

# Algorithms and Datastructures

## Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Algorithms and Datastructures, January 2017

## Balanced Trees

- Motivation

- AVL-Trees

- (a,b)-Trees

  - Introduction

  - Runtime Complexity

- Red-Black Trees



## Binary search tree:

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- Worst case:  $d \in O(n)$ , keys are inserted in ascending / descending order (20, 19, 18, ...)



## Gnarley trees:



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**Figure:** Binary search tree with random insert [Gna]



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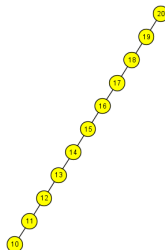


Figure: Binary search tree with descending insert [Gna]



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- We **rebalance** the tree from time to time



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- Used in C++ `std::map` and Java `SortedMap`

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(a,b)-Trees

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- Height difference of left and right subtree is at maximum one
- With that the height of the search tree is always  $O(\log n)$
- We can perform all basic operations in  $O(\log n)$



Figure: Example of an AVL-Tree

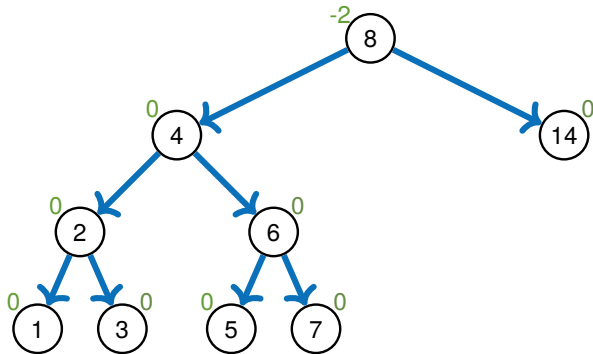


Figure: **Not** an AVL-Tree

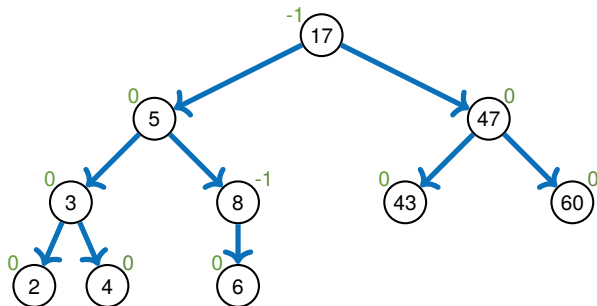


Figure: Another example of an AVL-Tree

# Balanced Trees

## AVL-Tree - Rebalancing

**Rotation:**

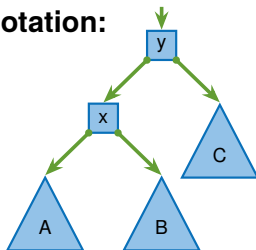


Figure: Before rotating

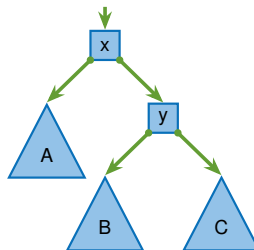


Figure: After rotating

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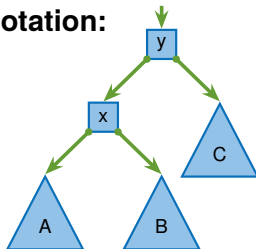


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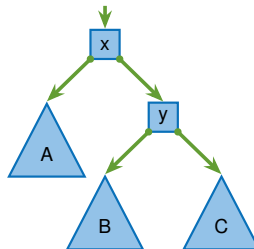


Figure: After rotating

- Central operation of **rebalancing**



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## AVL-Tree - Rebalancing



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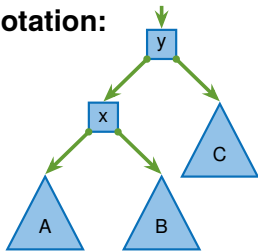


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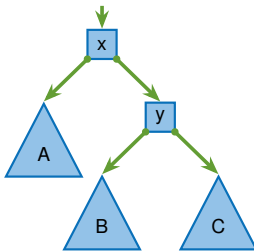


Figure: After rotating

- Central operation of **rebalancing**
- After rotation to the right:

