# Algorithmns and Datastructures

## Linked Lists, Binary Search Trees

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Algorithmns and Datastructures, January 2017

# Structure

Feedback
- Exercises
- Lecture

Sorted Sequences

Linked Lists

Binary Search Trees

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- Lecture recordings are now password protected

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  - `next()/previous()`: Returns the element with the next bigger/smaller key. This enables iteration over all elements.

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- How could we implement this?

**Static array:**

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  - We have to copy up to $n$ elements

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  - The order of the elements is independent of the order of the keys

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- Lets have a closer look

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- Single / Doubly linked lists possible

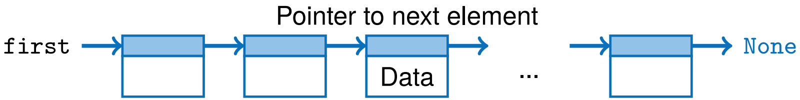


Figure: Linked list

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- The number of elements can be simply modified
- No direct access of elements
  $\Rightarrow$ We have to iterate over the list

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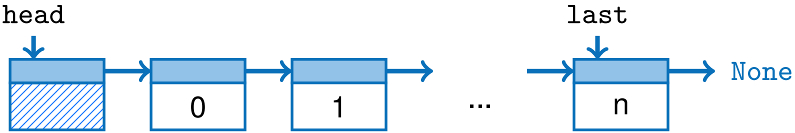


Figure: Singly linked list

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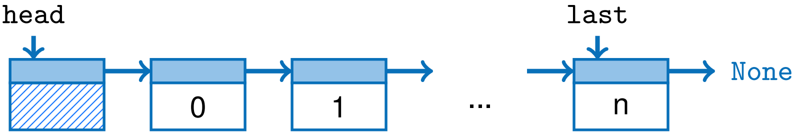


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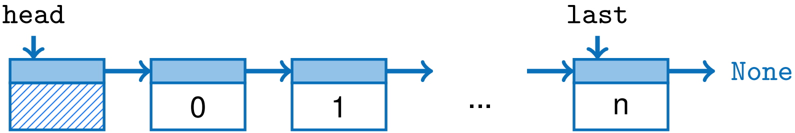


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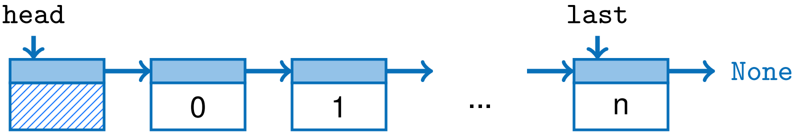


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- Head element has pointer to first list element
- May also hold additional information:
  - Number of elements

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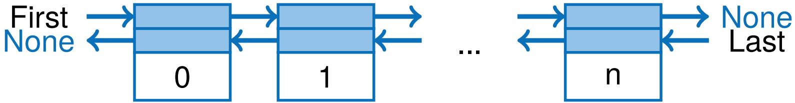


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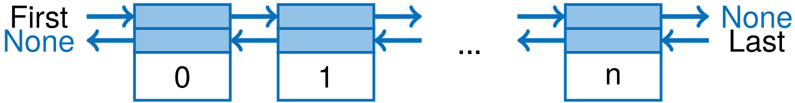


Figure: Doubly linked list

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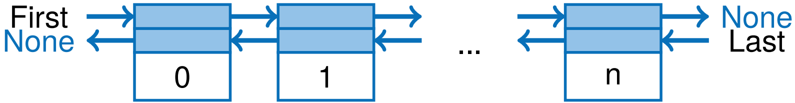


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- Pointer to successor element
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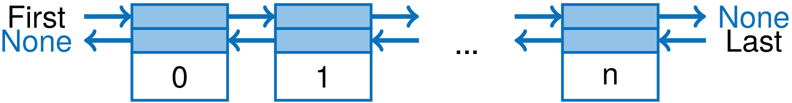


Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

```
public class Listelem
```

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public class Listelem
{   //2 fields: integer and reference
```

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public class Listelem
{   //2 fields: integer and reference
    //private only available in class
    private int data;
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```

# Linked Lists

## Implementation - Node/Element - Java

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```java
public class Listelem
{   //2 fields: integer and reference
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    //adopted from Mary K.Vernon
    //Introduction to Data Structures
```

