

Algorithms and Datastructures

Graphs, Depth-/Breadth-first Search, Graph-Connectivity

Albert-Ludwigs-Universität Freiburg



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Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Algorithms and Datastructures, January 2017

Feedback

Exercises

Lecture

Graphs

Introduction

Implementation

Application example

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The upcoming exercise sheet 12 and 13 will be merged together (finding largest connected component + Dijkstra)

Some people were asking for more solution sheets for the exercises

We are working on it.

Code in the lecture will be a little bit different from exercise sheet.

One person asked for additional explanations regarding proofs.

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Graphs - Overview:

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- Representation of graphs in the computer
- Breadth first search (BFS)
- Depth first search (DFS)
- Connected components of a graph

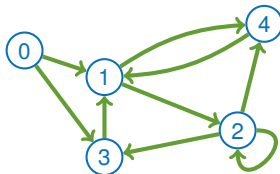
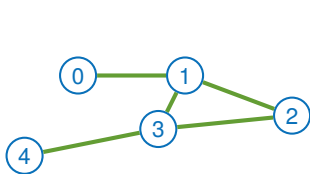


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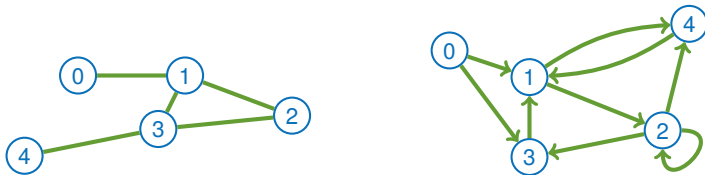
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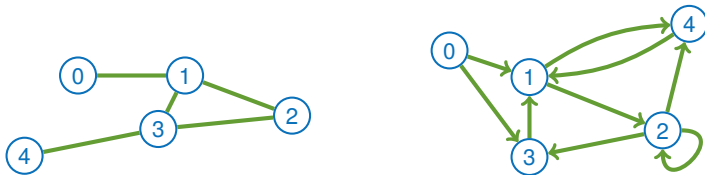
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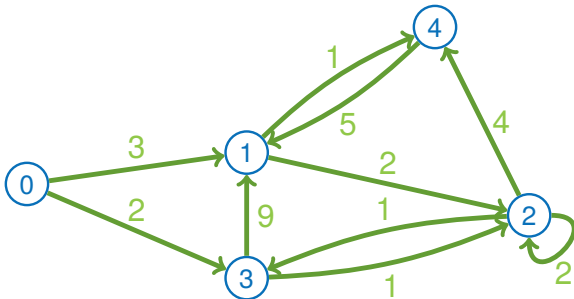


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- Each edge connects two vertices ($u, v \in V$)
 - Undirected edge: $e = \{u, v\}$ (set)
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- Self-loops are also possible: $e = (u, u)$ or $e = \{u, u\}$



Weighted graph:

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- Each edge is marked with a real number named **weight**

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- Each edge is marked with a real number named **weight**
- The **weight** is also named **length** or **cost** of the edge depending on the application

Example: Road network

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- Intersections:
vertices

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- Roads: edges

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Example: Road network

- Intersections: **vertices**
- Roads: **edges**
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Figure: Map of Freiburg © OpenStreetMap

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How to represent this graph computationally?



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- Two classic variants

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- 1 Adjacency matrix with space consumption $\Theta(|V|^2)$

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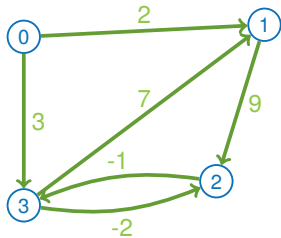


Figure: Weighted graph with
 $|V| = 4, |E| = 6$

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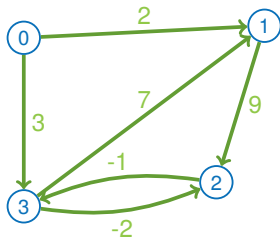


