

Pre-lecture brain teaser

Remembering the edit distance example we saw in class last time, we formaluted the processing of the recursion as a table:

	ϵ	<i>D</i>	<i>R</i>	<i>E</i>	<i>A</i>	<i>D</i>
ϵ						
<i>D</i>						
<i>E</i>						
<i>E</i>						
<i>D</i>						

Is there an easier way to get the minimum alignment without having to calculate all the values in the cell?

ECE-374-B: Lecture 14 - Graph search

Instructor: Nickvash Kani

October 16, 2025

University of Illinois Urbana-Champaign

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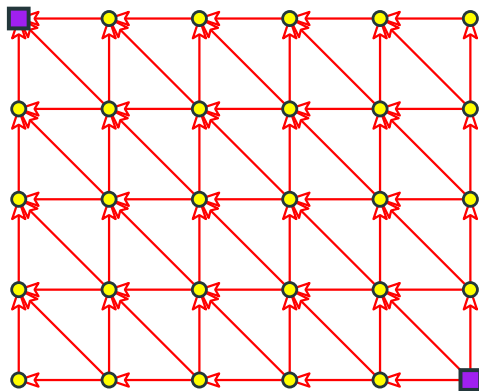
	ϵ	<i>D</i>	<i>R</i>	<i>E</i>	<i>A</i>	<i>D</i>
ϵ						
<i>D</i>						
<i>E</i>						
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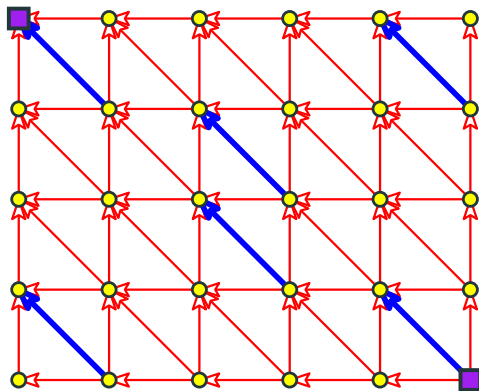
	ϵ	<i>D</i>	<i>R</i>	<i>E</i>	<i>A</i>	<i>D</i>
ϵ						
<i>D</i>						
<i>E</i>						
<i>E</i>						
<i>D</i>						



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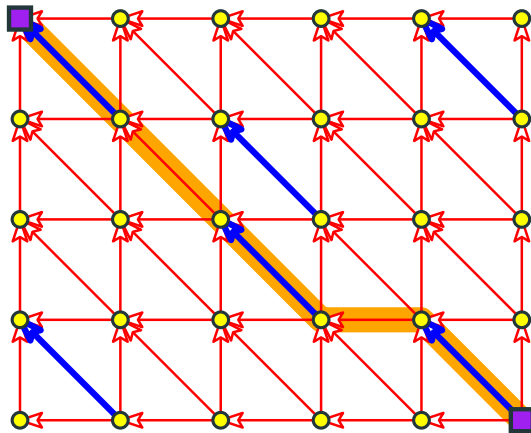
	ϵ	<i>D</i>	<i>R</i>	<i>E</i>	<i>A</i>	<i>D</i>
ϵ						
<i>D</i>						
<i>E</i>						
<i>E</i>						
<i>D</i>						



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ϵ						
<i>D</i>						
<i>E</i>						
<i>E</i>						
<i>D</i>						



Graph Basics

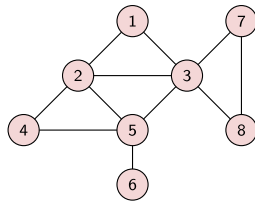
Why Graphs?

- Graphs help model networks which are ubiquitous: transportation networks (rail, roads, airways), social networks (interpersonal relationships), information networks (web page links), and many problems that don't even look like graph problems.
- Fundamental objects in Computer Science, Optimization, Combinatorics
- Many important and useful optimization problems are graph problems
- Graph theory: elegant, fun and deep mathematics

Graph

An undirected (simple) graph $G = (V, E)$ is a 2-tuple:

- V is a set of vertices (also referred to as nodes/points)
- E is a set of edges where each edge $e \in E$ is a set of the form $\{u, v\}$ with $u, v \in V$ and $u \neq v$.



Example

In figure, $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$.

Example: Modeling Problems as Search

State Space Search

Many search problems can be modeled as search on a graph.

The trick is figuring out what the vertices and edges are.

Missionaries and Cannibals

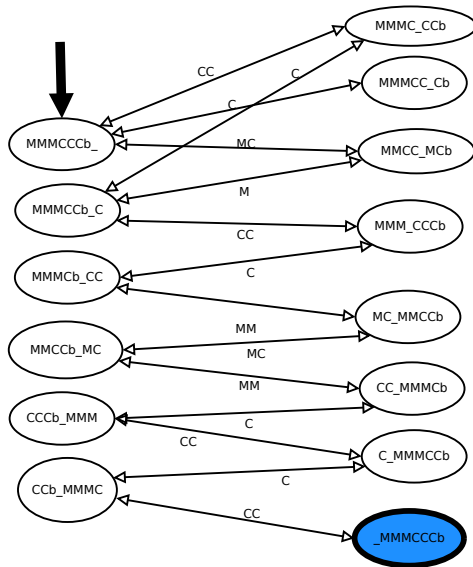
- Three missionaries, three cannibals, one boat, one river
- Boat carries two people, must have at least one person
- Must all get across
- At no time can cannibals outnumber missionaries

How is this a graph search problem?

