Algorithmns and Datastructures

Runtime analysis Minsort / Heapsort, Induction

Albert-Ludwigs-Universität Freiburg

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science

Algorithmns and Datastructures, March 2016

Structure

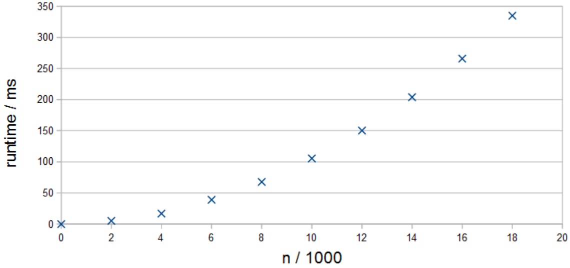
Structure

Feedback from the exercises

Feedback from the lecture

Structure

Runtime analysis - Minsort



How long does the program run?

- In the last lecture we had a schematic
- **Observation:** It is going to be “disproportionaly” slower the more numbers are being sorted
- How can we say more precisely what is happening?

Runtime analysis - Minsort

How can we analyze the runtime?

- Ideally we have a formula which provides the runtime of the program for an specific input

Runtime analysis - Minsort

How can we analyze the runtime?

- Ideally we have a formula which provides the runtime of the program for an specific input
- **Problem:** The runtime is also depending on many other influences, especially:
  - Which kind of computer is the code executed on
  - What is running in the background
  - Which compiler is used to compile the code

Runtime analysis - Minsort

How can we analyze the runtime?

- Ideally we have a formula which provides the runtime of the program for an specific input
- **Problem:** The runtime is also depending on many other influences, especially:
  - Which kind of computer is the code executed on
  - What is running in the background
  - Which compiler is used to compile the code
- **Abstraction 1:** Analyze the number of basic operations, rather than analyzing the runtime

Structure

Basic Operations

Incomplete list of basic operations:

- Arithmetic operation, for example: $a + b$
- Assignment of variables, for example: $x = y$
- Function call, for example: *Sorter.minSort(array)*

Basic Operations

Intuitive:

lines of code

Better:

lines of machine code

Best:

process cycles

Important:

The actual runtime has to be roughly proportional to the number of operations.

Structure

Runtime analysis - Minsort

How many operations does *Minsort* need?

- **Abstraction 2:** We calculate the upper (lower) bound, rather than counting the operations exactly

  Reason: Runtime is approximated by number of basic operations, but we can still infer:

  - Upper bound
  - Lower bound

- **Basic Assmuption:**
  - n is size of the input data (i.e. array)
  - $T(n)$ number of operations for input n

Runtime analysis - Minsort

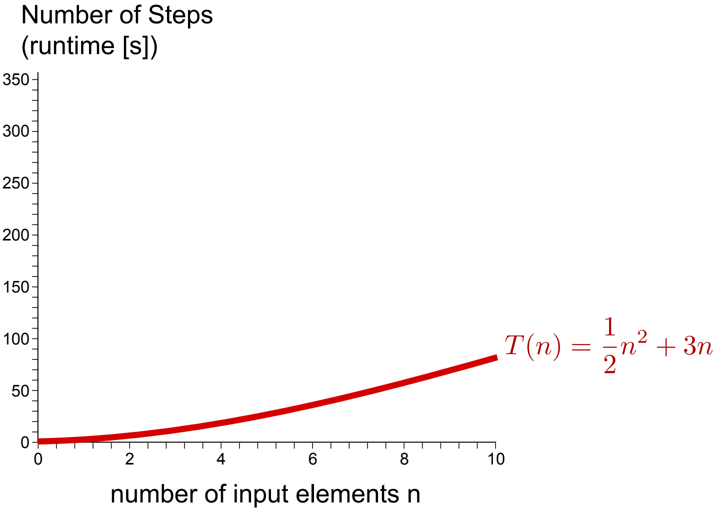
How many operations does *Minsort* need?

- **Observation:** The number of operations depends only on the size n of the array and not on the content!
- **Claim:** There are constants C_1 and C_2 such that:

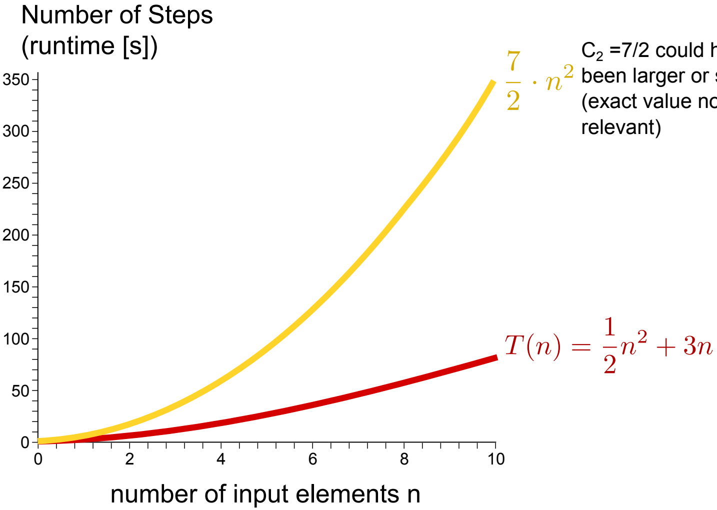
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- This is called "quadratic runtime" (due to n^2)

