Algorithms and Datastructures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

Albert-Ludwigs-Universität Freiburg

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Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, January 2017

Structure

Balanced Trees

- Motivation
- AVL-Trees
- (a,b)-Trees
  - Introduction
  - Runtime Complexity
- Red-Black Trees

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  - if the keys are inserted in ascending / descending order $(20, 19, 18, \ldots)$

Gnarley trees:

Balanced Trees
Motivation

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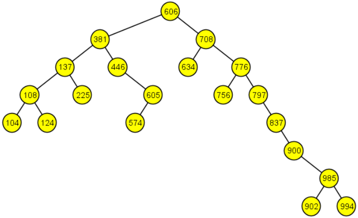

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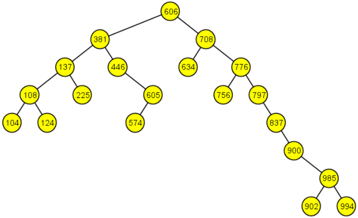

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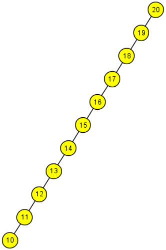

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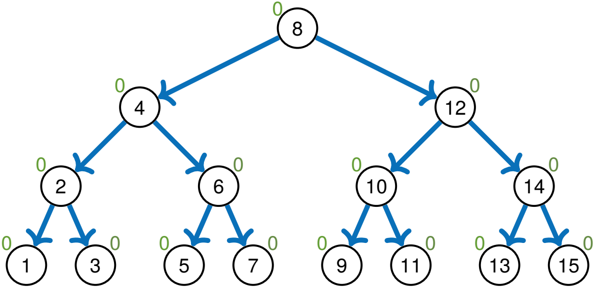
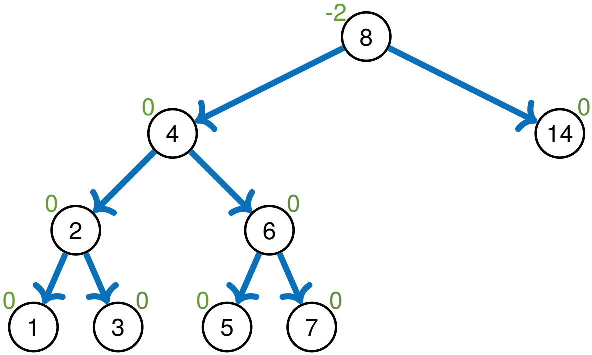


Abbildung: Example of an AVL-Tree



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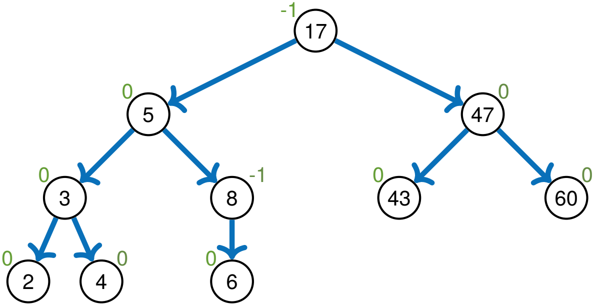


Abbildung: Another example of an AVL-Tree

Rotation:

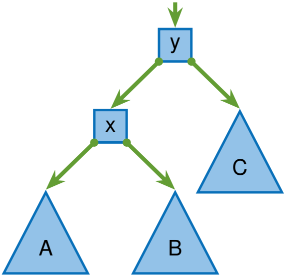
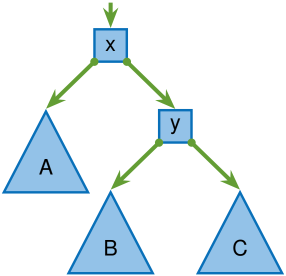


Abbildung: Before rotating

⇒



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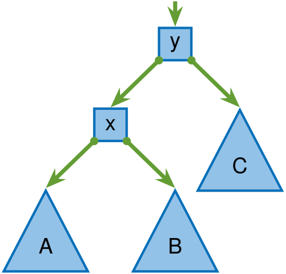

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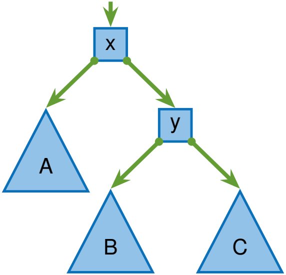
$\Rightarrow$



