Algorithmns and Datastructures

Divide and Conquer, Master theorem

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UNI FREIBURG

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Bioinformatics Group / Department of Computer Science

Algorithmns and Datastructures, March 2016

Structure

Structure

Divide and Conquer

- Concept
- Maximum Subtotal

Recursion Equations

- Substitution Method
- Recursion Tree Method
- Master theorem
  - Master theorem (Simple Form)
  - Master theorem (General Form)

Concept:

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- Recursive application of the algorithm on smaller subproblems
- Direct solving of small subproblems

Structure

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Input:

- Progression X of n integers

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Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

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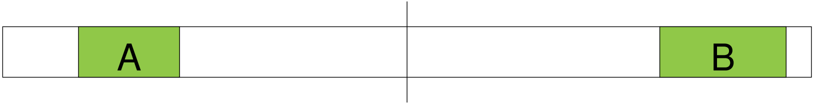
Table: Input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Idea:



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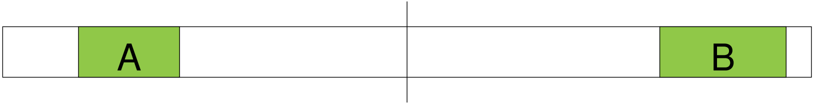

- Solve the left / right half of the problem recursive

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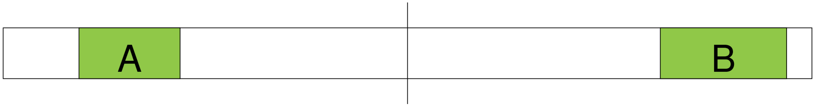

- Solve the left / right half of the problem recursive
- Combine both solutions into a overall solution

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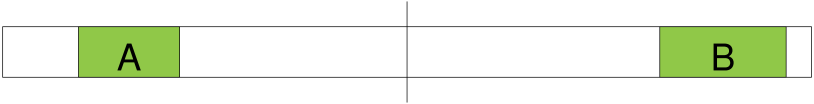

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- The maximum is located in the left half (A) or the right half (B)

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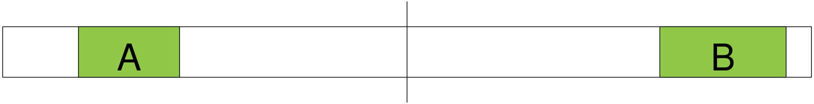
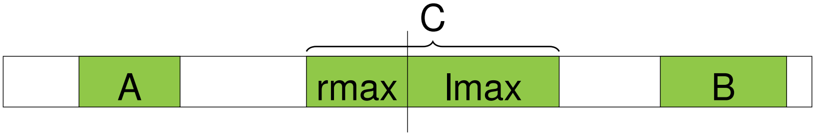


- Solve the left / right half of the problem recursive
- Combine both solutions into a overall solution
- The maximum is located in the left half (A) or the right half (B)
- The maximum interval can overlap with the border (C)

Principle:



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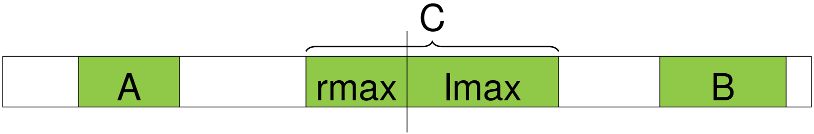

- Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$

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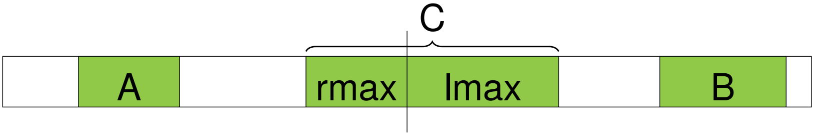

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- Big problems are decomposed into two subproblems and solved recursivly. Subsolutions A and B are returned.

