

# Algorithms and Datastructures

## Open Addressing, Priority Queue

Albert-Ludwigs-Universität Freiburg



**UNI  
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Algorithms and Datastructures, November 2017

## Hashing

- Recapitulation
- Treatment of hash collisions
- Open Addressing
- Summary

## Priority Queue

- Introduction



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- To find a good hash function for every key set universal hashing is needed
  - Then however, for a fixed set of keys not every hash function is suitable, but only some



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  - Look at **amortized analysis** in the next lecture

## Hashing

Recapitulation

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## Buckets as linked list:



### **Buckets as linked list:**

- Each bucket is a linked list

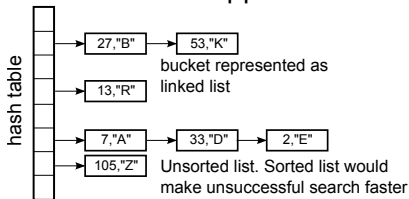


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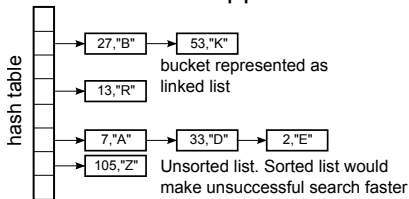
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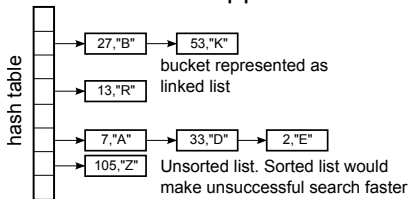


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- Dynamic number of elements is possible

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  - If an entry is already occupied, then iteratively the **following entry** can be checked. If a free entry is found the element is inserted
  - If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry have been found





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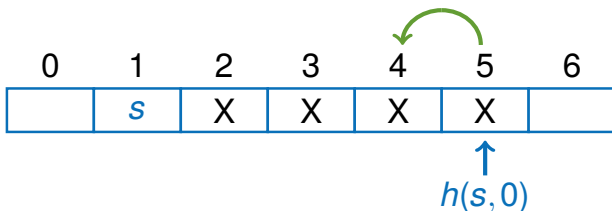
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$j \in \{0, \dots, m-1\}$  e.g.  $g(s,j)=j$

- The **probe sequence** is calculated by

$$h(s,j) = (h(s) - g(s,j)) \bmod m \in \{0, \dots, m-1\}$$



```
def insert(s, value):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
        j += 1  
  
    t[(h(s) - g(s, j)) mod m] \  
      = (s, value)
```

```
def lookup(s):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
        is not None:  
  
        if t[(h(s) - g(s, j)) mod m][0] == s:  
            return t[(h(s) - g(s, j)) mod m]  
  
        j += 1  
  
    return None
```

# Hashing

## Open Addressing - Linear Probing



Figure: Linear probe sequence



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- Check the element with lower index:  $g(s, j) := j$   
 $\Rightarrow$  Hash function:  $h(s, j) = (h(s) - j) \bmod m$





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⇒ Hash function:  $h(s, j) = (h(s) - j) \bmod m$
- This leads to the following probe sequence

$$h(s), h(s) - 1, h(s) - 2, \dots, \underbrace{0, m-1, m-2, \dots, h(s) + 1}_{\text{clipping}}$$

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## Open Addressing - Linear Probing

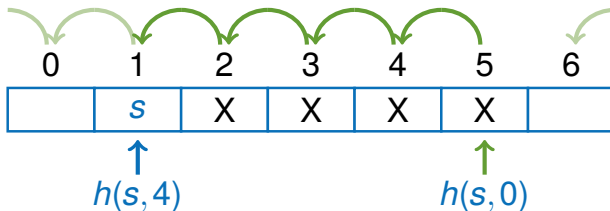


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- Can result in primary clustering
- Dealing with a hash collision will result in a higher probability of hash collisions in close entries



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0	1	2	3	4	5	6
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0	1	2	3	4	5	6
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- $t.\text{insert}(53, \text{"B"}), h(53, 0) = 4$

				53, B	12, A	
--	--	--	--	-------	-------	--

Figure: Probe/Insertion sequence on a hash map



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- t.insert (5, "C"),  $h(5, 0) = 5$ ,  $h(5, 1) = 4$ ,  $h(5, 2) = 3$

0	1	2	3	4	5	6
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0	1	2	3	4	5	6
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- t.insert (15, "D"),  $h(15, 0) = 1$