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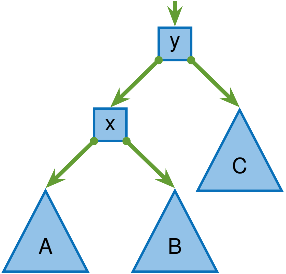

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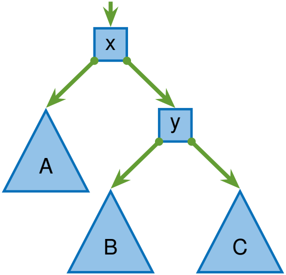

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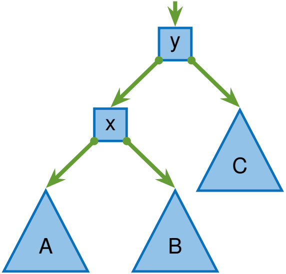

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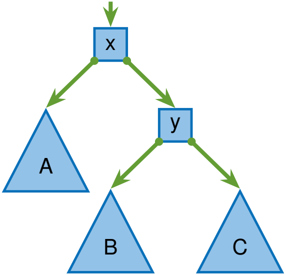

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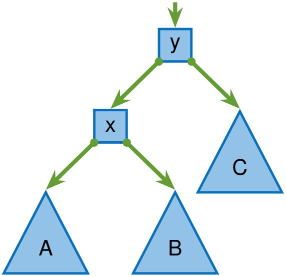

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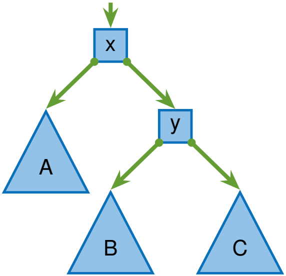

Abbildung: After rotating

- Central operation of rebalancing
- After rotation to the right:
  - Subtree *A* is a layer higher and subtree *C* a layer lower
  - The parent child relations between nodes *x* and *y* have been swapped

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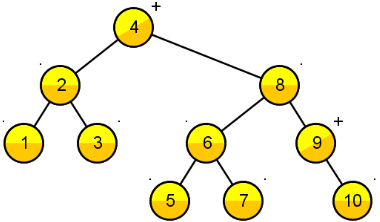

Abbildung: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

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- Better (and easier) to implement are (a,b)-trees

Structure

(a,b)-Tree:

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- So we have space for elements on an `insert` and balance operation

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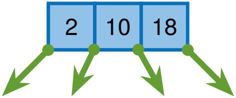

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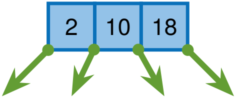

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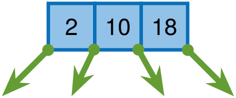

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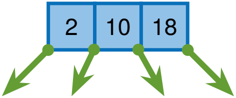

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- We require: $a \geq 2$ and $b \geq 2a-1$

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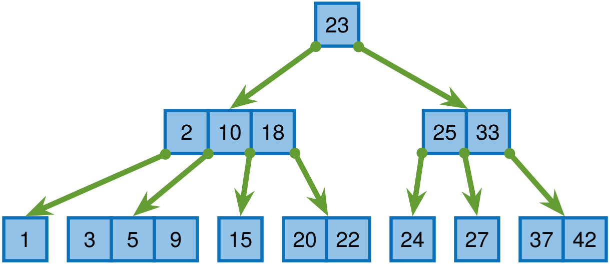

Abbildung: Example of an (2,4)-tree

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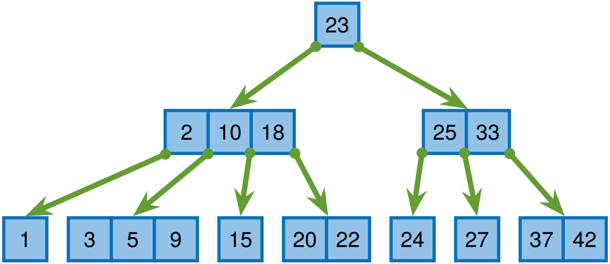