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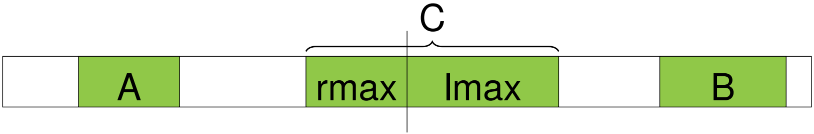

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Divide and Conquer
Maximum Subtotal

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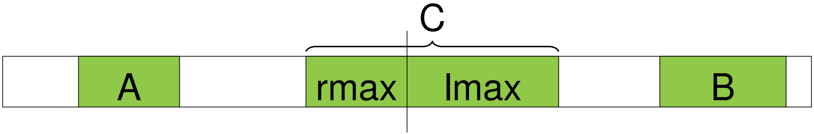

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- Big problems are decomposed into two subproblems and solved recursivly. Subsolutions A and B are returned.
- To solve C we have to calculate *rmax* and *lmax*
- Overall solution is maximum of A B C

```
def maxSubArray(X, i, j):
```

```python
def maxSubArray(X, i, j):
    if i == j: # trivial case
        return (X[i], i, i)
    m = (i + j) / 2
```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):
    if i == j: # trivial case
        return (X[i], i, i)
    m = (i + j) / 2
        #Solutions for A and B
    A = maxSubArray(X, i, m)
    B = maxSubArray(X, m + 1, j)
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def maxSubArray(X, i, j):
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        #Solutions for A and B
    A = maxSubArray(X, i, m)
    B = maxSubArray(X, m + 1, j)
        #rmax and lmax for bordercase C
    C1 = rmax(X, i, m)
    C2 = lmax(X, m + 1, j)
    C = (C1[0] + C2[0], C1[1], C2[1])
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    return max([A, B, C], \
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    C = (C1[0] + C2[0], C1[1], C2[1])
        #Solution is maximum of A,B,C
    return max([A, B, C], \
        key=lambda item: item[0])
        #Simplification: only maximum
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
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def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
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#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
    # trivial: only two elements
    if i + 1 = j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])
    ... # continue as before
```

```
#Implementation max
def max(a, b, c):
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
```

```python
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

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#Alternative implementation max
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def max(a, b):
    if a > b:
        return a
    else:
        return b
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```python
#Alternative implementation max

def max(a, b):
    if a > b:
        return a
    else:
        return b

def maxTripel(a, b, c):
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```python
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	i	$i+1$	$\cdots$	$\cdots$	$j-1$	j
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
lmax	58	58	58	90	90	90

Table: *lmax* example

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X	58	-53	26	59	-41	31
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- The *sum* and *lmax* are initialized with $X[i]$

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- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i+1$ to j and update *sum*

Table: *lmax* example

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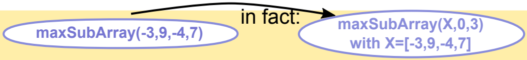
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
lmax	58	58	58	90	90	90

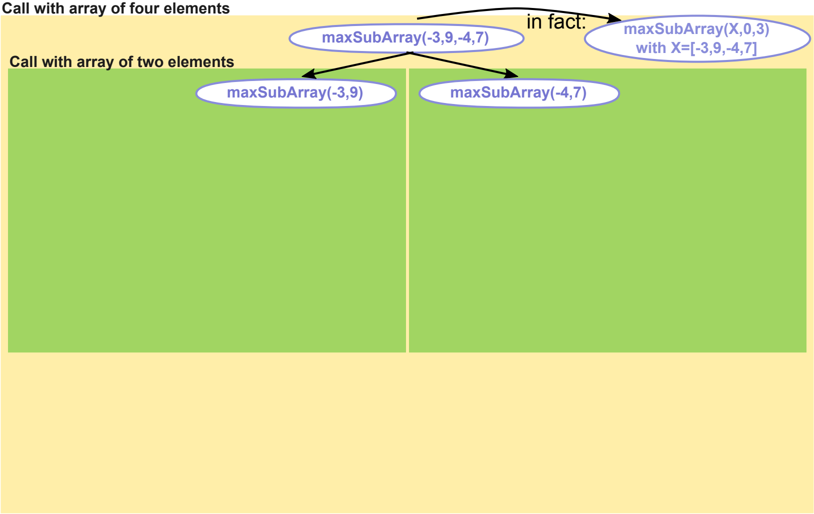
- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i+1$ to j and update *sum*
- If $s > lmax$ then *lmax* gets updated