	15, D		5, C	53, B	12, A	
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■ Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$

■ t.insert(2, "E"),  $h(2, 0) = 2$

0	1	2	3	4	5	6
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■ t.insert(19, "F"),  $h(19, 0) = 5$ ,  $h(19, 1) = 4$ ,  
 $h(19, 2) = 3$ ,  $h(19, 3) = 2$ ,  $h(19, 4) = 1$ ,  $h(19, 5) = 0$

19, F	15, D	2, E	5, C	53, B	12, A	
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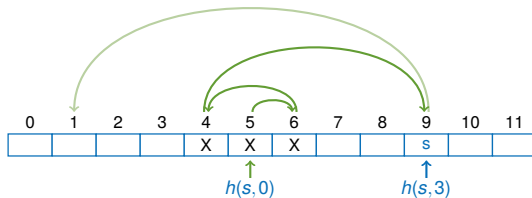


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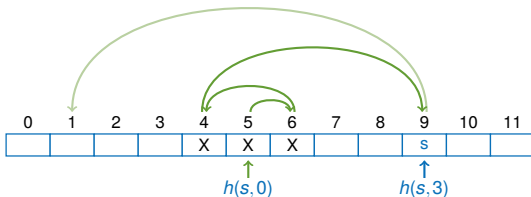


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- This leads to the following probe sequence

$$h(s), h(s) + 1, h(s) - 1, h(s) + 4, h(s) - 4, h(s) + 9, h(s) - 9, \dots$$

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- Alternatively:  $h(s, j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \bmod m$
- Problem of secondary clustering  
No local clustering anymore, but keys with same hash value have similar probe sequence



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- **Advantage:** Prevents clustering because different keys with the same hash value do not produce the same probe sequence
- **Disadvantage:** Hard to implement

Diagram illustrating the computation of  $h_2(s)$  for a sequence of values. The sequence is represented by a horizontal array of 13 cells, indexed 0 to 12. The cells contain the following values: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. The cells at indices 2, 4, 5, 6, 8, and 9 contain 'X'. The cell at index 11 contains 's'. Above the array, green arcs represent the sequence of values  $h_2(s)$  for indices 2 through 11. The arcs are labeled  $h_2(s)$  in green. A blue bracket above the first two arcs (from index 2 to 4) is labeled  $h_2(s)$  in blue. A blue bracket above the next two arcs (from index 5 to 7) is labeled  $h_2(s)$  in blue. Below the array, a green arrow points from the label  $h(s, 0) = h_1(s)$  to cell 2. A blue arrow points from the label  $h(s, 3)$  to cell 11.

November 2017

### Double Hashing:

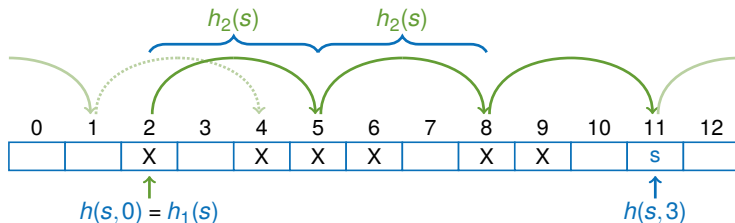


Figure: Double hashing probe sequence

- Motivation: Consider key  $s$  in probe sequence

Diagram illustrating a hash table with separate chaining. The table has 12 slots. Slots 2, 4, 5, 6, 8, and 9 contain 'X'. Slot 11 contains 's'. Arrows show the linked list structure: slot 2 points to slot 1 (dotted), slot 1 points to slot 4 (dotted), slot 4 points to slot 5 (solid), slot 5 points to slot 6 (solid), slot 6 points to slot 8 (solid), slot 8 points to slot 9 (solid), and slot 9 points to slot 11 (solid). Brackets above the table group slots 2-4 and 5-8 under the label  $h_2(s)$ . Below the table, an arrow points from the label  $h(s, 0) = h_1(s)$  to slot 2, and another arrow points from the label  $h(s, 3)$  to slot 11.

