Algorithms and Datastructures

Runtime analysis Minsort / Heapsort, Induction

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Algorithms and Datastructures, October 2018

Structure

Structure

Runtime Example

- Minsort

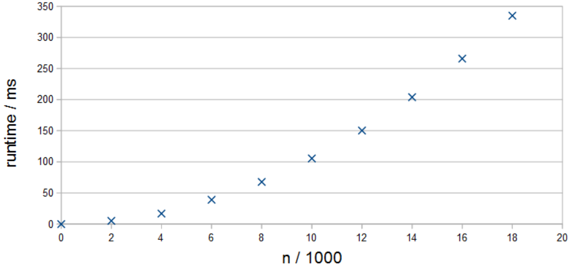
Basic Operations

Runtime analysis

- Minsort
- Heapsort
  - Introduction to Induction

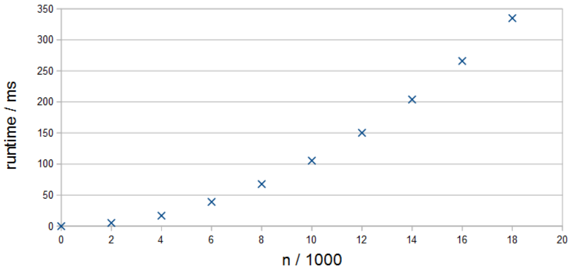
Logarithms

Runtime analysis - Minsort



How long does the program run?

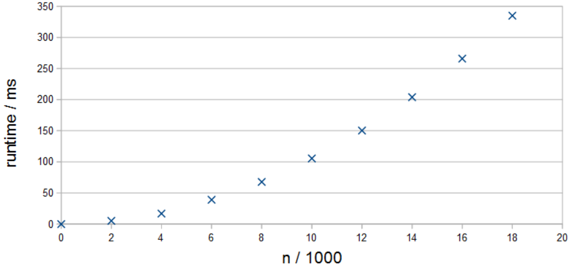
Runtime analysis - Minsort



How long does the program run?

- In the last lecture we had a schematic
- **Observation:** it is going to be "disproportionately" slower the more numbers are being sorted

Runtime analysis - Minsort



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- In the last lecture we had a schematic
- **Observation:** it is going to be "disproportionately" slower the more numbers are being sorted
- How can we say more precisely what is happening?

Runtime analysis - Minsort

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How can we analyze the runtime?

- Ideally we have a formula which provides the runtime of the program for a specific input

Runtime analysis - Minsort

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- **Problem:** the runtime is depends on many variables, especially:
  - What kind of computer the code is executed on
  - What is running in the background
  - Which compiler is used to compile the code

Runtime analysis - Minsort

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- Ideally we have a formula which provides the runtime of the program for a specific input
- **Problem:** the runtime is depends on many variables, especially:
  - What kind of computer the code is executed on
  - What is running in the background
  - Which compiler is used to compile the code
- **Abstraction 1:** analyze the number of basic operations, rather than analyzing the runtime

Structure

Basic Operations

Incomplete list of basic operations:

- Arithmetic operation, for example: $a + b$
- Assignment of variables, for example: $x = y$
- Function call, for example: *minsort(lst)*

Basic Operations

Intuitive:

lines of code

Better:

lines of machine code

Best:

process cycles

Important:

The actual runtime has to be roughly proportional to the number of operations.

Structure

Runtime analysis - Minsort

How many operations does *Minsort* need?

