Copulas
Student Presentation

13 November 2018

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Mathematics-Economics, 7th semester
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Introduction

A copula is any probability distribution function with uniform marginals. We define it below:

Definition (Copula)

Let $C : [0,1]^d \to [0,1]$, then C is a d-dimensional copula if:

$$C(u_1, \ldots, u_k, 0, u_{k+2}, \ldots, u_d) = 0$$
$$C(1, \ldots, 1, u_k, 1, \ldots, 1) = u_k$$
$$\int_{\{\times_{i=1}^{d}[x_i, y_i]\}} 1 dC \geq 0$$

where $0 \leq x_i \leq y_i \leq 1,\ \forall i \in \{1, 2, \ldots, d\}$

Probability transformation of Y

Theorem

Let Y be a random variable with continuous CDF, F. Then $F(Y) \sim \text{unif}\,(0,1)$.

Proof.

Assume F is strictly increasing. Then F^{-1} exists and

$$P(F(Y) \leq y) = P(Y \leq F^{-1}(y)) = F(F^{-1}(y)) = y$$

which corresponds to the CDF of a uniform distribution on $[0,1]$. The proof is also valid without assuming that F is strictly increasing which we do not show. $\square$

Copulas

Using Theorem 1, we can now specify the dependence between random variables with known distribution.

Let $X = (X_1, X_2, \dots, X_d)$ have joint distribution F_X and marginal distributions F_{X_i} which are continuous.

Note that this assumption may be relaxed to obtained mixed continuous and discrete distributions. However, we do not consider this case.

Copulas

Then we have:

$$\begin{aligned} C_X(u_1, u_2, \ldots u_d) &= P(F_{X_1}(X_1) \leq u_1, F_{X_2}(X_2) \leq u_2 \ldots F_{X_d}(X_d) \leq u_d) \\ &= F_X(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2), \ldots, F_{X_d}^{-1}(u_d)) \end{aligned}$$

Setting $u_i = F_{X_i}(X_i)$, we obtain:

$$C_X(F_{X_1}(X_1), F_{X_2}(X_2), \ldots F_{X_2}(X_2)) = F_X(X_1, X_2, \ldots, X_d)$$

Sklar's Theorem

Theorem

Given a d-dimensional random variable X with distribution function H and marginals $F_{X_i}, i \in \{1, 2, \ldots, d\}$, there exists a d-dimensional copula C such that $\forall x \in \overline{\mathbb{R}}^n$:

$$H(x) = H(x_1, x_2, \ldots, x_d) = C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_d}(x_d))$$

The copula is unique on $(\bigtimes_{i=1}^{d}$ range$(F_{X_i}))$, hence unique if the marginals are unique. Conversely, if $F_{X_1}, F_{X_2}, \ldots F_{X_d}$ are marginal distributions and C is a copula, then H defined as above is a distribution function.

Sklar's Theorem

We will not prove Theorem 2. However, arguments from before are the main reasoning of the theorem when considering more general inverse probability transforms.

Sklar's Theorem is not constructive; it does not describe how to construct or choose the appropriate copula. The important part of the theorem is the last one, i.e. that a copula applied to each of the marginal CDF's is a distribution.

Sklar's Theorem

Another relation, that holds under appropriate conditions, is:

$$H'(x_1, \dots, x_d) = C'(F_1(x_1), \dots, F_d(x_d)) \cdot f_1(x_1) \cdot \dots \cdot f_d(x_d),$$

In 2006, Patton also proved Sklar's Theorem for conditional marginals, i.e. allowing intertemporal analysis:

$$H_t(x, y|\mathcal{F}_{t-1}) = C_t(F_t(x|\mathcal{F}_{t-1}), G_t(y|\mathcal{F}_{t-1})|\mathcal{F}_{t-1}),$$

where X_t has CDF F_t and Y_t has CDF G_t.

Special Copulas

There are three especially interesting copulas.

- *The independence copula.*
- *The co-monotonicity copula.*
- *The counter-monotonicity copula.*

Special Copulas

The independence copula.

- CDF of *d* mutually independent $\mathrm{unif}\,(0,1)$ random variables.
- $C_0(u_1,\ldots,u_d) = u_1 \ldots u_d$

Special Copulas

The co-monotonicity copula.

