# Algorithms and Datastructures

## Shortest Path, Dijkstra Algorithm

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Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, March 2018

# Structure

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Graphs

Dijkstra Algorithm

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Graphs

Dijkstra Algorithm

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For a graph $G = (V, E)$:

- A path of $G$ is a sequence of edges $u_1, u_2, \ldots, u_i \in V$ with
  - Undirected graph: $\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{i-1}, u_i\} \in E$
  - Directed graph: $(u_1, u_2), (u_2, u_3), \ldots, (u_{i-1}, u_i) \in E$
- The length of a path is
  - Without weights: number of edges taken
  - With weights: sum of weigths of edges taken

For a graph $G = (V, E)$:

- The shortest path between two vertices $u, v$ is the path $P = (u, \ldots, v)$ with the shortest length $d(u, v)$ or lowest costs
- The diameter of a graph is the longest shortest path

# Structure

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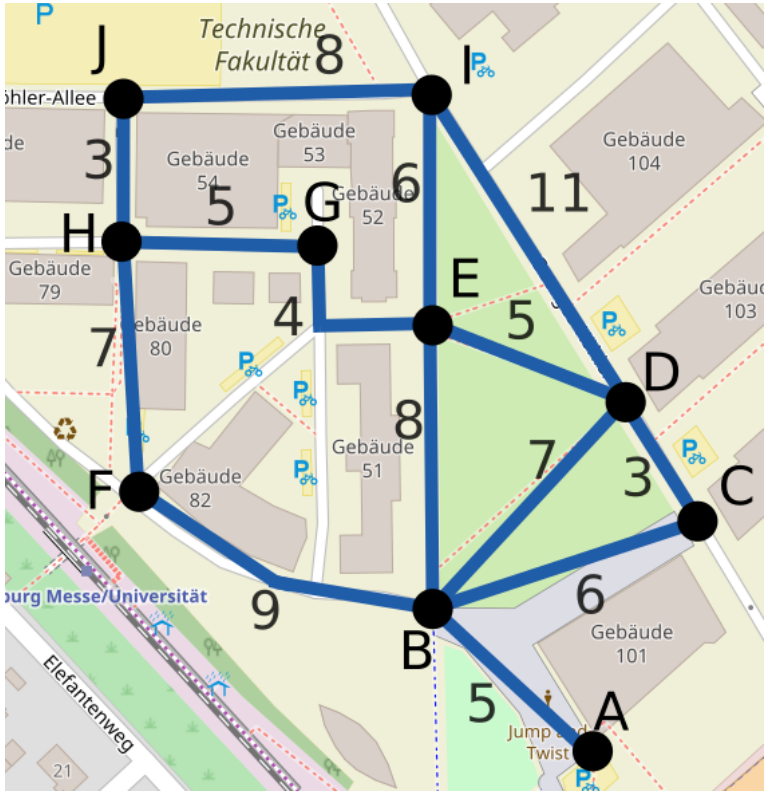
Graphs

Dijkstra Algorithm

# Dijkstra Algorithm

## Shortest Path without Computer

- Wanted: Shortest path from M to all other points
- Place pearls on crossings and clamp strings between them



# Dijkstra Algorithm

## Shortest Path without Computer

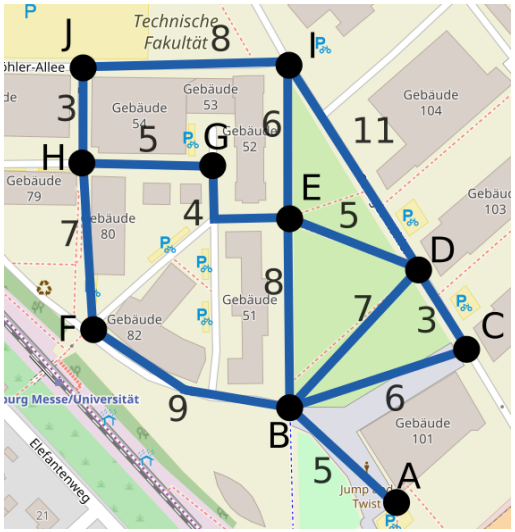
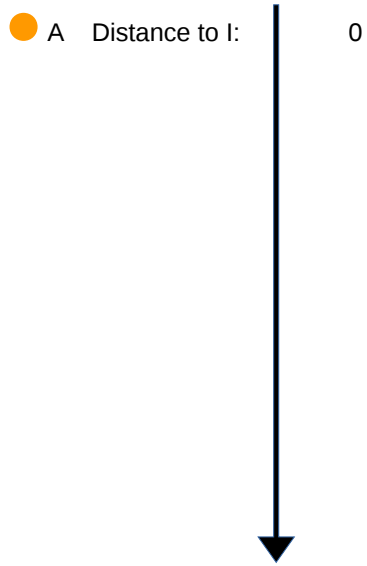



Figure: Based on OpenStreetMaps; CC BY-SA 2.0

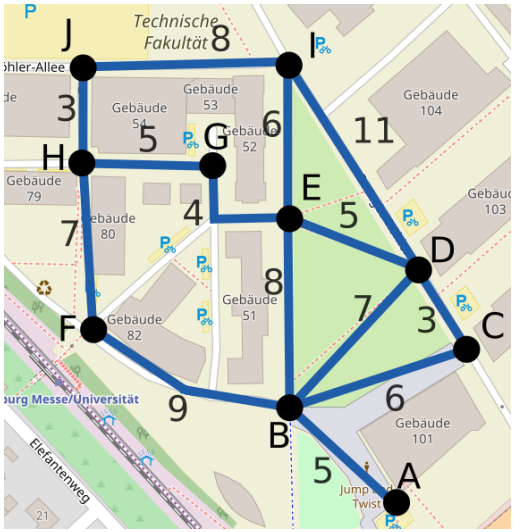
- Take the net and pull it slowly upwards until fully lifted



- Each node (pearl) now has a specific height
- The distance to M is exactly the shortest path

# Dijkstra Algorithm

## Shortest Path without Computer

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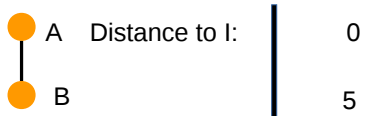

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# Dijkstra Algorithm

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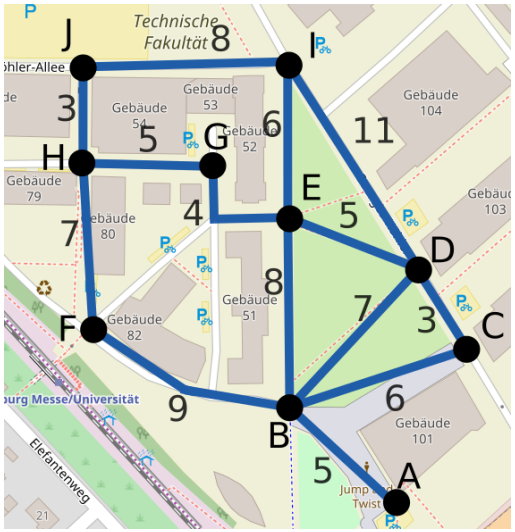


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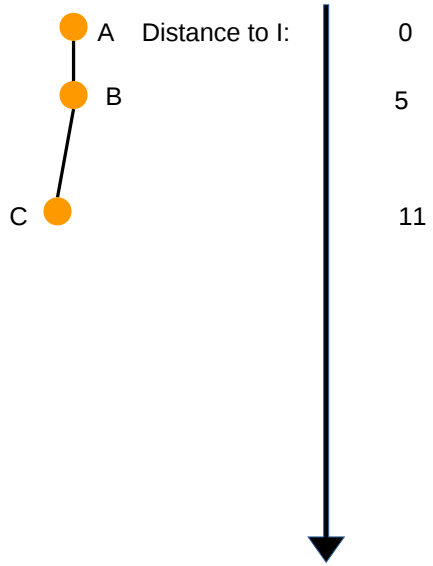

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# Dijkstra Algorithm

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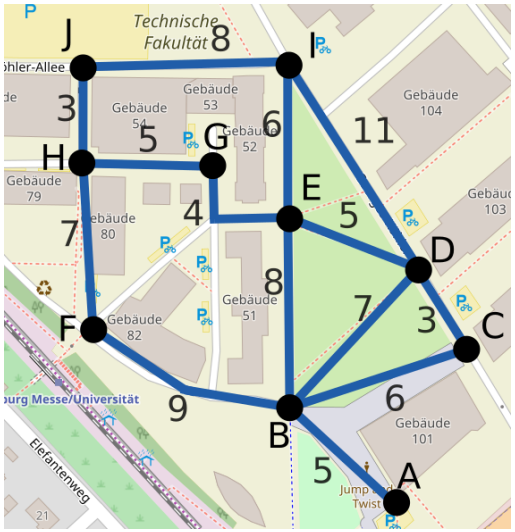


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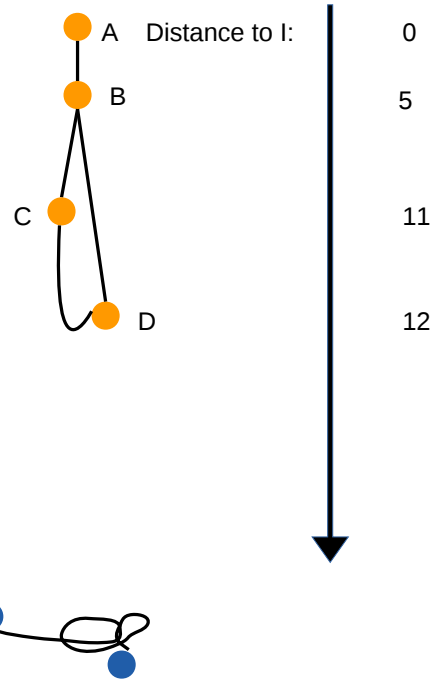


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# Dijkstra Algorithm

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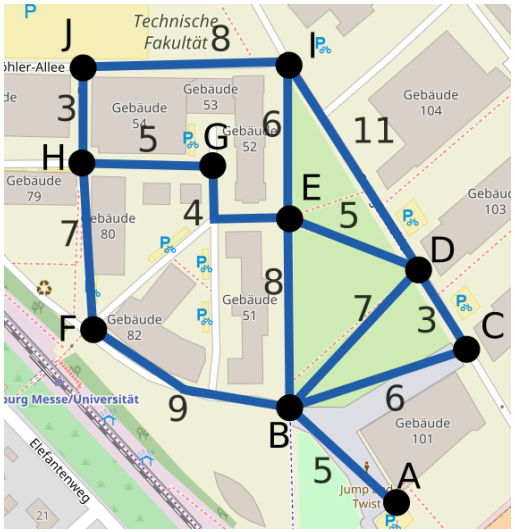


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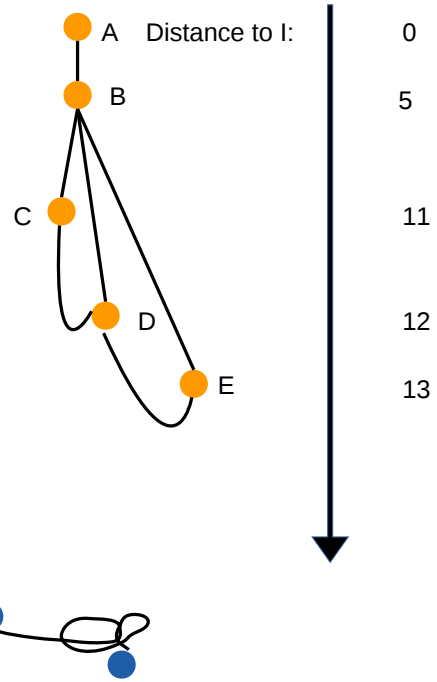

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# Dijkstra Algorithm

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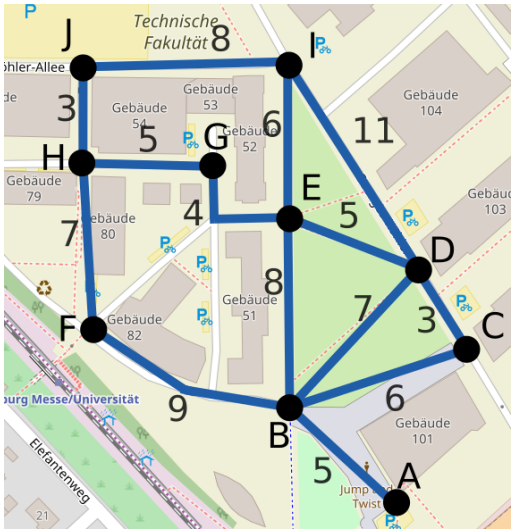


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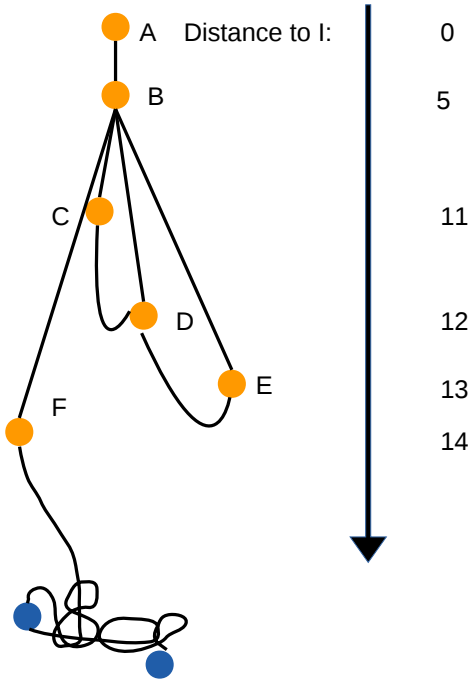


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# Dijkstra Algorithm

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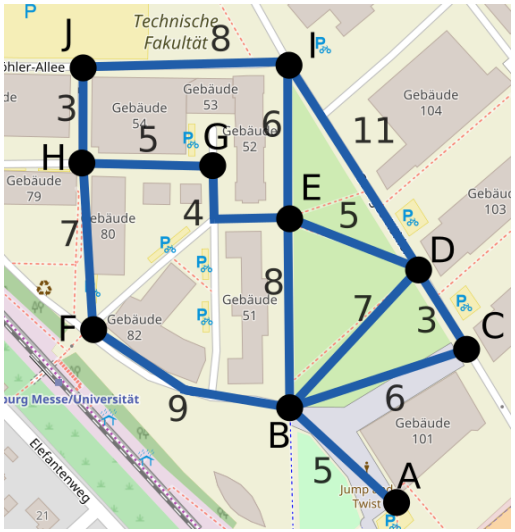


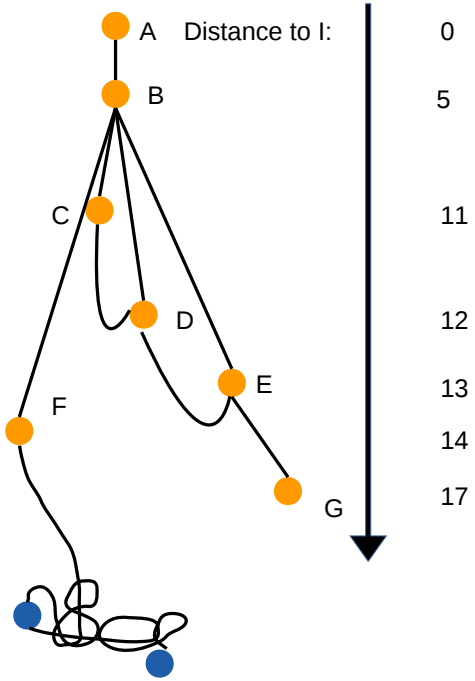


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# Dijkstra Algorithm

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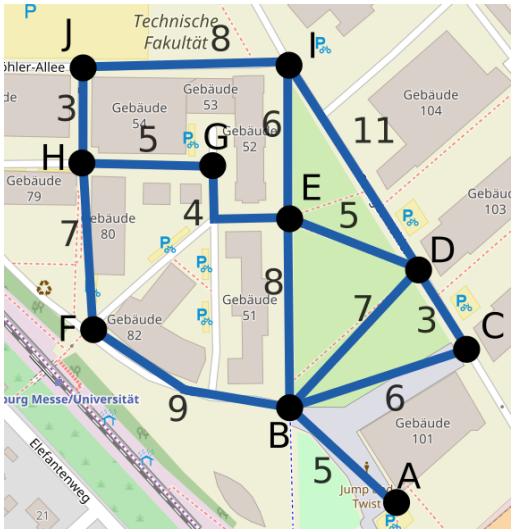


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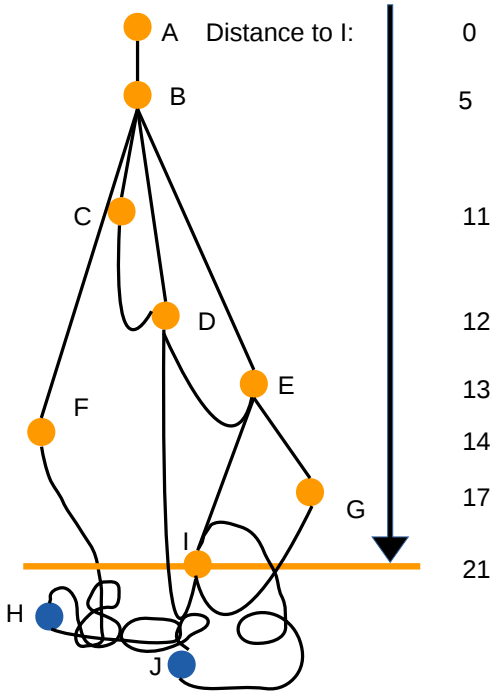


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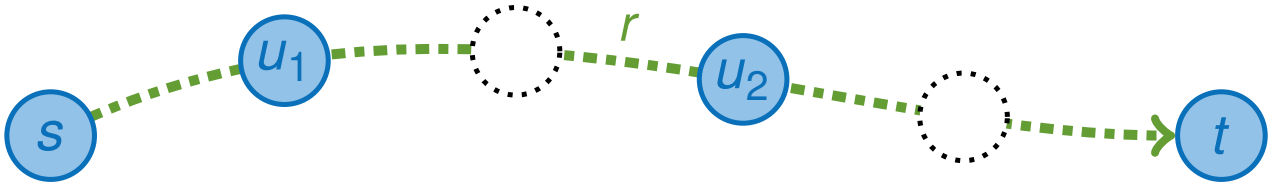



Figure: Shortest path from $s$ to $t$

- Let $r$ be the shortest path from $s$ to $t$
- For each node $u$ on path $r$ the path from $u$ to $t$ is the shortest path

**Proof:**

- If there was a shorter path from $s$ to $u$ then we could choose this path to get faster to $t$
- Then $r$ would not be the shortest path

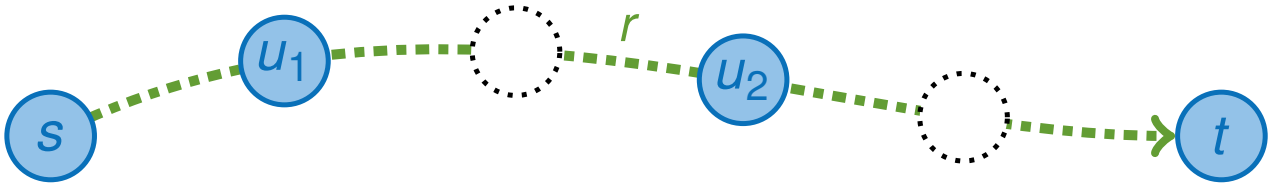



Figure: Shortest path from $s$ to $t$

- This is also correct for all sub paths on $r$
- If the shortest path from $s$ to $t$ passes $u_1$ and $u_2$ then the sub path $(u_1, u_2)$ is the shortest path from $u_1$ to $u_2$

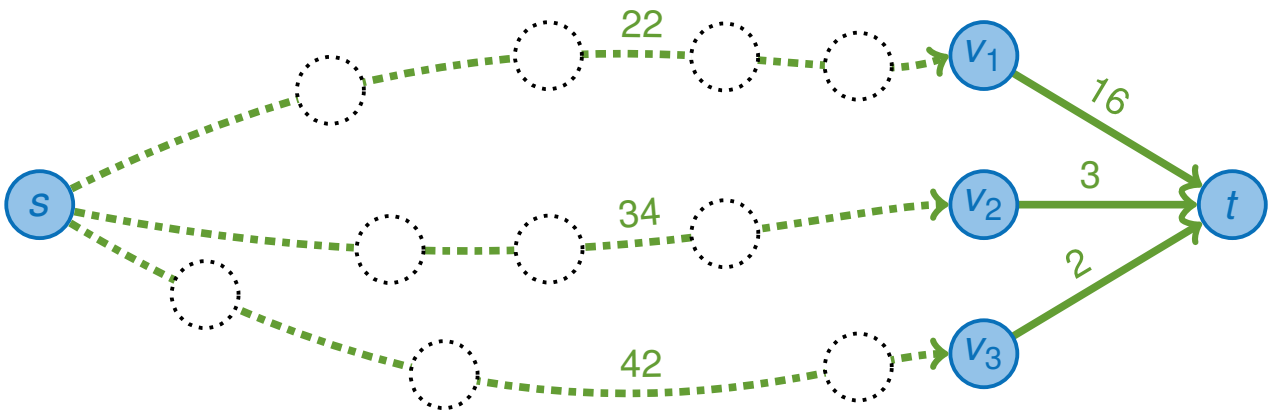



Figure: Shortest paths from $s$ to $t$

- If we know the shortest path form $s$ to the preceding nodes of $t$ ($v_1, v_2, v_3$) we can determine the shortest path to $t$

**Idea:**

- Attach the cost of the shortest path to each node
- Let the information travel over the edges (message passing)
- In which order should we process the nodes?