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- Works well in practical use
- This method is an approximation of uniform probing



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$$h_2(s) = (s \mod 5) + 1$$

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Table: Comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on  $h_1(s) \neq h_2(s)$



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### Double hashing by Brent:



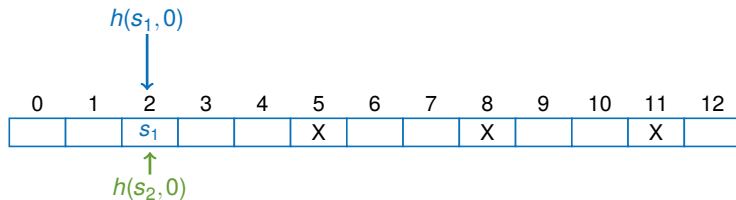


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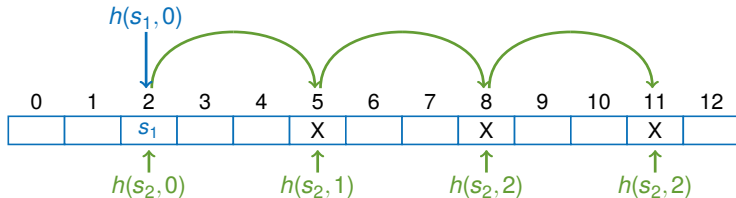


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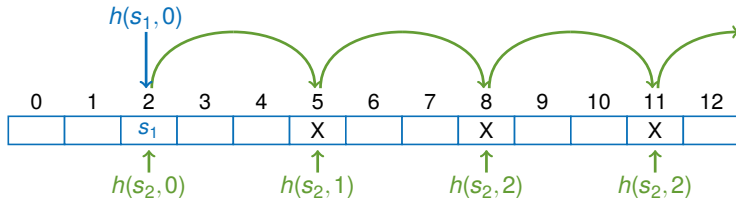


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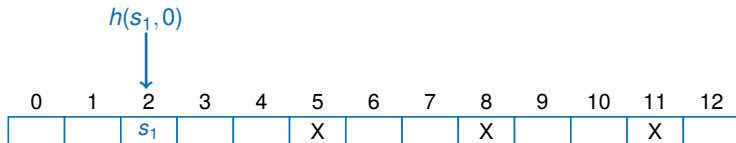


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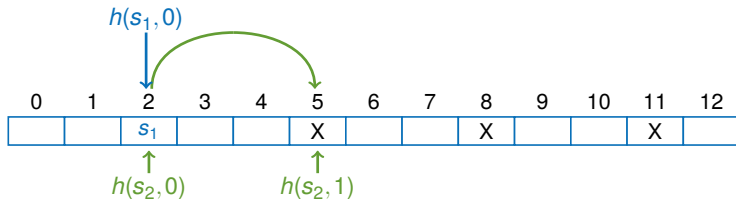


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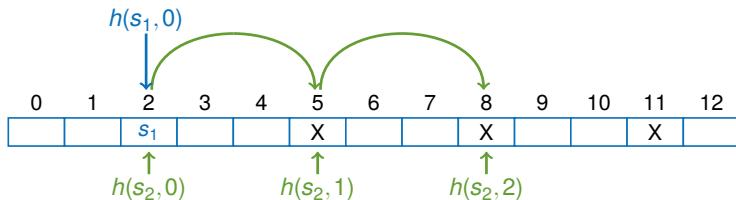


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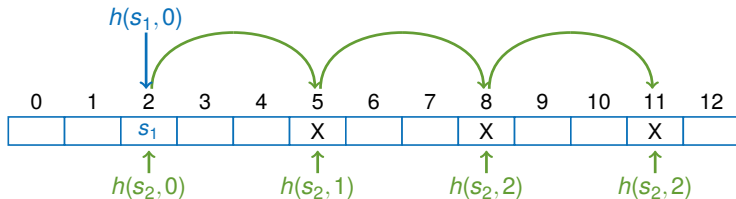


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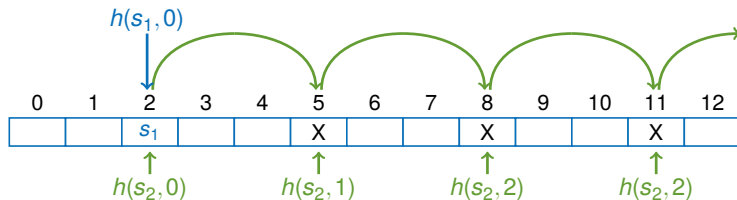


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- The locations  $h(s_2, j)$ ,  $j \in \{1, \dots, n\}$  are also occupied
- If we insert  $s_2$  at position  $h(s_2, n+1)$  the search will be inefficient