Figure: Weighted graph with $|V| = 4$, $|E| = 6$

		end-vertex			
		0	1	2	3
start-vertex	0		2		3
	1			9	
	2				-1
	3		7	-2	

Figure: Adjacency matrix



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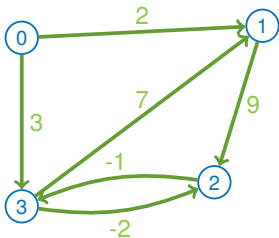


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2 Adjacency list / fields with space consumption $\Theta(|V| + |E|)$

- Each list item stores the **target vertex** and the **cost** of the edge



start-vertex	0	1, 2	3, 3
	1	2, 9	
	2	3, -1	
	3	1, 7	2, -2

Figure: Adjacency list

Figure: Weighted graph with
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Graph: Arrangement



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- Graph is fully defined through the [adjacency matrix / list](#)

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Figure: Weighted graph with
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Figure: Same graph ordered by number - outer planar graph

```
class Graph:
    def __init__(self):
        self.vertices = []
        self.edges = []

    def addVertice(self, vert):
        self.vertices.append(vert)

    def addEdge(self, fromVert, toVert):
        self.edges.append((fromVert, toVert))

    ...
```


...

```
def toString(self):  
    return '{'  
        + ', '.join( \  
            [str(len(self.vertices)), \  
              str(len(self.edges))] \  
        + ["(%s, %s)" % tup \  
          for tup in self.edges]) \  
        + '}'
```



Degree of a vertex: Directed graph: $G = (V, E)$

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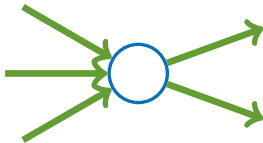


Figure: Vertex with in- / outdegree of 3 / 2

Degree of a vertex: Directed graph: $G = (V, E)$

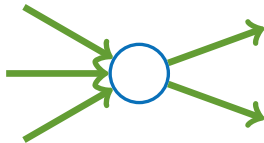


Figure: Vertex with in- / outdegree of 3 / 2

- **Indegree** of a vertex u is the number of **edge heads** adjacent to the vertex

$$\deg^+(u) = |\{(v, u) : (v, u) \in E\}|$$

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$$\deg^-(u) = |\{(u, v) : (u, v) \in E\}|$$

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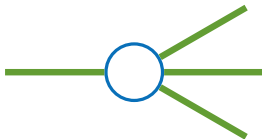


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Paths in a graph: $G = (V, E)$

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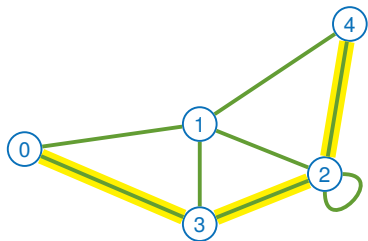


Figure: Undirected path of length 3
 $P = (0, 3, 2, 4)$

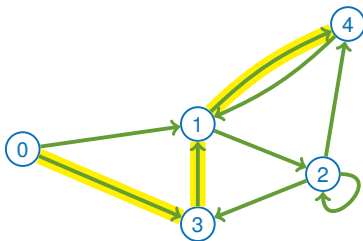


Figure: Directed path of length 3
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Paths in a graph: $G = (V, E)$

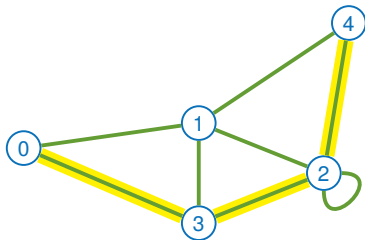


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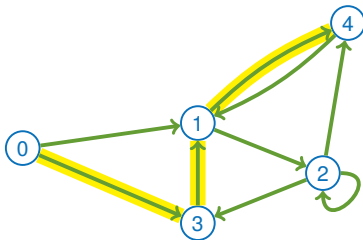