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## Implementation - Node/Element - Java

```java
//Function to read and write private fields
public int getData() {return data; }
public void setData(int d) { data = d; }
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public Listelem getNext() { return next; }
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    //Function to read and write private fields
    public int getData() {return data; }
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    //Integer represents possible data, e.g.
    //self defined refence datatypes
}
```

# Linked Lists

## Implementation - Node/Element - C++

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class Listelem
{
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```
class Listelem
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private:
  int data;
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  Listelem* getNext() { return next; }
  void setNext(Listelem* n) { next = n; }
}
```

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```

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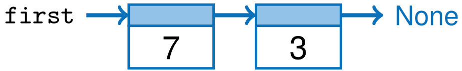
- `first.nextNode = Node(3)`



# Linked Lists
## Usage examples

### Creating linked lists - Python:

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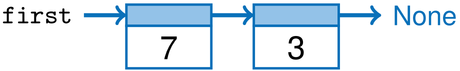
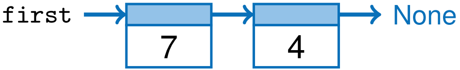
- `first.nextNode = Node(3)`




- `first.nextNode.value = 4`



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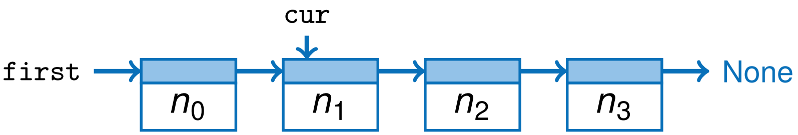

