

# Algorithms and Datastructures

## Cache Efficiency, Divide and Conquer

Albert-Ludwigs-Universität Freiburg



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Bioinformatics Group / Department of Computer Science  
Algorithms and Datastructures, March 2016

## Cache Efficiency

- Introduction

- Cache Organization

## Divide and Conquer

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- Assuming this is a good measure for the runtime of a algorithm/tool
- Today we will see examples where this is not suitable

### Example:

- We sum up all elements of a field  $a$  of size  $n$  in ...
  - natural order:

$$\text{sum}(a) = a[1] + a[2] + \dots + a[n]$$

- random order:

$$\text{sum}(a) = a[21] + a[5] + \dots + a[8]$$

### Python:

```
def init(size):  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```



### Python:

```
def run(param):  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```

# Cache Efficiency

## Linear Order

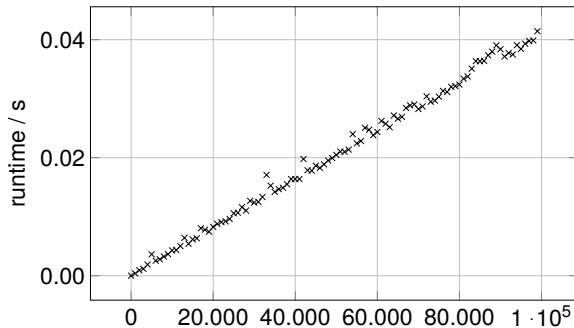


Figure: Summing elements in linear order

### Python:

```
def init(size):  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
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# Cache Efficiency

## Random Order



Figure: Summing elements in random order



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- Accessing elements in random order takes a lot longer (Factor 10)

Why?

- The costs in terms of memory access are very different

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# Cache Efficiency

## CPU Cache



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## Principle / organization:



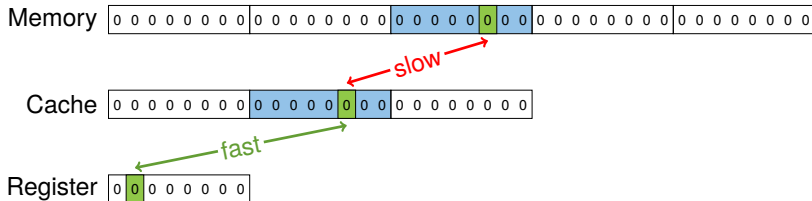
### Principle / organization:

- Accessing one byte of the main memory takes  $\approx 100$  ns



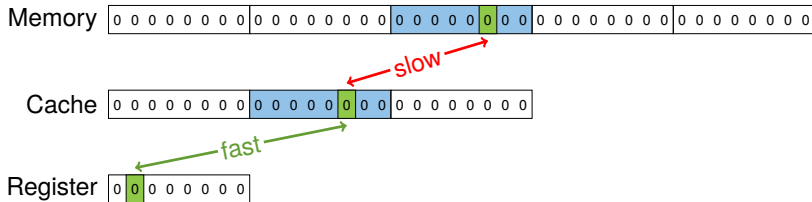
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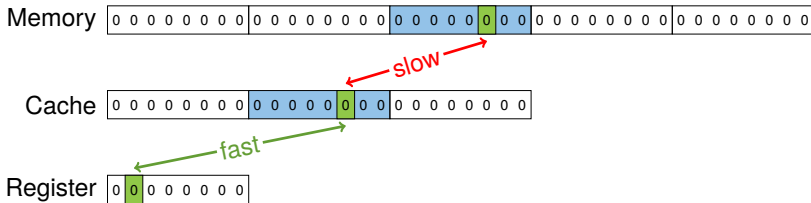


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- As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

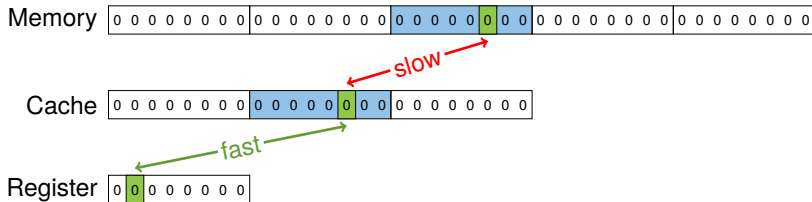
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**Cache organization:**





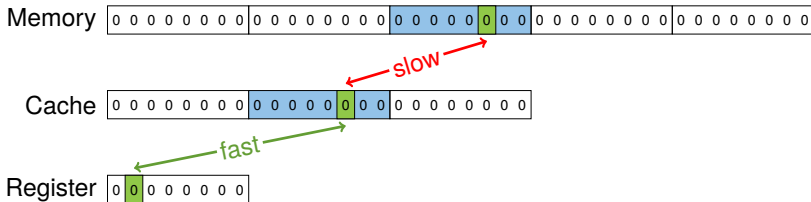
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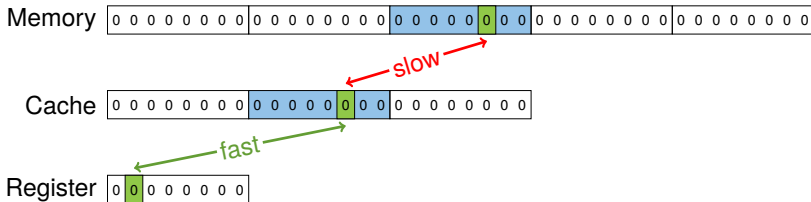
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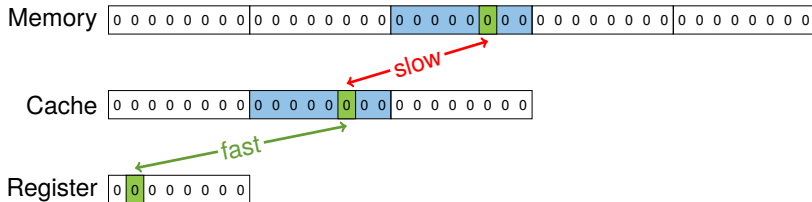
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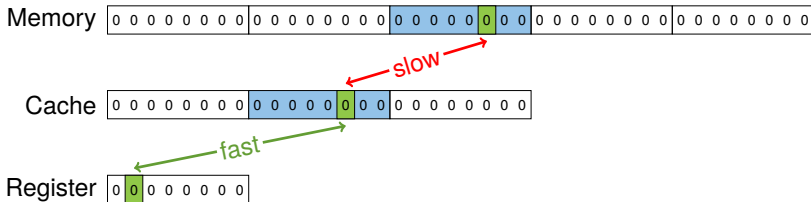
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- Details of discarding are not the topic for today

# Cache Efficiency

## Block Operations



## Terminology:





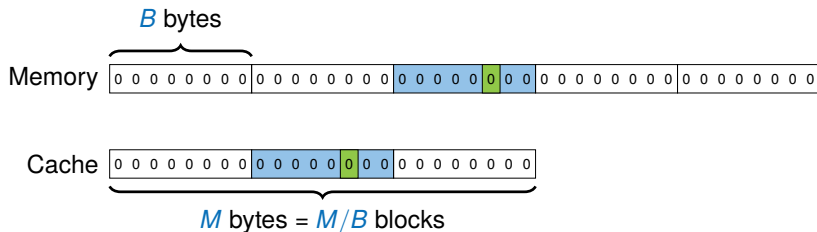
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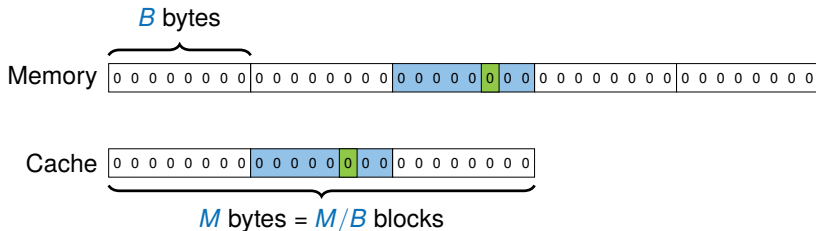
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- We use the number of **block operations** as runtime estimation