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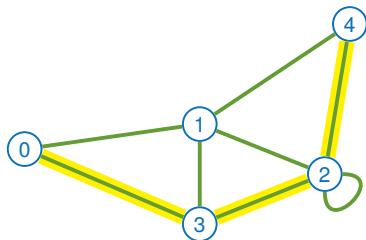


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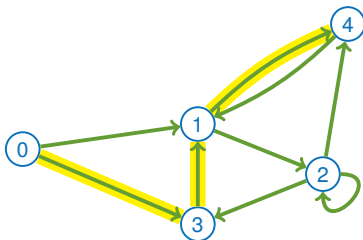


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 - Undirected graph: $\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{i-1}, u_i\} \in E$
 - Directed graph: $(u_1, u_2), (u_2, u_3), \dots, (u_{i-1}, u_i) \in E$

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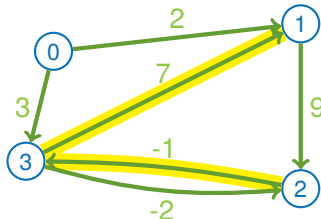


Figure: Weighted path with cost 6
 $P = (2, 3, 1)$

Paths in a graph: $G = (V, E)$



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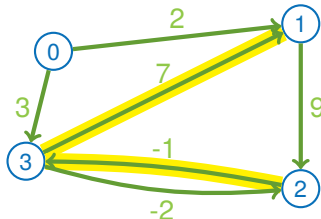


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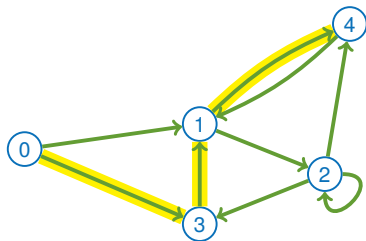


Figure: Directed path of length 3
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Paths in a graph: $G = (V, E)$



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Paths in a graph: $G = (V, E)$



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Figure: Weighted path with cost 6
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 - With weights: **sum of weights of edges** taken



Shortest path in a graph: $G = (V, E)$

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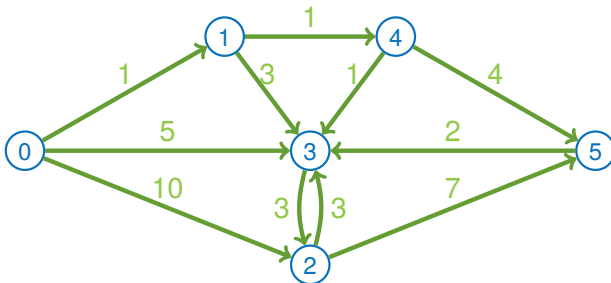


Figure: Shortest path from 0 to 2 with cost / distance $d(0,2) = ?$

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- The shortest path between two vertices u, v is the path $P = (u, \dots, v)$ with the shortest length $d(u, v)$ or lowest costs

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Figure: Shortest path from 0 to 2 with cost / distance $d(0,2) = 6$
 $P = (0, 1, 4, 3, 2)$

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Diameter of a graph: $G = (V, E)$

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Figure: Diameter of graph is $d = 10$, $P = (3, 2, 5)$

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Connected components: $G = (V, E)$

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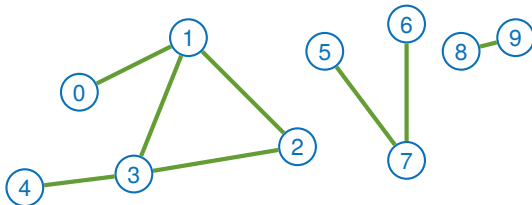


Figure: Three connected components

- Undirected graph:

Connected components: $G = (V, E)$



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- Undirected graph:
 - All connected components are a partition of V

$$V = V_1 \cup \dots \cup V_k$$

Connected components: $G = (V, E)$



Figure: Three connected components

- Undirected graph:
 - All connected components are a partition of V

$$V = V_1 \cup \dots \cup V_k$$

- Two vertices u, v are in the same connected component if a path between u and v exists