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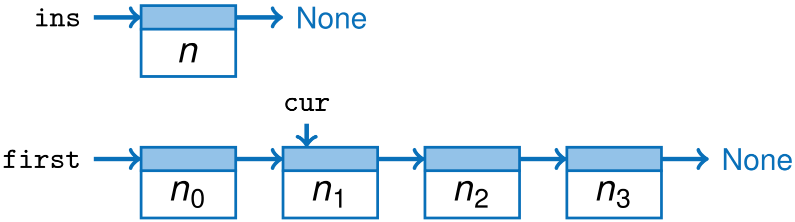

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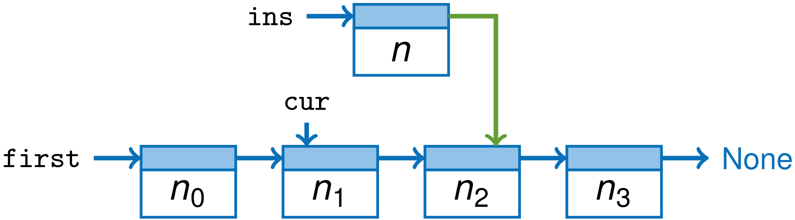

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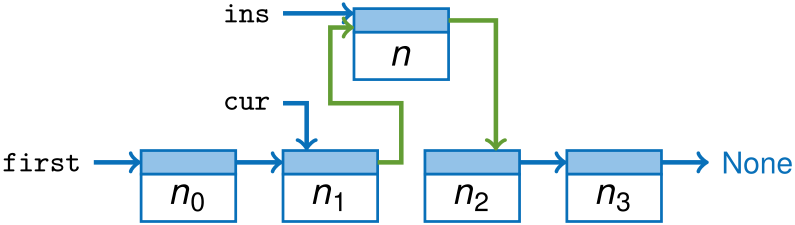

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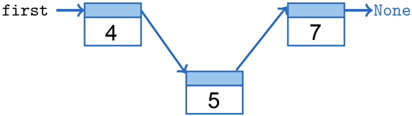

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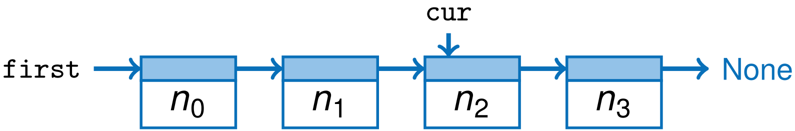

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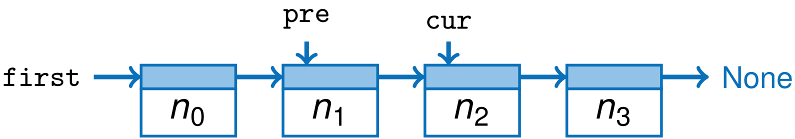

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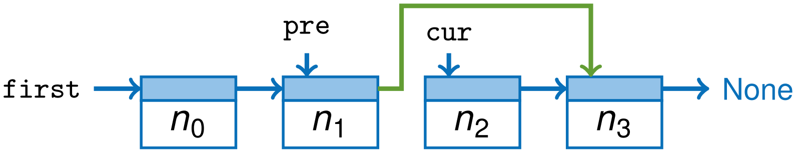

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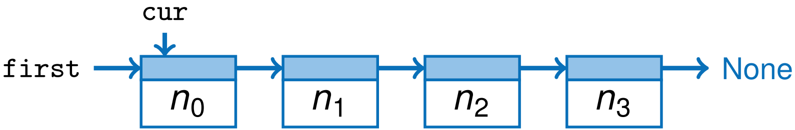

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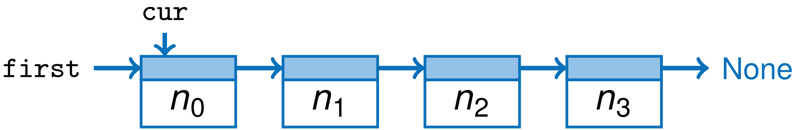


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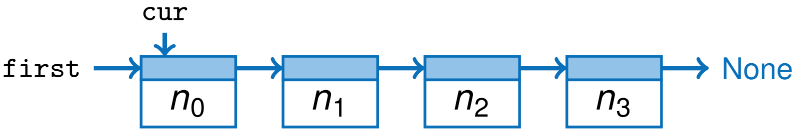


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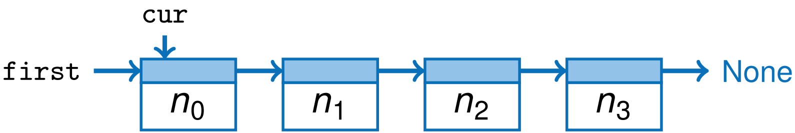
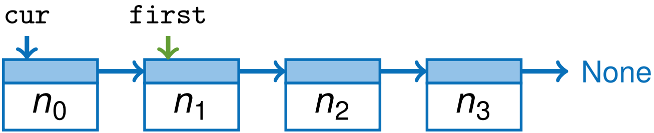
**Removing the first node:**




- Update the pointer to the next element:
  `first = first.nextNode`
- `cur` will get automaticly destroyed if no more references exist (`cur=None`)



**Removing a node `cur`:** (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```

**Using a `head` node:**

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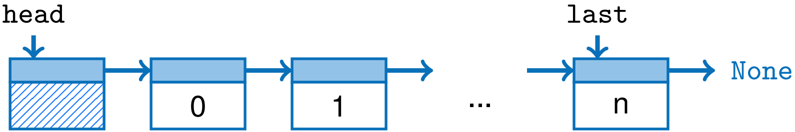