# Dijkstra Algorithm

**Inventor:**

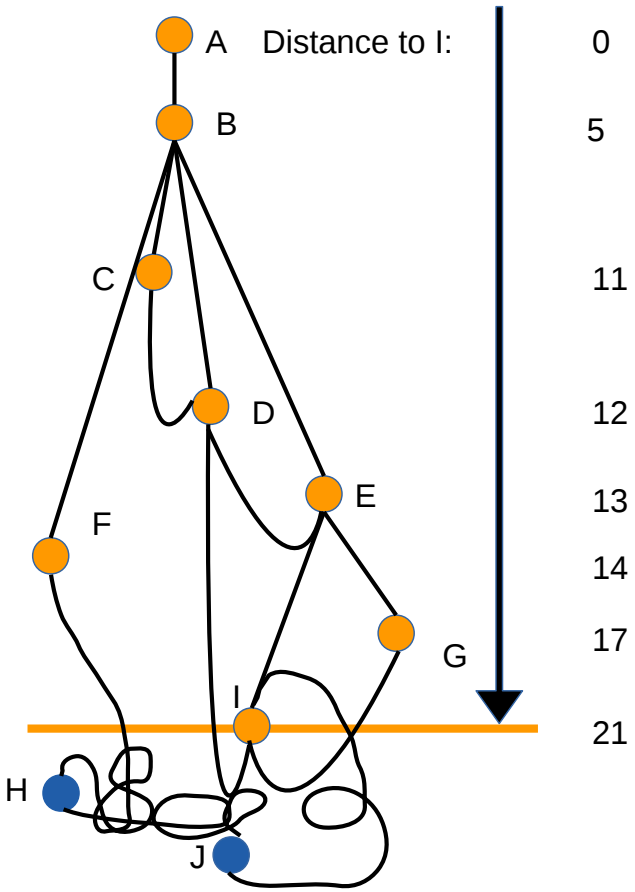
- Edsger Dijkstra (1930 - 2002)
- Computer scientist from Netherlands
- Won Turing-Award as one of few Europeans for his studies of structured programming
- Invented the Dijkstra-Algorithm in 1959

Figure: Portrait © Hamilton Richards - manuscripts of Edsger W. Dijkstra, University Texas at Austin

# Dijkstra Algorithm

**Example:**

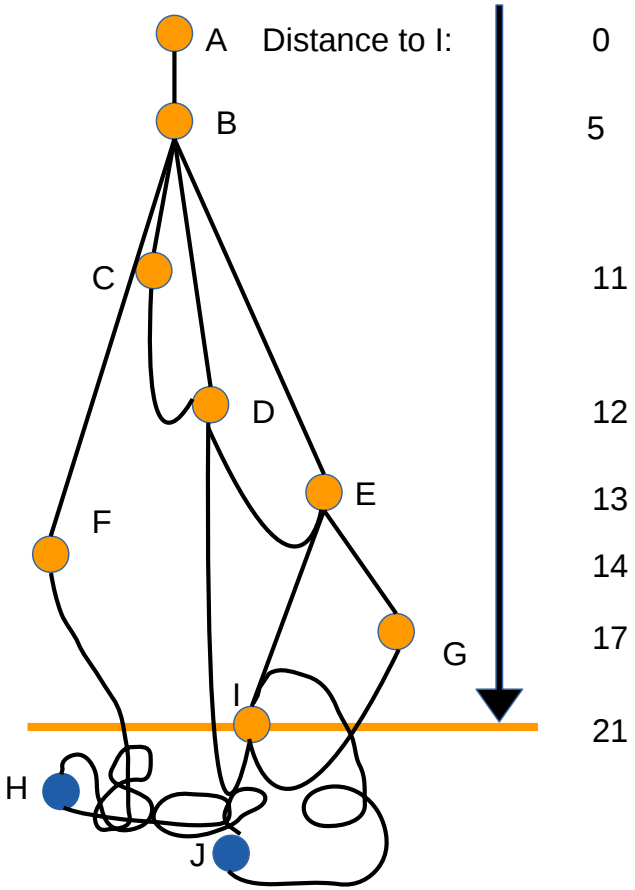
- Lift pearl *A* a little bit
- Connection to pearl *B* is hanging in the air
- Lift further until pearl *B* starts to lift at 5 m
- The shortest path to *B* is now known
- Lift further: The wires from *C*, *D*, *E* and *F* are now in the air
- One of the pearls *C*, *D*, *E* or *F* is the next one
  Which one?



# Dijkstra Algorithm

**Example:**

- At 11 m pearl *C* gets lifted
- The wire to *D* is now in the air
- One of the pearls *D*, *E* and *F* is the next one
  Which one?
- At 12 m pearl *D* gets lifted
  ...
- How to translate this into an computer algorithm?



# Dijkstra Algorithm

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**High level description:** Three types of nodes

- Settled: For node $u$ we know $\mathrm{dist}(s,u)$
  (Pearl example: This pearl is hanging in the air)

- Active: For node $u$ we know a tentative distance $\mathrm{td}(u) \geq \mathrm{dist}(s,u)$ (Can be optimal but doesn't have to)
  (Pearl example: This pearl is laying on the table but one connected wire is already in the air)

- Unreached: We have not reached the node yet
  (Pearl example: This preal is hanging in the air)

# Dijkstra Algorithm

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**High level description:**

- Each iteration take the active node $u$ with the smallest $\mathrm{td}(u)$
  (The pearl getting lifted next)
- We update the state of the node $u$ to settled
  (The pearl gets lifted)
- We check for each neighbor $v$ of node $u$ if we can reach $v$ faster than currently possible
  (Check all outgoing wires from this pearl: Activate all connected pearls, update tentative distance if smaller)
- Iterate until no active nodes exist anymore