What are the vertices?

What are the edges?

Cannibals and Missionaries: Is the language empty?



Problems goes back to 800 CE
Versions with brothers and sisters.

Jealous Husbands.

Lions and buffalo

All bad names to a simple problem...

Problems on DFAs and NFAs sometimes are just problems on graphs

- M : DFA/NFA is $L(M)$ empty?
- M : DFA is $L(M) = \Sigma^*$?
- M : DFA, and a string w . Does M accepts w ?
- N : NFA, and a string w . Does N accepts w ?

Graph notation and representation

Notation and Convention

Notation

An edge in an undirected graphs is an unordered pair of nodes and hence it is a set. Conventionally we use uv for $\{u, v\}$ when it is clear from the context that the graph is undirected.

- u and v are the end points of an edge $\{u, v\}$
- Multi-graphs allow
 - loops which are edges with the same node appearing as both end points
 - multi-edges: different edges between same pairs of nodes
- In this class we will assume that a graph is a simple graph unless explicitly stated otherwise.

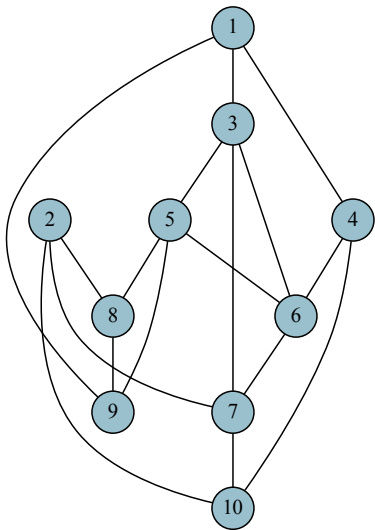
Graph Representation I

Adjacency Matrix

Represent $G = (V, E)$ with n vertices and m edges using a $n \times n$ adjacency matrix A where

- $A[i, j] = A[j, i] = 1$ if $\{i, j\} \in E$ and $A[i, j] = A[j, i] = 0$ if $\{i, j\} \notin E$.
- Advantage: can check if $\{i, j\} \in E$ in $O(1)$ time
- Disadvantage: needs $\Omega(n^2)$ space even when $m \ll n^2$

Graph adjacency matrix example [10 vertices]



	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	0	0	0	0	1	0
2	0	0	0	0	0	0	1	1	0	1
3	1	0	0	0	1	1	1	0	0	0
4	1	0	0	0	0	1	0	0	0	1
5	0	0	1	0	0	1	0	1	1	0
6	0	0	1	1	1	0	1	0	0	0
7	0	1	1	0	0	1	0	0	0	1
8	0	1	0	0	1	0	0	0	1	0
9	1	0	0	0	1	0	0	1	0	0
10	0	1	0	1	0	0	1	0	0	0

Graph Representation II

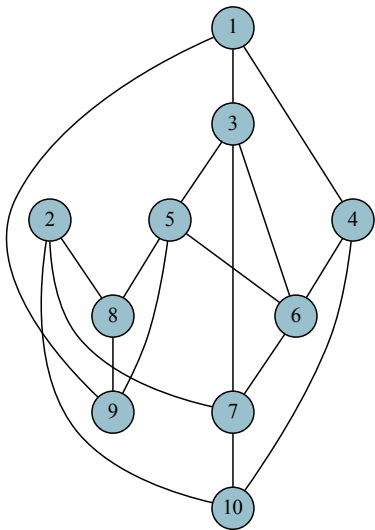
Adjacency Lists

Represent $G = (V, E)$ with n vertices and m edges using adjacency lists:

- For each $u \in V$, $\text{Adj}(u) = \{v \mid \{u, v\} \in E\}$, that is neighbors of u . Sometimes $\text{Adj}(u)$ is the list of edges incident to u .
- Advantage: space is $O(m + n)$
- Disadvantage: cannot “easily” determine in $O(1)$ time whether $\{i, j\} \in E$
 - By sorting each list, one can achieve $O(\log n)$ time
 - By hashing “appropriately”, one can achieve $O(1)$ time

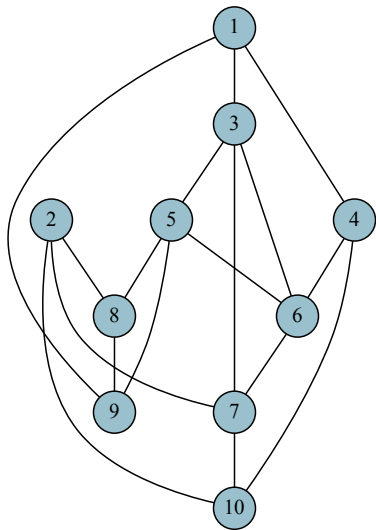
Note: In this class we will assume that by default, graphs are represented using plain vanilla (unsorted) adjacency lists.