- **Abstraction 2:** we calculate the upper (lower) bound, rather than exactly counting the number of operations

  Reason: runtime is approximated by number of basic operations, but we can still infer:

  - Upper bound
  - Lower bound

Runtime analysis - Minsort

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- **Basic Assmuption:**
  - n is size of the input data (i.e. array)
  - $T(n)$ number of operations for input n

Runtime analysis - Minsort

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$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

Runtime analysis - Minsort

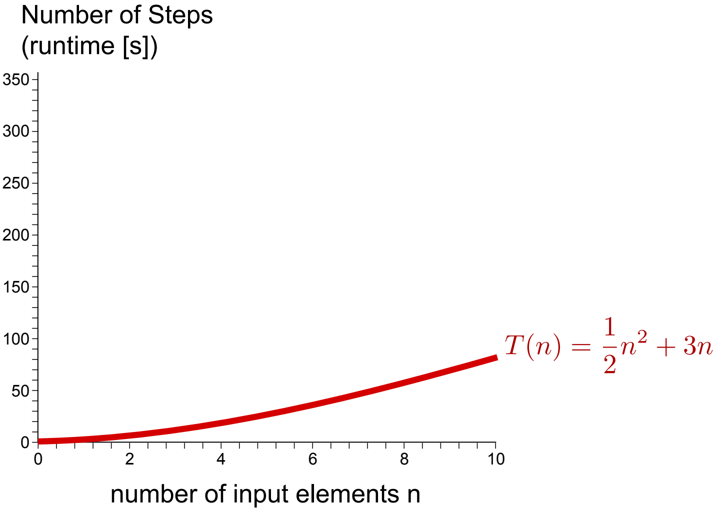
How many operations does *Minsort* need?

- **Observation:** the number of operations depends only on the size n of the array and not on the content!
- **Claim:** there are constants C_1 and C_2 such that:

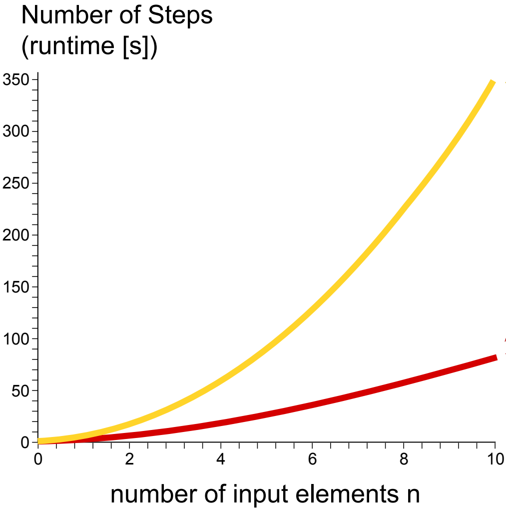
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- This is called "quadratic runtime" (due to n^2)

Runtime Example



Runtime Example

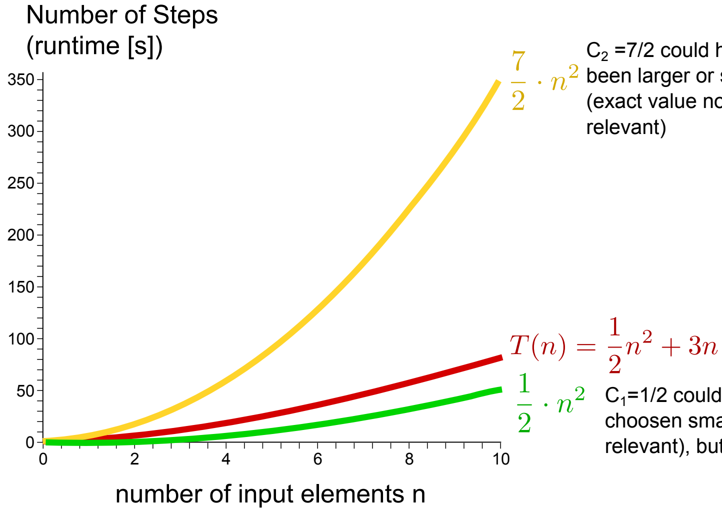


$$\frac{7}{2} \cdot n^2$$

C_2 =7/2 could have been larger or small (exact value not relevant)

$$T(n) = \frac{1}{2}n^2 + 3n$$

Runtime Example



C_2 =7/2 could have been larger or small (exact value not relevant)

C_1=1/2 could have been choosen smaller (not relevant), but not larger

Runtime analysis - Minsort

We declare:

