

# Algorithms and Datastructures

## Runtime analysis Minsort / Heapsort, Induction

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Algorithms and Datastructures, October 2018

## Runtime Example

Minsort

## Basic Operations

## Runtime analysis

Minsort

Heapsort

Introduction to Induction

## Logarithms

## Runtime Example

### Minsort

## Basic Operations

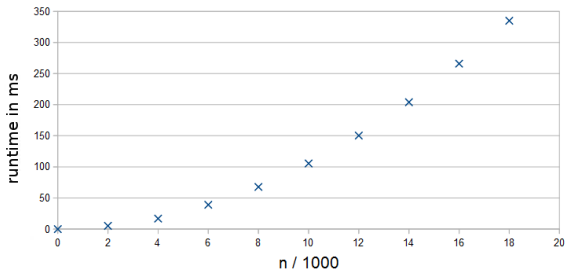
## Runtime analysis

Minsort

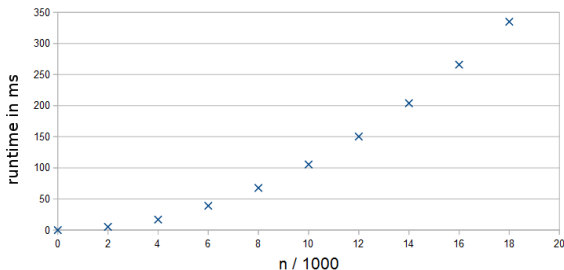
Heapsort

Introduction to Induction

## Logarithms

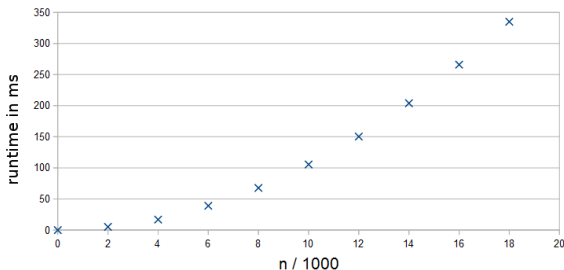


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- **Problem:** the runtime is depends on many variables, especially:
  - What kind of computer the code is executed on
  - What is running in the background
  - Which compiler is used to compile the code
- **Abstraction 1:** analyze the number of basic operations, rather than analyzing the runtime

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## Incomplete list of basic operations:

- Arithmetic operation, for example:  $a + b$
- Assignment of variables, for example:  $x = y$
- Function call, for example: *minsort(lst)*

## Intuitive:

lines of code

## Better:

lines of machine  
code

## Best:

process cycles

## Important:

The actual runtime has to be roughly proportional to the number of operations.

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- **Abstraction 2:** we calculate the upper (lower) bound, rather than exactly counting the number of operations

**Reason:** runtime is approximated by number of basic operations, but we can still infer:

- Upper bound
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**Reason:** runtime is approximated by number of basic operations, but we can still infer:

- Upper bound
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- **Basic Assumption:**
    - $n$  is size of the input data (i.e. array)
    - $T(n)$  number of operations for input  $n$

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$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

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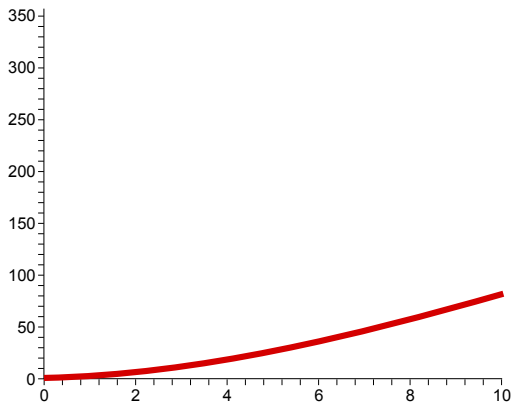
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- **Claim:** there are constants  $C_1$  and  $C_2$  such that:

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- This is called “quadratic runtime” (due to  $n^2$ )