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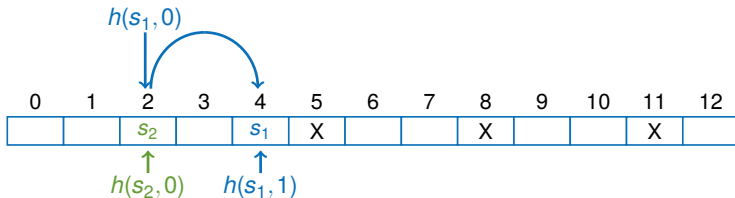


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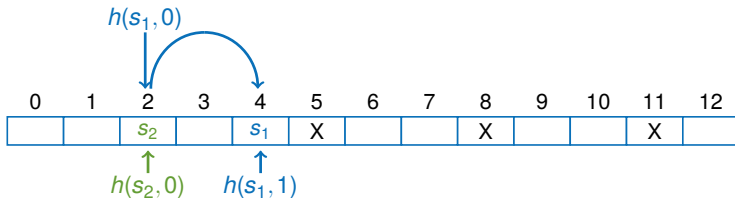


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  - Test if location  $h(s_1, 1)$  is free

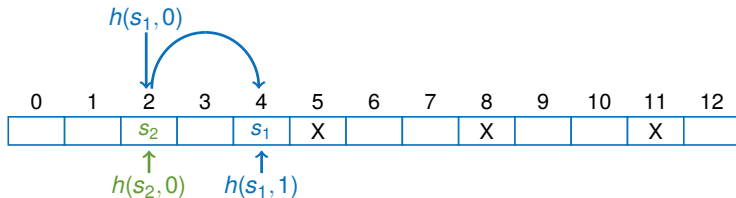


Figure: Double hashing by Brent

- Reversed sequence of keys would have been better
- **Brents Idea:**
  - Test if location  $h(s_1, 1)$  is free
  - If yes, move  $s_1$  from  $h(s_1, 0)$  to  $h(s_1, 1)$  and insert  $s_2$  at  $h(s_2, 0)$

### Idea:

- Motivation: Colliding elements are inserted in the hashtable sorted.
- Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is earlier possible because single probing steps have a fixed length

### Implementation:

- Compare both keys if a collision occurs
- Insert the smaller key at  $p_1$
- Search a position based on the diversion order for the bigger key

### Example:

- The key 12 is saved at position  $p_1 = h(12, 0)$
- We insert the key 5 into the hash map
- We assume  $h(5, 0)$  results in location  $p_1$
- Because  $5 < 12$  we insert the key 5 at position  $p_1$
- For the key 12 we iterate through the sequence

$h(12, 1), h(12, 2), h(12, 3), \dots$



### Motivation:

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Total costs stay the same, but they are distributed evenly.  
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- The key with the bigger search sequence is inserted at  $p_1$   
The other key is assigned a new location based on the sequence

### Example:

- The key 12 is saved at position  $p_1 = h(12, 7)$
- We insert the key 5 into the hash map
- We assume  $h(5, 0)$  results in location  $p_1$
- Because  $j_1 < j_2$  ( $0 < 7$ ) the key 12 stays at position  $p_1$
- For the key 5 we iterate through the sequence

$$h(5, 1), h(5, 2), h(5, 3), \dots$$

### Problem:

- The key  $s_1$  is inserted at position  $p_1$
- The key  $s_2$  returns the same hash value, but is inserted at position  $p_2$  because of the probing order
- If  $s_1$  is removed, it is impossible to find  $s_2$

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### Solution:

- **Remove:** Elements are marked as removed, but not deleted
- **Inserting:** Elements marked as removed will be overwritten

## Hashing

Recapitulation

Treatment of hash collisions

Open Addressing

**Summary**

## Priority Queue

Introduction

**Bucket as linked list:** (dynamic, number of elements variable)

- Save colliding elements as linked list

**Open hashing:** (static, number of elements fixed)

- Determine a probe sequence, permutation of all hash values
- Linear, quadratic probing:
  - Easy to implement
  - Raise the probability of collisions because probing order does not depend on the key



**Open hashing:** (static, number of elements fixed)

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### **Improving efficiency:** (Brent, Ordered Hashing)

- Improve search efficiency by sorting colliding insertions
  - Abortion of unsuccessful search
  - Search sequence length balancing



### Hashing:

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- Efficient for dictionary operations:

Insert:  $O(1) \dots O(n)$

Search:  $O(1) \dots O(n)$

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- Direct access of all elements in a hash table
- Using a hash function to find the position (hash value) in the hash table
- Hash function, size of the hash table and strategy to avoid hash collisions influence the efficiency of the datastructure



