

Algorithms and Datastructures

Open Addressing, Priority Queue

Albert-Ludwigs-Universität Freiburg



**UNI
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Hashing

- Recapitulation
- Treatment of hash collisions
- Open Addressing
- Summary

Priority Queue

- Introduction



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 - Then however, for a fixed set of keys not every hash function is suitable, but only some



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How to rehash?

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 - Expensive but happens not often
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 - Look at **amortized analysis** in the next lecture

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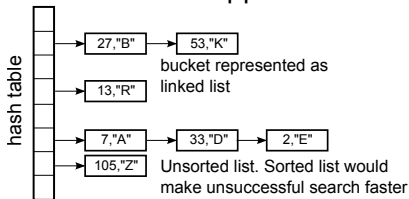


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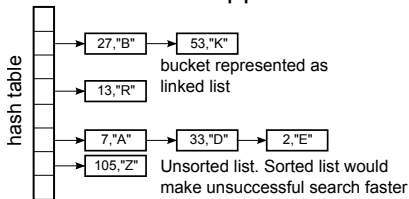
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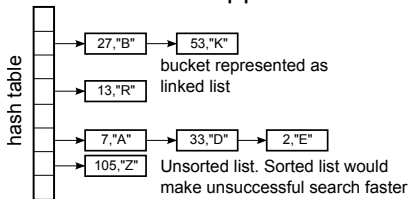
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- Worst case $O(n)$, e.g. table size of 1
- Dynamic number of elements is possible

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- The **probe sequence** determines for each key, in which sequence all hash table entries are searched for a free bucket
 - If a entry is already occupied, then iteratively the **following entry** can be checked. If a free entry is found the element is inserted
 - If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry have been found



Definitions:



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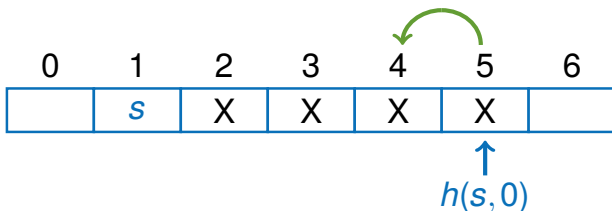
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$g(s,j)$ Probing function for key s with overflow positions

$j \in \{0, \dots, m-1\}$ e.g. $g(s,j)=j$

- The **probe sequence** is calculated by

$$h(s,j) = (h(s) - g(s,j)) \mod m \in \{0, \dots, m-1\}$$



```
def insert(s, value):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
        j += 1  
  
    t[(h(s) - g(s, j)) mod m] \  
      = (s, value)
```

```
def lookup(s):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
        is not None:  
  
        if t[(h(s) - g(s, j)) mod m][0] == s:  
            return t[(h(s) - g(s, j)) mod m]  
  
        j += 1  
  
    return None
```

Hashing

Open Addressing - Linear Probing

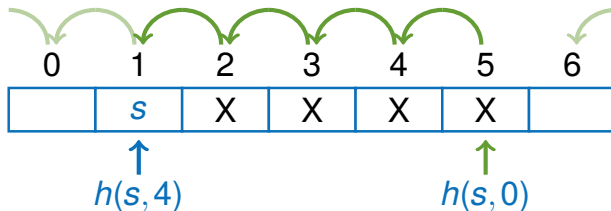


Figure: Linear probe sequence



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- Check the element with lower index: $g(s, j) := j$
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- Check the element with lower index: $g(s, j) := j$
⇒ Hash function: $h(s, j) = (h(s) - j) \bmod m$
- This leads to the following probe sequence

$$h(s), h(s) - 1, h(s) - 2, \dots, \underbrace{0, m-1, m-2, \dots, h(s) + 1}_{\text{clipping}}$$

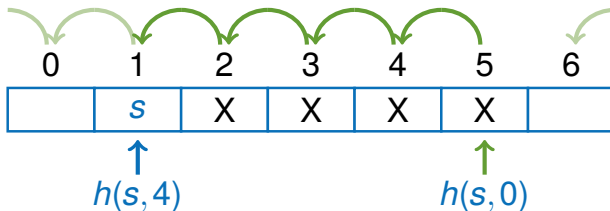


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- Can result in primary clustering



