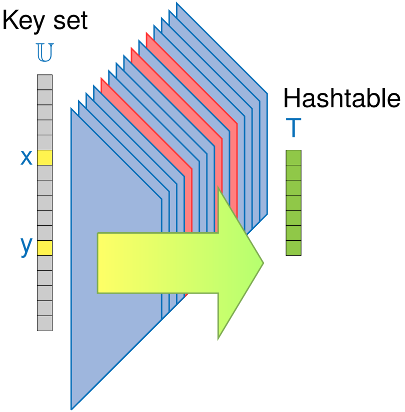
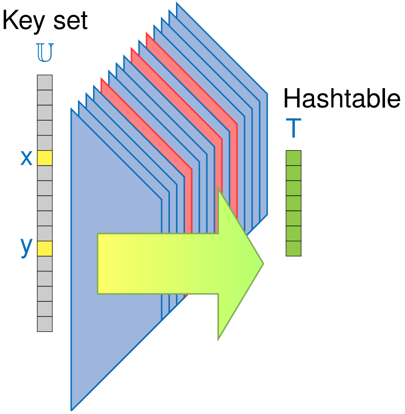


Figure: Set of hash functions $\mathbb{H}$

- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- An average of 3 out of 15 functions produce collisions



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$$\frac{\overbrace{|\{h \in \mathbb{H} : h(x) = h(y)\}|}^{\text{Number of hash functions that create collisions}}}{\underbrace{|\mathbb{H}|}_{\text{Number of hash functions}}} \leq c \cdot \frac{1}{m}, \qquad c \in \mathbb{R}$$

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$Prob(Collision) = \frac{1}{m} \Leftrightarrow c = 1$

- $\mathbb{U}$: Key universe
- $\mathbb{S}$: Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: Keys mapping to Bucket i ("synonyms")
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$

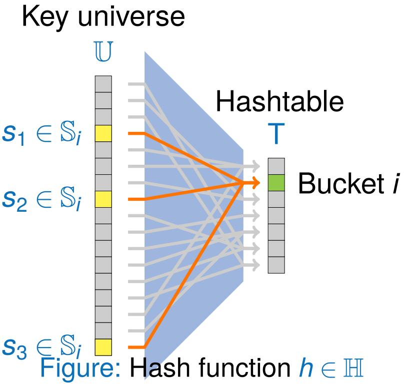

Figure: Hash function $h \in \mathbb{H}$

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$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m}$$

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- Particulary: If $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

Structure

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Table: Throwing a dice

e	$P(e)$
1	$^1/_6$
2	$^1/_6$
3	$^1/_6$
4	$^1/_6$
5	$^1/_6$
6	$^1/_6$

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$(1,3)$	${}^1/_{36}$
$\ldots$	$\ldots$
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$(6,6)$	${}^1/_{36}$

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- Rolling a dice twice ($\Omega = \{1, \ldots, 6\}^2$)
- Each event $e \in \Omega$ has the probability $P(e) = {}^1\!/_{36}$
- E = if both eye numbers even, then $P(E)$ =

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$(1,1)$	$^1/_{36}$	2
$(1,2)$	$^1/_{36}$	3
$(1,3)$	$^1/_{36}$	4
$\ldots$	$\ldots$	$\ldots$
$(6,5)$	$^1/_{36}$	11
$(6,6)$	$^1/_{36}$	12

Example:

- Random variable
  - Assigns a number to the result of an experiment
  - For example: X = Sum of eye numbers for rolling twice
  - $X = 12$ and $X \geq 7$ are then just events
  - Example 1: $P(X = 2) =$

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...	...	...
$(6,5)$	$^1/_{36}$	11
$(6,6)$	$^1/_{36}$	12

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Example:

- Random variable
  - Assigns a number to the result of an experiment
  - For example: X = Sum of eye numbers for rolling twice
  - $X = 12$ and $X \geq 7$ are then just events
  - Example 1: $P(X = 2) =$
  - Example 2: $P(X = 4) =$

Table: Throwing a dice twice

e	$P(e)$	X
$(1,1)$	$^1/_{36}$	2
$(1,2)$	$^1/_{36}$	3
$(1,3)$	$^1/_{36}$	4
...	...	...
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Expected value is defined as $\mathbb{E}(X) = \sum (k \cdot P(X = k))$

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- Example rolling once: $\mathbb{E}(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$

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Sum of expected values: For independent (discrete) result variables $X_1, \ldots, X_n$ we can write:

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n)$$

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- X_1: Expected number of eyes dice 1: $\mathbb{E}(X_1) = 3.5$
- X_2: Expected number of eyes dice 2: $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$: Expected total number of eyes:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

Corollary:

The probability of the event E is $p = P(E)$. Let X be the occurences of the event E and n be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurences of 6}) = \frac{1}{6} \cdot 60 = 10$$

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Def. $\mathbb{E}$-value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

Structure

Given:

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To proof:

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \forall i$$

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$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \qquad y \in \mathbb{S} \setminus \{x\}$$

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$$\Rightarrow \qquad \mathbb{E}\left(|\mathbb{S}_i|\right) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

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Auxiliary calculation:

$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1)\\ &= P(h(y) = i)\\ &= P(h(y) = h(x))\\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

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$$\leq 1 + |\mathbb{S}| \cdot c \cdot \frac{1}{m} = 1 + c \cdot \frac{|\mathbb{S}|}{m}$$

$$\mathbb{E}[|\mathbb{S}_i|] = 1 + \sum_{y \in \mathbb{S} \backslash \{x\}} \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \square$$

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Negative example:

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- The set of all h for which $h_a(x) = (a \cdot x) \mod m$, for a $a \in \mathbb{U}$

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- Which x, y lead to a relative collision count bigger than $\frac{c}{m}$?

Positive example:

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- Let p be a big prime number, $p > m$, and $p \geq |\mathbb{U}|$
- Let $\mathbb{H}$ be the set of all h for which:

$$h_{a,b}(x) = ((a \cdot x + b) \mod p) \mod m,$$
$$\text{where } 1 \leq a < p,\ 0 \leq b < p$$

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- Then $h(x) = ((47 \cdot x + 5) \mod 101) \mod m$
- Easy to implement but hard to proof
- Exercise: show empirically that it is 2-universal

Positive example:

- The set of hash functions is c-universal:

$$h_a(x) = a \bullet x \mod m, \qquad a \in \mathbb{U}$$

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- **We define:**

$$a = \sum_{0,\ldots,k-1} a_i \cdot m^i, \qquad k = \mathrm{ceil}(\log_m |\mathbb{U}|)$$

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- **Intuitive**: Scalar product with base m

$$a \bullet x = \sum_{0,\ldots,k-1} a_i \cdot x_i$$

Example ($\mathbb{U} = \{0, \ldots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

Further Literature

- **General for this Lecture**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Hash Map - Theory**

[Wik] Hash table
https://en.wikipedia.org/wiki/Hash_table

- **Hash Map - Implementations / API**

[Cpp] C++ - hash_map
http://www.sgi.com/tech/stl/hash_map.html

[Jav] Java - HashMap
https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html

[Pyt] Python - Dictionaries (Hash table)
https://en.wikipedia.org/wiki/Hash_table