- Runtime of operations: $T(n)$
- Number of Elements: n
- Constants: C_1 (lower bound), C_2 (upper bound)

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- Number of operations in round i: T_i

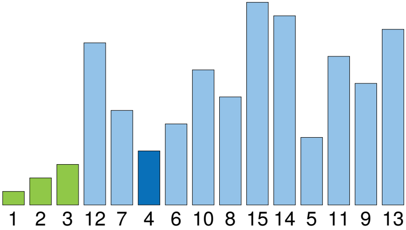


Figure: *Minsort* at iteration $i = 4$. We have to check $n - 3$ elements

Runtime analysis - Minsort

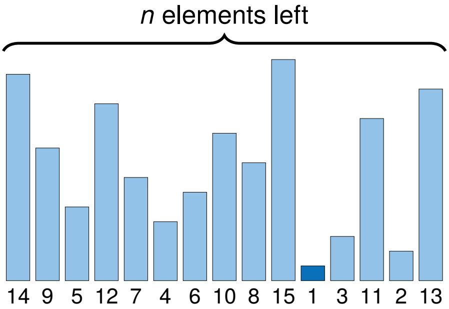


Figure: *Minsort* with start data

Runtime for each iteration:

Runtime analysis - Minsort

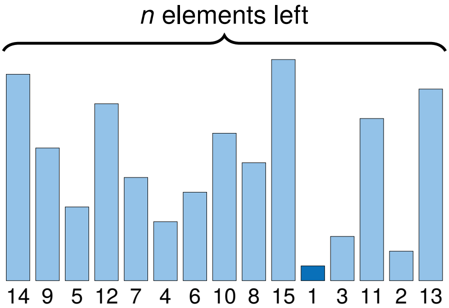


Figure: *Minsort* at iteration $i = 1$

Runtime for each iteration:

$$T_1 \leq C_2' \cdot (n - 0)$$

Runtime analysis - Minsort

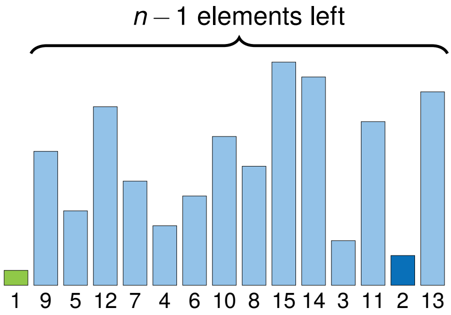


Figure: *Minsort* at iteration $i = 2$

Runtime for each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$

Runtime analysis - Minsort

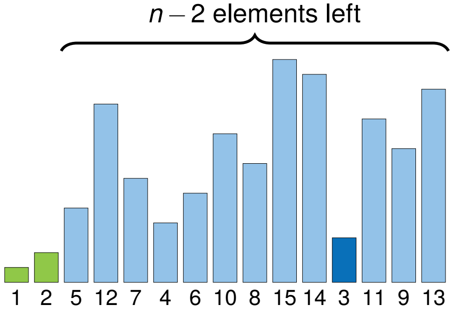


Figure: *Minsort* at iteration $i = 3$

Runtime for each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$
$$T_3 \leq C_2' \cdot (n-2)$$

Runtime analysis - Minsort

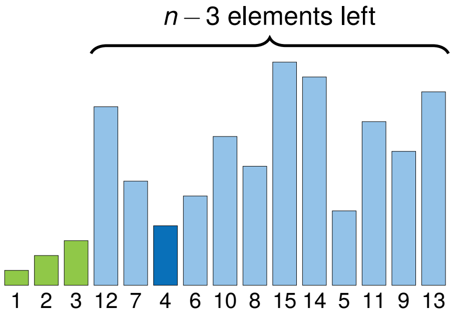


Figure: *Minsort* at iteration $i = 4$

Runtime for each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$
$$T_3 \leq C_2' \cdot (n-2)$$
$$T_4 \leq C_2' \cdot (n-3)$$