Graph adjacency list example [10 vertices]



vertex	adjacency list
1	3, 4, 9
2	7, 8, 10
3	1, 5, 6, 7
4	1, 6, 10
5	3, 6, 8, 9
6	3, 4, 5, 7
7	2, 3, 6, 10
8	2, 5, 9
9	1, 5, 8
10	2, 4, 7

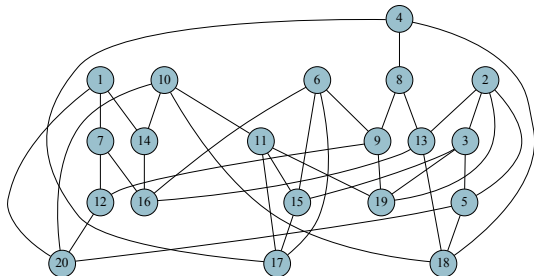
Graph adjacency matrix+list example [10 vertices]



vertex	adjacency list
1	3, 4, 9
2	7, 8, 10
3	1, 5, 6, 7
4	1, 6, 10
5	3, 6, 8, 9
6	3, 4, 5, 7
7	2, 3, 6, 10
8	2, 5, 9
9	1, 5, 8
10	2, 4, 7

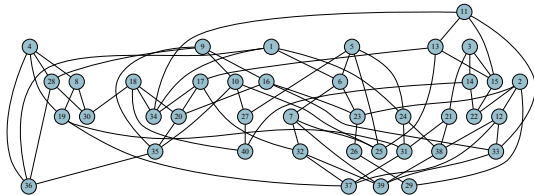
	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	0	0	0	0	1	0
2	0	0	0	0	0	0	1	1	0	1
3	1	0	0	0	1	1	1	0	0	0
4	1	0	0	0	0	1	0	0	0	1
5	0	0	1	0	0	1	0	1	1	0
6	0	0	1	1	1	0	1	0	0	0
7	0	1	1	0	0	1	0	0	0	1
8	0	1	0	0	1	0	0	0	1	0
9	1	0	0	0	1	0	0	1	0	0
10	0	1	0	1	0	0	1	0	0	0

Graph adjacency matrix example [20 vertices]

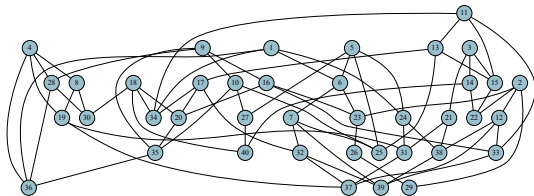


	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1
2	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
3	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0
5	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
6	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	0	0
7	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
8	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
9	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	1
11	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	1	0
12	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1
13	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0
14	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
15	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0
16	0	0	0	0	0	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0
17	0	0	0	1	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0
18	0	0	0	1	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
19	0	1	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
20	1	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0

Graph adjacency matrix example [40 vertices]

[illegible]

Graph adjacency list example [40 vertices]



vertex	adjacency list
1	6, 24, 34, 36
2	12, 22, 23, 29
3	14, 15, 21
4	8, 19, 28, 36
5	6, 24, 25, 27
6	1, 5, 7, 23
7	6, 25, 32, 39
8	4, 19, 30
9	10, 16, 28, 35
10	9, 25, 27, 35
11	13, 15, 33, 34
12	2, 33, 37, 38
13	11, 15, 17, 25
14	3, 22, 40
15	3, 11, 13, 22
16	9, 20, 23, 33
17	13, 20, 32, 34
18	20, 30, 34, 40
19	4, 8, 31, 37
20	16, 17, 18, 35
21	3, 31, 38
22	2, 14, 15
23	2, 6, 16, 26
24	1, 5, 31, 38
25	5, 7, 10, 13
26	23, 29
27	5, 10, 40
28	4, 9, 30, 36
29	2, 26
30	8, 18, 28
31	19, 21, 24, 37
32	7, 17, 37, 39
33	11, 12, 16, 39
34	1, 11, 17, 18
35	9, 10, 20, 36
36	1, 4, 28, 35
37	12, 19, 31, 32
38	12, 21, 24, 39
39	7, 32, 33, 38
40	14, 18, 27

A Concrete Representation

- Assume vertices are numbered arbitrarily as $\{1, 2, \dots, n\}$.
- Edges are numbered arbitrarily as $\{1, 2, \dots, m\}$.
- Edges stored in an array/list of size m . $E[j]$ is j^{th} edge with info on end points which are integers in range 1 to n .
- Array Adj of size n for adjacency lists. $Adj[i]$ points to adjacency list of vertex i . $Adj[i]$ is a list of edge indices in range 1 to m .

A Concrete Representation

Array of edges E



information including end point indices

Array of adjacency lists



List of edges (indices) that are incident to v_i



A Concrete Representation: Advantages

- Edges are explicitly represented/numbered. Scanning/processing all edges easy to do.
- Representation easily supports multigraphs including self-loops.
- Explicit numbering of vertices and edges allows use of arrays: $O(1)$ -time operations are easy to understand.
- Can also implement via pointer based lists for certain dynamic graph settings.