Connected components: $G = (V, E)$

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 - Named **strongly connected components**

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 - Direction of edge has to be regarded

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- Directed graph:
 - Named **strongly connected components**
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 - Not part of this lecture

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- Let $G = (V, E)$ be a graph and $s \in V$ a start vertex
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- **Depth-first search**: in sequence of the largest distance to s

Graph Exploration: (Informal definition)

- Let $G = (V, E)$ be a graph and $s \in V$ a start vertex
- We visit each reachable vertex connected to s
- **Breadth-first search**: in sequence of the smallest distance to s
- **Depth-first search**: in sequence of the largest distance to s
- Not a problem on its own but is often used as subroutine of other algorithms



Idea:



Idea:

- 1 We start with all vertices unmarked and **mark visited vertices**

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- 2 Mark the start vertex s (level 0)

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- 2 Mark the start vertex **s (level 0)**
- 3 Mark all unmarked **connected vertices (level 1)**
- 4 Mark all unmarked **vertices connected** to a **level 1-vertex (level 2)**
- 5 Iteratively mark reachable vertices for all levels
- 6 All connected nodes are now marked and in the same **connected component** as the start vertex **s**

Graphs

Connected Components - Breadth-First Search



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- The marked vertices create a “spanning tree” containing all reachable nodes

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Figure: spanning tree of a breadth-first search

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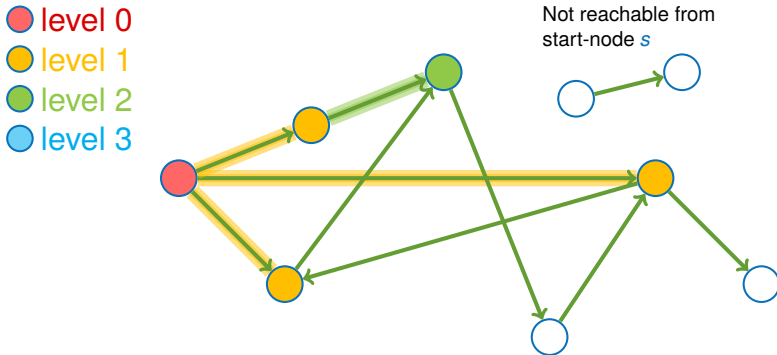


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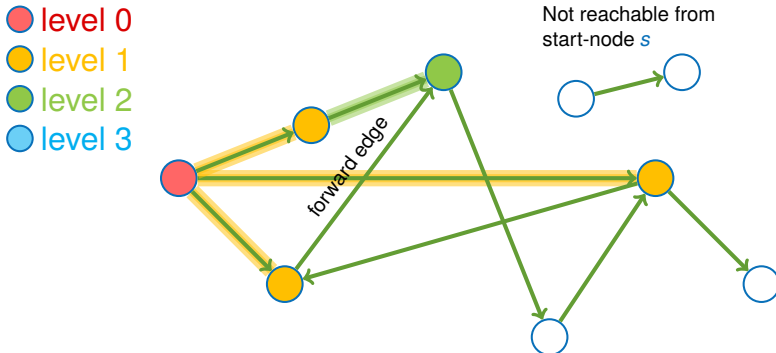


Figure: spanning tree of a breadth-first search

- 



- level 0
- level 1
- level 2
- level 3

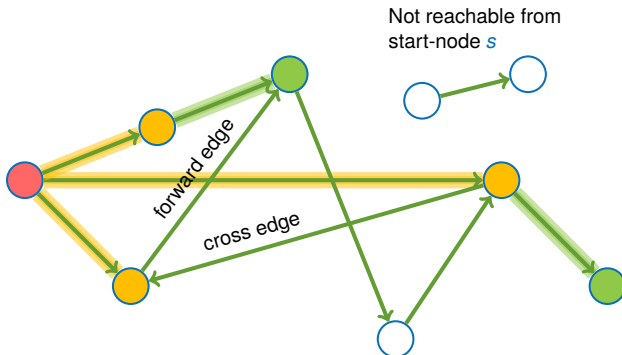


Figure: spanning tree of a breadth-first search

- Prof. Dr. Rolf Backofen – beamer-ufcd



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- 3 Pick an unmarked **connected vertex** and start a **recursive depth-first search** with the vertex as start vertex
(continue on step 2)

