

Algorithms and Datastructures

Linked Lists, Binary Search Trees

Albert-Ludwigs-Universität Freiburg



**UNI
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Algorithms and Datastructures, January 2018

Sorted Sequences

Linked Lists

Binary Search Trees



Structure:

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 - **next()/previous()**: Returns the element with the next bigger/smaller **key**. This enables iteration over all elements

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- How could we implement this?

Static array:

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Sorted Sequences

Implementation 2 (bad) - Hash Table



Hash map:



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Order of the elements is independent of the order of the keys

Sorted Sequences

Implementation 3 (good?) - Linked List

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- Let's have a closer look

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Binary Search Trees



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Figure: Linked list



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Properties in comparison to an array:

- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified
- No direct access of elements
 - ⇒ We have to iterate over the list



List with head / last element pointer:

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Figure: Singly linked list

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- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

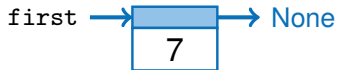
    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```




Creating linked lists - Python:

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■ first = Node(7)
```

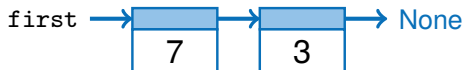


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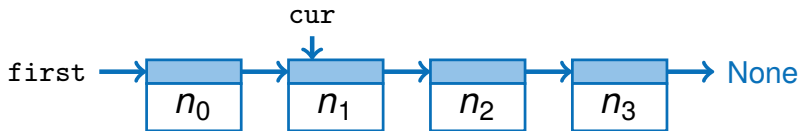
■ `first.nextNode = Node(3)`



■ `first.nextNode.value = 4`



Inserting a node after node `cur`:





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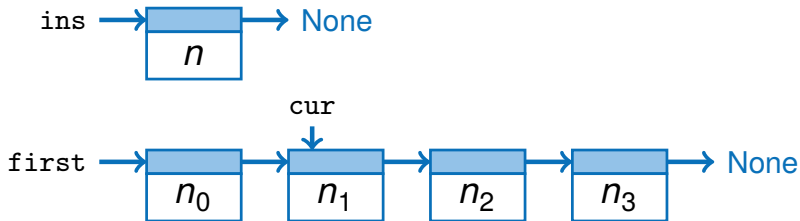


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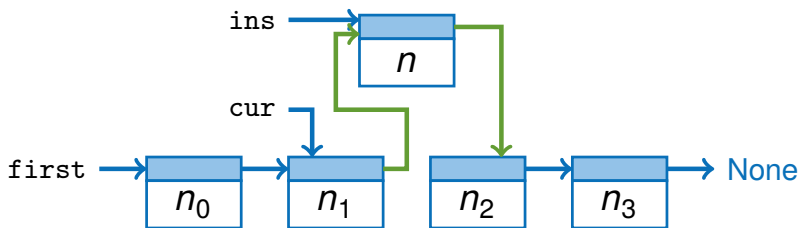


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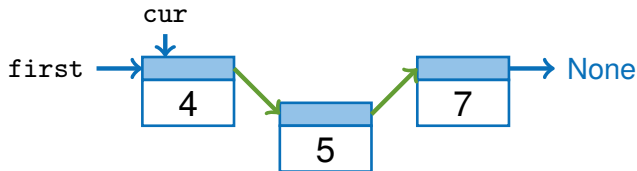


■ `cur.nextNode = Node(value, cur.nextNode)`

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pre = first
while pre.nextNode != cur:
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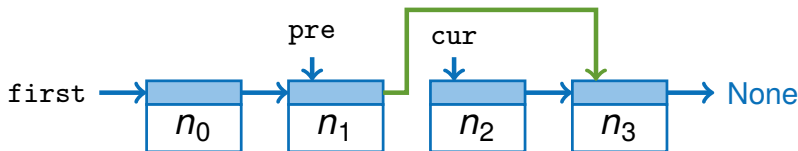
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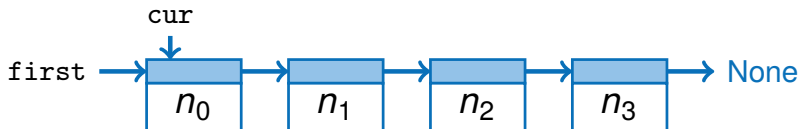
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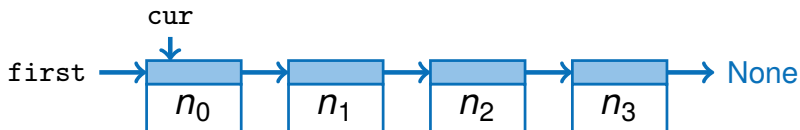
```
first = first.nextNode
```

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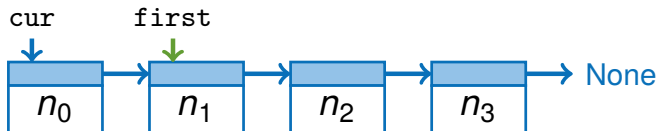


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Removing a node `cur`: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```




Using a head node:



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```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```



```
def append(self, value):  
    ...  
  
def insertAfter(self, cur, value):  
    ...  
  
def remove(self, cur):  
    ...  
  
def get(self, position):  
    ...  
  