- Perfect positive dependence.
- CDF of $(U, U, \dots, U)$.
- $C_+(u_1, \dots, u_d) = P(U \leq u_1, \dots, U \leq u_d) = \min(u_1, \dots, u_d)$.

Special Copulas

The co-monotonicity copula.

- Perfect positive dependence.
- CDF of $(U, U, \ldots, U)$.
- $C_+(u_1, \ldots, u_d) = P(U \leq u_1, \ldots, U \leq u_d) = \min(u_1, \ldots, u_d)$.
- For any copula C,
  $C(u_1, \ldots, u_d) \leq C_+(u_1, \ldots, u_d), \ \forall (u_1, \ldots, u_d) \in [0, 1]^d$

Special Copulas

The counter-monotonicity copula.

- Perfect negative dependence.
- CDF of $(U, 1-U)$.

$$
\begin{aligned}
C_{-}(u_1, u_2) &= P(U \leq u_1, 1 - U \leq u_2) \\
&= P(1 - u_2 \leq U \leq u_1) = \max(u_1 + u_2 - 1, 0).
\end{aligned}
$$

Special Copulas

The counter-monotonicity copula.

- Perfect negative dependence.
- CDF of $(U, 1-U)$.

$$
\begin{aligned}
C_-(u_1, u_2) &= P(U \leq u_1, 1 - U \leq u_2) \\
&= P(1 - u_2 \leq U \leq u_1) = \max(u_1 + u_2 - 1, 0).
\end{aligned}
$$

- For any 2-dimensional copula C,
  $C(u_1, u_2) \geq C_-(u_1, u_2),\ \forall (u_1, u_2) \in [0, 1]^2$

Gaussian Copula

Let $Y = (Y_1, Y_2, ..., Y_d)$ have a multivariate normal distribution and let C_Y be the copula of Y.

Since C_Y only depends only on the dependencies within Y, the copula only depends on the correlation matrix, Ω. The Gaussian Copula with correlation matrix Ω is defined as:

$$C_{\text{Gauss}}(u_1, u_2, \ldots, u_d|\Omega) = \Phi_\Omega\left(\phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_d)\right),$$

where Φ_Ω is a multivariate normal distribution with mean zero and covariance matrix Ω. If a random variable Y has a Gaussian copula, it is said to have a **meta-Gaussion distribution**.

Gaussian Copula

Note that a meta-Gaussion distributed variable is not necessary Gaussion distributed.

If Ω is the identity matrix, the Gaussian copula is equal to the independence copula.

A Gaussion copula will converge to the co-monotonicity copula, when all correlations in Ω converges to 1.

A bivariate Gaussion copula will converge to the counter-monotonicity copula, when the pairwise correlation in Ω converges to -1.

A variable is called meta t-distributed if it has a t-copula denoted by $C_t(u_1, u_2, \ldots, u_d|\Omega, \nu)$.

Archimedean Copulas

Archimedean copulas are copulas of the form:

$$C(u_1, \dots, u_d) = \varphi^{-1}\left(\varphi(u_1) + \cdots + \varphi(u_d)\right),$$

where φ is called a generator function and satisfies:

- $\varphi : [0,1] \to [0,\infty)$ is continuous, strictly decreasing, and convex.
- $\varphi(0) = \infty$.
- $\varphi(1) = 0$.

Archimedean Copulas

We consider the following four Archimedean copulas:

- Frank.
- Clayton.
- Gumbel.
- Joe.

Frank Copula

The Frank Copula has generator:

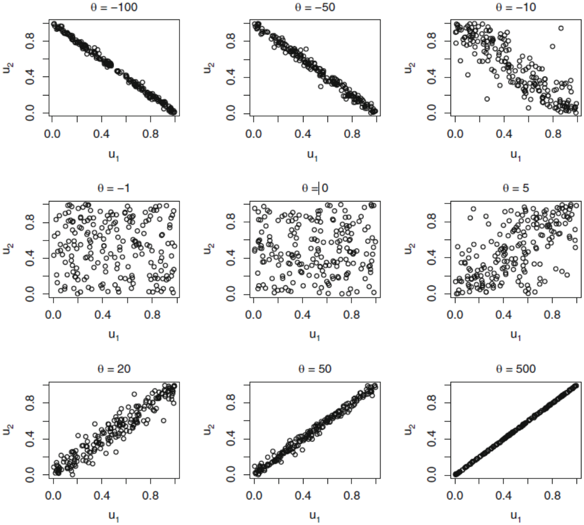
$$\varphi_{Fr}(u, |\theta) = -\log\left(\frac{e^{-\theta u} - 1}{e^{-\theta} - 1}\right), \ -\infty < \theta < \infty.$$

The inverse generator is:

$$\varphi_{Fr}^{-1}(y, |\theta) = -\frac{1}{\theta}\log\{(e^{-y}(e^{-\theta} - 1)\}$$

For $\theta = 0$, we set the Frank copula equal to the independence copula. Converges to the counter-monotonicity (co-monotonicity) copula as θ goes to $-\infty(\infty)$.