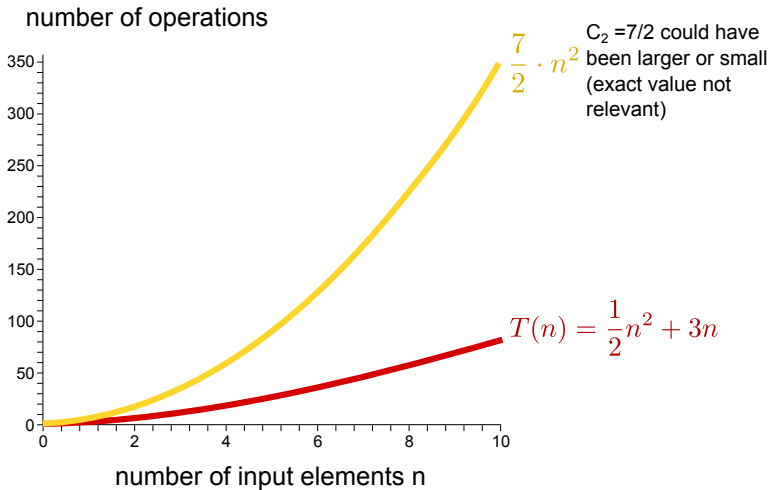
# Runtime Example

number of operations

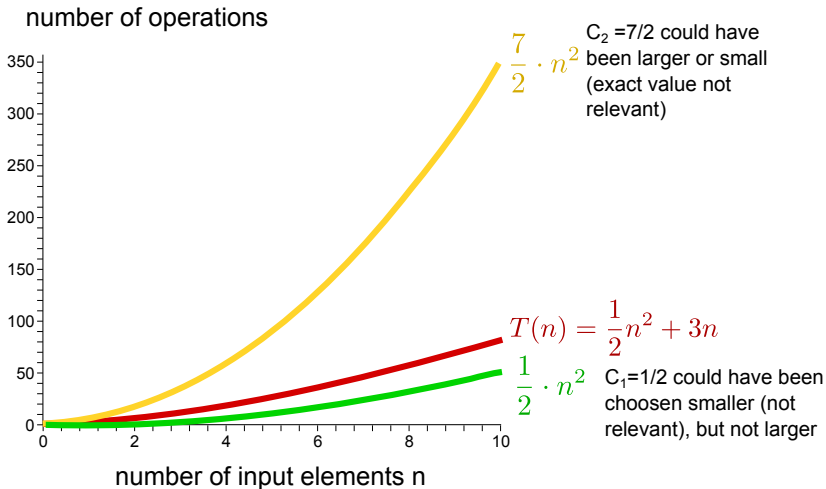


number of input elements n

# Runtime Example



# Runtime Example



## We declare:

- Runtime of operations:  $T(n)$
- Number of Elements:  $n$
- Constants:  $C_1$  (lower bound),  $C_2$  (upper bound)  
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$
- Number of operations in round  $i$ :  $T_i$



**Figure:** *Minsort* at iteration  $i = 4$ . We have to check  $n - 3$  elements

Runtime for each  
iteration:

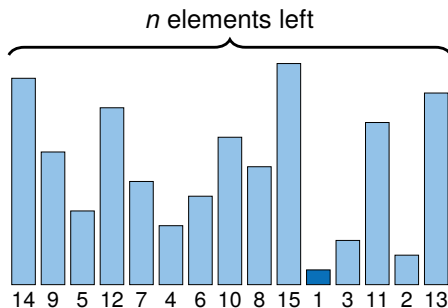


Figure: *Minsort* with start data

Runtime for each  
iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

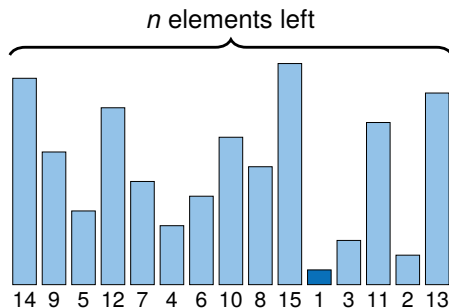


Figure: *Minsort* at iteration  $i = 1$





Runtime for each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

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Figure: *Minsort* at iteration  $i = 2$