```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```

```python
def append(self, value):
    ...

def insertAfter(self, cur, value):
    ...

def remove(self, cur):
    ...

def get(self, position):
    ...

def contains(self, value):
    ...
```

```java
/**
 * A singly linked list with data type int.
 */
public class LinkedList {

    private long itemCount;
    private Node head;
    private Node last;

    public LinkedList() {
        itemCount = 0;
        head = new Node();
        last = head;
    }
```

```java
        public int size() {
        return itemCount;
    }

    public boolean isEmpty() {
        return (itemCount == 0);
    }

public void add (int data) { ... }
    public void insertAfter(Node cur, int data)
        { ... }
    public void remove(Node cur) { ... }
    public Node get(int position) { ... }
    public boolean contains( int data) { ... }
}
```

**Head, last:**

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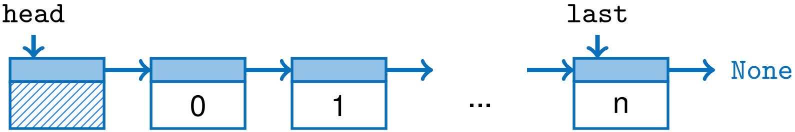
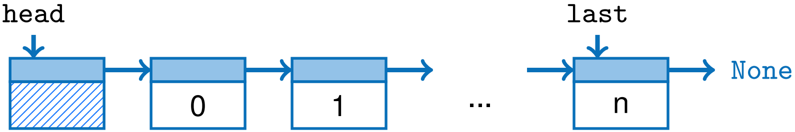


## Head, last:




- `Head` points to the first node, `last` to the last node

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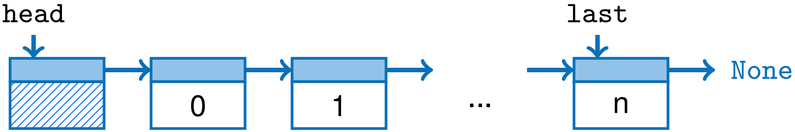


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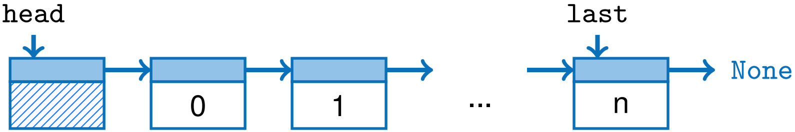


- `Head` points to the first node, `last` to the last node
- We can append elements to the end of the list in $O(1)$ through the `last` node
- We have to keep the pointer to `last` updated after all operations

**Appending an element:**

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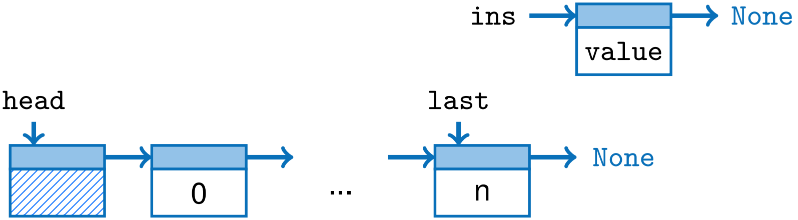
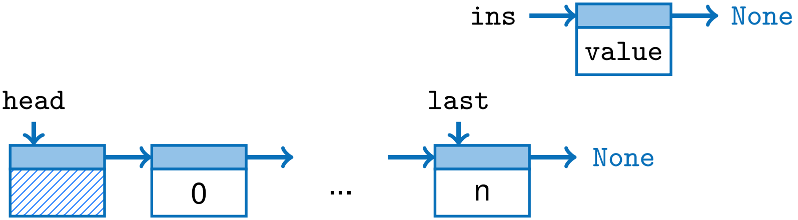


**Appending an element:**




- ```
  def append(self, value):
      last.nextNode = Node(value)
      last = last.NextNode
      itemCount += 1
  ```
  ```

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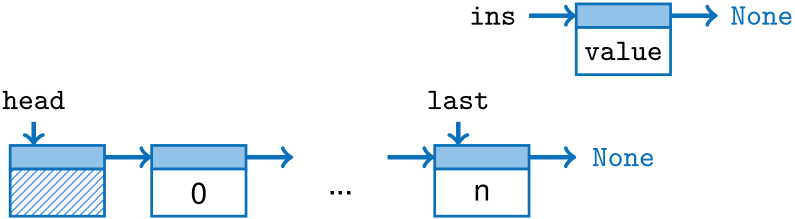
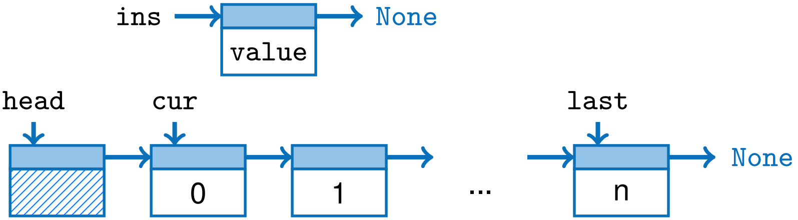



- ```
  def append(self, value):
      last.nextNode = Node(value)
      last = last.NextNode
      itemCount += 1
  ```
- The pointer to `last` avoids the iteration of the whole list
  ```