# Dijkstra Algorithm

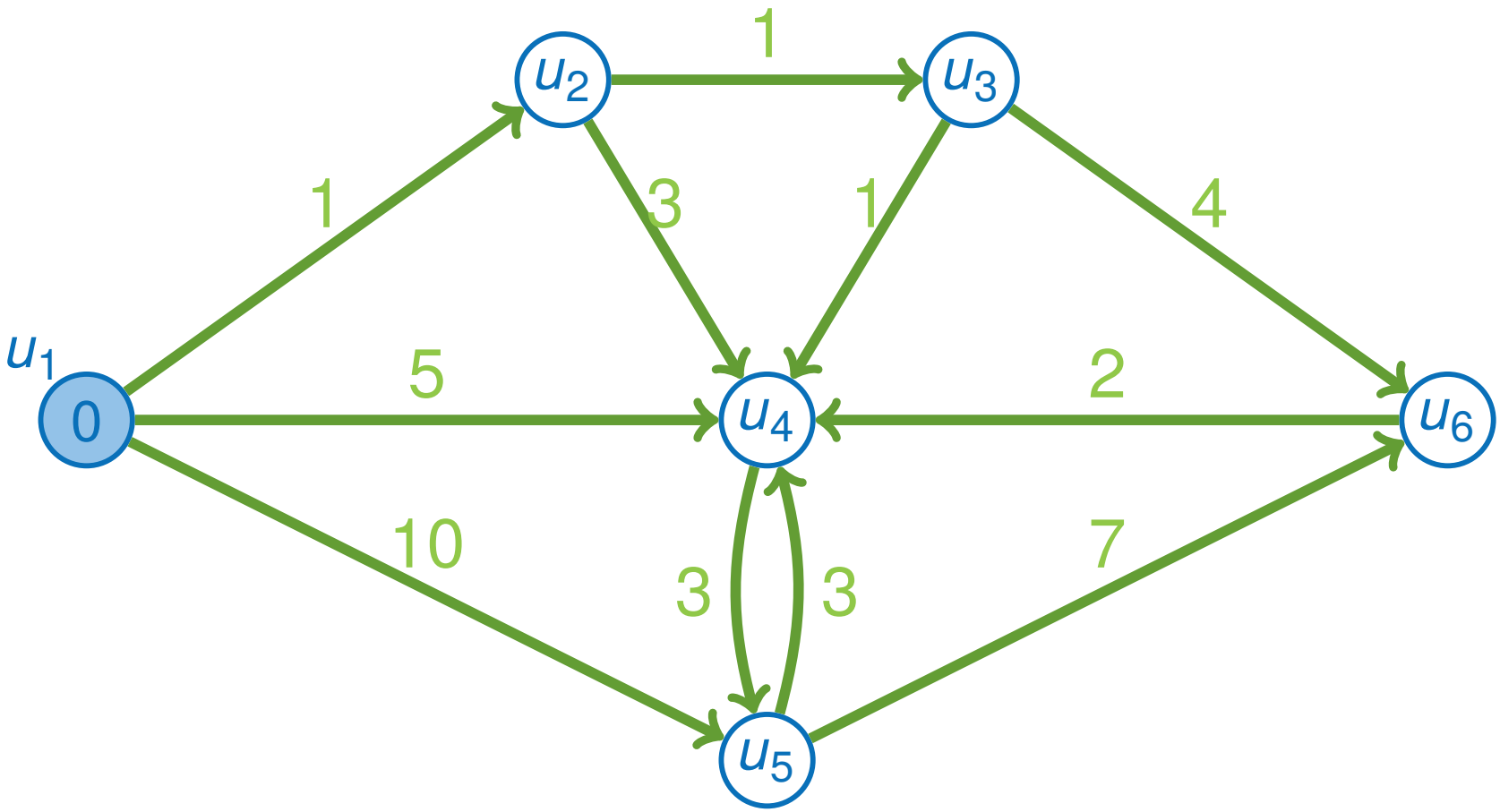



Figure: Start at $u_1$

# Dijkstra Algorithm

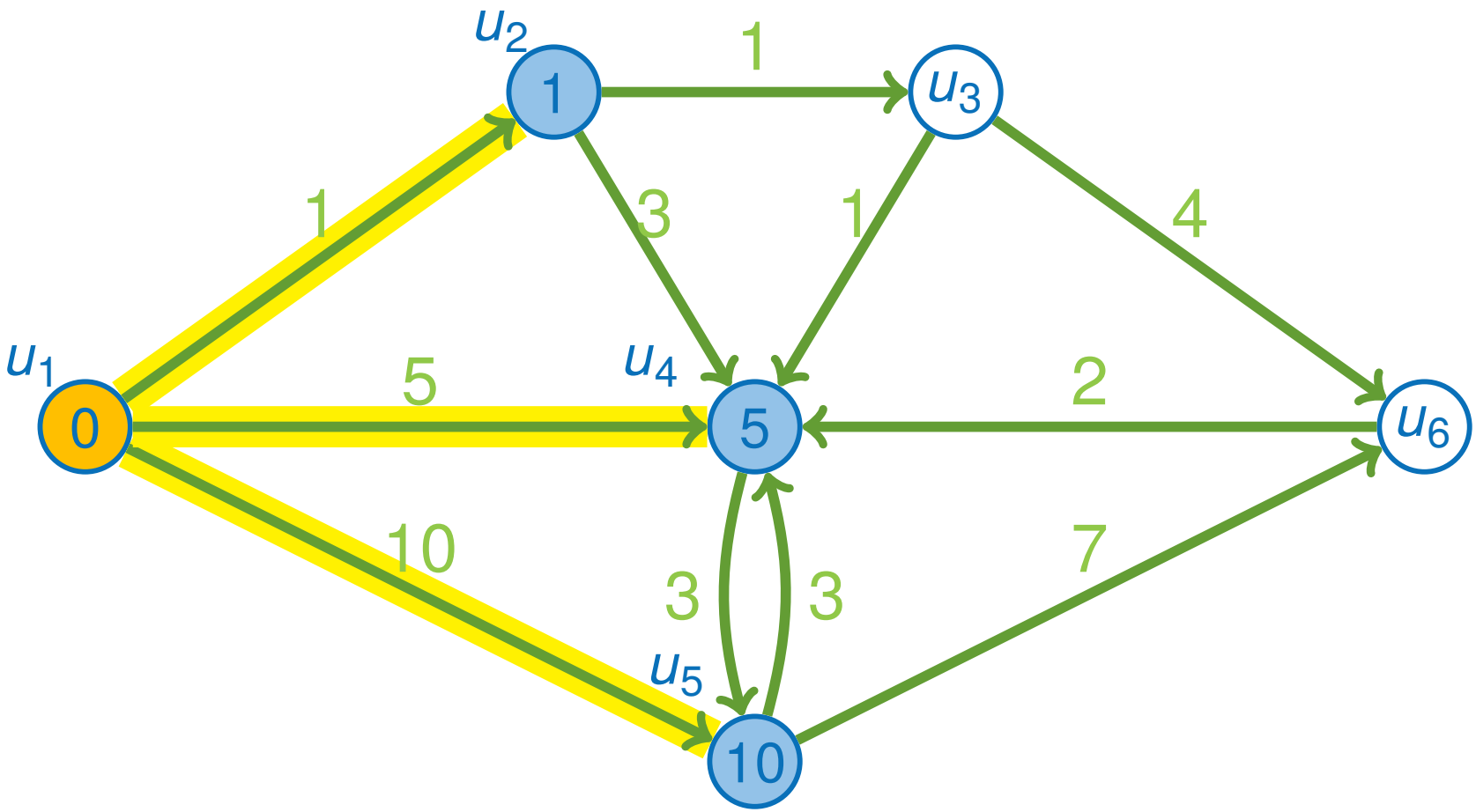



Figure: Iteration 1

# Dijkstra Algorithm

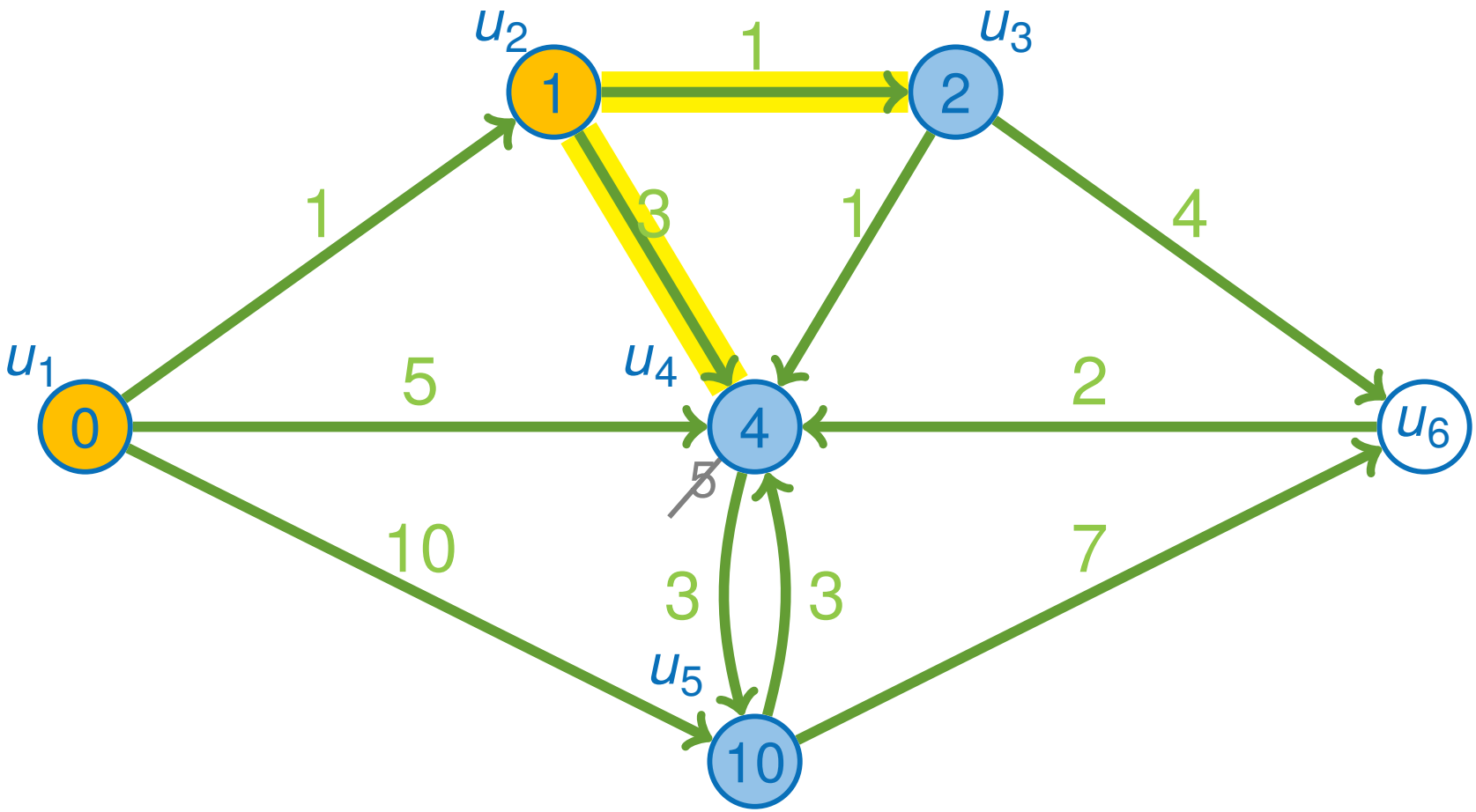



Figure: Iteration 2

# Dijkstra Algorithm

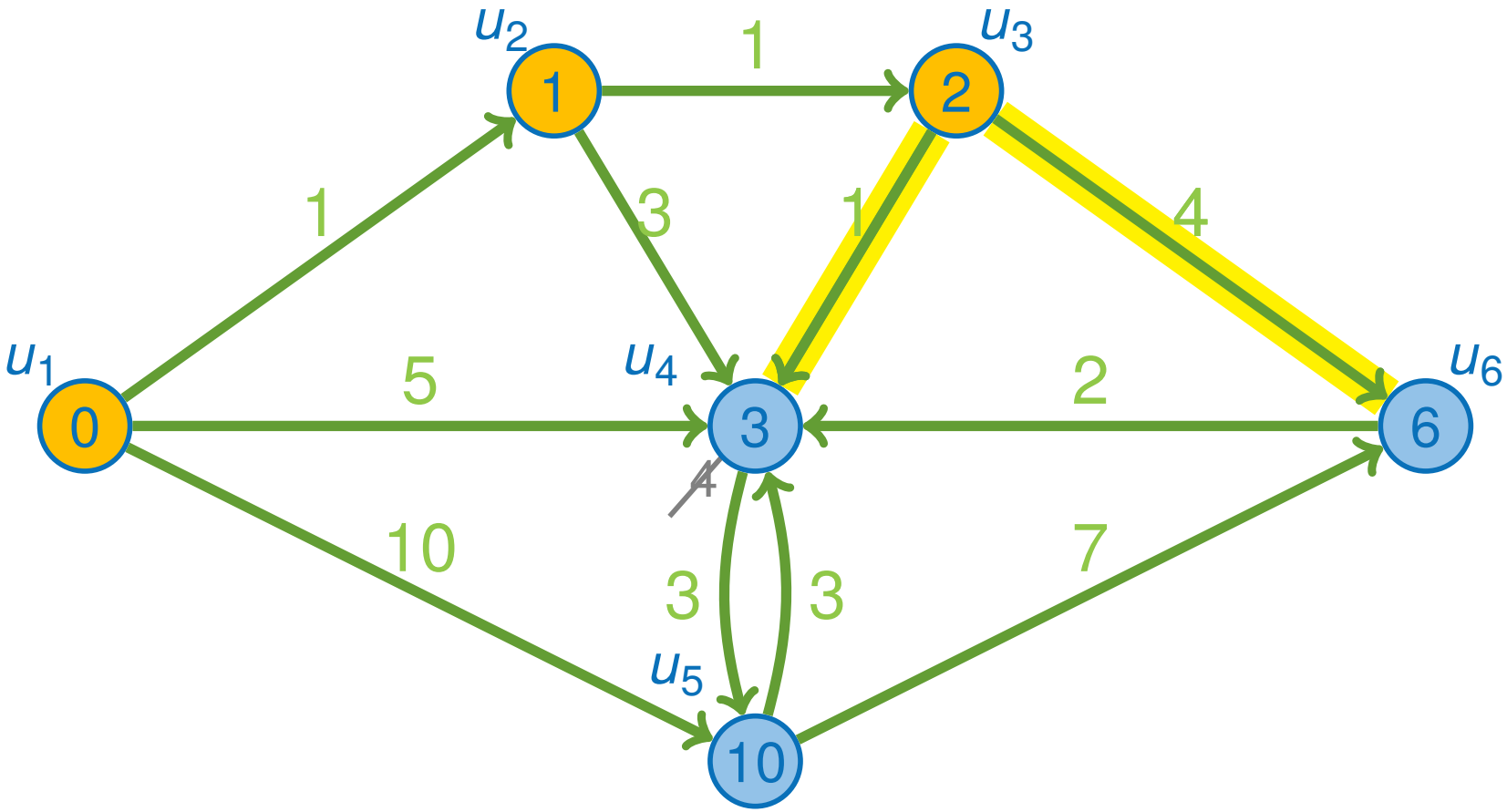



Figure: Iteration 3

# Dijkstra Algorithm

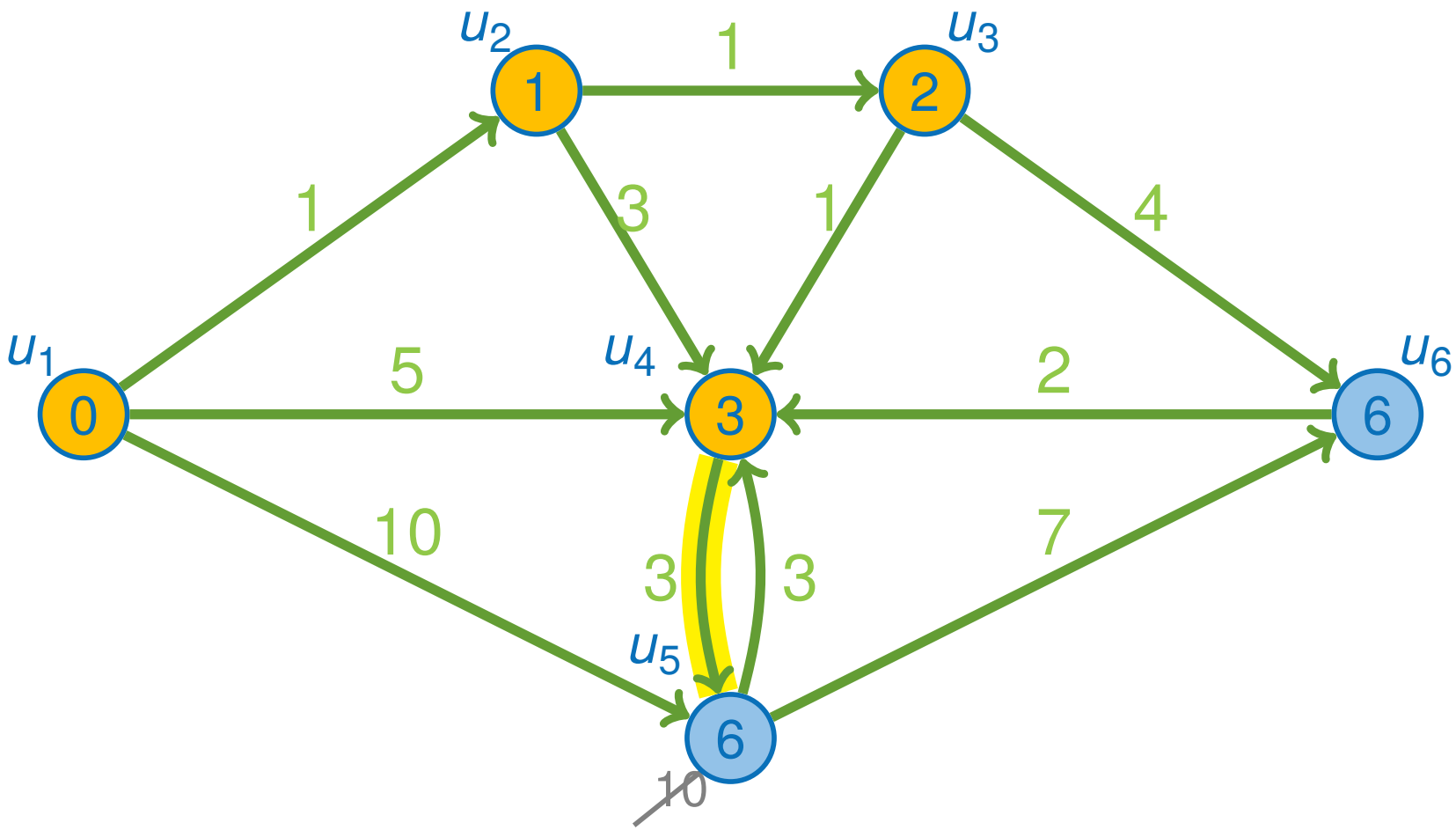



Figure: Iteration 4

# Dijkstra Algorithm

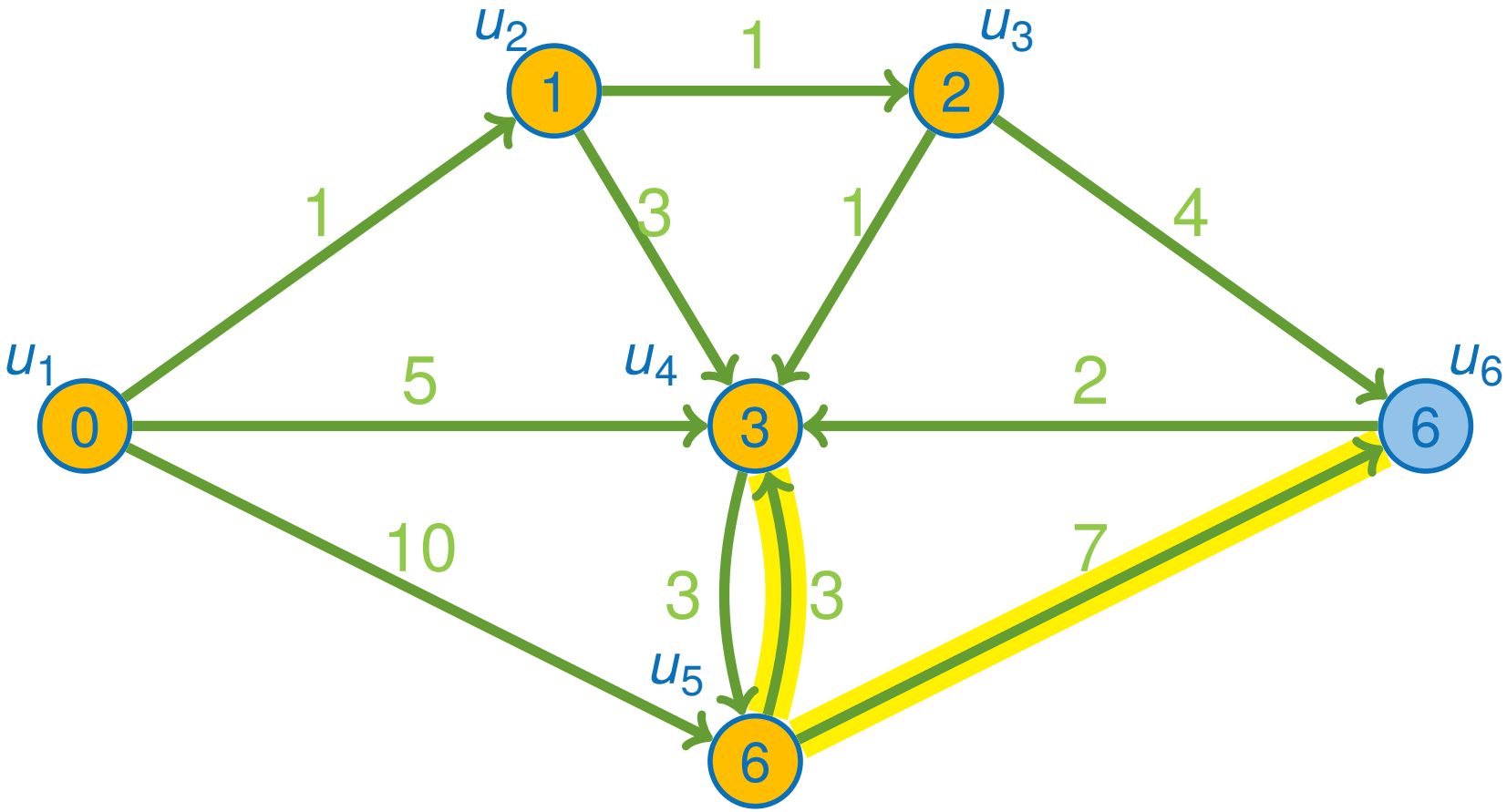



Figure: Iteration 5

# Dijkstra Algorithm

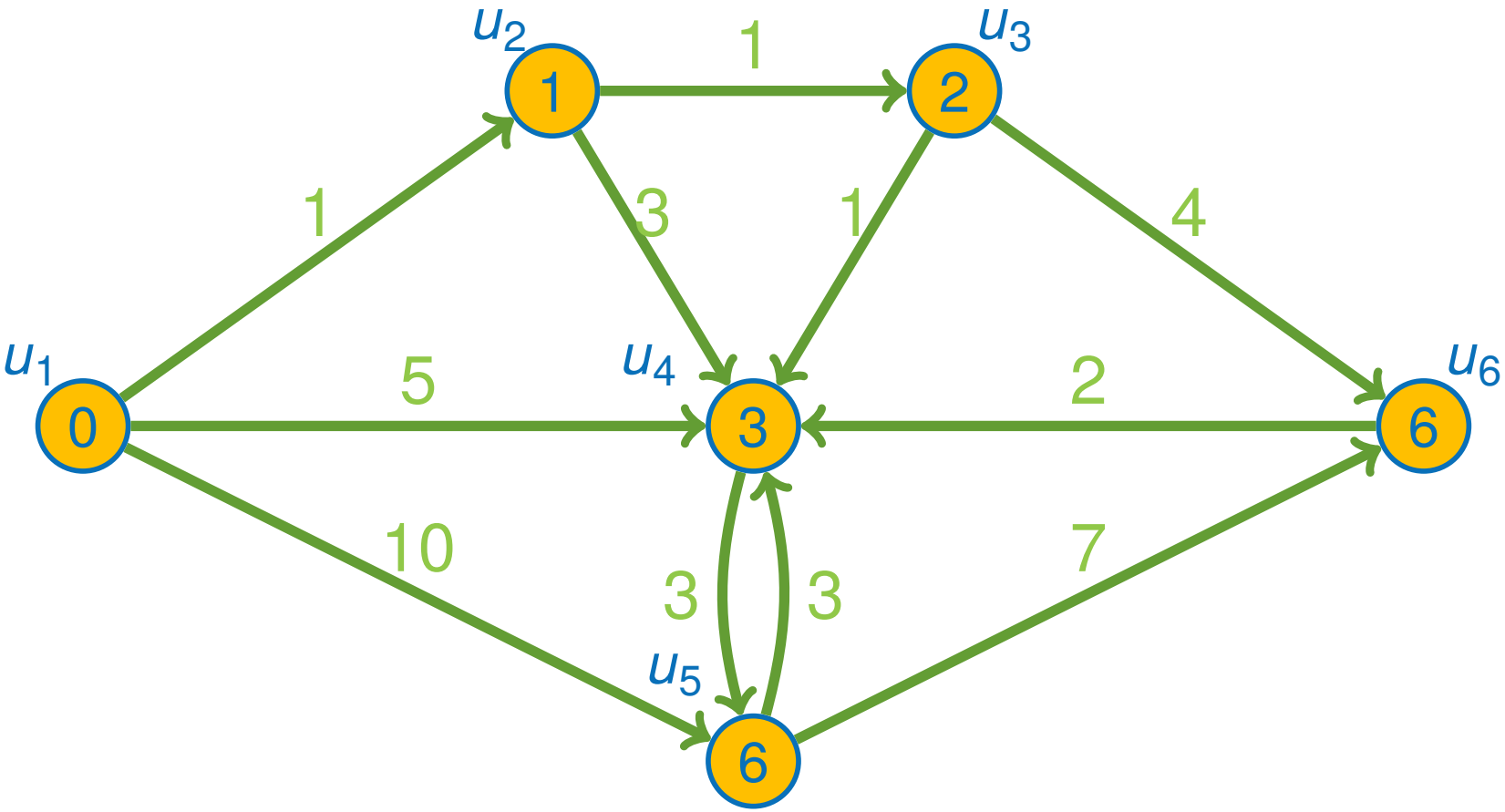



Figure: Iteration 6

**Proof:**

- **Assumption 1:** All edges have a positive length
- **Assumption 2:** Each node has a unique distance $\mathrm{dist}(s,u)$ to the start $s$
  (This was not the case on the previous slides)

  This results in an easy and intuitive proof.
  It is possible to show this without assumption 2. See references if interested

- With assumption 2 there exists a sorting $u_1, u_2, \ldots$ with that:

$$\mathrm{dist}(s,u_1) < \mathrm{dist}(s,u_2) < \mathrm{dist}(s,u_3) < \ldots$$

**Proof:**

- With **assumption 2** there exists a sorting $u_1$, $u_2$, ... with that:

$$\text{dist}(s, u_1) < \text{dist}(s, u_2) < \text{dist}(s, u_3) < \ldots$$

- We want to show that the *Dijkstra* algorithm finds the shortest path for each node $u_i$ so that $\text{td}(u_i) = \text{dist}(s, u_i)$ holds
- Additionally we show that each node gets solved in order of the distance: Node $u_i$ gets solved in iteration $i$

$$u_1, u_2, u_3, \ldots$$

**To show:** Node $u_i$ gets solved in round $i$

1. Node $u_i$ contains the correct distance ($\mathrm{td}(u_i) = \mathrm{dist}(s, u_i)$) and is active
2. Node $u_i$ has the smallest value for $\mathrm{td}(u_i)$ and gets selected by the algorithm

**Induction start:**

1. - Only the start node $s = u_1$ is active and $\mathrm{td}(s) = 0$
   - Node $u_1$ gets solved and $\mathrm{td}(u_1) = \mathrm{dist}(s, u_1) = 0$
2. Only the start node $u_1$ is active

**Induction step:** $i = i + 1$

1. **To show:** Node $u_{i+1}$ contains the correct distance ($\mathrm{td}(u_{i+1}) = \mathrm{dist}(s, u_{i+1})$) and is active
   - On the shortest path from $s$ to $u_{i+1}$ is a preceding node that:

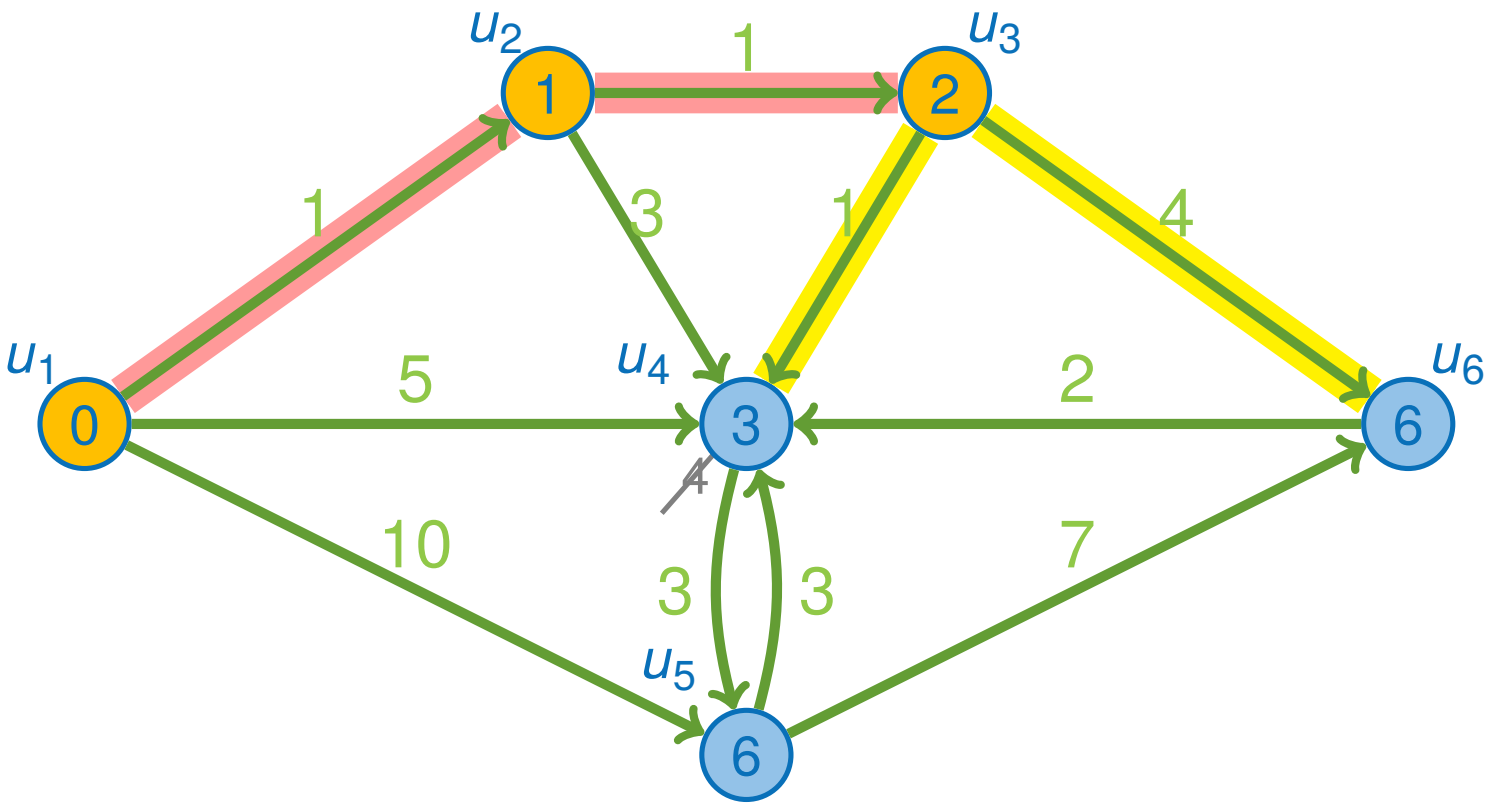
$$\mathrm{dist}(s, u_{i+1}) = \mathrm{dist}(s, v) + \mathrm{c}(v, u_{i+1})$$

($\mathrm{c}(v, u_{i+1})$ are the costs of the edge)

   - Hence $\mathrm{dist}(s, v) < \mathrm{dist}(s, u_{i+1})$ because $c > 0$ ($c$=cost of edge)
   - Because $u_{i+1}$ is currently settled, the node $v$ is one of the preceding nodes $u_1, \ldots, u_i$, hence $v = u_j$ with $0 \leq j \leq i$

# Dijkstra Algorithm

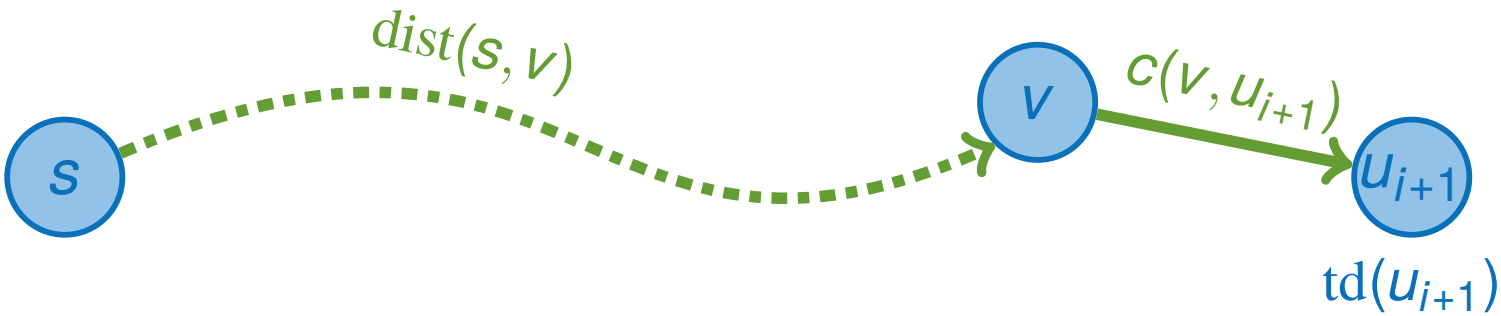
## Proof - Example of Iteration 6




- Preceding node of $u_6$ is $v = u_3$
- In round 3 $\mathrm{td}(u_6) = 2 + 4 = 6$ was already solved

# Dijkstra Algorithm

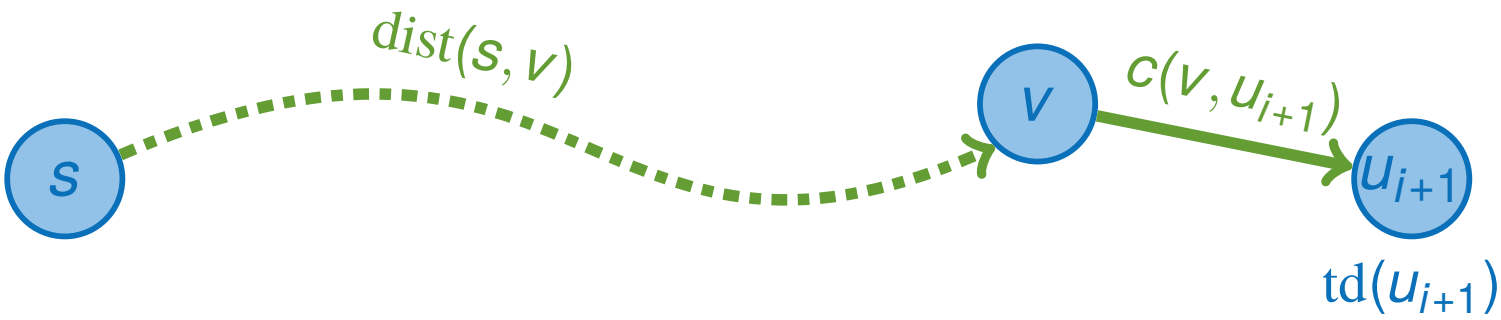
## Proof




1. **To show:** Node $u_i$ contains the correct distance $\mathrm{td}(u_i) = \mathrm{dist}(s, u_i)$ and is active
   - With **induction assumption**: $v$ already contains the correct distance which was evaluated in round $j$ (edge from $v$ to $u_{i+1}$) and is stored in $\mathrm{td}(u_{i+1})$
   - $u_{i+1}$ is active because the preceding node was solved

# Dijkstra Algorithm

## Proof




2. **To show:** Node $u_{i+1}$ has the smallest value for $\mathrm{td}(u_{i+1})$ and gets selected by the algorithm
   - All nodes with smaller $\mathrm{dist}$ are already solved
   - All other nodes $u_k$ with $k > i+1$ have a greater $\mathrm{dist}(s, u_k)$ and with that the $\mathrm{td}(u_k)$ is greater or equal

   $\Rightarrow$ $u_{i+1}$ is the node with the smallest $\mathrm{td}$ and gets selected by the algorithm

**Implementation:**

- We have to manage a set of active nodes
- We start with only the start node in our set
- At the start of each iteration we need the node $u$ with the smallest $\mathrm{td}(u)$

How to implement this?

**Implementation:**

- Using a priority queue with $\mathrm{td}(u)$ as keys
- The following problem occurs:
  - The tentative distance of an active node might change multiple times before it is settled
  - We have to change the key in our priority queue without removing the entry

**Limitations:**

- Often only `insert`, `getMin` and `deleteMin` are implemented

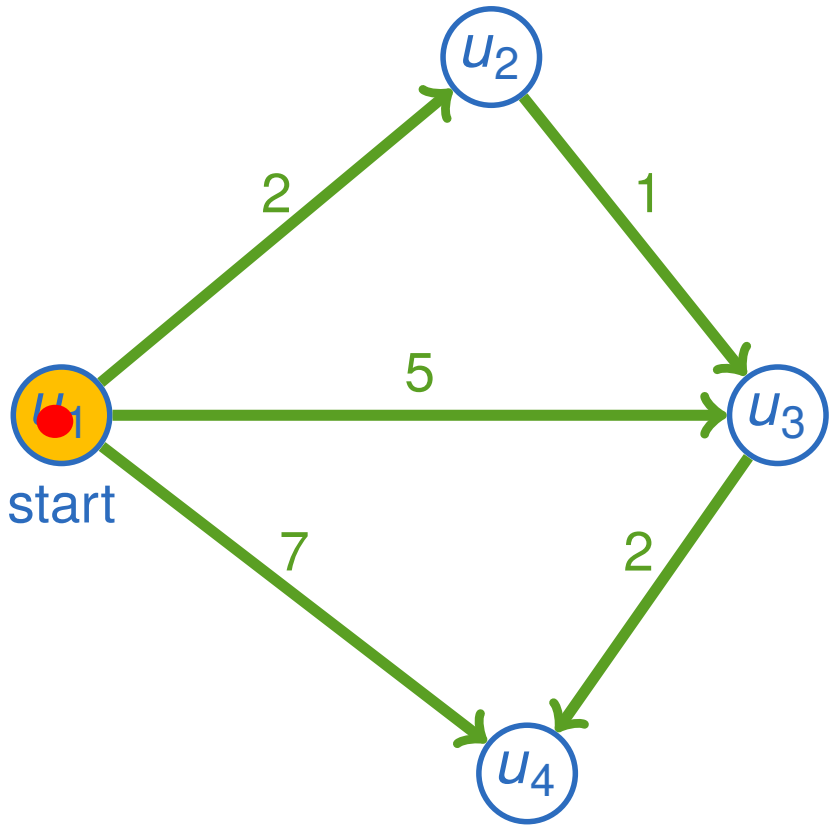
$\Rightarrow$ We only have access to the first element and not any desired one

**Alternative:**

- If a node reoccurs with a smaller dist we insert the element one more time into the priority queue
  (We do nothing if the distance is greater or equal)
- We do not remove the old entry
- The node always gets solved with the smallest distance because of the smaller key
- If a settled node reoccurs with a higher dist we remove it and do simply nothing

# Dijkstra Algorithm

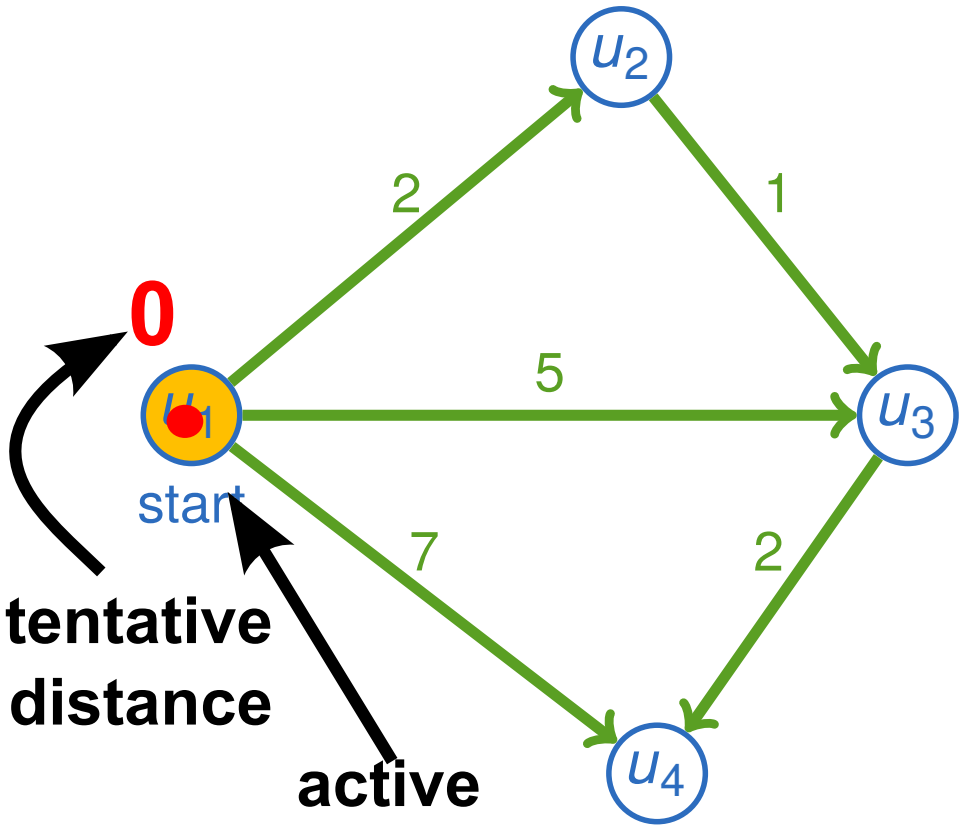
## Implementation - Example




**priority queue**

# Dijkstra Algorithm

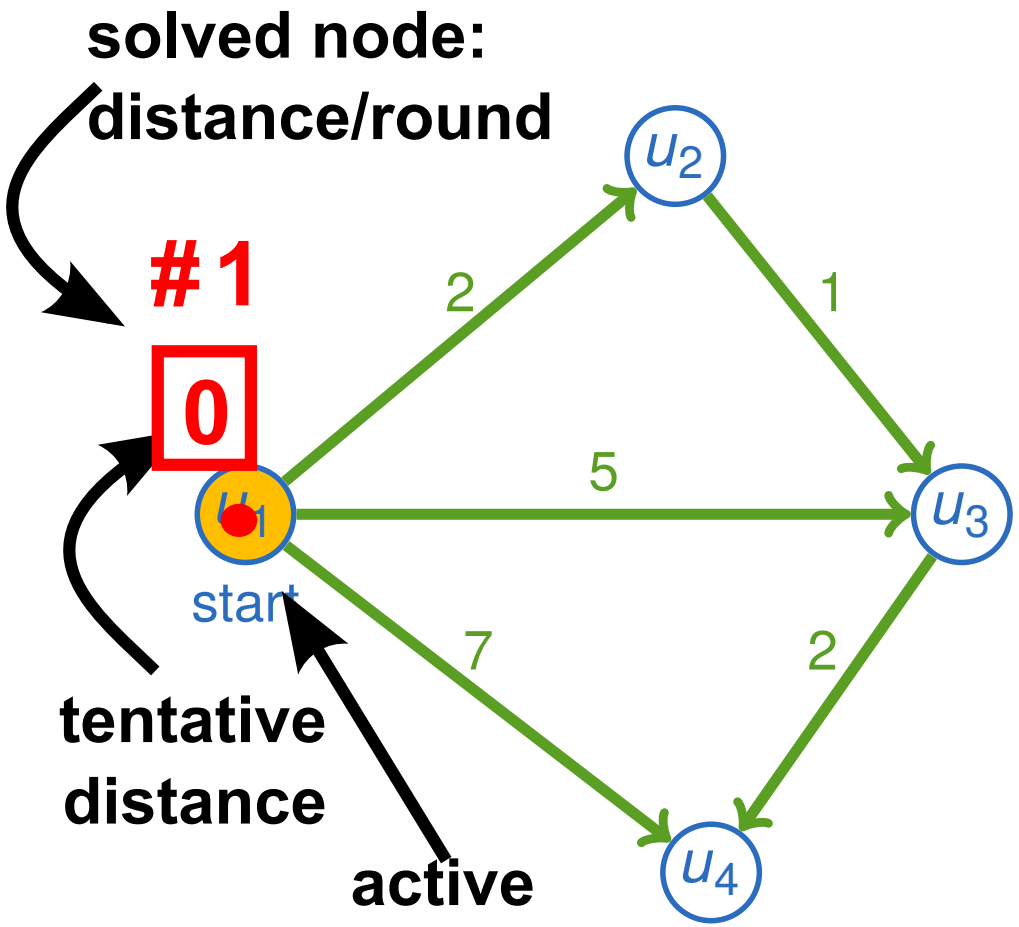
## Implementation - Example




**priority queue**

# Dijkstra Algorithm

## Implementation - Example




**priority queue**

# Dijkstra Algorithm

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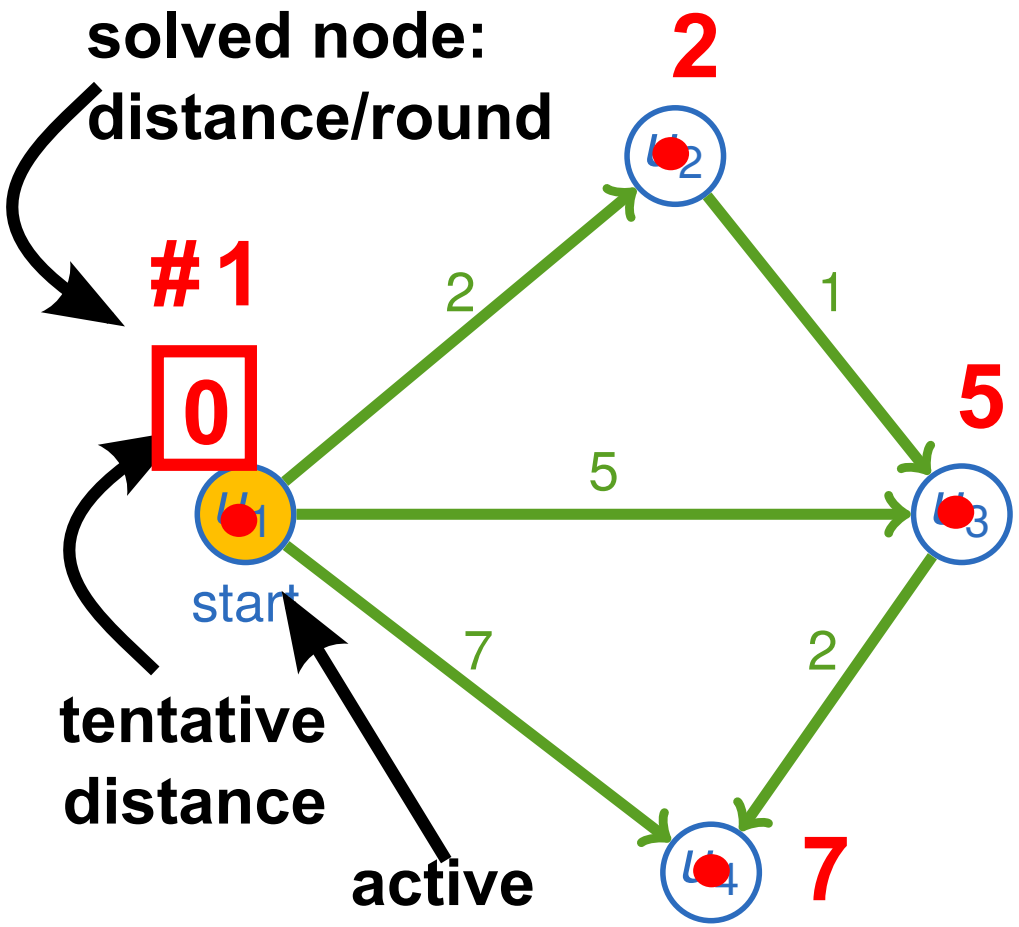


