# Algorithms and Datastructures

## Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Algorithms and Datastructures, January 2017

# Structure

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- Worst case: $d \in O(n)$, keys are inserted in ascending / descending order $(20, 19, 18, \ldots)$

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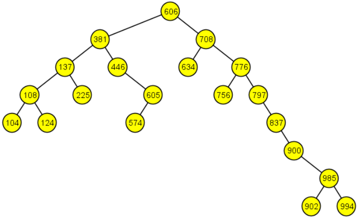


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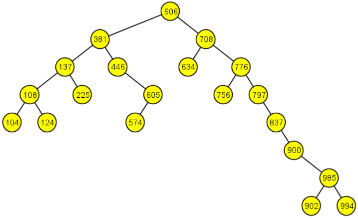


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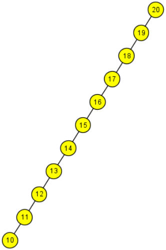


Figure: Binary search tree with descending insert [Gna]

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  - Used in C++ `std::map` and Java `SortedMap`

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- We can perform all basic operations in $O(\log n)$

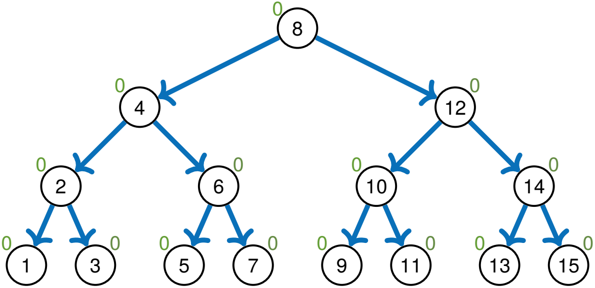


Figure: Example of an AVL-Tree

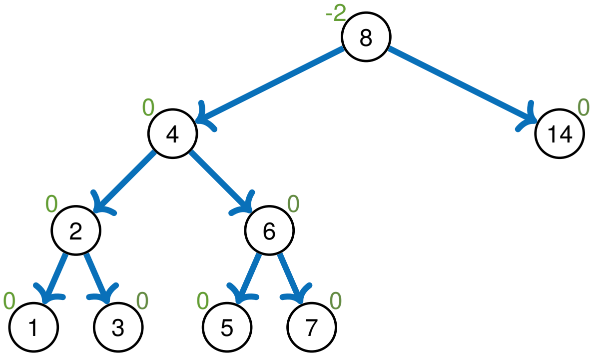


Figure: **Not** an AVL-Tree

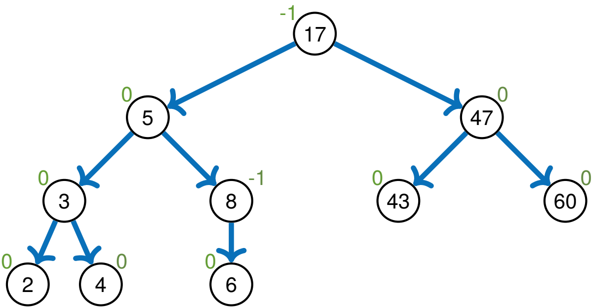


Figure: Another example of an AVL-Tree

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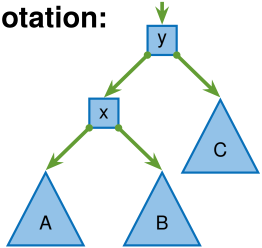


Figure: Before rotating

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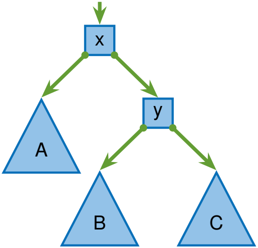


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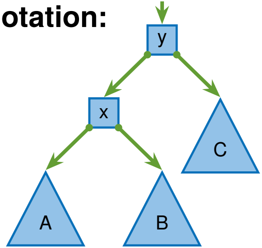


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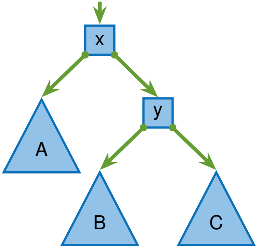