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- Can result in primary clustering
- Dealing with a hash collision will result in a higher probability of hash collisions in close entries



Example:



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- t.insert(12, "A"), $h(12, 0) = 5$

0	1	2	3	4	5	6
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Example:

- Keys: {12, 53, 5, 15, 2, 19}
- Hash function: $h(s, j) = (s \bmod 7 - j) \bmod 7$
- $t.\text{insert}(12, \text{"A"}), h(12, 0) = 5$

0	1	2	3	4	5	6
					12, A	

- $t.\text{insert}(53, \text{"B"}), h(53, 0) = 4$

				53, B	12, A	
--	--	--	--	-------	-------	--

Figure: Probe/Insertion sequence on a hash map



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- t.insert (5, "C"), $h(5, 0) = 5$, $h(5, 1) = 4$, $h(5, 2) = 3$

0	1	2	3	4	5	6
			5, C	53, B	12, A	

Example:

- Hash function: $h(s, j) = (s \bmod 7 - j) \bmod 7$
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- t.insert (15, "D"), $h(15, 0) = 1$

	15, D		5, C	53, B	12, A	
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- t.insert(2, "E"), $h(2, 0) = 2$

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■ Hash function: $h(s, j) = (s \bmod 7 - j) \bmod 7$

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0	1	2	3	4	5	6
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■ t.insert(19, "F"), $h(19, 0) = 5$, $h(19, 1) = 4$,
 $h(19, 2) = 3$, $h(19, 3) = 2$, $h(19, 4) = 1$, $h(19, 5) = 0$

19, F	15, D	2, E	5, C	53, B	12, A	
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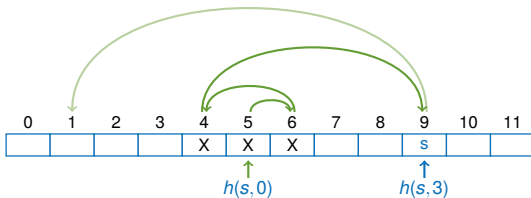


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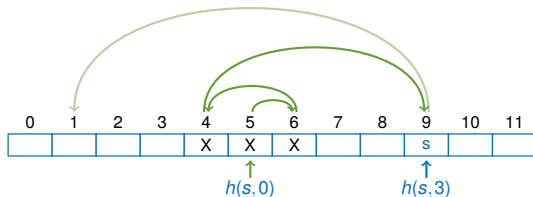


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- This leads to the following probe sequence

$$h(s), h(s) + 1, h(s) - 1, h(s) + 4, h(s) - 4, h(s) + 9, h(s) - 9, \dots$$

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- Alternatively: $h(s, j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \bmod m$
- Problem of secondary clustering
No local clustering anymore, but keys with same hash value have similar probe sequence



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- **Disadvantage:** Hard to implement

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Figure: Double hashing probe sequence

[illegible]

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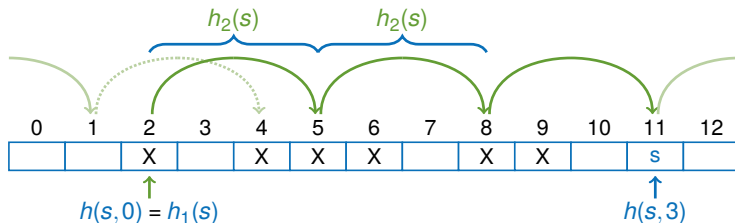


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- Works well in practical use
- This method is an approximation of uniform probing



Example:

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$$h_1(s) = s \mod 7$$

$$h_2(s) = (s \mod 5) + 1$$

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Table: Comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on $h_1(s) \neq h_2(s)$



Figure: Double hashing

Double hashing by Brent:

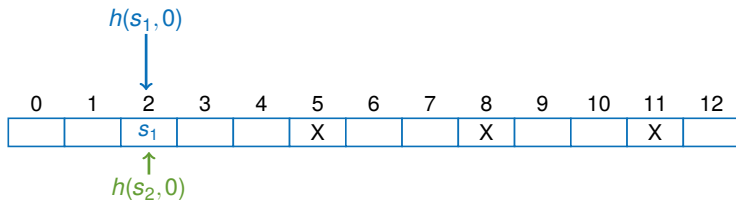


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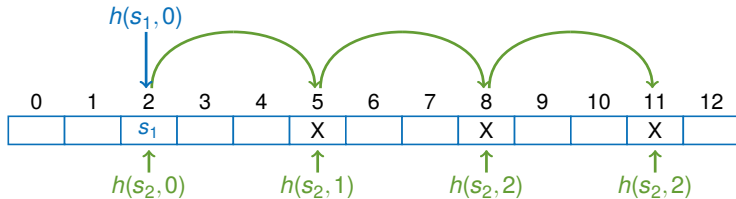