## Hashing

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- Argument of `changeKey` and `remove` operations
  - There is no **quick-access** to a element in the queue
  - That's why `insert` and `getMin` return a reference (handle, accessor object)
  - `changeKey` and `remove` take this reference as argument
  - Therefore each element has to store its current position in the heap.

```
from queue import PriorityQueue

q = PriorityQueue()

e1 = (5, "A") # element with priority 5
q.put(e1); # insert element e1

# remove and return the lowest item
e2 = q.get()
```

### Example 1:

- Calculation of the sorted union of  $k$  sorted lists  
(multi-way merge or  $k$ -way merge)



Figure: 3-way merge



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### Example 2:

- For example Dijkstra's algorithm for computing the shortest path ( $\leftarrow$  following lecture)
- Among other applications it can be used for sorting

# Priority Queue

## Implementation



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**Idea:**

### Idea:

- Save elements as tuples in a binary heap

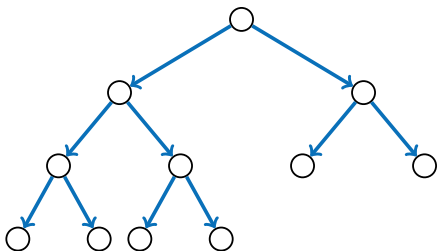


Figure: Heap with 11 nodes

### Idea:

- Save elements as tuples in a binary heap
- Summary from lecture 1 (*HeapSort*):
  - Nearly complete binary tree
  - **Heap condition:**  
The key of each node  $\leq$  the keys of the children



Figure: Heap with 11 nodes

# Priority Queue

## Implementation

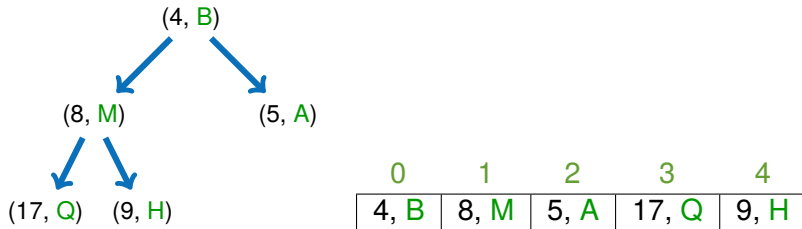


Figure: Min heap stored in array

# Priority Queue

## Implementation

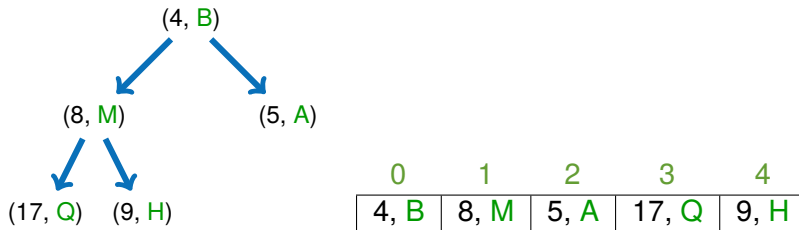


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## Storing a binary heap:



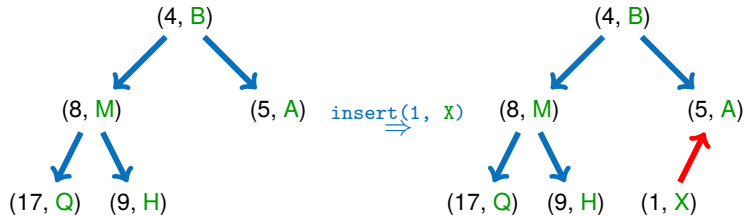


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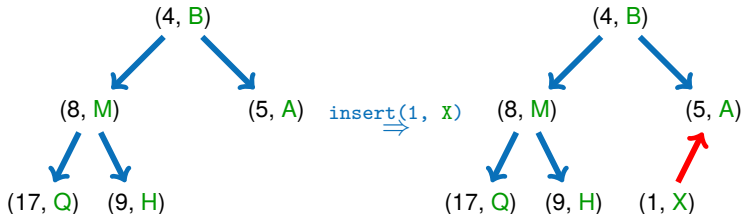
### Storing a binary heap:

- Number nodes from top to bottom and left to right starting with 0 and store entries in array
- Children of node  $i$  are the nodes  $2i+1$  and  $2i+2$
- Parent node of node  $i$  is  $\text{floor}((i-1)/2)$