Connectivity I

Connectivity

Given a graph $G = (V, E)$:

- path: sequence of distinct vertices v_1, v_2, \dots, v_k such that $v_i v_{i+1} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k . Note: a single vertex u is a path of length 0.
- cycle: sequence of distinct vertices v_1, v_2, \dots, v_k such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$ and $\{v_1, v_k\} \in E$. Single vertex not a cycle according to this definition.

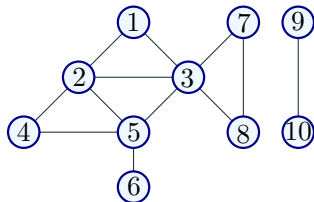
Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.

- A vertex u is connected to v if there is a path from u to v .
- The connected component of u , $\text{con}(u)$, is the set of all vertices connected to u . Is $u \in \text{con}(u)$?

Connectivity II

Define a relation C on $V \times V$ as uCv if u is connected to v

- In undirected graphs, connectivity is a reflexive, symmetric, and transitive relation. Connected components are the equivalence classes.
- Graph is connected if there is only one connected component.



Connectivity Problems

Algorithmic Problems

- Given graph G and nodes u and v , is u connected to v ?
- Given G and node u , find all nodes that are connected to u .
- Find all connected components of G .

Connectivity Problems

Algorithmic Problems

- Given graph G and nodes u and v , is u connected to v ?
- Given G and node u , find all nodes that are connected to u .
- Find all connected components of G .

Can be accomplished in $O(m + n)$ time using **BFS** or **DFS**.

BFS and **DFS** are refinements of a basic search procedure which is good to understand on its own.

Computing connected components
in undirected graphs using basic
graph search

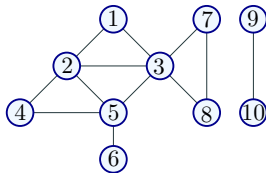
Basic Graph Search in Undirected Graphs

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore( $G, u$ ):  
    Visited[1 ..  $n$ ]  $\leftarrow$  FALSE  
    // ToExplore, S: Lists  
    Add  $u$  to ToExplore and to  $S$   
    Visited[ $u$ ]  $\leftarrow$  TRUE  
    while (ToExplore is non-empty) do  
        Remove node  $x$  from ToExplore  
        for each edge  $xy$  in Adj( $x$ ) do  
            if (Visited[ $y$ ] = FALSE)  
                Visited[ $y$ ]  $\leftarrow$  TRUE  
                Add  $y$  to ToExplore  
                Add  $y$  to  $S$   
  
    Output  $S$ 
```

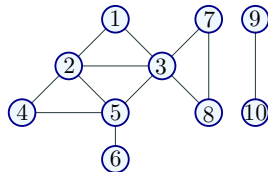
Example

```
Explore( $G, u$ ):  
  Visited[1 ..  $n$ ]  $\leftarrow$  FALSE  
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  Add  $u$  to ToExplore and to S  
  Visited[ $u$ ]  $\leftarrow$  TRUE  
  while (ToExplore is non-empty) do  
    Remove node  $x$  from ToExplore  
    for each edge  $xy$  in Adj( $x$ ) do  
      if (Visited[ $y$ ] = FALSE)  
        Visited[ $y$ ]  $\leftarrow$  TRUE  
        Add  $y$  to ToExplore  
        Add  $y$  to S  
  
  Output S
```



Example

```
Explore( $G, u$ ):  
  Visited[1 ..  $n$ ]  $\leftarrow$  FALSE  
  // ToExplore, S: Lists  
  Add  $u$  to ToExplore and to  $S$   
  Visited[ $u$ ]  $\leftarrow$  TRUE  
  while (ToExplore is non-empty) do  
    Remove node  $x$  from ToExplore  
    for each edge  $xy$  in Adj( $x$ ) do  
      if (Visited[ $y$ ] = FALSE)  
        Visited[ $y$ ]  $\leftarrow$  TRUE  
        Add  $y$  to ToExplore  
        Add  $y$  to  $S$   
  
  Output  $S$ 
```



Running Time:

Search Tree

One can create a natural search tree T rooted at u during search.

