

Algorithms and Datastructures

Graphs, Depth-/Breadth-first Search, Graph-Connectivity

Albert-Ludwigs-Universität Freiburg



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Algorithms and Datastructures, January 2017

Graphs

- Introduction

- Implementation

- Application example

Graphs - Overview:

- Besides arrays, lists and trees the most common datastructure
(Trees are a special type of graph)
- Representation of graphs in the computer
- Breadth first search (BFS)
- Depth first search (DFS)
- Connected components of a graph

Terminology:



- Each Graph $G = (V, E)$ consists of:
 - A set of vertices (nodes) $V = \{v_1, v_2, \dots\}$
 - A set of edges (arcs) $E = \{e_1, e_2, \dots\}$
- Each edge connects two vertices ($u, v \in V$)
 - Undirected edge: $e = \{u, v\}$ (set)
 - Directed edge: $e = (u, v)$ (tuple)
- Self-loops are also possible: $e = (u, u)$ or $e = \{u, u\}$

Weighted graph:



- Each edge is marked with a real number named **weight**
- The **weight** is also named **length** or **cost** of the edge depending on the application

Example: Road network

- Intersections: **vertices**
- Roads: **edges**
- Travel time: **costs of the edges**

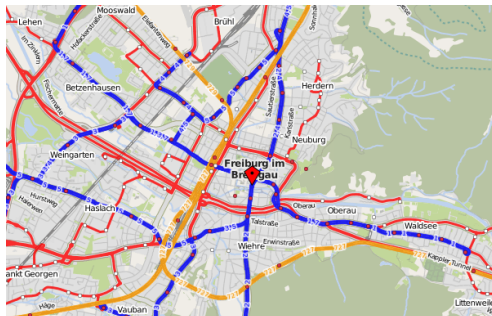


Figure: Map of Freiburg © OpenStreetMap

How to represent this graph computationally?

- 1 Adjacency matrix with space consumption $\Theta(|V|^2)$



Figure: Weighted graph with
 $|V| = 4$, $|E| = 6$

| | | end-vertex | | | |
|--------------|---|------------|---|----|----|
| | | 0 | 1 | 2 | 3 |
| start-vertex | 0 | | 2 | | 3 |
| | 1 | | | 9 | |
| | 2 | | | | -1 |
| | 3 | | 7 | -2 | |

Figure: Adjacency matrix

How to represent this graph computationally?

- 2 Adjacency list / fields with space consumption $\Theta(|V| + |E|)$
 Each list item stores the **target vertex** and the **cost** of the edge

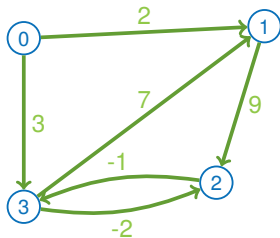


Figure: Weighted graph with
 $|V| = 4, |E| = 6$

| | | | |
|--------------|---|-------|-------|
| start-vertex | 0 | 1, 2 | 3, 3 |
| | 1 | 2, 9 | |
| | 2 | 3, -1 | |
| | 3 | 1, 7 | 2, -2 |

Figure: Adjacency list

Graph: Arrangement

- Graph is fully defined through the **adjacency matrix / list**
- The arrangement is not relevant for visualisation of the graph



Figure: Weighted graph with
 $|V| = 4$, $|E| = 6$



Figure: Same graph ordered by number - outer planar graph

```
class Graph:
    def __init__(self):
        self.vertices = []
        self.edges = []

    def addVertice(self, vert):
        self.vertices.append(vert)

    def addEdge(self, fromVert, toVert, cost):
        self.edges.append( \
            (fromVert, toVert, cost))

    ...
```