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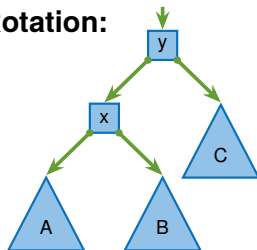


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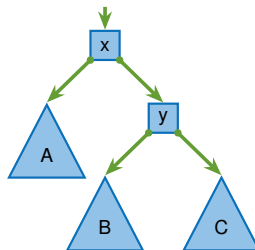


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- Central operation of **rebalancing**
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  - Subtree **A** is a layer higher and subtree **C** a layer lower

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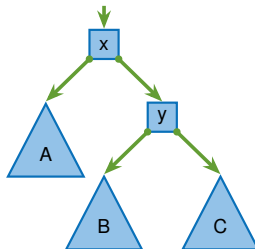


Figure: After rotating

- Central operation of **rebalancing**
- After rotation to the right:
  - Subtree **A** is a layer higher and subtree **C** a layer lower
  - The parent child relations between nodes **x** and **y** have been swapped

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- Many different cases of rebalancing
- **Example:** **insert** of 1,2,3,...



Figure: Inserting 1,...,10 into an AVL-tree [Gna]



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- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in  $O(\log n)$
- However not amortized update costs of  $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are  $(a,b)$ -trees

## Balanced Trees

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### **$(a,b)$ -Tree:**



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- So we have space for elements on an **insert** and balance operation



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- Each inner node has  $\geq a$  and  $\leq b$  nodes  
(Only the root node may have less nodes)



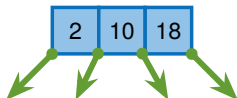
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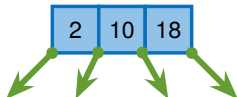
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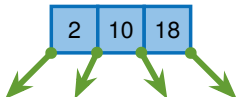
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- Subtrees are located “between” the elements
- We require:  $a \geq 2$  and  $b \geq 2a - 1$

### (2,4)-Tree:



Figure: Example of an (2,4)-tree

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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

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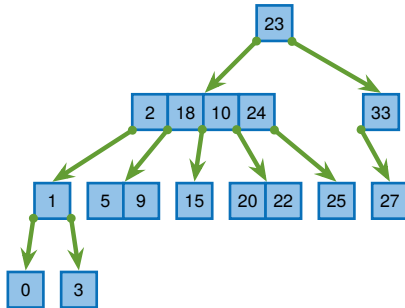


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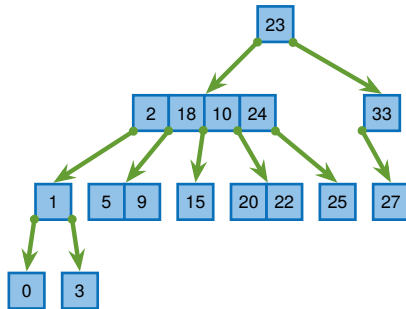


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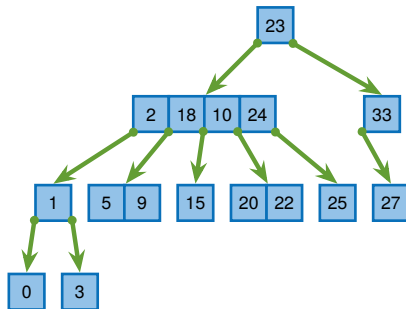


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- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels



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Figure: (3,5)-Tree [Gna]



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- Then we **split** the node

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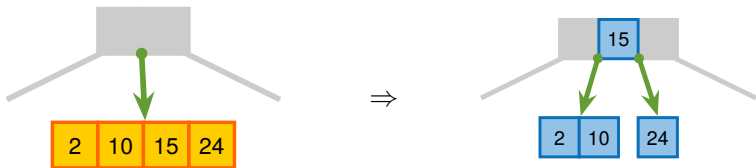


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- That's why we have the limit  $b \geq 2a - 1$

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- If we split the root node we create a new parent root node  
(The tree is now one level deeper)



**Removing an element:** (`remove`)





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  - Replace the element with its **successor** and delete the **successor** from the leaf