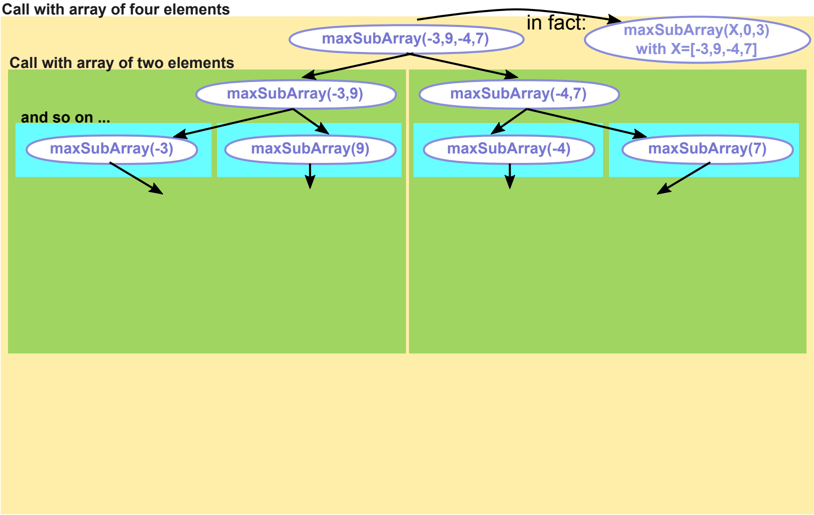
Call with array of four elements



Divide and Conquer

Maximum Subtotal





Call with array of four elements
maxSubArray(-3,9,-4,7)
in fact:
maxSubArray(X,0,3)
with X=[-3,9,-4,7]
Call with array of two elements
maxSubArray(-3,9)
maxSubArray(-4,7)
and so on ...
maxSubArray(-3)
maxSubArray(9)
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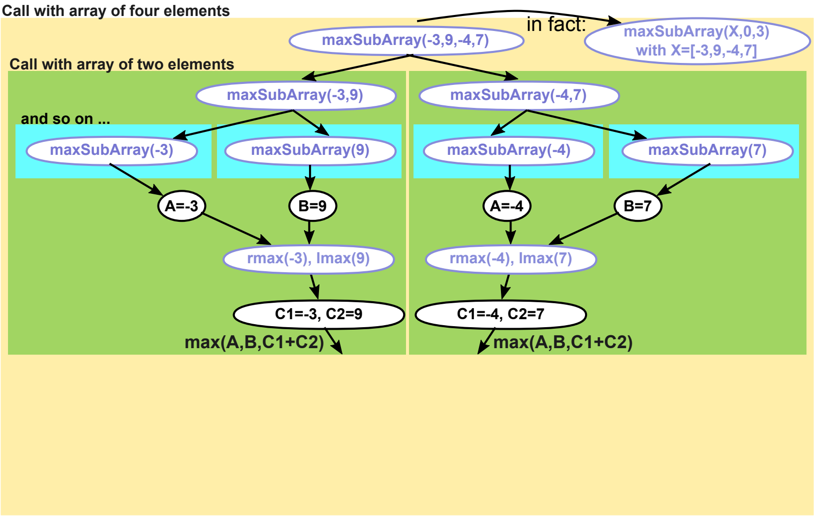
maxSubArray(7)



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Call with array of two elements
maxSubArray(-3,9)
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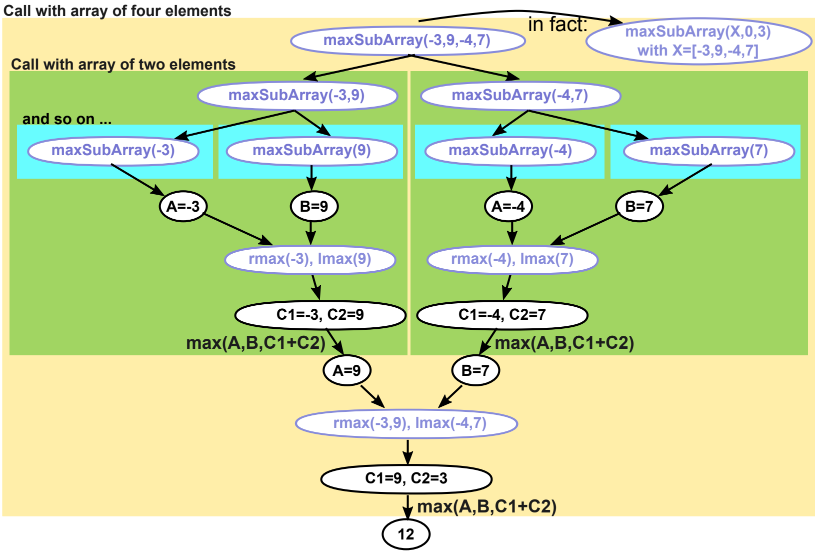
and so on ...
maxSubArray(-3)
maxSubArray(9)
maxSubArray(-4)
maxSubArray(7)
A=-3
B=9
A=-4
B=7
rmax(-3), lmax(9)
rmax(-4), lmax(7)
C1=-3, C2=9
C1=-4, C2=7
max(A,B,C1+C2)
max(A,B,C1+C2)