```
Explore( $G, u$ ):  
  array Visited[1.. $n$ ]  
  Initialize: Visited[ $i$ ]  $\leftarrow$  FALSE for  $i = 1, \dots, n$   
  List: ToExplore,  $S$   
  Add  $u$  to ToExplore and to  $S$ , Visited[ $u$ ]  $\leftarrow$  TRUE  
  Make tree  $T$  with root as  $u$   
  while (ToExplore is non-empty) do  
    Remove node  $x$  from ToExplore  
    for each edge  $(x, y)$  in Adj( $x$ ) do  
      if (Visited[ $y$ ] = FALSE)  
        Visited[ $y$ ]  $\leftarrow$  TRUE  
        Add  $y$  to ToExplore  
        Add  $y$  to  $S$   
        Add  $y$  to  $T$  with  $x$  as its parent  
  
  Output  $S$ 
```

Finding all connected components

Modify Basic Search to find all connected components of a given graph G in $O(m + n)$ time.

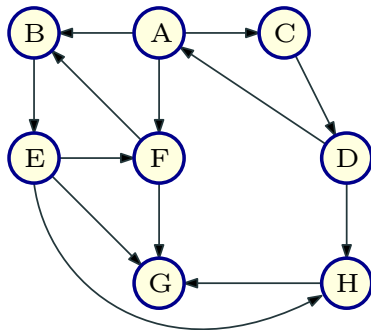
Directed Graphs and Directed Connectivity

Directed Graphs

Definition

A directed graph $G = (V, E)$ consists of:

- set of vertices/nodes V and
- a set of edges/arcs $E \subseteq V \times V$.



An edge is an ordered pair of vertices. (u, v) different from (v, u) .

Examples of Directed Graphs

In many situations relationship between vertices is asymmetric:

- Road networks with one-way streets.
- Web-link graph: vertices are web-pages and there is an edge from page p to page p' if p has a link to p' . Web graphs used by Google with PageRank algorithm to rank pages.
- Dependency graphs in variety of applications: link from x to y if y depends on x . Make files for compiling programs.
- Program Analysis: functions/procedures are vertices and there is an edge from x to y if x calls y .

Directed Graph Representation

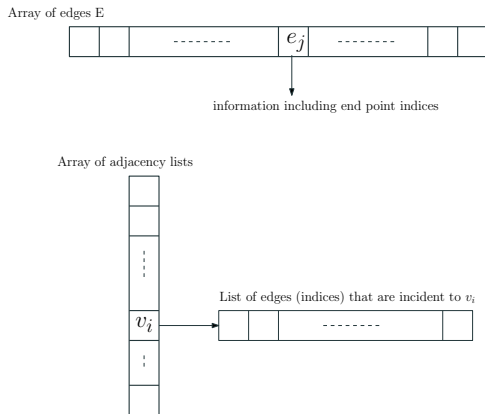
Graph $G = (V, E)$ with n vertices and m edges:

- Adjacency Matrix: $n \times n$ asymmetric matrix A . $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ if $(u, v) \notin E$. $A[u, v]$ is not same as $A[v, u]$.
- Adjacency Lists: for each node u , $Out(u)$ (also referred to as $Adj(u)$) and $In(u)$ store out-going edges and in-coming edges from u .

Default representation is adjacency lists.

A Concrete Representation for Directed Graphs

Concrete representation discussed previously for undirected graphs easily extends to directed graphs.



Directed Connectivity

Given a graph $G = (V, E)$:

- A (directed) path is a sequence of distinct vertices v_1, v_2, \dots, v_k such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from v_1 to v_k .

By convention, a single node u is a path of length 0.

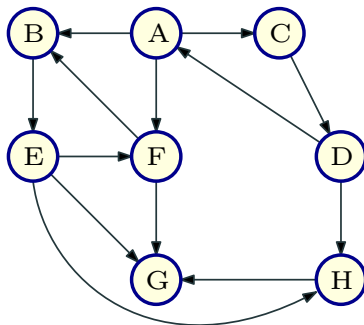
- A cycle is a sequence of distinct vertices v_1, v_2, \dots, v_k such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$.

By convention, a single node u is not a cycle.

