

Algorithms and Datastructures

Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg



**UNI
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Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
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Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
- **Recursive** application of the algorithm on smaller subproblems
- **Direct** solving of small subproblems

Input:

- Sequence X of n integers

Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

Table: Input values

| | | | | | | | | | | |
|-------|----|-----|----|----|-----|----|----|-----|-----|----|
| Index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Value | 31 | -41 | 59 | 26 | -53 | 58 | 97 | -93 | -23 | 84 |

Output: Sum: 187, Start: 2, End: 6

Idea:



- Solve the left / right half of the problem **recursive**
- Combine both solutions into a overall solution
- The maximum is located in the **left half (A)** or the **right half (B)**
- The maximum interval can **overlap with the border (C)**

Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned
- To solve C we have to calculate rmax and lmax
- Overall solution is maximum of A , B and C

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for cornercase C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```



```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

#Alternative implementation max

```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b  
  
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: $lmax$ example

| index | i | $i + 1$ | \dots | \dots | $j - 1$ | j |
|--------|-----|---------|---------|---------|---------|-----|
| X | 58 | -53 | 26 | 59 | -41 | 31 |
| sum | 58 | 5 | 31 | 90 | 49 | 80 |
| $lmax$ | 58 | 58 | 58 | 90 | 90 | 90 |

- The sum and $lmax$ are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update sum
- If $sum > lmax$ then $lmax$ gets updated

Divide and Conquer

Maximum Subtotal



Call with array of four elements

Call with array of two elements



```
def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)                             # O(n)  
    C2 = lmax(X, m + 1, j)                         # O(n)  
    C = (C1[0] + C2[0], C1[1], C2[1])              # O(1)  
  
    return max([A, B, C], \                       # O(1)  
               key=lambda item: item[0])
```

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

$\underbrace{\Theta(1)}_{\text{trivial case}}$

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^i nodes processing $\frac{n}{2^i}$ elements
 $\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$

n nodes processing 1 element
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$

The costs of merging the solutions and solving of the trivial problems are the same here

$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Summary:

- Direct solution is slow with $\mathcal{O}(n^3)$
- Better solution with incremental update of sum was $\mathcal{O}(n^2)$
- Divide and conquer approach results in $\mathcal{O}(n \log n)$
- There is an approach running in $\mathcal{O}(n)$ if you assume that all subtotals are positive



Figure: Scanning the array in linear time

#Implementation - linear runtime

```
def maxSubArray(X):  
    # sum, start index  
    rMax, irMax = 0, 0 # current maximum  
    tMax, itMax = 0, 0 # total maximum  
  
    for i in range(len(X)):  
        if rMax == 0:  
            irMax = i  
            rMax = max(0, rMax + X[i])  
  
        if rMax > tMax:  
            tMax, itMax = rMax, irMax  
  
    return (tMax, itMax)
```

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- n_0 is normally small, $f_0(n_0) \in \Theta(1)$
- Normally $a > 1$ and $b > 1$
- Dependent on the strategy of solving $T(n)$ f_0 is ignored
- $T(n)$ is only defined for integers of $\frac{n}{b}$ which is often ignored in benefit of a simpler solution

Substitution Method:

- Guess the solution and prove it with induction
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

- Assumption: $T(n) = n + n \cdot \log_2 n$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned}T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\&\stackrel{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \\&= 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot (\log_2 n - 1)\right) + n \\&= n + n \log_2 n - n + n \\&= n + n \log_2 n\end{aligned}$$

Substitution Method:

- Alternative assumption
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 0 \end{cases}$$

- Assumption: $T(n) \in O(n \log n)$
- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned}T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\&\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \\&= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\&= c \cdot n \log_2 n - c \cdot n + n \\&\leq c \cdot n \log_2 n, \quad c \geq 1\end{aligned}$$

Recursion tree method:

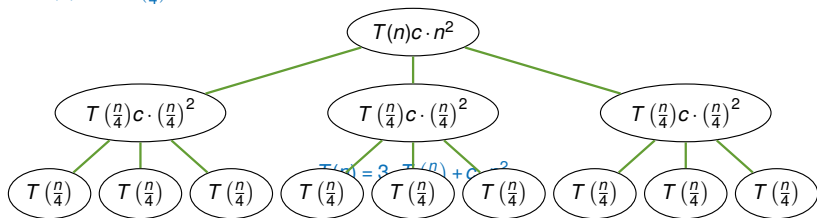
- Can be used to make assumptions about the runtime
- Example:

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

Figure: Recursion tree of example

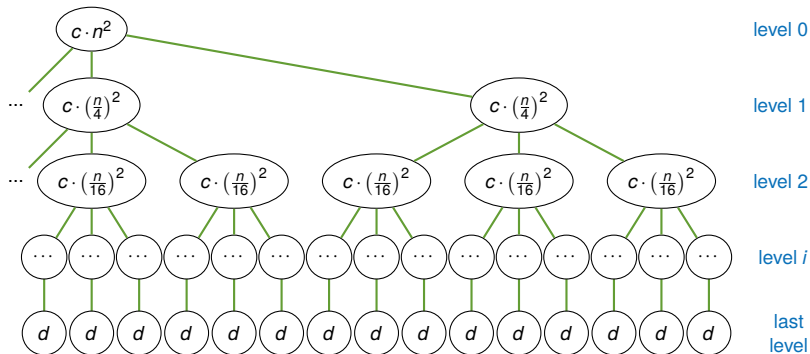


Figure: Levels of the recursion tree

Costs of connecting the partial solutions:

(excludes the last layer)

- Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- Costs of partial problem on level i :

$$T_{i_p}(n) = c \cdot \left(\left(\frac{1}{4} \right)^i \cdot n \right)^2$$

- Number of partial problems on level i : $n_i = 3^i$
- Costs on level i :

$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4} \right)^i \cdot n \right)^2 = \left(\frac{3}{16} \right)^i \cdot c \cdot n^2$$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the **last level**: $s_{i+1}(n) = 1$
- Costs of partial problem on the **last level**: $T_{i+1_p}(n) = d$
- With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \quad \Rightarrow n = 4^i \quad \Rightarrow i = \log_4 n$$

- Number of partial problems on the **last level**:

$$n_{i+1} = 3^{\log_4 n} = n^{\log_4 3} \quad \leftarrow \text{next slide}$$

- Costs on the **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

- Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right)$$

$$\text{uses } n = 3^{\log_3 n}$$

$$= \log_3 n \cdot \log_4 3$$

$$\text{uses } \log a^b = b \cdot \log a$$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3}$$

uses reformulation above

$$= \left(3^{\log_3 n} \right)^{\log_4 3}$$

$$\text{uses } x^{a \cdot b} = (x^a)^b$$

$$= n^{\log_4 3}$$

- This term will recur in the master theorem

Total costs:

- Costs of **level i**: $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- Costs of **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \text{even with} \\ \text{infinite elements}}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

- Here: The costs of connecting the partial problems dominate

- **Geometric progression:**

Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

- **Geometric series:**

The series (cumulative sum) of a geometric sequence

- For $|q| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \quad \Rightarrow \text{constant}$$

Proof of $\mathcal{O}(n^2)$:

- We know:

$$\begin{aligned}T(n) &= 3T\left(\frac{n}{4}\right) + \Theta(n^2) \\ &\leq 3T\left(\frac{n}{4}\right) + c \cdot n^2\end{aligned}$$

- Assumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) \leq k \cdot n^2$$

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16}k \cdot n^2 + c \cdot n^2 \\ &\leq k \cdot n^2 \qquad \text{for } k \geq \frac{16}{13}c \end{aligned}$$

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...
 - ... which divides a **problem of size n** in **a partial problems**
 - ... which solves each partial problem recursively with a **runtime of $T\left(\frac{n}{b}\right)$**
 - ... which takes **$f(n)$** steps to merge all partial solutions

Master theorem:

- In the examples we have seen that ...
 - Either the runtime of **connecting the solutions** dominates
 - Or the runtime of **solving the problems** dominates
 - Or both have **equal influence on runtime**
- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n) \text{ in general form}}, \quad a \geq 1, b > 1, c > 0$$

- This yields a runtime of:

Number of leaves

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$



Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- Three partial problems with $\frac{1}{2}$ the size
- Solving the partial problems dominates (last layer, leaves)
- Runtime of $\Theta(n^{\log_b a})$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- Two partial problems with $\frac{1}{2}$ the size
- Each layer has equal costs, $\log n$ layers
- Runtime of $\Theta(n \log n)$



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

- Two partial problems with $\frac{1}{3}$ the size
- Connecting all partial solutions dominates (first layer, root)
- Runtime of $\Theta(n)$

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \quad a \geq 1, b > 1, c > 0$$

- ... yields to a runtime of:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b \\ \Theta(n \log_b n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

Master theorem (general form):

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates
(last layer, leaves)

- **Case 2:** $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log_b n$ layers

Master theorem (general form):

- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root)
dominates

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates (last layer, leaves)

■ $T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

■ $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, b = 3, f(n) = 17 \cdot n, \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

■ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 10 \cdot n$

$$a = 2, b = 2, f(n) = 10 \cdot n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

■ $T(n) = T\left(\frac{2n}{3}\right) + 1$

$$a = 1, b = \frac{2}{3}, f(n) = 1, \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

■ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$

$$a = 2, b = 2, f(n) = n^2, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Omega(n^{1+\varepsilon})$$

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if **regularity condition** also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$

Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$
- **Case 2:** $f(n) \notin \Theta(n^1)$
- **Case 3:** $f(n) \notin \Omega(n^{1+\varepsilon})$

$n \log n$ is *asymptotically* larger than n ,
but not *polynomial* larger

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- Three cases depending on the dominance of the terms
- **Case 1:** Solving the partial problems is *polynomial* bigger than merging all solutions

$$T(n) \in \Theta(n^{\log_b a}), \quad T(n) \in \Theta(\text{number of leaves})$$

- **Case 2:** Each layer has equal costs

$$T(n) \in \Theta(n^{\log_b a} \log n), \quad \log n \text{ layers}$$

- **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems

$$T(n) \in \Theta(f(n))$$

■ General

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■ Master theorem

[Wik] [Master theorem](#)

https://en.wikipedia.org/wiki/Master_theorem