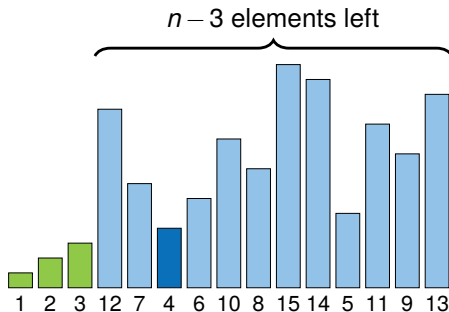
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Figure: *Minsort* at iteration  $i = 3$



Runtime for each iteration:

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Figure: Minsort at iteration  $i = 4$



Figure: *Minsort* at iteration  $i$

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$$T_{n-1} \leq C'_2 \cdot 2$$

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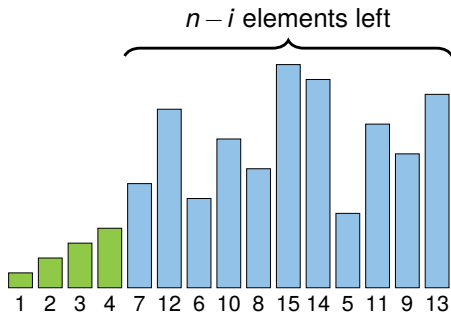


Figure: Minsort at iteration

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$$T(n) = C'_2 \cdot (T_1 + \dots + T_n) \leq \sum_{i=1}^n (C'_2 \cdot i)$$

## Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
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        runtime  
  
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$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2$$

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Diagram illustrating the runtime analysis of the Minsort algorithm:

- The inner loop (for j in range(i+1, len(elements))) is highlighted in a light blue box.
- The inner loop body (if elements[j] < elements[minimum]: minimum = j) is highlighted in a darker blue box.
- The inner loop body is annotated with "const. runtime".
- The inner loop is annotated with "n-i-1 times".
- The entire loop structure (for i in range(0, len(elements)-1)) is annotated with "n-1 times".

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2$$

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**Remark:**  $C'_2$  is cost of comparison  $\Rightarrow$  assumed constant

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# Excursion - Small Gauss Formula



**Proof of lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper bound there exists a  $C_1$ . Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i)$$

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How do we get to  $n^2$ ?

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## Summarized:

$$\frac{C'_1}{4} \cdot n^2 \leq T(n) \leq C'_2 \cdot n^2$$

## Quadratic runtime proven:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

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- **Quadratic runtime = “big” problems unsolvable**

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## Formal:

- Let  $T(n)$  be the runtime for the *Heapsort* algorithm with  $n$  elements
- On the next pages we will proof  $T(n) \leq C \cdot n \log_2 n$

## Depth of a binary tree:

- **Depth  $d$ :** longest path through the tree
- Complete binary tree has  $n = 2^d - 1$  nodes
- Example:  $d = 4$   
 $\Rightarrow n = 2^4 - 1 = 15$

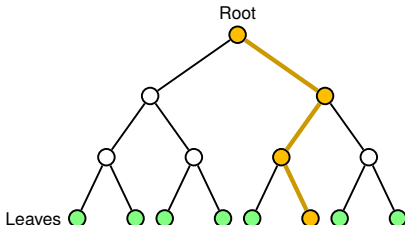


Figure: Binary tree with 15 nodes

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## Basics:

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- You want to show that assumption  $A(n)$  is valid  $\forall n \in \mathbb{N}$



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- If both has been proven, then  $A(n)$  holds for all natural numbers  $n$  by **induction**

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$\Rightarrow$  correct ✓

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# Induction - Example 1



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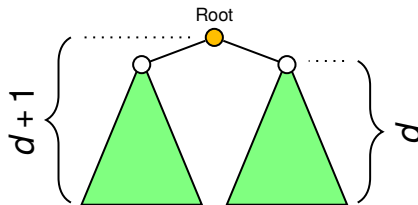
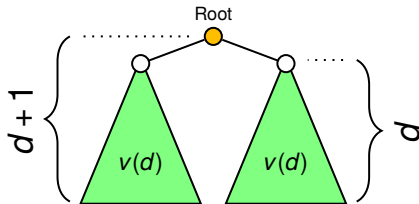


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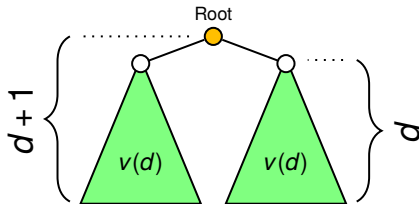


