

Algorithms and Datastructures

Static Arrays, Dynamic Arrays, Amortized Analysis

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Static Arrays

Dynamic Arrays

Introduction

Amortized Analysis

- Static arrays exist in nearly every programming language
- They are initialized with a fixed size n
- **Problem:** The needed size is not always clear at compile time

Table: Static array with size $n = 5$

Index	0	1	2	3	4
Value	"a"	"b"	"c"	"d"	"e"

Python:

- We have dynamic sized lists
- Python does automatic resizing when needed

```
# Creates a list of "0"s with init. size 10
numbers = [0] * 10
```

```
# Prints number at index 7 ("0")
print("%d" % numbers[7])
```

```
# Saves number 42 at index 8
numbers[8] = 42
```

```
# Prints the number at index 8 ("42")
print("%d" % numbers[8])
```

- The name “static array” has nothing to do with the keyword **static** from Java / C++
- Nor is the array allocated before the program starts
- The **size** of the array is static and can not be changed after creation
- The name “fixed-size array” would be more appropriate

Static Arrays

Dynamic Arrays

Introduction

Amortized Analysis

Dynamic arrays:

- The array is created with an initial size
- The size can be dynamically modified
- **Problem:** We need a dynamic structure to store the data

Python:

```
greetings = ["Good morning", "ohai"]

greetings.append("Guten morgen")
greetings.append("bonjour")

# Prints text at index 2 ("Guten morgen")
print("%s" % greetings[2])

# Removes all elements
greetings.clear();
```


- We store the data in a fixed-size array with the needed size
- **Append:**
 - Create fixed-size array with the needed size
 - Copy elements from the old to the new array
- **Remove:**
 - Create fixed-size array with the needed size
 - Copy elements from the old to the new array

First implementation:

- We resize the array before each append
- We choose the size exactly as needed

```
class DynamicArray:

    def __init__(self):
        self.size = 0
        self.elements = []

    def capacity(self):
        return len(self.elements)

    ...
```

```
class DynamicArray:
    ...

    def append(self, item):
        newElements = [0] * (self.size + 1)

        for i in range(0, self.size):
            newElements[i] = self.elements[i]

        self.elements = newElements

        newElements[self.size] = item
        self.size += 1
```

- Why is the runtime quadratic?

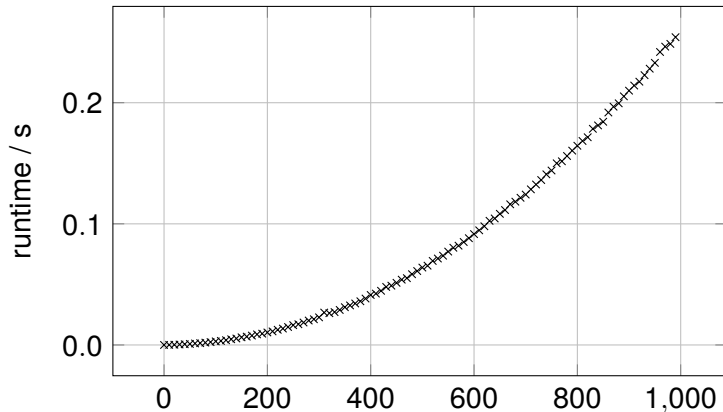



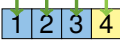




Figure: Runtime of *DynamicArray*

Runtime:

	$O(1)$	write 1 element
	$O(1 + 1)$	write 1 element, copy 1 element
	$O(1 + 2)$	write 1 element, copy 2 elements
	$O(1 + 3)$	write 1 element, copy 3 elements
	$O(1 + 4)$	write 1 element, copy 4 elements
	$O(1 + 5)$	write 1 element, copy 5 elements
...