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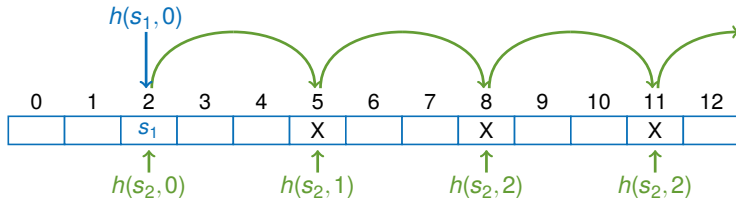


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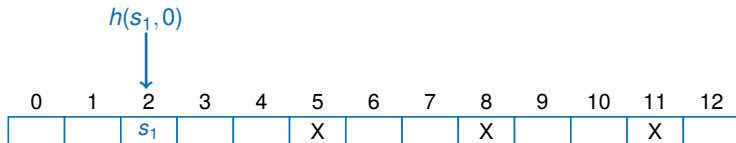


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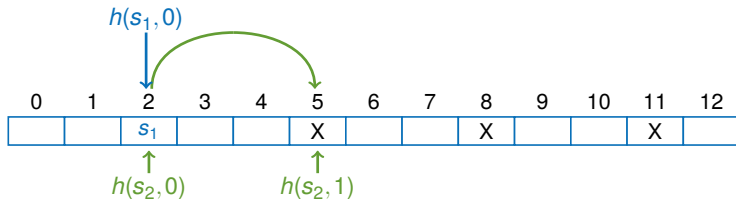


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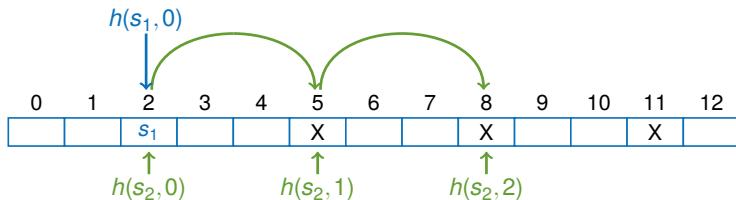


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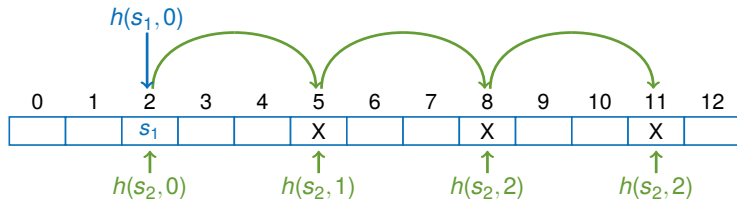


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- The locations $h(s_2, j)$, $j \in \{1, \dots, n\}$ are also occupied
- If we insert s_2 at position $h(s_2, n+1)$ the search will be inefficient



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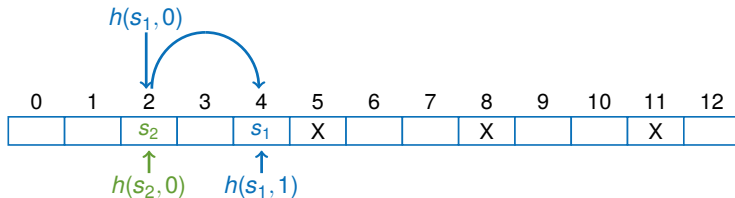


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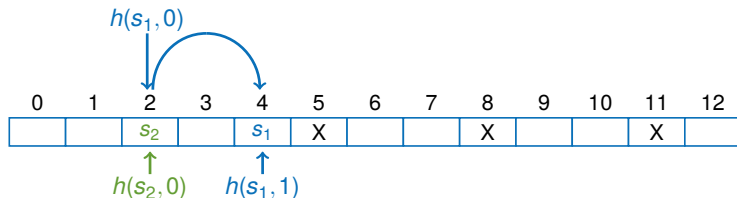


