

Algorithms and Datastructures

Linked Lists, Binary Search Trees

Albert-Ludwigs-Universität Freiburg



**UNI
FREIBURG**

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, January 2018

Sorted Sequences

Linked Lists

Binary Search Trees



Structure:

Structure:

- We have a set of **keys** mapped to **values**

Structure:

- We have a set of **keys** mapped to **values**
- We have a ordering $<$ applied to the keys

Structure:

- We have a set of **keys** mapped to **values**
- We have a ordering $<$ applied to the keys
- We need the following operations:

Structure:

- We have a set of **keys** mapped to **values**
- We have a ordering $<$ applied to the keys
- We need the following operations:
 - `insert(key, value)`: Insert the given pair

Structure:

- We have a set of **keys** mapped to **values**
- We have a ordering $<$ applied to the keys
- We need the following operations:
 - `insert(key, value)`: Insert the given pair
 - `remove(key)`: Remove the pair with the given **key**

Structure:

- We have a set of **keys** mapped to **values**
- We have a ordering $<$ applied to the keys
- We need the following operations:
 - **insert(key, value)**: Insert the given pair
 - **remove(key)**: Remove the pair with the given **key**
 - **lookup(key)**: Find the element with the given **key**, if it is not available find the element with the next smallest key

Structure:

- We have a set of **keys** mapped to **values**
- We have a ordering $<$ applied to the keys
- We need the following operations:
 - **insert(key, value)**: Insert the given pair
 - **remove(key)**: Remove the pair with the given **key**
 - **lookup(key)**: Find the element with the given **key**, if it is not available find the element with the next smallest key
 - **next()/previous()**: Returns the element with the next bigger/smaller **key**. This enables iteration over all elements



Application examples:

Application examples:

- Example: Database for books, products or apartments

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: Return all apartments with a monthly rent between 400€ and 600€

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: Return all apartments with a monthly rent between 400€ and 600€
 - This is called a **range query**

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: Return all apartments with a monthly rent between 400€ and 600€
 - This is called a **range query**
 - We can implement this with a combination of **lookup(key)** and **next()**

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: Return all apartments with a monthly rent between 400€ and 600€
 - This is called a **range query**
 - We can implement this with a combination of `lookup(key)` and `next()`
 - It's not essential if an apartments exists with **exactly** 400€ monthly rent

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: Return all apartments with a monthly rent between 400€ and 600€
 - This is called a **range query**
 - We can implement this with a combination of **lookup(key)** and **next()**
 - It's not essential if an apartments exists with **exactly** 400€ monthly rent
- We do not want to sort all elements every time on an **insert** operation

Application examples:

- Example: Database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: Return all apartments with a monthly rent between 400€ and 600€
 - This is called a **range query**
 - We can implement this with a combination of **lookup(key)** and **next()**
 - It's not essential if an apartments exists with **exactly** 400€ monthly rent
- We do not want to sort all elements every time on an **insert** operation
- How could we implement this?

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- **lookup** in time $O(\log n)$

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- **lookup** in time $O(\log n)$
 - With **binary search**

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- `lookup` in time $O(\log n)$
 - With **binary search**
 - Example: `lookup(41)`

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- `lookup` in time $O(\log n)$
 - With **binary search**
 - Example: `lookup(41)`
- `next` / `previous` in time $O(1)$

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- `lookup` in time $O(\log n)$
 - With **binary search**
 - Example: `lookup(41)`
- `next` / `previous` in time $O(1)$
 - They are next to each other

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- `lookup` in time $O(\log n)$
 - With **binary search**
 - Example: `lookup(41)`
- `next` / `previous` in time $O(1)$
 - They are next to each other
- `insert` and `remove` up to $\Theta(n)$

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

- `lookup` in time $O(\log n)$
 - With **binary search**
 - Example: `lookup(41)`
- `next` / `previous` in time $O(1)$
 - They are next to each other
- `insert` and `remove` up to $\Theta(n)$
 - We have to copy up to n elements

Sorted Sequences

Implementation 2 (bad) - Hash Table



Hash map:



Hash map:

- `insert` and `remove` in $O(1)$



Hash map:

- `insert` and `remove` in $O(1)$

If the hash table is big enough and we use a good hash function

Hash map:

- `insert` and `remove` in $O(1)$

If the hash table is big enough and we use a good hash function

- `lookup` in time $O(1)$

Hash map:

- `insert` and `remove` in $O(1)$

If the hash table is big enough and we use a good hash function

- `lookup` in time $O(1)$

If element with **exactly** this key exists, otherwise we get `None` as result

- `next` / `previous` in time up to $\Theta(n)$

Hash map:

- `insert` and `remove` in $O(1)$

If the hash table is big enough and we use a good hash function

- `lookup` in time $O(1)$

If element with **exactly** this key exists, otherwise we get `None` as result

- `next` / `previous` in time up to $\Theta(n)$

Order of the elements is independent of the order of the keys

Sorted Sequences

Implementation 3 (good?) - Linked List



Linked list:

Linked list:

- Runtimes for doubly linked lists:

Linked list:

- Runtimes for doubly linked lists:
 - `next` / `previous` in time $O(1)$

Linked list:

- Runtimes for doubly linked lists:
 - `next` / `previous` in time $O(1)$
 - `insert` and `remove` in $O(1)$

Linked list:

- Runtimes for doubly linked lists:
 - `next` / `previous` in time $O(1)$
 - `insert` and `remove` in $O(1)$
 - `lookup` in time $\Theta(n)$

Linked list:

- Runtimes for doubly linked lists:
 - `next` / `previous` in time $O(1)$
 - `insert` and `remove` in $O(1)$
 - `lookup` in time $\Theta(n)$
- Not yet what we want, but structure is related to binary search trees

Linked list:

- Runtimes for doubly linked lists:
 - `next` / `previous` in time $O(1)$
 - `insert` and `remove` in $O(1)$
 - `lookup` in time $\Theta(n)$
- Not yet what we want, but structure is related to binary search trees
- Let's have a closer look

Sorted Sequences

Linked Lists

Binary Search Trees



Linked list:



Linked list:

- Dynamic datastructure



Linked list:

- Dynamic datastructure
- Number of elements changeable

Linked list:

- Dynamic datastructure
- Number of elements changeable
- Data elements can be simple types or composed datastructures

Linked list:

- Dynamic datastructure
- Number of elements changeable
- Data elements can be simple types or composed datastructures
- **Elements are linked** through references / pointer to the predecessor / successor

Linked list:

- Dynamic datastructure
- Number of elements changeable
- Data elements can be simple types or composed datastructures
- **Elements are linked** through references / pointer to the predecessor / successor
- Single / doubly linked lists possible

Linked list:

- Dynamic datastructure
- Number of elements changeable
- Data elements can be simple types or composed datastructures
- Elements are linked through references / pointer to the predecessor / successor
- Single / doubly linked lists possible



Figure: Linked list



Properties in comparison to an array:



Properties in comparison to an array:

- Minimal extra space for storing pointer

Properties in comparison to an array:

- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`

Properties in comparison to an array:

- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified

Properties in comparison to an array:

- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified
- No direct access of elements
 - ⇒ We have to iterate over the list

List with head / last element pointer:

List with head / last element pointer:



Figure: Singly linked list

List with head / last element pointer:

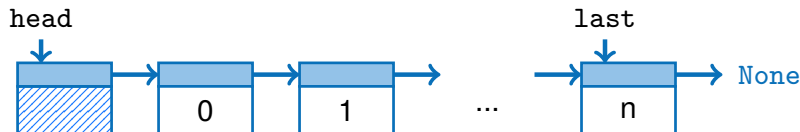


Figure: Singly linked list

- Head element has pointer to first list element

List with head / last element pointer:



Figure: Singly linked list

- Head element has pointer to first list element
- May also hold additional information:

List with head / last element pointer:



Figure: Singly linked list

- Head element has pointer to first list element
- May also hold additional information:
 - Number of elements



Doubly linked list:

Doubly linked list:



Figure: Doubly linked list

Doubly linked list:

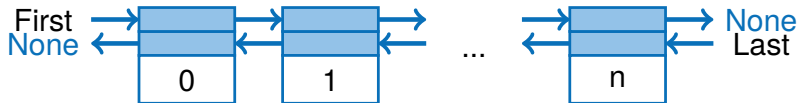


Figure: Doubly linked list

- Pointer to successor element

Doubly linked list:



Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element

Doubly linked list:



Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```




Creating linked lists - Python:

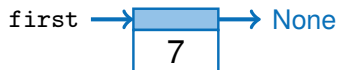
Creating linked lists - Python:

```
■ first = Node(7)
```



Creating linked lists - Python:

■ `first = Node(7)`



■ `first.nextNode = Node(3)`



Creating linked lists - Python:

■ `first = Node(7)`



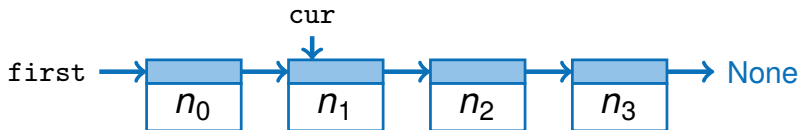
■ `first.nextNode = Node(3)`



■ `first.nextNode.value = 4`



Inserting a node after node `cur`:





Inserting a node after node `cur`:

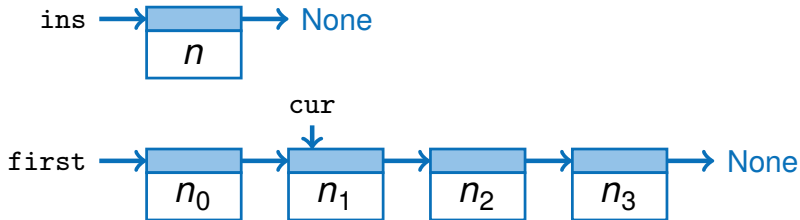


Inserting a node after node `cur`:

- `ins = Node(n)`

Inserting a node after node `cur`:

■ `ins = Node(n)`





Inserting a node after node `cur`:

- `ins.nextNode = cur.nextNode`

Inserting a node after node `cur`:

■ `ins.nextNode = cur.nextNode`



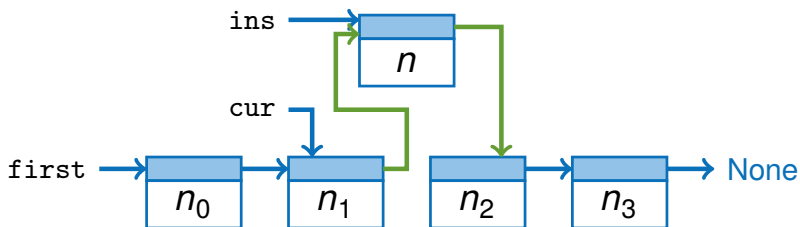


Inserting a node after node `cur`:

- `cur.nextNode = ins`

Inserting a node after node `cur`:

■ `cur.nextNode = ins`





Inserting a node after node `cur` - single line of code:

Inserting a node after node `cur` - single line of code:



Inserting a node after node `cur` - single line of code:

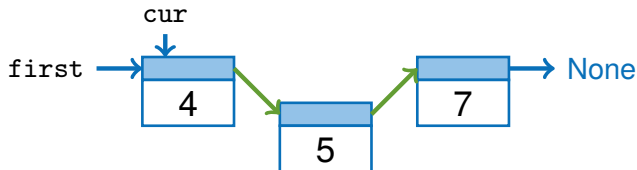


■ `cur.nextNode = Node(value, cur.nextNode)`

Inserting a node after node `cur` - single line of code:



■ `cur.nextNode = Node(value, cur.nextNode)`



Removing a node `cur`:





Removing a node `cur:`



Removing a node `cur`:

- Find the predecessor of `cur`:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```



Removing a node `cur`:

- Find the predecessor of `cur`:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

- Runtime of $O(n)$



Removing a node `cur`:

- Find the predecessor of `cur`:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

- Runtime of $O(n)$
- Does not work for first node!

Removing a node `cur`:

- Find the predecessor of `cur`:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

- Runtime of $O(n)$
- Does not work for first node!





