

# Algorithmns and Datastructures

## Hash Map, Universal Hashing

Albert-Ludwigs-Universität Freiburg



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Algorithmns and Datastructures, November 2016

## Feedback

- Exercises
- Lecture

## Associative Arrays

- Introduction
- Hash Map

## Universal Hashing

- Introduction
- Probability Calculation
- Proof
- Examples

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- For most using associative arrays (Python dictionary) tends to be a bit faster
  - Sorting the dictionary also takes time, depending on heterogeneity of the data (e.g. lots of locality names with

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### Problem:

- Quickly find a element with a specific key
- Naive solution: Store pairs of key and value in a normal field
- For  $n$  keys searching requires  $\Theta(n)$  time
- With a Hash Map this just requires  $\Theta(1)$  in the best case, ... regardless how many elements are in the map!

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### Idea:

- Mapping the keys onto indices with a [hash function](#)
- Store the values at the calculated indices in a normal array

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- Key set:  $x = \{3904433, 312692, 5148949\}$

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- We need an array **T** with **5** elements.  
A "hashtable" with 5 "buckets"

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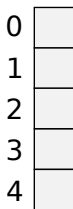
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- We need an array **T** with **5** elements.  
A "hashtable" with 5 "buckets"
- The element with the key **x** is stored in  $T[h(x)]$

### Storage:

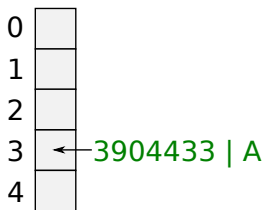
Figure: Hashtable T



### Storage:

- `insert(3904433,"A")`:  $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$

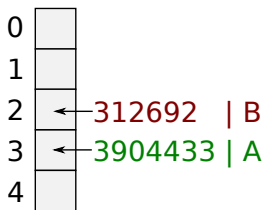
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### Storage:

- $\text{insert}(3904433, "A")$ :  $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- $\text{insert}(312692, "B")$ :  $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$

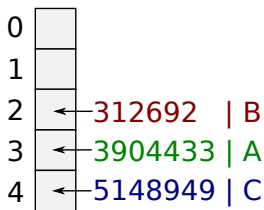
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- $\text{insert}(5148949, "C")$ :  $h(5148949) = 4 \Rightarrow T[4] = (5148949, "C")$

Figure: Hashtable T



### Searching:

- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$

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0	
1	
2	← 312692   B
3	← 3904433   A
4	← 5148949   C

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- $\text{search}(3904433): h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$
- $\text{search}(123459): h(123459) = 4 \Rightarrow T[4]$   
 $\Rightarrow$  Value with key 123459 does not exist

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- Search time for this example:  $\mathcal{O}(1)$

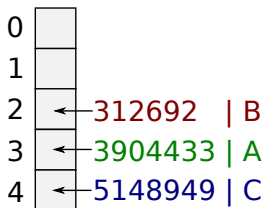
Figure: Hashtable T

0	
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### Further inserting:

- `insert(876543, "D")`:  $h(876543) = 3$

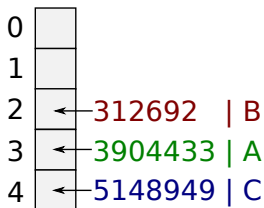
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- `insert(876543, "D")`:  $h(876543) = 3$   
 $\Rightarrow T[3] = (876543, "D")$  **COLLISION!**

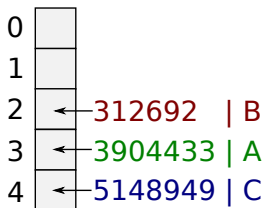
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### Further inserting:

- $\text{insert}(876543, "D")$ :  $h(876543) = 3$   
 $\Rightarrow T[3] = (876543, "D")$  **COLLISION!**
- This happens more often than expected
  - **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

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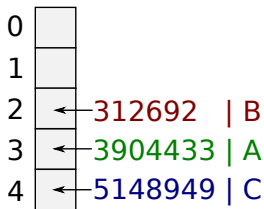
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Figure: Hashtable T



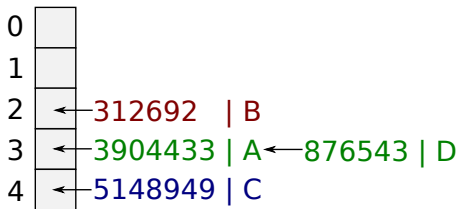
### Problem:

- Two keys are equal  $h(x) = h(y)$  but not the values  $x \neq y$

### Easiest Solution:

- Represent each bucket as list of key value pairs
- Append new values to the end of the list

Figure: Hashtable T

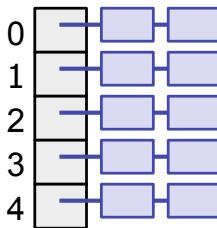




### Best case:

- We have  $n$  keys which are equally distributed over  $m$  buckets
- We have  $\approx \frac{n}{m}$  pairs per bucket