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- Reversed sequence of keys would have been better
- **Brents Idea:**
 - Test if location $h(s_1, 1)$ is free
 - If yes, move s_1 from $h(s_1, 0)$ to $h(s_1, 1)$ and insert s_2 at $h(s_2, 0)$

Idea:

- Motivation: Colliding elements are inserted in the hashtable sorted.
- Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is earlier possible because single probing steps have a fixed length

Implementation:

- Compare both keys if a collision occurs
- Insert the smaller key at p_1
- Search a position based on the diversion order for the bigger key

Example:

- The key 12 is saved at position $p_1 = h(12, 0)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $5 < 12$ we insert the key 5 at position p_1
- For the key 12 we iterate through the sequence

$h(12, 1), h(12, 2), h(12, 3), \dots$



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Implementation:

- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)

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Total costs stay the same, but they are distributed evenly.
Results in approximately similar search times for all elements

Implementation:

- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)
- The key with the bigger search sequence is inserted at p_1
The other key is assigned a new location based on the sequence

Example:

- The key 12 is saved at position $p_1 = h(12, 7)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $j_1 < j_2$ ($0 < 7$) the key 12 stays at position p_1
- For the key 5 we iterate through the sequence

$$h(5, 1), h(5, 2), h(5, 3), \dots$$

Problem:

- The key s_1 is inserted at position p_1
- The key s_2 returns the same hash value, but is inserted at position p_2 because of the probing order
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Solution:

- **Remove:** Elements are marked as removed, but not deleted
- **Inserting:** Elements marked as removed will be overwritten

Hashing

Recapitulation

Treatment of hash collisions

Open Addressing

Summary

Priority Queue

Introduction

Bucket as linked list: (dynamic, number of elements variable)

- Save colliding elements as linked list

Open hashing: (static, number of elements fixed)

- Determine a probe sequence, permutation of all hash values
- Linear, quadratic probing:
 - Easy to implement
 - Raise the probability of collisions because probing order does not depend on the key

Open hashing: (static, number of elements fixed)

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- Uniform probing, double hashing:
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 - Avoids clustering of elements

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Improving efficiency: (Brent, Ordered Hashing)

- Improve search efficiency by sorting colliding insertions
 - Abortion of unsuccessful search
 - Search sequence length balancing



Hashing:

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- Efficient for dictionary operations:

Insert: $O(1) \dots O(n)$

Search: $O(1) \dots O(n)$

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- Using a hash function to find the position (hash value) in the hash table
- Hash function, size of the hash table and strategy to avoid hash collisions influence the efficiency of the datastructure

Hashing

- Recapitulation

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Priority Queue

- Introduction



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- There is a total order (like \leq) defined on the keys



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- Argument of `changeKey` and `remove` operations
 - There is no **quick-access** to a element in the queue
 - That's why `insert` and `getMin` return a reference (handle, accessor object)
 - `changeKey` and `remove` take this reference as argument
 - Therefore each element has to store its current position in the heap.

```
from queue import PriorityQueue
```

```
q = PriorityQueue()
```

```
e1 = (5, "A") # element with priority 5
```

```
q.put(e1); # insert element e1
```

```
# remove and return the lowest item
```

```
e2 = q.get()
```

Example 1:

- Calculation of the sorted union of k sorted lists (multi-way merge or k -way merge)

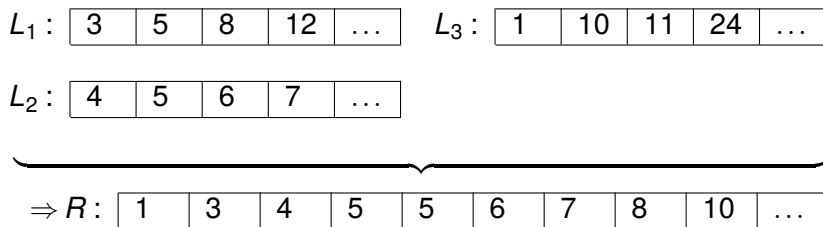


Figure: 3-way merge



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Example 2:

- For example Dijkstra's algorithm for computing the shortest path (\leftarrow following lecture)
- Among other applications it can be used for sorting

Priority Queue

Implementation



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Idea:

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- Save elements as tuples in a binary heap

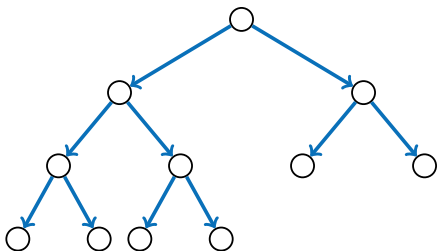


Figure: Heap with 11 nodes

Idea:

- Save elements as tuples in a binary heap
- Summary from lecture 1 (*HeapSort*):
 - Nearly complete binary tree
 - **Heap condition:**
The key of each node \leq the keys of the children



Figure: Heap with 11 nodes

Priority Queue

Implementation

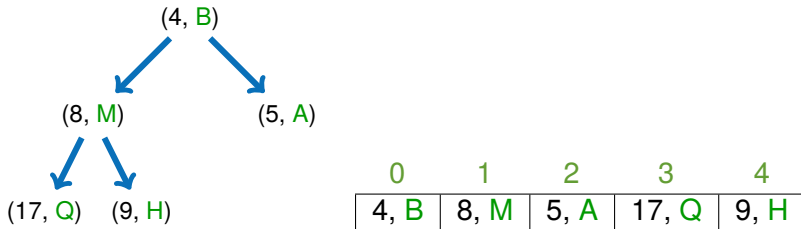


Figure: Min heap stored in array

Priority Queue

Implementation

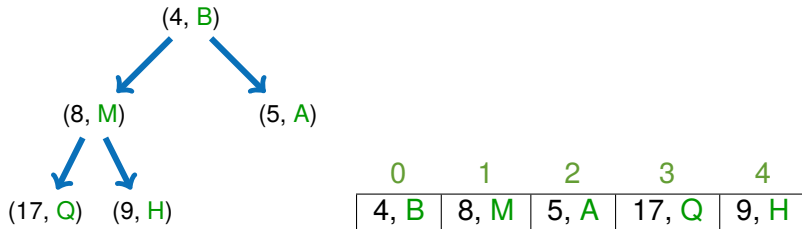


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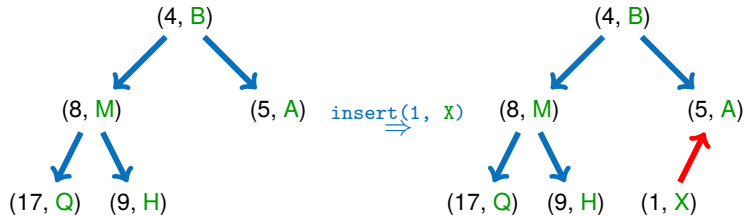


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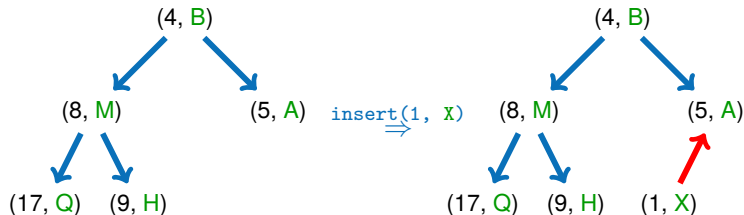
Storing a binary heap:

- Number nodes from top to bottom and left to right starting with 0 and store entries in array
- Children of node i are the nodes $2i+1$ and $2i+2$
- Parent node of node i is $\text{floor}((i-1)/2)$

Inserting an element: `insert(key, item)`

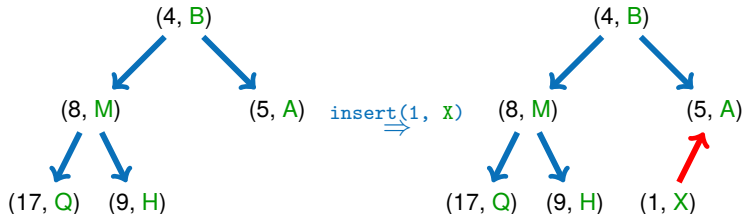


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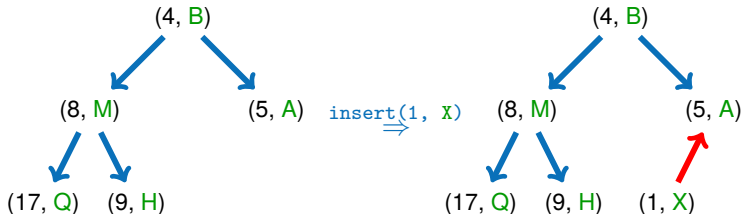
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- Repair **heap condition** \Rightarrow We will see later how to do this