Frank Copula



Clayton Copula

The Clayton Copula has generator:

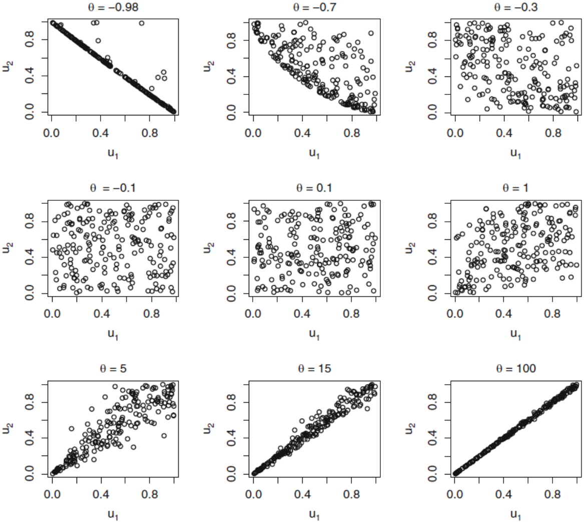
$$\varphi_{Cl}(u, |\theta) = \frac{1}{\theta}\left(u^{-\theta} - 1)\right),\ 0 < \theta$$

For $\theta = 0$, we set the Clayton copula equal to the independence copula.

Possible to extend the Clayton copula to include $-1 \leq \theta < 0$. However, $\varphi(0) \leq \infty$ and the copula is not strict which can cause some complications.

Converges to the counter-monotonicity (co-monotonicity) copula as θ goes to $-1(\infty)$.

Clayton Copula



Gumbel Copula

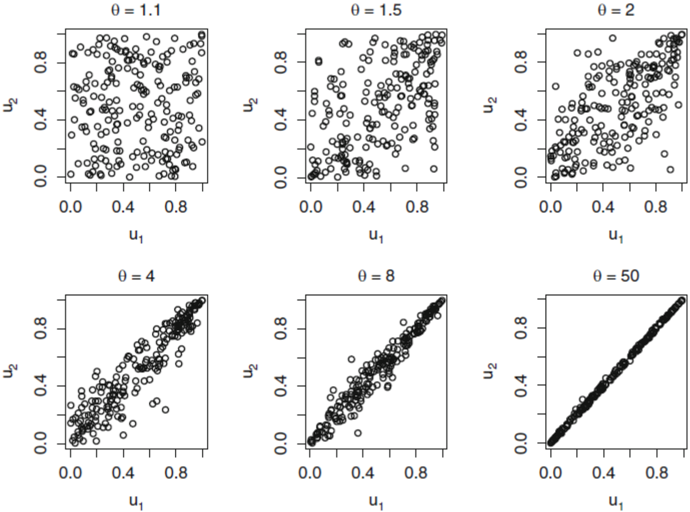
The Gumbel Copula has generator:

$$\varphi_{Gu}(u|\theta) = (-\log(u))^{-\theta}, \; 1 \leq \theta$$

Converges to the co-monotonicity (independence) copula as θ goes to $\infty(1)$.

The Gumbel copula cannot have negative dependence.

