

Algorithms and Datastructures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

Albert-Ludwigs-Universität Freiburg



**UNI
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Algorithms and Datastructures, January 2017

Balanced Trees

- Motivation

- AVL-Trees

- (a,b)-Trees

 - Introduction

 - Runtime Complexity

- Red-Black Trees



Binary search tree:

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- Worst case: $d \in O(n)$, keys are inserted in ascending / descending order (20, 19, 18, ...)



Gnarley trees:



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Figure: Binary search tree with random insert [Gna]



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Figure: Binary search tree with descending insert [Gna]



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- Used in C++ `std::map` and Java `SortedMap`

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- With that the height of the search tree is always $O(\log n)$
- We can perform all basic operations in $O(\log n)$



Figure: Example of an AVL-Tree

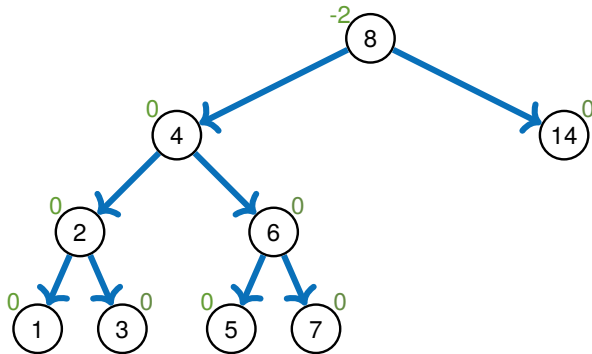


Figure: **Not** an AVL-Tree



Figure: Another example of an AVL-Tree

Balanced Trees

AVL-Tree - Rebalancing

Rotation:

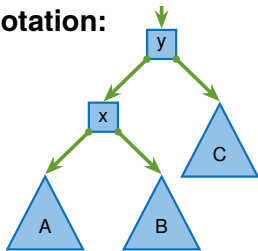


Figure: Before rotating

\Rightarrow

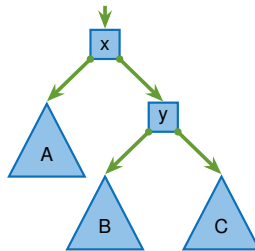


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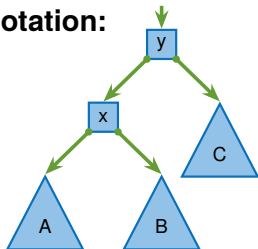


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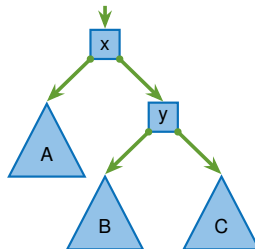


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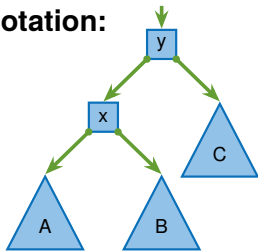


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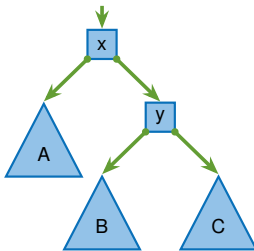


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- Central operation of **rebalancing**
- After rotation to the right:

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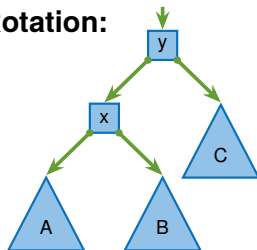


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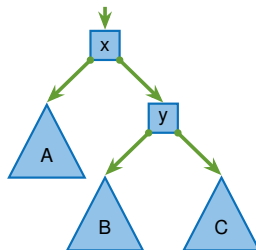


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 - Subtree **A** is a layer higher and subtree **C** a layer lower

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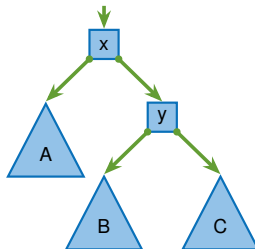


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- Central operation of **rebalancing**
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 - Subtree **A** is a layer higher and subtree **C** a layer lower
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Figure: Inserting 1,...,10 into an AVL-tree [Gna]



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- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in $O(\log n)$
- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b) -trees

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- So we have space for elements on an **insert** and balance operation



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- Each inner node has $\geq a$ and $\leq b$ nodes
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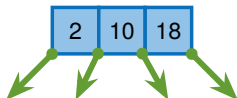
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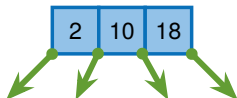
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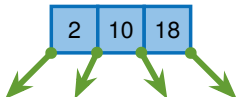
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- We require: $a \geq 2$ and $b \geq 2a - 1$