Analysis:

- Let $T(n)$ be the runtime of n sequential append operations
- Let T_i be the runtime of each i -th operation
 - Then $T_i = A \cdot i$ for a constant A
 - We have to copy $i - 1$ elements

$$\begin{aligned} T(n) &= \sum_{i=1}^n T_i = \sum_{i=1}^n (A \cdot i) = A \cdot \sum_{i=1}^n i = A \cdot \frac{n^2 + n}{2} \\ &= O(n^2) \end{aligned}$$



Idea:

- Better resize strategy
- We allocate more space than needed
- We over-allocate a constant amount of elements
 - Amount: $C = 3$ or $C = 100$


```
def append(self, item):  
    if self.size >= len(self.elements):  
        newElements = [0] * (self.size + 100)  
  
        for i in range(0, self.size - 1):  
            newElements[i] = self.elements[i]  
  
        self.elements = newElements  
  
    self.elements[self.size] = item  
    self.size += 1
```

- Why is the runtime still quadratic?

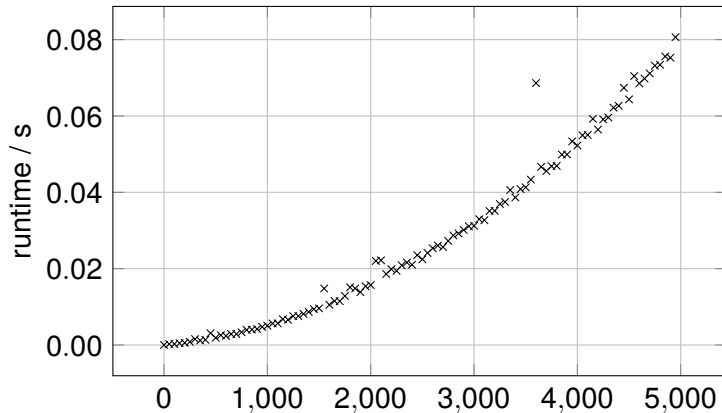




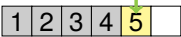
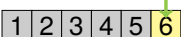
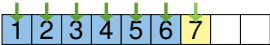


Figure: Runtime of *DynamicArray*

Runtime for $C = 3$:

	$O(1)$	write 1 element
	$O(1)$	write 1 element
	$O(1)$	write 1 element
	$O(1 + 3)$	write 1 element, copy 3 elements
	$O(1)$	write 1 element
	$O(1)$	write 1 element
	$O(1 + 6)$	write 1 element, copy 6 elements
...

Analysis:

- Most of the append operations now just cost $O(1)$
- Every C steps the costs for copying are added:
 $C, 2 \cdot C, 3 \cdot C, \dots$ this means:

$$\begin{aligned}T(n) &= \sum_{i=1}^n A \cdot 1 + \sum_{i=1}^{n/C} A \cdot i \cdot C \\&= A \cdot n + A \cdot C \cdot \sum_{i=1}^{n/C} i \\&= A \cdot n + A \cdot C \cdot \frac{\frac{n^2}{C^2} + \frac{n}{C}}{2} \\&= A \cdot n + \frac{A}{2 \cdot C} \cdot n^2 + \frac{A}{2} \cdot n = O(n^2)\end{aligned}$$

- The factor of n^2 is getting smaller

Idea:

- Double the size of the array

```
def append(self, item):
    if self.size >= len(self.elements):
        newElements = [0] \
            * max(1, 2 * self.size)

        for i in range(0, self.size):
            newElements[i] = self.elements[i]

        self.elements = newElements

    self.elements[self.size] = item
    self.size += 1
```

- Now the runtime is linear with some bumps. Why?