**Inserting after node `cur`:**



**Inserting after node `cur`:**

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- The pointer to `head` is not modified

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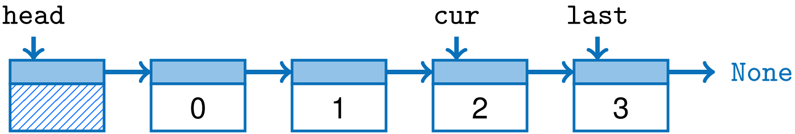
- ```
  def insertAfter(self, cur, value):
      if cur == last:
          # also update last node
          append(value)
      else:
          # last node is not modified
          cur.nextNode = Node(value, \
              cur.nextNode)
          itemCount += 1
  ```
  ```

**Remove node `cur`:**



**Remove node `cur`:**

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- Searching the predecessor in $O(n)$

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- Searching the predecessor in $O(n)$

```
def remove(self, cur):
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
    itemCount -= 1

    if pre.nextNode == None:
        last = pre
```

**Getting a reference to node at `pos`:**

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- Iterate the entries of the list until at position ($O(n)$)

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```
def get(self, pos):
    if pos < 0 or pos >= itemCount:
        return None

    cur = head
    for i in range(0, pos):
        cur = cur.nextNode

    return cur
```

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```
def contains(self, value):
    cur = head

    for i in range(0, itemCount):
        cur = cur.nextNode
        if cur.value == value:
            return true