### Removing an element: (`remove`)

- Search the element in  $O(\log n)$  time
- **Case 1:** The element is contained by a leaf
  - Remove element
- **Case 2:** The element is contained by an inner node
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  - Replace the element with its **successor** and delete the **successor** from the leaf
- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree



**Removing an element:** (`remove`)



### Removing an element: (remove)

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### Removing an element: (remove)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
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  - **Case a:** If the left or right neighbour node has a degree **greater than a** we **borrow** one element from this node

### Removing an element: (`remove`)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree
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Figure: Borrow an element



**Removing an element: (remove)**

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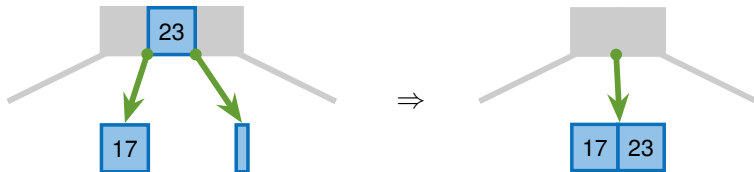


Figure: Merge two nodes



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- Now the parent node can be of degree  $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
  - Remove the root
  - Define sole child as new root
  - The tree shrinks by one level



**Runtime complexity of `lookup`, `insert` and `remove`:**



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- All operations in  $O(d)$  with  $d$  being the depth of the tree

### Runtime complexity of **lookup**, **insert** and **remove**:

- All operations in  $O(d)$  with  $d$  being the depth of the tree
- Each node (except the root) has **more than  $a$  children**  
 $\Rightarrow n \geq a^{d-1}$  and  $d \leq 1 + \log_a n = O(\log_a n)$

### Runtime complexity of **lookup**, **insert** and **remove**:

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- Therefore instead of  $b \geq 2a - 1$  we need  $b \geq 2a$



### Counter example $(2,3)$ -Tree:



### Counter example (2,3)-Tree:

- Before executing `delete(11)`

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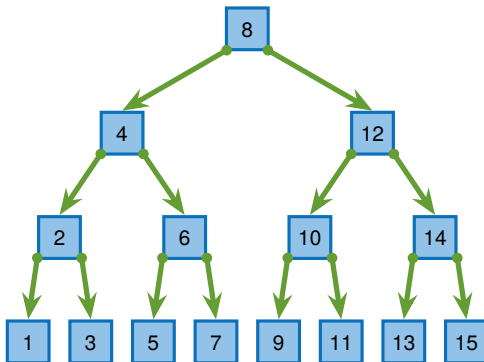


Figure: Normal (2,3)-Tree

### Counter example (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 1

### Counter example (2,3)-Tree:

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Figure: (2,3)-Tree - Delete step 2



### Counter example (2,3)-Tree:

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Figure: (2,3)-Tree - Delete step 3

### Counter example (2,3)-Tree:

- Executed `delete(11)`

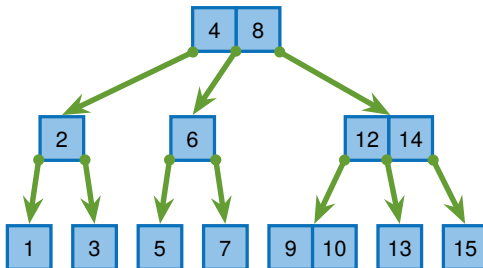


Figure: (2,3)-Tree - Delete step 4



### Counter example (2,3)-Tree:

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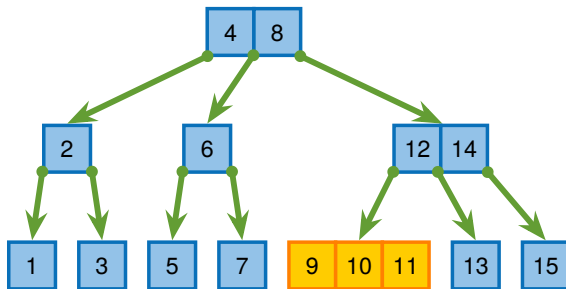


