

Algorithms and Datastructures

Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg



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Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, March 2018

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Divide and Conquer

Introduction



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Concept:



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- **Recursive** application of the algorithm on smaller subproblems

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- **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- **Connect** all subsolutions to solve the overall problem
- **Recursive** application of the algorithm on smaller subproblems
- **Direct** solving of small subproblems

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

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Master theorem

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Divide and Conquer

Maximum Subtotal



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Input:

Output:

Divide and Conquer

Maximum Subtotal

Input:

- Sequence X of n integers

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Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

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Output:

- Maximum sum of an uninterrupted subsequence of X and its index boundary

Table: Input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Idea:



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- Combine both solutions into a overall solution
- The maximum is located in the **left half (A)** or the **right half (B)**
- The maximum interval can **overlap with the border (C)**

Principle:

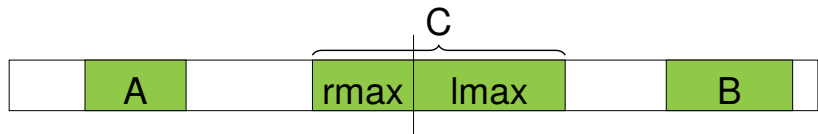


Principle:



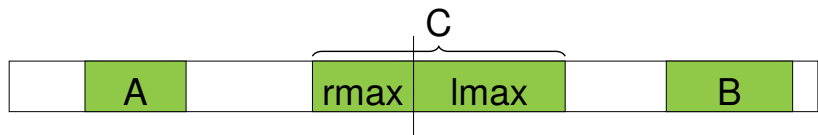
- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$

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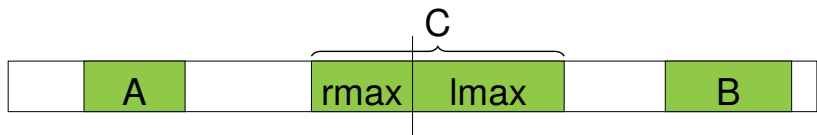
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Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions *A* and *B* are returned
- To solve *C* we have to calculate *rmax* and *lmax*
- Overall solution is maximum of *A* *B* and *C*

Divide and Conquer

Maximum Subtotal - Python



```
def maxSubArray(X, i, j):
```


Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2
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    A = maxSubArray(X, i, m)  
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    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])
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def maxSubArray(X, i, j):  
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    # recursive subsolutions for A, B  
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    # rmax and lmax for cornercase C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

Divide and Conquer

Maximum Subtotal - Python



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#Alternative trivial case  
def maxSubArray(X, i, j):
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def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

Divide and Conquer

Maximum Subtotal - Python



```
#Implementation max  
def max(a, b, c):
```


Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
```

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#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

Divide and Conquer

Maximum Subtotal - Python



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```
def max(a, b):  
    if a > b:  
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#Alternative implementation max

```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b  
  
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
<i>sum</i>	58	5	31	90	49	80
<i>lmax</i>	58	58	58	90	90	90

Table: *lmax* example

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- The *sum* and *lmax* are initialized with $X[i]$

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X	58	-53	26	59	-41	31
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<i>lmax</i>	58	58	58	90	90	90

- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update *sum*

Table: $lmax$ example

index	i	$i + 1$	\dots	\dots	$j - 1$	j
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
$lmax$	58	58	58	90	90	90

- The sum and $lmax$ are initialized with $X[i]$
- We iterate over X from $i + 1$ to j and update sum
- If $sum > lmax$ then $lmax$ gets updated

Divide and Conquer

Maximum Subtotal



Call with array of four elements

`maxSubArray(-3,9,-4,7)`

in fact:

`maxSubArray(X,0,3)`
with `X=[-3,9,-4,7]`

Divide and Conquer

Maximum Subtotal



Call with array of four elements

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in fact:

$\text{maxSubArray}(X, 0, 3)$
with $X = [-3, 9, -4, 7]$

Call with array of two elements

$\text{maxSubArray}(-3, 9)$

$\text{maxSubArray}(-4, 7)$

Divide and Conquer

Maximum Subtotal



Call with array of four elements

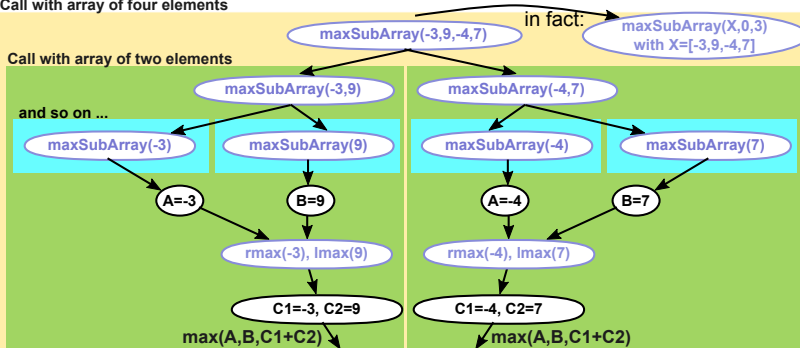


Divide and Conquer

Maximum Subtotal



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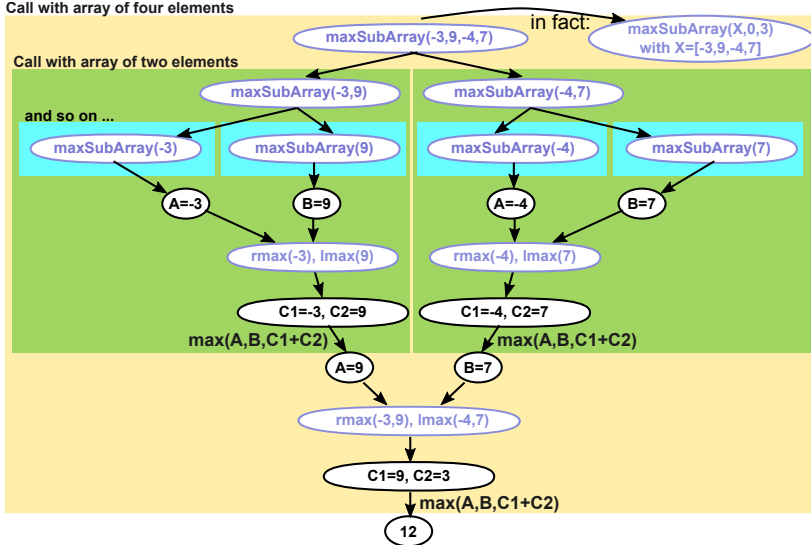
Divide and Conquer

Maximum Subtotal



Call with array of four elements

Call with array of two elements




```
def maxSubArray(X, i, j):  
    if i == j:  
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    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
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def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
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    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
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Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

trivial case

Recursion equation:

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$\underbrace{\Theta(1)}_{\text{trivial case}}$

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

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- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

Divide and Conquer

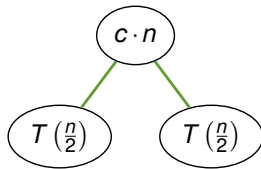
Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

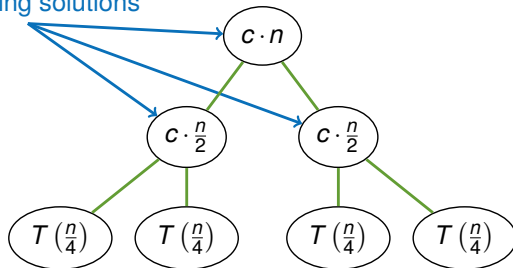
Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



combining solutions



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

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Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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Divide and Conquer

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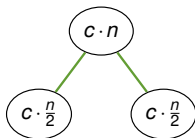

$$c \cdot n$$

1 node processing n elements
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

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Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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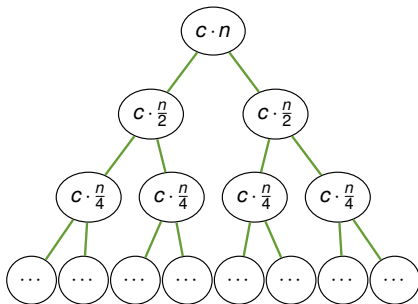
2 nodes processing $\frac{n}{2}$ elements
 $\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

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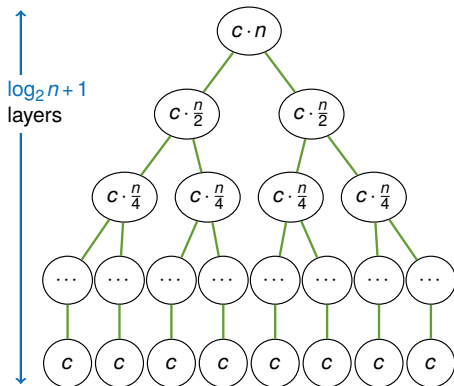
4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^j nodes processing $\frac{n}{2^j}$ elements
 $\Rightarrow 2^j c \cdot \frac{n}{2^j} = c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements
 $\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
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4 nodes processing $\frac{n}{4}$ elements
 $\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^i nodes processing $\frac{n}{2^i}$ elements
 $\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$

n nodes processing 1 element
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



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Depth:

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Depth:

- Top level with depth $i = 0$

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$
The costs of merging the solutions and solving of the trivial problems are the same here

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$

The costs of merging the solutions and solving of the trivial problems are the same here

$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Divide and Conquer

Maximum Subtotal - Summary



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- There is an approach running in $\mathcal{O}(n)$ if you assume that all subtotals are positive

Divide and Conquer

Maximum Subtotal

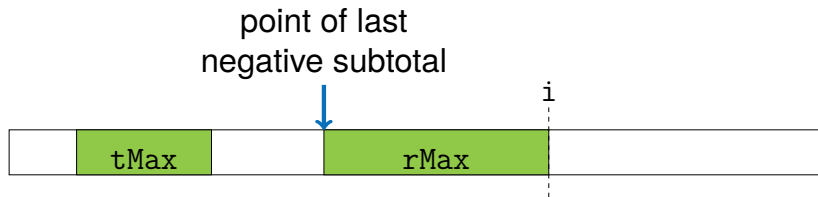


Figure: Scanning the array in linear time

Divide and Conquer

Maximum Subtotal - Python



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#Implementation - linear runtime  
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Divide and Conquer

Maximum Subtotal - Python



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    for i in range(len(X)):  
        if rMax == 0:  
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        if rMax > tMax:  
            tMax, itMax = rMax, irMax  
  
    return (tMax, itMax)
```


Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$



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- Normally $a > 1$ and $b > 1$
- Dependent on the strategy of solving $T(n)$ f_0 is ignored
- $T(n)$ is only defined for integers of $\frac{n}{b}$ which is often ignored in benefit of a simpler solution

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- Guess the solution and prove it with induction

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$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

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$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

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Figure: Recursion tree of example

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Figure: Recursion tree of example

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$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

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Figure: Levels of the recursion tree



Costs of connecting the partial solutions:
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$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4} \right)^i \cdot n \right)^2 = \left(\frac{3}{16} \right)^i \cdot c \cdot n^2$$

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- Transforming $3^{\log_4 n}$ uses general log rules

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uses $x^{a \cdot b} = (x^a)^b$

- This term will recur in the master theorem

Recursion Equations

Total costs



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$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left(\begin{array}{c} \text{even with} \\ \text{infinite elements} \end{array} \right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

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- Here: The costs of connecting the partial problems dominate

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Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

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- For $|q| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \quad \Rightarrow \text{constant}$$

Recursion Equations

Proof of $O(n^2)$



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Proof of $\mathcal{O}(n^2)$:

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Proof of $\mathcal{O}(n^2)$:

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- Assumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

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Recursion Equations

Proof of $O(n^2)$



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$$T(n) < k \cdot n^2$$

- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16}k \cdot n^2 + c \cdot n^2 \\ &\leq k \cdot n^2 \qquad \text{for } k \geq \frac{16}{13}c \end{aligned}$$

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)



Master theorem:

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- Approach to solve for a recursion equation of the form:

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 - ... which takes **$f(n)$** steps to merge all partial solutions



Master theorem:



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- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

Recursion Equations

Master theorem (Simple Form)



Simple form:

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$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n)}, \quad a \geq 1, b > 1, c > 0$$

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Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\substack{\text{Is any } f(n) \\ \text{in general form}}}, \quad a \geq 1, b > 1, c > 0$$

- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 3, b = 2$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

Recursion Equations

Master theorem (Simple Form)

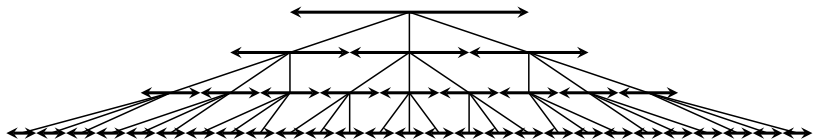


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Case 1: $a > b$

- Three partial problems with $\frac{1}{2}$ the size

Recursion Equations

Master theorem (Simple Form)

