Algorithmns and Datastructures

Levenshtein distance, Dynamic programming

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

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Bioinformatics Group / Department of Computer Science
Algorithmns and Datastructures, February 2017

Structure

Structure

Introduction

Edit distance

Edit distance:

Edit distance:

- Measurement for similarity of two words / strings

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- Algorithm for efficient calculation

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- Measurement for similarity of two words / strings
- Algorithm for efficient calculation
- General principle: dynamic programming

BioInfSearch

ejafjatlajökuk

eyjafjallajökull
eyjafjallajökull movie
eyjafjallajälull trailer

Search!

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Wikipedia.org:

"Der Eyjafjallajökull (['eɪjaˌfjatlaˌjœːkʏtl̥])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárþing eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."

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- Duplicates in databases:

```
Hein Blöd    27568 Bremerhaven
Hein Bloed   27568 Bremerhafen
Hein Doof    27478 Cuxhaven
```

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- Bioinformatics: Similarity of DNA-sequences

Search of similar proteins:

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- `BLAST` (**B**asic **L**ocal **A**lignment **S**earch **T**ool)

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- Changed life-science completely

Search of similar proteins:

- `BLAST` (**B**asic **L**ocal **A**lignment **S**earch **T**ool)
- Alignment $\hat{=}$ Edit distance
- Changed life-science completely
- Cited 63437 times on Google Scholar (Sep. 2017)

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Definition of edit distance: (*Levenshtein-distance*)

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- Let x, y be two strings
- Edit distance $\mathrm{ED}(x,y)$ of x and y:
  The minimal number of operations to transform x into y

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Definition of edit distance: (*Levenshtein-distance*)

- Let x, y be two strings
- Edit distance $\mathrm{ED}(x,y)$ of x and y:
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  - `Insert` a character

Edit distance

Definition of edit distance: (*Levenshtein-distance*)

- Let x, y be two strings
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  - `Insert` a character
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Definition of edit distance: (*Levenshtein-distance*)

- Let x, y be two strings
- Edit distance $\mathrm{ED}(x,y)$ of x and y:
  The minimal number of operations to transform x into y
  - `Insert` a character
  - `Replace` a character with another
  - `Delete` a character

1 2 3 4 5
DOOF

BLOED

Edit distance
Example

1 2 3 4 5
DOOF
$\downarrow$ `replace(1, B)`
BOOF

BLOED

1 2 3 4 5
DOOF
↓ `replace(1, B)`
BOOF
↓ `replace(2, L)`
BLOF

BLOED

1 2 3 4 5
DOOF

$\downarrow$ `replace(1, B)`

BOOF

$\downarrow$ `replace(2, L)`

BLOF

$\downarrow$ `insert(4, E)`

BLOEF

BLOED

1 2 3 4 5
DOOF

$\downarrow$ `replace(1, B)`

BOOF

$\downarrow$ `replace(2, L)`

BLOF

$\downarrow$ `insert(4, E)`

BLOEF

$\downarrow$ `replace(5, D)`

BLOED

```
12345
DOOF
  ↓     replace(1, B)
BOOF
  ↓     replace(2, L)
BLOF
  ↓     insert(4, E)
BLOEF
  ↓     replace(5, D)
BLOED
```

$\underbrace{\hspace{6em}}_{\text{ED}=4}$

1 2 3 4 5
DOOF
↓ `replace(1, B)`
BOOF
↓ `replace(2, L)`
BLOF
↓ `insert(4, E)`
BLOEF
↓ `replace(5, D)`
BLOED

$\underbrace{\texttt{replace(1, B) replace(2, L) insert(4, E) replace(5, D)}}_{\text{ED}=4}$

1 2 3 4 5
BLOED

Edit distance
Example

1 2 3 4 5
DOOF
↓ `replace(1, B)`
BOOF
↓ `replace(2, L)`
BLOF
↓ `insert(4, E)`
BLOEF
↓ `replace(5, D)`
BLOED

$\underbrace{\hphantom{replace(5, D)}}_{\text{ED}=4}$

1 2 3 4 5
B LOED

DOOF

Edit distance
Example

1 2 3 4 5
DOOF
↓ `replace(1, B)`
BOOF
↓ `replace(2, L)`
BLOF
↓ `insert(4, E)`
BLOEF
↓ `replace(5, D)`
BLOED

$\underbrace{\hspace{6em}}_{\text{ED}=4}$

1 2 3 4 5
BLOED
↓ `replace(5, F)`
BLOEF

DOOF

```
12345
DOOF
  ↓     replace(1, B)
BOOF
  ↓     replace(2, L)
BLOF
  ↓     insert(4, E)
BLOEF
  ↓     replace(5, D)
BLOED
```

$\underbrace{\hspace{5em}}_{\text{ED=4}}$

```
12345
BLOED
  ↓     replace(5, F)
BLOEF
  ↓     delete(4)
BLOF

DOOF
```

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	`replace(1, B)`
BOOF	
↓	`replace(2, L)`
B L OF	
↓	`insert(4, E)`
B L OEF	
↓	`replace(5, D)`
B L OED	

ED=4

1 2 3 4 5	
B LOED	
↓	`replace(5, F)`
B LOEF	
↓	`delete(4)`
B LOF	
↓	`replace(2, O)`
B OOF	
DOOF	

Edit distance
Example

1 2 3 4 5
DOOF
↓ `replace(1, B)`
BOOF
↓ `replace(2, L)`
BLOF
↓ `insert(4, E)`
BLOEF
↓ `replace(5, D)`
BLOED

ED=4

1 2 3 4 5
BLOED
↓ `replace(5, F)`
BLOEF
↓ `delete(4)`
BLOF
↓ `replace(2, O)`
BOOF
↓ `replace(1, D)`
DOOF

1 2 3 4 5		1 2 3 4 5	
DOOF		BLOED	
↓	`replace(1, B)`	↓	`replace(5, F)`
BOOF		BLOEF	
↓	`replace(2, L)`	↓	`delete(4)`
BLOF		BLOF	
↓	`insert(4, E)`	↓	`replace(2, O)`
BLOEF		BOOF	
↓	`replace(5, D)`	↓	`replace(1, D)`
BLOED		DOOF	
	ED=4		ED=4

Edit distance

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- $x[i..j]$ is the slice of x from i to j where $1 \leq i \leq j \leq |x|$

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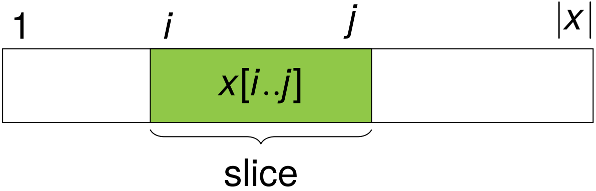

Edit distance

Trivial facts:

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- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$

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- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$
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- $\mathrm{ED}(x,y) \geq \mathrm{abs}(|x| - |y|)$

$$\mathrm{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

Trivial facts:

- $\mathrm{ED}(x,y) = \mathrm{ED}(y,x)$
- $\mathrm{ED}(x,\varepsilon) = |x|$
- $\mathrm{ED}(x,y) \geq \mathrm{abs}(|x| - |y|)$ $\quad \mathrm{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$
- $\mathrm{ED}(x,y) \leq \mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ $\quad n = |x|,\ m = |y|$

Solutions based on examples:

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- From `VERIEN` to `FERIEN`?

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- From VERIEN to FERIEN?
- From MEXIKO to AMERIKA?

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- From `VERIEN` to `FERIEN`?
- From `MEXIKO` to `AMERIKA`?
- From `AAEBEAABEAREEAEBA` to `RBEAAEEBAAAEBBAEAE`?

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- Searching biggest substrings can yield the solution but doesn't have to

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Recursive approach:

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Recursive approach:

- Dividing in two halves? Not a good idea:

$\mathrm{ED}(GRAU, RAUM) = 2$ but $\mathrm{ED}(GR, RA) + \mathrm{ED}(AU, UM) = 4$

Edit distance
Solving examples

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- From `VERIEN` to `FERIEN`?
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Recursive approach:

- Dividing in two halves? Not a good idea:

  $\mathrm{ED}(GRAU, RAUM) = 2$ but $\mathrm{ED}(GR, RA) + \mathrm{ED}(AU, UM) = 4$

- Finding "smaller" sub problems?
  Let's try it!