- We ignore runtime costs of cache accesses / management



## Block Operations



**Figure:** Comparison good / bad locality

**Accessing the cache  $B$  times:**

- **Best case:** 1 block operation  $\rightarrow$  good locality
- **Worst case:**  $B$  block operations  $\rightarrow$  bad locality



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### Note:

- If the input size is smaller than  $M$  we load the complete data chunk directly into the cache
- Cache handling is only interesting when the input size is greater than  $M$



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  - Many operating systems use free system memory as disk cache



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- Block operations on disk-cache are called **IOs**  
(input / output operations)
- These also fall under the term **cache efficiency** or **IO efficiency**



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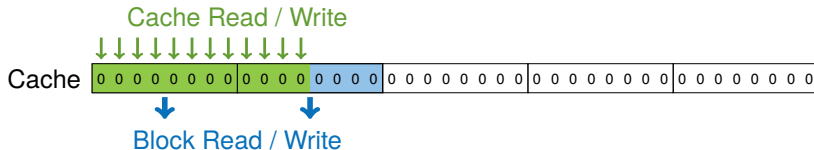


Figure: Good locality of sum operation



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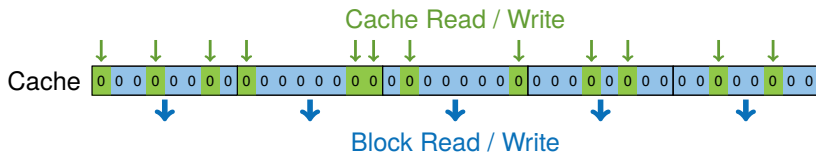


Figure: Bad locality of sum operation

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- If **not  $n \gg M$**  the next element might already with a high probability loaded in cache



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Figure: QuickSort with pivot-element

- **at start:** pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- do required changes *in place*



- **end point:**  $k$  is left to left-most element greater than pivot  
*swap position 0 (pivot) with  $k$  (smaller than pivot)*

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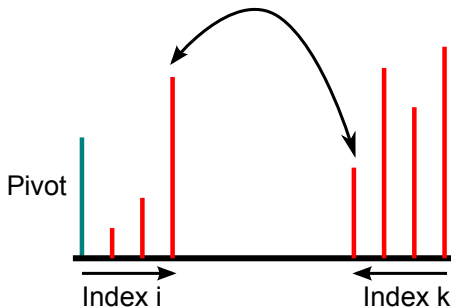


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### Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```



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$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left( A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

# Cache Efficiency

## Block Operations - QuickSort

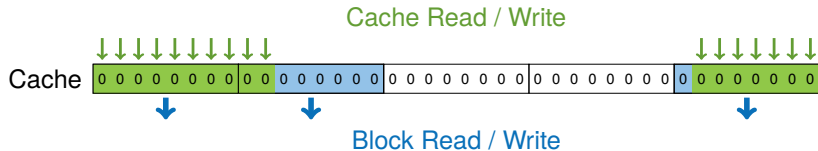


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- Assumptions as before but recursion depth is  $k = \log_2 \frac{n}{B}$

Why?

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in O\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

## Cache Efficiency

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Cache Organization

## Divide and Conquer

Introduction



# Divide and Conquer

## Introduction



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# Divide and Conquer

Introduction - Python



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```
def solve(problem):  
    if n < threshold:  
        # solve directly  
        return solution  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
    return S1 + S2 + ... + Sk
```

