

Algorithms and Datastructures

Hash Map, Universal Hashing

Albert-Ludwigs-Universität Freiburg



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Algorithms and Datastructures, November 2017

Associative Arrays

- Introduction

- Hash Map

Universal Hashing

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- Probability Calculation

- Proof

- Examples

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- Naive solution: Store pairs of key and value in a normal field
- For n keys searching requires $\Theta(n)$ time
- With a **hash map** this just requires $\Theta(1)$ in the best case, ... regardless how many elements are in the map!

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Idea:

- Mapping the keys onto indices with a [hash function](#)
- Store the values at the calculated indices in a normal array

Example:

- Key set: $x = \{3904433, 312692, 5148949\}$

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A “hashtable” with 5 “buckets”

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- We need an array **T** with **5** elements.
A “hashtable” with 5 “buckets”
- The element with the key **x** is stored in $T[h(x)]$

Storage:

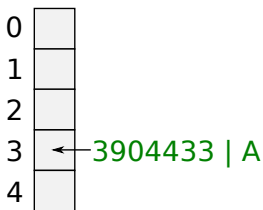
Figure: Hashtable T



Storage:

■ `insert(3904433, "A")`: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$

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- `insert(3904433, "A")`: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$

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- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$
- `insert(5148949, "C")`: $h(5148949) = 4 \Rightarrow T[4] = (5148949, "C")$

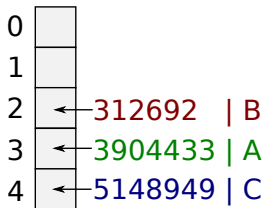
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Searching:

■ `search(3904433): $h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$`

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- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$
- $\text{search}(123459): h(123459) = 4 \Rightarrow T[4]$
 \Rightarrow Value with key 123459 does not exist

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0	
1	
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- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$
- $\text{search}(123459): h(123459) = 4 \Rightarrow T[4]$
 \Rightarrow Value with key 123459 does not exist
- Search time for this example: $\mathcal{O}(1)$

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Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$

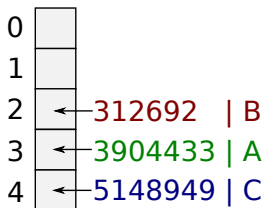
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Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$
 $\Rightarrow T[3] = (876543, "D") \Rightarrow$ Collision

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Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$
 $\Rightarrow T[3] = (876543, "D") \Rightarrow$ Collision
- This happens more often than expected
 - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

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- Represent each bucket as list of key value pairs

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Easiest Solution:

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

Figure: Hashtable T



Best case:

- We have n keys which are equally distributed over m buckets
- We have $\approx \frac{n}{m}$ pairs per bucket
- The runtime for searching is nearly $\mathcal{O}(1)$ when **not** $n \gg m$

Best case
($m = 5, n = 10$)



Worst case:

- All n keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching

Worst case
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Thought Experiment:

- A **hash function** is defined for a given **key set**
- Find a **set of keys** resulting in a degenerated **hash table**
 - *The **hash function** stays fixed*
 - *For table size of 100: Try $100 \times (99 + 1)$ different numbers*
 - *Worst case: All 100 **key sets** map to one bucket*
- **Now:** Find a solution to avoid that problem

Solution: universal hashing

- Out of a set of hash functions we randomly choose one



Figure: Hash func. 1



Figure: Hash func. 2



Figure: Hash func.
coll.

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Solution: universal hashing

- Out of a set of hash functions we randomly choose one
 - The **expected result** of the hash function is an equal distribution over the buckets
 - This hash function stays fixed for the lifetime of table
- Optional: copy table with new hash when degenerated



Figure: Hash func. 1



Figure: Hash func. 2



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Definition:

- We call \mathcal{U} the set (universum) of possible keys

Key universe \mathcal{U}



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T (Hashtable)



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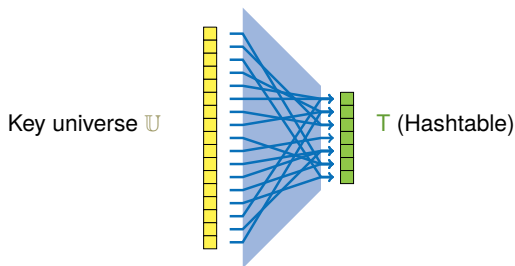


Figure: Hash function h_1

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- Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load



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Figure: Set of hash functions \mathbb{H}

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- An average of 3 out of 15 functions produce collisions



Figure: Set of hash functions H

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Number of hash functions that create collisions

$$\frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}, \quad c \in \mathbb{R}$$

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$$\text{Prob}(\text{Collision}) = \frac{1}{m} \Leftrightarrow c = 1$$

- \mathbb{U} : Key universe
- \mathbb{S} : Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: Keys mapping to Bucket i (“synonyms”)
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$



Figure: Hash function $h \in \mathbb{H}$

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- Particularity: If $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

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- Rolling a dice twice ($\Omega = \{1, \dots, 6\}^2$)
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- $E =$ if both results are even, then $P(E) =$

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- Example rolling twice: $\mathbb{E}(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \dots + 12 \cdot \frac{1}{36} = 7$

Sum of expected values: For arbitrary discrete random variables X_1, \dots, X_n we can write:

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Example: Throwing two dice

- X_1 : Expected result of dice 1: $\mathbb{E}(X_1) = 3.5$
- X_2 : Expected result of dice 2: $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$: Expected total number:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$



Corollary:

The probability of the event E is $p = P(E)$. Let X be the occurrences of the event E and n be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurrences of } 6) = \frac{1}{6} \cdot 60 = 10$$



Proof Corollary:

Indicator variable: X_i



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Indicator variable: X_i

$$X_i = \begin{cases} 1, & \text{if event occurs} \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow X = \sum_{i=1}^n X_i$$

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Indicator variable: X_i

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$$\Rightarrow X = \sum_{i=1}^n X_i$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{E}(X_i) \stackrel{\text{def. } \mathbb{E}\text{-value}}{=} \sum_{i=1}^n p = n \cdot p$$



Proof Corollary:

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Def. \mathbb{E} -value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

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To proof:

$$\mathbb{E}[|S_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \quad \forall i$$



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$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \quad y \in \mathbb{S} \setminus \{x\}$$

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Auxiliary calculation:

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- Which x, y lead to a relative collision count bigger than $\frac{c}{m}$?



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- Let \mathcal{H} be the set of all h for which:

$$h_{a,b}(x) = ((a \cdot x + b) \bmod p) \bmod m,$$

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- Easy to implement but hard to proof
- Exercise: show empirically that it is 2-universal

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- **Intuitive:** Scalar product with base m

$$a \bullet x = \sum_{0, \dots, k-1} a_i \cdot x_i$$

Example ($\mathbb{U} = \{0, \dots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

■ General for this Lecture

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Hash Map - Theory

[Wik] [Hash table](#)

https://en.wikipedia.org/wiki/Hash_table

■ Hash Map - Implementations / API

[Cpp] [C++ - hash_map](#)

http://www.sgi.com/tech/stl/hash_map.html

[Jav] [Java - HashMap](#)

<https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html>

[Pyt] [Python - Dictionaries \(Hash table\)](#)

https://en.wikipedia.org/wiki/Hash_table