Pre-lecture brain teaser

We talked a lot about languages representing problems. Consider the problem of adding two numbers. What language class does it belong to?

ECE-374-B: Lecture 9 - Recursion, Sorting and Recurrences

Lecturor: Nickvash Kani

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University of Illinois Urbana-Champaign

Pre-lecture brain teaser

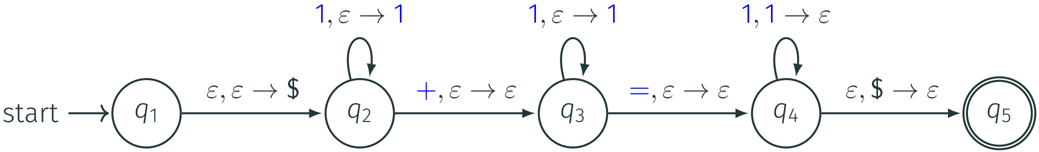
We talked a lot about languages representing problems. Consider the problem of adding two numbers. What language class does it belong to?

Pre-lecture brain teaser

Let's say we are adding two unary numbers.

$$3 + 4 = 7 \rightarrow 111 + 1111 = 1111111 \tag{1}$$

Seems like we can make a PDA that considers

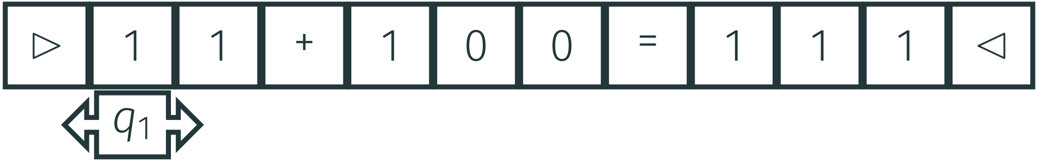


Pre-lecture brain teaser

What if we wanted add two binary numbers?

$$3 + 4 = 7 \rightarrow 11 + 100 = 111 \tag{2}$$

At least context-sensitive b/c we can build a finite Turing machine which takes in the encoding

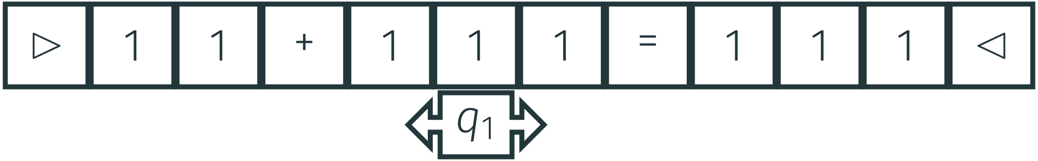


Pre-lecture brain teaser

What if we wanted add two binary numbers?

$$3 + 4 = 7 \rightarrow 11 + 100 = 111 \tag{3}$$

Computes value on left hand side



Pre-lecture brain teaser

What if we wanted add two binary numbers?

$$3 + 4 = 7 \rightarrow 11 + 100 = 111 \tag{4}$$

And compares it to the value on the right..

New Course Section: Introductory algorithms

Brief intro to the RAM model

Algorithms and Computing

- Algorithm solves a specific problem.
- Steps/instructions of an algorithm are simple/primitive and can be executed mechanically.
- Algorithm has a finite description; same description for all instances of the problem
- Algorithm implicitly may have state/memory

A computer is a device that

- implements the primitive instructions
- allows for an automated implementation of the entire algorithm by keeping track of state

Models of Computation vs Computers

- Model of Computation: an idealized mathematical construct that describes the primitive instructions and other details
- Computer: an actual physical device that implements a very specific model of computation

In this course section: design algorithms in a high-level model of computation.

Question: What model of computation will we use to design algorithms?

Models of Computation vs Computers

- Model of Computation: an idealized mathematical construct that describes the primitive instructions and other details
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In this course section: design algorithms in a high-level model of computation.

Question: What model of computation will we use to design algorithms?

The standard programming model that you are used to in programming languages such as Java/C++. We have already seen the Turing Machine model.

Unit-Cost RAM Model

Informal description:

- Basic data type is an integer number
- Numbers in input fit in a word
- Arithmetic/comparison operations on words take constant time
- Arrays allow random access (constant time to access $A[i]$)
- Pointer based data structures via storing addresses in a word

Example

Sorting: input is an array of n numbers

- input size is n (ignore the bits in each number),
- comparing two numbers takes $O(1)$ time,
- random access to array elements,
- addition of indices takes constant time,
- basic arithmetic operations take constant time,
- reading/writing one word from/to memory takes constant time.

We will usually do not allow (or be careful about allowing):

- bitwise operations (and, or, xor, shift, etc).
- floor function.
- limit word size (usually assume unbounded word size).

What is an algorithmic problem?

What is an algorithmic problem?

An algorithmic problem is simply to compute a function $f : \Sigma^* \to \Sigma^*$ over strings of a finite alphabet.

Algorithm $\mathcal{A}$ solves f if for all **input strings** w, $\mathcal{A}$ outputs $f(w)$.

Types of Problems

We will broadly see three types of problems.

- Decision Problem: Is the input a YES or NO input?
  Example: Given graph G, nodes s, t, is there a path from s to t in G?
  Example: Given a CFG grammar G and string w, is $w \in L(G)$?
- Search Problem: Find a solution if input is a YES input.
  Example: Given graph G, nodes s, t, find an s-t path.
- Optimization Problem: Find a best solution among all solutions for the input.
  Example: Given graph G, nodes s, t, find a shortest s-t path.