- A vertex u can reach v if there is a path from u to v . Alternatively v can be reached from u
- Let $\text{rch}(u)$ be the set of all vertices reachable from u .

Directed Connectivity II

Asymmetry: *D* can reach *B* but *B* cannot reach *D*



Questions:

- Is there a notion of connected components?
- How do we understand connectivity in directed graphs?

Strong connected components

Connectivity and Strong Connected Components

Definition

Given a directed graph G , u is strongly connected to v if u can reach v and v can reach u . In other words $v \in \text{rch}(u)$ and $u \in \text{rch}(v)$.

Connectivity and Strong Connected Components

Definition

Given a directed graph G , u is strongly connected to v if u can reach v and v can reach u . In other words $v \in \text{rch}(u)$ and $u \in \text{rch}(v)$.

Define relation C where uCv if u is (strongly) connected to v .

Connectivity and Strong Connected Components

Definition

Given a directed graph G , u is strongly connected to v if u can reach v and v can reach u . In other words $v \in \text{rch}(u)$ and $u \in \text{rch}(v)$.

Define relation C where uCv if u is (strongly) connected to v .

Proposition

C is an equivalence relation, that is reflexive, symmetric and transitive.

Connectivity and Strong Connected Components

Definition

Given a directed graph G , u is strongly connected to v if u can reach v and v can reach u . In other words $v \in \text{rch}(u)$ and $u \in \text{rch}(v)$.

Define relation C where uCv if u is (strongly) connected to v .

Proposition

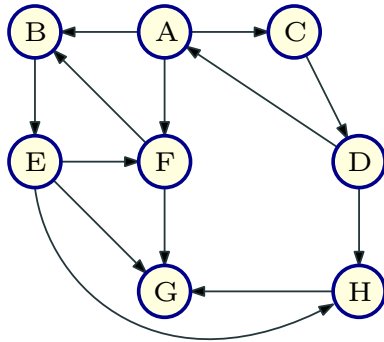
C is an equivalence relation, that is reflexive, symmetric and transitive.

Equivalence classes of C : strong connected components of G .

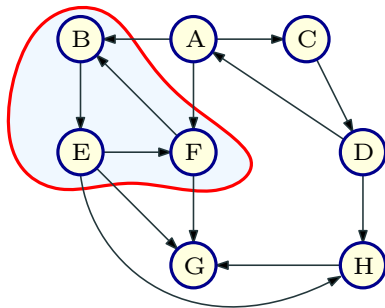
They partition the vertices of G .

$\text{SCC}(u)$: strongly connected component containing u .

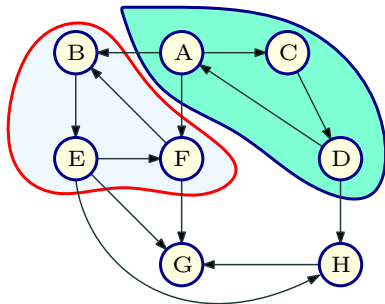
Strongly Connected Components: Example



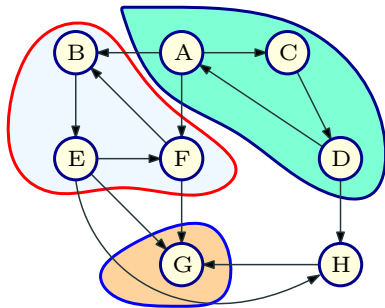
Strongly Connected Components: Example



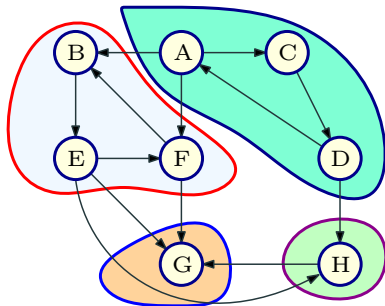
Strongly Connected Components: Example



Strongly Connected Components: Example



Strongly Connected Components: Example



Directed Graph Connectivity Problems

- Given G and nodes u and v , can u reach v ?
- Given G and u , compute $\text{rch}(u)$.
- Given G and u , compute all v that can reach u , that is all v such that $u \in \text{rch}(v)$.
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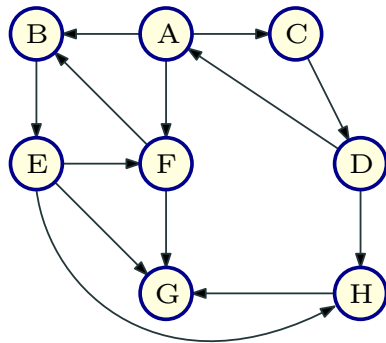
Graph exploration in directed graphs

Basic Graph Search in Directed Graphs

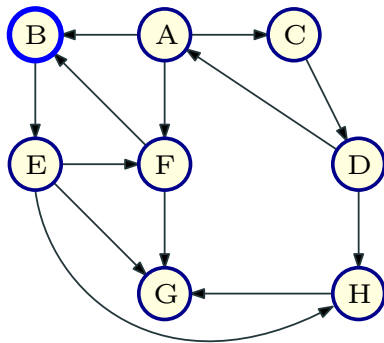
Given $G = (V, E)$ a directed graph and vertex $u \in V$. Let $n = |V|$.