Removing a node `cur:`



Removing a node `cur`:

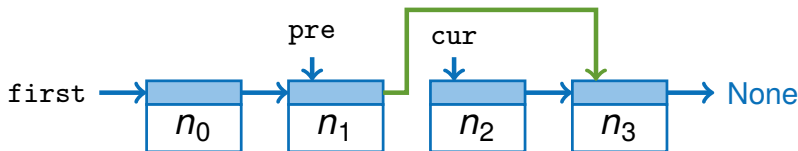
- Update the pointer to the next element:
`pre.nextNode = cur.nextNode`

Removing a node `cur`:

- Update the pointer to the next element:
`pre.nextNode = cur.nextNode`
- `cur` will get automatically destroyed if no more references exist (`cur=None`)

Removing a node `cur`:

- Update the pointer to the next element:
`pre.nextNode = cur.nextNode`
- `cur` will get automatically destroyed if no more references exist (`cur=None`)





Removing the first node:

Removing the first node:



Removing the first node:



- Update the pointer to the next element:

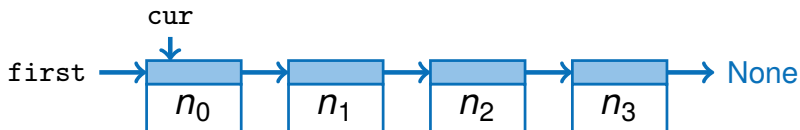
```
first = first.nextNode
```

Removing the first node:

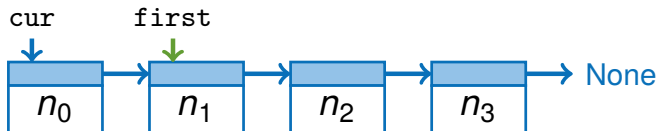


- Update the pointer to the next element:
`first = first.nextNode`
- `cur` will get automatically destroyed if no more references exist (`cur=None`)

Removing the first node:



- Update the pointer to the next element:
`first = first.nextNode`
- `cur` will get automatically destroyed if no more references exist (`cur=None`)



Removing a node `cur`: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```




Using a head node:



Using a head node:

- Advantage:



Using a head node:

- Advantage:
 - Deleting the first node is no special case



Using a head node:

- Advantage:
 - Deleting the first node is no special case
- Disadvantage
 - We have to consider the first node at other operations

Using a head node:

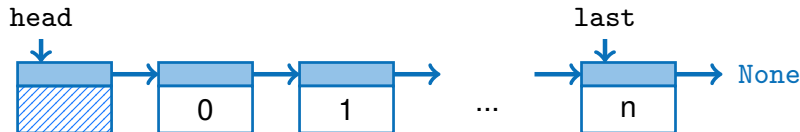
- Advantage:
 - Deleting the first node is no special case
- Disadvantage
 - We have to consider the first node at other operations
 - Iterating all nodes
 - Counting of all nodes

Using a head node:

- Advantage:
 - Deleting the first node is no special case
- Disadvantage
 - We have to consider the first node at other operations
 - Iterating all nodes
 - Counting of all nodes
 - ...

Using a head node:

- Advantage:
 - Deleting the first node is no special case
- Disadvantage
 - We have to consider the first node at other operations
 - Iterating all nodes
 - Counting of all nodes
 - ...



```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```



```
def append(self, value):  
    ...  
  
def insertAfter(self, cur, value):  
    ...  
  
def remove(self, cur):  
    ...  
  
def get(self, position):  
    ...  
  