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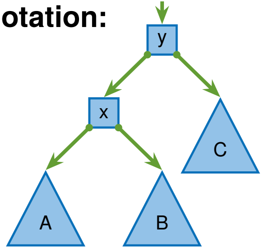


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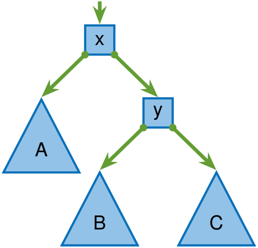


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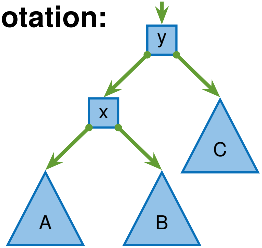


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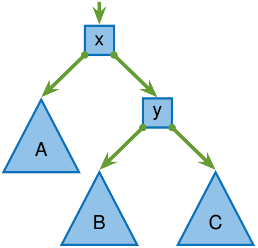


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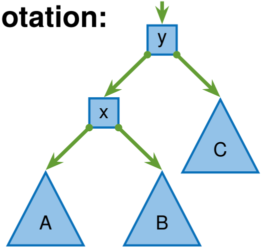


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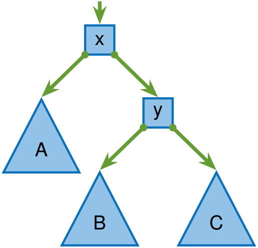


Figure: After rotating

- Central operation of rebalancing
- After rotation to the right:
  - Subtree $A$ is a layer higher and subtree $C$ a layer lower
  - The parent child relations between nodes $x$ and $y$ have been swapped

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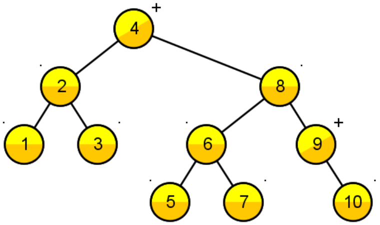


Figure: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

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- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in $O(\log n)$
- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b)-trees

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- So we have space for elements on an `insert` and balance operation

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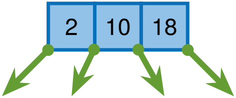

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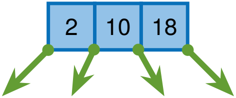


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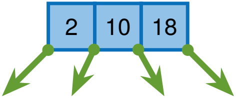


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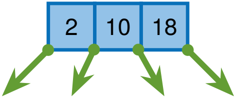


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- We require: $a \geq 2$ and $b \geq 2a - 1$

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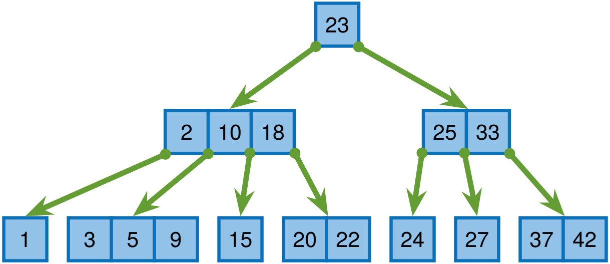


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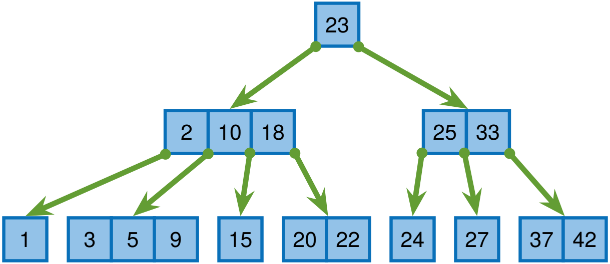


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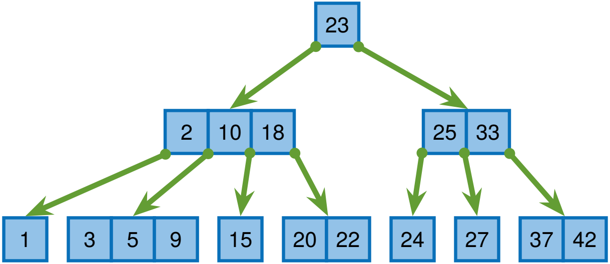


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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