Degree of a vertex: Directed graph: $G = (V, E)$

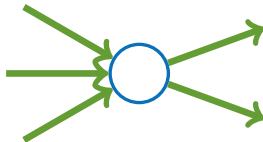


Figure: Vertex with in- / outdegree of 3 / 2

- **Indegree** of a vertex u is the number of **edge head ends** adjacent to the vertex

$$\deg^+(u) = |\{(v, u) : (v, u) \in E\}|$$

- **Outdegree** of a vertex u is the number of **edge tail ends** adjacent to the vertex

$$\deg^-(u) = |\{(u, v) : (u, v) \in E\}|$$

Degree of a vertex: Undirected graph: $G = (V, E)$

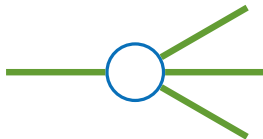


Figure: Vertex with degree of 4

- **Degree** of a vertex u is the number of **vertices** adjacent to the vertex

$$\deg(u) = |\{\{v, u\} : \{v, u\} \in E\}|$$

Paths in a graph: $G = (V, E)$



Figure: Undirected path of length 3
 $P = (0, 3, 2, 4)$

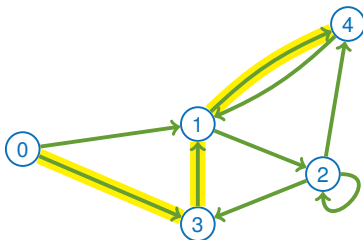


Figure: Directed path of length 3
 $P = (0, 3, 1, 4)$

- A path of G is a sequence of edges $u_1, u_2, \dots, u_i \in V$ with
 - Undirected graph: $\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{i-1}, u_i\} \in E$
 - Directed graph: $(u_1, u_2), (u_2, u_3), \dots, (u_{i-1}, u_i) \in E$

Paths in a graph: $G = (V, E)$



Figure: Directed path of length 3
 $P = (0, 3, 1, 4)$

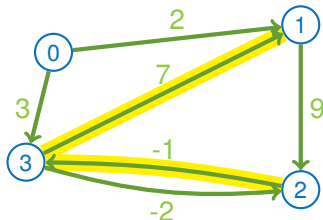


Figure: Weighted path with cost 6
 $P = (2, 3, 1)$

- The **length of a path** is: (also costs of a path)
 - Without weights: **number of edges** taken
 - With weights: **sum of weights of edges** taken

Shortest path in a graph: $G = (V, E)$



Figure: Shortest path from 0 to 2 with cost / distance $d(0,2) = ?$

- The **shortest path** between two vertices u, v is the path $P = (u, \dots, v)$ with the shortest length $d(u, v)$ or lowest costs

Shortest path in a graph: $G = (V, E)$



Figure: Shortest path from 0 to 2 with cost / distance $d(0, 2) = 6$
 $P = (0, 1, 4, 3, 2)$

- The shortest path between two vertices u, v is the path $P = (u, \dots, v)$ with the shortest length $d(u, v)$ or lowest costs

Diameter of a graph: $G = (V, E)$

$$d = \max_{u,v \in V} d(u,v)$$



Figure: Diameter of graph is $d = ?$

- The **diameter** of a graph is the length / the costs of the longest shortest path

Diameter of a graph: $G = (V, E)$

$$d = \max_{u,v \in V} d(u,v)$$



Figure: Diameter of graph is $d = 10$, $P = (3, 2, 5)$

- The **diameter** of a graph is the length / the costs of the longest shortest path

Connected components: $G = (V, E)$



Figure: Three connected components

- Undirected graph:
 - All connected components are a partition of V

$$V = V_1 \cup \dots \cup V_k$$

- Two vertices u, v are in the same connected component if a path between u and v exists

Connected components: $G = (V, E)$

- Directed graph:
 - Named **strongly connected components**
 - Direction of edge has to be regarded
 - Not part of this lecture

Graph Exploration: (Informal definition)

- Let $G = (V, E)$ be a graph and $s \in V$ a start vertex
- We visit each reachable vertex connected to s
- **Breadth-first search**: in order of the smallest distance to s
- **Depth-first search**: in order of the largest distance to s
- Not a problem on its own but is often used as subroutine of other algorithms
 - Searching of connected components
 - Flood fill in drawing programmes