Runtime Example

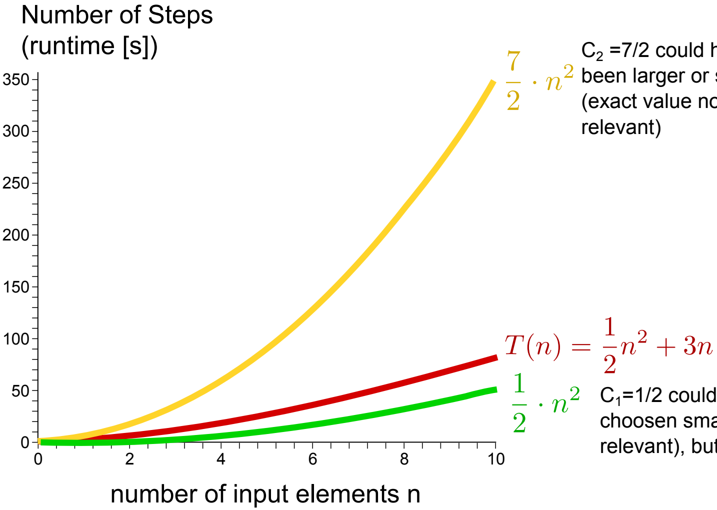


Runtime Example



C_2 =7/2 could have been larger or small (exact value not relevant)

Runtime Example



C_2 =7/2 could have been larger or small (exact value not relevant)

C_1=1/2 could have been choosen smaller (not relevant), but not larger

Runtime analysis - Minsort

We declare:

- Runtime of opertations: $T(n)$
- Number of Elements: n
- Constants: C_1 (lower bound), C_2 (upper bound)

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- Number of operations in round i: T_i

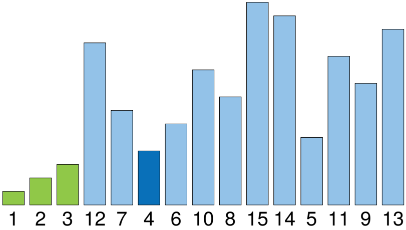


Figure: *Minsort* at the iteration $i = 4$. We have to check $n - 3$ elements

Runtime analysis - Minsort

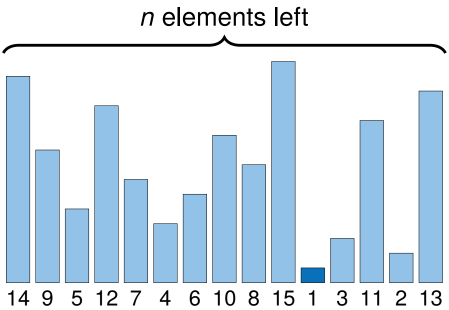


Figure: *Minsort* with start data

Compares at each iteration:

Runtime analysis - Minsort

n elements left

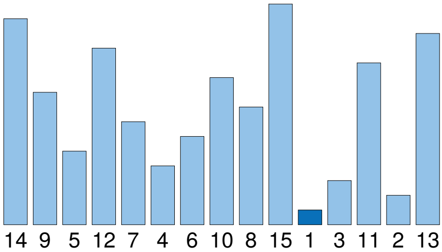


Figure: *Minsort* at iteration $i = 1$

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n - 0)$$

Runtime analysis - Minsort

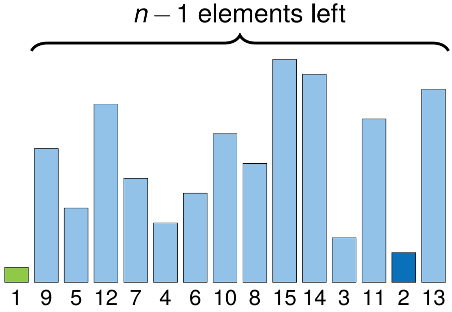


Figure: *Minsort* at iteration $i = 2$

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$

Runtime analysis - Minsort

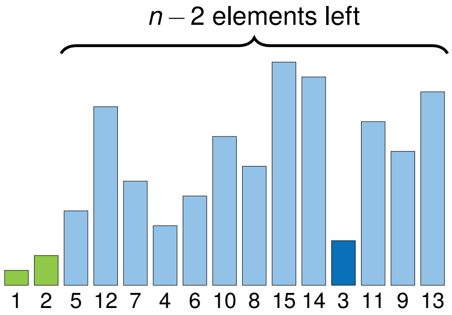


Figure: *Minsort* at iteration $i = 3$

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$
$$T_3 \leq C_2' \cdot (n-2)$$