(2,4)-Tree:



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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

Not an (2,4)-Tree:

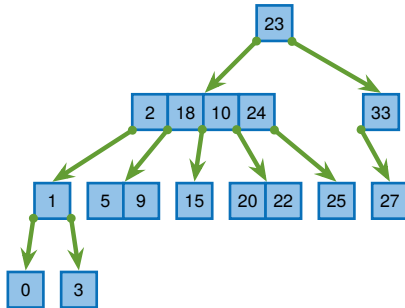


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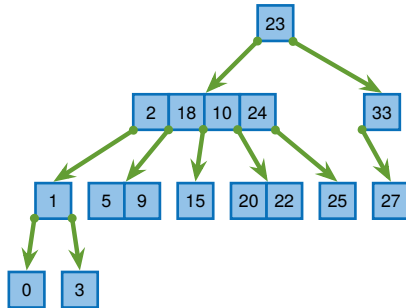


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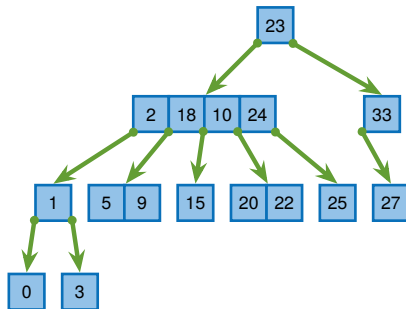


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- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

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Figure: (3,5)-Tree [Gna]



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- Insert the element into the tree
- **Attention:** As a result node can overflow by one element (Degree $b+1$)
- Then we **split** the node

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- If the degree is higher than $b + 1$ we split the node
- This results in a node with $\text{ceil}\left(\frac{b-1}{2}\right)$ elements, a node with $\text{floor}\left(\frac{b-1}{2}\right)$ elements and one element for the parent node

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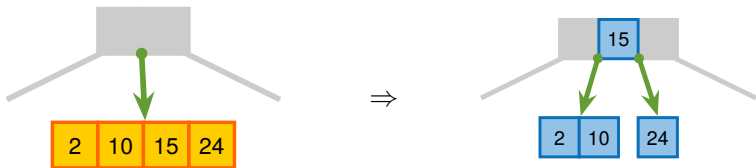


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(The tree is now one level deeper)



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Figure: Borrow an element



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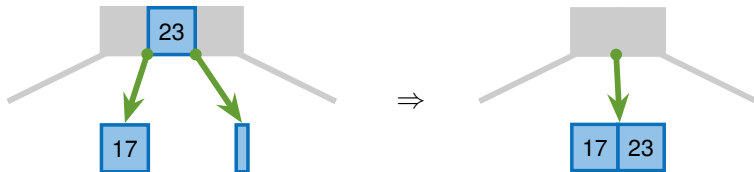


Figure: Merge two nodes



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Removing an element: (remove)

- Now the parent node can be of degree $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
 - Remove the root
 - Define sole child as new root
 - The tree shrinks by one level



Runtime complexity of `lookup`, `insert` and `remove`:



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In detail:

- `lookup` always takes $\Theta(d)$
- `insert` and `remove` often require only $O(1)$ time
- **Worst case:** `split` or `merge` all nodes on path up to the root
- Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$



Counter example $(2,3)$ -Tree:



Counter example (2,3)-Tree:

- Before executing `delete(11)`

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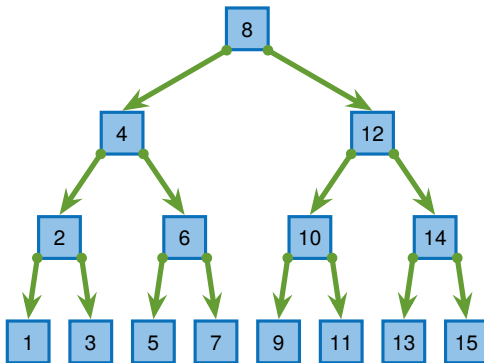


Figure: Normal (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 1

Counter example (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 2

Counter example (2,3)-Tree:

- Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 3

Counter example (2,3)-Tree:

- Executed `delete(11)`

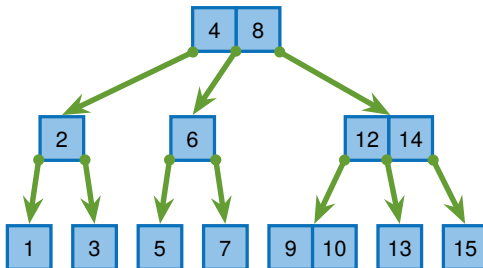


Figure: (2,3)-Tree - Delete step 4



Counter example (2,3)-Tree:

- Executing `insert(11)`



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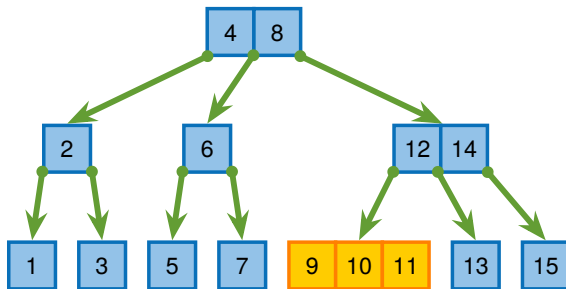


Figure: (2,3)-Tree - Insert step 1

Counter example (2,3)-Tree:

- Executing `insert(11)`

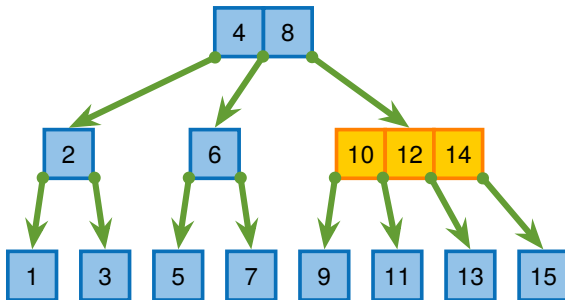


Figure: (2,3)-Tree - Insert step 2

Counter example (2,3)-Tree:

- Executing `insert(11)`



Figure: (2,3)-Tree - Insert step 3

Counter example (2,3)-Tree:

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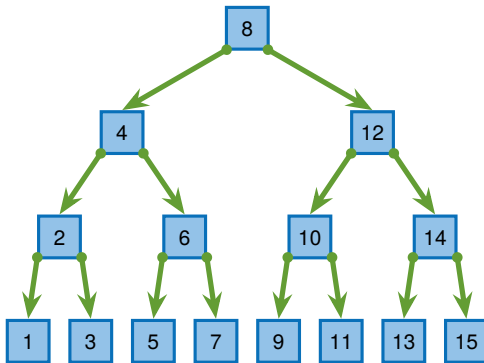


Figure: (2,3)-Tree - Insert step 4

Counter example (2,3)-Tree:

- We are exactly where we started

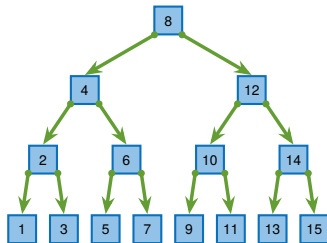


Figure: (2,3)-Tree

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- We are exactly where we started
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- We need $b \geq 2a$ instead of $b \geq 2a - 1$

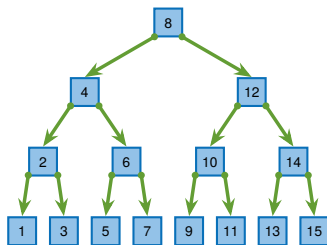


Figure: (2,3)-Tree



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(2,4)-Tree:

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⇒ **Nodes of degree 3 are stable**

Neither an insert nor a remove operation trigger rebalancing operations

$(2,4)$ -Tree:

- Idea:

(2,4)-Tree:

- **Idea:**

- After an expensive operation the tree is in a stable state

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■ Idea:

- After an expensive operation the tree is in a stable state
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- Like with dynamic arrays:
 - **Reallocation** is expensive but it takes some time until the next expensive operation occurs
 - If we **overallocate** clever we have an amortized runtime of $O(1)$

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- Empty tree has 0 nodes: $\phi = 0$



Example:



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- Nodes of degree 3 are highlighted

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Figure: Tree with potential $\phi = 4$



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- The costs for operation i are coupled to the difference of the potential levels

$$c_i \leq A \cdot (\underbrace{\phi_i - \phi_{i-1}}_{\text{difference of potential levels}}) + B, \quad A > 0 \text{ and } B > A$$

Number of gained stable nodes (degree 3) ≥ -1

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$$c_i \leq A \cdot (\underbrace{\Phi_i - \Phi_{i-1}}_{\text{change in potential}}) + B, \quad A > 0 \text{ and } B > A$$

Number of gained stable nodes (degree 3) ≥ -1

- Each operation has an amortized cost of $O(1)$ summing up to $O(n)$ in total

Case 1: *i*-th operation is an `insert` operation on a full node

Case 1: i -th operation is an `insert` operation on a full node



Figure: Splitting a node on `insert`

Case 1: i -th operation is an **insert** operation on a full node



Figure: Splitting a node on **insert**

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- Each splitted node creates a node of **degree 3**
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too