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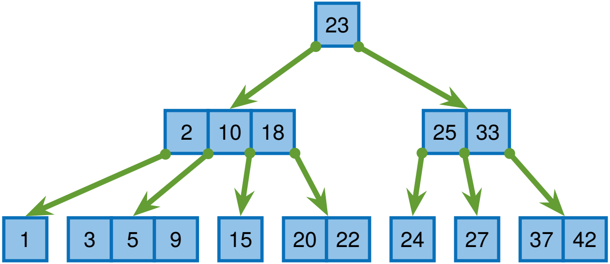

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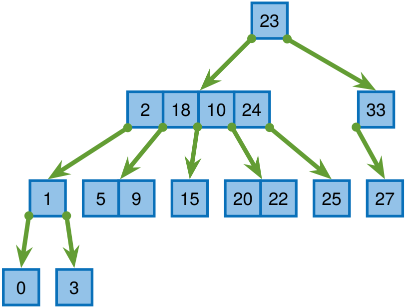
- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

Not an (2,4)-Tree:



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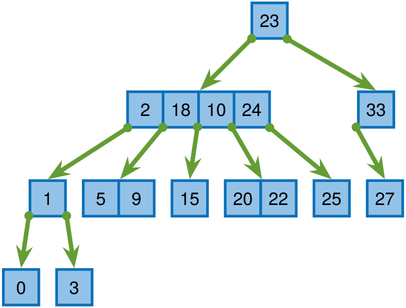

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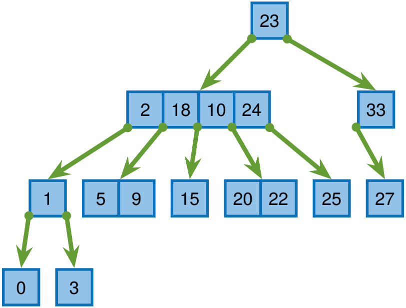

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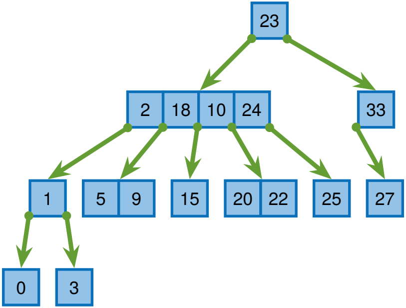

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- Degree of node too large / too small
- Leaves on different levels

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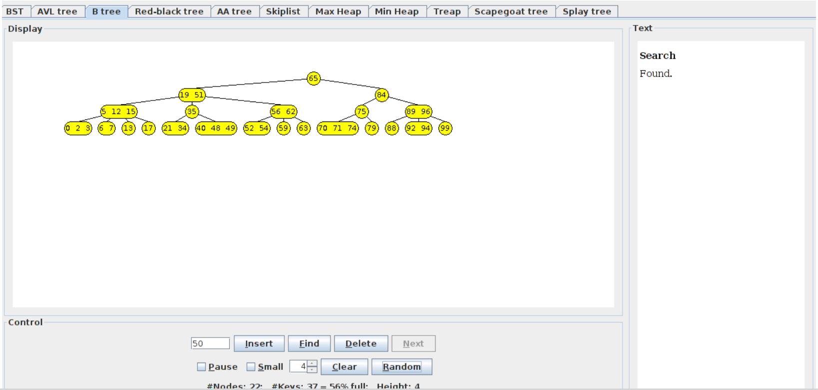

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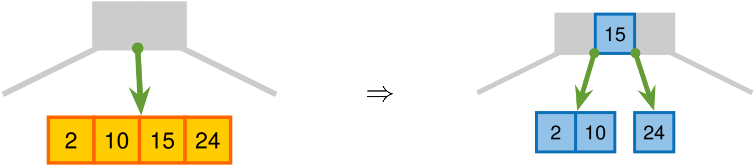

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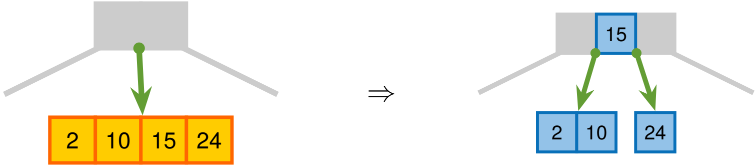