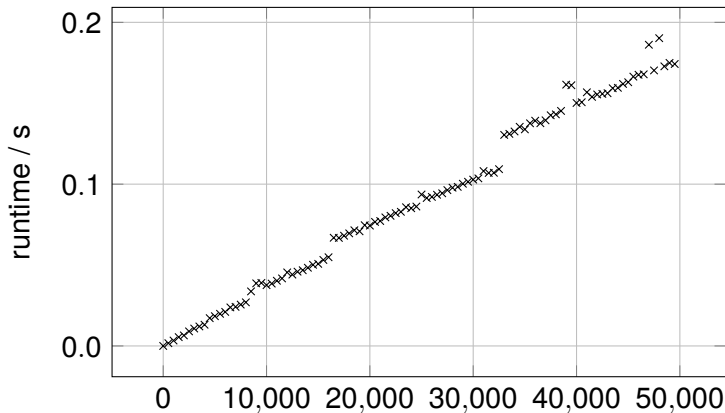


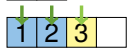
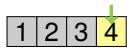

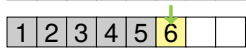
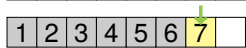




Figure: Runtime of *DynamicArray*

Runtime for $C = 2$ (Double the size):

	$O(1)$	write 1
	$O(1 + 1)$	write 1, copy 1 element
	$O(1 + 2)$	write 1, copy 2 elements
	$O(1)$	write 1
	$O(1 + 4)$	write 1, copy 4 elements
	$O(1)$	write 1
	$O(1)$	write 1
	$O(1)$	write 1
	$O(1 + 8)$	write 1, copy 8 elements
...

Analysis:

- Now all appends cost $O(1)$
- Every 2^i steps we have to add the cost $A \cdot 2^i$ (for $i = 0, 1, 2, \dots, k$ with $k = \text{floor}(\log_2(n-1))$)
- In total that accounts to:

$$\begin{aligned} T(n) &= n \cdot A + A \cdot \sum_{i=0}^k 2^i = n \cdot A + A(2^{k+1} - 1) \\ &\leq n \cdot A + A \cdot 2^{(k+1)} \\ &= n \cdot A + 2 \cdot A \cdot 2^{(k)} \\ &\leq n \cdot A + 2 \cdot A \cdot n \\ &= 3 \cdot A \cdot n \\ &= O(n) \end{aligned}$$

How do we shrink the array?

- If the array is half-full, we can shrink it by half, like for the append operation
 - If we *append* directly after *shrinking* we have to extend the array again
 - We leave some space for following append operations
- ⇒ We only shrink the array to 75%

Analysis:

- **Difficult:** We have a random number of *append* / *remove* operations
- We can not exactly predict when resizing is happening

Static Arrays

Dynamic Arrays

Introduction

Amortized Analysis



Figure: Static array with capacity c_i

Notation:

- We have n instructions $O = \{O_1, \dots, O_n\}$
- The **size** after operation i is s_i , with $s_0 := 0$
- The **capacity** after operation i is c_i , with $c_0 := 0$
- The **cost** of operation i is $\text{cost}(O_i)$ (previously named T_i)

Reallocation: $\text{cost}(O_i) \leq A \cdot s_i$,

Insert / Delete (Update): $\text{cost}(O_i) \leq A$,

Dynamic Arrays

Amortized Analysis - Example



Operation			Size s_i	Capacity c_i	Costs $\text{cost}(O_i)$
O_1	append	realloc.	s_1	c_1	$A \cdot s_1$
O_2	append		s_2	$c_2 = c_1$	$A \cdot 1$
O_3	append		s_3	$c_3 = c_1$	$A \cdot 1$
O_4	remove		s_4	$c_4 = c_1$	$A \cdot 1$
O_5	remove	realloc.	s_5	c_5	$A \cdot s_5$
O_6	append		s_6	$c_6 = c_5$	$A \cdot 1$
O_7	remove		s_7	$c_7 = c_5$	$A \cdot 1$
O_8	append		s_8	$c_8 = c_5$	$A \cdot 1$
O_9	append	realloc.	s_9	c_9	$A \cdot s_9$
...
O_n	append		s_n	c_n	$A \cdot 1$

Implementation:

- If O_i is an *append* operation and $s_{i-1} = c_{i-1}$:
⇒ Resize array to $c_i = \left\lfloor \frac{3}{2} s_i \right\rfloor = \text{floor} \left(\frac{3}{2} s_i \right)$
⇒ $\text{cost}(O_i) = A \cdot s_i$



Figure: *Append* operation with reallocation

Result: after operation we have $c_i = \frac{3}{2} \cdot s_i$

Implementation:

- If O_i is an *remove* operation and $s_{i-1} \leq \frac{1}{3}c_{i-1}$:
 \Rightarrow Resize array to $c_i = \left\lfloor \frac{3}{2}s_i \right\rfloor = \text{floor}\left(\frac{3}{2}s_i\right)$
 $\Rightarrow \text{cost}(O_i) = A \cdot s_i$



Figure: Remove operation with reallocation

Result: after operation we have again $c_i = \frac{3}{2} \cdot s_i$

Idea for proof:

- Expensive are only operations where reallocations are necessary
- If we just reallocated, it takes some time until the next reallocation is required.
- **Assumption:** After a costly *reallocation* of size X we have at least X operations of runtime $O(1)$
- **Then:** Total cost of n operations is maximally $2 \cdot n$

Table: Dynamic Array with $C_{\text{ext}} = \frac{3}{2}$

Operation			Size s_i	Capacity c_i	Costs $\text{cost}(O_i)$	
O_1	app.	realloc.	s_1	$c_1 = 4$	$C_1 \cdot s_1$	$\left\{ \begin{array}{l} \text{distance} \\ 4 \geq \left\lfloor \frac{s_1}{2} \right\rfloor \end{array} \right.$
O_2	app.		s_2	$c_2 = c_1$	C_2	
O_3	app.		s_3	$c_3 = c_1$	C_2	
O_4	app.		s_4	$c_4 = c_1$	C_2	
O_5	app.	realloc.	s_5	$c_5 = \left\lfloor \frac{3}{2} s_5 \right\rfloor = 7$	$C_1 \cdot s_5$	$\left\{ \begin{array}{l} \text{distance} \\ 3 \geq \left\lfloor \frac{s_5}{2} \right\rfloor \end{array} \right.$
O_6	app.		s_6	$c_6 = c_5$	C_2	
O_7	app.		s_7	$c_7 = c_5$	C_2	
O_8	app.	realloc.	s_8	$c_8 = \frac{3}{2} s_8 = 12$	$C_1 \cdot s_8$	
...	

To show:

- **Lemma:** If a *reallocation* occurs at O_i the nearest *reallocation* is at O_j with $j - i > \frac{s_i}{2}$
- **Corollary:** $\text{cost}(O_1) + \dots + \text{cost}(O_n) \leq 4A \cdot n$

Dynamic Arrays

Proof: Worst Case Same Operation

Table: Case 1: $\frac{1}{2}s_j$ appends

Array

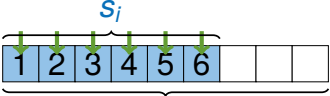
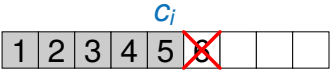


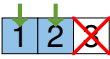
Costs

O_i :		reallocation $A \cdot s_j$ (linear)
O_{i+1} :		A (constant)
O_{i+2} :		A (constant)
O_{i+3} :		A (constant)
O_j :	 ...	reallocation $A \cdot s_j$ (earliest realloc)

Table: Case 2: $\frac{1}{2}s_j$ removes

Array

Costs

O_i : 	<p>reallocation $A \cdot s_j$ (linear)</p>	$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \frac{s_j}{2} \text{ times}$
O_{i+1} : 	<p>A (constant)</p>	
O_{i+2} : 	<p>A (constant)</p>	
O_{i+3} : 	<p>A (constant)</p>	
O_j : 	<p>reallocation $A \cdot s_j$ (earliest reallocation)</p>	

Proof of lemma:

- If a reallocation happens at O_i and then again at O_j , then $j - i \geq s_i/2$
- After operation O_i the capacity is

$$c_i = \left\lfloor \frac{3}{2} \cdot s_i \right\rfloor$$

- Lets consider a operation O_i to O_k with $k - i \leq \frac{s_i}{2}$:
 - Case 1: Since the *reallocation* we have inserted at maximum $\text{floor}\left(\frac{1}{2} \cdot s_i\right)$ elements

$$s_k \leq s_i + \left\lfloor \frac{s_i}{2} \right\rfloor = \left\lfloor \frac{3}{2} s_i \right\rfloor = c_i \quad \text{no reallocation needed}$$

Proof of lemma - continued:

- Case 2: Since the *reallocation* we have removed at maximum $\left\lfloor \frac{1}{2}s_j \right\rfloor$ elements

$$s_k \geq s_j - \left\lfloor \frac{s_j}{2} \right\rfloor = \left\lceil \frac{1}{2}s_j \right\rceil$$

no reallocation needed

$$\Rightarrow 3 \cdot s_k \geq \left\lceil \frac{3}{2}s_j \right\rceil \geq \left\lfloor \frac{3}{2}s_j \right\rfloor = c_j$$

Corollary:

$$\text{cost}(O_1) + \dots + \text{cost}(O_n) \leq 4A \cdot n$$

- Let the *reallocations* be at operations $\text{cost}(O_{i_1}), \dots, \text{cost}(O_{i_m})$
- The **cost** of all *reallocations* are $A \cdot (s_{i_1} + \dots + s_{i_m})$
- With the lemma we know:

$$i_2 - i_1 > \frac{s_{i_1}}{2}, \quad i_3 - i_2 > \frac{s_{i_2}}{2}, \quad \dots, \quad i_m - i_{m-1} > \frac{s_{i_{m-1}}}{2}$$

- We can conclude that:

$$i_2 - i_1 > \frac{s_{i_1}}{2} \quad \Rightarrow \quad s_{i_1} < 2(i_2 - i_1)$$

$$i_3 - i_2 > \frac{s_{i_2}}{2} \quad \Rightarrow \quad s_{i_2} < 2(i_3 - i_2)$$

\vdots

$$i_m - i_{m-1} > \frac{s_{i_{m-1}}}{2} \quad \Rightarrow \quad s_{i_{m-1}} < 2(i_m - i_{m-1})$$
$$s_{i_m} \leq n \quad (\text{trivial})$$

- The **costs** of all reallocations are:

$$\begin{aligned}\text{cost}(\text{realloc.}) &= A \cdot (s_{i_1} + \dots + s_{i_m}) \\ &< A \cdot (2(i_2 - i_1) + 2(i_3 - i_2) + \dots + 2(i_m - i_{m-1}) + n) \\ &= A \cdot (2(i_m - i_1) + n) \\ &\leq A \cdot (2n + n) = 3A \cdot n\end{aligned}$$

- Additionally we have to consider the respective constant costs for a normal append or remove ($\leq A \cdot n$) therefore in total $\leq 4 \cdot A \cdot n$

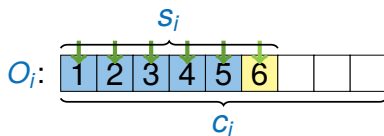
Dynamic Arrays

Amortized Analysis - Alternate Proof of Corollary

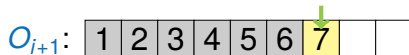
Table: Case 1: $\frac{1}{2}s_j$ appends

Array

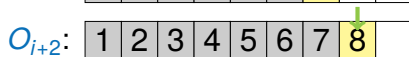
Costs



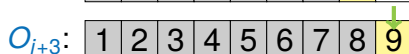
reallocation
 $A \cdot s_j$ (linear)



A (constant)

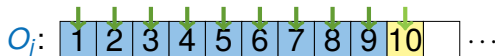


A (constant)



A (constant)

$\left. \begin{array}{l} A \text{ (constant)} \\ A \text{ (constant)} \\ A \text{ (constant)} \end{array} \right\} \frac{s_j}{2} \text{ times}$



reallocation
 $A \cdot s_j$
(earliest realloc.)