Returning the minimum: `getMin()`



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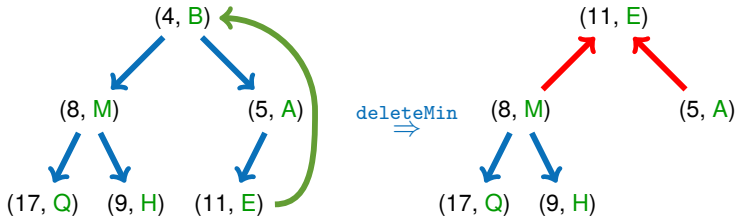
- Else return the first element

Returning the minimum: `getMin()`



- Else return the first element
- If the heap is empty return `None`

Removing the minimum: `deleteMin()`



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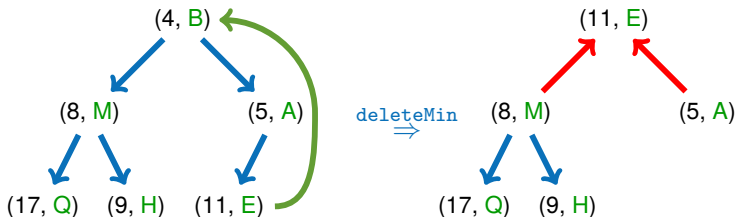
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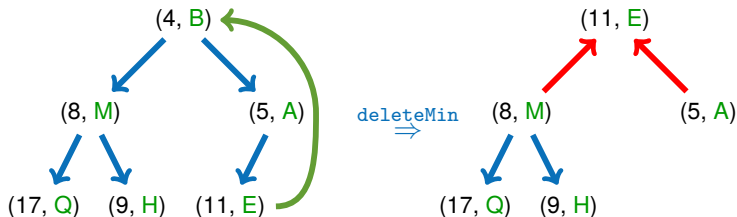
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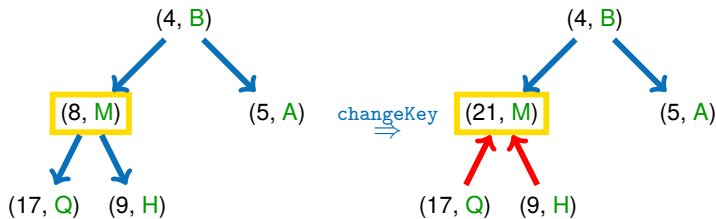
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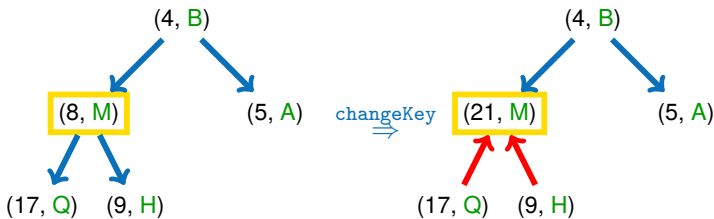
- The element (queue item) is given as argument
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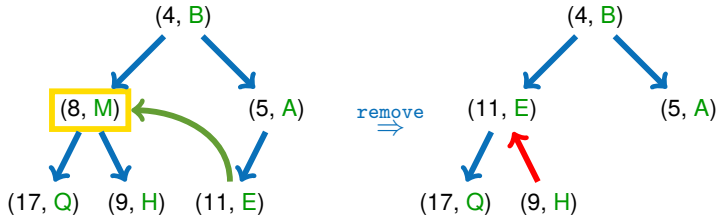
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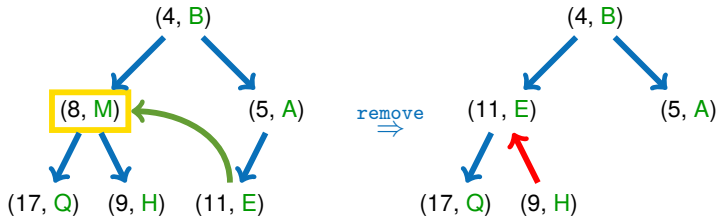