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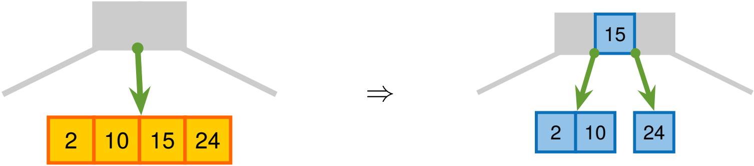

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- If the degree is higher than $b+1$ we split the node
  - This results in a node with $\mathrm{ceil}\left(\frac{b-1}{2}\right)$ elements, a element for the parent node, and a node with $\mathrm{floor}\left(\frac{b-1}{2}\right)$ elements

Inserting an element: (`insert`)

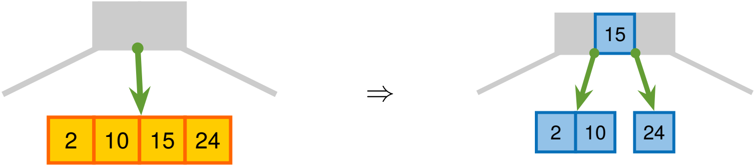

Abbildung: Splitting a node

- If the degree is higher than $b+1$ we split the node
  - This results in a node with $\text{ceil}\left(\frac{b-1}{2}\right)$ elements, a element for the parent node, and a node with $\text{floor}\left(\frac{b-1}{2}\right)$ elements
  - Thats why we have the limit $b \geq 2a-1$

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- We split the parent nodes the same way
- If the node to split is the root we split it and create a new root node
  (The tree is now one level deeper)

Removing an element: (`remove`)

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Removing an element: (`remove`)

- Search the element in $O(\log n)$ time

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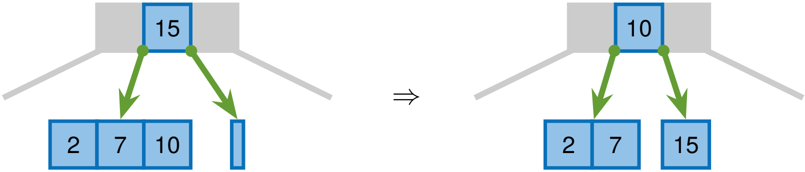

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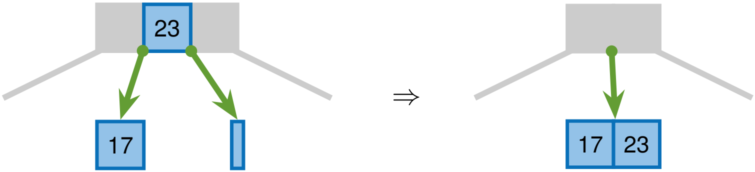

Abbildung: Combining two nodes

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- Now the parent node can be of degree $a - 1$

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- Now the parent node can be of degree $a - 1$
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Removing an element: (`remove`)

- Now the parent node can be of degree $a-1$
- We combine parent nodes the same way
- If the root has only one child left we take the child as new root
  (The tree shrinks one level)

Runtime complexity of `lookup`, `insert` and `remove`:

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- All operations in $O(d)$ with d being the depth of the tree

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  - Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$.

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  - Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$.
  - Here is a counter-example for (2,3)-trees, analysis of (2,4)-trees

(2,3)-Tree:

(2,3)-Tree:

- Before executing `delete(11)`

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(2,3)-Tree:

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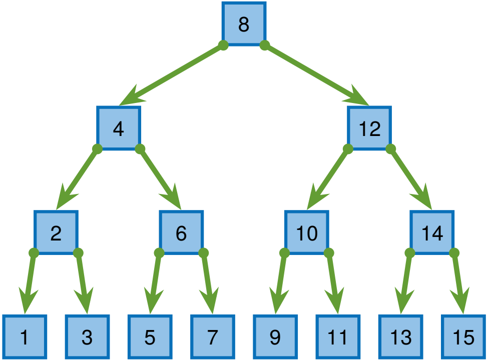

Abbildung: Normal (2,3)-Tree

(2,3)-Tree:

- Executing `delete(11)`

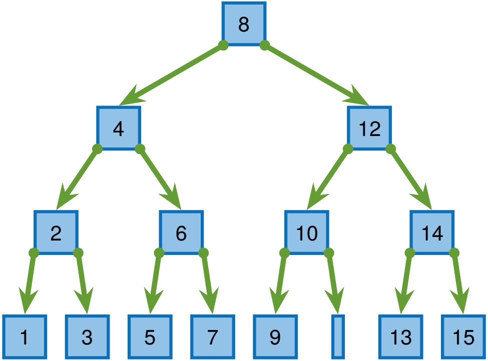

Abbildung: (2,3)-Tree - Delete step 1

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(2,3)-Tree:

- Executing `delete(11)`

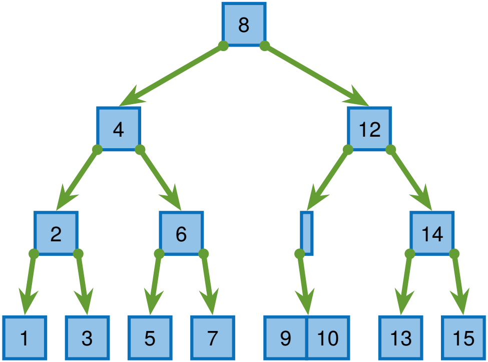

Abbildung: (2,3)-Tree - Delete step 2

(2,3)-Tree:

- Executing `delete(11)`

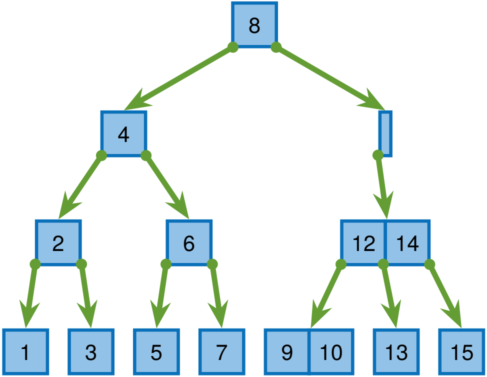

Abbildung: (2,3)-Tree - Delete step 3

(2,3)-Tree:

- Executed `delete(11)`

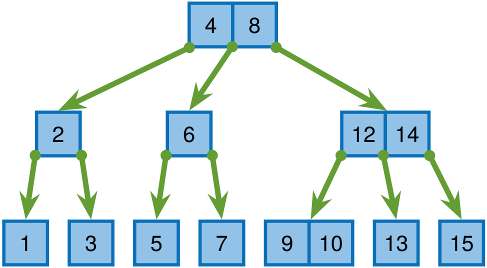

Abbildung: (2,3)-Tree - Delete step 4

(2,3)-Tree:

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- Executing `insert(11)`

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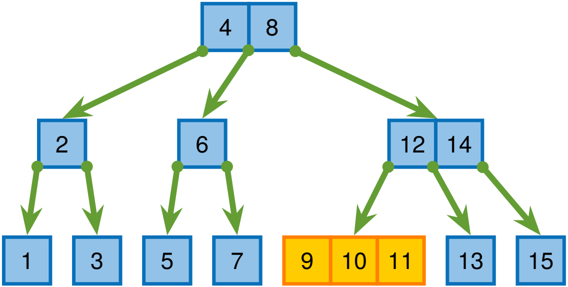

Abbildung: (2,3)-Tree - Insert step 1

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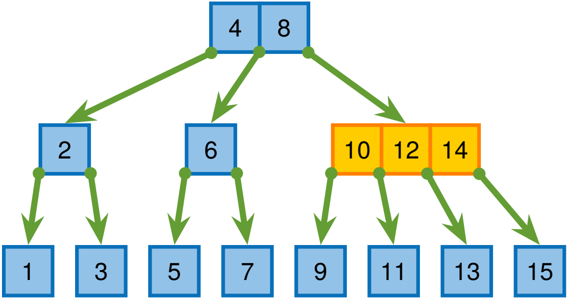

Abbildung: (2,3)-Tree - Insert step 2

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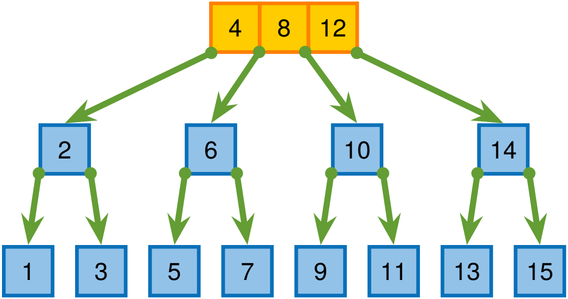