- Total costs of $A \cdot \frac{3}{2} \cdot s_i$ for $\frac{s_i}{2} + 1$ operations
- Cost per operation:

$$\frac{\frac{3}{2}A \cdot s_i}{\frac{1}{2}s_i + 1} \leq \frac{\frac{3}{2}A \cdot s_i}{\frac{1}{2}s_i} = 3 \cdot A = \text{const.}$$

Dynamic Arrays

Amortized Analysis - Alternate Proof of Corollary



Array

Costs



- Runtime analysis for local worst-case sequence
- Same total cost as previous slide

Bank account paradigm:

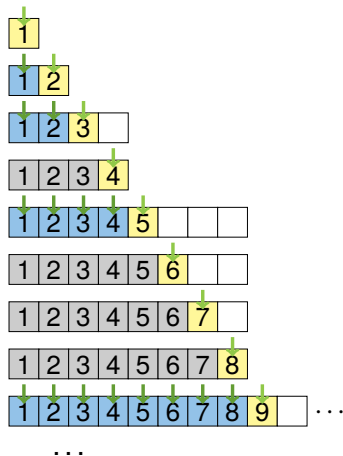
- **Idea:** “Save first, spend later”
- For each operation we deposit some coins on an “bank account”
⇒ We still have **constant costs**
- When we have a **linear operation** (reallocation) we pay with the coins from our “bank account”
- For the “double the size” strategy we have to pay two coins per operation

Dynamic Arrays

Amortized Analysis - Yet Another Proof of Corollary



Double the size:



$\text{cost}(O_i)$	deposit / withdraw	account value
$O(1)$	+2	2
$O(1 + 1)$	+2 -1	3
$O(1 + 2)$	+2 -2	3
$O(1)$	+2	5
$O(1 + 4)$	+2 -4	3
$O(1)$	+2	5
$O(1)$	+2	7
$O(1)$	+2	9
$O(1 + 8)$	+2 -8	3
...



Figure: Array after realloc. (insert) operation

Why do we need to deposit 2 coins per operation?

- 1 Each newly inserted element has to be copied later (first coin)
- 2 Due to the factor of two there is for each new element also an old one in the array that also has to be copied (second coin)

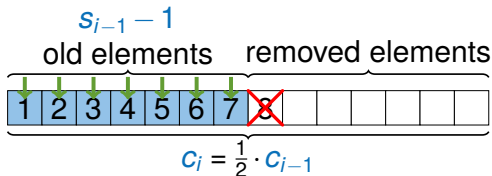


Figure: Array after realloc. (remove) operation

Shrinking strategy: If array 1/4 full shrink by half

- How many coins do we need per *remove* operation?
- **Worst case:** The previous remove operation triggered a *reallocation*

⇒ Array is half full

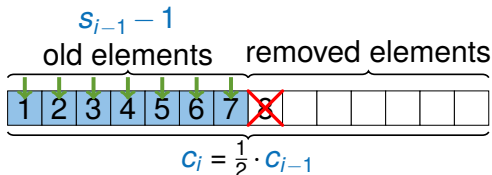


Figure: Array after realloc. (remove) operation

Shrinking strategy: If array 1/4 full shrink by half

- Array is half full
 - The nearest *reallocation* is after removing $\frac{1}{4}c_i$ elements
 - We have to copy $\frac{1}{4}c_i$ elements
- ⇒ 1 coin per operation is enough

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Amortized Analysis

[Wik] [Amortized analysis](https://en.wikipedia.org/wiki/Amortized_analysis)

https:

`//en.wikipedia.org/wiki/Amortized_analysis`