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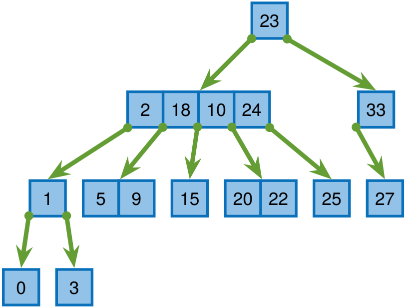


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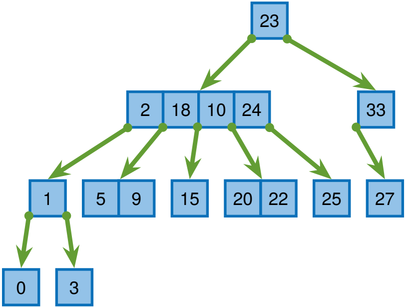


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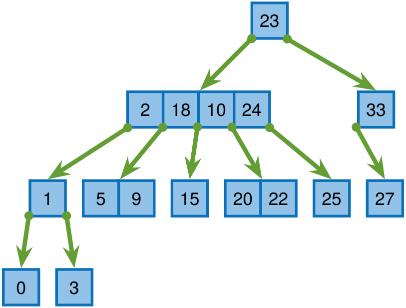


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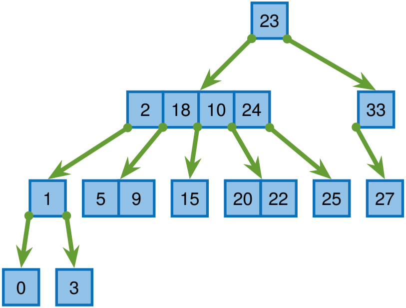


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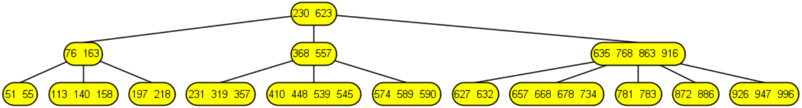


Figure: (3,5)-Tree [Gna]

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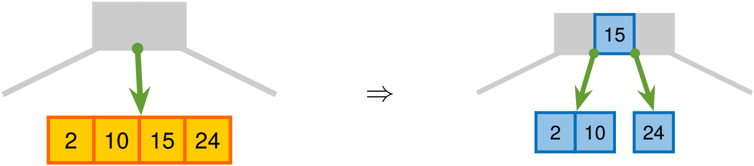


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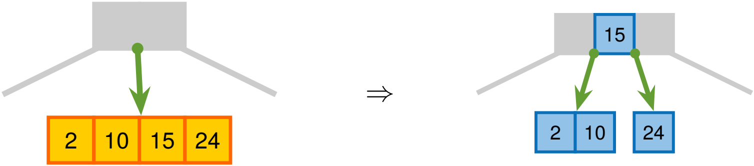


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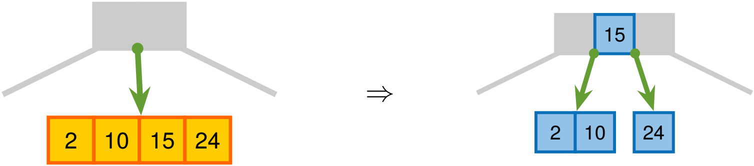


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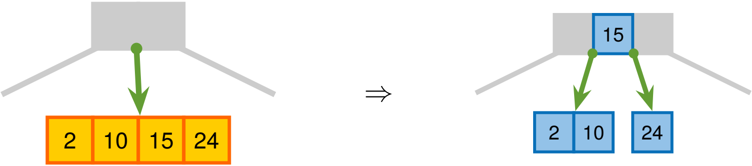


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- Thats why we have the limit $b \geq 2a - 1$

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- If the degree is higher than $b+1$ we split the node
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  (The tree is now one level deeper)

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  - Replace the element with its successor and delete the successor from the leaf

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  - The successor is always contained by a leaf
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- **Attention:** The leaf might be too small (degree of $a - 1$)
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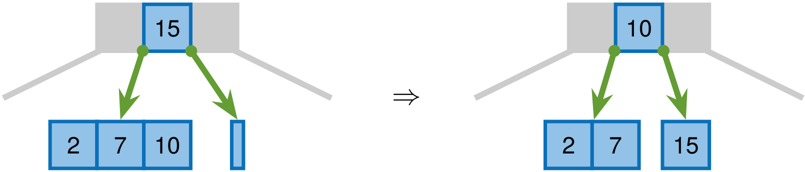


