Algorithms and Datastructures Runtime Complexity, Associative Arrays

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Associative Arrays

Introduction
Practical Example
Sorting
Associative Array



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- The runtime does not entirely depend on the size of the problem, but also on the type of input
- This results in:
 - Best runtime: Lowest possible runtime complexity for an input of size n
 - Worst runtime:
 Highest possible runtime complexity for an input of size *n*
 - Average / Expected runtime:
 The average of all runtime complexities for an input of size n

- Input: Field a with n elements $a[i] \in \mathbb{N}$, 0 < a[i] < n, 0 < i < n
- Output: Field a with n elements $a[0] \neq 1$

$$\begin{array}{c} \text{if } a[0] == 0: \\ a[0] = 2 \end{array} \qquad \begin{array}{c} \mathcal{O}(1) \\ \mathcal{O}(1) \end{array} \right\} \qquad \mathcal{O}(1) \\ \text{else:} \\ \text{for i in range}(0, \, n): \\ a[i] = 2 \qquad \begin{array}{c} \mathcal{O}(n) \\ \mathcal{O}(1) \end{array} \right\} \qquad \mathcal{O}(n) \cdot \mathcal{O}(1) \\ = \mathcal{O}(n) \end{array}$$

- Best runtime: $\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1)$
- Worst runtime: $\mathcal{O}(1) + \mathcal{O}(n) = \mathcal{O}(n)$

- The average runtime is determined by the average runtime for all input instances of size *n*
- Every element of a can have n different values $\Rightarrow n \cdot ... \cdot n = n^n$ different input instances of size n a[i] == 1 in n^{n-1} instances
 - **a**[i] != 1 in $n^n n^{n-1} = n^{n-1} \cdot (n-1)$ instances
- Sum of all runtime complexities:

$$\underbrace{n^{n-1} \cdot \mathcal{O}(1)}_{a[i] == 1} + \underbrace{n^{n-1} \cdot (n-1) \cdot \mathcal{O}(n)}_{a[i] != 1}$$

Average runtime:

$$\frac{n^{n-1}+n^{n-1}\cdot(n-1)\cdot n}{n^n}=\frac{1}{n}+n-1\in\mathcal{O}(n)$$

- Input: n digit dual number a
- Output: n digit dual number a + 1
- Runtime of the algorithm is determined by the number of bits getting changed (steps)

- Best runtime: $1 \text{ step} = \mathcal{O}(1)$
- Worst runtime: n steps = $\mathcal{O}(n)$

Table: Binary addition

Digits (n)	Input	Output	Steps
10	1011100100	1011100101	1
4	1011	1100	3
8	11111111	00000000	8

Example 2 - Average Steps



Table: Binary addition with n = 1

Input	Output	Steps
0	1	1
1	0	1

Table: Binary addition with n = 2

Input	Output	Steps
00	01	1
01	10	2
10	11	1
11	00	2

$$\overline{\text{steps}} = \frac{1+1}{2} = 1$$
$$= 2 - \frac{1}{1} = 2 - \frac{1}{2^{n-1}}$$

$$\overline{\text{steps}} = \frac{1+2+1+2}{4} = \frac{3}{2}$$
$$= 2 - \frac{1}{2} = 2 - \frac{1}{2^{n-1}}$$

Example 2 - Average Steps



Table: Binary addition with n = 3

Input	Output	Steps
000	001	1
001	010	2
010	011	1
011	100	3
100	101	1
101	110	2
110	111	1
111	000	3

$$\overline{\text{steps}} = \frac{1 + 2 + 1 + 3 + 1 + 2 + 1 + 3}{8} = \frac{7}{4}$$

$$= 2 - \frac{1}{4} = 2 - \frac{1}{2^{n-1}}$$

$$\Rightarrow \text{Average runtime:}$$

$$\Rightarrow \text{Average runtime:} \\ 2 - \frac{1}{2^{n-1}} \in \mathscr{O}(1)$$





Table: Case analysis for instances of size <i>n</i>			
Input	Output	Instances	Steps
0	1	2 ⁿ⁻¹	1
01	10	2^{n-2}	2
011	100	2^{n-3}	3
:	÷	÷	:
_011111	_100000	2 ¹	n-1
0111111	1000000	2^{0}	n
1111111	0000000	1	n

Average steps:

$$\frac{1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + \dots + (n-1) \cdot 2^1 + n \cdot 2^0 + n \cdot 1}{2^{n-1} + 2^{n-2} + \dots + 2^1 + 2^0 + 1} = \frac{\sum_{i=1}^{n} (i \cdot 2^{n-i}) + n}{\sum_{i=0}^{n-1} 2^i + 1}$$

geometric

$$\sum_{i=0}^{n-1} 2^i + 1 = 2^n = 2^n - 1 + 1 = 2^n$$

Counter:

$$\sum_{i=1}^{n} (i \cdot 2^{n-i}) + n^{a=2a-a} 2 \sum_{i=1}^{n} (i \cdot 2^{n-i}) - \sum_{i=1}^{n} (i \cdot 2^{n-i}) + n$$

$$= 1 \cdot 2^{n} + 2 \cdot 2^{n-1} + 3 \cdot n^{n-2} + \dots + (n-1) \cdot 2^{2} + n \cdot 2^{1}$$

$$- 1 \cdot 2^{n-1} - 2 \cdot 2^{n-2} - \dots - (n-2) \cdot 2^{2} - (n-1) \cdot 2^{1} - n \cdot 2^{0} + n$$

$$= \underbrace{2^{n} + 2^{n-1} + \dots + 2^{1} + 2^{0}}_{2^{n+1} - 1} - 1 = 2^{n+1} - 2$$