Idea:

- 1 We start with all vertices unmarked and **mark visited vertices**
- 2 Mark the start vertex **s**
- 3 Pick an unmarked **connected vertex** and start a **recursive depth-first search** with the vertex as start vertex (continue on step 2)
- 4 If no unmarked connected vertex exists go one vertex back (reduce the recursion level by one)



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- Search starts with **long paths** (searching with depth)
- Marks like **breadth-first search** all connected vertices
- If the graph is acyclic we get a **topological sorting**
 - Each newly visited vertex gets marked by an increasing number
 - The numbers increase with path from the start vertex

Graphs

Connected Components - Depth-First Search



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- The marked vertices create a different spanning tree containing all reachable nodes

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● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

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● path 2

● path 3

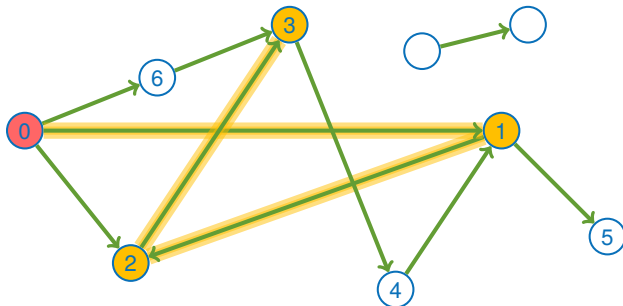


Figure: spanning tree of a depth-first search

- The marked vertices create a different spanning tree containing all reachable nodes

● start-node

● path 1

● path 2

● path 3

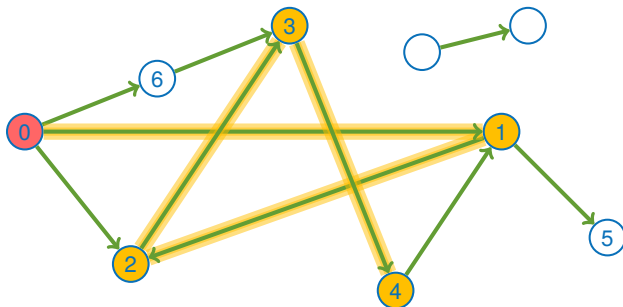


Figure: spanning tree of a depth-first search

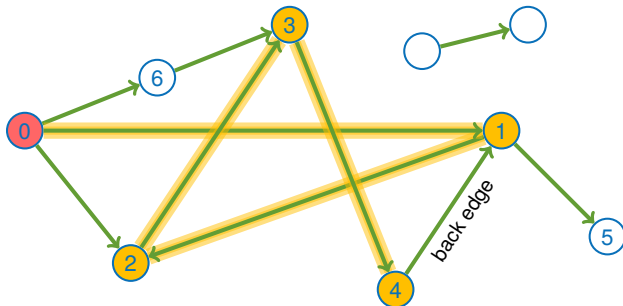
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● path 1

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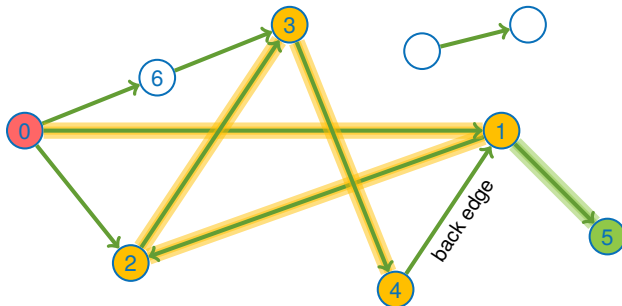