Divide and Conquer

Maximum Subtotal



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    m = (i + j) / 2
    A = maxSubArray(X, i, m)
    B = maxSubArray(X, m + 1, j)

    C1 = rmax(X, i, m)
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    C = (C1[0] + C2[0], C1[1], C2[1])

    return max([A, B, C], \
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def maxSubArray(X, i, j):
    if i == j:                                      # O(1)
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    m = (i + j) / 2                                     # O(1)
    A = maxSubArray(X, i, m)                            # T(n/2)
    B = maxSubArray(X, m + 1, j)

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Divide and Conquer

Maximum Subtotal - Python

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    return max([A, B, C], \                   # O(1)
        key=lambda item: item[0])
```

Divide and Conquer

Maximum Subtotal - Number of steps $T(n)$

Recursion equation:

$$T(n) = \begin{cases} \underbrace{\Theta(1)}_{\text{trivial case}} & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{cobination of solutions}} & n > 1 \end{cases}$$

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- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

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- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

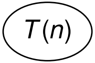
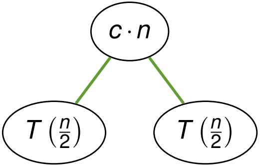
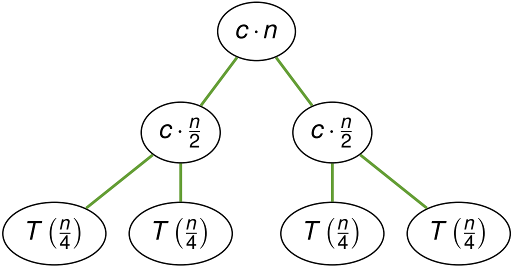


Figure: Illustration of the runtime



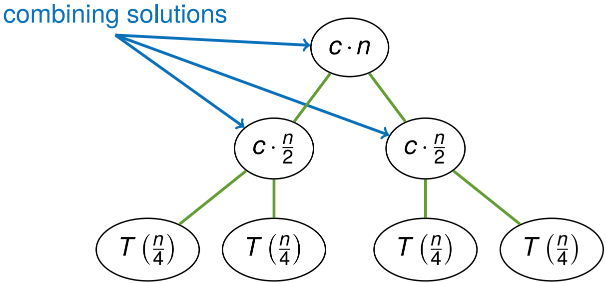
$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime



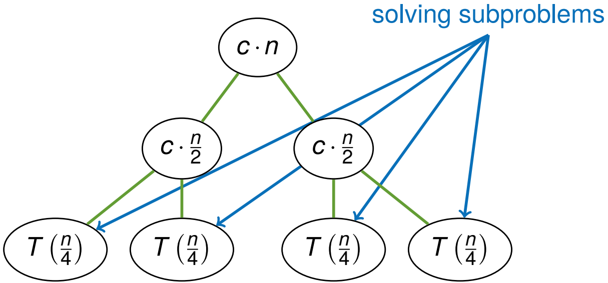
$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime



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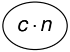

1 node processing n elements
$\Rightarrow c \cdot n$

Figure: Recursion tree method

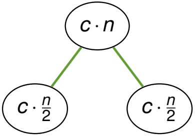

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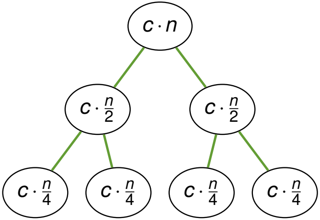
Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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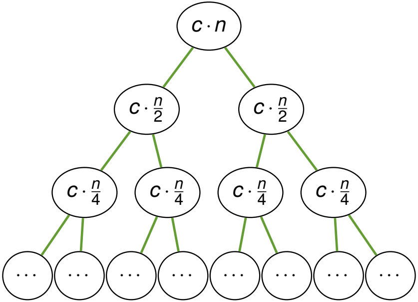

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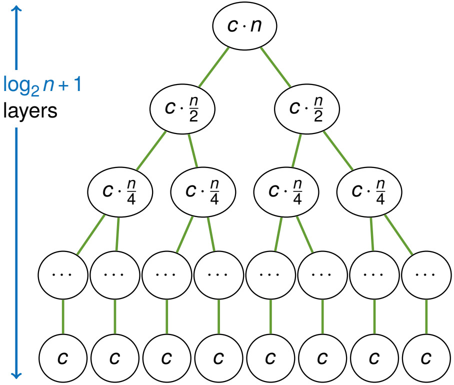

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n nodes processing 1 element
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- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$
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$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Summary:

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- Direct solution is slow with $O(n^3)$

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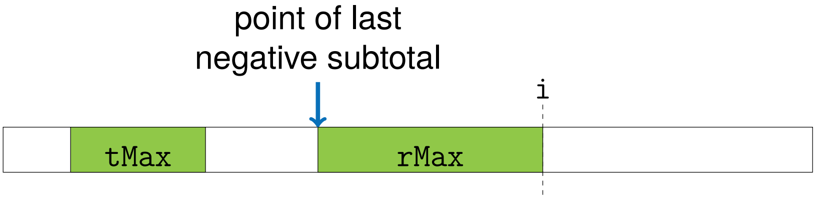

Figure: Scanning the array in linear time