def contains(self, value):  
    ...
```

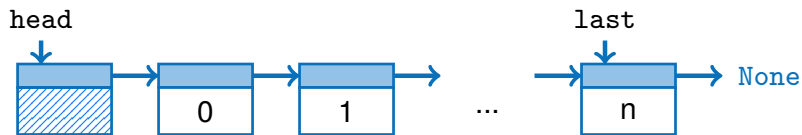


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Head, last:



- Head points to the first node, `last` to the last node
- We can append elements to the end of the list in $O(1)$ through the `last` node
- We have to keep the pointer to `last` updated after all operations



Appending an element:

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```
def append(self, value):  
    last.nextNode = Node(value)  
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- The pointer to `last` avoids the iteration of the whole list

Inserting after node `cur`:





Inserting after node `cur`:

- The pointer to `head` is not modified

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```
def insertAfter(self, cur, value):  
    if cur == last:  
        # also update last node  
        append(value)  
    else:  
        # last node is not modified  
        cur.nextNode = Node(value, \  
                             cur.nextNode)  
        itemCount += 1
```

Remove node cur:





Remove node `cur`:

- Searching the predecessor in $O(n)$

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```
def remove(self, cur):  
    pre = first  
    while pre.nextNode != cur:  
        pre = pre.nextNode  
  
    pre.nextNode = cur.nextNode  
    itemCount -= 1  
  
    if pre.nextNode == None:  
        last = pre
```




Getting a reference to node at `pos`:

- Iterate the entries of the list until at position in $O(n)$

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```
def get(self, pos):  
    if pos < 0 or pos >= itemCount:  
        return None  
  
    cur = head  
    for i in range(0, pos):  
        cur = cur.nextNode  
  
    return cur
```



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```
def contains(self, value):  
    cur = head  
  
    for i in range(0, itemCount):  
        cur = cur.nextNode  
        if cur.value == value:  
            return True  
  
    return False
```



Runtime:



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- Better with `doubly linked lists`



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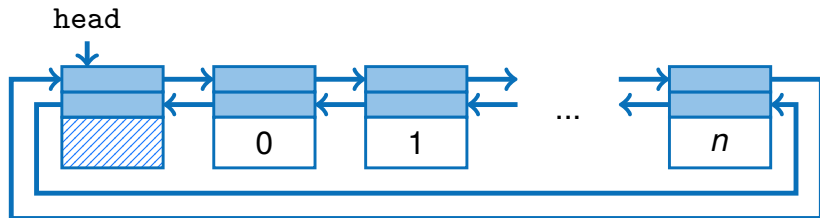
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Even if the elements are sorted we can only retrieve them in $\Theta(n)$ Why?

Linked list in book:



Linked Lists

List in real program



Linked list in memory:



Sorted Sequences

Linked Lists

Binary Search Trees

Runtime of a search tree:

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The structure helps searching efficiently

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- We define a total order for the search tree
- All nodes of the left subtree have **smaller keys** than the current node
- All nodes of the right subtree have **bigger keys** than the current node

- Edge direction indicates ordering

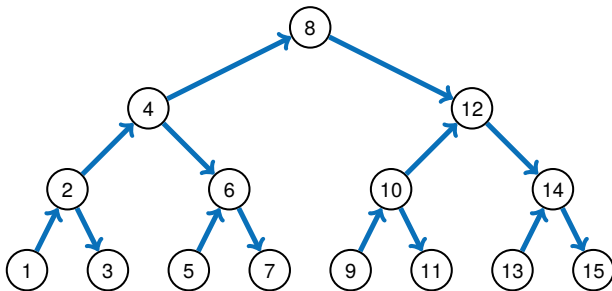


Figure: A binary search tree



Figure: Another binary search tree



Figure: **Not** a binary search tree



Implementation:

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- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists



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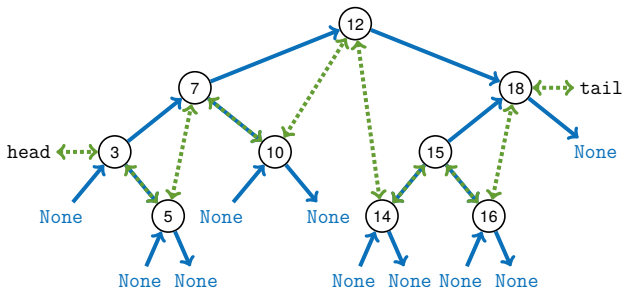


Figure: Binary search tree with links



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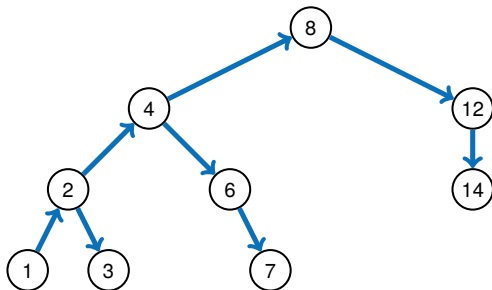


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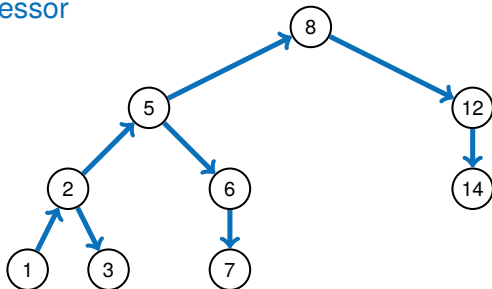
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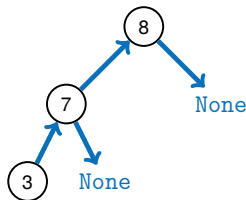


Figure: Degenerated binary tree $d = n$

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Figure: Degenerated binary tree $d = n$



Figure: Complete binary tree $d = \log n$

■ General

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■ **Linked List**

[Wik] [Linked list](#)

`https://en.wikipedia.org/wiki/Linked_list`

■ **Binary Search Tree**

[Wik] [Binary search tree](#)

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