Gumbel Copula



Joe Copula

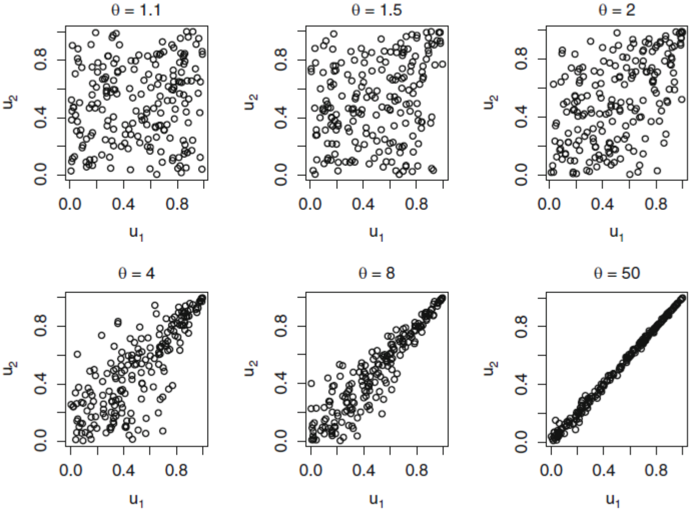
The Joe Copula has generator:

$$\varphi_{Joe}(u|\theta) = -\log(1-(1-u))^{\theta},\ 1 \leq \theta$$

Converges to the co-monotonicity (independence) copula as θ goes to $\infty(1)$.

The Joe copula cannot have negative dependence.

Joe Copula



Rank correlation

Let $y_1, y_2, \ldots, y_n$ be a sample of a random variable. Then

$$\text{Rank}(y_i) = \sum_{j=1}^{n} I(y_j \leq y_i)$$

Thus, the smallest observation has rank 1 and the largest rank n.

Ranks are unchanged by strictly monotonic transformations. Therefore, transforming a variable by its CDF does not change the ranks, meaning that the distribution of a rank statistic does not depend on the marginal distributions but only on the copula of the observations.

Kendall's Tau

Let (Y_1, Y_2) and (Y_1^*, Y_2^*) be independent random vectors with identical distributions.

- *Concordant pair:* $(Y_1 - Y_1^*)(Y_2 - Y_2^*) > 0$
- *Discordant pair:* $(Y_1 - Y_1^*)(Y_2 - Y_2^*) < 0$

Then Kendall's Tau is defined as:

$$\begin{aligned}\rho_\tau &= P\big[(Y_1 - Y_1^*)(Y_2 - Y_2^*) > 0\big] - P\big[(Y_1 - Y_1^*)(Y_2 - Y_2^*) < 0\big] \\ &= \mathbb{E}\left[\text{sign}\{(Y_1 - Y_1^*)(Y_2 - Y_2^*)\}\right]\end{aligned}$$

Invariant to monotonically increasing transformations such as marginal CDF's. Kendall's Tau estimated from a sample of two variables Y_1 and Y_2 is given by

$$\widehat{\rho}_\tau(Y_1, Y_2) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \text{sign}\left((Y_{i,1} - Y_{j,1})(Y_{i,2} - Y_{j,2})\right)$$

Spearman's Rho

Let (Y_1, Y_2) be a random vector, then Spearman's Rho is given by:

$$\rho_S(Y_1, Y_2) = \text{Corr}(F_{Y_1}(Y_1), F_{Y_2}(Y_2))$$

That is, the standard pearson correlation coefficient of the variables transformed by their CDF. Transforming the variables by their CDF is analogous to ranking observations and thus Spearmans Rho estimated from a sample of two variables Y_1 and Y_2 is given by:

$$\widehat{\rho}_S(Y_1, Y_2) = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left(\text{rank}(Y_{i,1}) - \frac{n+1}{2}\right) \left(\text{rank}(Y_{i,2}) - \frac{n+1}{2}\right)$$

Kendall's Tau and Spearman's Rho

Both can be used for non parametric tests, meaning we need not make any assumptions about the data.

$\mathcal{H}_0$: *No association between the variables*.

They assess whether there is any monotone relationship between the variables (Not just linear relationships).

It is possible to define a correlation matrix using both rank statistics:

$$[\Omega_t(Y)]_{jk} = \widehat{\rho}_\tau(Y_j, Y_k)$$
$$[\Omega_s(Y)]_{jk} = \widehat{\rho}_s(Y_j, Y_k).$$

Tail Dependence

Tail dependence measures association between the extreme values of two random variables and depends only on their copula.