Runtime analysis - Minsort

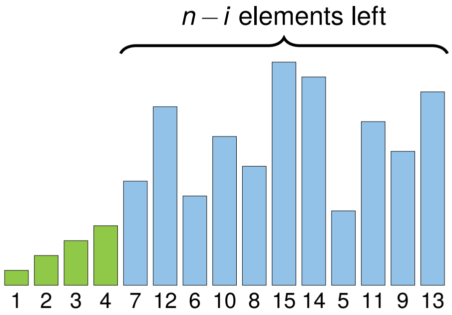


Figure: *Minsort* at iteration i

Runtime for each iteration:

$$
\begin{aligned}
T_1 &\leq C_2' \cdot (n-0) \\
T_2 &\leq C_2' \cdot (n-1) \\
T_3 &\leq C_2' \cdot (n-2) \\
T_4 &\leq C_2' \cdot (n-3) \\
&\vdots \\
T_{n-1} &\leq C_2' \cdot 2 \\
T_n &\leq C_2' \cdot 1
\end{aligned}
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Runtime analysis - Minsort

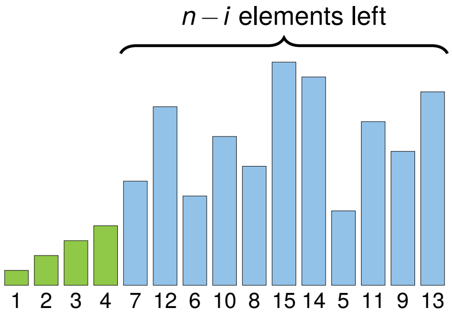

Figure: *Minsort* at iteration

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$$T(n) = C_2' \cdot (T_1 + \cdots + T_n) \leq \sum_{i=1}^{n} \left(C_2' \cdot i\right)$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
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        if minimum != i:
            elements[i], elements[minimum] = \
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    return elements
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Remark: C_2' is cost of comparison $\Rightarrow$ assumed constant

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$T(n) \leq \sum_{i=1}^{n} C_2' \cdot i$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\ &= C_2' \cdot \sum_{i=1}^{n} i \end{aligned}$$

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

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\begin{aligned}
T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\
&= C_2' \cdot \sum_{i=1}^{n} i \\
&\quad \Downarrow \quad \text{Small Gauss sum} \\
&= C_2' \cdot \frac{n(n+1)}{2}
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Runtime analysis - Minsort

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&\leq C_2' \cdot \frac{n(n+n)}{2}, \; 1 \leq n \\
&= C_2' \cdot \frac{2 \cdot n^2}{2}
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Runtime analysis - Minsort

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&= C_2' \cdot \frac{2 \cdot n^2}{2} \quad = \quad C_2' \cdot n^2
\end{aligned}
$$

Excursion - Small Gauss Formula

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper bound there exists a C_1. Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C_1' \cdot (n-i)$$

Runtime analysis - Minsort

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&\geq \quad C_1' \cdot \frac{(n-1) \cdot n}{2}
\end{aligned}
$$

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Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

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How do we get to n^2?

Runtime analysis - Minsort

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How do we get to n^2?

$$\Downarrow \quad n - 1 \geq \frac{n}{2} \text{ for } n \geq 2$$

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$$

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&\geq C_1' \cdot \frac{n \cdot n}{2 \cdot 2} = \frac{C_1'}{4} \cdot n^2
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$$

Runtime analysis - Minsort

Runtime Analysis:

- Upper bound: $T(n) \leq C_2' \cdot n^2$

Runtime analysis - Minsort

Runtime Analysis:

- Upper bound: $T(n) \leq C'_2 \cdot n^2$
- Lower bound: $\frac{C'_1}{4} \cdot n^2 \leq T(n)$

Runtime analysis - Minsort

Runtime Analysis:

- Upper bound: $T(n) \leq C_2' \cdot n^2$
- Lower bound: $\frac{C_1'}{4} \cdot n^2 \leq T(n)$