def contains(self, value):  
    ...
```

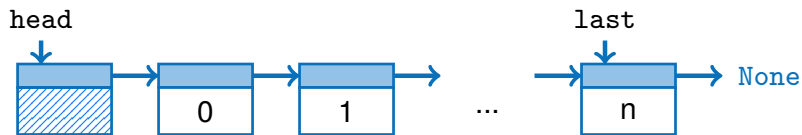


Head, last:

Head, last:



Head, last:



- Head points to the first node, last to the last node

Head, last:



- Head points to the first node, last to the last node
- We can append elements to the end of the list in $O(1)$ through the last node

Head, last:



- Head points to the first node, `last` to the last node
- We can append elements to the end of the list in $O(1)$ through the `last` node
- We have to keep the pointer to `last` updated after all operations

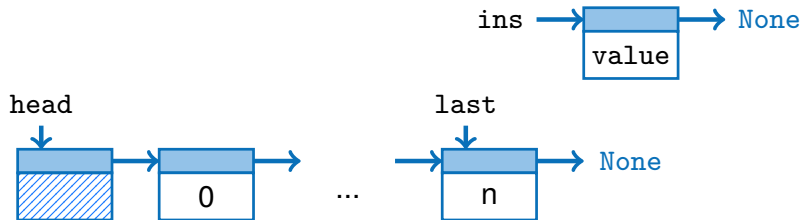


Appending an element:

Appending an element:



Appending an element:



```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

Appending an element:



```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

- The pointer to `last` avoids the iteration of the whole list

Inserting after node `cur`:





Inserting after node `cur`:

- The pointer to `head` is not modified

Inserting after node `cur`:

- The pointer to head is not modified

```
def insertAfter(self, cur, value):  
    if cur == last:  
        # also update last node  
        append(value)  
    else:  
        # last node is not modified  
        cur.nextNode = Node(value, \  
                             cur.nextNode)  
        itemCount += 1
```

Remove node cur:





Remove node `cur`:

- Searching the predecessor in $O(n)$

Remove node cur:

- Searching the predecessor in $O(n)$

```
def remove(self, cur):  
    pre = first  
    while pre.nextNode != cur:  
        pre = pre.nextNode  
  
    pre.nextNode = cur.nextNode  
    itemCount -= 1  
  
    if pre.nextNode == None:  
        last = pre
```




Getting a reference to node at `pos`:

- Iterate the entries of the list until at position in $O(n)$

Getting a reference to node at pos:

- Iterate the entries of the list until at position in $O(n)$

```
def get(self, pos):  
    if pos < 0 or pos >= itemCount:  
        return None  
  
    cur = head  
    for i in range(0, pos):  
        cur = cur.nextNode  
  
    return cur
```



Searching a value:



Searching a value:

- First element is head without an assigned value

Searching a value:

- First element is head without an assigned value
- Iterate the entries of the list until value found in $O(n)$

Searching a value:

- First element is head without an assigned value
- Iterate the entries of the list until value found in $O(n)$

```
def contains(self, value):  
    cur = head  
  
    for i in range(0, itemCount):  
        cur = cur.nextNode  
        if cur.value == value:  
            return True  
  