Breadth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s (level 0)**
- 3 Mark all unmarked **connected vertices (level 1)**
- 4 Mark all unmarked **vertices connected** to a **level 1-vertex (level 2)**
- 5 Iteratively mark reachable vertices for all levels
- 6 All connected nodes are now marked and in the same **connected component** as the start vertex **s**

- The marked vertices create a “spanning tree” containing all reachable nodes

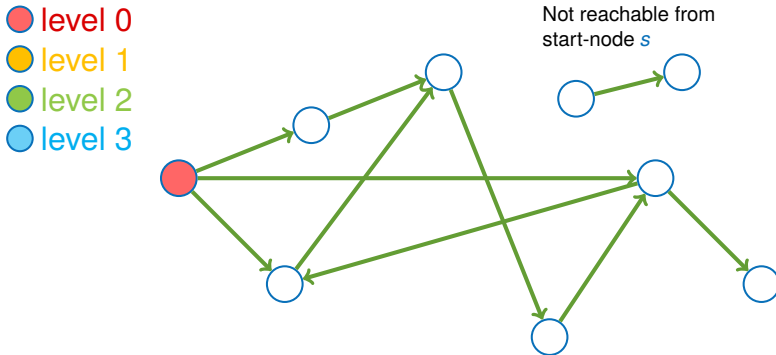


Figure: spanning tree of a breadth-first search

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- The marked vertices create a “spanning tree” containing all reachable nodes