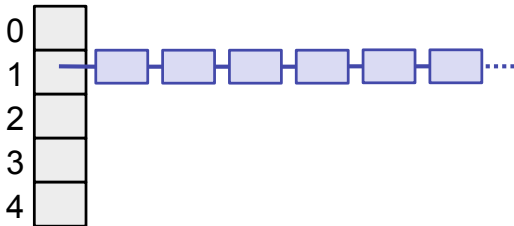
### Best case ( $m = 5, n = 10$ )



### Worst case:

- All  $n$  keys are mapped onto the same bucket
- The runtime is  $\Theta(n)$  for searching

**Worst case**  
( $m = 5, n = 10$ )



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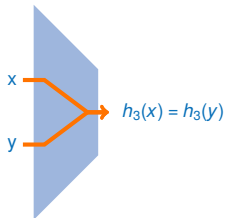
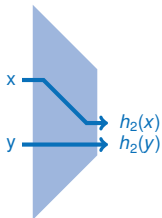
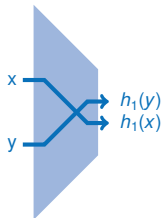
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  - *you may use the hash function*
  - *for table size 100: try  $100 \times 99 + 1$  different numbers*
  - *worst case: still 100 must have same hash bucket*
- **Now:** Find a solution to avoid that problem



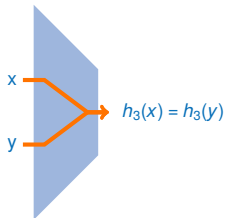
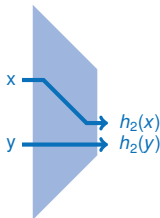
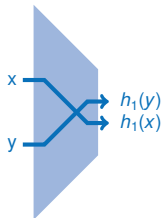
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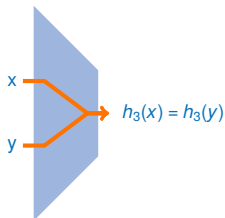
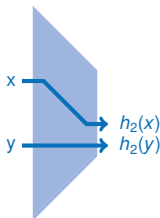
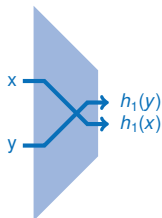
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- We choose a random hash function so that the **expected result** is an equal distribution over the buckets  
*this is fixed for the lifetime of table*  
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- We use a set of hash functions
- We choose a random hash function so that the **expected result** is an equal distribution over the buckets  
*this is fixed for the lifetime of table*  
*optional: copy table with new hash when degenerated*
- This is called **universal hashing**



### Definition:

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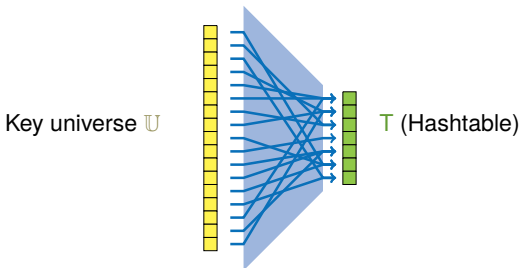


$T$  (Hashtable)



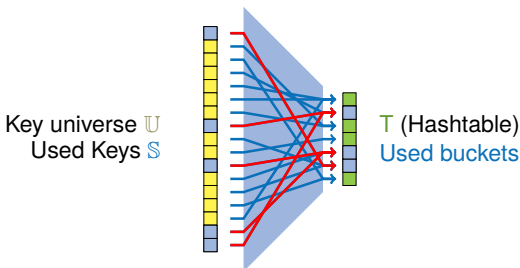
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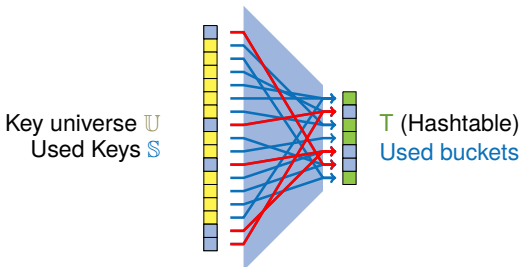
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- Idea: runtime should be  $O(1 + \frac{|\mathcal{S}|}{m})$ , where  $\frac{|\mathcal{S}|}{m}$  is the table load





- We choose two random keys  $x, y \in \mathbb{U} \mid x \neq y$

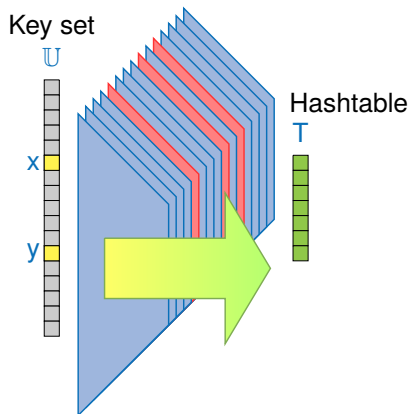


Figure: Set of hash functions  $\mathcal{H}$

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- An average of 3 out of 15 functions produce collisions