Summarized:

$$\frac{C_1'}{4} \cdot n^2 \leq T(n) \leq C_2' \cdot n^2$$

Quadratic runtime proven:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

Runtime Example

- The runtime is growing quadratically with the number of elements n in the list

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  - $C = 1\,\text{ns}$ (1 simple instruction $\approx 1\,\text{ns}$)
  - $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{12} = 10^3\,\text{s} = 16.7\,\text{min}$

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  - $n = 10^9$ (1 billion numbers = 4 GB)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{18} = 10^9\,\text{s} = 31.7$ years
- **Quadratic runtime = “big” problems unsolvable**

Structure

Runtime - Heapsort

Intuitive to extract minimum:

- **Minsort:** to determine the minimum value we have to iterate through all the unsorted elements.

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Formal:

- Let $\mathrm{T}(n)$ be the runtime for the *Heapsort* algorithm with n elements
- On the next pages we will proof $T(n) \leq C \cdot n \log_2 n$

Depth of a binary tree:

- **Depth d:** longest path through the tree
- Complete binary tree has $n = 2^d - 1$ nodes
- Example: $d = 4$
  $\Rightarrow n = 2^4 - 1 = 15$

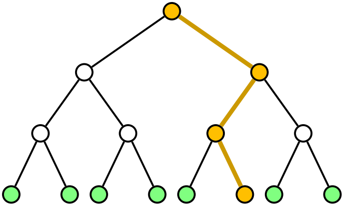

Figure: Binary tree with 15 nodes

Structure

Induction

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- If both has been proven, then $A(n)$ holds for all natural numbers n by **induction**

Induction - Example 1

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

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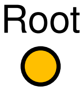

$n(1) = 2^1 - 1 = 1$

Figure: Tree of depth 1 has 1 node

Induction - Example 1

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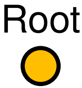

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$n(1) = 2^1 - 1 = 1$

$\Rightarrow$ correct ✓

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

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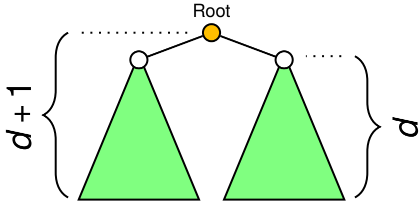

Figure: binary tree with subtrees

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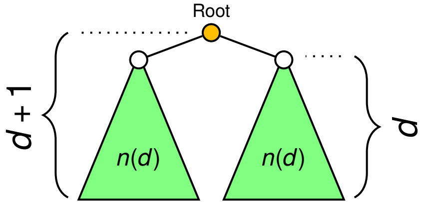

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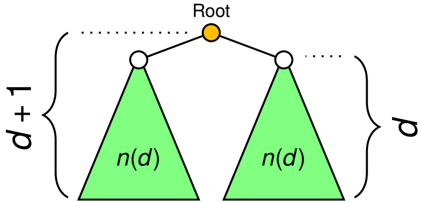

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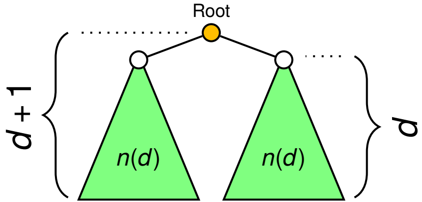

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$$\begin{aligned} n(d+1) &= 2 \cdot n(d) + 1 \\ &= 2 \cdot \left(2^d - 1\right) + 1 \\ &= 2^{d+1} - 2 + 1 \end{aligned}$$

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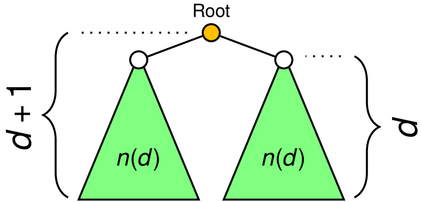

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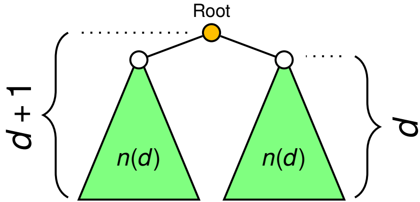