    return false
```

**Runtime:**

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**Runtime:**

- Singly linked list:

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- Better with doubly linked lists

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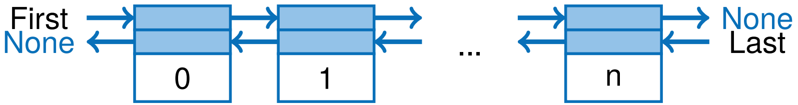

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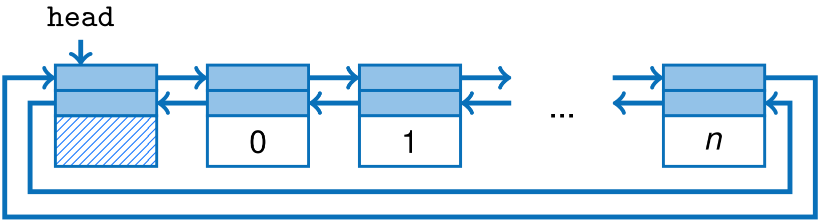

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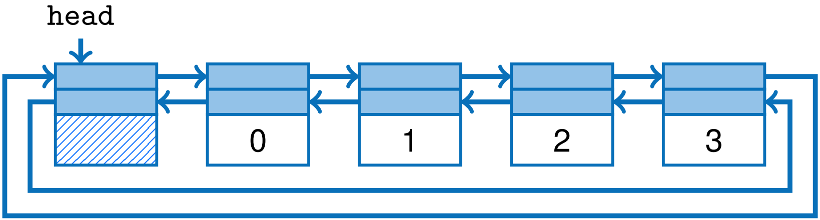
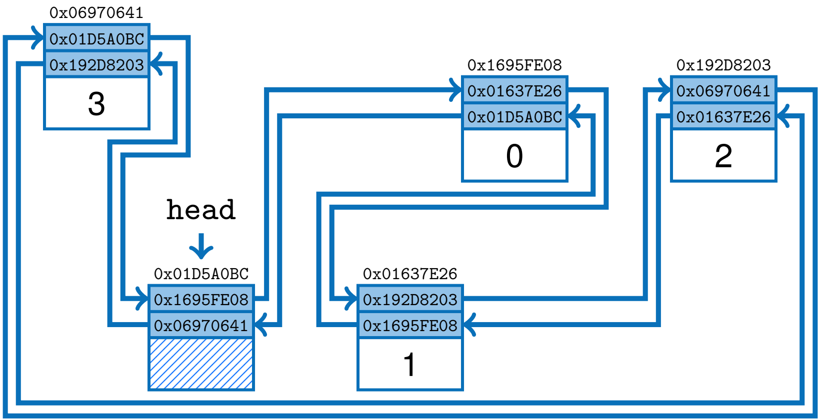
        - a constant number of pointers needs to be modified
    - `lookup` in $\Theta(n)$
        - Even if the elements are sorted we can only retrieve them in $\Theta(n)$.
          Why?

**Linked list in book:**



## Linked list in memory:



# Structure

Feedback

- Exercises
- Lecture

Sorted Sequences

Linked Lists

Binary Search Trees

**Runtime of a search tree:**

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  - We will see why
- `lookup` in $O(\log n)$
  - The structure helps searching efficiently

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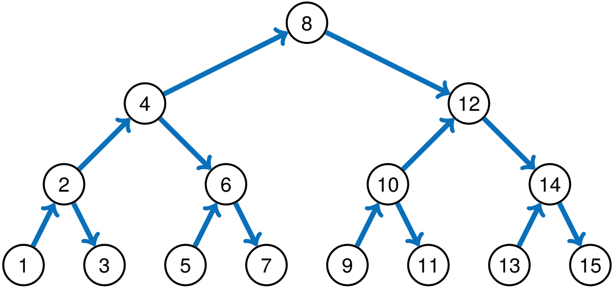


Figure: A binary search tree

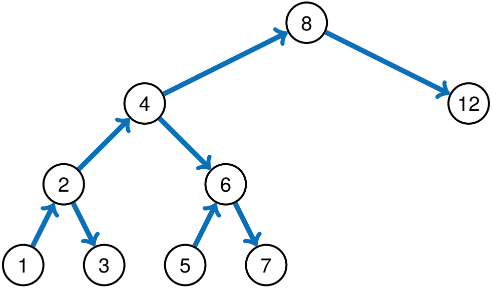


Figure: Another binary search tree

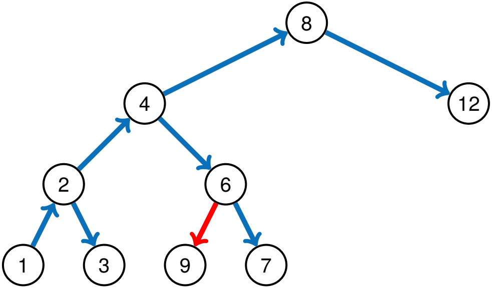


Figure: **Not** a binary search tree

**Implementation:**

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**Implementation:**

- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists

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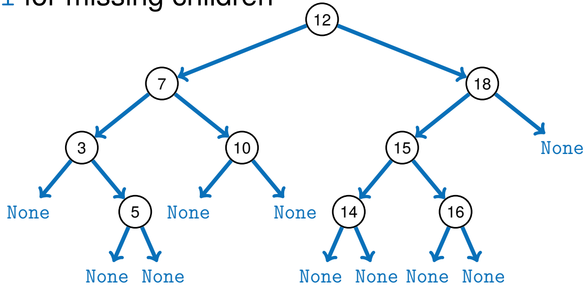

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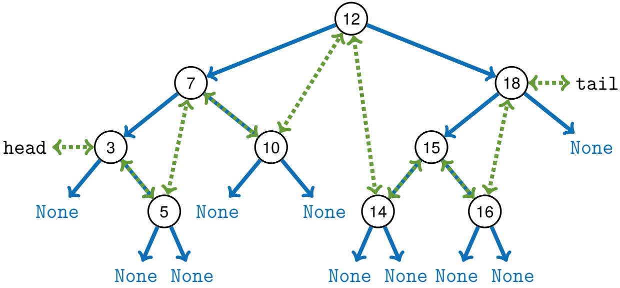


Figure: Binary search tree with links