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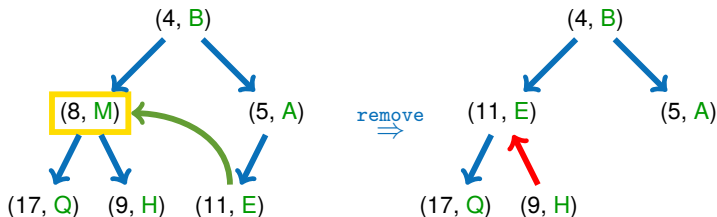
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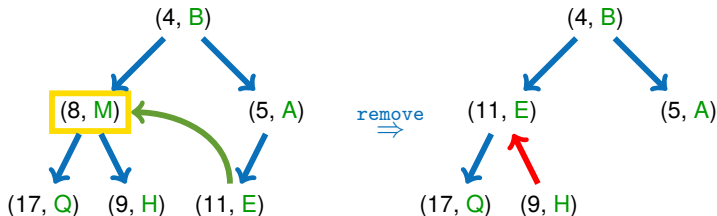
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- Heap conditions can be violated in two directions:
 - Downwards: The key at index i is not \leq than the value of its children
 - Upwards: The key at index i is not \geq than the value of its parent
- We need two repair methods: `repairHeapUp`, `repairHeapDown`

`repairHeapDown:`

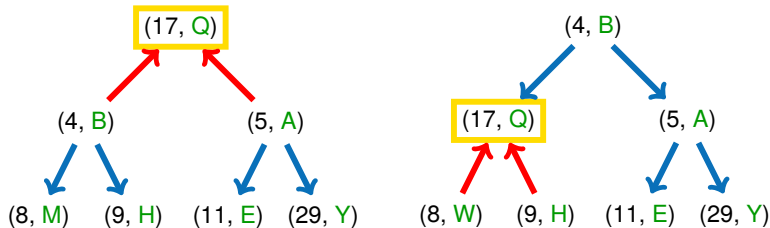


Figure: Repairing the heap downwards

`repairHeapDown:`

- Sift the element until the **heap condition** is valid

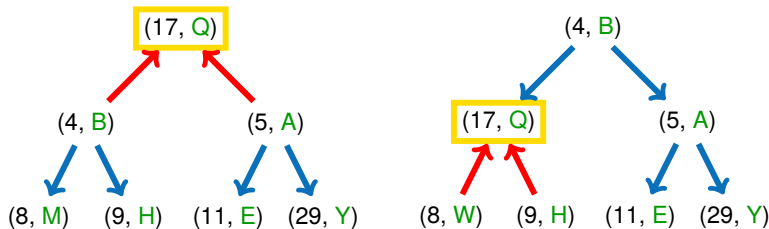


Figure: Repairing the heap downwards

repairHeapDown:

- Sift the element until the **heap condition** is valid
- Change node with child, which has the lower key of both children

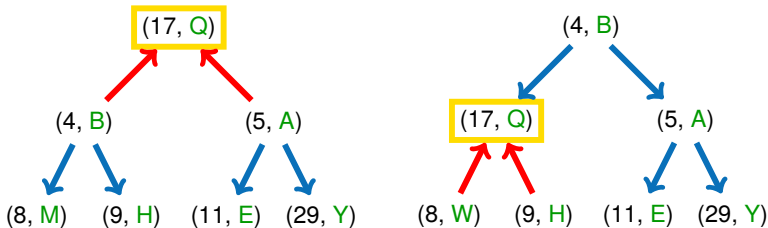


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repairHeapDown:

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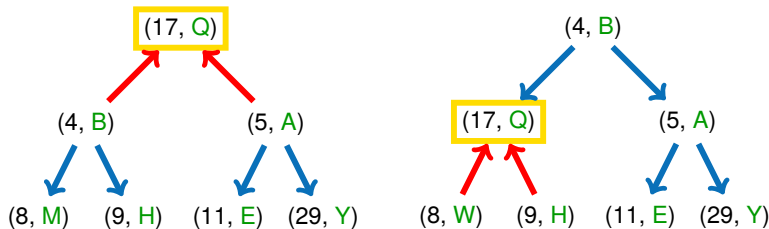