Figure: spanning tree of a breadth-first search

- The marked vertices create a “spanning tree” containing all reachable nodes

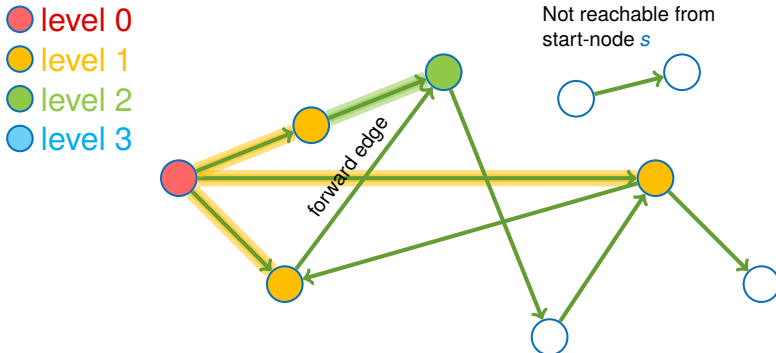


Figure: spanning tree of a breadth-first search

- The marked vertices create a “spanning tree” containing all reachable nodes

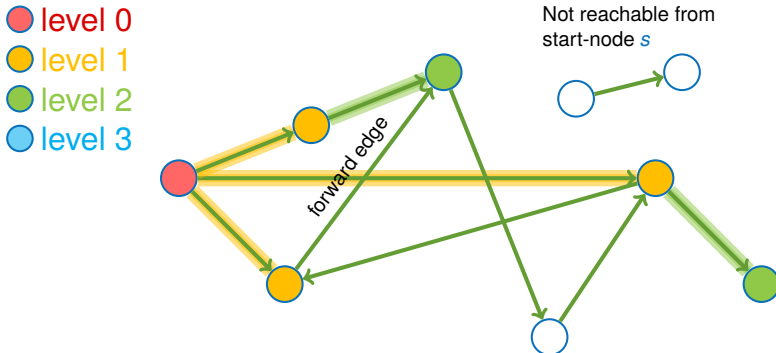


Figure: spanning tree of a breadth-first search

- 



- The marked vertices create a “spanning tree” containing all reachable nodes

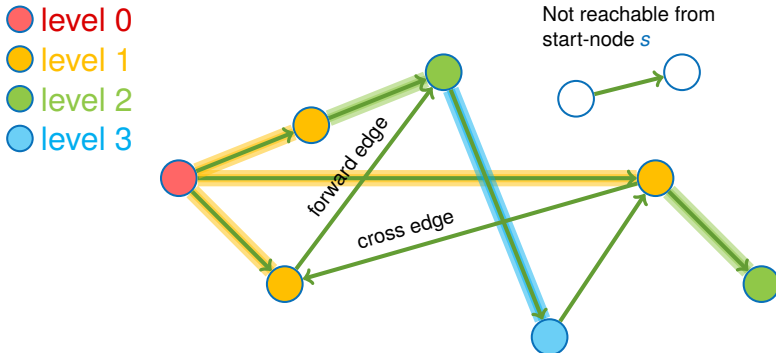


Figure: spanning tree of a breadth-first search

- The marked vertices create a “spanning tree” containing all reachable nodes



Figure: spanning tree of a breadth-first search

Depth-First Search:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s**
- 3 Pick an unmarked **connected vertex** and start a **recursive depth-first search** with the vertex as start vertex (continue on step 2)
- 4 If no unmarked connected vertex exists go one vertex back and continue recursive search (reduce the recursion level by one)

Depth-first search:

- Search starts with **long paths** (searching with depth)
- Marks like **breadth-first search** all connected vertices
- If the graph is acyclic we get a **topological sorting**
 - Each newly visited vertex gets marked by an increasing number
 - The numbers increase with path length from the start vertex

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

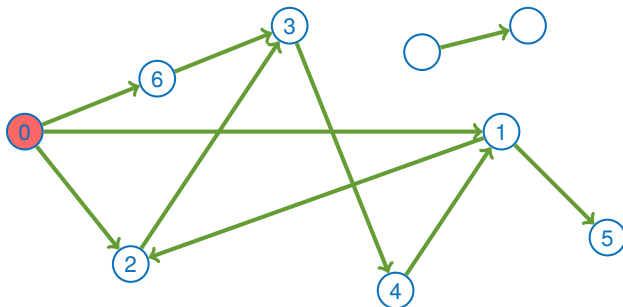


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

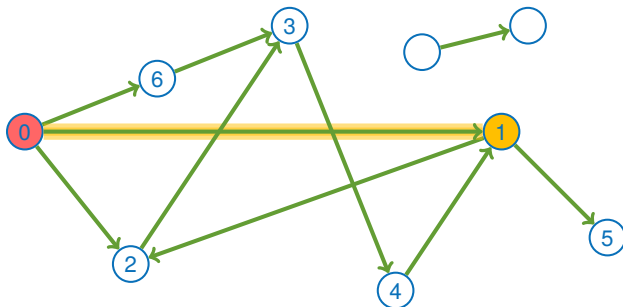


Figure: spanning tree of a depth-first search

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Figure: spanning tree of a depth-first search

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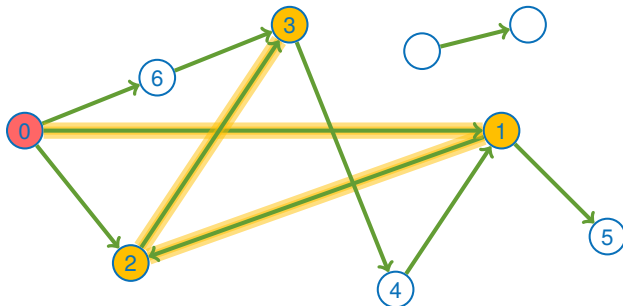