**Lookup:**

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  " Search the element with the given key. If no element is found return the element with the next (bigger) key. "

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keys of left subtree < `node.key` < keys of right subtree

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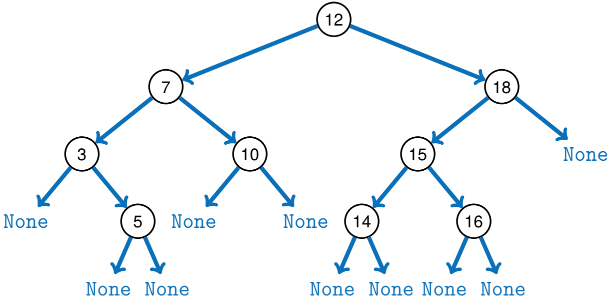


Figure: Binary search tree with total order "<"

**Examples:**

**For each node applies the total order:**

keys of left subtree < `node.key` < keys of right subtree

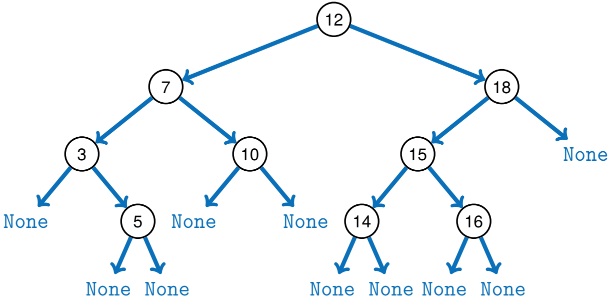


Figure: Binary search tree with total order "<"

**Examples:**

- `lookup(14)`

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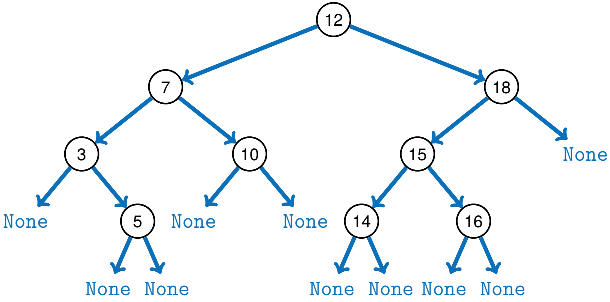


Figure: Binary search tree with total order "<"

**Examples:**

- `lookup(14)`
- `lookup(6)`

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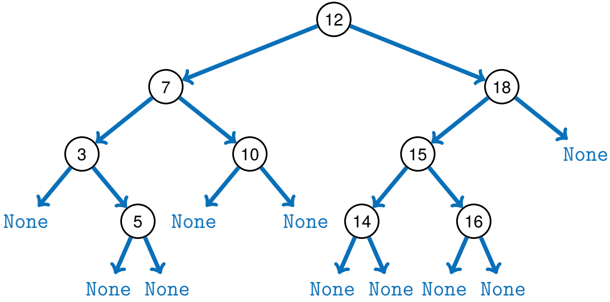


**Examples:**

- `lookup(14)`
- `lookup(6)`
- `lookup(19)`

Figure: Binary search tree with total order "<"

**Insert:**

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- We search for the key in our search tree

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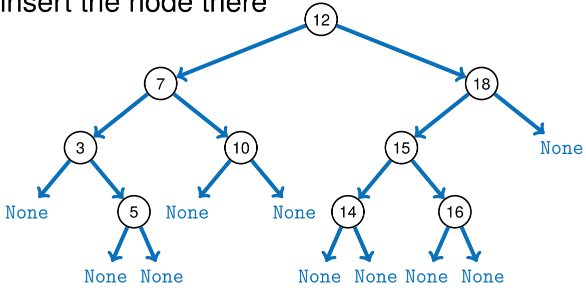


Figure: Binary search tree with total order "<"

**Remove:** Case 1: The node "5" has no children

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- Find parent of node “5” (“6”)

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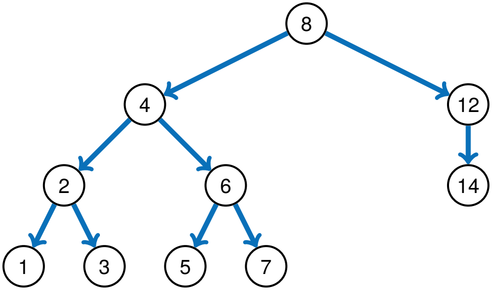


Figure: Binary search tree with total order “<”

**Remove:** Case 1: The node “5” has no children

- Find parent of node “5” (“6”)
- Set left / right child of node “6” to `None` depending on position of node “5”

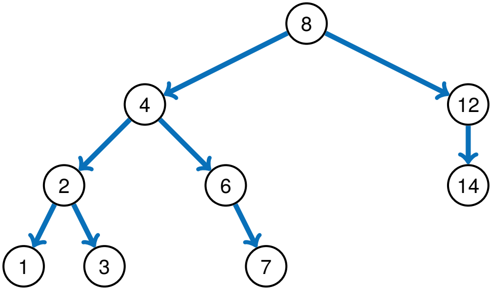


Figure: Binary search tree after deleting node “5”