    return False
```



Runtime:



Runtime:

- Singly linked list:

Runtime:

- Singly linked list:
 - `next` in $O(1)$



Runtime:

- Singly linked list:
 - `next` in $O(1)$
 - `previous` in $\Theta(n)$



Runtime:

- Singly linked list:
 - `next` in $O(1)$
 - `previous` in $\Theta(n)$
 - `insert` in $O(1)$

Runtime:

- Singly linked list:
 - `next` in $O(1)$
 - `previous` in $\Theta(n)$
 - `insert` in $O(1)$
 - `remove` in $\Theta(n)$

Runtime:

- Singly linked list:
 - `next` in $O(1)$
 - `previous` in $\Theta(n)$
 - `insert` in $O(1)$
 - `remove` in $\Theta(n)$
 - `lookup` in $\Theta(n)$

Runtime:

- Singly linked list:
 - `next` in $O(1)$
 - `previous` in $\Theta(n)$
 - `insert` in $O(1)$
 - `remove` in $\Theta(n)$
 - `lookup` in $\Theta(n)$
- Better with `doubly linked lists`



Doubly linked list:

Doubly linked list:

- Each node has a reference to its successor and its predecessor

Doubly linked list:

- Each node has a reference to its successor and its predecessor
- We can iterate the list forward and backward

Doubly linked list:

- Each node has a reference to its successor and its predecessor
- We can iterate the list forward and backward





Doubly linked list:



Doubly linked list:

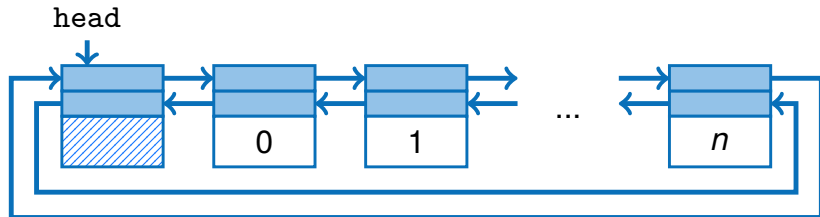
- It is helpful to have a **head** node

Doubly linked list:

- It is helpful to have a **head** node
- We only need **one head** node if we connect the list cyclic

Doubly linked list:

- It is helpful to have a **head** node
- We only need **one head** node if we connect the list cyclic





Runtime of doubly linked list:



Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Each element has a pointer to pred-/sucessor



Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Each element has a pointer to pred-/sucessor

- `insert` and `remove` in $O(1)$

Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Each element has a pointer to pred-/sucessor

- `insert` and `remove` in $O(1)$

A constant number of pointers needs to be modified

Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Each element has a pointer to pred-/sucessor

- `insert` and `remove` in $O(1)$

A constant number of pointers needs to be modified

- `lookup` in $\Theta(n)$

Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Each element has a pointer to pred-/sucessor

- `insert` and `remove` in $O(1)$

A constant number of pointers needs to be modified

- `lookup` in $\Theta(n)$

Even if the elements are sorted we can only retrieve them in $\Theta(n)$

Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

Each element has a pointer to pred-/sucessor

- `insert` and `remove` in $O(1)$

A constant number of pointers needs to be modified

- `lookup` in $\Theta(n)$

Even if the elements are sorted we can only retrieve them in $\Theta(n)$ Why?

Linked list in book:



Linked Lists

List in real program



Linked list in memory:



Sorted Sequences

Linked Lists

Binary Search Trees



Runtime of a search tree:

Runtime of a search tree:

- `next` and `previous` in $O(1)$

Runtime of a search tree:

- `next` and `previous` in $O(1)$

Pointers corresponding to linked list

Runtime of a search tree:

- `next` and `previous` in $O(1)$

Pointers corresponding to linked list

- `insert` and `remove` in $O(\log n)$

Runtime of a search tree:

- `next` and `previous` in $O(1)$

Pointers corresponding to linked list

- `insert` and `remove` in $O(\log n)$

- `lookup` in $O(\log n)$

Runtime of a search tree:

- `next` and `previous` in $O(1)$

Pointers corresponding to linked list

- `insert` and `remove` in $O(\log n)$

- `lookup` in $O(\log n)$

The structure helps searching efficiently



Idea:

Idea:

- We define a total order for the search tree

Idea:

- We define a total order for the search tree
- All nodes of the left subtree have **smaller keys** than the current node

Idea:

- We define a total order for the search tree
- All nodes of the left subtree have **smaller keys** than the current node
- All nodes of the right subtree have **bigger keys** than the current node

- Edge direction indicates ordering

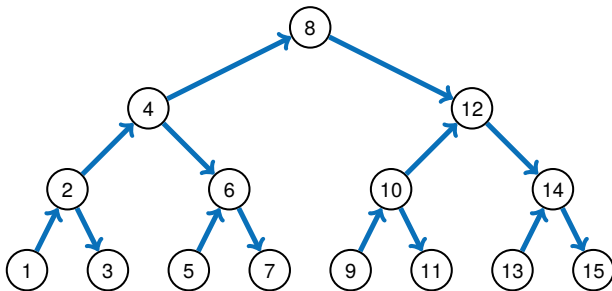


Figure: A binary search tree



Figure: Another binary search tree



Figure: **Not** a binary search tree



Implementation:

Implementation:

- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists

Implementation:

- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists
- Each node has a pointer / reference to its children (`leftChild` / `rightChild`)

Implementation:

- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists
- Each node has a pointer / reference to its children (`leftChild` / `rightChild`)
- `None` for missing children

Implementation:

- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists
- Each node has a pointer / reference to its children (`leftChild` / `rightChild`)
- `None` for missing children





Implementation:



Implementation:

- We create a sorted doubly linked list of all elements

Implementation:

- We create a sorted doubly linked list of all elements
- This enables an efficient implementation of (`next` / `previous`)

Implementation:

- We create a sorted doubly linked list of all elements
- This enables an efficient implementation of (`next` / `previous`)



Figure: Binary search tree with links



Lookup:

Lookup:

- Definition:
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”

Lookup:

- Definition:
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”
- We search from the root downwards:

Lookup:

- Definition:
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”
- We search from the root downwards:
 - Compare the searched key with the key of the node

Lookup:

- Definition:
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”
- We search from the root downwards:
 - Compare the searched key with the key of the node
 - Go to the left / right until the child is **None** or the key is found

Lookup:

- Definition:
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”
- We search from the root downwards:
 - Compare the searched key with the key of the node
 - Go to the left / right until the child is **None** or the key is found
 - If the key is not found return the next bigger one



For each node applies the total order:



For each node applies the total order:

keys of left subtree < `node.key` < keys of right subtree

For each node applies the total order:

keys of left subtree < `node.key` < keys of right subtree



Examples:

Figure: Binary search tree with total order “<”

For each node applies the total order:

keys of left subtree < `node.key` < keys of right subtree



Examples:

lookup(14)

Figure: Binary search tree with total order “<”

For each node applies the total order:

keys of left subtree < `node.key` < keys of right subtree



Examples:

lookup(14)

lookup(6)

Figure: Binary search tree with total order “<”

For each node applies the total order:

keys of left subtree < `node.key` < keys of right subtree



Examples:

lookup(14)

lookup(6)

lookup(19)

Figure: Binary search tree with total order “<”

Binary Search Trees

Implementation - Insert



UNI
FREIBURG

Insert:



Insert:

- We search for the key in our search tree



Insert:

- We search for the key in our search tree
- If a node is found we replace the value with the new one

Insert:

- We search for the key in our search tree
- If a node is found we replace the value with the new one
- Else we insert a new node



Insert:

- We search for the key in our search tree
- If a node is found we replace the value with the new one
- Else we insert a new node
- If the key was not present we get a `None` entry

Insert:

- We search for the key in our search tree
- If a node is found we replace the value with the new one
- Else we insert a new node
- If the key was not present we get a `None` entry
- We insert the node there

Insert:

- We search for the key in our search tree
- If a node is found we replace the value with the new one
- Else we insert a new node
- If the key was not present we get a **None** entry
- We insert the node there



Figure: Binary search tree with total order “<”



Remove: Case 1: The node “5” has no children



Remove: Case 1: The node “5” has no children

- Find **parent** of node “5” (“6”)



Remove: Case 1: The node “5” has no children

- Find **parent** of node “5” (“6”)
- Set left / right child of node “6” to **None** depending on position of node “5”

Remove: Case 1: The node “5” has no children

- Find **parent** of node “5” (“6”)
- Set left / right child of node “6” to **None** depending on position of node “5”



Figure: Binary search tree with total order “<”