Runtime analysis - Minsort

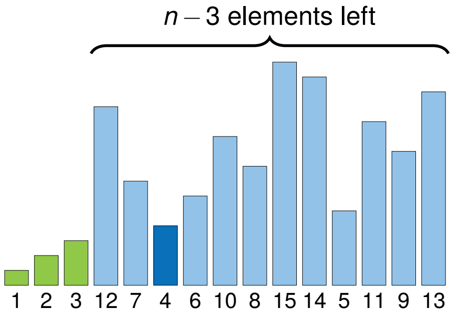


Figure: *Minsort* at iteration $i = 4$

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$
$$T_3 \leq C_2' \cdot (n-2)$$
$$T_4 \leq C_2' \cdot (n-3)$$

Runtime analysis - Minsort

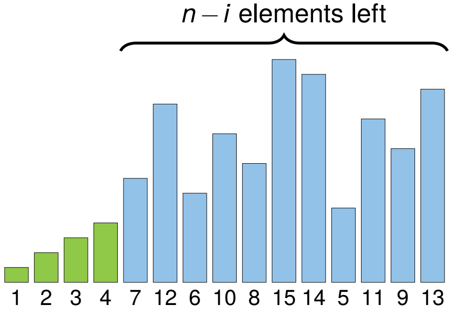


Figure: *Minsort* at iteration *i*

Compares at each iteration:

$$
\begin{aligned}
T_1 &\leq C_2' \cdot (n-0) \\
T_2 &\leq C_2' \cdot (n-1) \\
T_3 &\leq C_2' \cdot (n-2) \\
T_4 &\leq C_2' \cdot (n-3) \\
&\vdots \\
T_{n-1} &\leq C_2' \cdot 2 \\
T_n &\leq C_2' \cdot 1
\end{aligned}
$$

Runtime analysis - Minsort

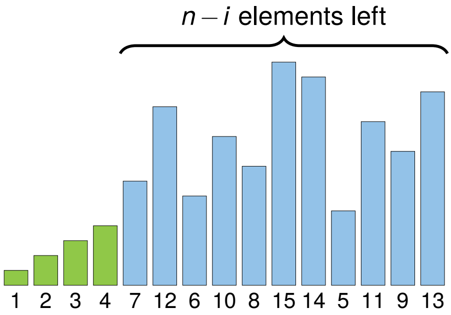


Figure: *Minsort* at iteration

Compares at each iteration:

$$T_1 \leq C_2' \cdot (n-0)$$
$$T_2 \leq C_2' \cdot (n-1)$$
$$T_3 \leq C_2' \cdot (n-2)$$
$$T_4 \leq C_2' \cdot (n-3)$$
$$\vdots$$
$$T_{n-1} \leq C_2' \cdot 2$$
$$T_n \leq C_2' \cdot 1$$

$$T(n) = C_2' \cdot (T_1 + \cdots + T_n) \leq \sum_{i=1}^{n} \left(C_2' \cdot i\right)$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```python
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

Runtime analysis - Minsort

Alternative: Analyse the Code:

```python
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const.
runtime

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i
        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j
        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

n-1 times

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C_2'$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```python
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C_2' = \sum_{i=0}^{n-2} (n-i-1) \cdot C_2'$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C_2' = \sum_{i=0}^{n-2} (n-i-1) \cdot C_2' = \sum_{i=1}^{n-1} (n-i) \cdot C_2'$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C_2' = \sum_{i=0}^{n-2} (n-i-1) \cdot C_2' = \sum_{i=1}^{n-1} (n-i) \cdot C_2' \leq \sum_{i=1}^{n} i \cdot C_2'$$

Alternative: Analyse the Code:

```
def minsort(elements):
    for i in range(0, len(elements)-1):
        minimum = i

        for j in range(i+1, len(elements)):
            if elements[j] < elements[minimum]:
                minimum = j

        if minimum != i:
            elements[i], elements[minimum] = \
                elements[minimum], elements[i]