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n(d+1) &= 2 \cdot n(d) + 1 \\
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$$

$\Rightarrow$ **By induction:** $n(d) = 2^d - 1 \ \forall n \in \mathbb{N} \ \square$

Structure

Runtime - Heapsort

Heapsort has the following steps:

- **Initially:** `heapify` list of n elements

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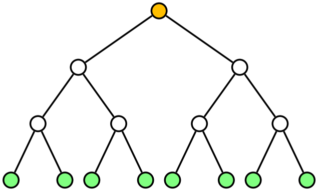
  - Repair heap by `sifting`

Runtime of `heapify` depends on depth d:



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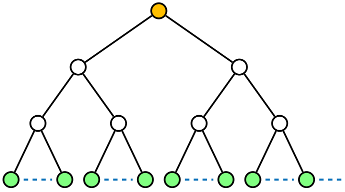

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- No costs at depth d with 2^{d-1} (or less) nodes

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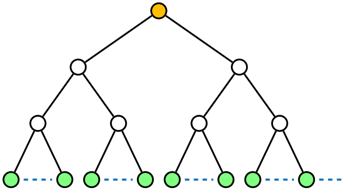

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Runtime - Heapsort
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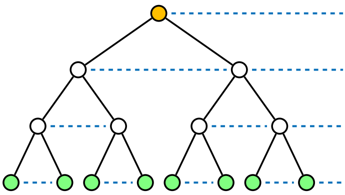
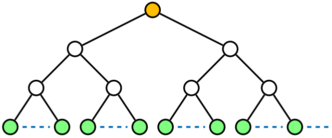


Runtime of `heapify` with depth of d:

- No costs at depth d with 2^{d-1} (or less) nodes
- The cost for `sifting` with depth 1 is at most $1C$ per node
- In general: `Sifting` costs are linear with path length **and** number of nodes

`Heapify` total runtime:



Depth	Nodes	Path length	Costs per node
d	2^{d-1}	0	$\leq C \cdot 0$

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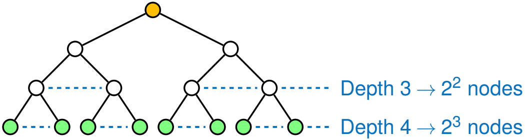

Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
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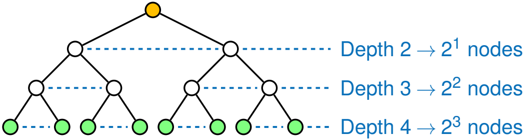

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d	2^{d-1}	0	$\leq C \cdot 0$
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$d-2$	2^{d-3}	2	$\leq C \cdot 2$

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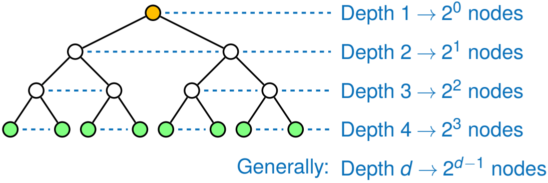

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$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

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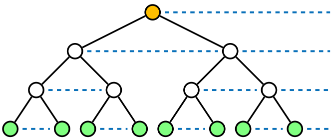

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$d-3$	2^{d-4}	3	$\leq C \cdot 3$

In total: $$T(d) \leq \sum_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right)$$

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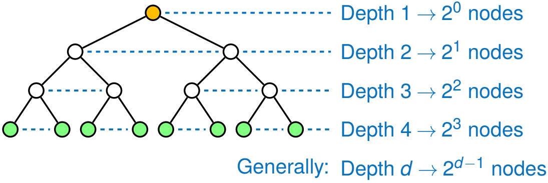

Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	Standard
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	Equation
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

In total: $T(d) \leq \sum_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^{d} \left(C \cdot i \cdot 2^{d-i} \right)$