# Divide and Conquer

## Features



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  - The runtime is  $\in O(n \cdot \log n)$

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  - The runtime is  $\in O(n \cdot \log n)$
- Suitable for parallel processing

- Can help with conceptual hard problems
  - **Solution** of the trivial problems has to be known
  - **Dividing** in subproblems has to be possible
  - **Combination** of solutions has to be possible
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  - Only needed input for each subproblem has to be known



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- Otherwise the efficiency will be improved if relative big subproblems can be solved directly
- Recursion depth should not get too big (stack / memory overhead)

# Divide and Conquer

## Implementation



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- Choosing the number of subproblems and the concrete allocation can be demanding

### **Combination of solutions:**

- Typically conceptual demanding

# Divide and Conquer

## Example - Maximum Subtotal



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# Divide and Conquer

## Example - Maximum Subtotal



## Example - Maximum Subtotal Input:



# Divide and Conquer

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Value	31	-41	59	26	-53	58	97	-93	-23	84

### Example - Maximum Subtotal Input:

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Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

**Output:** Sum: 187, Start: 2, End: 6

### Application:

- Maximum profit of buying and selling shares





### **Naive solution (brute force)**



### Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

Example - Maximum Subtotal - Python



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## Runtime - Upper bound

### Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

## Example - Maximum Subtotal



**Upper bound:**



### Upper bound:

- Three interleaved loops

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- Each loop with runtime  $O(n)$