```
#Implementation - linear runtime
def maxSubArray(X):
```

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def maxSubArray(X):
    # sum, start index
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def maxSubArray(X):
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        if rMax == 0:
            irMax = i
        rMax = max(0, rMax + X[i])
```

```python
#Implementation - linear runtime
def maxSubArray(X):
    # sum, start index
    rMax, irMax = 0, 0 # current maximum
    tMax, itMax = 0, 0 # total maximum

    for i in range(len(X)):
        if rMax == 0:
            irMax = i
        rMax = max(0, rMax + X[i])

        if rMax > tMax:
            tMax, itMax = rMax, irMax

    return (tMax, itMax)
```

Structure

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$

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- $T(n)$ is only defined for integers of $\frac{n}{b}$ which is often ignored in benefit of a simpler solution

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Substitution Method:

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- Assumption: $T(n) = n + n \cdot \log_2 n$

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&= c \cdot n \log_2 n - c \cdot n + n \\
&\leq c \cdot n \log_2 n, \quad c \geq 1
\end{aligned}
$$

Structure

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- Can be used to make assumptions about the runtime

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$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

Recursion Equations

Recursion Tree Method

$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$

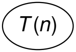

Figure: Recursion tree of example

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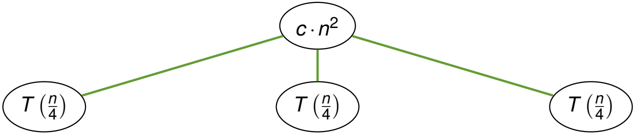

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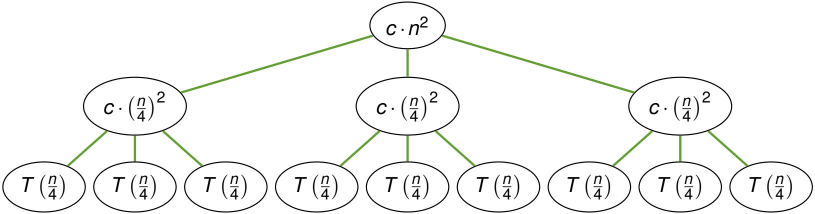

$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$

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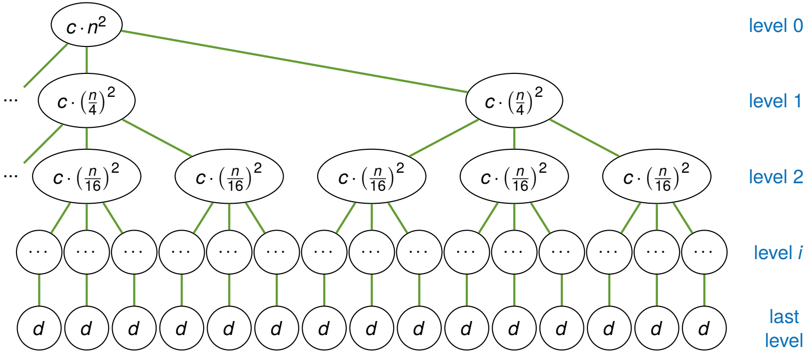

Figure: Levels of the recursion tree

Costs of connecting the partial solutions:
(excludes the last layer)