**priority queue**

**(u2, 2)**
**(u3, 5)**
**(u4, 7)**

# Dijkstra Algorithm

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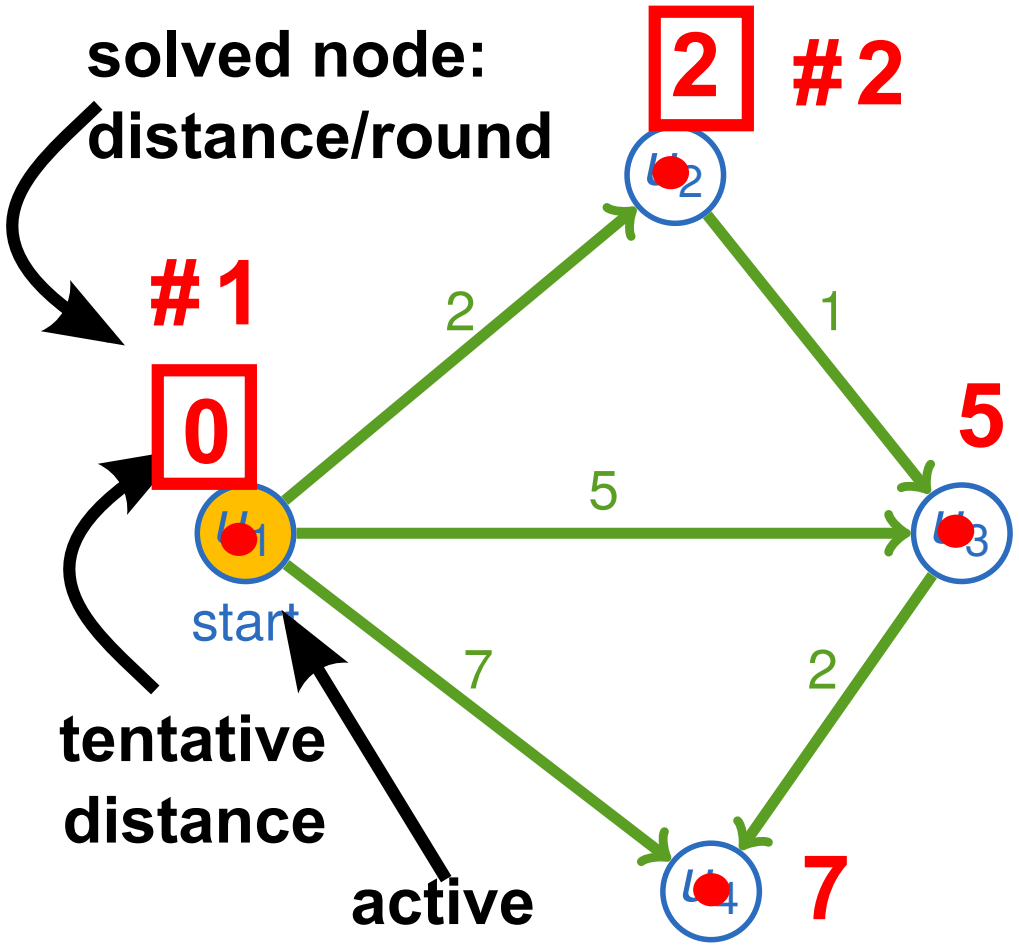



**priority queue**

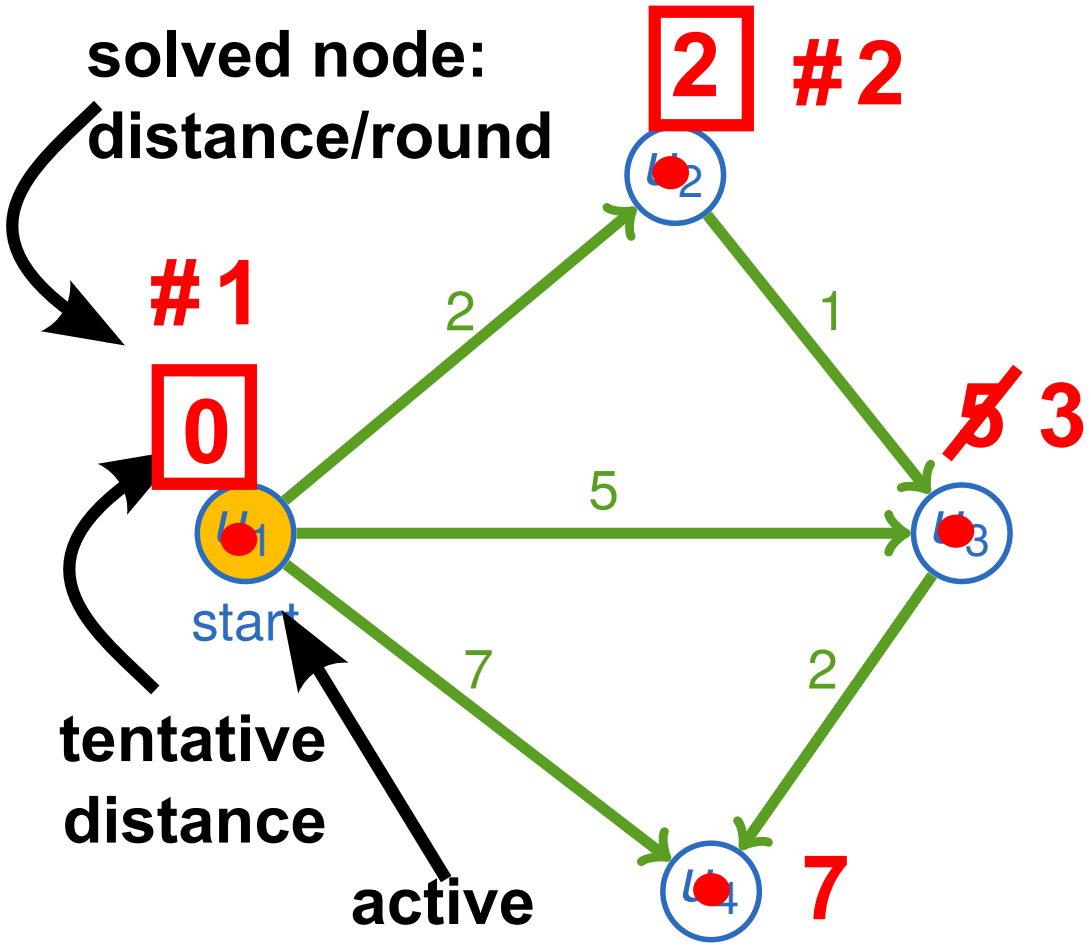
**(u2, 2) → solved #2**

**(u3, 5)**

**(u4, 7)**

# Dijkstra Algorithm

## Implementation - Example

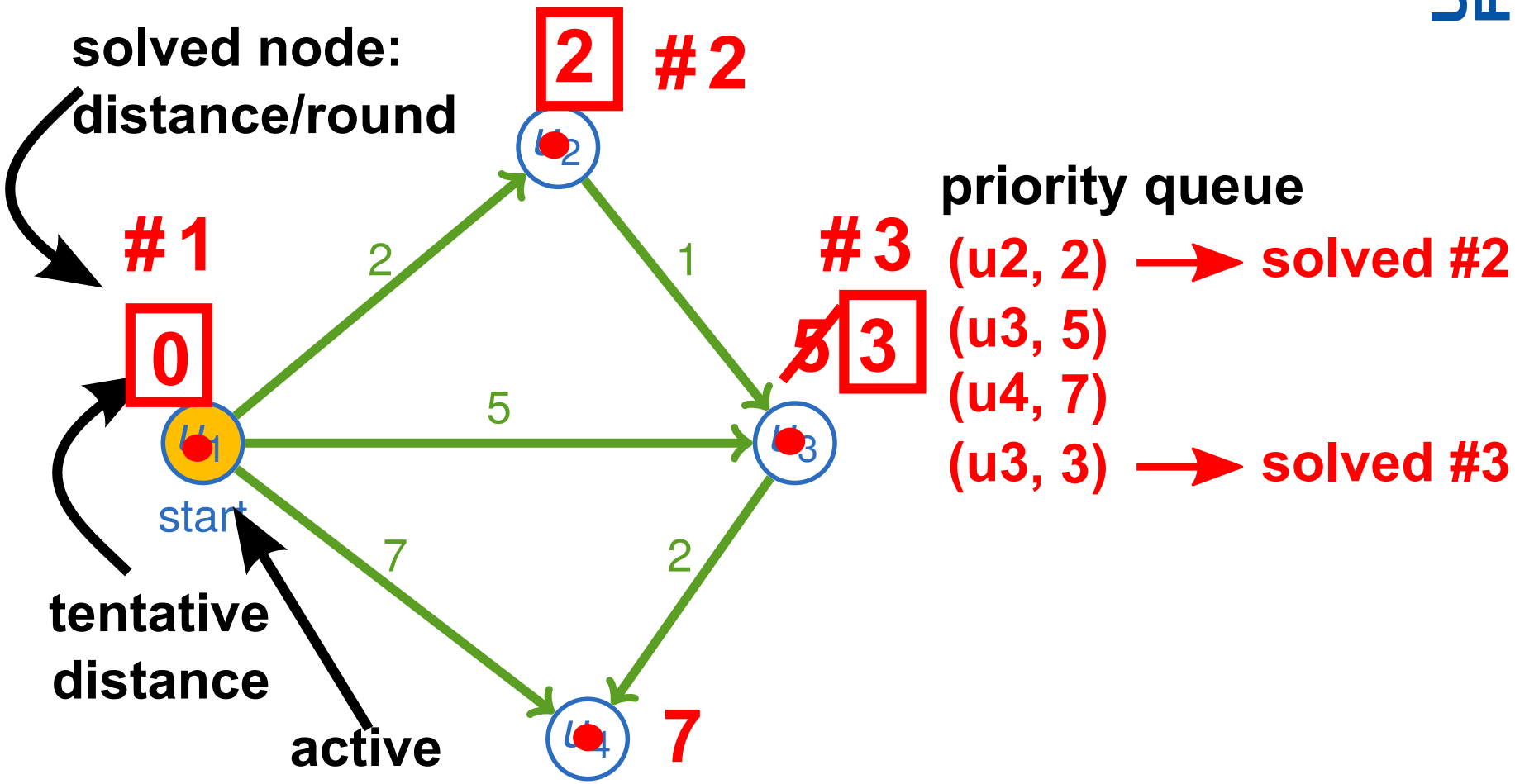



**priority queue**

**(u2, 2) → solved #2**
**(u3, 5)**
**(u4, 7)**
**(u3, 3)**

# Dijkstra Algorithm

## Implementation - Example



# Dijkstra Algorithm

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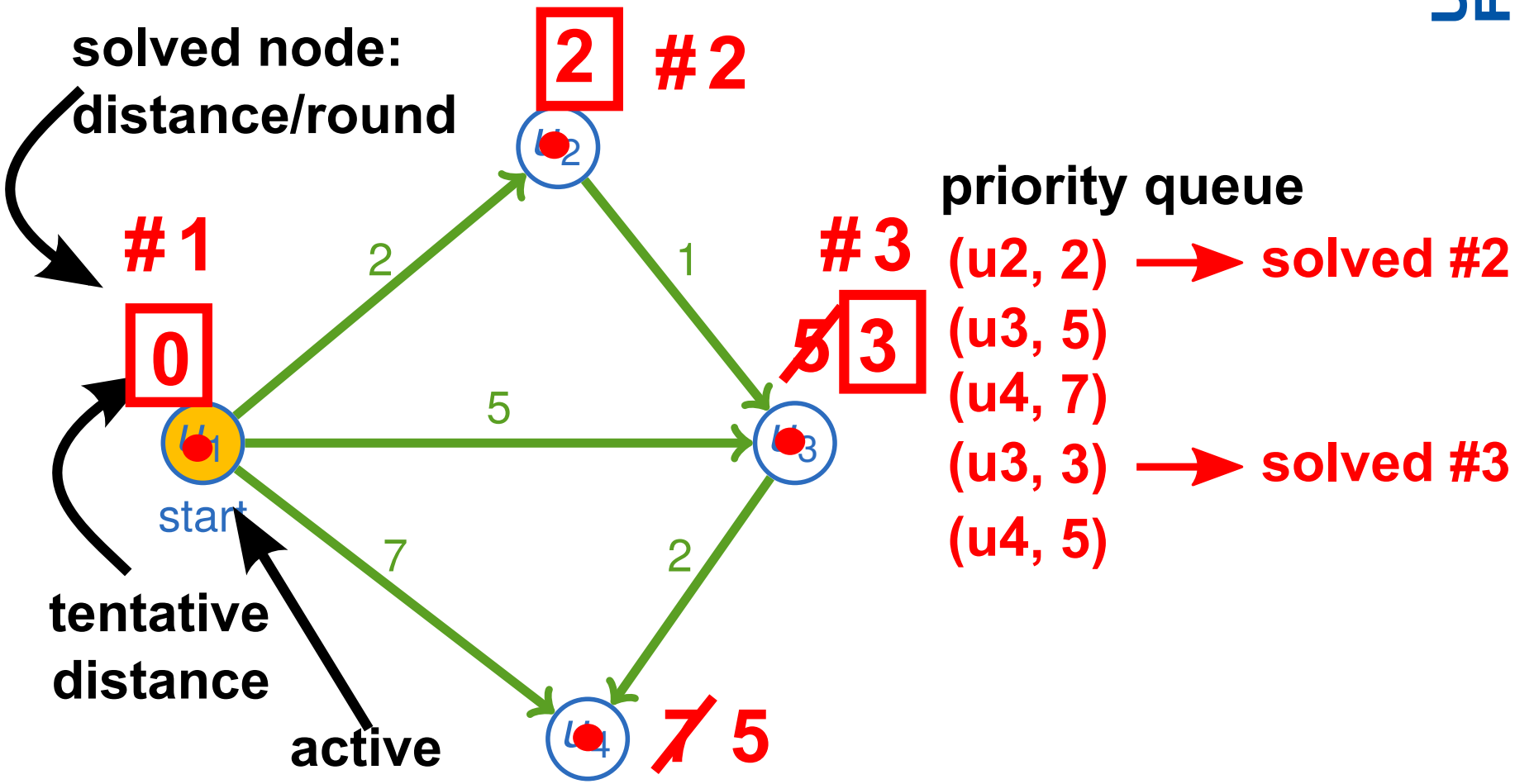

# Dijkstra Algorithm

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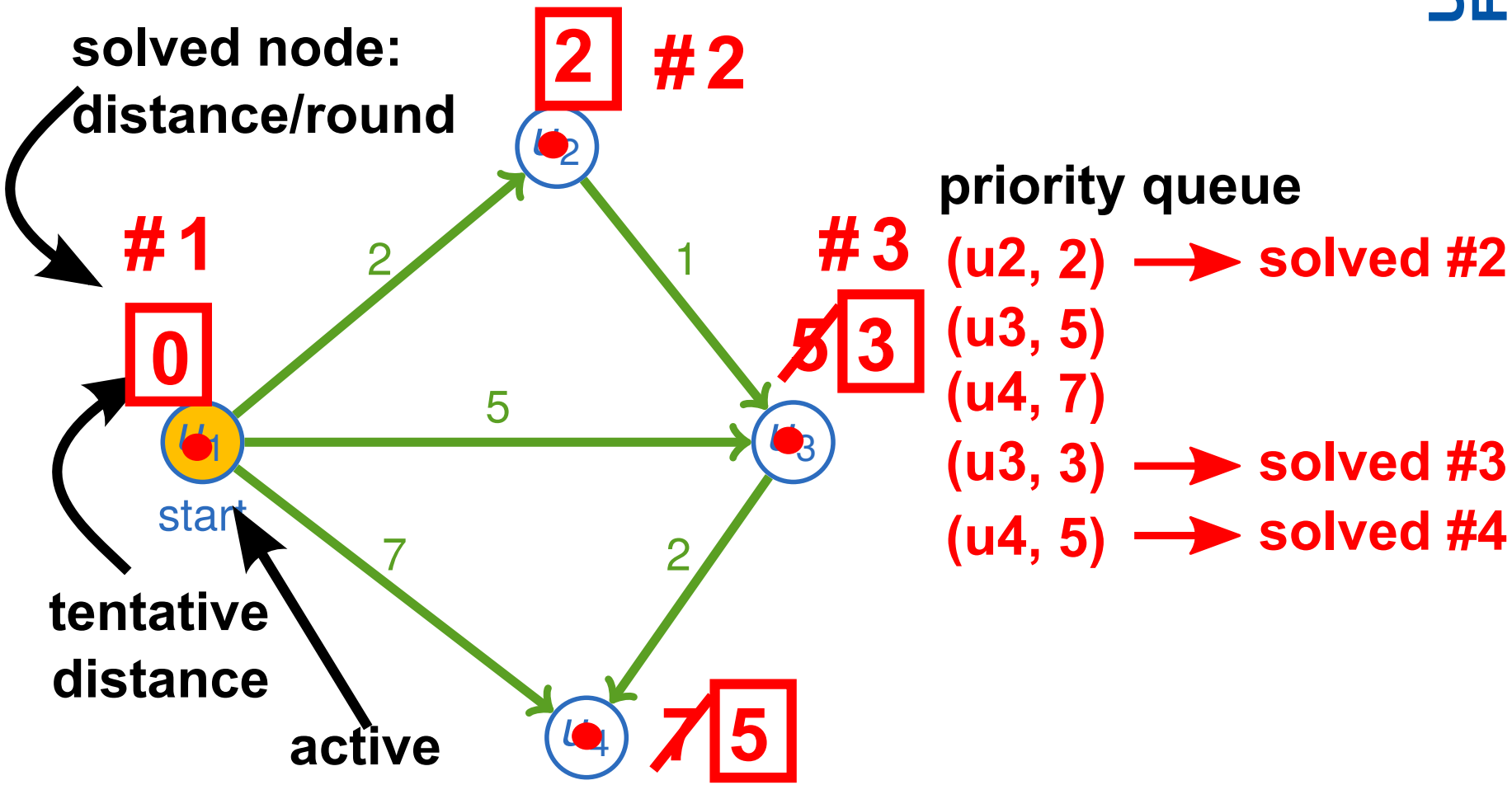
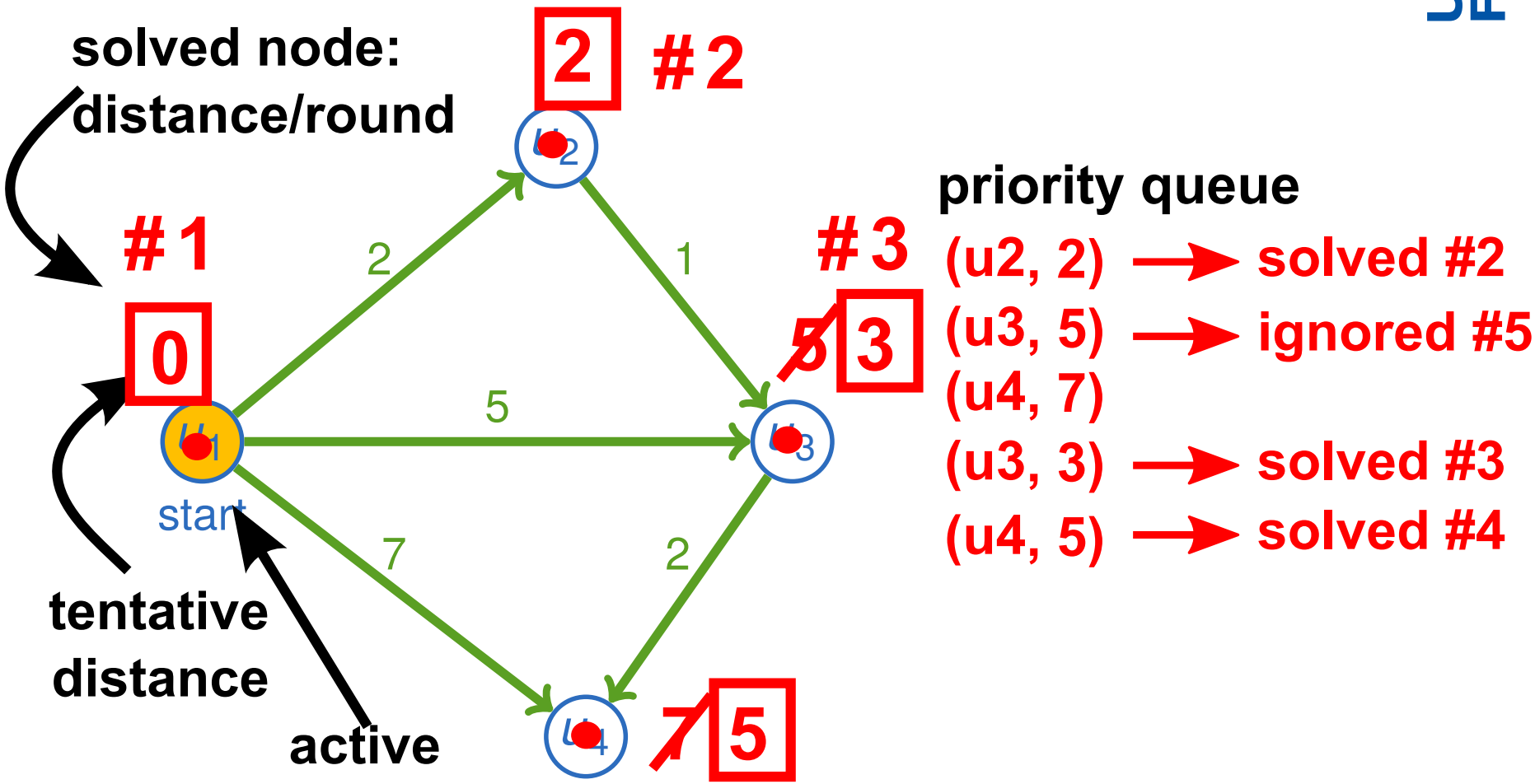


