

# Algorithms and Datastructures

## Runtime analysis Minsort / Heapsort, Induction

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## Runtime Example

Minsort

## Basic Operations

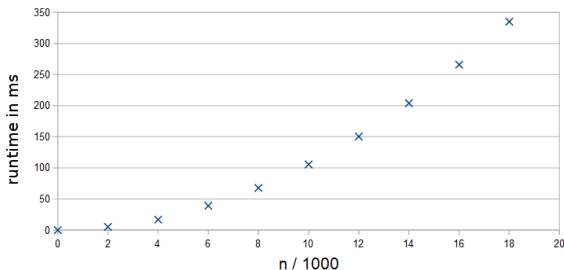
## Runtime analysis

Minsort

Heapsort

Introduction to Induction

## Logarithms



## How long does the program run?

- In the last lecture we had a schematic
- **Observation:** it is going to be “disproportionately” slower the more numbers are being sorted
- How can we say more precisely what is happening?

## How can we analyze the runtime?

- Ideally we have a formula which provides the runtime of the program for a specific input
- **Problem:** the runtime is depends on many variables, especially:
  - What kind of computer the code is executed on
  - What is running in the background
  - Which compiler is used to compile the code
- **Abstraction 1:** analyze the number of basic operations, rather than analyzing the runtime

## Incomplete list of basic operations:

- Arithmetic operation, for example:  $a + b$
- Assignment of variables, for example:  $x = y$
- Function call, for example: *minsort(lst)*

## Intuitive:

lines of code

## Better:

lines of machine  
code

## Best:

process cycles

## Important:

The actual runtime has to be roughly proportional to the number of operations.

How many operations does *Minsort* need?

- **Abstraction 2:** we calculate the upper (lower) bound, rather than exactly counting the number of operations

**Reason:** runtime is approximated by number of basic operations, but we can still infer:

- Upper bound
  - Lower bound
- 
- **Basic Assumption:**
    - $n$  is size of the input data (i.e. array)
    - $T(n)$  number of operations for input  $n$

How many operations does *Minsort* need?

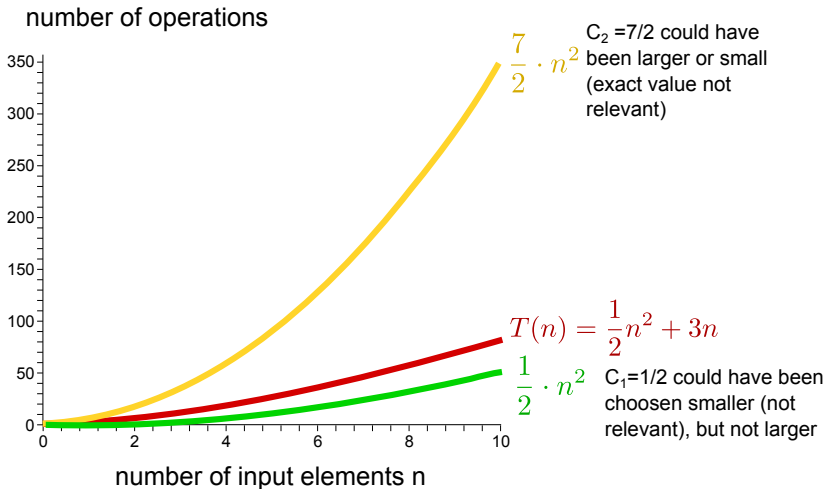
- **Observation:** the number of operations depends only on the size  $n$  of the array and not on the content!
- **Claim:** there are constants  $C_1$  and  $C_2$  such that:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- This is called “quadratic runtime” (due to  $n^2$ )



# Runtime Example



## We declare:

- Runtime of operations:  $T(n)$
- Number of Elements:  $n$
- Constants:  $C_1$  (lower bound),  $C_2$  (upper bound)  
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$
- Number of operations in round  $i$ :  $T_i$



**Figure:** *Minsort* at iteration  $i = 4$ . We have to check  $n - 3$  elements

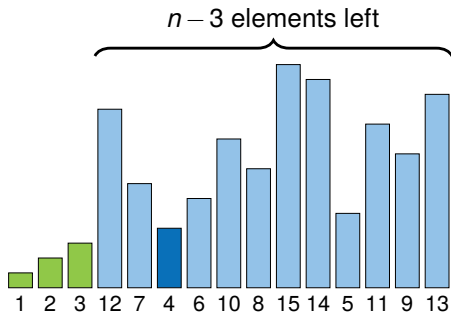


Figure: Minsort at iteration  $i = 4$

Runtime for each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

$\vdots$

$$T_{n-1} \leq C'_2 \cdot 2$$

$$T_n \leq C'_2 \cdot 1$$

$$T(n) = C'_2 \cdot (T_1 + \dots + T_n) \leq \sum_{i=1}^n (C'_2 \cdot i)$$

## Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j  
  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
  