**Inserting an element:** `insert(key, item)`

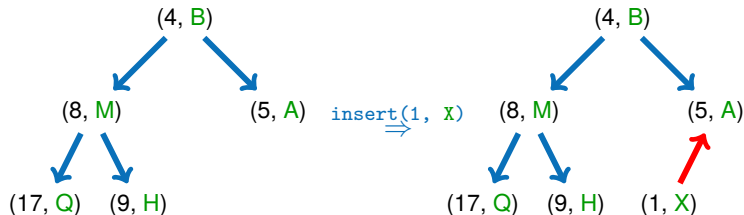


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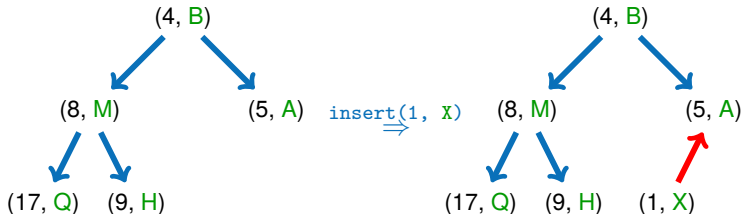
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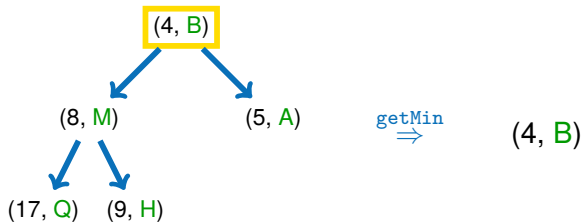
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**Inserting an element:** `insert(key, item)`



- Append the element at the end of the array
- The **heap condition** may be violated, but only at the last index
- Repair **heap condition**  $\Rightarrow$  We will see later how to do this

Returning the minimum: `getMin()`



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- Else return the first element

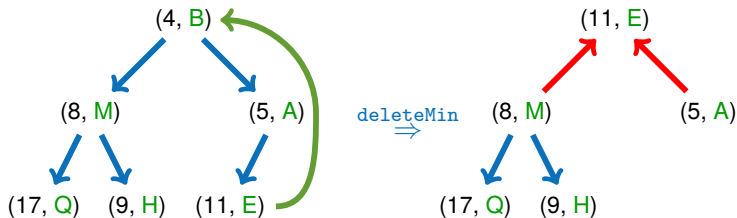
Returning the minimum: `getMin()`



- Else return the first element
- If the heap is empty return `None`



### Removing the minimum: `deleteMin()`



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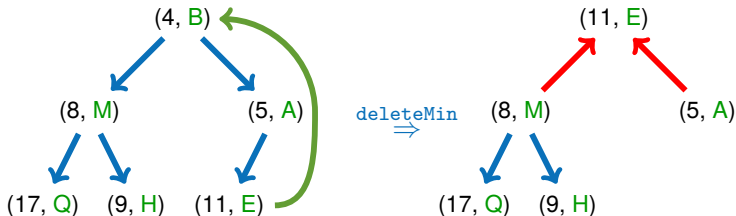
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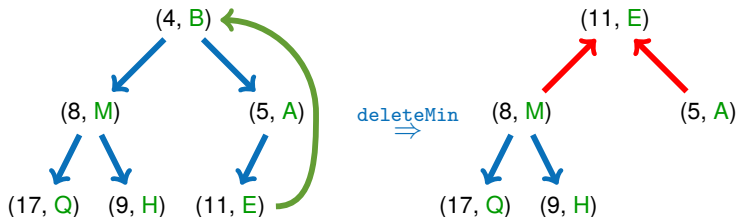
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### Changing the key (priority): `changeKey(item, key)`