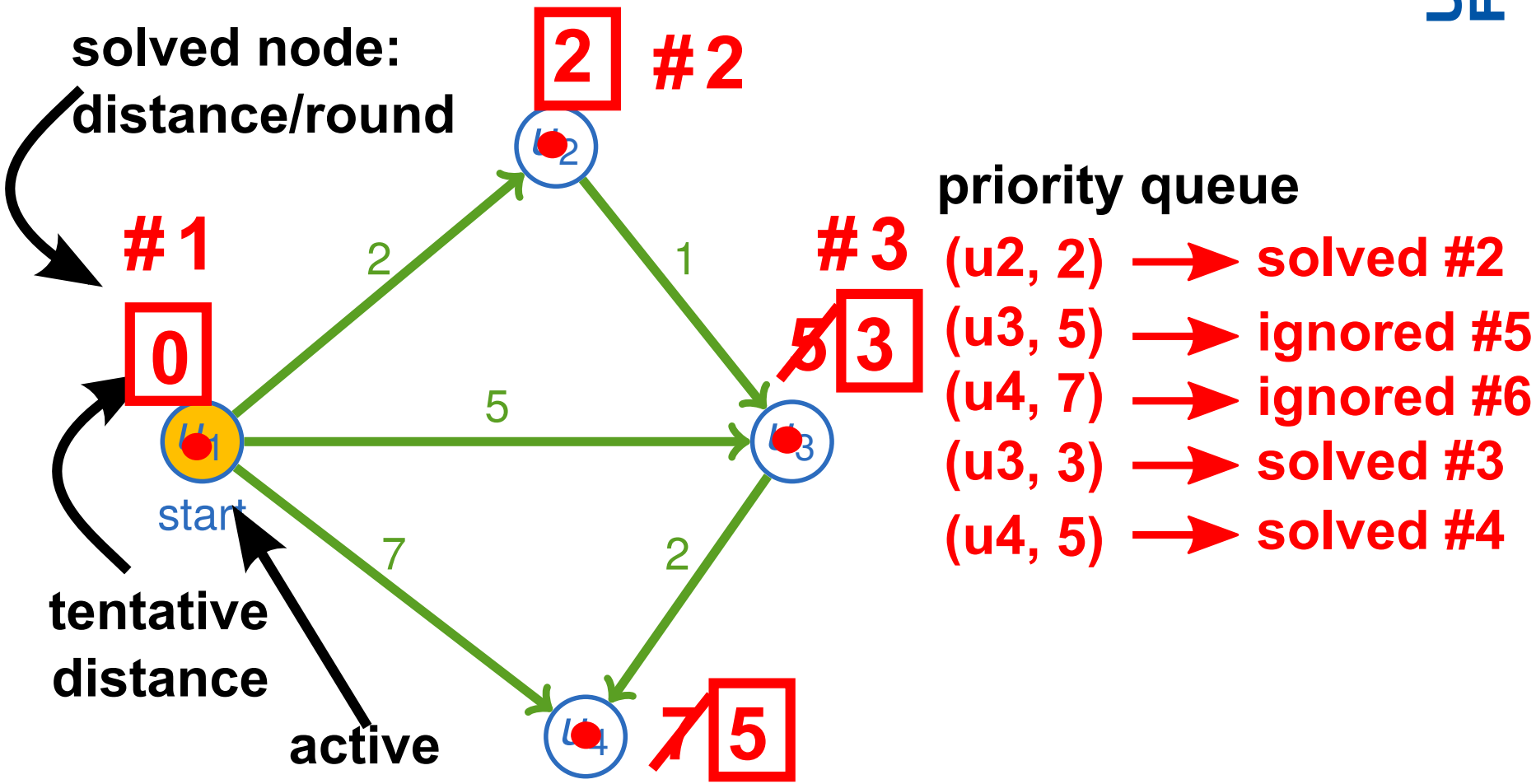
# Dijkstra Algorithm

## Implementation - Example



# Dijkstra Algorithm

## Implementation - Example



# Dijkstra Algorithm

## Runtime analysis

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Graph with $n$ nodes and $m$ edges: $(m \geq n)$

- Each node gets solved exactly one time
- When solving a node it's outgoing edges are taken into account
- Each edge triggers at maximum one `insert` operation
- The number of operations on the priority queue is at maximum $O(m)$
- This results in a runtime of $O(m \cdot \log m)$
  ($\log m$ because of at max. $m$ elements in the priority queue)

Runtime of $O(m \cdot \log m)$:

- Because of $m \leq n^2$ we have a maximum runtime of $O(m \cdot \log n)$, because $\log n^2 = 2 \log n$
- With a complex priority queue the runtime can be reduced to $O(m + n \log n)$
  - For example with a **Fibonacci heap**
  - This results in a better runtime for complex graphs $m \sim n^2$
  - Complex heaps create a management overhead

  $\Rightarrow$ In practice $m \in O(n)$ with a **binary heap** being faster (See lecture 6)

**Termination criteria:**

- Terminate as soon as the target node $t$ is settled
  ... never before because tentative distance might change:

$$\mathrm{td}(t) \geq \mathrm{dist}(s,t)$$

- Before the node $t$ is solved all nodes $u$ with $\mathrm{dist}(s,u) \leq \mathrm{dist}(s,t)$ are settled

**Termination criteria:**

- Not only the single source **single** target shortest path problem is solved by the Dijkstra algorithm but also the single source **all** targets problem
- This sounds wasteful but there is not a (much) better method for general graphs
  **Intuitive:** We only know that there is no shorter path if all nodes in the distance of $\mathrm{dist}(s,t)$ are evaluated

**Calculate the shortest path:**

- With the current implementation of the Dijkstra algorithm we only get the length of the path
  How to get the path itself too?
- If we save the preceding node of the current shortest path on settling of each node we can reconstruct the path