```
Explore( $G, u$ ):  
    array Visited[1.. $n$ ]  
    Initialize: Set  $Visited[i] \leftarrow \text{FALSE}$  for  $1 \leq i \leq n$   
    List:  $ToExplore, S$   
    Add  $u$  to  $ToExplore$  and to  $S$ ,  $Visited[u] \leftarrow \text{TRUE}$   
    Make tree  $T$  with root as  $u$   
    while ( $ToExplore$  is non-empty) do  
        Remove node  $x$  from  $ToExplore$   
        for each edge  $(x, y)$  in  $Adj(x)$  do  
            if ( $Visited[y] = \text{FALSE}$ )  
                 $Visited[y] \leftarrow \text{TRUE}$   
                Add  $y$  to  $ToExplore$   
                Add  $y$  to  $S$   
                Add  $y$  to  $T$  with edge  $(x, y)$   
  
    Output  $S$ 
```

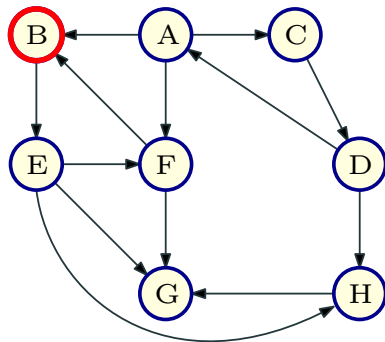
Example



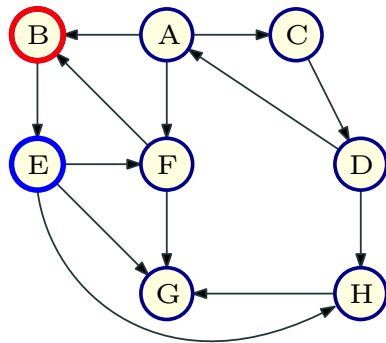
Example



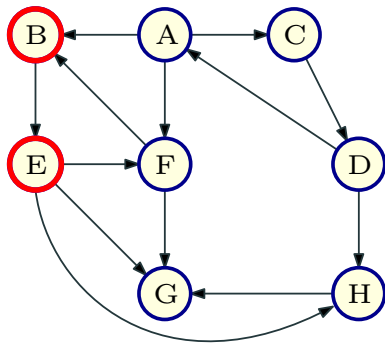
Example



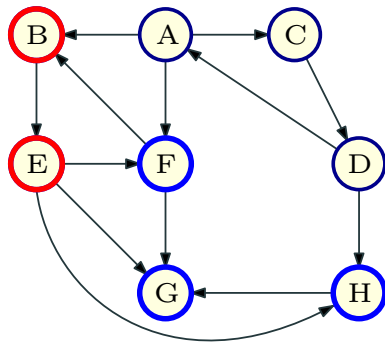
Example



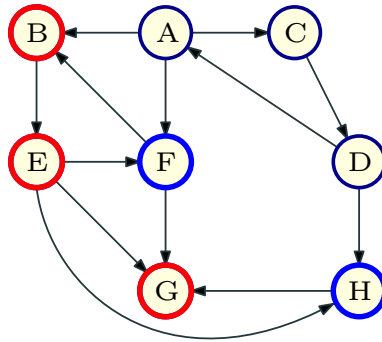
Example



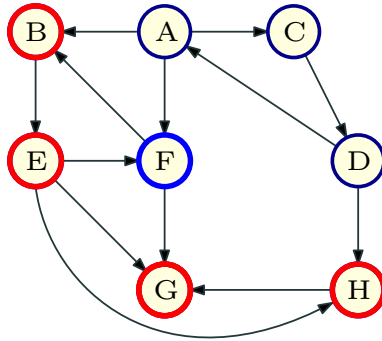
Example



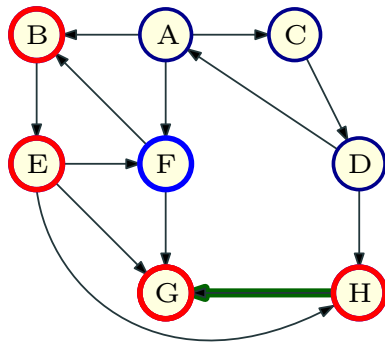
Example



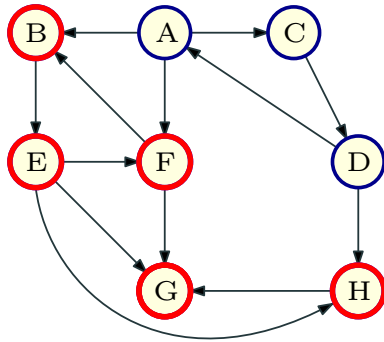
Example



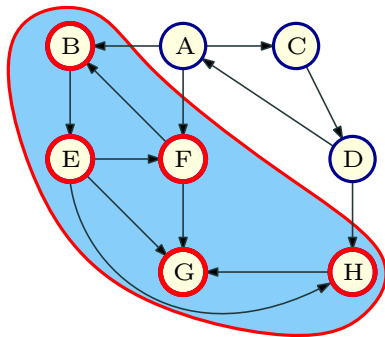
Example



Example



Example



Properties of Basic Search

Proposition

Explore(G, u) terminates with $S = \text{rch}(u)$.

Proof Sketch.

- Once $Visited[i]$ is set to $TRUE$ it never changes. Hence a node is added only once to $ToExplore$. Thus algorithm terminates in at most n iterations of while loop.
- By induction on iterations, can show $v \in S \Rightarrow v \in \text{rch}(u)$
- Since each node $v \in S$ was in $ToExplore$ and was explored, no edges in G leave S . Hence no node in $V - S$ is in $\text{rch}(u)$. Caveat: In directed graphs edges can enter S .
- Thus $S = \text{rch}(u)$ at termination.

Directed Graph Connectivity Problems

- Given G and nodes u and v , can u reach v ?
- Given G and u , compute $\text{rch}(u)$.
- Given G and u , compute all v that can reach u , that is all v such that $u \in \text{rch}(v)$.
- Find the strongly connected component containing node u , that is $\text{SCC}(u)$.
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First five problems can be solved in $O(n + m)$ time by via Basic Search (or **BFS/DFS**). The last one can also be done in linear time but requires a rather clever **DFS** based algorithm (next lecture).

Algorithms via Basic Search

Algorithms via Basic Search - I

- Given G and nodes u and v , can u reach v ?
- Given G and u , compute $\text{rch}(u)$.

Algorithms via Basic Search - I

- Given G and nodes u and v , can u reach v ?
- Given G and u , compute $\text{rch}(u)$.

Use $\text{Explore}(G, u)$ to compute $\text{rch}(u)$ in $O(n + m)$ time.

Algorithms via Basic Search - II

- Given G and u , compute all v that can reach u , that is all v such that $u \in \text{rch}(v)$.

Algorithms via Basic Search - II

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Definition (Reverse graph.)

Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed

$G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

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Definition (Reverse graph.)

Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed

$G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

Compute $\text{rch}(u)$ in G^{rev} !