Figure: Borrow an element

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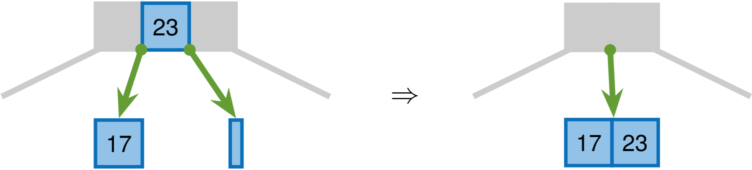


Figure: Merge two nodes

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- Now the parent node can be of degree $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
  - Remove the root
  - Define sole child as new root
  - The tree shrinks by one level

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- All operations in $O(d)$ with $d$ being the depth of the tree

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- `lookup` always takes $\Theta(d)$
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- **Worst case:** split or merge all nodes on path up to the root
- Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$

**Counter example (2,3)-Tree:**

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- Before executing `delete(11)`

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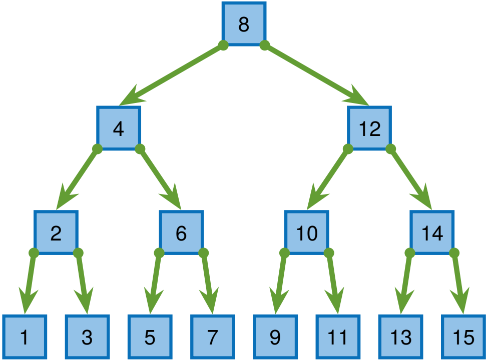


Figure: Normal (2,3)-Tree

**Counter example (2,3)-Tree:**

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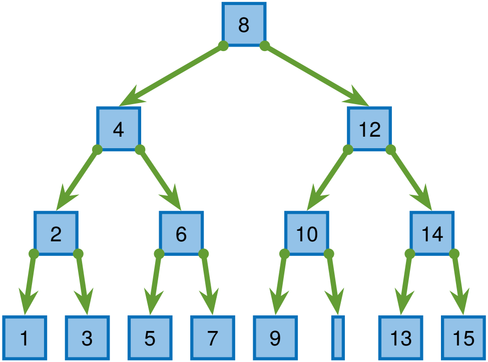


Figure: (2,3)-Tree - Delete step 1

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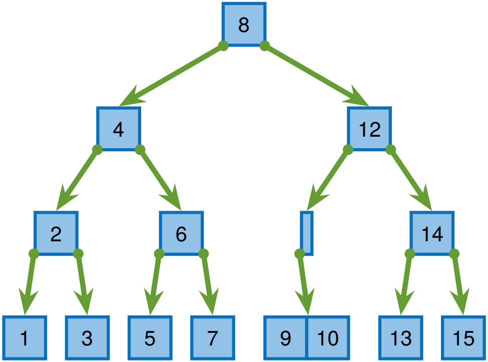


Figure: (2,3)-Tree - Delete step 2

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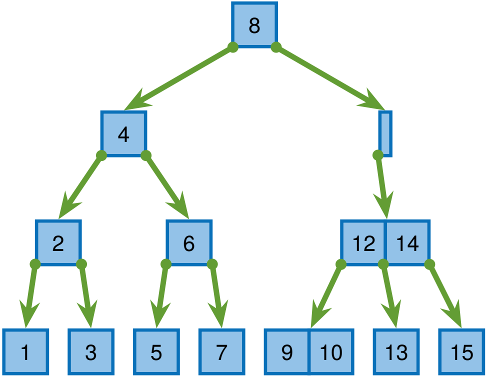


Figure: (2,3)-Tree - Delete step 3

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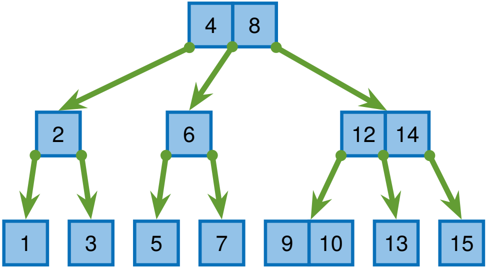