    return elements
```

const. runtime } n-i-1 times } n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2 = \sum_{i=1}^{n-1} (n-i) \cdot C'_2 \leq \sum_{i=1}^n i \cdot C'_2$$

**Remark:**  $C'_2$  is cost of comparison  $\Rightarrow$  assumed constant

**Proof of upper bound:**  $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \\ &\leq C'_2 \cdot \frac{n(n+n)}{2}, \quad 1 \leq n \\ &= C'_2 \cdot \frac{2 \cdot n^2}{2} = C'_2 \cdot n^2 \end{aligned}$$

**Proof of lower bound:**  $C_1 \cdot n^2 \leq T(n)$

Like for the upper bound there exists a  $C_1$ . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n-i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \\ &\geq C'_1 \cdot \frac{n \cdot n}{2 \cdot 2} = \frac{C'_1}{4} \cdot n^2 \end{aligned}$$

## Runtime Analysis:

- Upper bound:  $T(n) \leq C'_2 \cdot n^2$
- Lower bound:  $\frac{C'_1}{4} \cdot n^2 \leq T(n)$

## Summarized:

$$\frac{C'_1}{4} \cdot n^2 \leq T(n) \leq C'_2 \cdot n^2$$

## Quadratic runtime proven:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- The runtime is growing quadratically with the number of elements  $n$  in the list
- With constants  $C_1$  and  $C_2$  for which  $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- $3 \times$  elements  $\Rightarrow 9 \times$  runtime
  - $C = 1$  ns (1 simple instruction  $\approx 1$  ns)
  - $n = 10^6$  (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{12} = 10^3 \text{ s} = 16.7 \text{ min}$
  - $n = 10^9$  (1 billion numbers = 4 GB)
    - $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{18} = 10^9 \text{ s} = 31.7 \text{ years}$
- **Quadratic runtime = “big” problems unsolvable**



## Intuitive to extract minimum:

- **Minsort:** to determine the minimum value we have to iterate through all the unsorted elements.
- **Heapsort:** the root node is always the smallest (minheap). We only need to repair a part of the full tree after the delete operation.

## Formal:

- Let  $T(n)$  be the runtime for the *Heapsort* algorithm with  $n$  elements
- On the next pages we will proof  $T(n) \leq C \cdot n \log_2 n$

## Depth of a binary tree:

- **Depth  $d$ :** longest path through the tree
- Complete binary tree has  $n = 2^d - 1$  nodes
- Example:  $d = 4$   
 $\Rightarrow n = 2^4 - 1 = 15$



Figure: Binary tree with 15 nodes

## Basics:

- You want to show that assumption  $A(n)$  is valid  $\forall n \in \mathbb{N}$
- We show induction in two steps:
  - 1 **Induction basis:** we show that our assumption is valid for one value (for example:  $n = 1, A(1)$ ).
  - 2 **Induction step:** we show that the assumption is valid for all  $n$  (normally one step forward:  $n = n + 1, A(1), \dots, A(n)$ ).
- If both has been proven, then  $A(n)$  holds for all natural numbers  $n$  by **induction**

Claim:

A **complete** binary tree of depth  $d$  has  $v(d) = 2^d - 1$  nodes

- **Induction basis:** assumption holds for  $d = 1$

Root



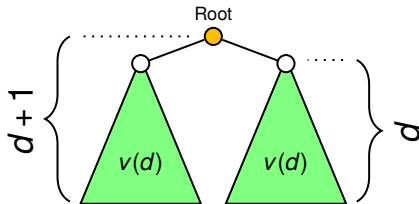
$$v(1) = 2^1 - 1 = 1$$

$\Rightarrow$  correct ✓

Figure: Tree of depth 1 has 1 node

Number of nodes  $v(d)$  in a binary tree with depth  $d$ :

- **Induction assumption:**  $v(d) = 2^d - 1$
- **Induction basis:**  $v(1) = 2^1 - 1 = 2^1 - 1 = 1$  ✓
- **Induction step:** to show for  $d := d + 1$



$$\begin{aligned} v(d+1) &= 2 \cdot v(d) + 1 \\ &= 2 \cdot (2^d - 1) + 1 \\ &= 2^{d+1} - 2 + 1 \\ &= 2^{d+1} - 1 \quad \checkmark \end{aligned}$$

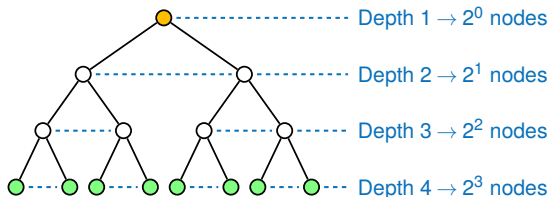
Figure: binary tree with subtrees

⇒ **By induction:**  $v(d) = 2^d - 1 \quad \forall d \in \mathbb{N} \quad \square$

## Heapsort has the following steps:

- **Initially:** heapify list of  $n$  elements
- **Then:** until all  $n$  elements are sorted
  - Remove root (=minimum element)
  - Move last leaf to root position
  - Repair heap by sifting

Runtime of heapify depends on depth  $d$ :



Runtime of heapify with depth of  $d$ :

- No costs at depth  $d$  with  $2^{d-1}$  (or less) nodes
- The cost for sifting with depth 1 is at most  $1C$  per node
- In general: sifting costs are linear with path length **and** number of nodes

Heapify total runtime:



Depth	Nodes	Path length	Costs per node	Upper bound
$d$	$2^{d-1}$	0	$\leq C \cdot 0$	$\leq C \cdot 1$
$d-1$	$2^{d-2}$	1	$\leq C \cdot 1$	Standard $\leq C \cdot 2$
