

# Algorithms and Datastructures

## Cache Efficiency, Divide and Conquer

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Algorithms and Datastructures, March 2018

## Cache Efficiency

- Introduction

- Cache Organization

## Divide and Conquer

- Introduction

### Background:

- Up to now we always counted the **number of operations**
- Assuming this is a good measure for the runtime of a algorithm/tool
- Today we will see examples where this is not suitable

### Example:

- We sum up all elements of a field  $a$  of size  $n$  in ...
  - natural order:

$$\text{sum}(a) = a[1] + a[2] + \dots + a[n]$$

- random order:

$$\text{sum}(a) = a[21] + a[5] + \dots + a[8]$$

### Python:

```
def init(size):  
    """Creates the dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

### Python:

```
def run(param):  
    """Processes the dataset."""  
  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```



Figure: Summing elements in linear order

```
def init(size):  
    """Creates a randomly ordered dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```





Figure: Summing elements in random order

### Conclusion:

- The number of operations are identical for both algorithms
- Accessing elements in random order takes a lot longer (Factor 10)
- The costs in terms of memory access are very different



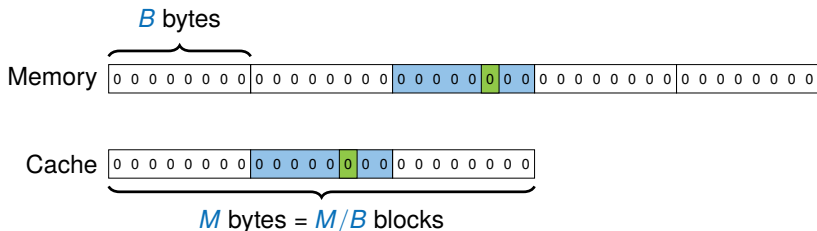
### Principle / organization:

- Accessing one byte of the main memory takes  $\approx 100$  ns
- Accessing one byte of (L1-)cache takes  $\approx 1$  ns
- Accessing one or more byte/s of main memory loads a whole block  $\approx 100$  B into the cache
- As long as this block is in the cache, it is not necessary to access the memory for bytes of this block



### Cache organization:

- The (L1-)cache can hold multiple memory blocks
  - Cache lines  $\approx 100$  kB
- If the capacity is reached unused blocks are discarded
  - Least recently used (LRU)
  - Least frequently used (LFU)
  - First in first out (FIFO)
- Details of discarding are not the topic for today



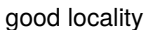
### Terminology:

- The system consists of slow and fast memory
- The **slow memory** is divided in **blocks of size  $B$**
- The **fast cache** has size  $M$  and can store  $M/B$  blocks
- If data is not in fast memory, the corresponding block is loaded into the **cache**



### Terminology:

- The program defines which blocks are held in the **cache**
- We use the number of **block operations** as runtime estimation
- We ignore runtime costs of cache accesses / management



bad locality

- **Best case:** 1 block operation  $\rightarrow$  good locality
- **Worst case:**  $B$  block operations  $\rightarrow$  bad locality

### Additional factors:

- The following settings change only a small constant factor in number of block operations
  - Partitioning of the slow memory into blocks
  - Regardless of the block size: 1 Bytes or 4 Bytes or 8 Bytes

### Note:

- If the input size is smaller than  $M$  we load the complete data chunk directly into the cache
- Cache handling is only interesting when the input size is greater than  $M$



**Typical values:** (Intel© i7-4770 Haswell, WD© Blue 2TB)

- CPU L1 Cache:  $B = 64\text{ B}$ ,  $M = 4 \times (32\text{ kB} + 32\text{ kB})$
- CPU L2 Cache:  $B = 64\text{ B}$ ,  $M = 4 \times 256\text{ kB}$
- CPU L3 Cache:  $B = 64\text{ B}$ ,  $M = 8\text{ MB}$
- Disk Cache:  $B = 64\text{ kB}$ ,  $M = 64\text{ MB}$ 
  - Many operating systems use free system memory as disk cache

### Terminology:

- Block loads on CPU-cache are called **cache misses**
- Block operations on disk-cache are called **IOs**  
(input / output operations)
- These also fall under the term **cache efficiency** or **IO efficiency**

### Example 1 - Linear order:

- We sum up all elements in **natural order**

$$\text{sum}(a) = a[1] + a[2] + \dots + a[n]$$

- The number of block operations is  $\text{ceil}(\frac{n}{B})$



Figure: Good locality of sum operation

### Example 2 - Random order:

- We sum up all elements in **random order**

$$\text{sum}(a) = a[21] + a[5] + \dots + a[8]$$

- The number of block operations is  $n$  in the **worst case**
- This leads to a runtime factor difference of  $B$



Figure: Bad locality of sum operation

**Generally the factor is substantially  $< B$**

- Even with a **random order** we access 4 neighboring bytes at once per `int` (`int32_t`)
- The next element might already be loaded in the cache
- If **not**  $n \gg M$  this might occur with a high probability

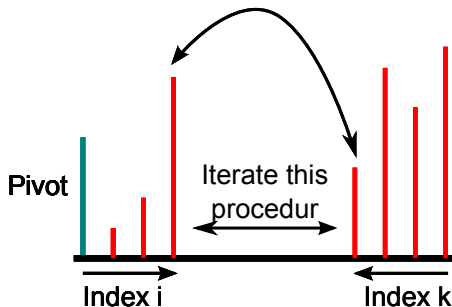
### QuickSort:

- **Strategy:** Divide and conquer
- Divide the data into two parts where the “left” part contains all values  $\leq$  those in the right part
- Choose one element (e.g the first one) as “pivot”-element
- Ideally both parts are the same size
- Both parts are sorted recursively



Figure: *QuickSort* with pivot-element

- **At start:** Pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- Do required changes *in place*



- **End point:**  $k$  is left to left-most element greater than pivot  
*swap position 0 (pivot) with  $k$  (smaller than pivot)*

### Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```