Figure: (2,3)-Tree - Delete step 4

## Counter example (2,3)-Tree:

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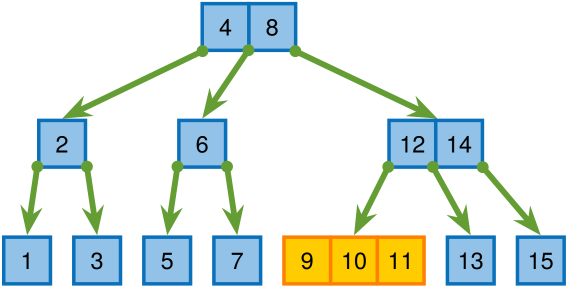


Figure: (2,3)-Tree - Insert step 1

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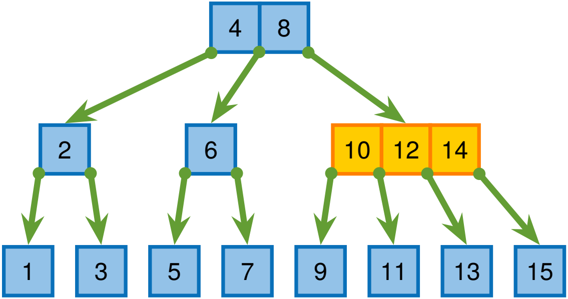


Figure: (2,3)-Tree - Insert step 2

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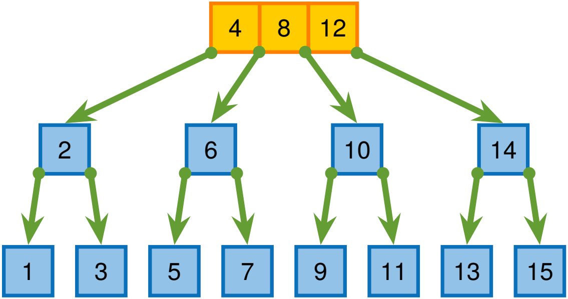


Figure: (2,3)-Tree - Insert step 3

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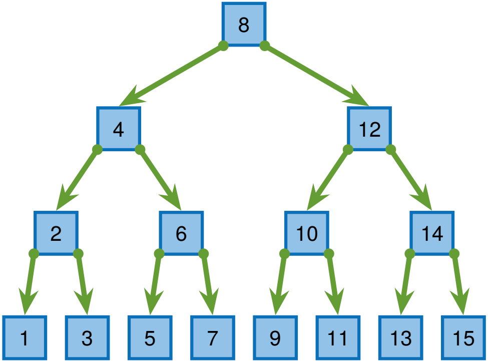
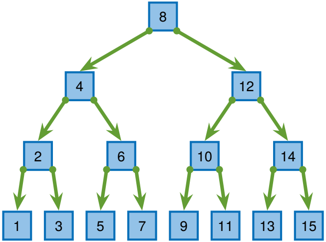



Figure: (2,3)-Tree - Insert step 4

**Counter example (2,3)-Tree:**

- We are exactly where we started

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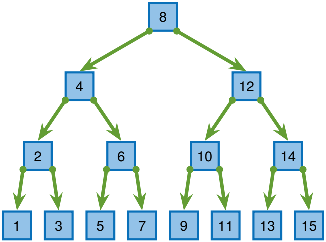
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$

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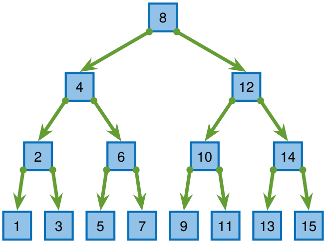
- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$
- We need $b \geq 2a$ instead of $b \geq 2a - 1$

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$\Rightarrow$ **Nodes of degree 3 are stable**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

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**(2,4)-Tree:**

- **Idea:**
  - After an expensive operation the tree is in a stable state
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- Like with dynamic arrays:
  - Reallocation is expensive but it takes some time until the next expensive operation occurs
  - If we overallocate clever we have an amortized runtime of $O(1)$

**Terminology:**

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**Example:**

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- Nodes of degree 3 are highlighted