- Running time:** $O(n + m)$ to obtain G^{rev} from G and $O(n + m)$ time to compute $\text{rch}(u)$ via Basic Search. If both $\text{Out}(v)$ and $\text{In}(v)$ are available at each v then no need to explicitly compute G^{rev} . Can do $\text{Explore}(G, u)$ in G^{rev} implicitly.

$$SCC(G, u) = \{v \mid u \text{ is strongly connected to } v\}$$

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- Find the strongly connected component containing node u . That is, compute $SCC(G, u)$.

Algorithms via Basic Search - III

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- Find the strongly connected component containing node u . That is, compute $SCC(G, u)$.

$$SCC(G, u) = \text{rch}(G, u) \cap \text{rch}(G^{rev}, u)$$

Algorithms via Basic Search - III

$$SCC(G, u) = \{v \mid u \text{ is strongly connected to } v\}$$

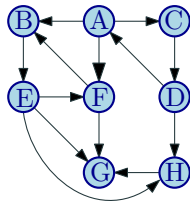
- Find the strongly connected component containing node u . That is, compute $SCC(G, u)$.

$$SCC(G, u) = \text{rch}(G, u) \cap \text{rch}(G^{rev}, u)$$

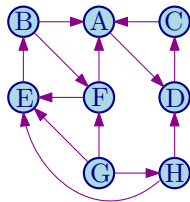
Hence, $SCC(G, u)$ can be computed with $Explore(G, u)$ and $Explore(G^{rev}, u)$. Total $O(n + m)$ time.

Why can $\text{rch}(G, u) \cap \text{rch}(G^{rev}, u)$ be done in $O(n)$ time?

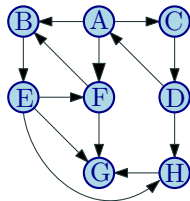
Graph G and its reverse graph G^{rev}



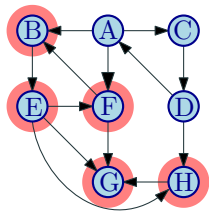
Graph G



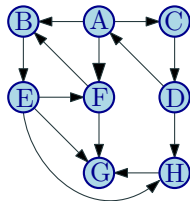
Graph G a vertex F and its reachable set (G, F)



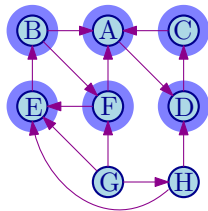
Graph G



Graph G a vertex F and the set of vertices that can reach it in $\text{G:rch}(G^{rev}, F)$

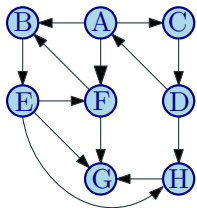


Graph G

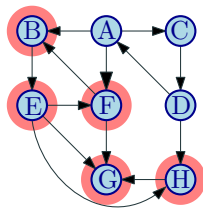


SCC IV: ...

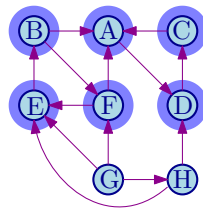
Graph G a vertex F and its strong connected component in G : $\text{SCC}(G, F)$



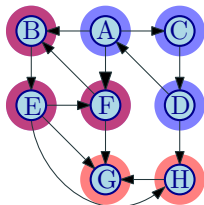
Graph G



$\text{rch}(G, F)$



$\text{rch}(G^{\text{rev}}, F)$



$$\text{SCC}(G, F) = \text{rch}(G, F) \cap \text{rch}(G^{\text{rev}}, F)$$

- Is G strongly connected?

- Is G strongly connected?

Pick arbitrary vertex u . Check if $SCC(G, u) = V$.

- Find all strongly connected components of G .

Algorithms via Basic Search - V

- Find all strongly connected components of G .

```
While  $G$  is not empty do  
    Pick arbitrary node  $u$   
    find  $S = \text{SCC}(G, u)$   
    Remove  $S$  from  $G$ 
```

Algorithms via Basic Search - V

- Find all strongly connected components of G .

```
While  $G$  is not empty do  
    Pick arbitrary node  $u$   
    find  $S = \text{SCC}(G, u)$   
    Remove  $S$  from  $G$ 
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

Algorithms via Basic Search - V

- Find all strongly connected components of G .

```
While  $G$  is not empty do  
    Pick arbitrary node  $u$   
    find  $S = \text{SCC}(G, u)$   
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```

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Running time: $O(n(n + m))$.

Algorithms via Basic Search - V

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```
While  $G$  is not empty do  
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```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

Running time: $O(n(n + m))$.

Question: Can we do it in $O(n + m)$ time?

Find out next time....