Terminology:

Edit distance

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- Let x, y be two strings

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- Let x, y be two strings
- Let $\sigma_1, \dots, \sigma_k$ be a sequence of k operations where $k = \mathrm{ED}(x, y)$ for $x \to y$ (transform x into y)
  (We do not know this sequence but we assume it exists)

Edit distance

Terminology:

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Terminology:

- We only consider monotonous sequences:
  The posititon of σ_{i+1} is $\geq$ the position of σ_i where we only allow the positions to be equal on a `delete` operation

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1 2 3 4 5		1 2 3 4 5 6 7	
DOOF		SAUDOOF	
$\downarrow$	`replace(1, B)`	$\downarrow$	`delete(1)`
BOOF		AUDOOF	
$\downarrow$	`replace(2, L)`	$\downarrow$	`delete(1)`
BLOF		UDOOF	
$\downarrow$	insert(4, E)	$\downarrow$	`delete(1)`
BLOEF		DOOF	
$\downarrow$	`replace(5, D)`	$\downarrow$	`insert(4, O)`
BLOED		DOOOF	

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Terminology:

- We only consider monotonous sequences:
  The posititition of σ_{i+1} is $\geq$ the position of σ_i where we only allow the positions to be equal on a `delete` operation

$$
\begin{array}{ll}
\text{1 2 3 4 5} & \\
\text{DOOF} & \\
\downarrow & \texttt{replace(1, B)} \\
\text{BOOF} & \\
\downarrow & \texttt{replace(2, L)} \\
\text{BLOF} & \\
\downarrow & \text{insert(4, E)} \\
\text{BLOEF} & \\
\downarrow & \texttt{replace(5, D)} \\
\text{BLOED} &
\end{array}
\qquad
\begin{array}{ll}
\text{1 2 3 4 5 6 7} & \\
\text{SAUDOOF} & \\
\downarrow & \texttt{delete(1)} \\
\text{AUDOOF} & \\
\downarrow & \texttt{delete(1)} \\
\text{UDOOF} & \\
\downarrow & \texttt{delete(1)} \\
\text{DOOF} & \\
\downarrow & \texttt{insert(4, O)} \\
\text{DOOOF} &
\end{array}
$$

Edit distance

Terminology:

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- **Lemma:** For any x and y with $k = \mathrm{ED}(x,y)$ exists a monotonous sequence of k operations for $x \to y$

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- **Lemma:** For any x and y with $k = \mathrm{ED}(x, y)$ exists a monotonous sequence of k operations for $x \to y$
- **Intuition:** The order of our sequence is not relevant (Therefore we can also sort them monotonously)

1	2	3	4	5		1	2	3	4	5	6	7
D	O	O	F			S	A	U	D	O	O	F
B	L	O	E	D		D	O	O	O	F		

Consider the last operation:

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Consider the last operation:

- Solve blue part recursively

Edit distance

Recursive approach

Consider the last operation:

- Solve blue part recursively

DOOF
↓↓↓↓
BLOE
↓insert
BLOED

Figure: Case 1a

DOOF
↓↓↓↓↓↓
BLOEDF
↓delete
BLOED

Figure: Case 1b

DOOF
↓↓↓↓↓
BLOEF
↓replace
BLOED

Figure: Case 1c

Consider the last operation:

Consider the last operation:

- Solve blue part recursively

Edit distance

Recursive approach

Consider the last operation:

- Solve blue part recursively

WINTER
↓↓↓↓↓↓
SOMMER
↓nothing
SOMMER

Figure: Case 2

Display of solution:

- Alignment
- Example:

_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

Dynamic programming:

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- Instances of Bellman's principle of optimality:

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Figure: Richard Bellman (1920 - 1984)

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Figure: Richard Bellman (1920 - 1984)

- Optimal solutions consist of optimal partial solutions
  - Shortest paths: Each partial path has to be optimal
  - Edit distance: Each partial alignment has to be optimal

Dynamic programming:

- Instances of Bellman's principle of optimality:
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Figure: Richard Bellman (1920 - 1984)

- Optimal solutions consist of optimal partial solutions
    - Shortest paths: Each partial path has to be optimal
    - Edit distance: Each partial alignment has to be optimal

- Always solvable through dynamic programming (Caching of optimal partial solutions)

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Case analysis:

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  - $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow z$ and σ_k: $z \rightarrow y$
    Example:

$$x = \text{DOOF},\ z = \text{SAUBLOEF},\ y = \text{SAUBLOED}$$

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    Example:

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  - Let $n = |x|$, $m = |y|$, $m' = |z|$

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  - $\sigma_1, \dots, \sigma_{k-1}$: $x \to z$ and σ_k: $z \to y$
    Example:

$$x = \text{DOOF},\ z = \text{SAUBLOEF},\ y = \text{SAUBLOED}$$

  - Let $n = |x|$, $m = |y|$, $m' = |z|$
  - We note $m' \in \{m-1, m, m+1\}$ why?

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Edit distance

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- Case 2: σ_k does nothing at the outer end:

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  - Case 1a: $\sigma_k = \mathtt{insert}(m'+1, y[m])$ [then $m' = m-1$]
  - Case 1b: $\sigma_k = \mathtt{delete}(m')$ [then $m' = m+1$]
  - Case 1c: $\sigma_k = \mathtt{replace}(m', y[m])$ [then $m' = m$]

- Case 2: σ_k does nothing at the outer end:
  - Then $z[m'] = y[m]$ and $x[n'] = z[m']$ and with that $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \to y[1..m-1]$ and $x[n] = y[m]$

Edit distance

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- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \quad \rightarrow y[1..m-1]$

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1, \ldots, \sigma_{k-1}$: $x \rightarrow y[1..m-1]$
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- Case 1c (`replace`): $\sigma_1, \ldots, \sigma_{k-1}$: $x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1, \ldots, \sigma_k$: $x[1..n-1] \rightarrow y[1..m-1]$

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Edit distance

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  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and

Edit distance

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Case analysis:

- Case 1a (`insert`): $\sigma_1,\ldots,\sigma_{k-1}\colon x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1,\ldots,\sigma_{k-1}\colon x[1..n-1] \rightarrow y$
- Case 1c (`replace`): $\sigma_1,\ldots,\sigma_{k-1}\colon x[1..n-1] \rightarrow y[1..m-1]$
- Case 2 (`nothing`): $\sigma_1,\ldots,\sigma_k\colon x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- For $|x| > 0$ and $|y| > 0$ is $\mathrm{ED}(x,y)$ the minimum of
  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y) + 1$ and

Edit distance

Case analysis:

- Case 1a (`insert`): $\sigma_1,\ldots,\sigma_{k-1}: x \rightarrow y[1..m-1]$
- Case 1b (`delete`): $\sigma_1,\ldots,\sigma_{k-1}: x[1..n-1] \rightarrow y$
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  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$

Edit distance

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  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $\mathrm{ED}(x,y) = |y|$

Edit distance

Case analysis:

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- Case 1b (`delete`): $\sigma_1,\ldots,\sigma_{k-1}$: $x[1..n-1] \rightarrow y$
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  - $\mathrm{ED}(x, y[1..m-1]) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y) + 1$ and
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 1$ if $x[n] \neq y[m]$
  - $\mathrm{ED}(x[1..n-1], y[1..m-1]) + 0$ if $x[n] = y[m]$
- For $|x| = 0$ is $\mathrm{ED}(x,y) = |y|$
- For $|y| = 0$ is $\mathrm{ED}(x,y) = |x|$

Edit distance

Implementation - Python

```python
def edit_distance(x, y):
    if len(x) == 0:
        return len(y)
    if len(y) == 0:
        return len(x)

    ed1 = edit_distance(x, y[:-1]) + 1
    ed2 = edit_distance(x[:-1], y) + 1
    ed3 = edit_distance(x[:-1], y[:-1])
    if x[-1] != y[-1]:
        ed3 += 1

    return min(ed1, ed2, ed3)
```

Recursive program:

Recursive program:

- The algorithm results in the following recursive formular:

$$\begin{aligned}
T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\
&\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\
&= 3 \cdot T(n-1,m-1)
\end{aligned}$$

Recursive program:

- The algorithm results in the following recursive formular:

$$\begin{aligned} T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\ &\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\ &= 3 \cdot T(n-1,m-1) \end{aligned}$$

- This results in $T(n,n) \geq 3^n$

Recursive program:

- The algorithm results in the following recursive formular:

$$
\begin{aligned}
T(n,m) &= T(n-1,m) + T(n,m-1) + T(n-1,m-1) + 1 \\
&\geq T(n-1,m-1) + T(n-1,m-1) + T(n-1,m-1) \\
&= 3 \cdot T(n-1,m-1)
\end{aligned}
$$