    return elements
```

const. runtime

n-i-1 times

n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C_2' = \sum_{i=0}^{n-2} (n-i-1) \cdot C_2' = \sum_{i=1}^{n-1} (n-i) \cdot C_2' \leq \sum_{i=1}^{n} i \cdot C_2'$$

Remark: C_2' is cost of comparison $\Rightarrow$ assumed constant

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$T(n) \leq \sum_{i=1}^{n} C_2' \cdot i$$

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\ &= C_2' \cdot \sum_{i=1}^{n} i \end{aligned}$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$
\begin{aligned}
T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\
&= C_2' \cdot \sum_{i=1}^{n} i \\
&\Downarrow \quad \text{Small Gauss sum} \\
&= C_2' \cdot \frac{n(n+1)}{2}
\end{aligned}
$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$
\begin{aligned}
T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\
&= C_2' \cdot \sum_{i=1}^{n} i \\
&\Downarrow \quad \text{Small Gauss sum} \\
&= C_2' \cdot \frac{n(n+1)}{2} \\
&\leq C_2' \cdot \frac{n(n+n)}{2}, \ 1 \leq n
\end{aligned}
$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$
\begin{aligned}
T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\
&= C_2' \cdot \sum_{i=1}^{n} i \\
&\Downarrow \quad \text{Small Gauss sum} \\
&= C_2' \cdot \frac{n(n+1)}{2} \\
&\leq C_2' \cdot \frac{n(n+n)}{2}, \; 1 \leq n \\
&= C_2' \cdot \frac{2 \cdot n^2}{2}
\end{aligned}
$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$
\begin{aligned}
T(n) &\leq \sum_{i=1}^{n} C_2' \cdot i \\
&= C_2' \cdot \sum_{i=1}^{n} i \\
&\Downarrow \quad \text{Small Gauss sum} \\
&= C_2' \cdot \frac{n(n+1)}{2} \\
&\leq C_2' \cdot \frac{n(n+n)}{2}, \; 1 \leq n \\
&= C_2' \cdot \frac{2 \cdot n^2}{2} \quad = \quad C_2' \cdot n^2
\end{aligned}
$$

Excursion - Small Gauss Formula

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C_1' \cdot (n-i)$$

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C_1' \cdot (n-i) = C_1' \sum_{i=1}^{n-1} i$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same

$$
\begin{aligned}
T(n) &\geq \sum_{i=1}^{n-1} C_1' \cdot (n-i) = C_1' \sum_{i=1}^{n-1} i \\
&\geq C_1' \cdot \frac{(n-1) \cdot n}{2}
\end{aligned}
$$

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same, only final approximation differs

$$
\begin{aligned}
T(n) &\geq \sum_{i=1}^{n-1} C_1' \cdot (n-i) = C_1' \sum_{i=1}^{n-1} i \\
&\geq C_1' \cdot \frac{(n-1) \cdot n}{2}
\end{aligned}
$$

How do we get to n^2?

Runtime analysis - Minsort

UNI FREIBURG

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same, only final approximation differs

$$
\begin{aligned}
T(n) &\geq \sum_{i=1}^{n-1} C_1' \cdot (n-i) = C_1' \sum_{i=1}^{n-1} i \\
&\geq C_1' \cdot \frac{(n-1) \cdot n}{2}
\end{aligned}
$$

How do we get to n^2?

$$\Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same, only final approximation differs

$$
\begin{aligned}
T(n) &\geq \sum_{i=1}^{n-1} C_1' \cdot (n-i) = C_1' \sum_{i=1}^{n-1} i \\
&\geq C_1' \cdot \frac{(n-1)\cdot n}{2} \qquad \text{How do we get to } n^2\text{?} \\
&\qquad \Downarrow \qquad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \\
&\geq C_1' \cdot \frac{n \cdot n}{2 \cdot 2}
\end{aligned}
$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1. Summation analysis is the same, only final approximation differs

$$T(n) \geq \sum_{i=1}^{n-1} C_1' \cdot (n-i) = C_1' \sum_{i=1}^{n-1} i$$

$$\geq C_1' \cdot \frac{(n-1)\cdot n}{2}$$

How do we get to n^2?

$$\Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2$$

$$\geq C_1' \cdot \frac{n \cdot n}{2 \cdot 2} = \frac{C_1'}{4} \cdot n^2$$

Runtime analysis - Minsort

Runtime Analysis:

- Upper bound: $T(n) \leq C_2' \cdot n^2$

UNI FREIBURG

Runtime Analysis:

- Upper bound: $T(n) \leq C_2' \cdot n^2$
- Lower bound: $\frac{C_1'}{4} \cdot n^2 \leq T(n)$

Runtime analysis - Minsort

Runtime Analysis:

- Upper bound: $T(n) \leq C_2' \cdot n^2$
- Lower bound: $\frac{C_1'}{4} \cdot n^2 \leq T(n)$

Summarized:

$$\frac{C_1'}{4} \cdot n^2 \leq T(n) \leq C_2' \cdot n^2$$

Quadratic runtime proven:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list
- Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list
- Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $2 \times$ elements $\Rightarrow 4 \times$ runtime