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

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● path 1

● path 2

● path 3

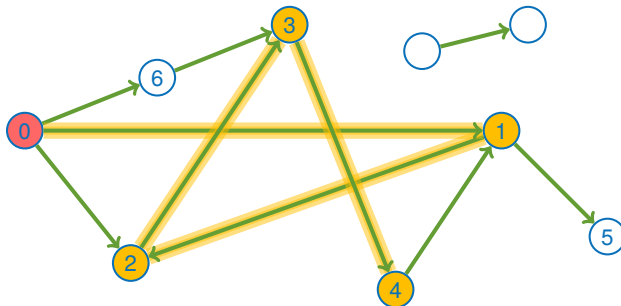


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

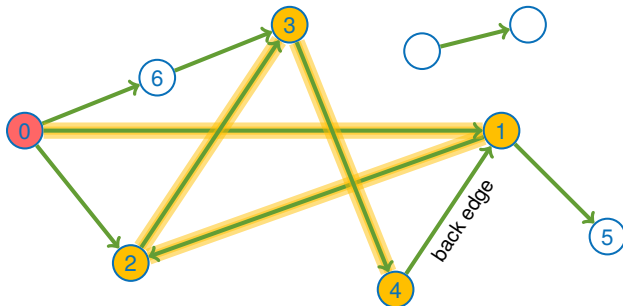


Figure: spanning tree of a depth-first search

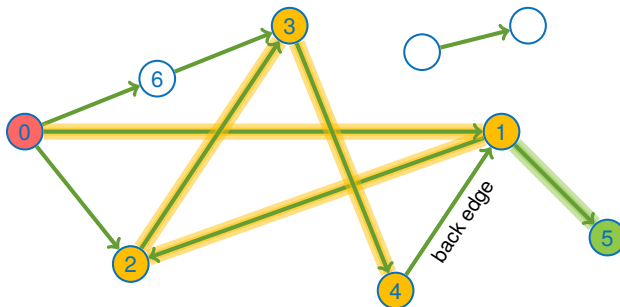
- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

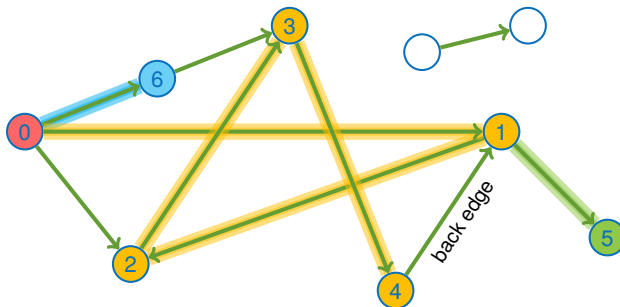


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

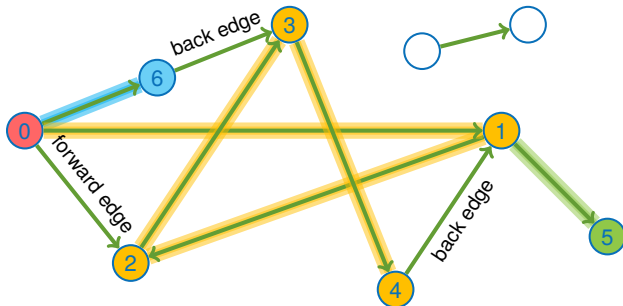


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

Graphs

Why is this called Breadth - and Depth First Search?



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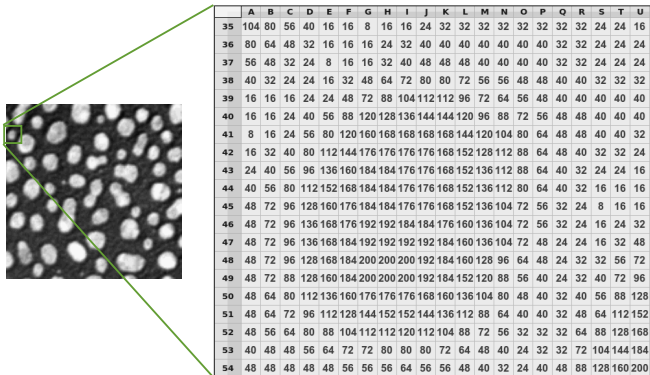
Runtime complexity:

- Constant costs for each visited vertex and edge
- We get a runtime complexity of $\Theta(|V'| + |E'|)$
- Let V' and E' be the reachable vertices and edges
- All vertices of V' are in the same connected component as our start vertex s
- This can only be improved by a constant factor

- Connected component labeling
- Counting of objects in an image



What's object, what's background?