- This results in $T(n,n) \geq 3^n$

$\Rightarrow$ The runtime is at least exponential

Edit distance

Dynamic programming:

Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

Edit distance

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Visualization on the next slide:

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Dynamic programming:

- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

Visualization on the next slide:

- Operations always refer to the last position (indices are omitted)

Edit distance

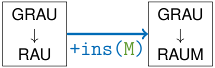
Dynamic programming:

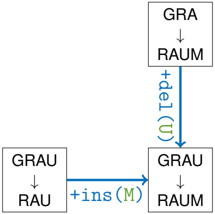
- We create a table with all possible combination of substrings and save calculated entries
- This results in a runtime and space consumption of $O(n \cdot m)$

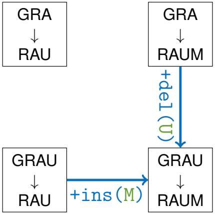
Visualization on the next slide:

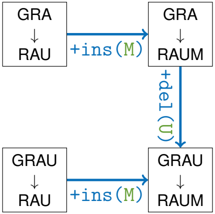
- Operations always refer to the last position (indices are omitted)
- We also display the replaced character on a `replace` operation to visualize operations without costs
  $\Rightarrow$ `repl(A, A)`

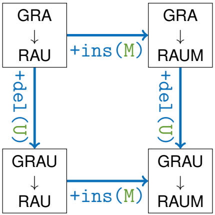
GRAU
$\downarrow$
RAUM

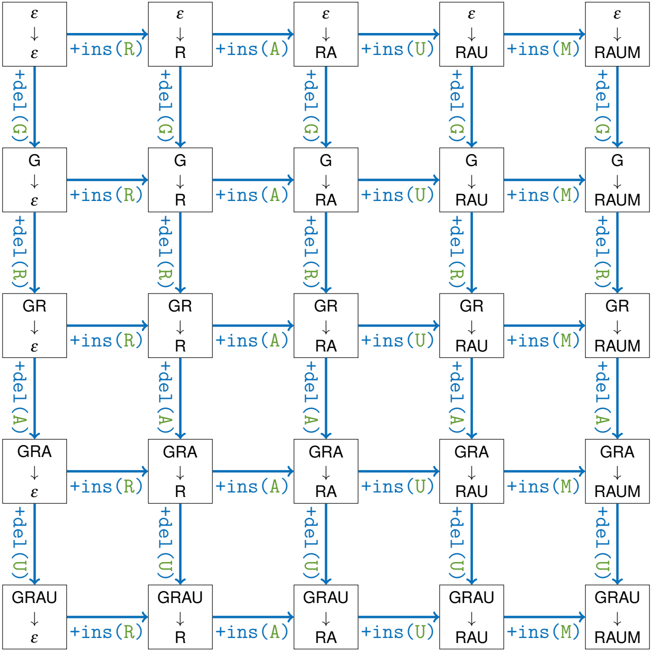

GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM


GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM

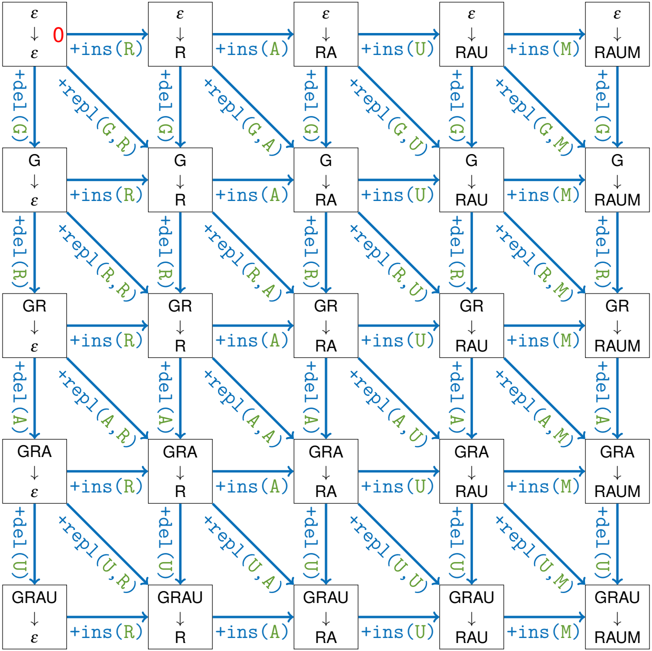

GRA
↓
RAU
GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM

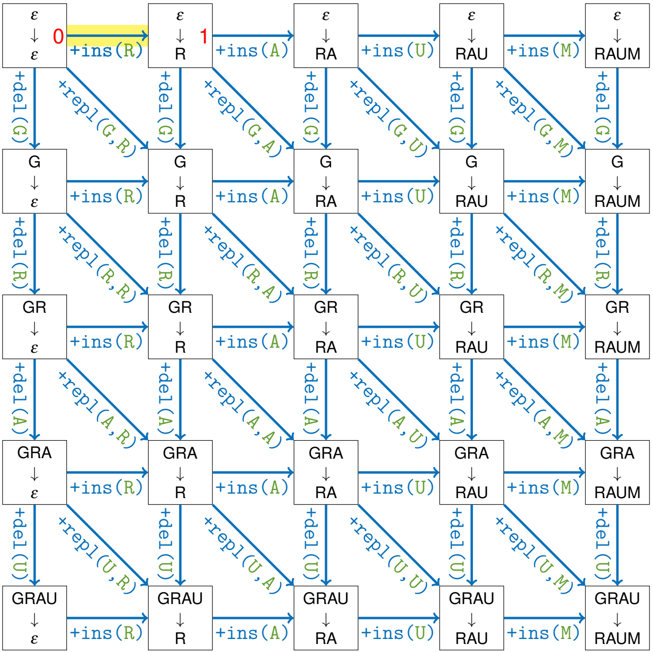

GRA
↓
RAU
+ins(M)
GRA
↓
RAUM
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM


GRA
↓
RAU
+ins(M)
GRA
↓
RAUM
+del(U)
+del(U)
GRAU
↓
RAU
+ins(M)
GRAU
↓
RAUM

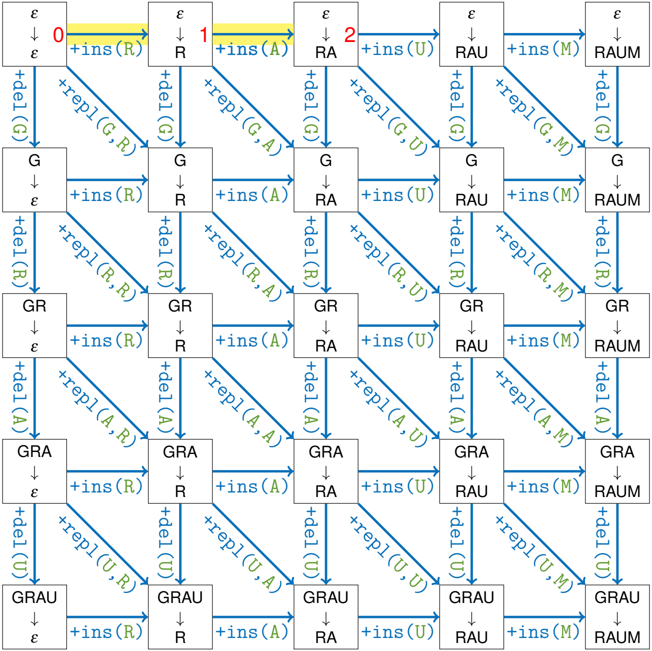

ε ↓ ε
+ins(R)
ε ↓ R
+ins(A)
ε ↓ RA
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

Fast algorithm:
We can determine the edit distance for all combination of partial strings from the top left to bottom right.