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list
- Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $2 \times$ elements $\Rightarrow 4 \times$ runtime
  - $C = 1\,\text{ns}$ (1 simple instruction $\approx 1\,\text{ns}$)

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list
- Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $2 \times$ elements $\Rightarrow$ $4 \times$ runtime
  - $C = 1\,\text{ns}$ (1 simple instruction $\approx 1\,\text{ns}$)
  - $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{12} = 10^3\,\text{s} = 16.7\,\text{min}$

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list
- Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $2 \times$ elements $\Rightarrow$ $4 \times$ runtime
  - $C = 1\,\text{ns}$ (1 simple instruction $\approx 1\,\text{ns}$)
  - $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{12} = 10^3\,\text{s} = 16.7\,\text{min}$
  - $n = 10^9$ (1 billion numbers = 4 GB)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{18} = 10^9\,\text{s} = 31.7$ years

Runtime Example

- The runtime is growing quadratic with the number of elements n in the list
- Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $2\times$ elements $\Rightarrow$ $4\times$ runtime
  - C = 1 ns (1 simple instruction $\approx$ 1 ns)
  - $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{12} = 10^3\,\text{s} = 16.7\,\text{min}$
  - $n = 10^9$ (1 billion numbers = 4 GB)
    - $C \cdot n^2 = 10^{-9}\,\text{s} \cdot 10^{18} = 10^9\,\text{s} = 31.7$ years
- **Quadratic runtime = “big” problems unsolvable**

Structure

Runtime - Heapsort

Intuitive to extract minimum:

- **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.
- **Heapsort:** The root node is always the smallest (minheap). We only need to repair a part of the full tree after an delete operation.

Formal:

- Let $\mathrm{T}(n)$ be the runtime for the *Heapsort* algorithm with n elements.
- On the next pages we will proof $T(n) \leq C \cdot n \log_2 n$

Runtime - Heapsort

Depth of a binary tree:

- **Depth d:** longest path through the tree
- Complete binary tree has $n = 2^d - 1$ nodes
- Example: $d = 4$
  $\Rightarrow n = 2^4 - 1 = 15$

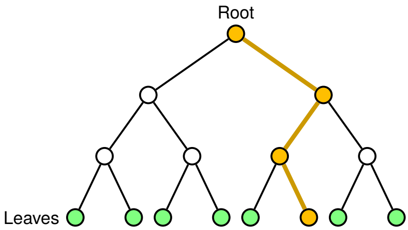


Figure: Binary tree with 15 nodes

Structure

Induction

Basics:

Induction

Basics:

- You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$

Induction

Basics:

- You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- We show induction in two steps:

Induction

Basics:

- You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- We show induction in two steps:
  1. **Induction basis:** we show that our assumption is valid at one point (for example: $n = 1, A(1)$).

Induction

Basics:

- You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- We show induction in two steps:
  1. **Induction basis:** we show that our assumption is valid at one point (for example: $n = 1, A(1)$).
  2. **Induction step:** we show that the assumption is valid for all n (normally one step forward: $n = n+1, A(1), \ldots, A(n)$).

Basics:

- You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- We show induction in two steps:
  1. **Induction basis:** we show that our assumption is valid at one point (for example: $n = 1, A(1)$).
  2. **Induction step:** we show that the assumption is valid for all n (normally one step forward: $n = n+1, A(1), \ldots, A(n)$).
- If both has been proven, then $A(n)$ holds for all natural numbers n by **induction**

Induction - Example 1

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

- **Induction basis:** Assumption holds for $d = 1$

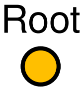


Figure: Tree of depth 1 has 1 node

$n(1) = 2^1 - 1 = 1$

$\Rightarrow$ correct ✓

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓
- **Induction step:** to show for $d = d + 1$

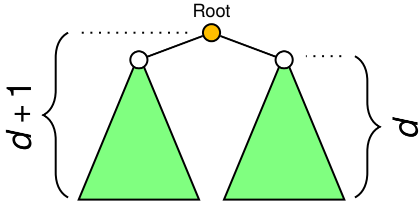


Figure: Binary tree with subtrees

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓
- **Induction step:** to show for $d = d + 1$

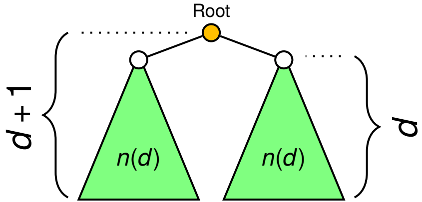


Figure: Binary tree with subtrees

$$n(d+1) = 2 \cdot n(d) + 1$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓
- **Induction step:** to show for $d = d + 1$

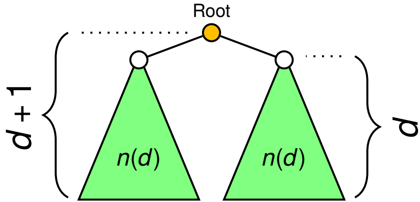


Figure: Binary tree with subtrees

$$
\begin{aligned}
n(d+1) &= 2 \cdot n(d) + 1 \\
&= 2 \cdot \left(2^d - 1\right) + 1
\end{aligned}
$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓
- **Induction step:** to show for $d = d + 1$

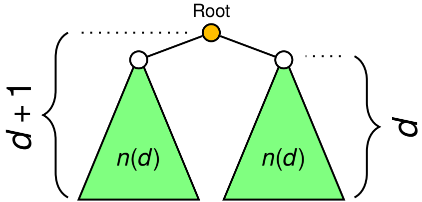


Figure: Binary tree with subtrees

$$
\begin{aligned}
n(d+1) &= 2 \cdot n(d) + 1 \\
&= 2 \cdot \left(2^d - 1\right) + 1 \\
&= 2^{d+1} - 2 + 1
\end{aligned}
$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓
- **Induction step:** to show for $d = d + 1$

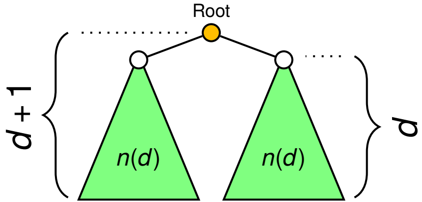


Figure: Binary tree with subtrees

$$
\begin{aligned}
n(d+1) &= 2 \cdot n(d) + 1 \\
&= 2 \cdot \left(2^d - 1\right) + 1 \\
&= 2^{d+1} - 2 + 1 \\
&= 2^{d+1} - 1 \checkmark
\end{aligned}
$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d:

- **Induction assumption:** $n(d) = 2^d - 1$
- **Induction basis:** $n(1) = 2^d - 1 = 2^1 - 1 = 1$ ✓
- **Induction step:** to show for $d = d + 1$

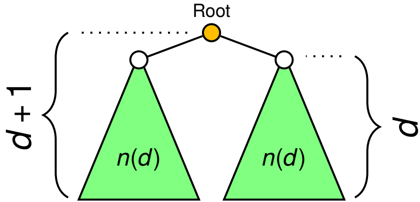


Figure: Binary tree with subtrees

$$
\begin{aligned}
n(d+1) &= 2 \cdot n(d) + 1 \\
&= 2 \cdot \left(2^d - 1\right) + 1 \\
&= 2^{d+1} - 2 + 1 \\
&= 2^{d+1} - 1 \checkmark
\end{aligned}
$$

$\Rightarrow$ **By induction:** $n(d) = 2^d - 1 \ \forall n \in \mathbb{N}$ $\square$

Structure

Runtime - Heapsort

Heapsort has the following steps:

- **Initially:** `heapify` list of n elements

Runtime - Heapsort

Heapsort has the following steps:

- **Initially:** `heapify` list of *n* elements
- **Then:** until all *n* elements are sorted

Runtime - Heapsort

Heapsort has the following steps:

- **Initially:** `heapify` list of *n* elements
- **Then:** until all *n* elements are sorted
  - Remove root as minimal element

Runtime - Heapsort

Heapsort has the following steps:

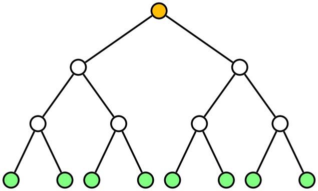
- **Initially:** `heapify` list of n elements
- **Then:** until all n elements are sorted
  - Remove root as minimal element
  - Move last leaf to root position

Runtime - Heapsort

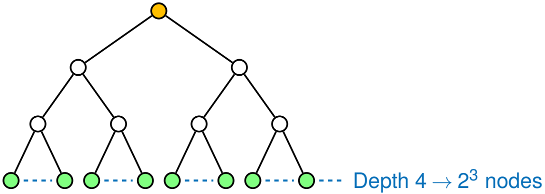
Heapsort has the following steps:

- **Initially:** `heapify` list of n elements
- **Then:** until all n elements are sorted
  - Remove root as minimal element
  - Move last leaf to root position
  - Repair heap by `sifting`

Runtime of `heapify` depends on depth d:



Runtime of `heapify` depends on depth d:



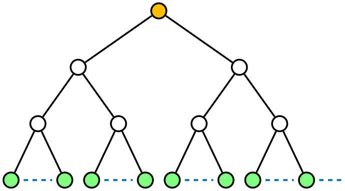
Runtime of `heapify` with depth of d:

- No costs at depth d with 2^{d-1} (or less) nodes

Runtime - Heapsort

Heapify

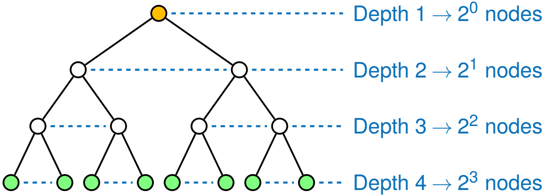
Runtime of `heapify` depends on depth d:



Runtime of `heapify` with depth of d:

- No costs at depth d with 2^{d-1} (or less) nodes
- The cost for `sifting` with depth 1 is at most $1C$ per node

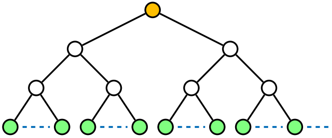
Runtime of `heapify` depends on depth d:



Runtime of `heapify` with depth of d:

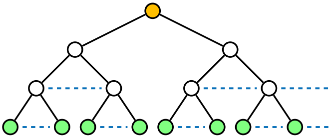
- No costs at depth d with 2^{d-1} (or less) nodes
- The cost for `sifting` with depth 1 is at most $1C$ per node
- In general: `Sifting` costs are linear with path length **and** number of nodes

`Heapify` total runtime:



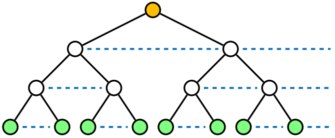
Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	

`Heapify` total runtime:



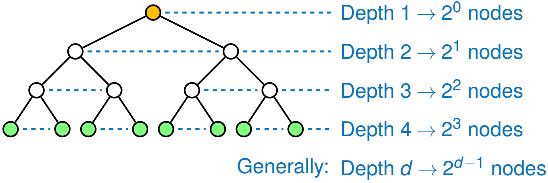
Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	

`Heapify` total runtime:



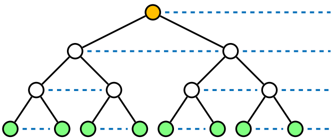
Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	

`Heapify` total runtime:



Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

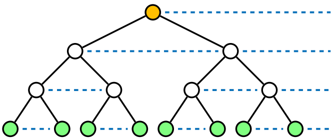
`Heapify` total runtime:



Depth	Nodes	Path length	Costs per node
d	2^{d-1}	0	$\leq C \cdot 0$
$d-1$	2^{d-2}	1	$\leq C \cdot 1$
$d-2$	2^{d-3}	2	$\leq C \cdot 2$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$

In total: $$T(d) \leq \sum_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right)$$

`Heapify` total runtime:



Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	$\leq C \cdot 1$
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	$\leq C \cdot 2$