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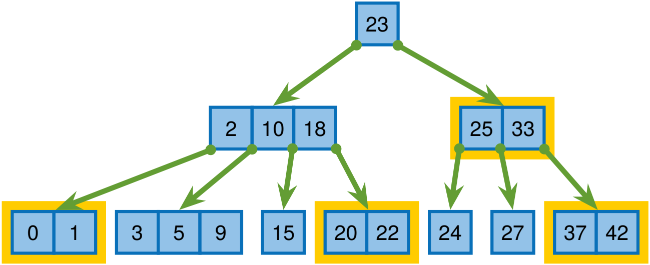


Figure: Tree with potential $\Phi = 4$

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Number of gained stable nodes (degree 3) $\geq -1$

- Each operation has an amortitzed cost of $O(1)$ summing up to $O(n)$ in total

**Case 1:** *i-th* operation is an `insert` operation on a full node

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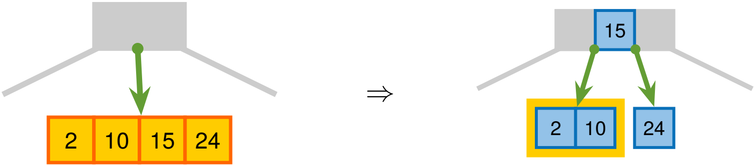


Figure: Splitting a node on `insert`

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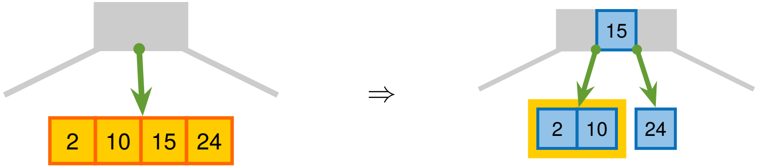


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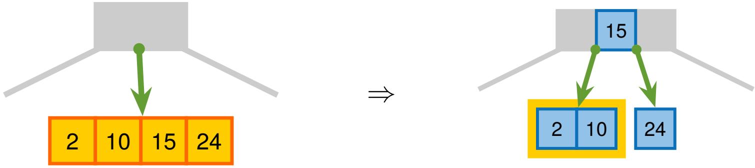


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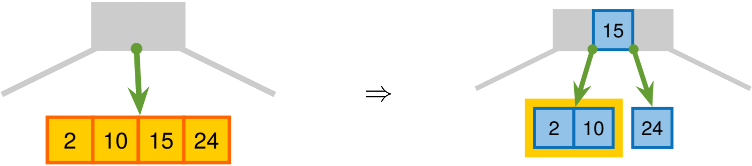


Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
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Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

**Case 2:** *i-th* operation is an `remove` operation

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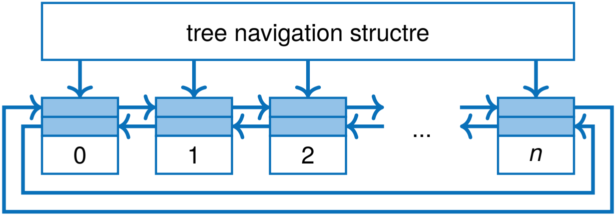


Figure: Tree with doubly linked list

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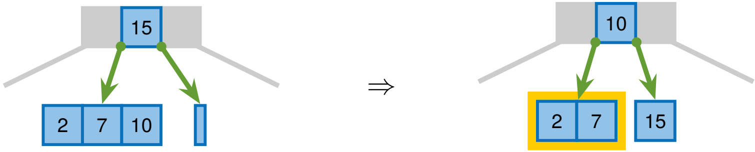


Figure: Case 2.1.1: Borrow an element

**Case 2:** *$i$-th* operation is an `remove` operation

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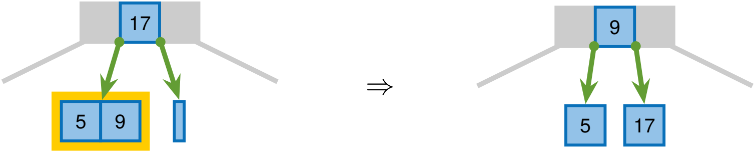


Figure: Case 2.1.2: Borrow an element

**Case 2:** *i-th* operation is an `remove` operation

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- **Case 2.2:** Merging two node