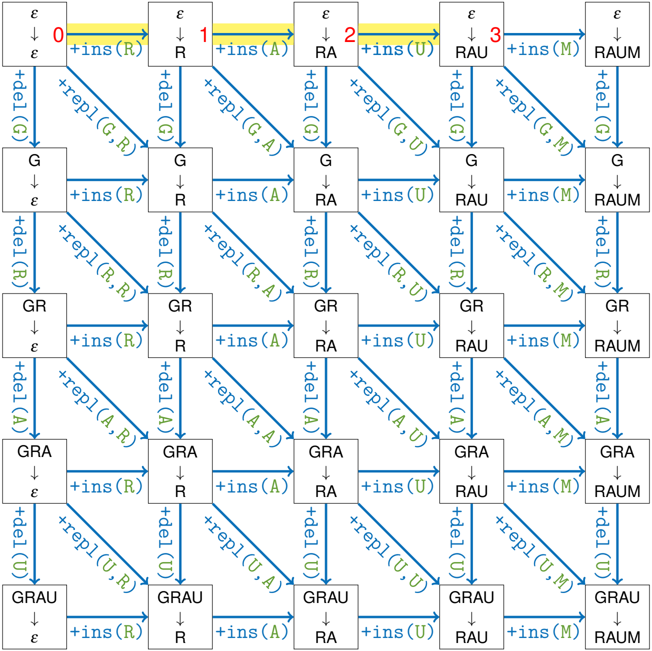

ε ↓ ε
0
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
ε ↓ R
ε ↓ RA
ε ↓ RAU
ε ↓ RAUM
G ↓ ε
GR ↓ ε
GRA ↓ ε
GRAU ↓ ε
GRAU ↓ RAUM

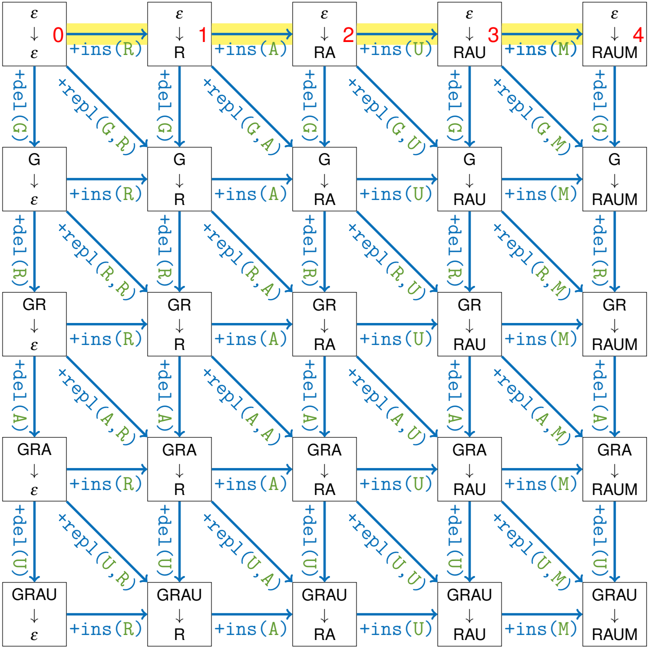


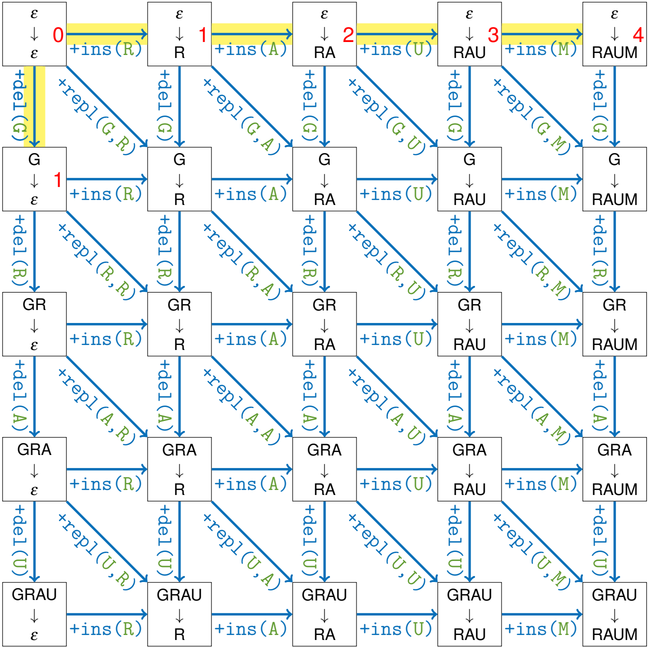
ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
+ins(U)
ε ↓ RAU
+ins(M)
ε ↓ RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

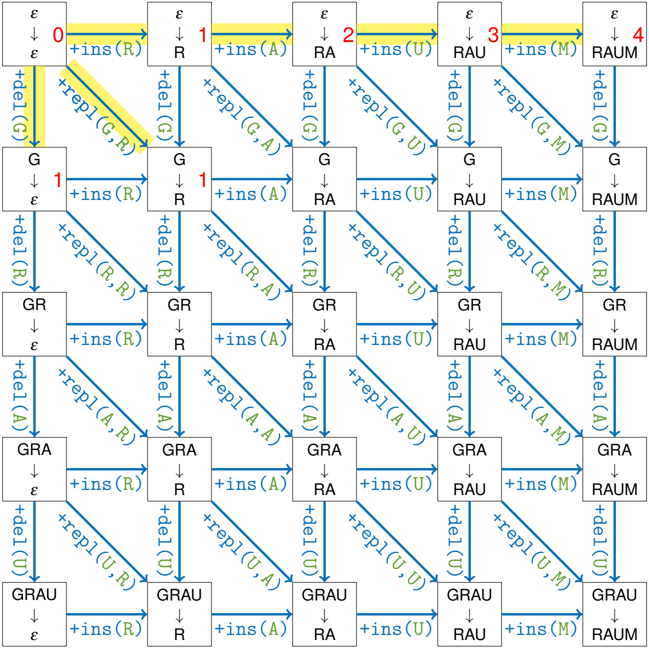


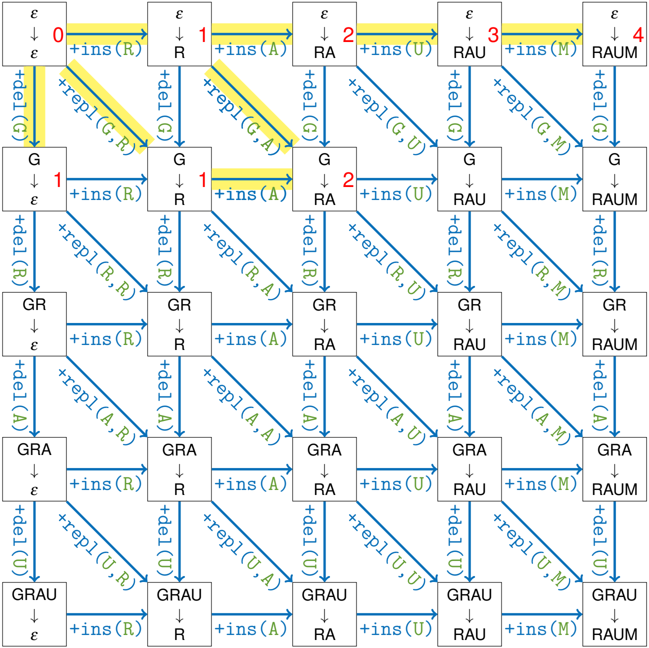
ε → ε
0
+ins(R)
ε → R
1
+ins(A)
ε → RA
2
+ins(U)
ε → RAU
+ins(M)
ε → RAUM
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G → ε
G → R
G → RA
G → RAU
G → RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM


ε ↓ ε
ε ↓ R
ε ↓ RA
ε ↓ RAU
ε ↓ RAUM
0
1
2
3
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM

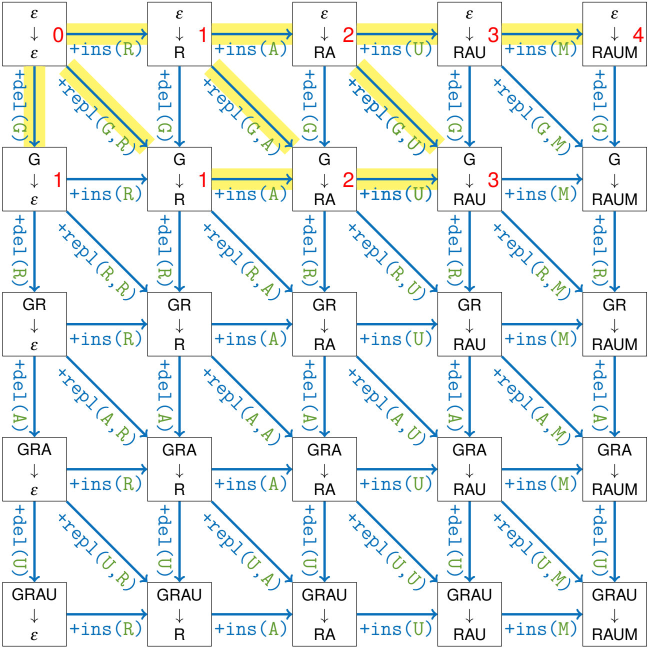

ε → ε
ε → R
ε → RA
ε → RAU
ε → RAUM
0
1
2
3
4
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
G → ε
G → R
G → RA
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM

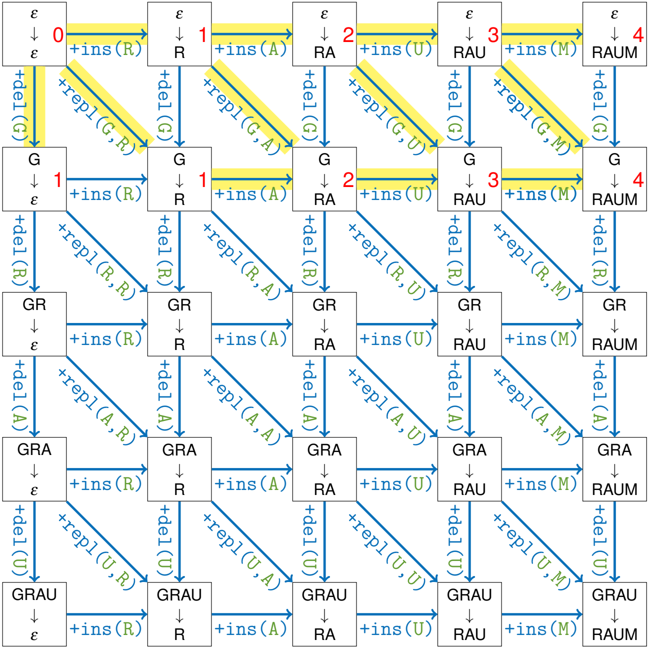

ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
3
+ins(M)
ε ↓ RAUM
4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
1
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA
G ↓ RAU
G ↓ RAUM
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)



ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
G → RAU
G → RAUM
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM


ε ↓ ε
0
+ins(R)
ε ↓ R
1
+ins(A)
ε ↓ RA
2
+ins(U)
ε ↓ RAU
3
+ins(M)
ε ↓ RAUM
4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε
G ↓ R
G ↓ RA
G ↓ RAU
G ↓ RAUM
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε
GR ↓ R
GR ↓ RA
GR ↓ RAU
GR ↓ RAUM
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
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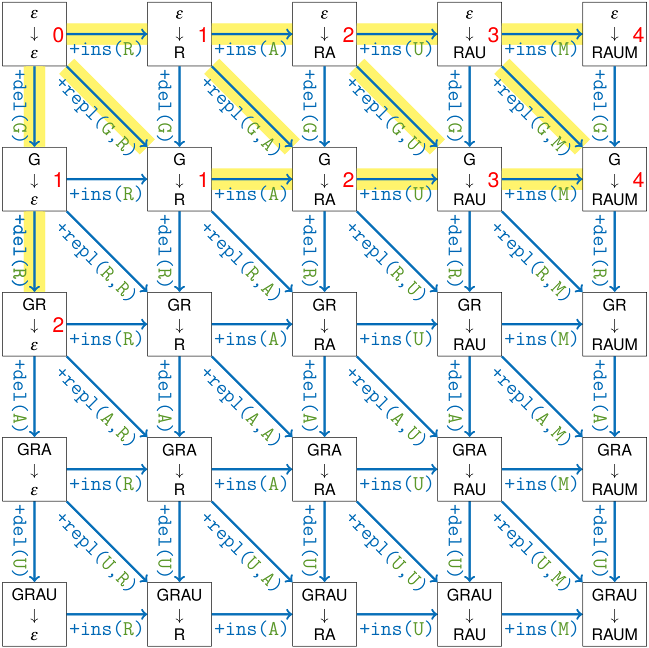
+repl(U,M)


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
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+del(U)
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+repl(A,M)
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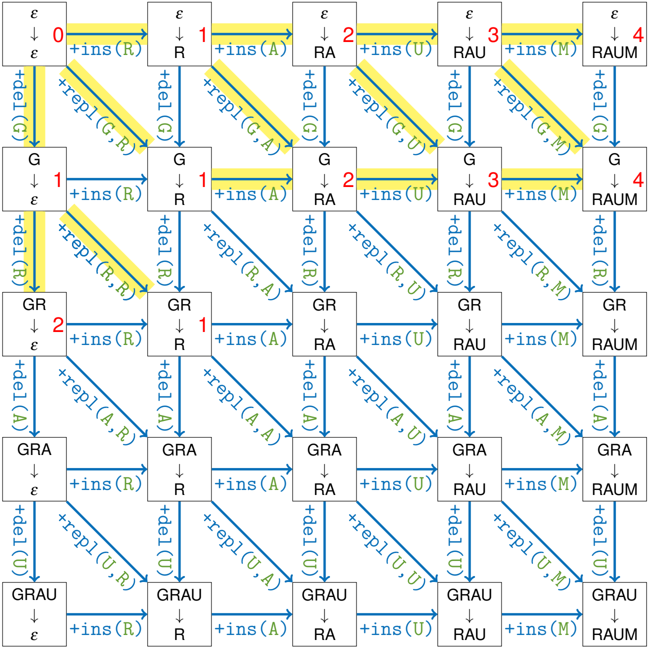
+repl(U,U)
+repl(U,M)


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA
GR → RAU
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
+ins(R)
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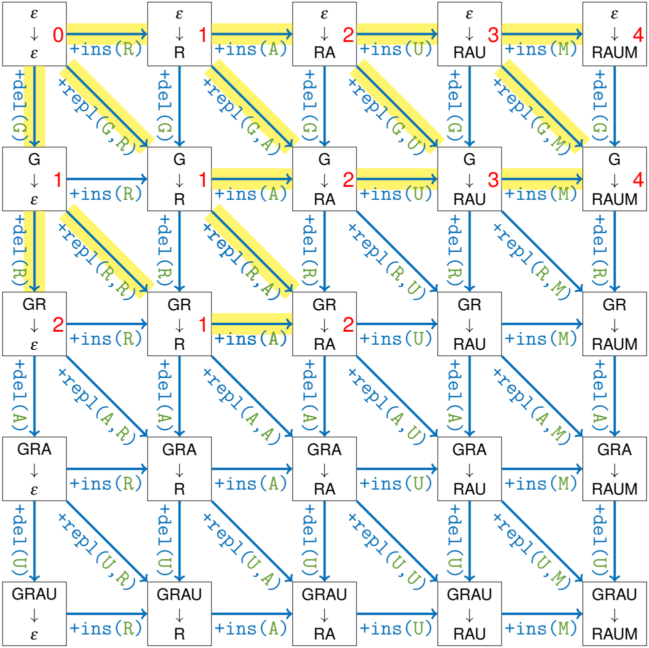
ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU
GR ↓ RAUM
GRA ↓ ε
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
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+repl(R,R)
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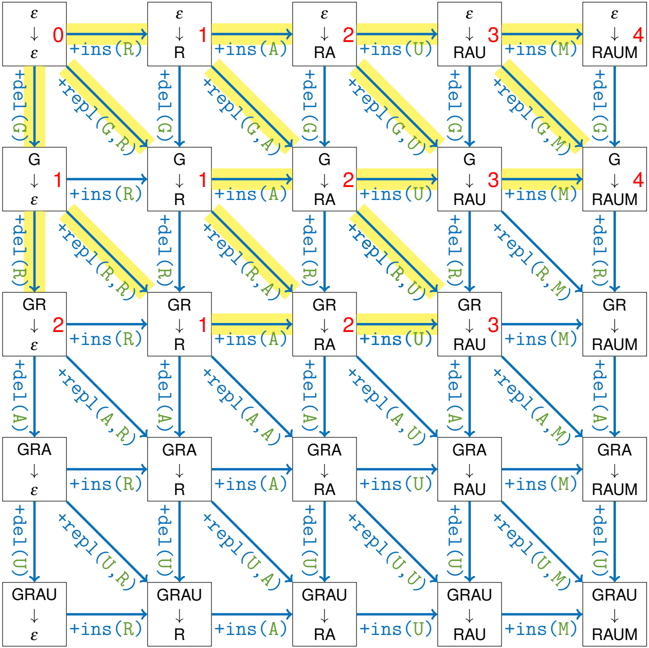
ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
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GRAU → RAUM
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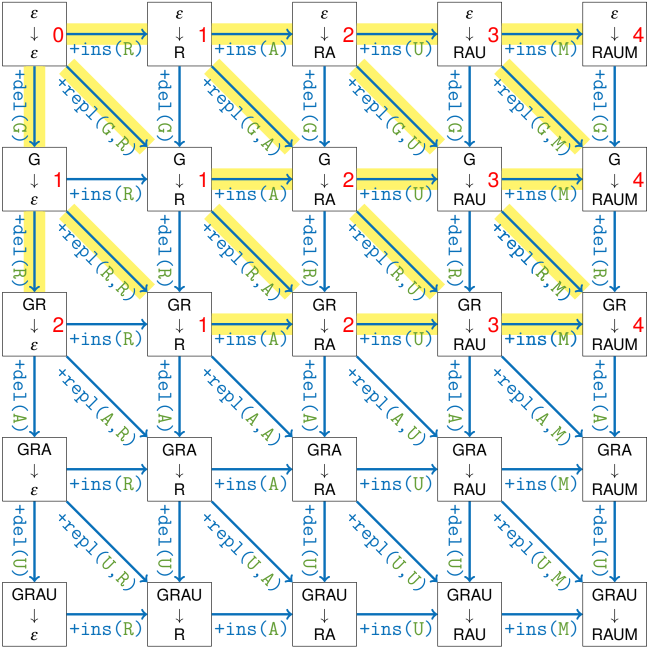
+repl(U,M)


ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM 4
GRA → ε
GRA → R
GRA → RA
GRA → RAU
GRA → RAUM
GRAU → ε
GRAU → R
GRAU → RA
GRAU → RAU
GRAU → RAUM
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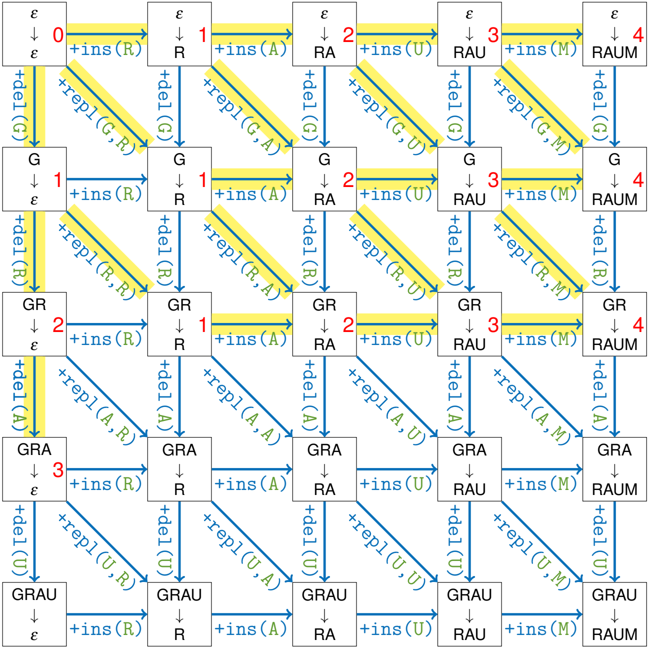
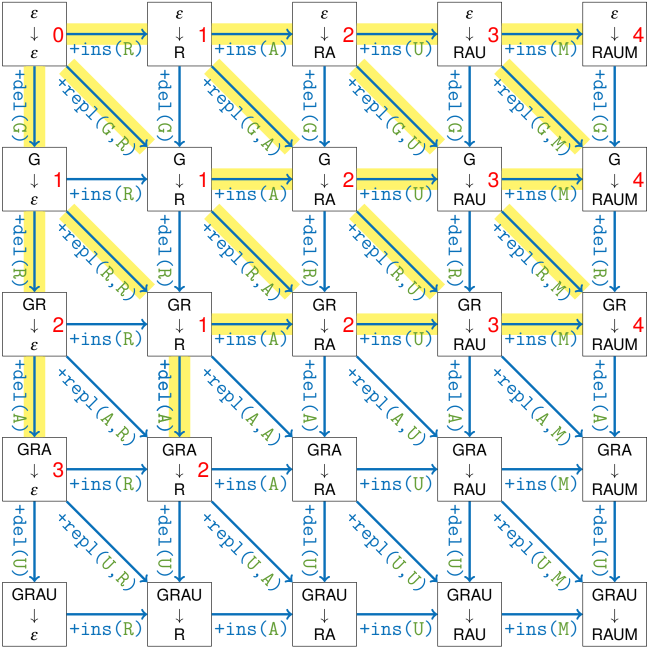

ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM



ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA
GRA ↓ RAU
GRA ↓ RAUM
GRAU ↓ ε
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
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+del(A)
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+repl(U,R)
+repl(U,A)
+repl(U,U)
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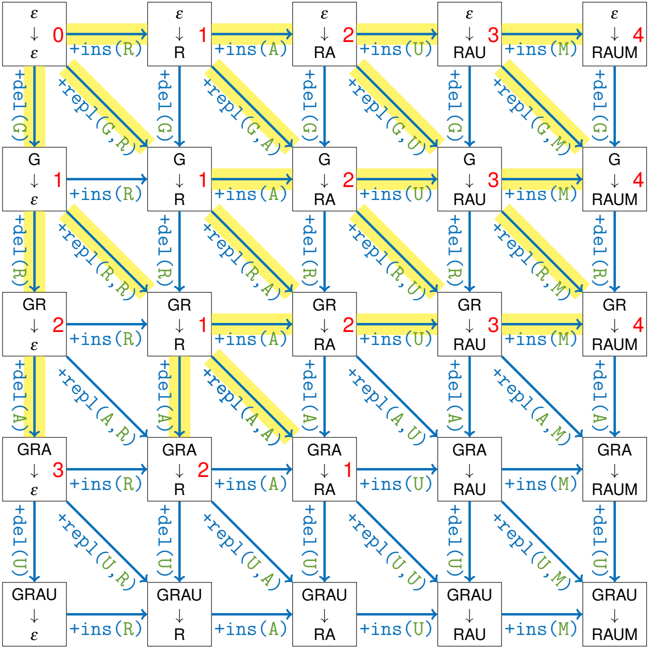
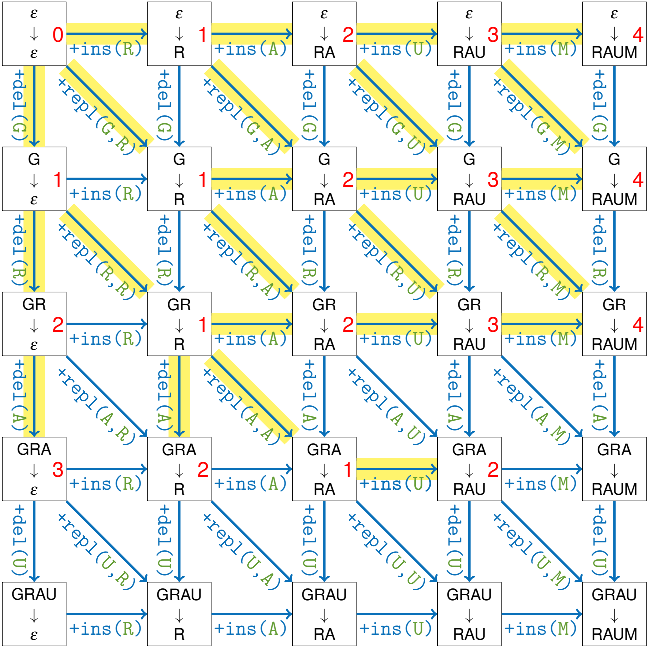
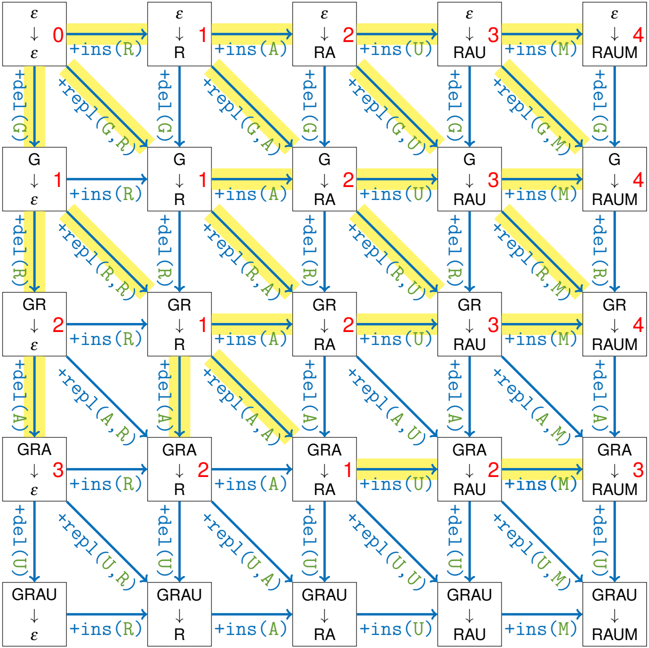
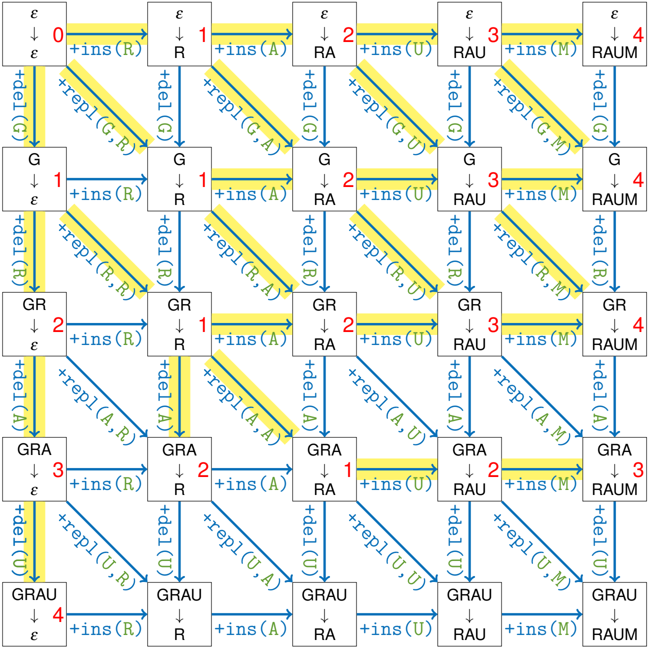