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- Costs on level i:

$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n\right)^2 = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$$

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- transforming $3^{\log_4 n}$ uses general log rules

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&= n^{\log_4 3}
\end{aligned}
$$

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- transforming $3^{\log_4 n}$ uses general log rules

$$
\begin{aligned}
\log_4 n &= \log_4\left(3^{\log_3 n}\right) && \text{uses } n = 3^{\log_3 n} \\
&= \log_3 n \cdot \log_4 3 && \text{uses } \log a^b = b \cdot \log a
\end{aligned}
$$

- this proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- now the whole expression:

$$
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- This term will recur in the master theorem

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- Here: The costs of connecting the partial problems dominate

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- Therefore constant

Proof of $O(n^2)$:

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- We know:

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Structure

Master theorem:

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- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

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- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

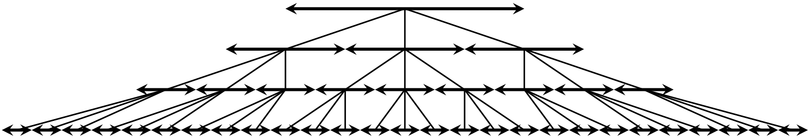
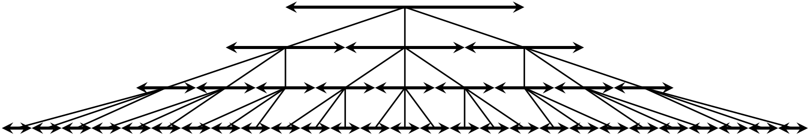


Figure: Simple recursion equation with $a = 3, b = 2$



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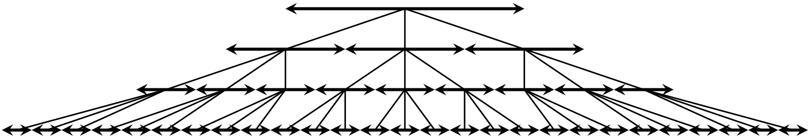

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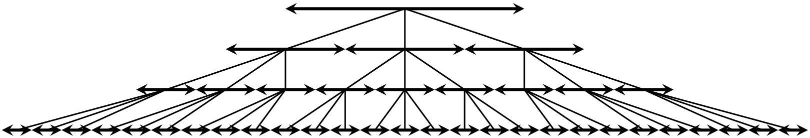

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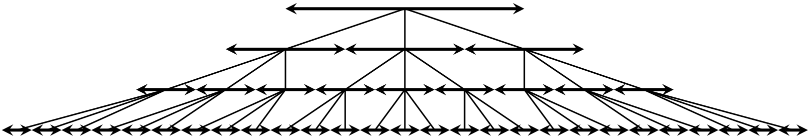

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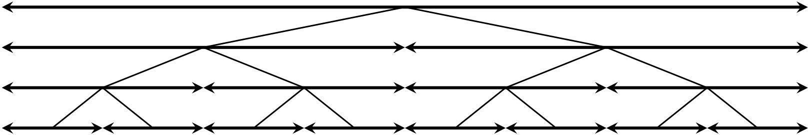

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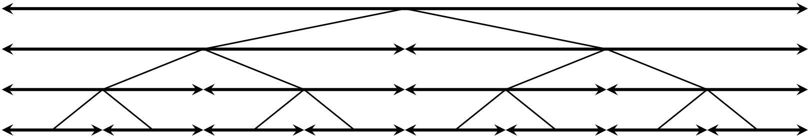

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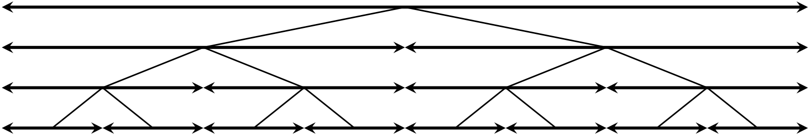

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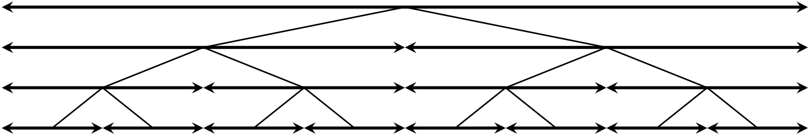