Average steps:

$$\overline{steps} = \frac{\sum_{i=1}^{n} (i \cdot 2^{n-i}) + n}{\sum_{i=0}^{n-1} 2^{i} + 1} = \frac{2^{n+1} - 2}{2^{n}} = 2 - \frac{1}{2^{n-1}}$$

$$\lim_{n \to \infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2 \in \mathcal{O}(1)$$

Structure



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Normal array:

Introduction

$$A = [0 \Rightarrow A_0, 1 \Rightarrow A_1, 2 \Rightarrow A_2, 3 \Rightarrow A_3, \ldots]$$

- Searching elements by index
- Lookup of element with index "3":

$$\Rightarrow A[3] = A_3$$

Associative array:

$$A = \begin{bmatrix} "Europa" \Rightarrow A_0, "Amerika" \Rightarrow A_1, \\ "Asien" \Rightarrow A_2, "Afrika" \Rightarrow A_3, \\ \dots \end{bmatrix}$$

- Searching elements by key
- The keys can be of any type with unique elements
- Lookup of element with key "Afrika":

$$\Rightarrow$$
 A [" A frika"] = A 3

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Practical Example



Table: Country data query from http://geonames.org

ISO	ISO3	Country	Continent	
AD	AND	Andorra	EU	
ΑE	ARE	United Arab Emirates	AS	
AF	AFG	Afghanistan	AS	
AG	ATG	Antigua and Barbuda	NA	
ΑI	AIA	Anguilla	NA	
AL	ALB	Albania	EU	
AM	ARM	Armenia	AS	
AO	AGO	Angola	AF	
AQ	ATA	Antarctica	AN	
÷	÷	:	:	٠.



Associative Arrays Practical Example



Task: How many countries belong to each continent?

- We are interested in column 2 (country) and 3 (continent)
- There are two typical ways to solve this:
 - Using Sorting
 - Using an associative array

Structure



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Idea using sorting:

- We sort the columns 2 and 3 by continent, so that all countries with the same continent are grouped in one block
- We count the size of the blocks

Disadvantage:

- Runtime of $\Theta(n \log n)$
- We have to iterate the array twice (sort and then count)

Advantage:

Easy to implement (even with simple linux / unix commands)

Input:

- The data is saved as tab seperated text (countryInfo.txt)
- Comments begin with a hash sign #

Commands:

```
■ grep: Selects a specific set of lines (filter by ...)
grep -v '^#' countryInfo.txt
```

-v: not

^#: # at start of line

cut: Selects specific columns of each line (tab separated)

cut -f5,9

-f5,9: columns 5 and 9 (columns 2, 3 of Table 6)



Commands:

sort: Sorts lines by a key

```
sort -t ' '-k2,2
```

-t ': Separator: Tab (Insert with CTRL-V TAB)

-k2,2: Key from column 2 to 2

uniq: Finds or counts unique keys

uniq -c

-c: count occurences of keys

■ head: Displays a provided number of lines

head -n30

-n30: Displays the first 30 lines

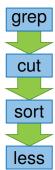
less: Displays the file page wise



Sort countries by continent:

Table: Resulting data

Figure: Data pipeline



Associative Arrays

Practical Example - Sorting With Linux / Unix Commands



Count countries per continent:

Table: Resulting data

58 AF

54 EU

52 AS

42 NA

27 OC

14 SA

5 AN

Figure: Data pipeline



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Idea using associative arrays:

- Take the continent as key
- Use a counter (occurences) or a list with all countries found belonging to this continent as value

Advantage:

■ Runtime $\mathcal{O}(n)$, implied we can find an element in $\mathcal{O}(1)$ like in an normal array

Python:

```
# creates a new map (called dictionary)
countries = {"DE" : "Deutschland", \
    "EN" : "England"}
# check if element exists
if "EN" in countries:
    print("Found %s!" % countries["EN"])
# map key "DE" to value 0
countries["DE"] = "Germany"
# delete key "DE"
del countries["DE"]
```

Efficiency:

- Depends on implementation
- Two typical implementations:
 - Hashing: Calculates a checksum of the key and uses as

key of a normal array search: $\mathcal{O}(1) \dots \mathcal{O}(n)$

insert: $\mathcal{O}(1) \dots \mathcal{O}(n)$

delete: $\mathcal{O}(1) \dots \mathcal{O}(n)$

■ (Binary-)Tree: Creates a sorted (binary) tree

search: $\mathcal{O}(\log n) \dots \mathcal{O}(n)$

insert: $\mathcal{O}(\log n) \dots \mathcal{O}(n)$

delete: $\mathcal{O}(\log n) \dots \mathcal{O}(n)$

Table: Map implementions of programming languages

	Hashing	(Binary-)Tree
Python	all dictionaries	
Java	java.util.HashMap	java.util.TreeMap
C++11/14	std::unordered_map	std::map
C++98	gnu_cxx::hash_map	std::map

General

- [CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson. Introduction to Algorithms. MIT Press, Cambridge, Mass, 2001.
- Kurt Mehlhorn and Peter Sanders. [MS08] Algorithms and data structures, 2008. https://people.mpi-inf.mpg.de/~mehlhorn/

ftp/Mehlhorn-Sanders-Toolbox.pdf.

■ Map - Implementations / API

- [Java] Java HashMap
 - https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html
- [Javb] Java TreeMap
 - https://docs.oracle.com/javase/7/docs/api/java/util/TreeMap.html
- [Pyt] Python Dictionaries (Hash table)
 https://docs.python.org/3/tutorial/
 datastructures.html#dictionaries

■ Map - Implementations / API

```
[Cppa] C++ - hash_map
        http://www.sgi.com/tech/stl/hash_map.html
[Cppb] C++ - map
        http://www.sgi.com/tech/stl/Map.html
```