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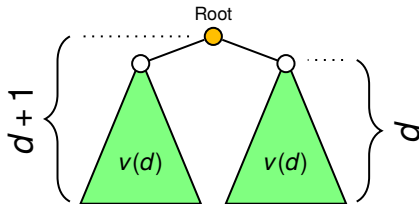


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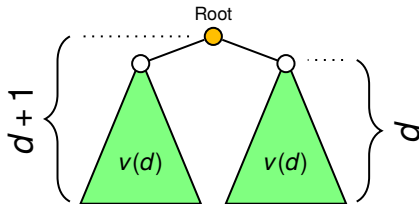
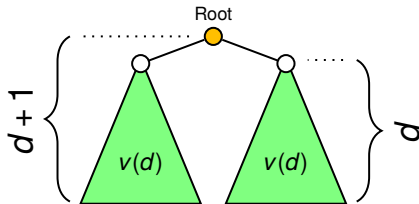


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⇒ **By induction:**  $v(d) = 2^d - 1 \quad \forall d \in \mathbb{N} \quad \square$

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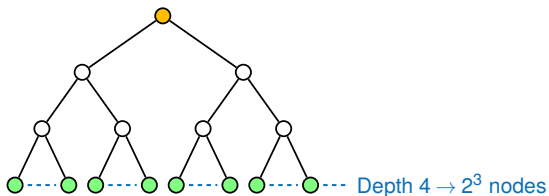
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  - Repair heap by sifting

Runtime of heapify depends on depth  $d$ :



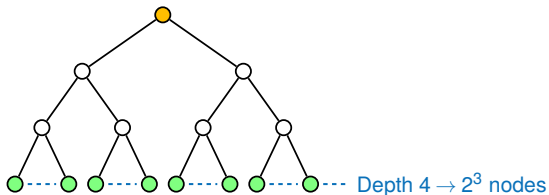
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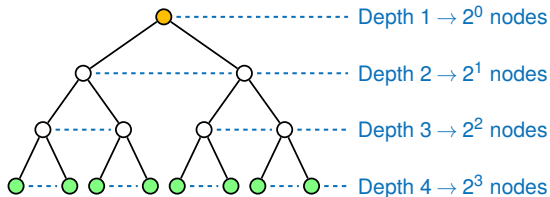


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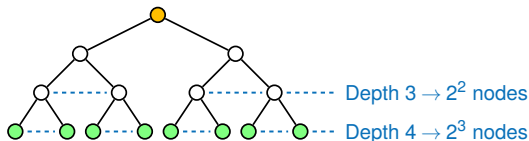
- No costs at depth  $d$  with  $2^{d-1}$  (or less) nodes
- The cost for sifting with depth 1 is at most  $1C$  per node
- In general: sifting costs are linear with path length **and** number of nodes

Heapify total runtime:



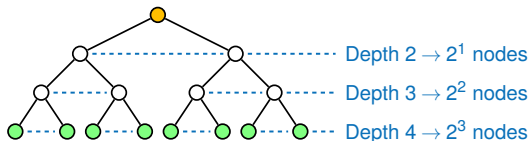
Depth	Nodes	Path length	Costs per node
$d$	$2^{d-1}$	0	$\leq C \cdot 0$

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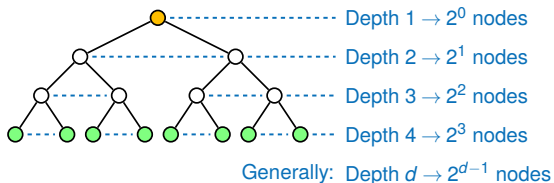
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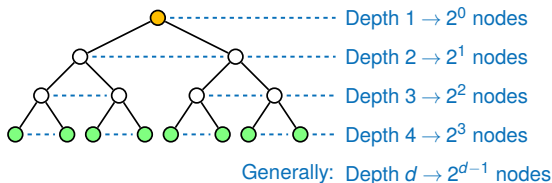
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**In total:** 
$$T(d) \leq \sum_{i=1}^d (C \cdot (i-1) \cdot 2^{d-i})$$