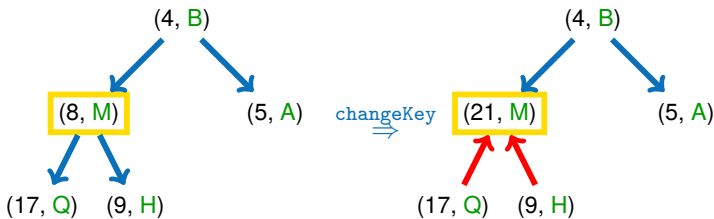
- The element (queue item) is given as argument
- Replace the key of the element
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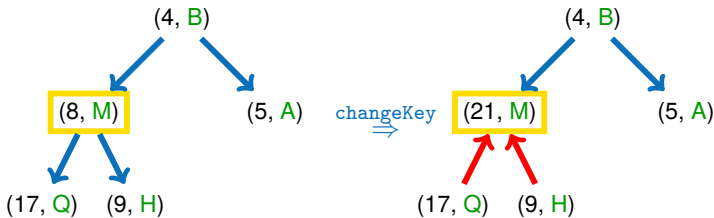
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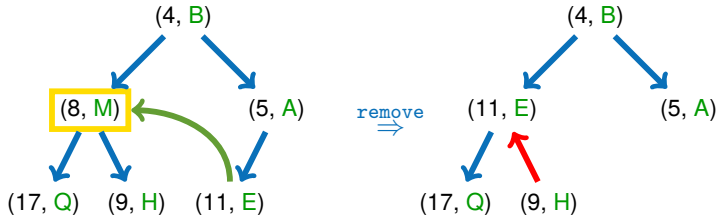


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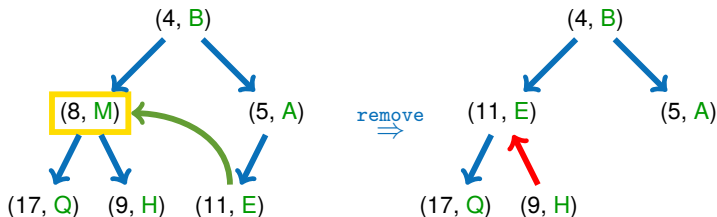
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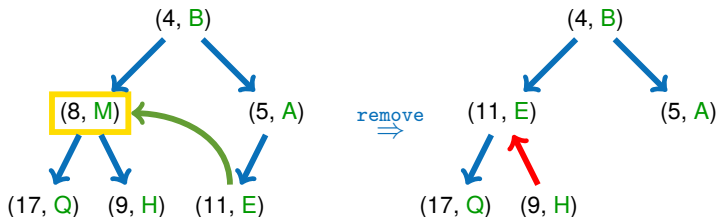
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### **Repairing after modifying operations:**

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  - Downwards: The key at index  $i$  is not  $\leq$  than the value of its children
  - Upwards: The key at index  $i$  is not  $\geq$  than the value of its parent
- We need two repair methods: `repairHeapUp`, `repairHeapDown`

`repairHeapDown:`

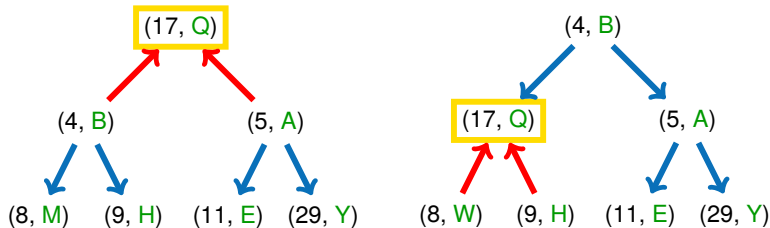


Figure: Repairing the heap downwards

`repairHeapDown:`

- Sift the element until the **heap condition** is valid

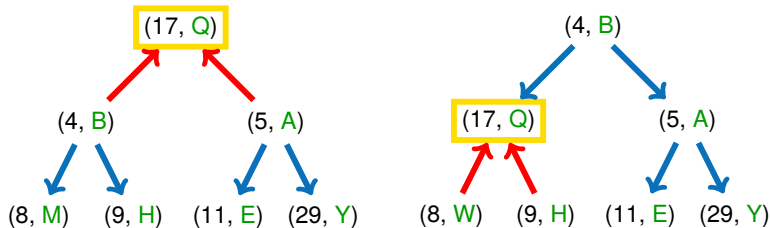


Figure: Repairing the heap downwards

### repairHeapDown:

- Sift the element until the **heap condition** is valid
- Change node with child, which has the lower key of both children

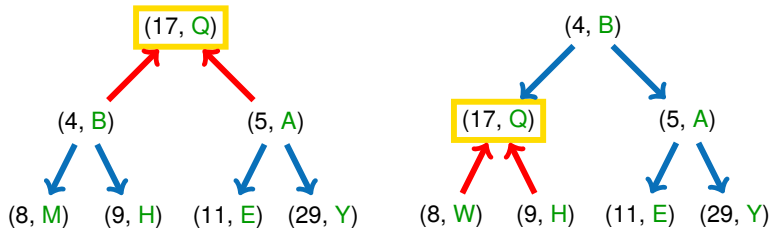


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### repairHeapDown:

- Sift the element until the **heap condition** is valid
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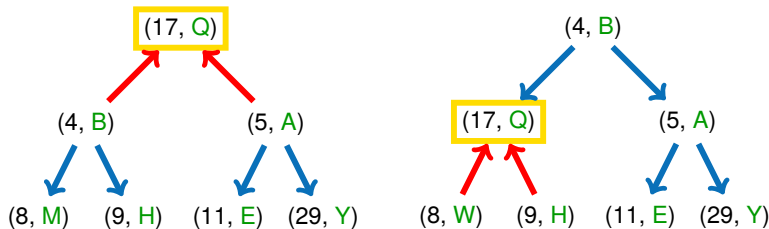


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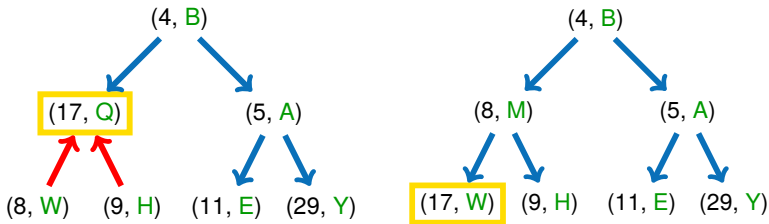


Figure: Repairing the heap downwards

`repairHeapUp:`



Figure: Repairing the heap upwards

`repairHeapUp:`

- Change node with parent

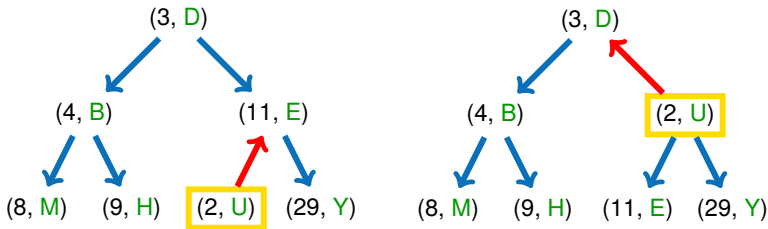


Figure: Repairing the heap upwards

### repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node

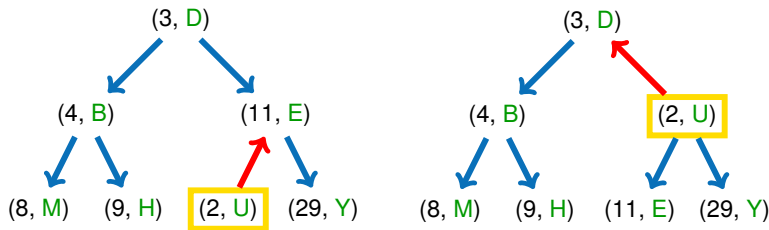


Figure: Repairing the heap upwards

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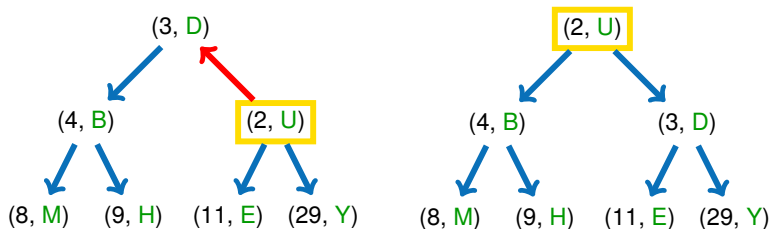


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**Index of a priority queue item:**

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- **Attention:** For `changeKey` and `remove` the item has to “know” where it is located in the heap
- Remember for `repairHeapUp` and `repairHeapDown`:  
Update the index if moving an heap element



```
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```



## Summary lecture 1:

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- **getMin**: Return the element at index 0:  $O(1)$





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- Example:
  - For  $n = 2^{10} \approx 1,000$  is the the `depth`  $\log_2 n$  only 10
  - For  $n = 2^{20} \approx 1,000,000$  is the `depth`  $\log_2 n$  only 20

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.



## ■ Priority Queue - Implementations / API

[Cpp] [C++ - priority\\_queue](#)

`http:`

`//www.sgi.com/tech/stl/priority_queue.html`

[Jav] [Java - PriorityQueue](#)

`https://docs.oracle.com/javase/7/docs/api/  
java/util/PriorityQueue.html`

[Pyt] [Python - PriorityQueue](#)

`https://docs.python.org/3/library/queue.  
html#queue.PriorityQueue`