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	$\leq C \cdot 3$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	$\leq C \cdot 4$

In total: $$T(d) \leq \sum_{i=1}^{d} \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^{d} \left(C \cdot i \cdot 2^{d-i} \right)$$

`Heapify` total runtime:

$$T(d) \leq C \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq C \cdot 2^{d+1}$$

`Heapify` total runtime:

$$T(d) \leq C \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq C \cdot 2^{d+1}$$

- **Hence:** Resulting costs for `heapify`:

$$T(d) \leq C \cdot 2^{d+1}$$

`Heapify` total runtime:

$$T(d) \leq C \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq C \cdot 2^{d+1}$$

- **Hence:** Resulting costs for `heapify`:

$$T(d) \leq C \cdot 2^{d+1}$$

- **However:** We want costs in relation to n

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes

Runtime - Heapsort

Heapify

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes Why?

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes Why?

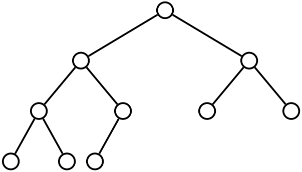


Figure: Partial binary tree

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes Why?
- $2^{d-1} - 1$ nodes in full tree till layer $d - 1$

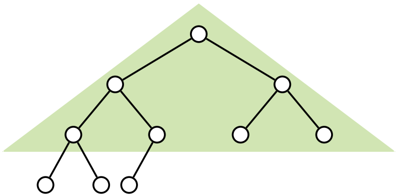


Figure: Partial binary tree

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes Why?
- $2^{d-1} - 1$ nodes in full tree till layer $d - 1$
- At least 1 node in layer d

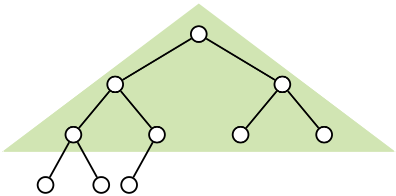


Figure: Partial binary tree

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes Why?

- $2^{d-1} - 1$ nodes in full tree till layer $d - 1$
- At least 1 node in layer d

- Equation multiplied by 2^2
  $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$

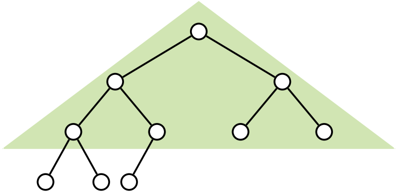


Figure: Partial binary tree

Runtime - Heapsort

Heapify

`Heapify` total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- A binary tree of depth d has $2^{d-1} \leq n$ nodes Why?

- $2^{d-1} - 1$ nodes in full tree till layer $d-1$
- At least 1 node in layer d

- Equation multiplied by 2^2
  $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$
- Cost for `heapify`:
  $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

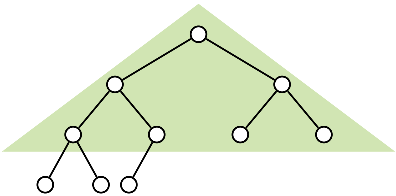


Figure: Partial binary tree

Structure

Induction - Example 2

- **We want to proof (induction assumption):**

$$\underbrace{\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right)}_{} \leq \underbrace{2^{d+1}}_{}$$

$$A(d) \leq B(d)$$

  - We denote the left side with A, the right side with B

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

Induction - Example 2

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq 2^{d+1}$$

UNI FREIBURG

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq 2^{d+1}$$

$$\sum_{i=1}^{1} \left(i \cdot 2^{1-i} \right) \leq 2^{1+1}$$

UNI FREIBURG

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq 2^{d+1}$$

$$\sum_{i=1}^{1} \left(i \cdot 2^{1-i} \right) \leq 2^{1+1}$$

$$2^0 \leq 2^2 \checkmark$$

Induction - Example 2

Induction step: $(d := d + 1)$:

- **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

Induction step: $(d := d + 1)$:

- **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \qquad \Rightarrow \qquad A(d+1) \leq B(d+1)$$

$$\sum_{i=1}^{d+1} \left(i \cdot 2^{d+1-i}\right) \leq 2^{d+1+1}$$

Induction step: $(d := d + 1)$:

- **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \qquad \Rightarrow \qquad A(d+1) \leq B(d+1)$$

$$\sum_{i=1}^{d+1} \left(i \cdot 2^{d+1-i} \right) \leq 2^{d+1+1}$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

$$\vdots$$

Induction - Example 2

Induction step: $(d := d + 1)$:

$$\vdots$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

Induction - Example 2

Induction step: $(d := d+1)$:

$$\vdots$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot B(d)$$

Induction - Example 2

Induction step: $(d := d + 1)$:

$$\vdots$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

Induction - Example 2

Induction step: $(d := d + 1)$:

$$\vdots$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i}\right) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i}\right) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i}\right) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