Abbildung: (2,3)-Tree - Insert step 3

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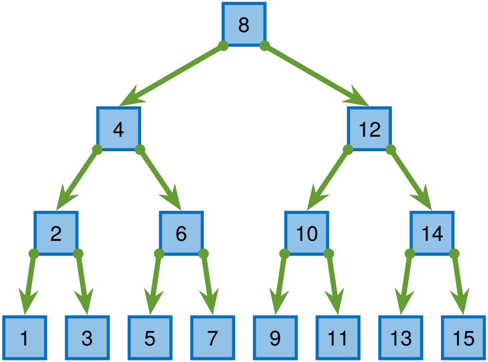

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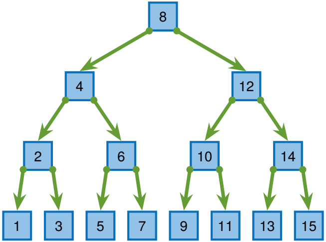

Abbildung: (2,3)-Tree

(2,3)-Tree:

- We are exactly where we started

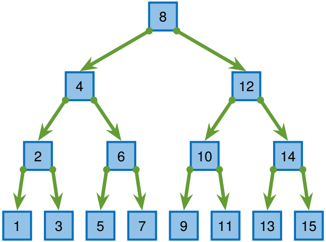

Abbildung: (2,3)-Tree

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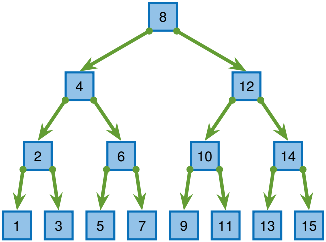
- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$



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(2,3)-Tree:

- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$
- We need $b \geq 2a$ instead of $b \geq 2a - 1$

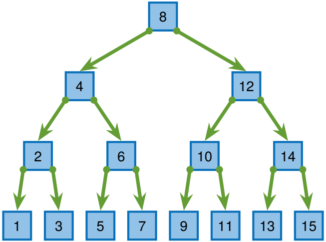

Abbildung: (2,3)-Tree

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⇒ **Nodes of degree 3 are harmless**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

(2,4)-Tree:

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- **Idea:**

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  - After an expensive operation the tree is in a stable state

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(2,4)-Tree:

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  - After an expensive operation the tree is in a stable state
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- Like with dynamic arrays:
  - Reallocation is expensive but it takes some time until the next expensive operation occurs
  - If we overallocate clever we have an amortized runtime of $O(1)$

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Example:

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- Nodes of degree 3 are highlighted

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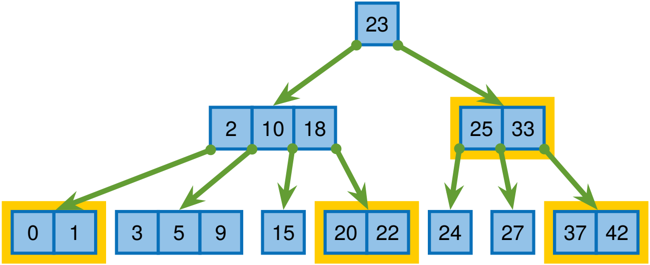

Abbildung: Tree with potential $\Phi = 4$

Terminology:

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- Let c_i be the costs = runtime of the i-th operation

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- The costs for operation i are coupled to the difference of the potential levels

$$c_i \leq A \cdot \underbrace{(\Phi_i - \Phi_{i-1})} + B, \qquad A > 0 \text{ and } B > A$$

Number of harmless (degree 3) nodes at operation i. Can be -1, but not smaller than -1

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- With that each operation has an amortitzed cost of $O(1)$

Case 1: *i-th* operation is an `insert` operation on a full node

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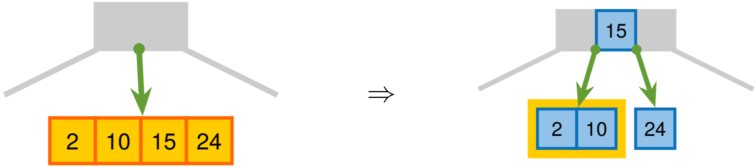