**Remove:** Case 2: The node “12” has one child

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**Remove:** Case 2: The node “12” has one child

- Find the child of node “12” (“14”)
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- Set left / right child of node “8” to “14” depending on position of node “12” (skip node “14”)

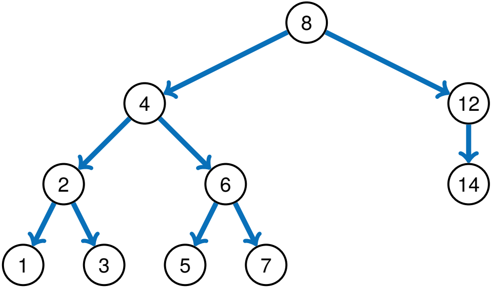


Figure: Binary search tree with total order “$<$”

**Remove:** Case 2: The node "12" has one child

- Find the child of node "12" ("14")
- Find the parent of node "12" ("8")
- Set left / right child of node "8" to "14" depending on position of node "12" (skip node "14")

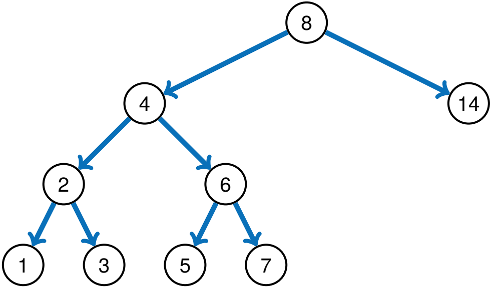


Figure: Binary search tree after delting node "12"

**Remove:** Case 3: The node “4” has two children

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- Find the successor of node “4” (“5”)

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- Find the successor of node "4" ("5")
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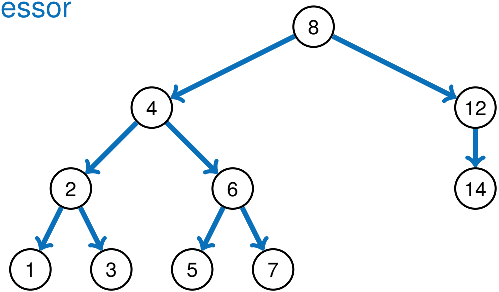

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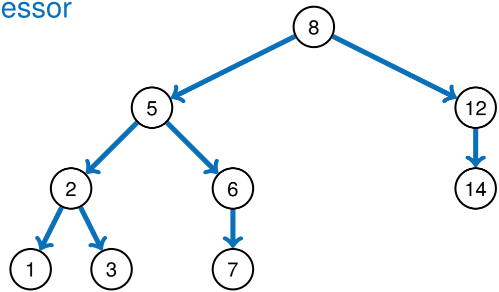

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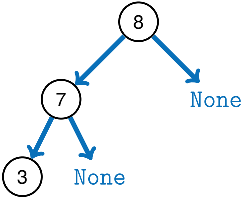


Figure: Degenerated binary tree $d = n$

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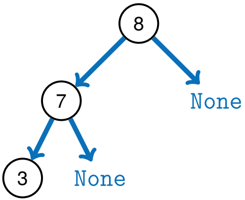


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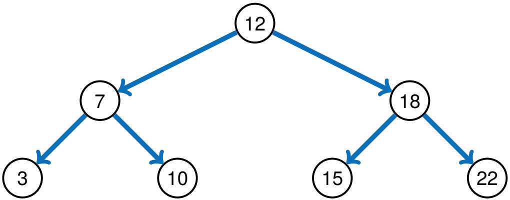


Figure: Complete binary tree $d = \log n$

- **General**

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- **Linked List**

[Wik] Linked list
`https://en.wikipedia.org/wiki/Linked_list`

- **Binary Search Tree**

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