$d-2$	$2^{d-3}$	2	$\leq C \cdot 2$	Equation $\leq C \cdot 3$
$d-3$	$2^{d-4}$	3	$\leq C \cdot 3$	$\leq C \cdot 4$

**In total:** 
$$T(d) \leq \sum_{i=1}^d \left( C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^d \left( C \cdot i \cdot 2^{d-i} \right)$$



Heapify total runtime:

$$T(d) \leq \underbrace{C \cdot \sum_{i=1}^d (i \cdot 2^{d-i})}_{\text{See next slides}} \leq C \cdot 2^{d+1}$$

- **Hence:** Resulting costs for heapify:

$$T(d) \leq C \cdot 2^{d+1}$$

- **However:** We want costs in relation to  $n$

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

■ A binary tree of depth  $d$  has  $2^{d-1} \leq n$  nodes

■  $2^{d-1} - 1$  nodes in full tree  
till layer  $d - 1$

■ At least 1 node in layer  $d$

■ Equation multiplied by  $2^2$   
 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$

■ Cost for heapify:  
 $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

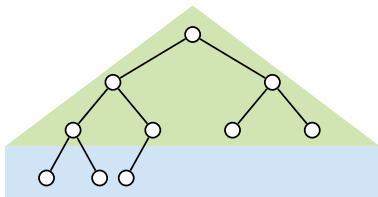


Figure: Partial binary tree

- We want to proof (induction assumption):

$$\underbrace{\sum_{i=1}^d (i \cdot 2^{d-i})}_{A(d)} \leq \underbrace{2^{d+1}}_{B(d)}$$

- We denote the left side with  $A$ , the right side with  $B$

■ **Induction basis:**  $d := 1$ :

$$A(d) \leq B(d)$$

$$\sum_{i=1}^d (i \cdot 2^{d-i}) \leq 2^{d+1}$$

$$\sum_{i=1}^1 (i \cdot 2^{1-i}) \leq 2^{1+1}$$

$$2^0 \leq 2^2 \quad \checkmark$$

**Induction step:** ( $d := d + 1$ ):

- **Idea:** Write down right-hand formula and try to get  $A(d)$  and  $B(d)$  out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

$$\sum_{i=1}^{d+1} (i \cdot 2^{d+1-i}) \leq 2^{d+1+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$\vdots$

**Induction step:** ( $d := d + 1$ ):

$\vdots$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^d (i \cdot 2^{d-i}) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

■ **Problem:** does not work but claim still holds

## Working proof:

- Show a **little bit stronger** claim

$$\sum_{i=1}^d (i \cdot 2^{d-i}) \leq 2^{d+1} - d - 2 \leq 2^{d+1}$$

- **Advantage:** results in a stronger induction assumption  
 $\Rightarrow$  **exercise**

## Runtime of the other operations:

- Constant costs for taking out  $n \times$  maximum
- Maximum of  $d$  steps repairing the heap  $n$  times
- Depth of heap at the start is  $d \leq 1 + \log_2 n$

$$2^{d-1} \leq n \Rightarrow d-1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

- **Recall:** the depth and number of elements is decreasing
  - **Hence:**  $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
  - We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \quad (\text{holds for } n > 2)$$



## Runtime costs:

- Heapify:  $T(n) \leq 4 \cdot n \cdot C$
- Remove:  $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- Total runtime:  $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- Constraints:
  - **Upper bound:**  $C_2 \cdot n \log_2 n \geq T(n)$  (for  $n \geq 2$ )
  - **Lower bound:**  $C_1 \cdot n \log_2 n \leq T(n)$  (for  $n \geq 2$ )
  - $\Rightarrow C_1$  and  $C_2$  are constant

## Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient  $\frac{1}{\log_b a}$

### Examples:

- $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602 \dots \cdot 3.322 \dots = 2 \checkmark$
- $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3 \checkmark$

## Runtime of $n \log_2 n$ :

- Assume we have constants  $C_1$  and  $C_2$  with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- $2 \times$  elements  $\Rightarrow$  only slightly larger than  $2 \times$  runtime
  - $C = 1$  ns (1 simple instruction  $\approx 1$  ns)
  - $n = 2^{20}$  (1 million numbers = 4 MB with 4 B/number)
    - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{20} \cdot 20 = 21.0 \text{ ms}$
  - $n = 2^{30}$  (1 billion numbers = 4 GB)
    - $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{30} \cdot 30 = 32 \text{ s}$
- **Runtime  $n \log_2 n$  is nearly as good as linear!**

## ■ Course literature

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

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[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Mathematical Induction

[Wik] [Mathematical induction](https://en.wikipedia.org/wiki/Mathematical_induction)

`https://en.wikipedia.org/wiki/Mathematical\_induction`