+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)




ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+del(G)
+repl(G,A)
+del(G)
+repl(G,U)
+del(G)
+repl(G,M)
+del(G)
G ↓ ε 1
+ins(R)
G ↓ R 1
+ins(A)
G ↓ RA 2
+ins(U)
G ↓ RAU 3
+ins(M)
G ↓ RAUM 4
+del(R)
+repl(R,R)
+del(R)
+repl(R,A)
+del(R)
+repl(R,U)
+del(R)
+repl(R,M)
+del(R)
GR ↓ ε 2
+ins(R)
GR ↓ R 1
+ins(A)
GR ↓ RA 2
+ins(U)
GR ↓ RAU 3
+ins(M)
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+del(A)
+repl(A,A)
+del(A)
+repl(A,U)
+del(A)
+repl(A,M)
+del(A)
GRA ↓ ε 3
+ins(R)
GRA ↓ R 2
+ins(A)
GRA ↓ RA 1
+ins(U)
GRA ↓ RAU 2
+ins(M)
GRA ↓ RAUM 3
+del(U)
+repl(U,R)
+del(U)
+repl(U,A)
+del(U)
+repl(U,U)
+del(U)
+repl(U,M)
+del(U)
GRAU ↓ ε
+ins(R)
GRAU ↓ R
+ins(A)
GRAU ↓ RA
+ins(U)
GRAU ↓ RAU
+ins(M)
GRAU ↓ RAUM



ε ↓ ε 0
ε ↓ R 1
ε ↓ RA 2
ε ↓ RAU 3
ε ↓ RAUM 4
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
GRAU ↓ ε 4
GRAU ↓ R
GRAU ↓ RA
GRAU ↓ RAU
GRAU ↓ RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
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+del(U)
+repl(G,R)
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+repl(G,U)
+repl(G,M)
+repl(R,R)
+repl(R,A)
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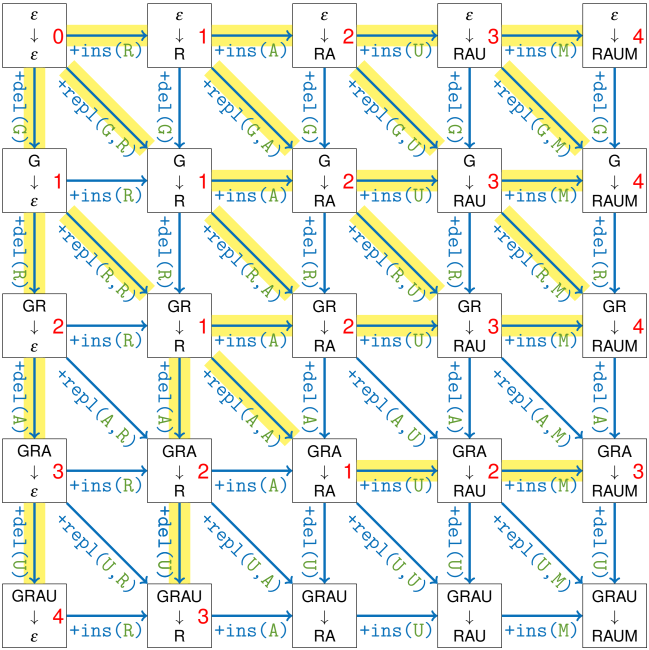

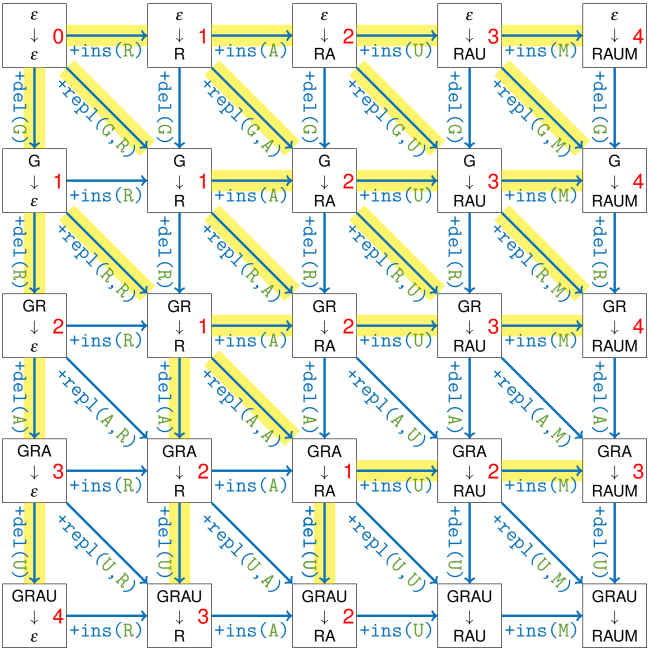

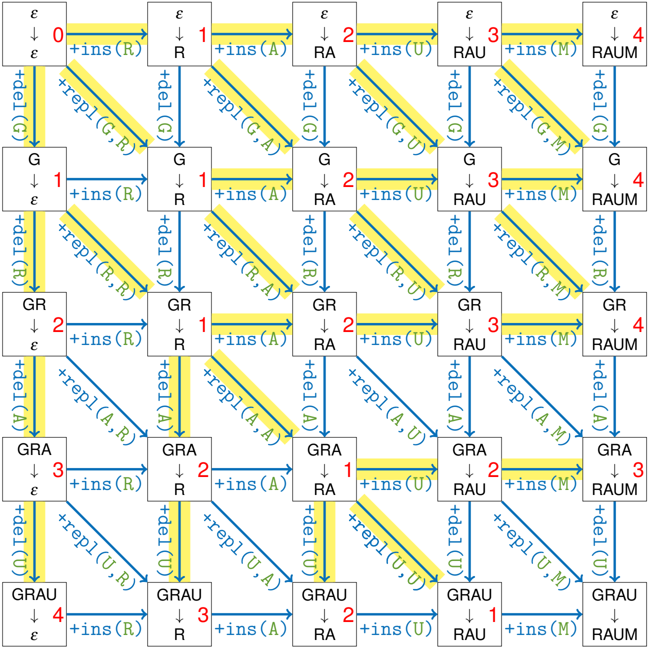

ε → ε 0
ε → R 1
ε → RA 2
ε → RAU 3
ε → RAUM 4
G → ε 1
G → R 1
G → RA 2
G → RAU 3
G → RAUM 4
GR → ε 2
GR → R 1
GR → RA 2
GR → RAU 3
GR → RAUM 4
GRA → ε 3
GRA → R 2
GRA → RA 1
GRA → RAU 2
GRA → RAUM 3
GRAU → ε 4
GRAU → R 3
GRAU → RA 2
GRAU → RAU 1
GRAU → RAUM
+ins(R)
+ins(A)
+ins(U)
+ins(M)
+del(G)
+del(R)
+del(A)
+del(U)
+repl(G,R)
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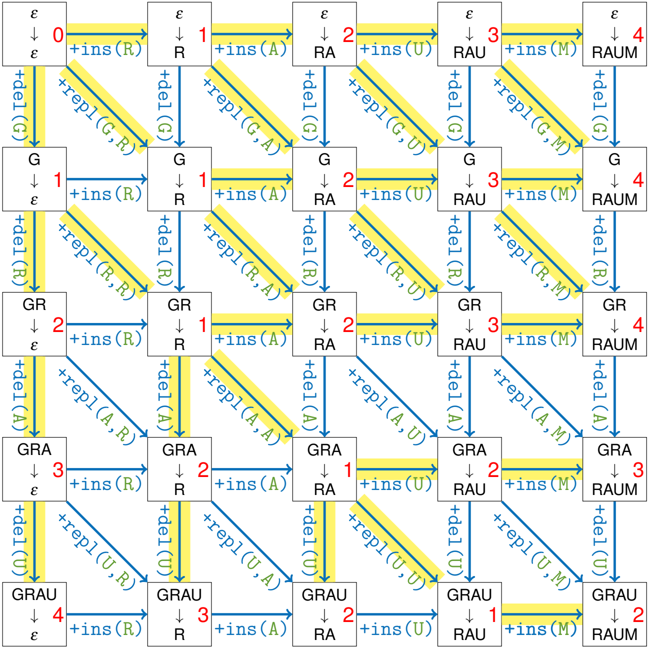