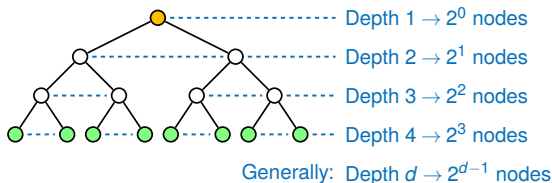
Heapify total runtime:



Depth	Nodes	Path length	Costs per node	Upper bound
$d$	$2^{d-1}$	0	$\leq C \cdot 0$	Standard Equation
$d-1$	$2^{d-2}$	1	$\leq C \cdot 1$	
$d-2$	$2^{d-3}$	2	$\leq C \cdot 2$	
$d-3$	$2^{d-4}$	3	$\leq C \cdot 3$	

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- **However:** We want costs in relation to  $n$

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Figure: Partial binary tree

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- A binary tree of depth  $d$  has  $2^{d+1} - 1$  nodes **Why?**
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- At least 1 node in layer  $d$

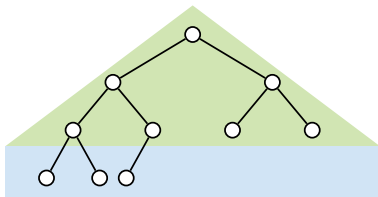


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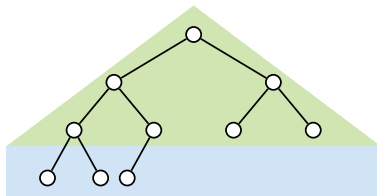


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 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$
- Cost for heapify:  
 $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

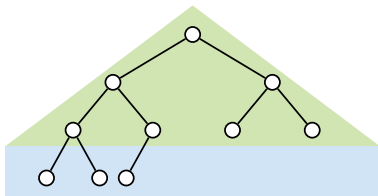


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- We want to proof (induction assumption):

$$\underbrace{\sum_{i=1}^d (i \cdot 2^{d-i})}_{A(d)} \leq \underbrace{2^{d+1}}_{B(d)}$$

- We denote the left side with  $A$ , the right side with  $B$

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$$2^0 \leq 2^2 \quad \checkmark$$

**Induction step:** ( $d := d + 1$ ):

- **Idea:** Write down right-hand formula and try to get  $A(d)$  and  $B(d)$  out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

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$$\sum_{i=1}^{d+1} (i \cdot 2^{d+1-i}) \leq 2^{d+1+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$\vdots$

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■ **Problem:** does not work but claim still holds

## Working proof:

- Show a **little bit stronger** claim

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- **Advantage:** results in a stronger induction assumption  
 $\Rightarrow$  **exercise**

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## **Runtime of the other operations:**

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- **Recall:** the depth and number of elements is decreasing
  - **Hence:**  $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
  - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \quad (\text{holds for } n > 2)$$

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- Remove:  $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- Total runtime:  $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - Upper bound:  $C_2 \cdot n \log_2 n \geq T(n)$  (for  $n \geq 2$ )
  - Lower bound:  $C_1 \cdot n \log_2 n \leq T(n)$  (for  $n \geq 2$ )

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- Heapify:  $T(n) \leq 4 \cdot n \cdot C$
- Remove:  $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- Total runtime:  $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - **Upper bound:**  $C_2 \cdot n \log_2 n \geq T(n)$  (for  $n \geq 2$ )
  - **Lower bound:**  $C_1 \cdot n \log_2 n \leq T(n)$  (for  $n \geq 2$ )
  - $\Rightarrow C_1$  and  $C_2$  are constant



Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

## Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient  $\frac{1}{\log_b a}$

### Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602 \dots \cdot 3.322 \dots = 2 \checkmark$
- $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3 \checkmark$

## Runtime of $n \log_2 n$ :

- Assume we have constants  $C_1$  and  $C_2$  with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

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- **Runtime  $n \log_2 n$  is nearly as good as linear!**



## ■ Course literature

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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MIT Press, Cambridge, Mass, 2001.

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## ■ Mathematical Induction

[Wik] [Mathematical induction](https://en.wikipedia.org/wiki/Mathematical_induction)

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