Figure: (2,3)-Tree - Insert step 1

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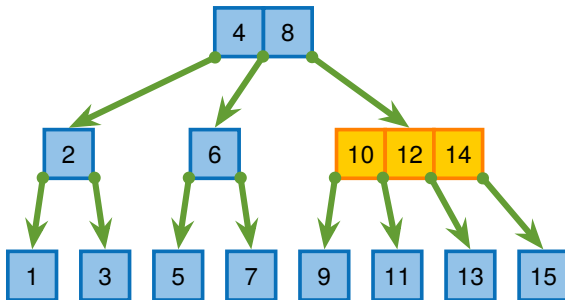


Figure: (2,3)-Tree - Insert step 2

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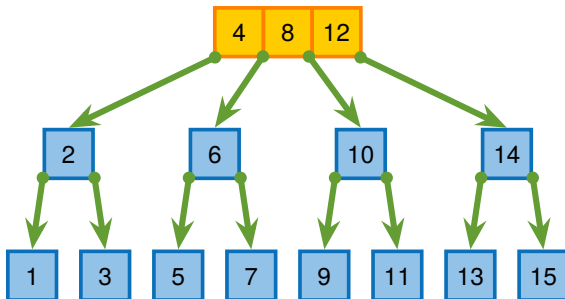


Figure: (2,3)-Tree - Insert step 3

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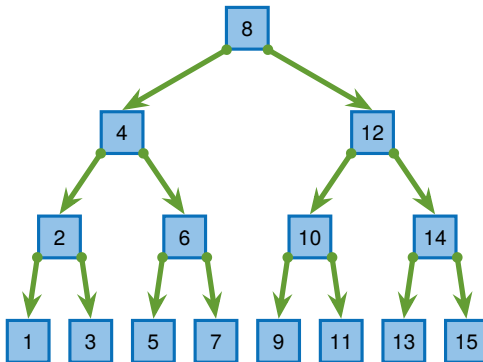


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- We are exactly where we started

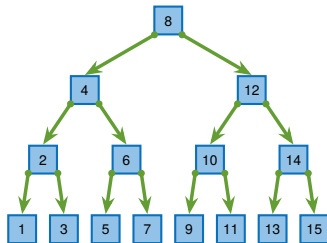


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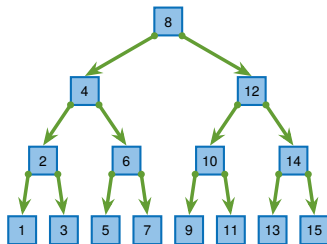


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⇒ **Nodes of degree 3 are stable**

Neither an insert nor a remove operation trigger rebalancing operations



### $(2,4)$ -Tree:

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- Like with dynamic arrays:
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  - If we **overallocate** clever we have an amortized runtime of  $O(1)$

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- Empty tree has 0 nodes:  $\phi = 0$



### Example:



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Figure: Tree with potential  $\phi = 4$



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- The costs for operation  $i$  are coupled to the difference of the potential levels

$$c_i \leq A \cdot (\underbrace{\phi_i - \phi_{i-1}}_{\text{difference of potential levels}}) + B, \quad A > 0 \text{ and } B > A$$

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Number of gained stable nodes (degree 3)  $\geq -1$

- Each operation has an amortized cost of  $O(1)$  summing up to  $O(n)$  in total

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- Each splitted node creates a node of **degree 3**
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$$\begin{aligned}\phi_i &\geq \phi_{i-1} + m - 1 \\ \Rightarrow m &\leq \phi_i - \phi_{i-1} + 1\end{aligned}$$

Costs:  $c_i \leq A \cdot m + B$

$$\begin{aligned}\Rightarrow c_i &\leq A \cdot (\phi_i - \phi_{i-1} + 1) + B \\ c_i &\leq A \cdot (\phi_i - \phi_{i-1}) + \underbrace{A + B}_{B'}\end{aligned}$$





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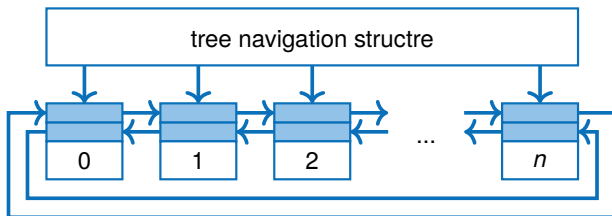


Figure: Tree with doubly linked list



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Figure: Case 2.1.2: Borrow an element