ε ↓ ε 0
+ins(R)
ε ↓ R 1
+ins(A)
ε ↓ RA 2
+ins(U)
ε ↓ RAU 3
+ins(M)
ε ↓ RAUM 4
+del(G)
+repl(G,R)
+repl(G,A)
+repl(G,U)
+repl(G,M)
G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM 2

How to get the sequence of operations?

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- We save at each recursion the most efficient previous entry (the highlighted arrows in our image)

Edit distance

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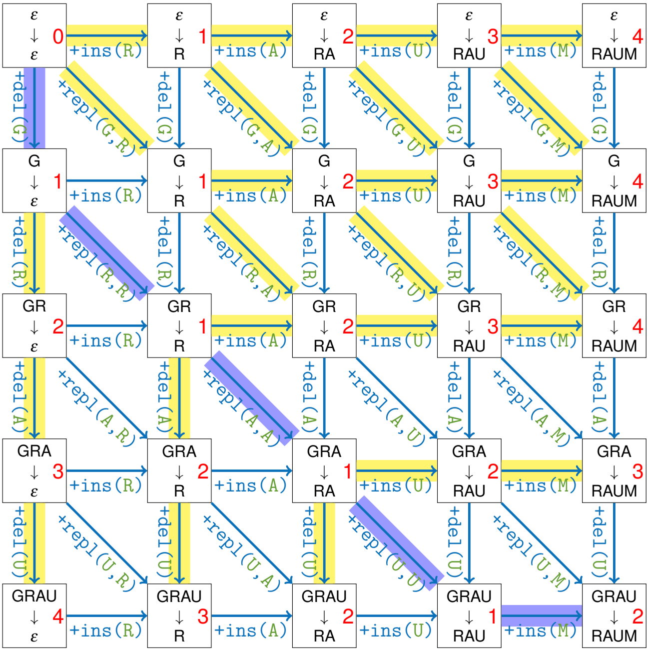
- There can be more than one arrows to the three previous entries
- If we follow the highlighted path from (n,m) to $(1,1)$ we get the optimum operations to transform x into y
  - If we can follow more than one path there exist more than one ideal sequence


ε ↓ ε 0
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+ins(M)
ε ↓ RAUM 4
+del(G)
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G ↓ ε 1
G ↓ R 1
G ↓ RA 2
G ↓ RAU 3
G ↓ RAUM 4
+del(R)
+repl(R,R)
+repl(R,A)
+repl(R,U)
+repl(R,M)
GR ↓ ε 2
GR ↓ R 1
GR ↓ RA 2
GR ↓ RAU 3
GR ↓ RAUM 4
+del(A)
+repl(A,R)
+repl(A,A)
+repl(A,U)
+repl(A,M)
GRA ↓ ε 3
GRA ↓ R 2
GRA ↓ RA 1
GRA ↓ RAU 2
GRA ↓ RAUM 3
+del(U)
+repl(U,R)
+repl(U,A)
+repl(U,U)
+repl(U,M)
GRAU ↓ ε 4
GRAU ↓ R 3
GRAU ↓ RA 2
GRAU ↓ RAU 1
GRAU ↓ RAUM 2

General principle:

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- Recursive computation of ...
  - ... the same reoccuring partial problems
  - ... a limited number of partial problems

Edit distance

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- Recursive computation of ...
  - ... the same reoccuring partial problems
  - ... a limited number of partial problems
- Computation of the solutions for all partial problems
- In a order that unsolved partial problems consist of already solved partial problems
- The “path” to our solution normally gets computed while searching the best solution
- Dijkstra algorithm is basically dynamic programming!

Additional applications:

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- *Edit distance*: global alignment with $O(n^2)$ space and time consumption

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_	_	_	B	L	O	E	D
S	A	U	B	L	O	E	D

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- *Edit distance*: global alignment with $O(n^2)$ space and time consumption
- But: Model for deletition unrealistic
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$$\begin{array}{cccccccc} \_ & \_ & \_ & B & L & O & E & D \\ S & A & U & B & L & O & E & D \end{array}$$

  - Solution in $O(n^3)$ time or $O(n^2)$ affine

$O(n^2)$ space consumption might be problematic:

Hirschberg algorithm:

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Hirschberg algorithm:

- Divide-and-conquer approach

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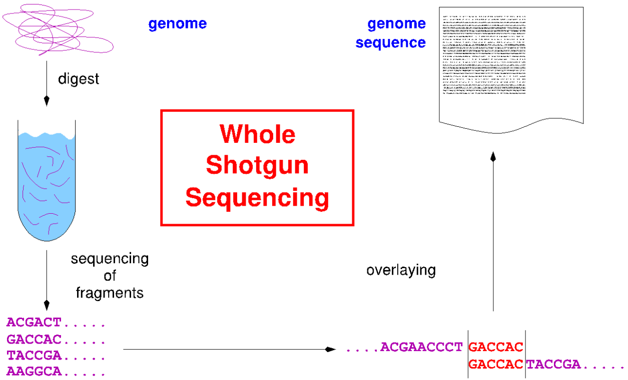
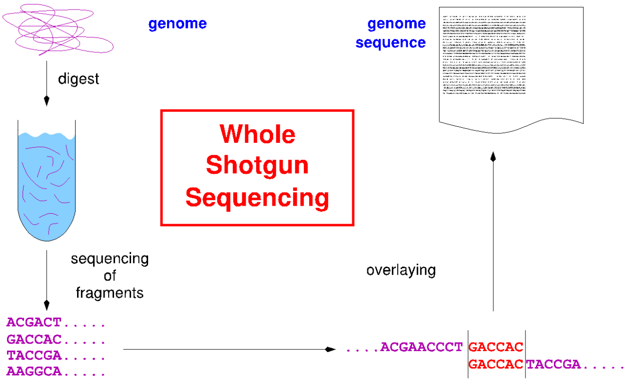
Hirschberg algorithm:

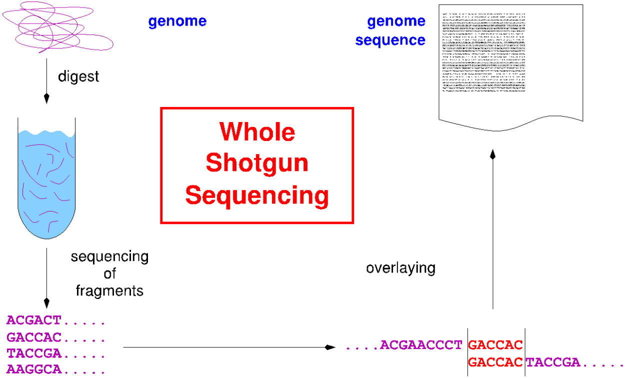
- Divide-and-conquer approach
- $O(n)$ space and $O(n^2)$ time consumption



genome
digest
sequencing of fragments
ACGACT.....
GACCAC.....
TACCGA.....
AAGGCA.....
Whole Shotgun Sequencing
genome sequence
overlaying
....ACGAACCCT GACCAC
GACCAC TACCGA.....



- Sequencing: $O(n^2)$ is too much



- Sequencing: $O(n^2)$ is too much
- Index: suffixtree, suffixarray, burrow-wheeler-transform

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Dynamic programming**

  [Wik] Dynamic programming
  https://en.wikipedia.org/wiki/Dynamic_programming

- **Edit distance**

  [Wik] Levenshtein distance
  https://en.wikipedia.org/wiki/Levenshtein_distance