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- Three interleaved loops
- Each loop with runtime  $O(n)$
- Algorithm runtime of  $O(n^3)$

### Lower bound:

Table: Operations

$i$	Additions	$j$
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$



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- For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$

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- For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$
- For each  $j$  we have at least  $\frac{n}{3}$  additions
- We need at least  $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$  steps

# Divide and Conquer

Example - Maximum Subtotal - Runtime



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**Runtime:**

### Runtime:

- With  $T(n) \in O(n^3)$  and  $T(n) \in \Omega(n^3)$  we know:

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- It is hard to solve the problem in a worse way ...

# Divide and Conquer

Example - Maximum Subtotal - Runtime



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**Current approach:**



### Current approach:

- Calculating the sum for range from  $i$  to  $j$  with loop

$$S_{i,j} = X[i] + X[i+1] + \dots + X[j]$$

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$$S_{i,j} = X[i] + X[i+1] + \dots + X[j]$$

### Better approach:

- Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i+1] + \dots + X[j] + X[j+1]$$

$$S_{i,j+1} = S_{i,j} + X[j+1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

# Divide and Conquer

Example - Maximum Subtotal - Python



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**Better solution:**

### Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
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### Better solution:

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■ Runtime  $\in O(n^2)$

# Divide and Conquer

Example - Maximum Subtotal



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## Divide and Conquer:



## Divide and Conquer Idea to solve:

- split the sequence in the middle

# Divide and Conquer

Example - Maximum Subtotal



## Divide and Conquer:



## Divide and Conquer Idea to solve:

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- Solve the left half of the problem



# Divide and Conquer

Example - Maximum Subtotal

## Divide and Conquer:



## Divide and Conquer Idea to solve:

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- Solve the left half of the problem
- Solve right half and combine both solutions into a total solution

# Divide and Conquer

Example - Maximum Subtotal

## Divide and Conquer:



## Divide and Conquer Idea to solve:

- split the sequence in the middle
- Solve the left half of the problem
- Solve right half and combine both solutions into a total solution
- OK if maximum is located in left half (*A*) or right half (*B*)

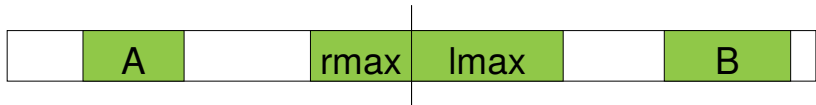
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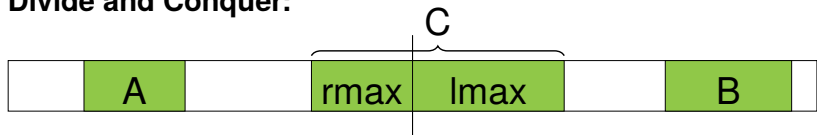
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### Divide and Conquer:



### Divide and Conquer Idea to solve:

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- Solve the left half of the problem
- Solve right half and combine both solutions into a total solution
- OK if maximum is located in **left half (A)** or **right half (B)**
- Problem: Maximum can **overlap split**
- To solve this case we have to calculate *rmax* and *lmax*
- The overall solution is the **maximum of A, B and C**

# Divide and Conquer

## Example - Maximum Subtotal



## Principle - Divide and Conquer:

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- Small problems are solved directly:  $n = 1 \Rightarrow \text{max} = X[0]$

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### Principle - Divide and Conquer:

- Small problems are solved directly:  $n = 1 \Rightarrow \text{max} = X[0]$
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- To determine subsolution C, rmax and lmax for the subproblems are computed.

### Principle - Divide and Conquer:

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- The overall solution is the maximum of A, B and C

### Divide and conquer solution

```
def maxSubArray(X, i, j):  
    if i == j: #trivial case  
        return (X[i], i, i)  
    m = (i + j) / 2  
    #recursive Subsolutions for A,B  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
    #rmax and lmax for bordercase C  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
    #Solution results from A,B,C  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

## ■ General

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Caching

[Wik] [Cache](https://en.wikipedia.org/wiki/Cache)

`https://en.wikipedia.org/wiki/Cache`