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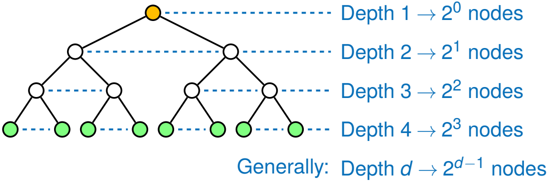

Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	$\leq C \cdot 1$
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	$\leq C \cdot 2$
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	$\leq C \cdot 3$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	$\leq C \cdot 4$

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- **Hence:** Resulting costs for `heapify`:

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- **However:** We want costs in relation to n

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Heapify

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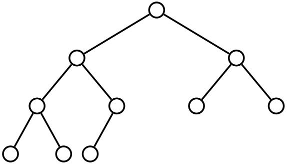

Figure: Partial binary tree

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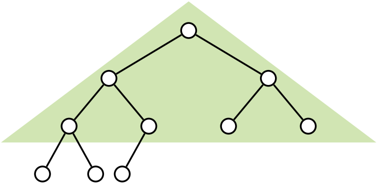

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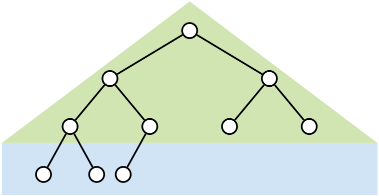

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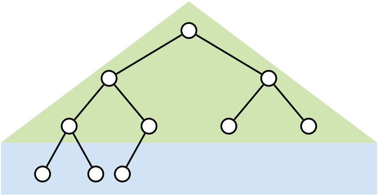

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- Equation multiplied by 2^2
  $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$
- Cost for `heapify`:
  $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

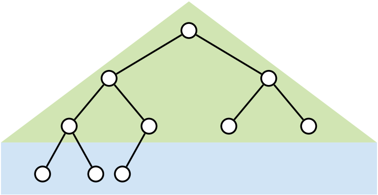

Figure: Partial binary tree

Structure

Induction - Example 2

- **We want to proof (induction assumption):**

$$\underbrace{\sum_{i=1}^{d} \left(i \cdot 2^{d-i}\right)}_{} \leq \underbrace{2^{d+1}}_{}$$

$$A(d) \leq B(d)$$

  - We denote the left side with A, the right side with B

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

Induction - Example 2

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$$2^0 \leq 2^2 \ \checkmark$$

Induction - Example 2

Induction step: $(d := d + 1)$:

- **Idea:** Write down right-hand formula and try to get $A(d)$ and $B(d)$ out of it

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$$\sum_{i=1}^{d+1} \left(i \cdot 2^{d+1-i} \right) \leq 2^{d+1+1}$$

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$$\sum_{i=1}^{d+1} \left(i \cdot 2^{d+1-i}\right) \le 2^{d+1+1}$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i}\right) \le 2 \cdot 2^{d+1}$$

$$\vdots$$

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$$\vdots$$

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Induction step: $(d := d+1)$:

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- **Problem:** does not work but claim still holds

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Working proof:

- Show a little bit stronger claim

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- **Advantage:** results in a stronger induction assumption

$\Rightarrow$ exercise

Structure

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  - **Hence**: $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
  - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \qquad \text{(holds for } n > 2\text{)}$$

Runtime - Heapsort

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  - $\Rightarrow C_1$ and C_2 are constant

Structure

Base of Logarithms

Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient $\frac{1}{\log_b a}$

Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602\ldots \cdot 3.322\ldots = 2$ ✓
- $\log_{10} 1000 = \log_{\mathrm{e}} 1000 \cdot \frac{1}{\log_{\mathrm{e}} 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3$ ✓

Runtime Example

Runtime of $n \log_2 n$:

- Assume we have constants C_1 and C_2 with

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- **Runtime $n \log_2 n$ is nearly as good as linear!**

Further Literature

- **Course literature**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
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Further Literature

- **Mathematical Induction**

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  https://en.wikipedia.org/wiki/Mathematical_induction