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- **Case 2.2:** Merging two node

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Figure: Merging two nodes

■ Potential rises by one

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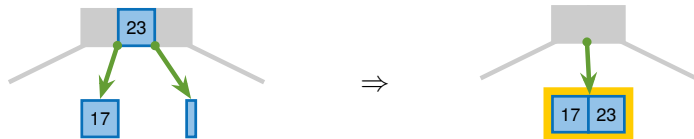


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Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree  $> 2$  or a node of degree 2, which can borrow from a neighbour

**Case 2:**  $i$ -th operation is an **remove** operation

■ **Case 2.2:** Merging two nodes

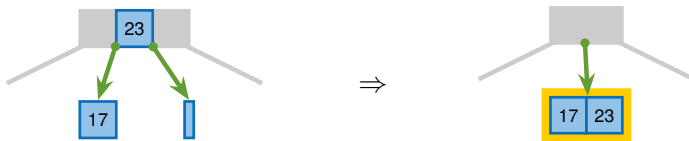


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Figure: Merging two nodes

- The potential rises by  $m$
- If the “stop-node” is of **degree 2** then the potential eventually goes down by one
- Same costs as **insert**





### **Lemma:**

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- With that we can conclude:

$$\sum_{i=0}^n c_i \in O(n)$$

### Proof:

$$\begin{aligned}\sum_{i=0}^n c_i &\leq \underbrace{A \cdot (\phi_1 - \phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\phi_2 - \phi_1) + B}_{\leq c_2} + \dots + \underbrace{A \cdot (\phi_n - \phi_{n-1}) + B}_{\leq c_n} \\ &= A \cdot (\phi_n - \phi_0) + B \cdot n && | \text{ telescope sum} \\ &= A \cdot \phi_n + B \cdot n && | \text{ we start with an empty tree} \\ &< A \cdot n + B \cdot n \in O(n) && | \text{ number of degree 3 nodes} \\ &&& < \text{ number of nodes}\end{aligned}$$

## Balanced Trees

Motivation

AVL-Trees

(a,b)-Trees

Introduction

Runtime Complexity

Red-Black Trees



## Red-Black Tree:



### Red-Black Tree:

- Binary tree with red and black nodes

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- Binary tree with **red** and **black** nodes
- Number of **black** nodes on path to leaves is equal



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- Binary tree with **red** and **black** nodes
- Number of **black** nodes on path to leaves is equal
- Can be interpreted as **(2,4)-tree** (also named 2-3-4-tree)
- Each **(2,4)-tree**-node is a small red-black-tree with a **black** root node

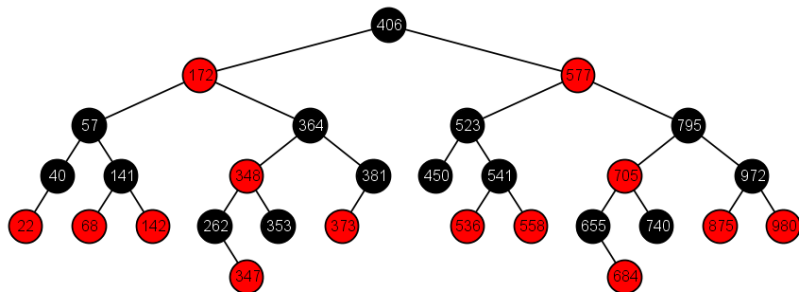


Figure: Example of an red-black-tree [Gna]

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[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

### **Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Gnarley Trees

[Gna] **Gnarley Trees**

<https://people.ksp.sk/~kuko/gnarley-trees/>

## ■ AVL-Tree

[Wik] [AVL tree](#)

`https://en.wikipedia.org/wiki/AVL\_tree`

## ■ (a,b)-Tree

[Wika] [2-3-4 tree](#)

`https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934\_tree`

[Wikb] [\(a,b\)-tree](#)

`https://en.wikipedia.org/wiki/\(a,b\)-tree`

## ■ Red-Black-Tree

[Wik] [Red-black tree](https://en.wikipedia.org/wiki/Red%E2%80%93black_tree)

`https://en.wikipedia.org/wiki/Red%E2%80%93black\_tree`