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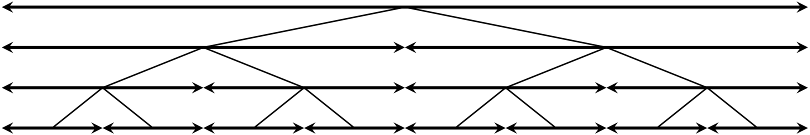

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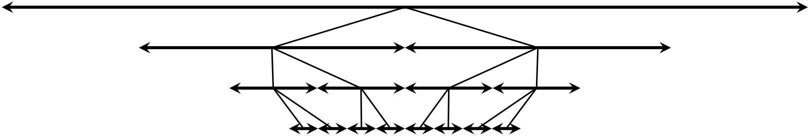
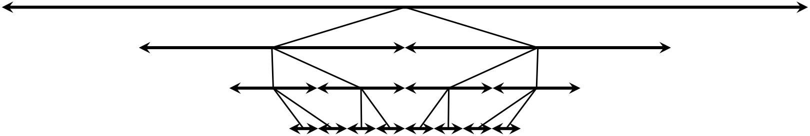


Figure: Simple recursion equation with $a = 2, b = 3$



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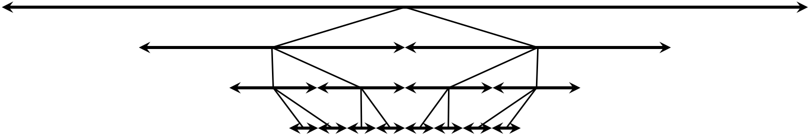

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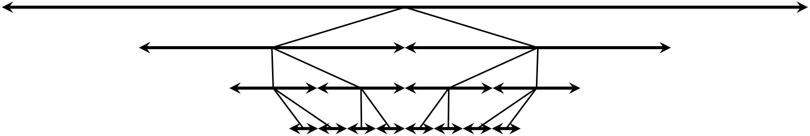

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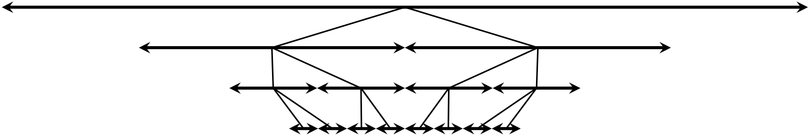

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- Runtime of $\Theta(n)$

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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

Structure

Master theorem (general form):

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- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon}),\ \varepsilon > 0$
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  Each layer has equal costs, $\log_b n$ layers

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  Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \qquad 0 \leq c \leq 1,$$

$$n > n_0$$

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Case 1 - Example: if $f(n) \in O(n^{\log_b a - \varepsilon}),\ \varepsilon > 0$

Solving the partial problems dominates (last layer, leaves)

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Solving the partial problems dominates (last layer, leaves)

- $T(n) = 8 \cdot T(\frac{n}{2}) + 1000 \cdot n^2$

$$a = 8,\ b = 2,\ f(n) = 1000 \cdot n^2,\ \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$

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- $T(n) = 9 \cdot T(\frac{n}{3}) + 17 \cdot n$

$$a = 9,\ b = 3,\ f(n) = 17 \cdot n,\ \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$

Case 2: if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (General Form) - Case 2

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

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- $T(n) = 2 \cdot T(\frac{n}{2}) + 10 \cdot n$

$$a = 2,\ b = 2,\ f(n) = 10 \cdot n,\ \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

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$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

- $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1,\ b = \frac{2}{3},\ f(n) = 1,\ \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

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Case 3: if $f(n) \in \Omega(n^{\log_b a+\varepsilon}),\ \varepsilon > 0$

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$f(n) \in \Omega(n^{1+\varepsilon})$

Check if regularity condition also holds:

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow c \geq \frac{1}{2}$$

$\Rightarrow T(n) \in \Theta(n^2)$

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Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

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- **Case 2:** $f(n) \notin \Theta(n^1)$

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- **Case 2:** $f(n) \notin \Theta(n^1)$
- **Case 3:** $f(n) \notin \Omega(n^{1+\varepsilon})$

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Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

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$n \log n$ is *asymptotically* larger than n,
but not *polynominal* larger

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  $T(n) \in \Theta(n^{\log_b a} \log n)$, $\log n$ layers
- **Case 3:** Connecting all partial solutions is *polynominal* bigger than solving all partial porblems
  $T(n) \in \Theta(f(n))$

- **General**

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Further Literature

- **Master theorem**

[Wik] Master theorem
https://en.wikipedia.org/wiki/Master_theorem