Remove: Case 1: The node “5” has no children

- Find **parent** of node “5” (“6”)
- Set left / right child of node “6” to **None** depending on position of node “5”

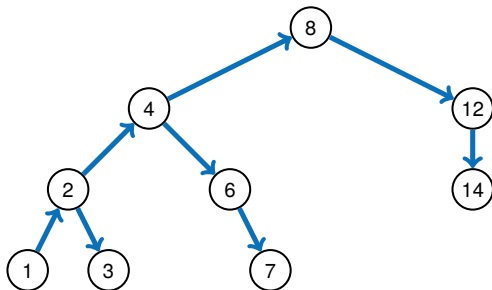


Figure: Binary search tree after deleting node “5”



Remove: Case 2: The node “12” has one child

Remove: Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)

Remove: Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)
- Find the **parent** of node “12” (“8”)



Remove: Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)
- Find the **parent** of node “12” (“8”)
- Set left / right **child** of node “8” to “14” depending on position of node “12” (skip node “14”)

Remove: Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)
- Find the **parent** of node “12” (“8”)
- Set left / right **child** of node “8” to “14” depending on position of node “12” (skip node “14”)

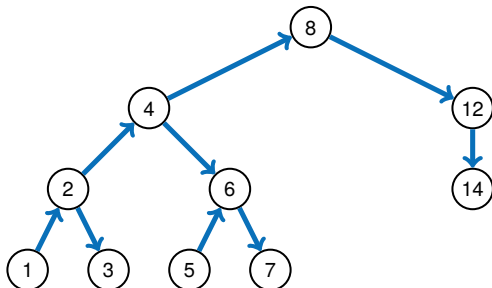


Figure: Binary search tree with total order “<”

Remove: Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)
- Find the **parent** of node “12” (“8”)
- Set left / right **child** of node “8” to “14” depending on position of node “12” (skip node “14”)



Figure: Binary search tree after deleting node “12”



Remove: Case 3: The node “4” has two children



Remove: Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)



Remove: Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”

Remove: Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the **successor** of node “4”) with remove-case 1 or 2

Remove: Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the **successor** of node “4”) with remove-case 1 or 2
- There is no left node because we are deleting the **predecessor**

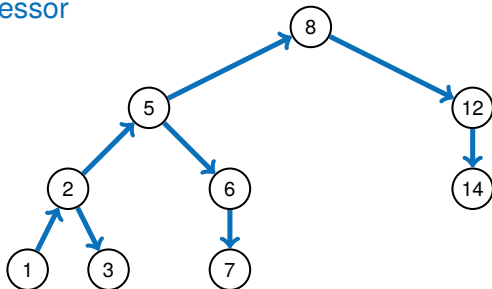
Remove: Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the **successor** of node “4”) with remove-case 1 or 2
- There is no left node because we are deleting the **predecessor**



Remove: Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the **successor** of node “4”) with remove-case 1 or 2
- There is no left node because we are deleting the **predecessor**





How long takes `insert` and `lookup`?

How long takes **insert** and **lookup**?

- Up to $\Theta(d)$, with d being the **depth of the tree**
(The longest path from the root to a leaf)

How long takes **insert** and **lookup**?

- Up to $\Theta(d)$, with d being the **depth of the tree**
(The longest path from the root to a leaf)
- **Best case** with $d = \log n$ the runtime is $\Theta(\log n)$

How long takes **insert** and **lookup**?

- Up to $\Theta(d)$, with d being the **depth of the tree**
(The longest path from the root to a leaf)
- **Best case** with $d = \log n$ the runtime is $\Theta(\log n)$
- **Worst case** with $d = n$ the runtime is $\Theta(n)$

How long takes **insert** and **lookup**?

- Up to $\Theta(d)$, with d being the **depth of the tree**
(The longest path from the root to a leaf)
- **Best case** with $d = \log n$ the runtime is $\Theta(\log n)$
- **Worst case** with $d = n$ the runtime is $\Theta(n)$
- If we **always** want to have a runtime of $\Theta(\log n)$ then we have to **rebalance** the tree

How long takes **insert** and **lookup**?

- Up to $\Theta(d)$, with d being the **depth of the tree**
(The longest path from the root to a leaf)
- **Best case** with $d = \log n$ the runtime is $\Theta(\log n)$
- **Worst case** with $d = n$ the runtime is $\Theta(n)$
- If we **always** want to have a runtime of $\Theta(\log n)$ then we have to **rebalance** the tree

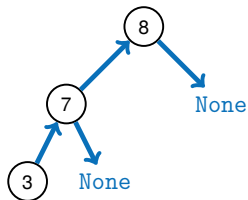


Figure: Degenerated binary tree $d = n$

How long takes **insert** and **lookup**?

- Up to $\Theta(d)$, with d being the **depth of the tree**
(The longest path from the root to a leaf)
- **Best case** with $d = \log n$ the runtime is $\Theta(\log n)$
- **Worst case** with $d = n$ the runtime is $\Theta(n)$
- If we **always** want to have a runtime of $\Theta(\log n)$ then we have to **rebalance** the tree



Figure: Degenerated binary tree $d = n$



Figure: Complete binary tree $d = \log n$

■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ **Linked List**

[Wik] [Linked list](#)

`https://en.wikipedia.org/wiki/Linked_list`

■ **Binary Search Tree**

[Wik] [Binary search tree](#)

`https://en.wikipedia.org/wiki/Binary_search_tree`