Convert to black white using threshold:

value = 255 if value > 100 else 0



Interpret image as graph:

- Each white pixel is a node
- Edges between adjacent pixels (normally 4 or 8 neighbors)
- Edges are not saved externally, algorithm works directly on array
- Breadth- / depth-first search find all connected components (particles)

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

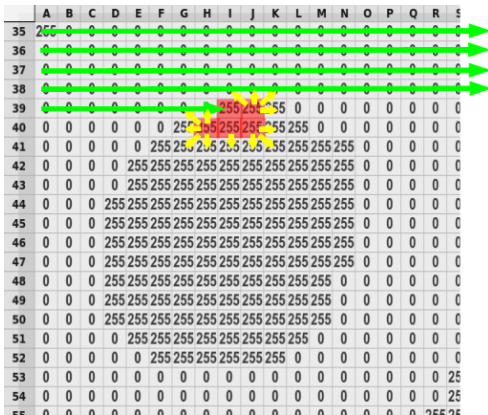
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



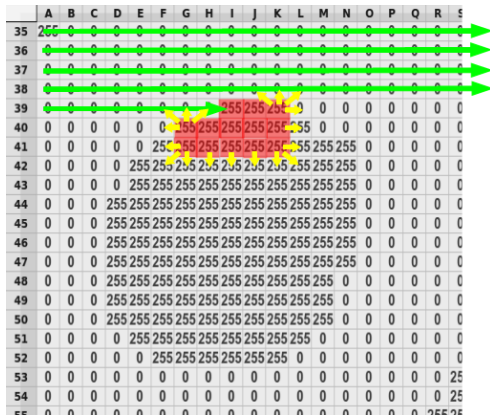
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Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



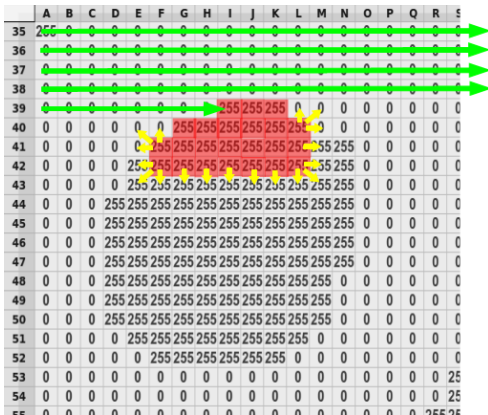
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



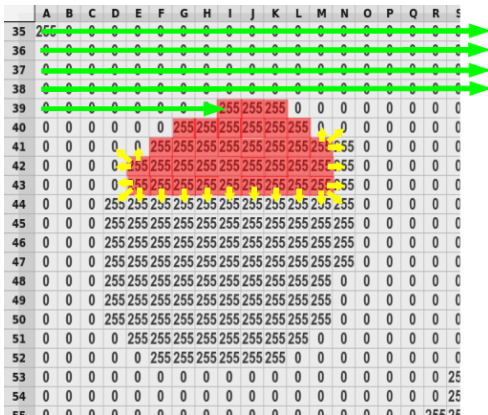
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 255 | 255 | 255 | 255 | 255 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|----|-----|---|---|---|---|---|---|---|---|----|---|---|---|---|---|---|---|-----|----|
| 35 | 255 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 255 | 25 |

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as **component 2**
- ...

Result of connected component labeling:



Figure: Result: particle indices instead of intensities

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Graph-Search

[Wika] [Breadth-first search](#)

`https://en.wikipedia.org/wiki/
Breadth-first_search`

[Wikb] [Depth-first search](#)

`https:
//en.wikipedia.org/wiki/Depth-first_search`

■ Graph-Connectivity

[Wik] [Connectivity \(graph theory\)](#)

`https://en.wikipedia.org/wiki/Connectivity_
\(graph_theory\)`