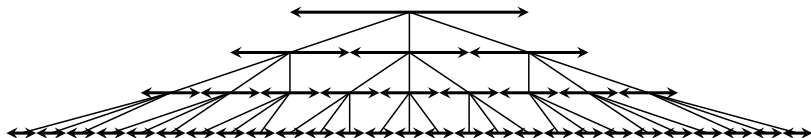


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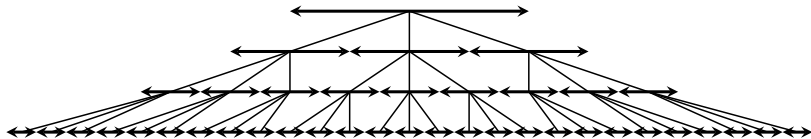


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Recursion Equations

Master theorem (Simple Form)

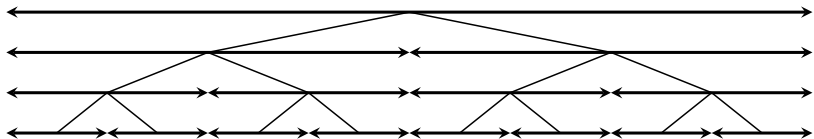


Figure: Simple recursion equation with $a = 2, b = 2$

Recursion Equations

Master theorem (Simple Form)

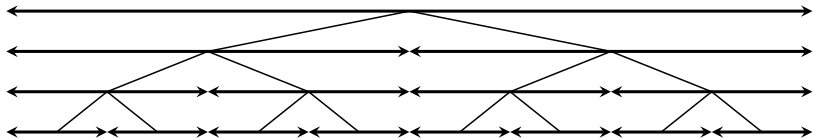


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Master theorem (Simple Form)



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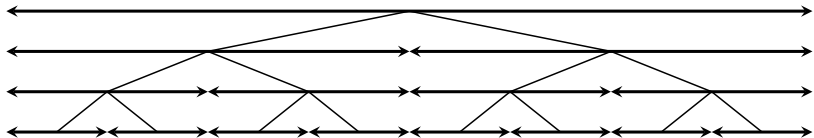


Figure: Simple recursion equation with $a = 2, b = 2$

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Recursion Equations

Master theorem (Simple Form)



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- Runtime of $\Theta(n \log n)$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

Recursion Equations

Master theorem (Simple Form)

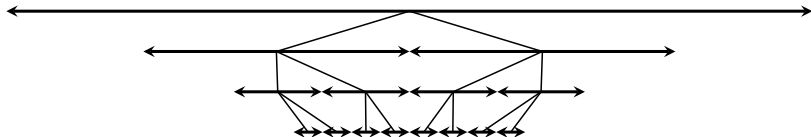


Figure: Simple recursion equation with $a = 2, b = 3$

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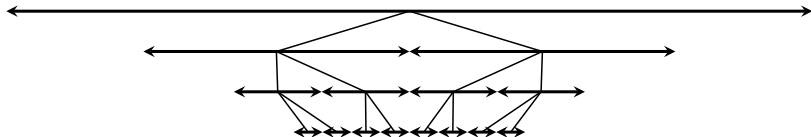


Figure: Simple recursion equation with $a = 2, b = 3$

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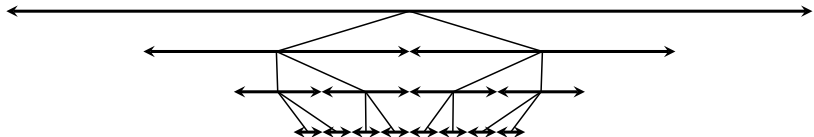


Figure: Simple recursion equation with $a = 2, b = 3$

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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

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Master theorem (Simple Form)

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- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

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- **Case 2:** $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log_b n$ layers

Master theorem (general form):

- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$
Connecting all partial solutions in first layer (root)
dominates

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$



Case 1 - Example:

if $f(n) \in O(n^{\log_b a - \epsilon})$, $\epsilon > 0$

Solving the partial problems dominates (last layer, leaves)



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$$\blacksquare T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

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■ $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, b = 3, f(n) = 17 \cdot n, \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$



Case 2: if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (General Form) - Case 2

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$$a = 2, b = 2, f(n) = 10 \cdot n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

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■ $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1, b = \frac{2}{3}, f(n) = 1, \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Case 3: if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$
Connecting all partial solutions in first layer (root) dominates

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

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$$f(n) \in \Omega(n^{1+\varepsilon})$$



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Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if **regularity condition** also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$

Master theorem:



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$n \log n$ is *asymptotically* larger than n ,
but not *polynomial* larger



Master theorem:



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- **Case 2:** Each layer has equal costs

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- **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems

$$T(n) \in \Theta(f(n))$$

■ General

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■ Master theorem

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https://en.wikipedia.org/wiki/Master_theorem