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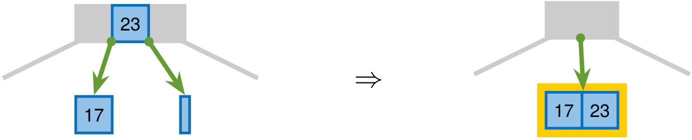


Figure: Merging two nodes

- Potential rises by one

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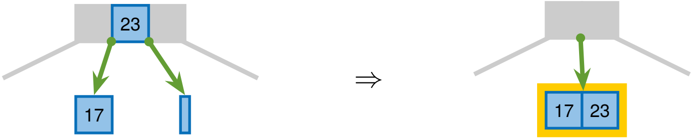


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation

**Case 2:** *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

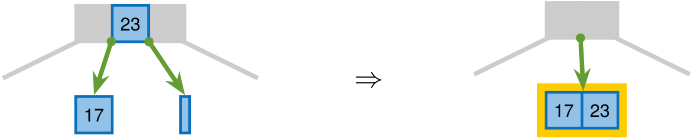


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree $> 2$ or a node of degree 2, which can borrow from a neighbour

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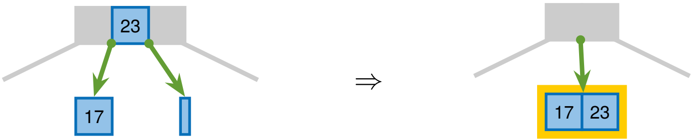


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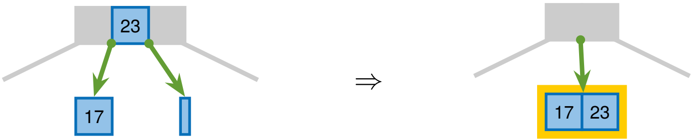


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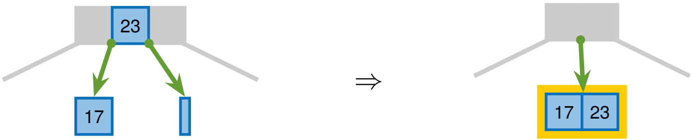


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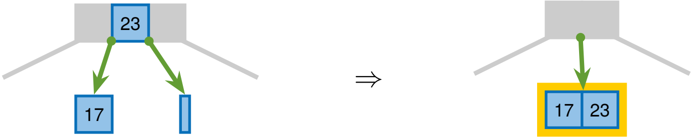


Figure: Merging two nodes

- The potential rises by *m*
- If the “stop-node” is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

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**Lemma:**

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- With that we can conclude:

$$\sum_{i=0}^{n} c_i \in O(n)$$

**Proof:**

$$\sum_{i=0}^{n} c_i \leq \underbrace{A \cdot (\Phi_1 - \Phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\Phi_2 - \Phi_1) + B}_{\leq c_2} + \cdots + \underbrace{A \cdot (\Phi_n - \Phi_{n-1}) + B}_{\leq c_n}$$

$$= A \cdot (\Phi_n - \Phi_0) + B \cdot n \qquad \text{| telescope sum}$$

$$= A \cdot \Phi_n + B \cdot n \qquad \text{| we start with an empty tree}$$

$$< A \cdot n + B \cdot n \in O(n) \qquad \text{| number of degree 3 nodes}$$

$$< \text{number of nodes}$$

# Structure

**Red-Black Tree:**

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- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

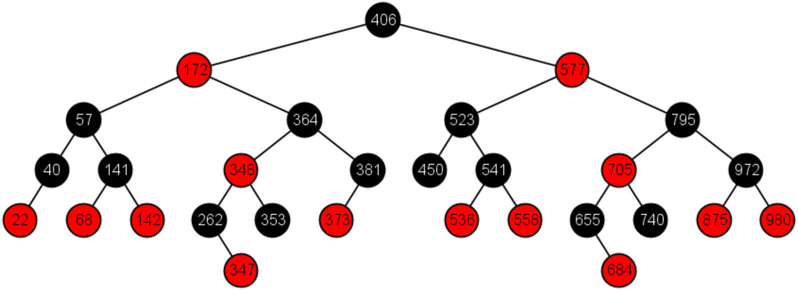


Figure: Example of an red-black-tree [Gna]

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- **AVL-Tree**

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- **(a,b)-Tree**

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