

Algorithms and Datastructures

Levenshtein distance, Dynamic programming

Albert-Ludwigs-Universität Freiburg



**UNI
FREIBURG**

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Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, February 2018

Introduction

Edit distance

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Edit distance:

Edit distance:

- Measurement for similarity of two words / strings

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- Algorithm for efficient calculation

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- Measurement for similarity of two words / strings
- Algorithm for efficient calculation
- General principle: dynamic programming

BioInfSearch

| |
|--------------------------|
| ejafjatljökuk |
| eyjafjallajökull |
| eyjafjallajökull movie |
| eyjafjallajälull trailer |

Search!



Ulrich Latzenhofer; CC BY-SA 2.0

Wikipedia.org:

"Der Eyjafjallajökull ([ˈeɪjaˌfjatlaˌjœːkʏtʃ])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárpíng eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."



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- Duplicates in databases:

| | | |
|------------|-------|-------------|
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| Hein Bloed | 27568 | Bremerhafen |
| Hein Doof | 27478 | Cuxhaven |

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uniwersität verien 2017

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- Bioinformatics: Similarity of DNA-sequences

Introduction

Example: Bioinformatics DNA-matching

Search of similar proteins:

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- Cited 63437 times on Google Scholar (Sep. 2017)

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Edit distance

Definition of edit distance: (*Levenshtein-distance*)

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Edit distance

Example



1 2 3 4 5
DOOF

BLOED

Edit distance

Example



1 2 3 4 5

DOOF



replace(1, B)

BOOF

BLOED

Edit distance

Example

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replace(1, B)

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replace(2, L)

BLOF

BLOED

Edit distance

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replace(2, L)

BLOF



insert(4, E)

BLOEF

BLOED

Edit distance

Example

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BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

Edit distance

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1 2 3 4 5

DOOF



replace(1, B)

BOOF



replace(2, L)

BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

ED=4

Edit distance

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1 2 3 4 5

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BOOF



BLOF



BLOEF



BLOED

replace(1, B)

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replace(5, D)

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BLOED

⏟
ED=4

Edit distance

Example

1 2 3 4 5

DOOF



BOOF



BLOF



BLOEF



BLOED

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1 2 3 4 5

B L O E D

DOOF

Edit distance

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DOOF



BOOF



BLOF



BLOEF



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⏟
ED=4

1 2 3 4 5

B LOED



B LOEF

replace(5, F)

DOOF

Edit distance

Example

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DOOF



BOOF



BLOF



BLOEF



BLOED

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B LOED



B LOEF



B LOF

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DOOF

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DOOF



BOOF



BLOF



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B LOF



BOOF

DOOF

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Edit distance

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DOOF



BOOF



BLOF



BLOEF



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B LOF



BOOF



DOOF

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Edit distance

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BOOF



BLOF



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$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

- $ED(x, y) \leq ED(x[1..n-1], y[1..m-1]) + 1 \quad n = |x|, m = |y|$



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Recursive approach:

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Recursive approach:

- Dividing in two halves? Not a good idea:

$$ED(\textit{GRAU}, \textit{RAUM}) = 2 \quad \text{but} \quad ED(\textit{GR}, \textit{RA}) + ED(\textit{AU}, \textit{UM}) = 4$$

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- Finding “smaller” sub problems?
Let's try it!



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- Let $\sigma_1, \dots, \sigma_k$ be a sequence of k operations where $k = \text{ED}(x, y)$ for $x \rightarrow y$ (transform x into y)
(We do not know this sequence but we assume it exists)



Terminology:

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- We only consider **monotonous** sequences:
The position of σ_{i+1} is \geq the position of σ_i where we only allow the positions to be equal on a delete operation

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1 2 3 4 5

DOOF



replace(1, B)

BOOF



replace(2, L)

BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

1 2 3 4 5 6 7

SAUDOOF



delete(1)

AUDOOF



delete(1)

UDOOF



delete(1)

DOOF



insert(4, O)

DOOOF

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The position of σ_{i+1} is \geq the position of σ_i where we only allow the positions to be equal on a delete operation

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- **Intuition:** The order of our sequence is not relevant (Therefore we can also sort them monotonously)

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| D | O | O | F | |

B L O E D

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| S | A | U | D | O | O | F |

D O O O F



Consider the last operation:

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- Solve **blue** part recursively

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DOOF

↓↓↓↓

BLOE

↓ insert

BLOED

Figure: Case 1a

DOOF

↓↓↓↓↓

BLOEDF

↓ delete

BLOED

Figure: Case 1b

DOOF

↓↓↓↓↓

BLOEF

↓ replace

BLOED

Figure: Case 1c



Consider the last operation:

Consider the last operation:

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Consider the last operation:

- Solve **blue** part recursively

W I N T E R



S O M M E R

↓ nothing

S O M M E R

Display of solution:

- Alignment

- Example:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| — | — | — | B | L | O | E | D |
| S | A | U | B | L | O | E | D |

Figure: Case 2



Dynamic programming:

Dynamic programming:

- Instances of Bellman's principle of optimality:

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Figure: Richard Bellman
(1920 - 1984)

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- Optimal solutions consist of optimal partial solutions
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 - Edit distance: Each partial alignment has to be optimal

Dynamic programming:

- Instances of Bellman's principle of optimality:
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 - Edit distance
- Optimal solutions consist of optimal partial solutions
 - Shortest paths: Each partial path has to be optimal
 - Edit distance: Each partial alignment has to be optimal
- Always solvable through dynamic programming
(Caching of optimal partial solutions)



Figure: Richard Bellman
(1920 - 1984)



Case analysis:

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 - $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$ and $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

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- Let $n = |x|$, $m = |y|$, $m' = |z|$

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Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- Let $n = |x|, m = |y|, m' = |z|$
- We note $m' \in \{m-1, m, m+1\}$ why?



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 - Case 1c: $\sigma_k = \text{replace}(m', y[m])$ [then $m' = m$]
- Case 2: σ_k does nothing at the outer end:
 - Then $z[m'] = y[m]$ and $x[n'] = z[m']$ and with that
 $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$ and $x[n] = y[m]$



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 - $ED(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$

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 - $ED(x, y[1..m-1]) + 1$ and
 - $ED(x[1..n-1], y) + 1$ and
 - $ED(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
 - $ED(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $ED(x, y) = |y|$

Case analysis:

- Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- Case 1b (delete): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
- Case 1c (replace): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (nothing): $\sigma_1, \dots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

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```
def edit_distance(x, y):  
    if len(x) == 0:  
        return len(y)  
    if len(y) == 0:  
        return len(x)  
  
    ed1 = edit_distance(x, y[:-1]) + 1  
    ed2 = edit_distance(x[:-1], y) + 1  
    ed3 = edit_distance(x[:-1], y[:-1])  
    if x[-1] != y[-1]:  
        ed3 += 1  
  