# Dijkstra Algorithm

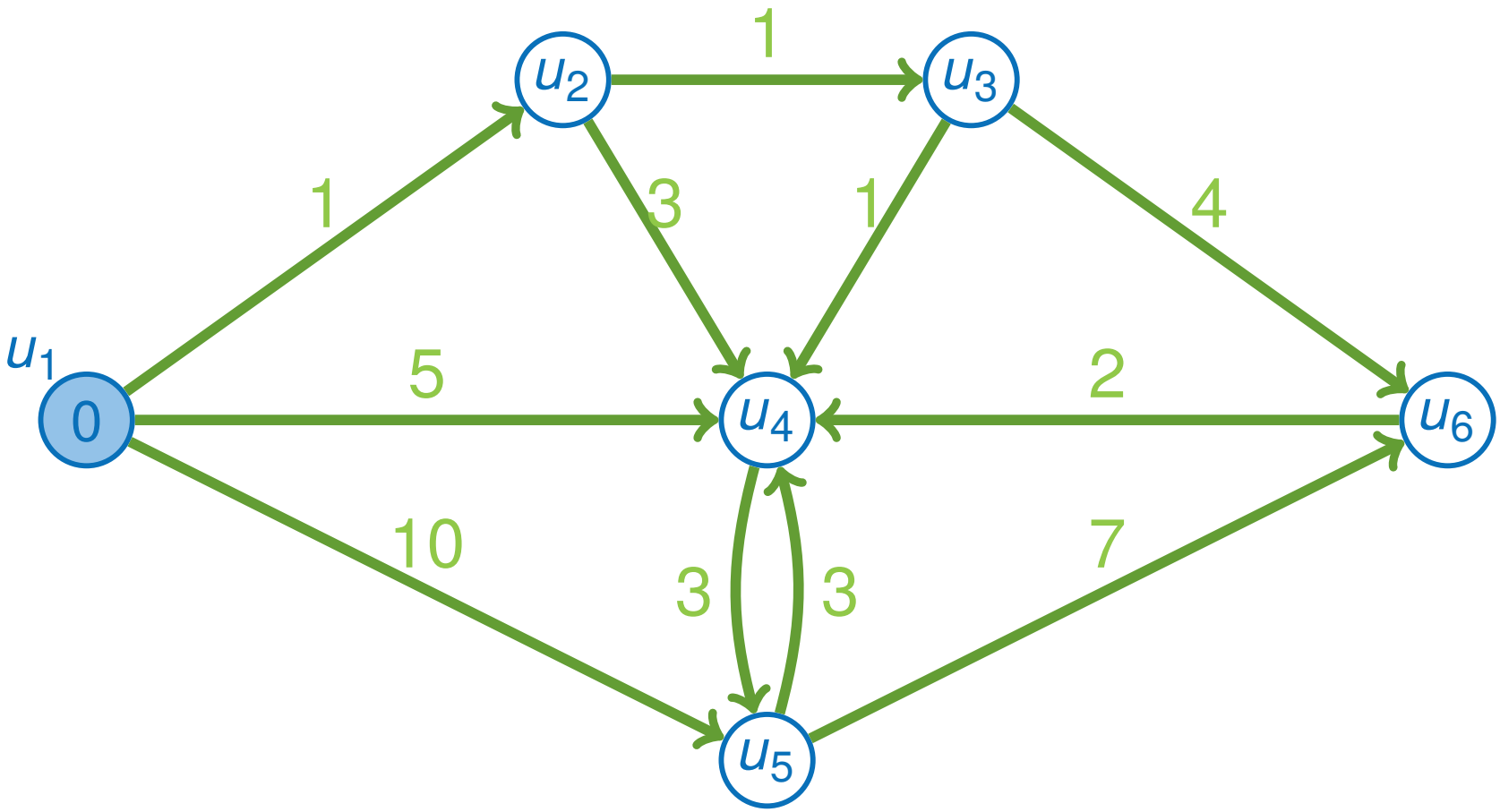



Figure: Start at $u_1$

# Dijkstra Algorithm

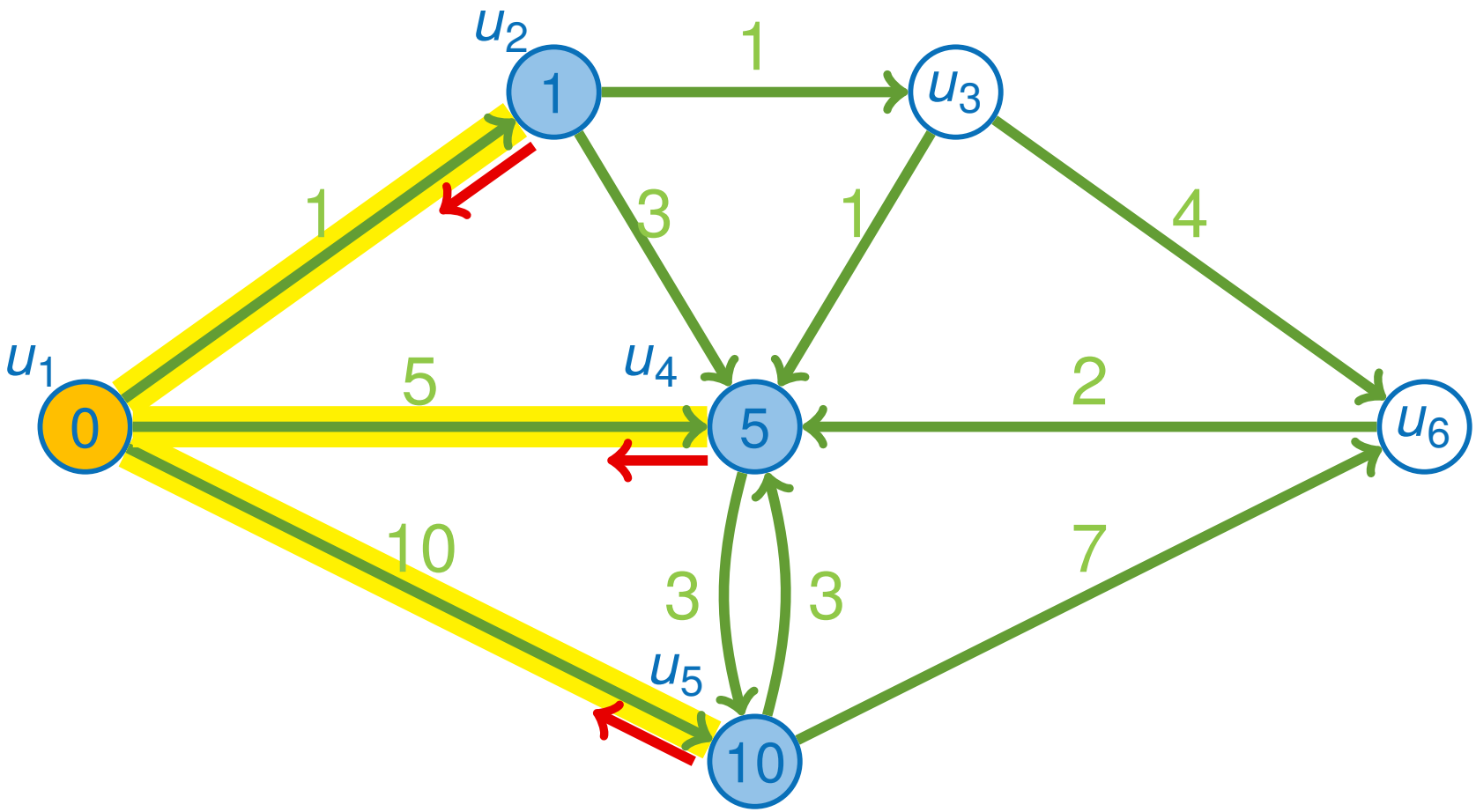



Figure: Iteration 1

# Dijkstra Algorithm

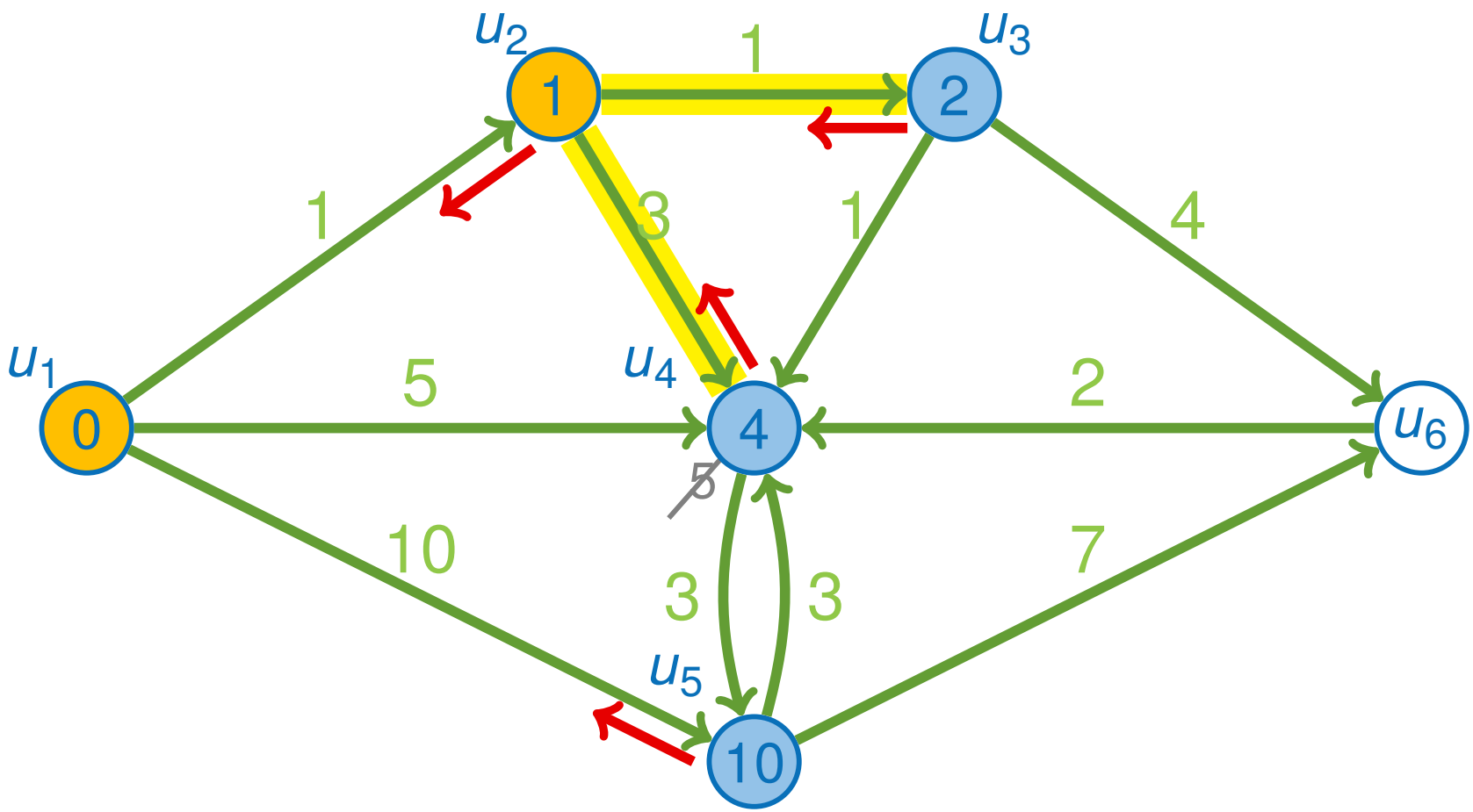



Figure: Iteration 2

# Dijkstra Algorithm

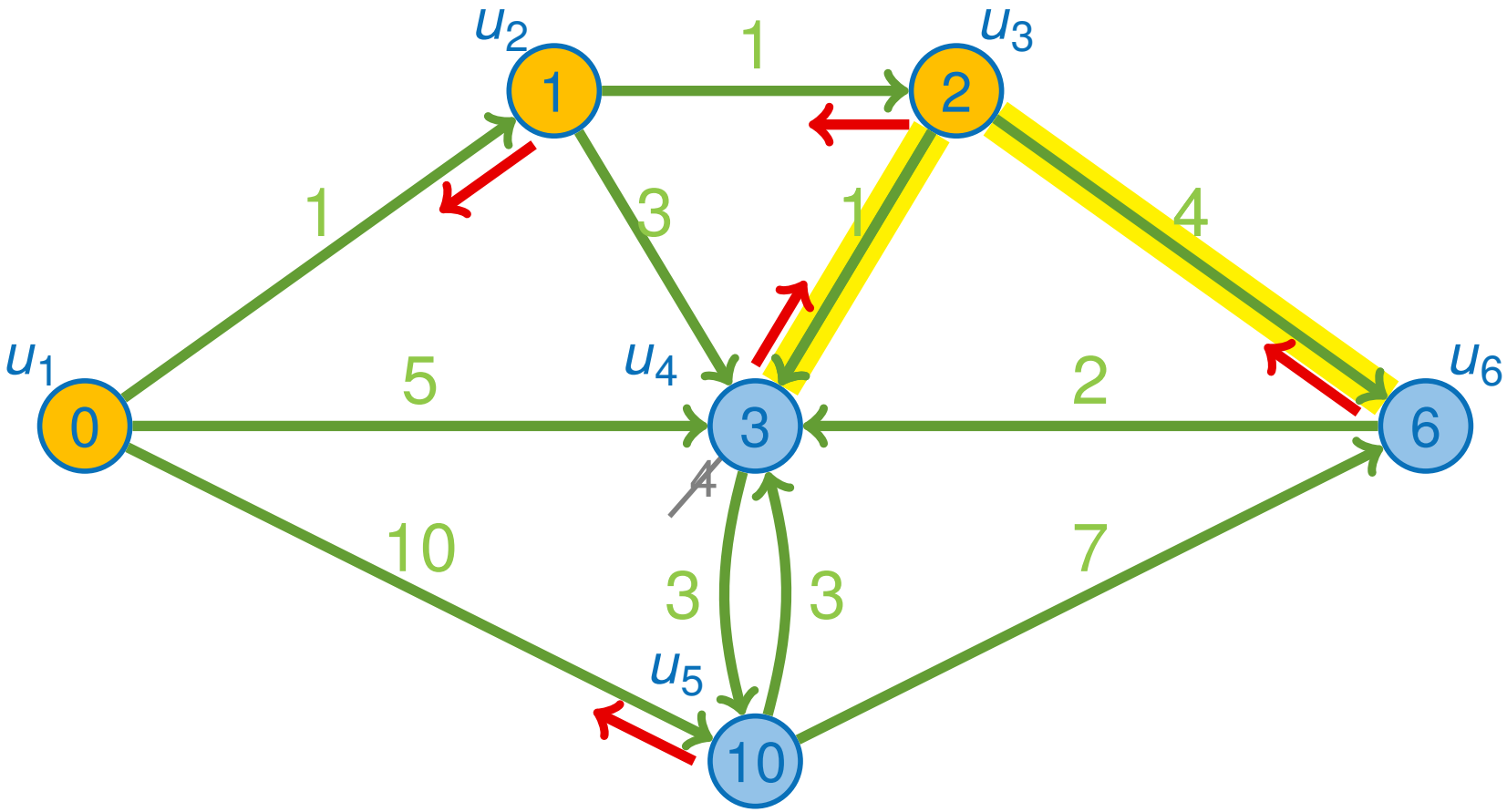



Figure: Iteration 3

# Dijkstra Algorithm

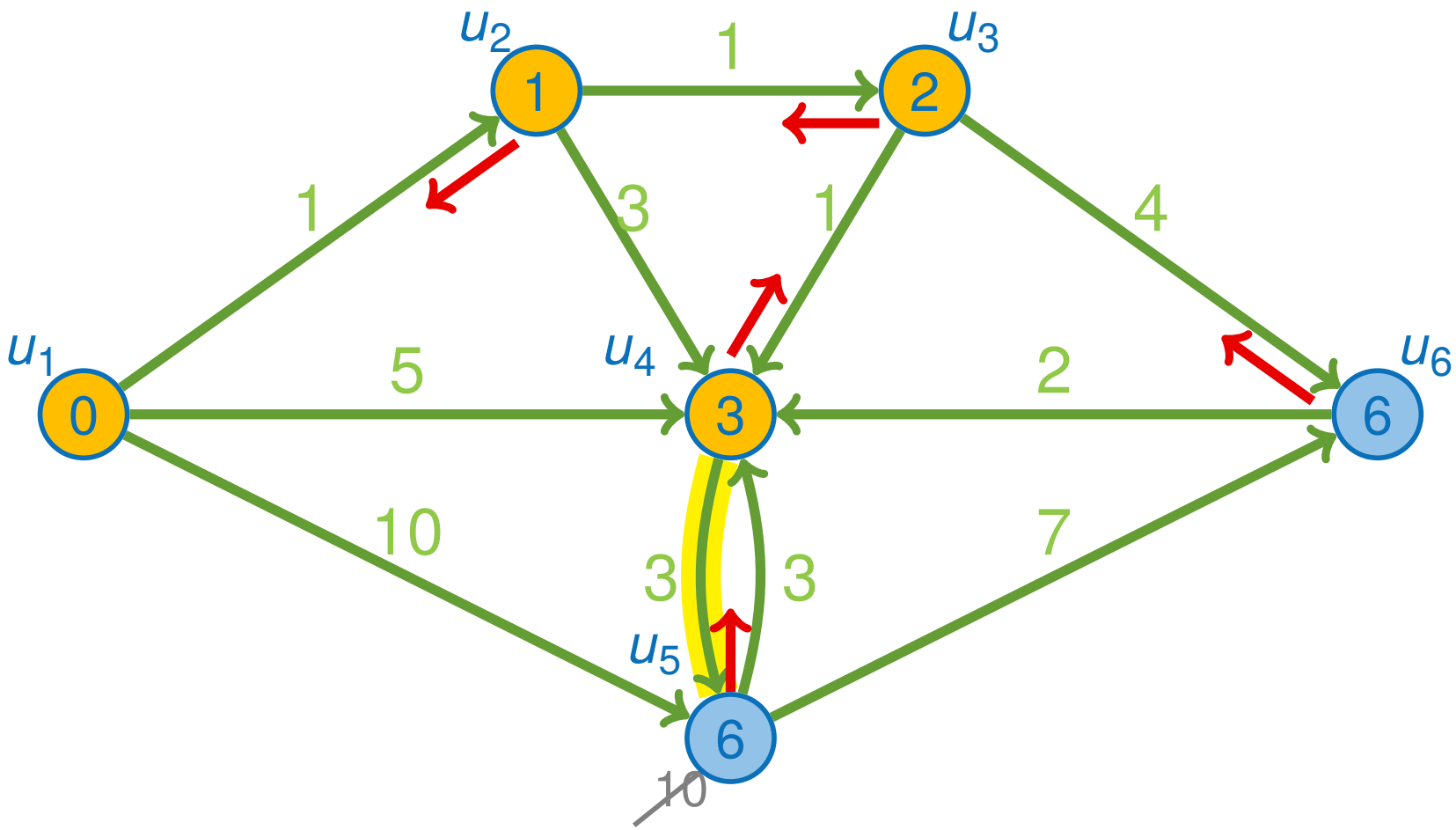



Figure: Iteration 4

# Dijkstra Algorithm

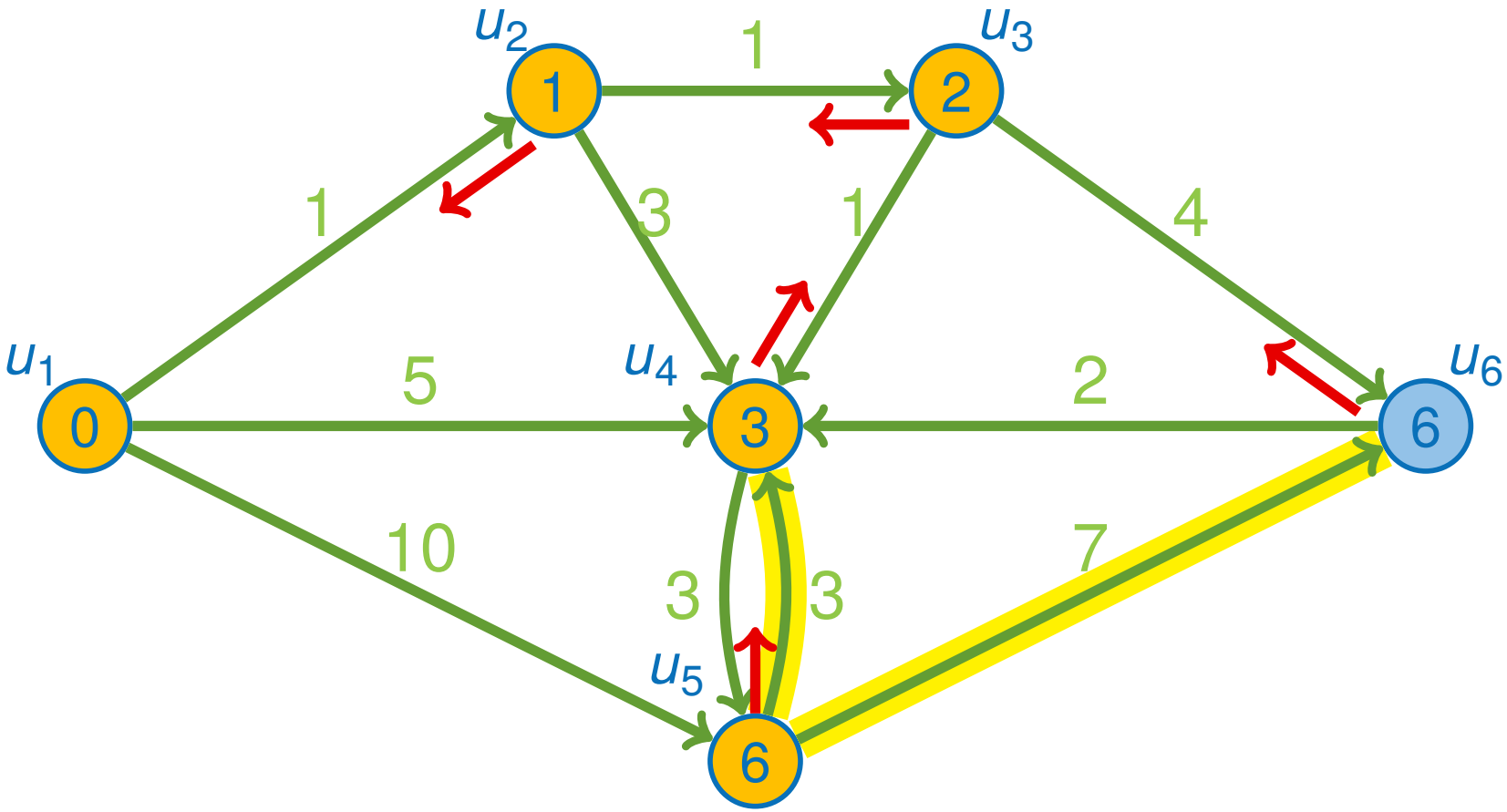



Figure: Iteration 5

# Dijkstra Algorithm

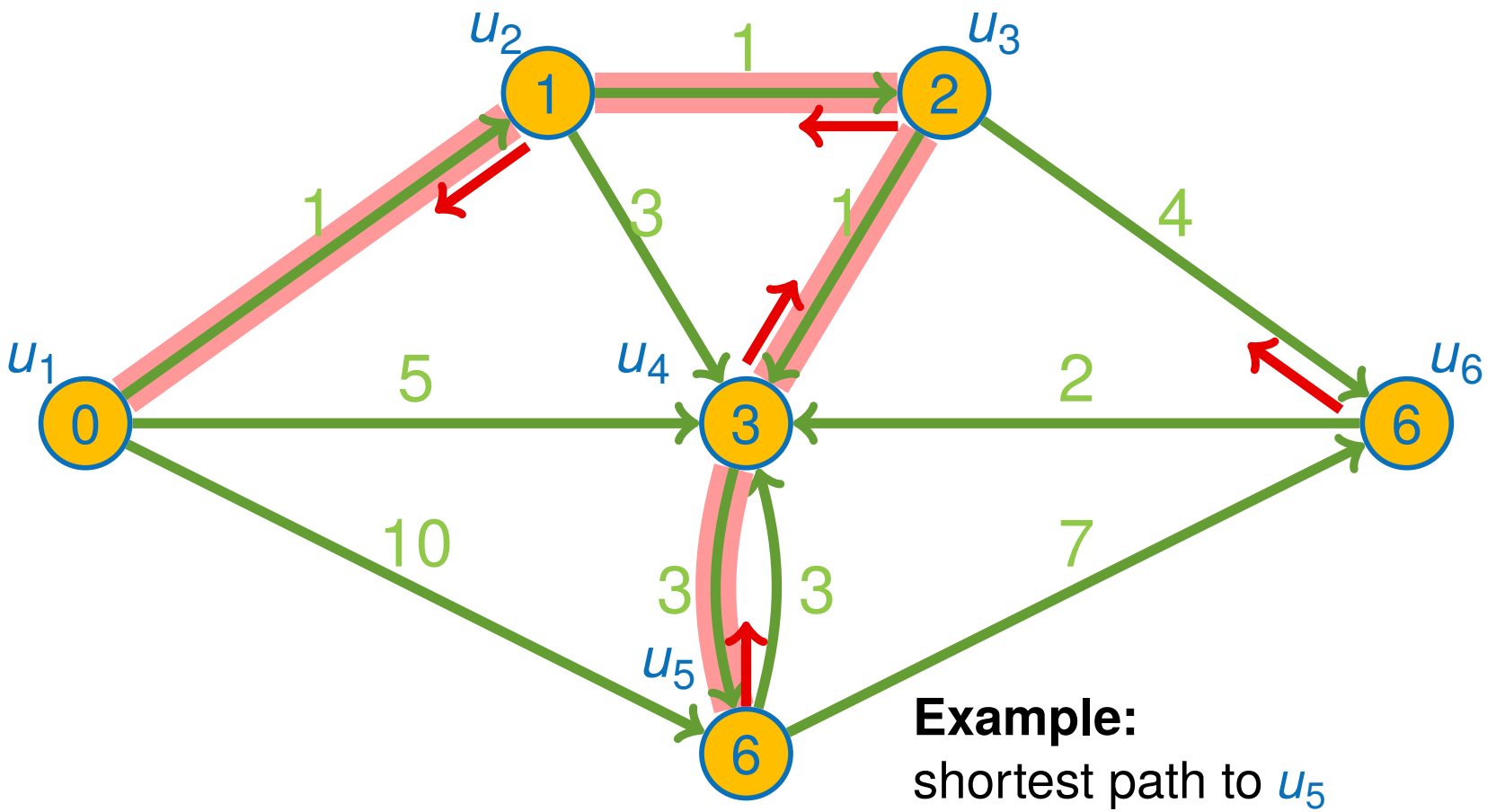



Figure: Iteration 6

**Enhancement:**

- In our proof we used the assumption that all costs are not negative (even $> 0$)
- With negative costs there might be negative cycles:

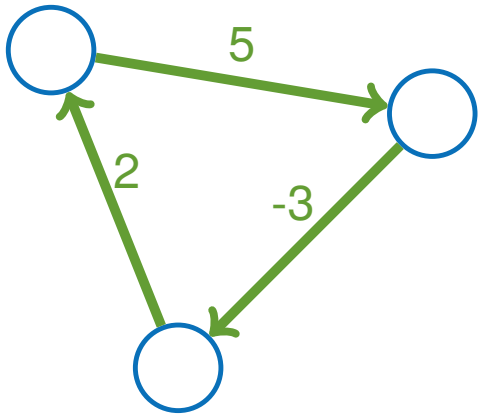



Figure: Here no problem ...