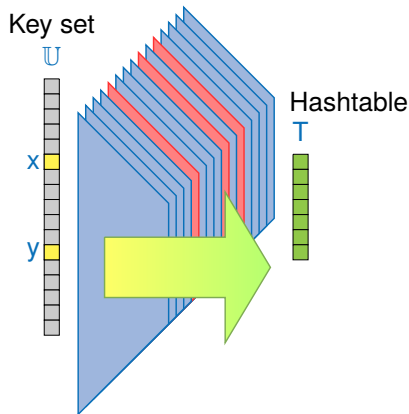


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Number of hash functions that create collisions

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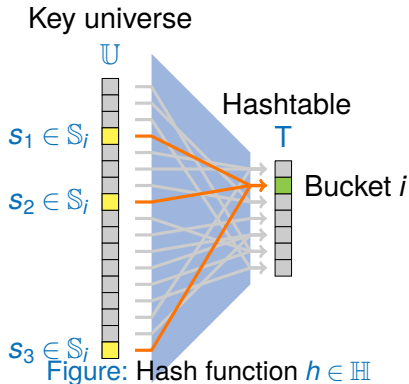
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$$\text{Prob}(\text{Collision}) = \frac{1}{m} \Leftrightarrow c = 1$$

- $\mathbb{U}$ : Key universe
- $\mathbb{S}$ : Used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$ : Keys mapping to Bucket  $i$  (“synonyms”)
- Ideal would be  $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$







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- Particularity: If  $(m = \Omega(|\mathbb{S}|))$  then  $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

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# Universal Hashing

## Probability Calculation



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- Probability space  $\Omega$  with elementary (simple) events



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Table: Throwing a dice

$e$	$P(e)$
1	$1/6$
2	$1/6$
3	$1/6$
4	$1/6$
5	$1/6$
6	$1/6$



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(1, 3)	$1/36$
...	...
(6, 5)	$1/36$
(6, 6)	$1/36$

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- Each event  $e \in \Omega$  has the probability  $P(e) = 1/36$
- $E =$  if both eye numbers even, then  $P(E) =$

Table: Throwing a dice twice

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...	...
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(6, 6)	$1/36$



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- Random variable

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- Random variable
  - Assigns a number to the result of an experiment

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$e$	$P(e)$	$X$
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**Sum of expected values:** For independent (discrete) result variables  $X_1, \dots, X_n$  we can write:

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n)$$

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- $X = X_1 + X_2$ : Expected total number of eyes:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$



### Corollary:

The probability of the event  $E$  is  $p = P(E)$ . Let  $X$  be the occurrences of the event  $E$  and  $n$  be the number of executions of the experiment. Then  $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurrences of } 6) = \frac{1}{6} \cdot 60 = 10$$



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Def.  $\mathbb{E}$ -value:  $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$



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### To proof:

$$\mathbb{E}[|S_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \quad \forall i$$



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$$\Rightarrow \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

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$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1) \\ &= P(h(y) = i) \\ &= P(h(y) = h(x)) \\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

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**Negative example:**

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- Which  $x, y$  lead to a relative collision count bigger than  $\frac{c}{m}$ ?



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- Exercise: show empirically that it is 2-universal



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- **Intuitive:** Scalar product with base  $m$

$$a \bullet x = \sum_{0, \dots, k-1} a_i \cdot x_i$$



Example ( $\mathbb{U} = \{0, \dots, 999\}$ ,  $m = 10$ ,  $a = 348$ )

With  $a = 348$ :  $a_2 = 3$ ,  $a_1 = 4$ ,  $a_0 = 8$

$$\begin{aligned}h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10\end{aligned}$$

With  $x = 127$ :  $x_2 = 1$ ,  $x_1 = 2$ ,  $x_0 = 7$

$$\begin{aligned}h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7\end{aligned}$$

## ■ General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

**Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Hash Map - Theory

[Wik] [Hash table](#)

[https://en.wikipedia.org/wiki/Hash\\_table](https://en.wikipedia.org/wiki/Hash_table)

## ■ Hash Map - Implementations / API

[Cpp] [C++ - hash\\_map](#)

[http://www.sgi.com/tech/stl/hash\\_map.html](http://www.sgi.com/tech/stl/hash_map.html)

[Jav] [Java - HashMap](#)

<https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html>

[Pyt] [Python - Dictionaries \(Hash table\)](#)

[https://en.wikipedia.org/wiki/Hash\\_table](https://en.wikipedia.org/wiki/Hash_table)