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Costs: $c_i \leq A \cdot m + B$

$$\begin{aligned}\Rightarrow c_i &\leq A \cdot (\phi_i - \phi_{i-1} + 1) + B \\ c_i &\leq A \cdot (\phi_i - \phi_{i-1}) + \underbrace{A + B}_{B'}\end{aligned}$$



Case 2: *i*-th operation is an **remove** operation



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- **Case 2.1:** Inner node



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⇒ We can find the successor in $O(1)$

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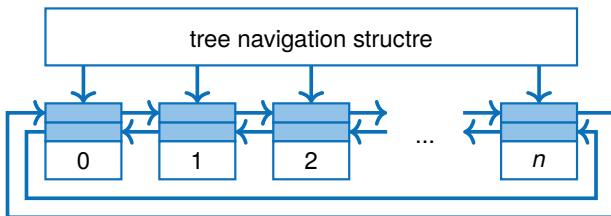


Figure: Tree with doubly linked list



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Figure: Case 2.1.1: Borrow an element



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Figure: Case 2.1.2: Borrow an element

Case 2: *i*-th operation is an `remove` operation



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■ **Case 2.2:** Merging two nodes



Figure: Merging two nodes

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Case 2: i -th operation is an **remove** operation

■ **Case 2.2:** Merging two nodes

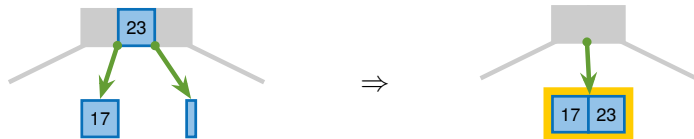


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation

Case 2: i -th operation is an **remove** operation

■ **Case 2.2:** Merging two nodes



Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree > 2 or a node of degree 2, which can borrow from a neighbour

Case 2: i -th operation is an **remove** operation

■ **Case 2.2:** Merging two nodes

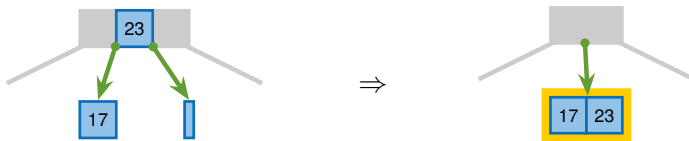


Figure: Merging two nodes

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■ **Case 2.2:** Merging two nodes



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- If the “stop-node” is of **degree 2** then the potential eventually goes down by one

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Figure: Merging two nodes

- The potential rises by m
- If the “stop-node” is of **degree 2** then the potential eventually goes down by one
- Same costs as **insert**



Lemma:

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$$c_i \leq A \cdot (\phi_i - \phi_{i-1}) + B, \quad A > 0 \text{ and } B > A$$

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- With that we can conclude:

$$\sum_{i=0}^n c_i \in O(n)$$

Proof:

$$\begin{aligned}\sum_{i=0}^n c_i &\leq \underbrace{A \cdot (\phi_1 - \phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\phi_2 - \phi_1) + B}_{\leq c_2} + \dots + \underbrace{A \cdot (\phi_n - \phi_{n-1}) + B}_{\leq c_n} \\ &= A \cdot (\phi_n - \phi_0) + B \cdot n \quad | \text{ telescope sum} \\ &= A \cdot \phi_n + B \cdot n \quad | \text{ we start with an empty tree} \\ &< A \cdot n + B \cdot n \in O(n) \quad | \text{ number of degree 3 nodes} \\ &\quad < \text{ number of nodes}\end{aligned}$$

Balanced Trees

Motivation

AVL-Trees

(a,b)-Trees

Introduction

Runtime Complexity

Red-Black Trees



Red-Black Tree:



Red-Black Tree:

- Binary tree with red and black nodes

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Red-Black Tree:

- Binary tree with **red** and **black** nodes
- Number of **black** nodes on path to leaves is equal
- Can be interpreted as **(2,4)-tree** (also named 2-3-4-tree)
- Each **(2,4)-tree**-node is a small red-black-tree with a **black** root node



Figure: Example of an red-black-tree [Gna]

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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■ Gnarley Trees

[Gna] **Gnarley Trees**

<https://people.ksp.sk/~kuko/gnarley-trees/>

■ AVL-Tree

[Wik] [AVL tree](#)

`https://en.wikipedia.org/wiki/AVL_tree`

■ (a,b)-Tree

[Wika] [2-3-4 tree](#)

`https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934_tree`

[Wikb] [\(a,b\)-tree](#)

`https://en.wikipedia.org/wiki/\(a,b\)-tree`

■ Red-Black-Tree

[Wik] [Red-black tree](https://en.wikipedia.org/wiki/Red%E2%80%93black_tree)

`https://en.wikipedia.org/wiki/Red%E2%80%93black_tree`