    return min(ed1, ed2, ed3)
```



Recursive program:

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- The algorithm results in the following recursive formular:

$$\begin{aligned}T(n, m) &= T(n-1, m) + T(n, m-1) + T(n-1, m-1) + 1 \\&\geq T(n-1, m-1) + T(n-1, m-1) + T(n-1, m-1) \\&= 3 \cdot T(n-1, m-1)\end{aligned}$$

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⇒ The runtime is at least exponential



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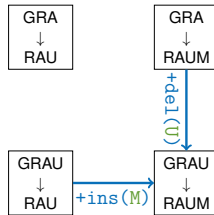
Visualization on the next slide:

- Operations always refer to the last position (indices are omitted)
- We also display the replaced character on a `replace` operation to visualize operations without costs
 $\Rightarrow \text{repl}(\text{A}, \text{A})$















Fast algorithm:

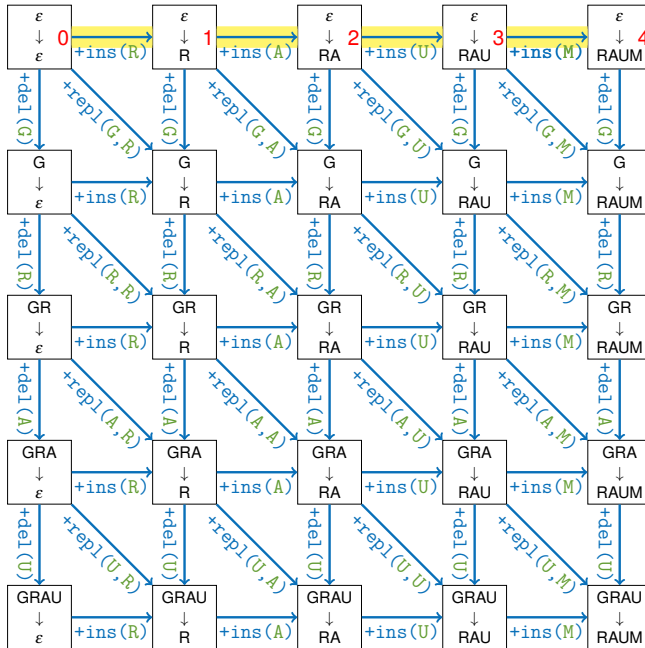
We can determine the **edit distance** for all combination of partial strings from the top left to bottom right.









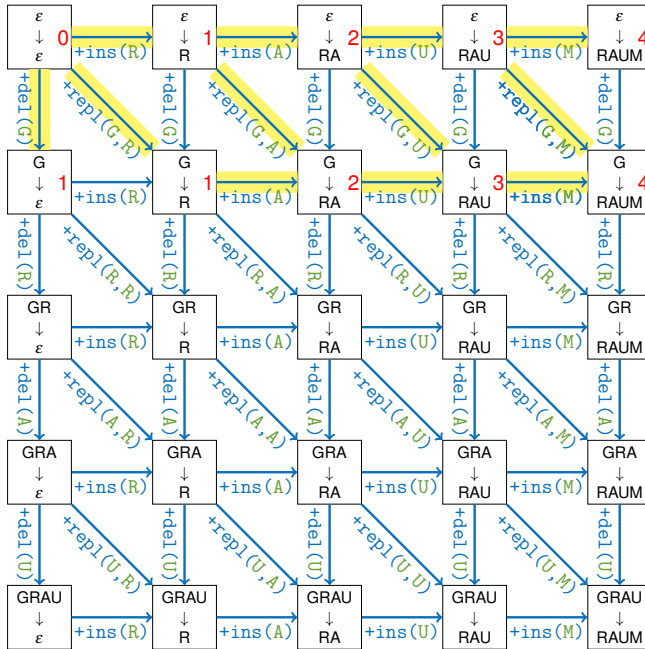


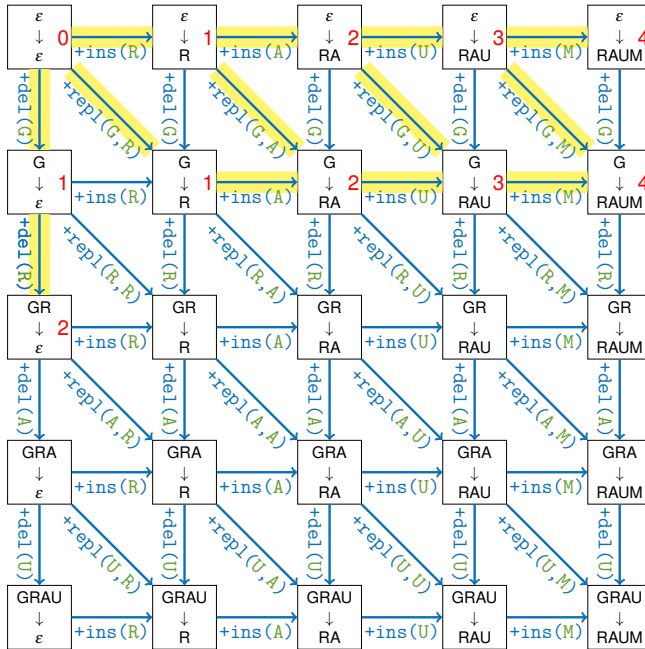










































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 - If we can follow **more than one path** there exist more than one ideal **sequence**



General principle:

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- Recursive computation of ...
 - ... the same reoccurring partial problems
 - ... a limited number of partial problems

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- Recursive computation of ...
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- Computation of the solutions for all partial problems
- In a order that unsolved partial problems consist of already solved partial problems
- The “path” to our solution normally gets computed while searching the best solution
- Dijkstra algorithm is basically dynamic programming!



Additional applications:



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| | | | | | | | |
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- Solution in $O(n^3)$ time or $O(n^2)$ affine

$O(n^2)$ space consumption might be problematic:

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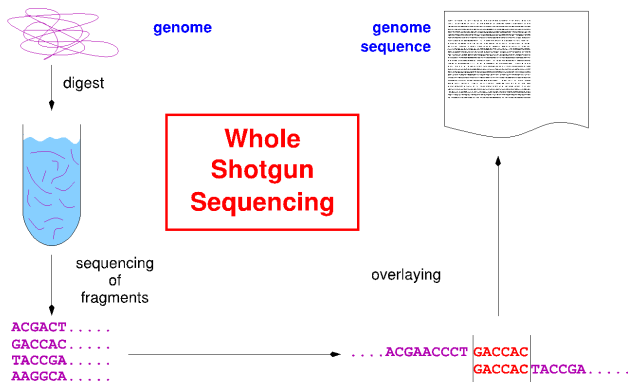
Edit distance

Additional applications (III)





- Sequencing: $O(n^2)$ is too much



- Sequencing: $O(n^2)$ is too much
- Index: suffixtree, suffixarray, burrow-wheeler-transform

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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[MS08] Kurt Mehlhorn and Peter Sanders.

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■ **Dynamic programming**

[Wik] [Dynamic programming](https://en.wikipedia.org/wiki/Dynamic_programming)

`https:`

`//en.wikipedia.org/wiki/Dynamic_programming`

■ **Edit distance**

[Wik] [Levenshtein distance](https://en.wikipedia.org/wiki/Levenshtein_distance)

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