```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

### Number of operations for Quicksort:

- Let  $T(n)$  be the runtime for the input size  $n$

### Assumptions:

- Fields are always separated perfectly in the middle
- $n$  is a power of two and recursion depth is  $k = \log_2 n$

$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left( A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$



Figure: Locality of quicksort

- Let  $IO(n)$  be the number of **block operations** for input size  $n$
- Assumptions as before but recursion depth is  $k = \log_2 \frac{n}{B}$

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in \mathcal{O}\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

### Concept:

- **Divide** the problem into smaller subproblems
- **Conquer** the subproblems through recursive solving.  
If subproblems are small enough solve them directly
- **Connect** all solutions of the subproblems to a solution of the full problem
- **Recursive** application of the algorithm to ever smaller subproblems
- **Direct** solving of sufficiently small subproblems

- Function `solve` for solving a problem of size  $n$

```
def solve(problem):  
    if n < threshold:  
        return solution # solve directly  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
    return S1 + S2 + ... + Sk
```

### Divide and Conquer:

- Can help with conceptual hard problems
- **Solution** of the trivial problems has to be known
- **Dividing** in subproblems has to be possible
- **Combination** of solutions has to be possible



### Features:

- Realization of **efficient solutions**
  - If trivial solution is  $\in O(1)$
  - And separation / combination of subproblems is  $\in O(n)$
  - And the number of subproblems is limited
  - The runtime is  $\in O(n \cdot \log n)$
- Suitable for parallel processing
  - Subproblems are **independent** of each other
  - Only needed input for each subproblem has to be known

### Definition of the trivial case:

- Smaller subproblems are elegant and simple
- Otherwise the efficiency will be improved if relative big subproblems can be solved directly
- Recursion depth should not get too big (stack / memory overhead)

### **Division in subproblems:**

- Choosing the number of subproblems and the concrete allocation can be demanding

### **Combination of solutions:**

- Typically conceptual demanding

### Example - Maximum Subtotal Input:

- Sequence  $X$  of  $n$  integers

### Output:

- Maximum sum of related subsequence and its index boundary

|       |    |     |    |    |     |    |    |     |     |    |
|-------|----|-----|----|----|-----|----|----|-----|-----|----|
| Index | 0  | 1   | 2  | 3  | 4   | 5  | 6  | 7   | 8   | 9  |
| Value | 31 | -41 | 59 | 26 | -53 | 58 | 97 | -93 | -23 | 84 |

**Output:** Sum: 187, Start: 2, End: 6

### Application:

- Maximum profit of buying and selling shares



Figure: Stock value over time

### Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

### Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

### Upper bound:

- Three interleaved loops
- Each loop with runtime  $O(n)$
- Algorithm runtime of  $O(n^3)$



### Lower bound:

Table: Operations

| $i$                    | Additions              | $j$                    |
|------------------------|------------------------|------------------------|
| $\frac{n}{3} \in O(n)$ | $\frac{n}{3} \in O(n)$ | $\frac{n}{3} \in O(n)$ |

- We iterate at least  $\frac{n}{3}$  values for  $i$
- For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$
- For each  $j$  we have at least  $\frac{n}{3}$  additions
- We need at least  $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$  steps

### Runtime:

- With  $T(n) \in O(n^3)$  and  $T(n) \in \Omega(n^3)$  we know:

$$T(n) \in \Theta(n^3)$$

- It is hard to solve the problem in a worse way ...

### Current approach:

- Calculating the sum for range from  $i$  to  $j$  with loop

$$S_{i,j} = X[i] + X[i+1] + \dots + X[j]$$

### Better approach:

- Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i+1] + \dots + X[j] + X[j+1]$$

$$S_{i,j+1} = S_{i,j} + X[j+1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

### Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

■ Runtime  $\in O(n^2)$

### Divide and Conquer:



### Divide and Conquer Idea to solve:

- Split the sequence in the middle
- Solve left half of the problem
- Solve right half and combine both solutions into one
- Maximum might be located in **left half (A)** or **right half (B)**
- Problem: Maximum can **overlap the split**
- To solve this case we have to calculate **rmax** and **lmax**
- The overall solution is the **maximum of A, B and C**

### Principle - Divide and Conquer:

- Small problems are solved directly:  $n = 1 \Rightarrow \text{max} = X[0]$
- Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions **A** and **B** are returned
- To determine subsolution **C**, **rmax** and **lmax** for the subproblems are computed
- The overall solution is the **maximum of A, B and C**

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for cornercase C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.



## ■ Caching

[Wik] [Cache](https://en.wikipedia.org/wiki/Cache)

`https://en.wikipedia.org/wiki/Cache`