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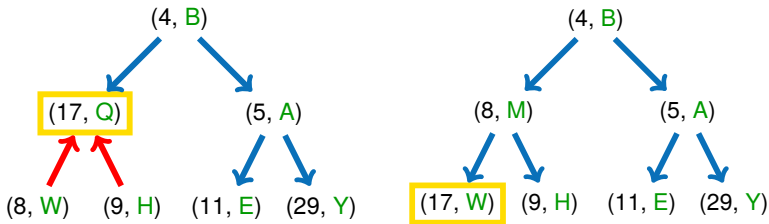


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`repairHeapUp:`



Figure: Repairing the heap upwards

`repairHeapUp:`

- Change node with parent

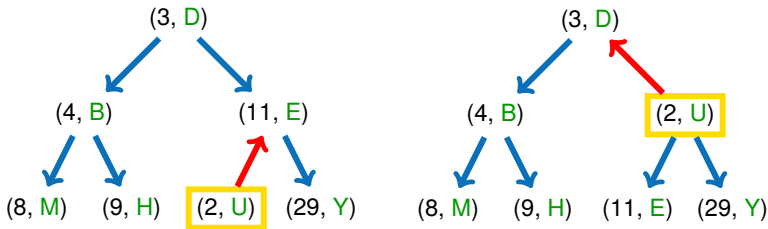


Figure: Repairing the heap upwards

repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node

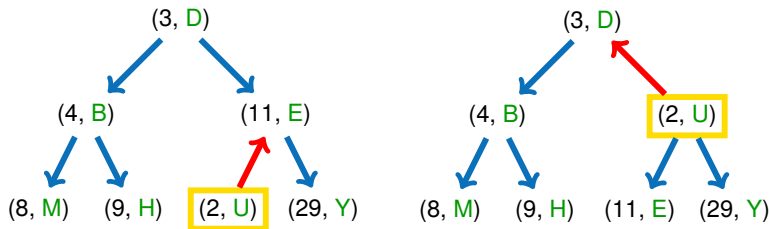


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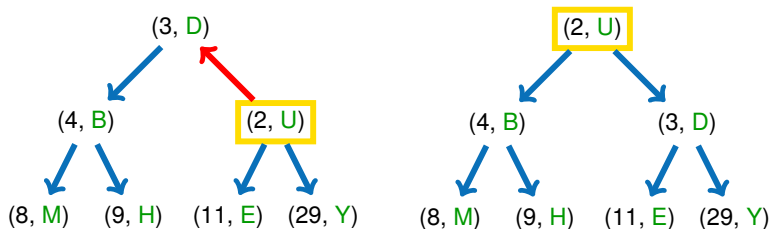


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- **Attention:** For `changeKey` and `remove` the item has to “know” where it is located in the heap
- Remember for `repairHeapUp` and `repairHeapDown`:
Update the index if moving an heap element


```
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```



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Runtime for methods

- **insert**, **deleteMin**, **changeKey**, **remove**:
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- **getMin**: Return the element at index 0: $O(1)$



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- `getMin`, `insert` and `decreaseKey` in amortized time of $O(1)$

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Practical experience:

- The binary heap is simpler: Costs for managing the structure are low
- If the number of elements is relatively small so the difference is negligible
- Example:
 - For $n = 2^{10} \approx 1,000$ is the the `depth` $\log_2 n$ only 10
 - For $n = 2^{20} \approx 1,000,000$ is the `depth` $\log_2 n$ only 20

■ General

- [CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

- [MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Priority Queue - Implementations / API

[Cpp] [C++ - priority_queue](#)

`http:`

`//www.sgi.com/tech/stl/priority_queue.html`

[Jav] [Java - PriorityQueue](#)

`https://docs.oracle.com/javase/7/docs/api/
java/util/PriorityQueue.html`

[Pyt] [Python - PriorityQueue](#)

`https://docs.python.org/3/library/queue.
html#queue.PriorityQueue`