Abbildung: Splitting a node on `insert`

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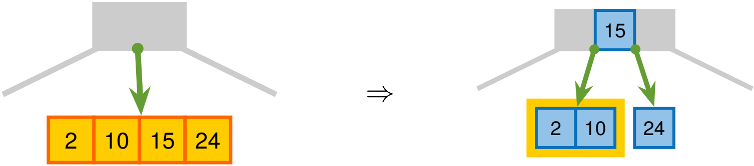

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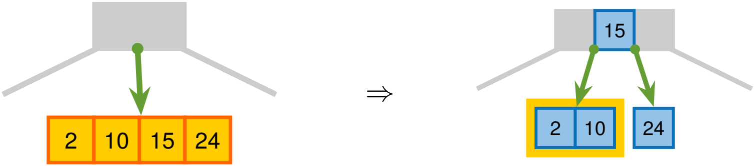

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- Each splitted node creates a node of degree 3
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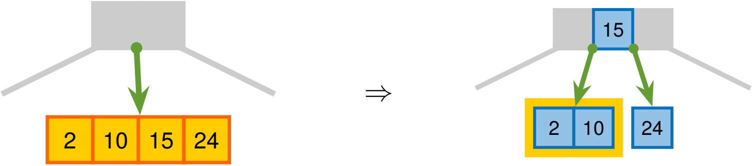

Abbildung: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

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Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

Case 2: *i-th* operation is an `remove` operation

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- **Case 2.1:** Inner node

Case 2: *i-th* operation is an `remove` operation

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  - Searching the successor in a tree is $O(d) = O(\log n)$

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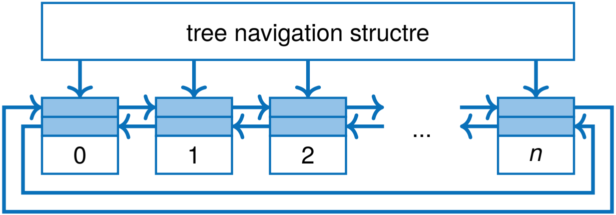

Abbildung: Tree with doubly linked list

Case 2: *i-th* operation is an `remove` operation

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- **Case 2.1:** Borrowing a node

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- **Case 2.1:** Borrowing a node
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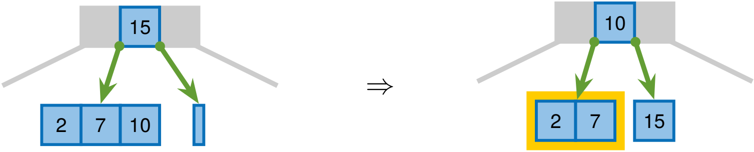

Abbildung: Borrowing an element case 2.1.1

Case 2: *i-th* operation is an `remove` operation

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- **Case 2.1:** Borrowing a node

Case 2: *i-th* operation is an `remove` operation

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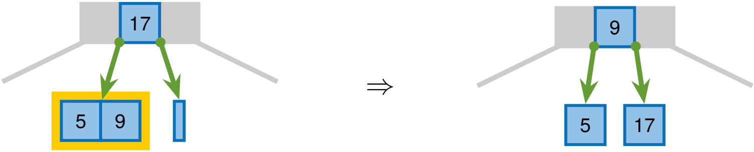

Abbildung: Borrowing an element case 2.1.2

Case 2: *i-th* operation is an `remove` operation

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- **Case 2.2:** Merging a node

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- **Case 2.2:** Merging a node

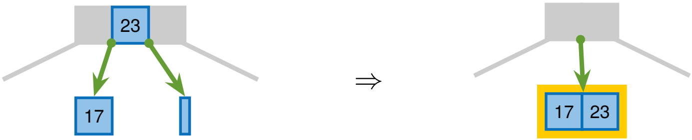

Abbildung: Merging two nodes

  - Potential rises by one

Case 2: *i-th* operation is an `remove` operation

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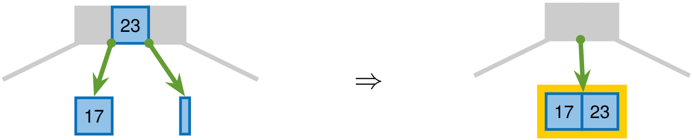

Abbildung: Merging two nodes

  - Potential rises by one
  - Parent node has one element less after the operation