Induction - Example 2

Induction step: $(d := d + 1)$:

$$\vdots$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

- **Problem:** Does not work but claim still holds

Induction - Example 2

Working proof:

- Show a little bit stronger claim

$$\sum_{i=1}^{d}\left(i \cdot 2^{d-i}\right) \leq 2^{d+1} - d - 2 \leq 2^{d+1}$$

Induction - Example 2

Working proof:

- Show a little bit stronger claim

$$\sum_{i=1}^{d} \left(i \cdot 2^{d-i} \right) \leq 2^{d+1} - d - 2 \leq 2^{d+1}$$

- **Advantage:** Results in a stronger induction assumption

$\Rightarrow$ exercise

Structure

Runtime of the other operations:

Runtime of the other operations:

- Constant costs for taking out $n \times$ maximum

Runtime of the other operations:

- Constant costs for taking out $n \times$ maximum
- Maximum of d steps repairing the heap n times

Runtime of the other operations:

- Constant costs for taking out $n \times$ maximum
- Maximum of d steps repairing the heap n times
- Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

Runtime of the other operations:

- Constant costs for taking out $n \times$ maximum
- Maximum of d steps repairing the heap n times
- Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \;\Rightarrow\; d-1 \leq \log_2 n \;\Rightarrow\; d \leq 1 + \log_2 n$$

- **Recall**: The depth and number of elements is decreasing

Runtime of the other operations:

- Constant costs for taking out $n \times$ maximum
- Maximum of d steps repairing the heap n times
- Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

- **Recall**: The depth and number of elements is decreasing
  - **Hence**: $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$

Runtime of the other operations:

- Constant costs for taking out $n \times$ maximum
- Maximum of d steps repairing the heap n times
- Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

- **Recall**: The depth and number of elements is decreasing
  - **Hence**: $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
  - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \qquad \text{(holds for } n > 2\text{)}$$

Runtime - Heapsort

Runtime costs:

- `Heapify`: $T(n) \leq 4 \cdot n \cdot C$

Runtime - Heapsort

Runtime costs:

- `Heapify`: $T(n) \leq 4 \cdot n \cdot C$
- `Remove`: $T(n) \leq 2 \cdot n \log_2 n \cdot C$

Runtime - Heapsort

Runtime costs:

- `Heapify`: $T(n) \leq 4 \cdot n \cdot C$
- `Remove`: $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$

Runtime - Heapsort

Runtime costs:

- `Heapify`: $T(n) \leq 4 \cdot n \cdot C$
- `Remove`: $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - Upper bound: $C_2 \cdot n \log_2 n \geq T(n)$ (for $n \geq 2$)
  - Lower bound: $C_1 \cdot n \log_2 n \leq T(n)$ (for $n \geq 2$)

Runtime - Heapsort

Runtime costs:

- `Heapify`: $T(n) \leq 4 \cdot n \cdot C$
- `Remove`: $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - Upper bound: $C_2 \cdot n \log_2 n \geq T(n)$ (for $n \geq 2$)
  - Lower bound: $C_1 \cdot n \log_2 n \leq T(n)$ (for $n \geq 2$)
  - $\Rightarrow C_1$ and C_2 are constant

Structure

Base of Logaritms

UNI FREIBURG

Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient $\frac{1}{\log_b a}$

Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602\ldots \cdot 3.322\ldots = 2$ ✓
- $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3$ ✓

Runtime Example

Runtime of $n \log_2 n$:

- Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot log_2 n \qquad \text{for } n \geq 2$$

- $2 \times$ elements $\Rightarrow$ only slightly larger than $2 \times$ runtime
  - $C = 1\,\text{ns}$ (1 simple instruction $\approx 1\,\text{ns}$)
  - $n = 2^{20}$ (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{20} \cdot 20 = 21.0\,\text{ms}$
  - $n = 2^{30}$ (1 billion numbers = 4 GB)
    - $C \cdot n \cdot log_2 n = 10^{-9}\,\text{s} \cdot 2^{30} \cdot 30 = 32\,\text{s}$
- **Runtime $n \log_2 n$ is nearly as good as linear!**

Further Literature

- **General for this Lecture**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Mathematical Induction**

  [Wik] Mathematical induction
  https://en.wikipedia.org/wiki/Mathematical_induction