Figure: spanning tree of a depth-first search

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● start-node

● path 1

● path 2

● path 3

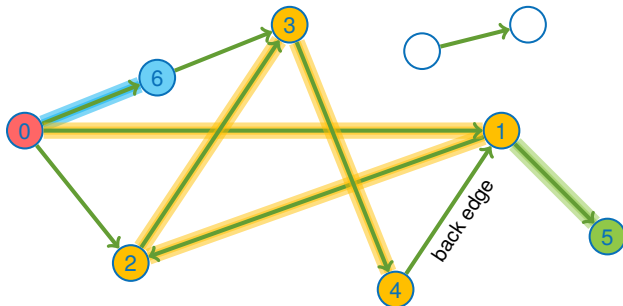


Figure: spanning tree of a depth-first search

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● path 1

● path 2

● path 3

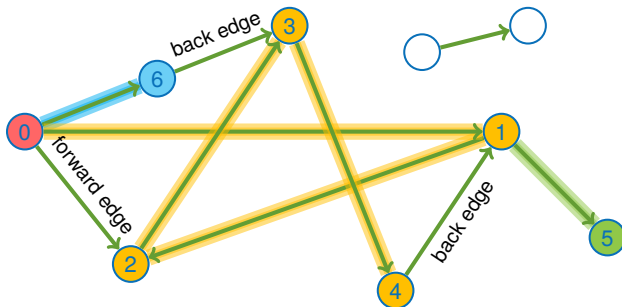


Figure: spanning tree of a depth-first search

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● start-node

● path 1

● path 2

● path 3



Figure: spanning tree of a depth-first search

Graphs

Why is this called Breadth - and Depth First Search?



**UNI
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Runtime complexity:

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- Constant costs for each visited vertex and edge

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- All vertices of V' are in the same connected component as our start vertex s
- This can only be improved by a constant factor

Feedback

Exercises

Lecture

Graphs

Introduction

Implementation

Application example

Application example

Image processing



**UNI
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- Connected component labeling

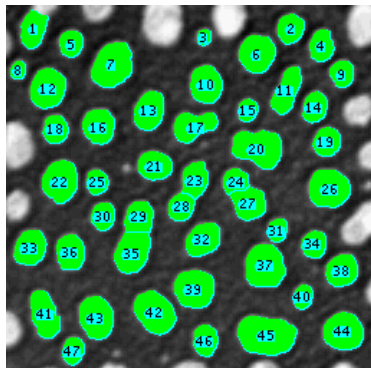
Application example

Image processing

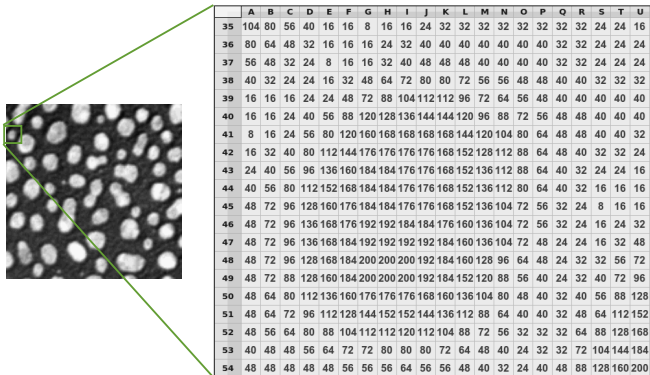


- Connected component labeling
- Counting of objects in an image

- Connected component labeling
- Counting of objects in an image



What's object, what's background?