Case 2: *i-th* operation is an `remove` operation

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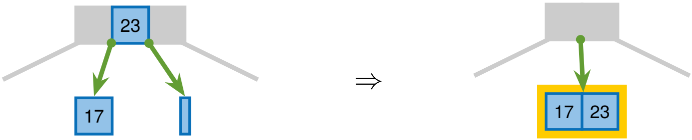

Abbildung: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree $>$ 2 or a degree 2 node, which can borrow from a neighbour

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging a node

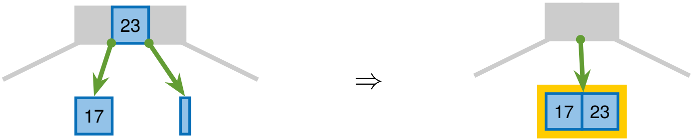

Abbildung: Merging two nodes

- Potential rises by one
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- This operation propagates upwards until a node of degree > 2 or a degree 2 node, which can borrow from a neighbour
- The potential rises by m

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Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging a node

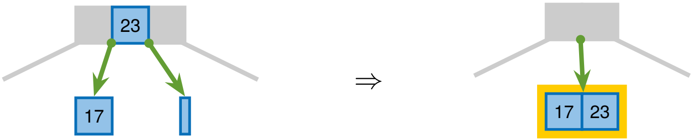

Abbildung: Merging two nodes

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- **Case 2.2:** Merging a node

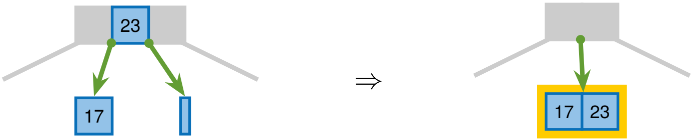

Abbildung: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree > 2 or a degree 2 node, which can borrow from a neighbour
- The potential rises by m
- If the „stop-node“ is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

Lemma:

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- We know:

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$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^{n} c_i = O(n)$$

Proof:

$$\sum_{i=0}^{n} c_i \leq \underbrace{A \cdot (\Phi_1 - \Phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\Phi_2 - \Phi_1) + B}_{\leq c_1} + \cdots + \underbrace{A \cdot (\Phi_n - \Phi_{n-1}) + B}_{\leq c_n}$$

$$= A \cdot (\Phi_n - \Phi_0) + B \cdot n \qquad | \text{ telescope sum}$$

$$= A \cdot \Phi_n + B \cdot n \qquad | \text{ we start with an empty tree}$$

$$< A \cdot n + B \cdot n = O(n) \qquad | \text{ number of degree 3 nodes} < \text{number of nodes}$$

Structure

Balanced Trees

- Motivation
- AVL-Trees
- (a,b)-Trees
  - Introduction
  - Runtime Complexity
- Red-Black Trees

Red-Black Tree:

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- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)

Red-Black Tree:

- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

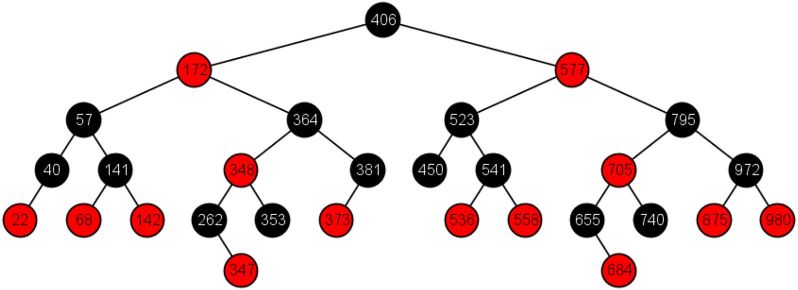

Abbildung: Example of an red-black-tree [Gna]

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
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`https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf`.

- **Gnarley Trees**

[Gna] Gnarley Trees
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- **AVL-Tree**

[Wik] AVL tree
https://en.wikipedia.org/wiki/AVL_tree

- **(a,b)-Tree**

[Wika] 2-3-4 tree
https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934_tree

[Wikb] (a,b)-tree
https://en.wikipedia.org/wiki/(a,b)-tree

Red-Black-Tree

[Wik] Red-black tree
https://en.wikipedia.org/wiki/Red%E2%80%93black_tree