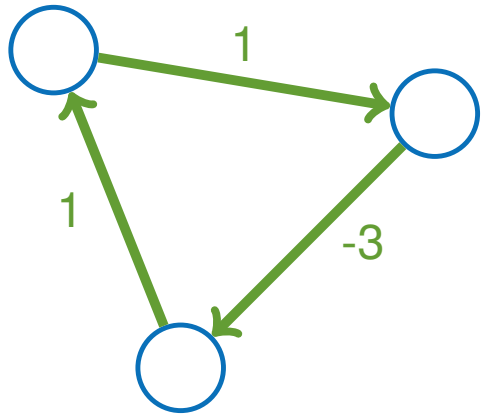
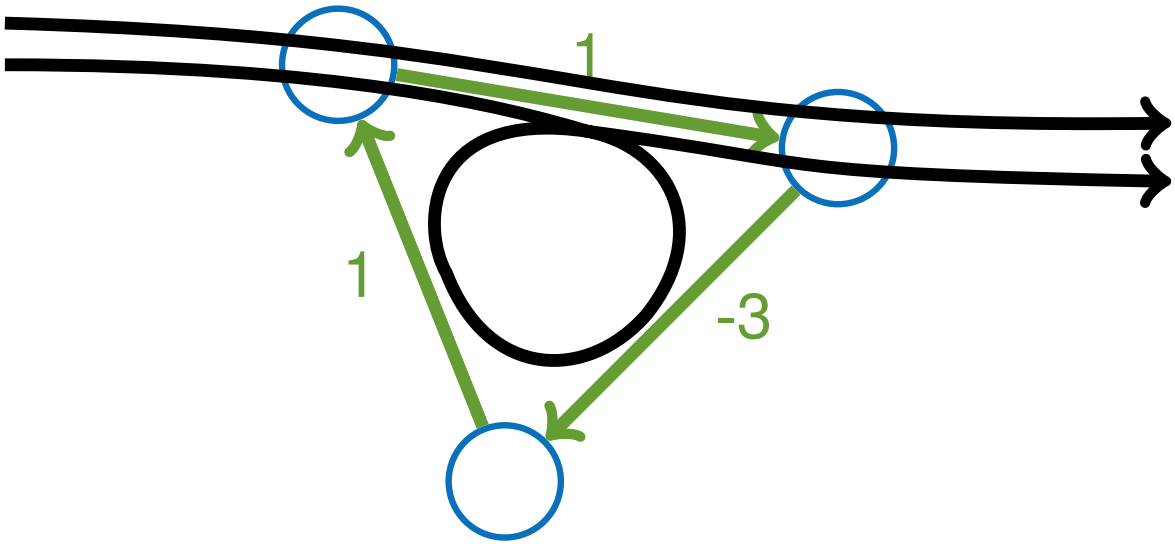



Figure: ... but here

## Negative cycles:



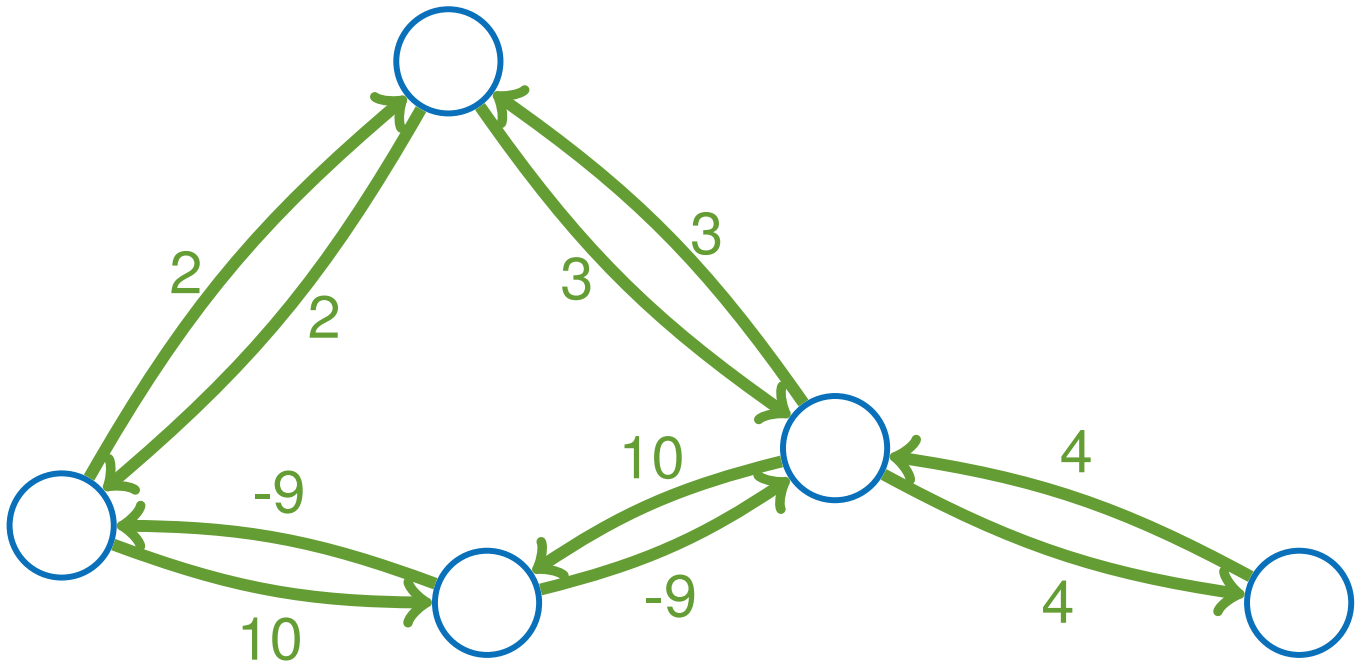

- No cycle:
  cost of 1
- 1 cycle:
  cost of 0
- 2 cycles:
  cost of -1
- 3 cycles:
  cost of -2
- ...

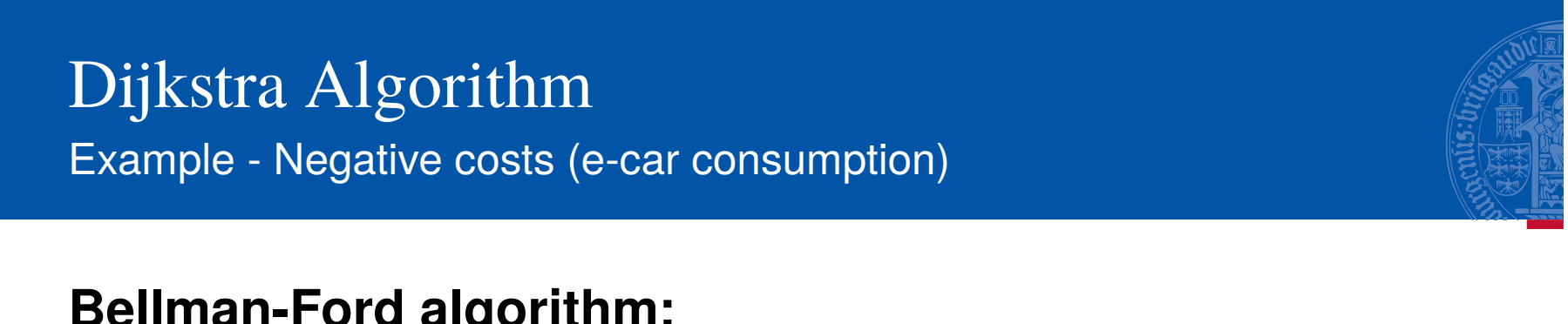
**Enhancement:**

- We need a different algorithm to deal with negative edges
  - For example the **Bellman-Ford** algorithm
  - If the graph is acyclic we can simply use a topological sorting (with DFS) and settling the nodes in order of this sorting
- Another (not only) in artificial intelligence used variant of the Dijkstra algorithm is the A* algorithm
  Additional information given:
  $h(u)$ = estimated value for $\mathrm{dist}(u,t)$

**Dijkstra algorithm:**

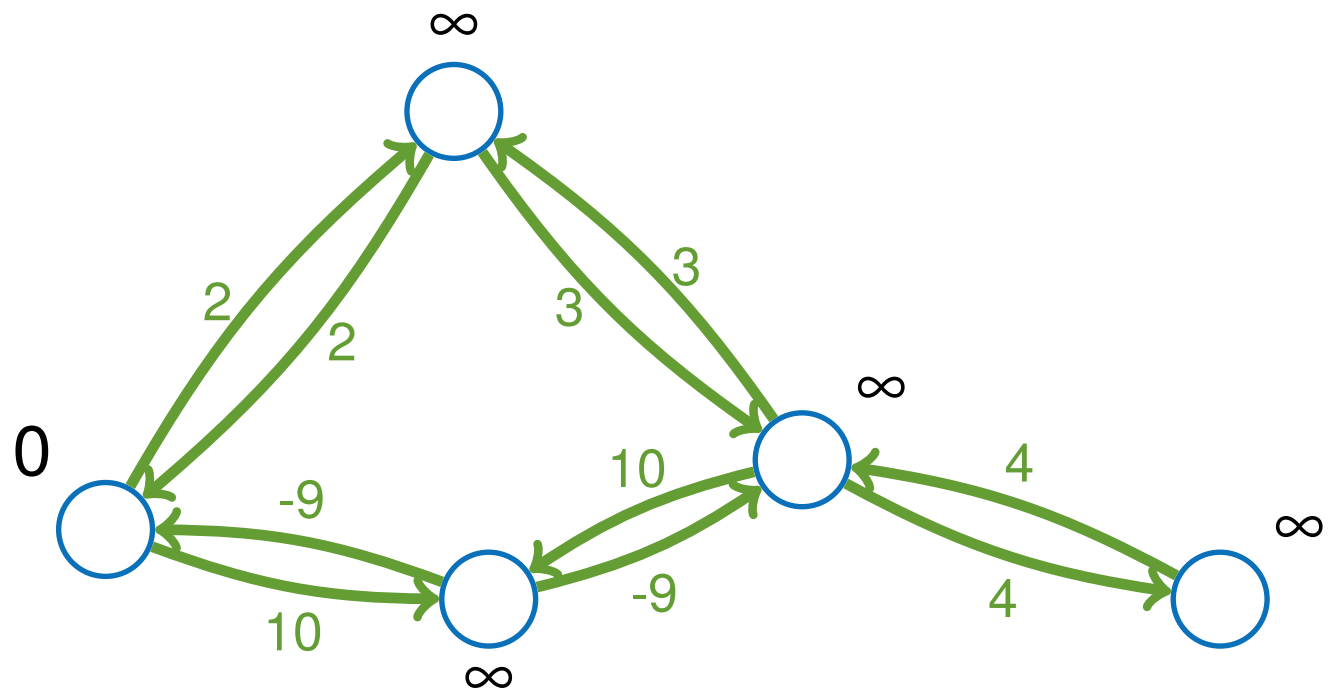
Message passing only from solved nodes





**Bellman-Ford algorithm:**

Message passing from all nodes until the path lengths are stable



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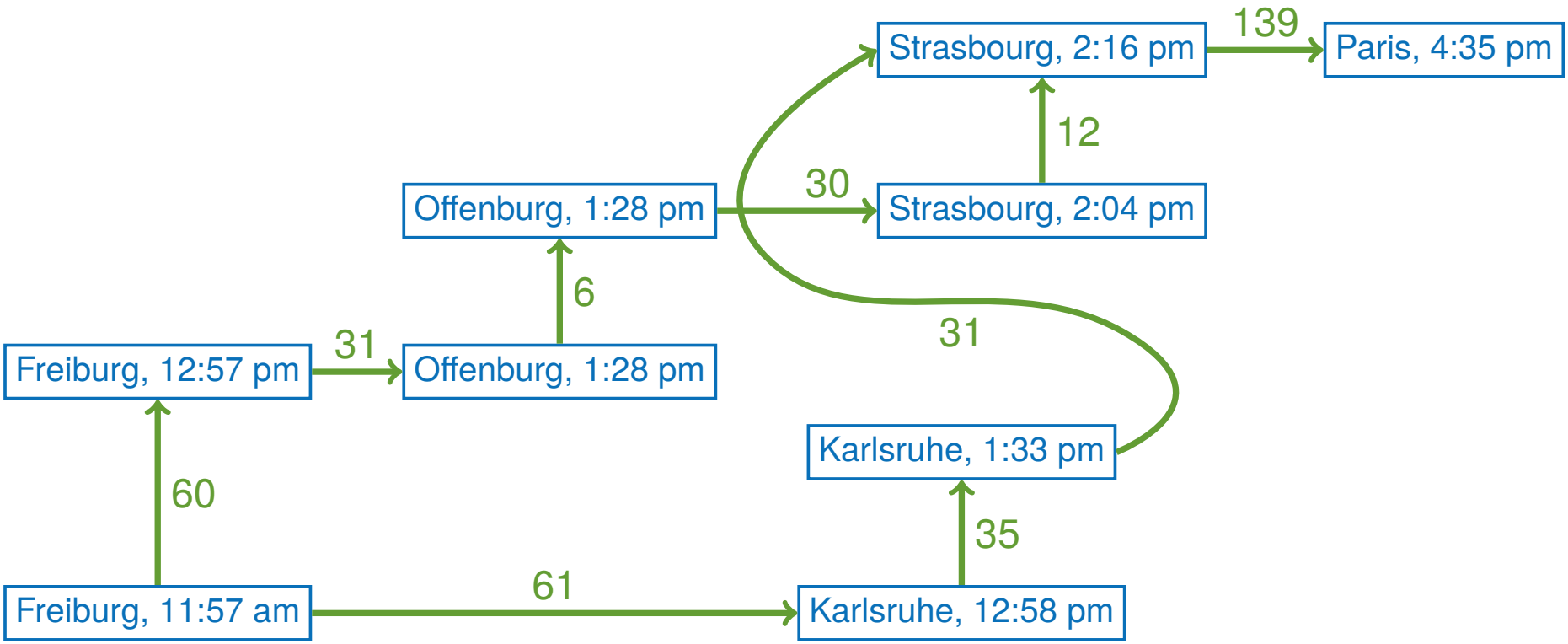
**Application example:**

- Route planner for car trips (exercise sheet)
- Route planner for bus / train connections
  What could the graph look like?

## Space-time graph:



# Dijkstra Algorithm

## Application in image processing

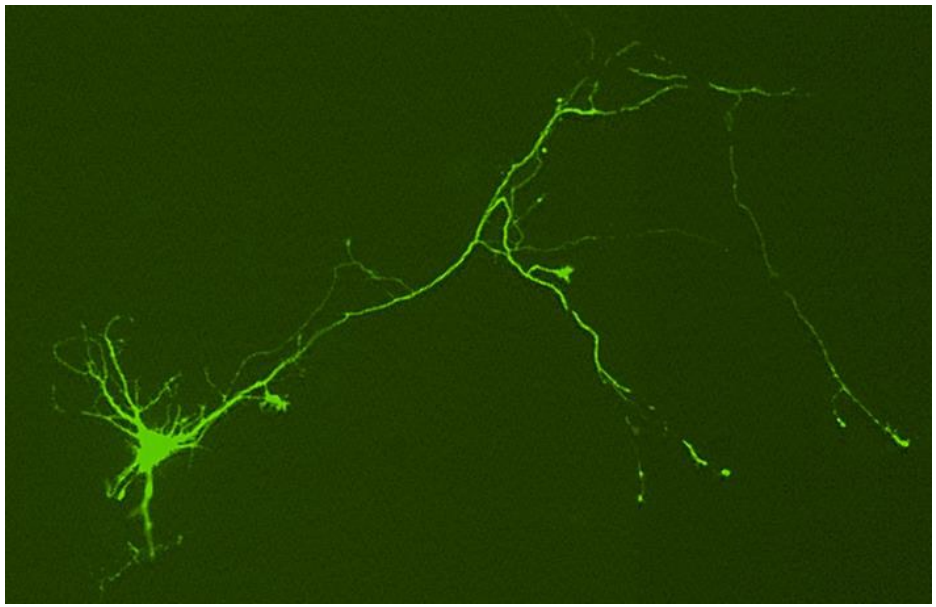
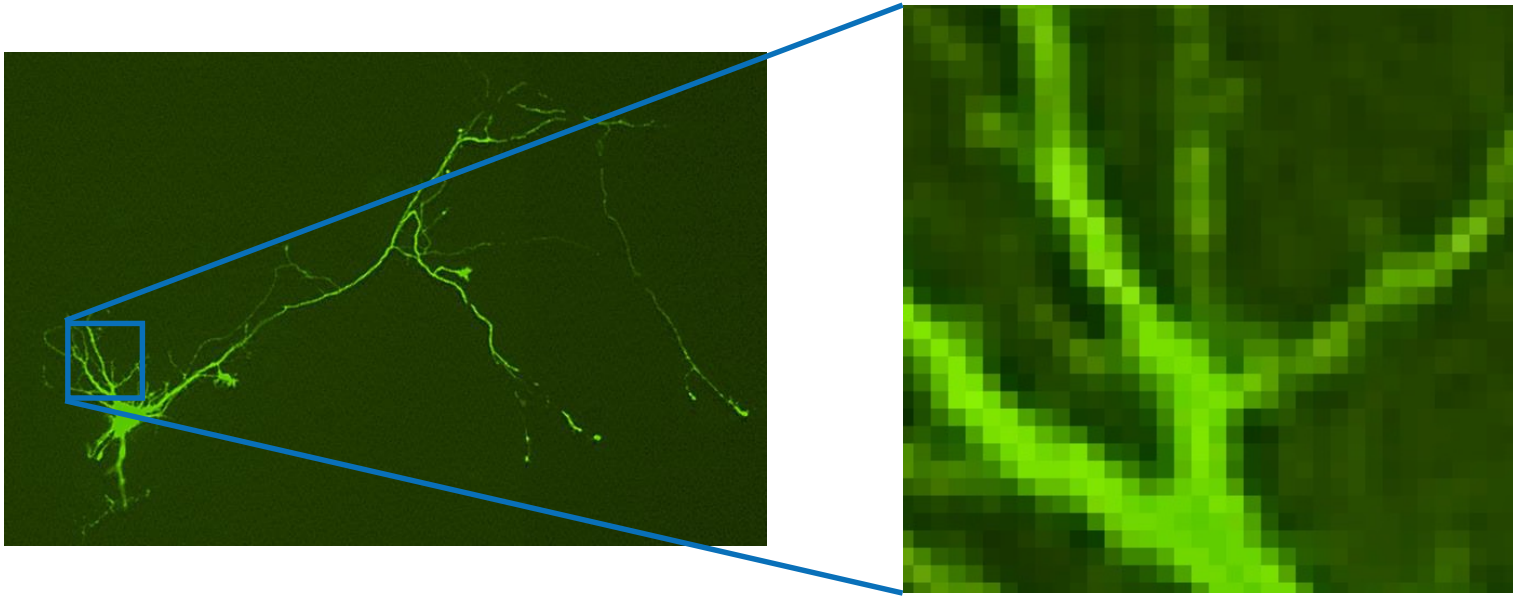


Figure: Neurons under fluorescence microscope

- **Task:** Measure length of axons (connections of neurons)
- Demo with ImageJ plugin NeuronJ
  `http://www.imagescience.org/meijering/software/neuronj/`

# Dijkstra Algorithm

## Application: Trace axons



- Image as graph: Each pixel is a node
- Implicit edges: Each pixel has an edge to it's 8 neighbours (no need to save the edges)
- Costs for nodes (not edges): bright pixels are cheap, dark pixels are costly

# Further Literature

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- **General**


[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
**Introduction to Algorithms**.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

# Further Literature

UNI FREIBURG

- **Dijkstra's algorithm**

  [Wik] Dijkstra's algorithm
  `https://en.wikipedia.org/wiki/Dijkstra's_algorithm`

- **Shortest path problem**

  [Wik] Shortest path problem
  `https://en.wikipedia.org/wiki/Shortest_path_problem`