Let $Y = (Y_1, Y_2)$ have copula C_y and $q \in (0, 1)$. We define the coefficient of lower tail dependence as:

$$
\begin{aligned}
\lambda_l := &\lim_{q \to 0} P(Y_2 < F_{Y_2}^{-1}(q) \mid Y_1 < F_{Y_1}^{-1}(q)) \\
= &\lim_{q \to 0} \frac{P(Y_2 < F_{Y_2}^{-1}(q)\ ,\ Y_1 < F_{Y_1}^{-1}(q))}{P(Y_1 < F_{Y_1}^{-1}(q))} \\
= &\lim_{q \to 0} \frac{P(F_{Y_2}(Y_2) < q\ ,\ F_{Y_1}(Y_1) < q)}{P(F_{Y_1}(Y_1) < q)} \\
= &\lim_{q \to 0} \frac{C_y(q, q)}{q}
\end{aligned}
$$

Upper Tail Dependence

Upper tail dependence is defined as:

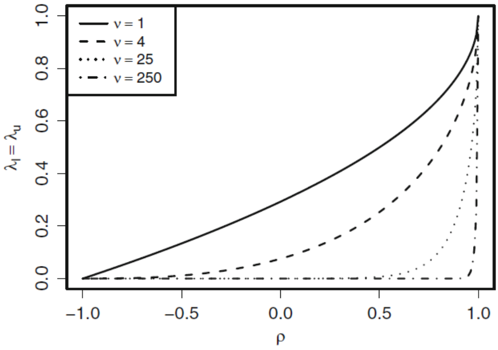
$$\begin{aligned} \lambda_u &:= \lim_{q \to 1} P(Y_2 > F_{Y_2}^{-1}(q) \mid Y_1 > F_{Y_1}^{-1}(q)) \\ &= 2 - \lim_{q \to 1} \frac{1 - C_y(q, q)}{1 - q} \end{aligned}$$

For both the Gaussian and t copula, $\lambda_l = \lambda_u$.

For any bivariate Gaussian Copula where $\rho \neq 1$, λ_l is equal to zero.

Tail Dependence of t copula

For a t copula, we have the following connection between λ_l, ρ, and ν:



Relevance in Finance

Tail dependence is important for risk management.

If there is no tail dependence among the returns on the assets in a portfolio, then there is little risk of simultaneous large negative returns.

Conversely, tail dependence in a portfolio means that extreme negative returns occurring simultaneously is more likely.

Tail dependencies should therefore be considered when assessing the diversification or risk of a portfolio.

Misuse of Gaussian copulas during financial crisis.

Calibrating Copulas

Assume that we have an i.i.d sample $Y_{1:n} = \{(Y_{i,d}, \dots, Y_{i,d}) | i = 1, \dots, n\}$, and we wish to estimate the copula of Y and maybe its univariate marginal distributions as well.

Maximum likelihood
Initially, a copula must be chosen. Suppose we have parametric models $F_{Y_1}(\cdot|\theta_1), \dots, F_{Y_d}(\cdot|\theta_d)$ for the marginal CDFs and a parametric model $C_Y(\cdot|\theta_C)$ for the copula density.

$$\ell(\theta_1, \dots, \theta_d, \theta_C) = \sum_{i=1}^{n} (\log\left(C'_Y\left(F_{Y_1}(Y_{i,1}|\theta_1), \dots, F_{Y_d}(Y_{i,d}|\theta_d)\right)|\theta_C\right)$$
$$+ \log(f_{Y_1}(Y_1, i|\theta_1) + \cdots + \log(f_{Y_d}(Y_d, i|\theta_d))$$

Pseudo-Maximum Likelihood

Maximum likelihood can be numerically challenging due to the large number of parameters when d is large. Optimization works better when the algorithms are initialized close to the solution.

Pseudo Maximum likelihood can be used either instead of maximum likelihood or to initialize the maximum likelihood.

Pseudo Maximum likelihood is a two step procedure. Initially, the uniform distributions are estimated separately and afterwards, the copula will have a pseudo log likelihood function given by:

$$\ell(\theta_C) = \sum_{i=1}^{n} (\log \left(C'_Y \left(\widehat{F}_{Y_1}(Y_{i,1}|\theta_1), \ldots, \widehat{F}_{Y_d}(Y_{i,d}|\theta_d) \right) |\theta_C \right)$$

The first step can be done either parametric by assuming a model or non parametric by using the EDF.

Pseudo-Maximum Likelihood

In the second step, the maximization can still be difficult when d is large.

For example, the Gaussian and t copula contains $d(d-1)/2$ correlation parameters.

Sometimes, we assume that all off diagonal elements of Ω have a common value.

Spearmanns correlation and Kendalls tau can be used for Gaussian and t copulas respectively.