Convert to black white using threshold:

value = 255 if value > 100 else 0





Interpret image as graph:

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Interpret image as graph:

- Each white pixel is a node
- Edges between adjacent pixels (normally 4 or 8 neighbors)
- Edges are not saved externally, algorithm works directly on array
- Breadth- / depth-first search find all connected components (particles)



Find connected components:

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	255

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255
40	0	0	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
51	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255
52	0	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	255	255
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	255

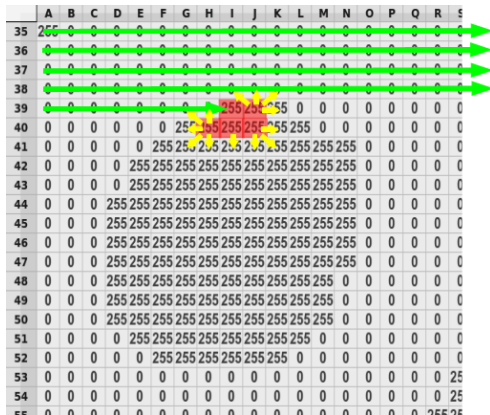
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



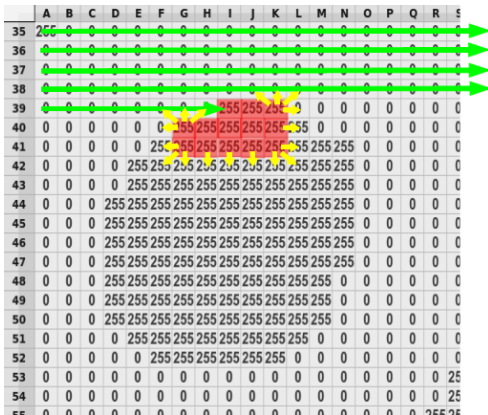
- Search pixel-by-pixel for non-zero intensity
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Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	255	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

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Find connected components:



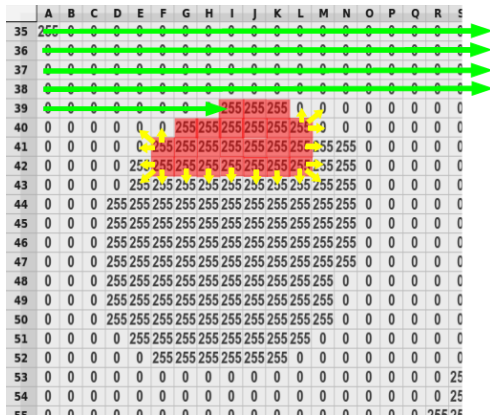
- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	255

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:



- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Find connected components:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	25	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as **component 2**
- ...

Result of connected component labeling:

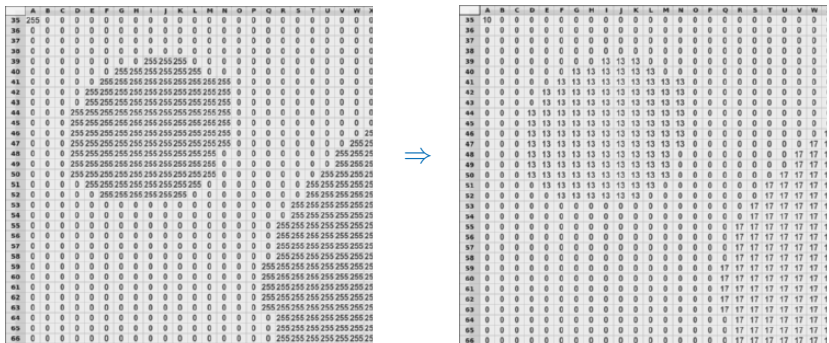


Figure: Result: particle indices instead of intensities

■ General

- [CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

- [MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

■ Graph-Search

[Wika] [Breadth-first search](#)

`https://en.wikipedia.org/wiki/
Breadth-first_search`

[Wikb] [Depth-first search](#)

`https:
//en.wikipedia.org/wiki/Depth-first_search`

■ Graph-Connectivity

[Wik] [Connectivity \(graph theory\)](#)

`https://en.wikipedia.org/wiki/Connectivity_
(graph_theory)`