Analysis of Algorithms

Given a problem P and an algorithm $\mathcal{A}$ for P we want to know:

- Does $\mathcal{A}$ **correctly** solve problem P?
- What is the **asymptotic worst-case running time** of $\mathcal{A}$?
- What is the **asymptotic worst-case space** used by $\mathcal{A}$.

Asymptotic running-time analysis: $\mathcal{A}$ runs in $O(f(n))$ time if:

"for all n and for all inputs I of size n, $\mathcal{A}$ on input I terminates after $O(f(n))$ primitive steps."

Algorithmic Techniques

- Reduction to known problem/algorithm
- Recursion, divide-and-conquer, dynamic programming
- Graph algorithms to use as basic reductions
- Greedy

Some advanced techniques not covered in this class:

- Combinatorial optimization
- Linear and Convex Programming, more generally continuous optimization method
- Advanced data structure
- Randomization
- Many specialized areas

Reductions: Reducing problem A to problem B:

UNIQUENESS: Distinct Elements Problem

Problem Given an array A of n integers, are there any duplicates in A?

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Naive algorithm:

```
DistinctElements(A[1..n])
    for i = 1 to n − 1 do
        for j = i + 1 to n do
            if (A[i] = A[j])
                return YES
    return NO
```

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Running time:

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```

Running time: $O(n^2)$

Reduction to Sorting

```
DistinctElements(A[1..n])
    Sort A
    for i = 1 to n − 1 do
        if (A[i] = A[i + 1]) then
            return YES
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Running time: $O(n)$ plus time to sort an array of n numbers

Important point: algorithm uses sorting as a black box

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Running time: $O(n)$ plus time to sort an array of n numbers

Important point: algorithm uses sorting as a black box

Advantage of naive algorithm: works for objects that cannot be "sorted". Can also consider hashing but outside scope of current course.

Two sides of Reductions

Suppose problem A reduces to problem B

- Positive direction: Algorithm for B implies an algorithm for A
- Negative direction: Suppose there is no "efficient" algorithm for A then it implies no efficient algorithm for B (technical condition for reduction time necessary for this)

Two sides of Reductions

Suppose problem A reduces to problem B

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Example: Distinct Elements reduces to Sorting in $O(n)$ time

- An $O(n \log n)$ time algorithm for Sorting implies an $O(n \log n)$ time algorithm for Distinct Elements problem.
- If there is no $o(n \log n)$ time algorithm for Distinct Elements problem then there is no $o(n \log n)$ time algorithm for Sorting.

Recursion as self reductions

Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

- reduce problem to a <u>smaller</u> instance of <u>itself</u>
- self-reduction

Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

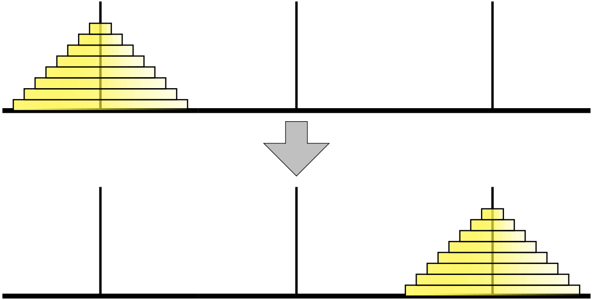
- reduce problem to a smaller instance of itself
- self-reduction

- Problem instance of size n is reduced to one or more instances of size $n - 1$ or less.
- For termination, problem instances of small size are solved by some other method as base cases

Recursion

- Recursion is a very powerful and fundamental technique
- Basis for several other methods
    - Divide and conquer
    - Dynamic programming
    - Enumeration and branch and bound etc
    - Some classes of greedy algorithms
- Makes proof of correctness easy (via induction)
- Recurrences arise in analysis

Tower of Hanoi



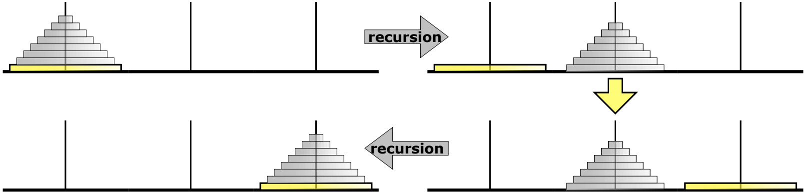
The Tower of Hanoi puzzle

Move stack of n disks from peg 0 to peg 2, one disk at a time.

Rule: cannot put a larger disk on a smaller disk.

Question: what is a strategy and how many moves does it take?

Tower of Hanoi via Recursion



The Tower of Hanoi algorithm; ignore everything but the bottom disk

Recursive Algorithm

```
Hanoi(n, src, dest, tmp):
    if (n > 0) then
        Hanoi(n − 1, src, tmp, dest)
        Move disk n from src to dest
        Hanoi(n − 1, tmp, dest, src)
```

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```

$T(n)$: time to move n disks via recursive strategy

$$T(n) = 2T(n-1) + 1 \qquad n > 1 \qquad \text{and } T(1) = 1$$

Analysis

$$
\begin{aligned}
T(n) &= 2T(n-1)+1 \\
&= 2^2T(n-2)+2+1 \\
&= \ldots \\
&= 2^iT(n-i)+2^{i-1}+2^{i-2}+\ldots+1 \\
&= \ldots \\
&= 2^{n-1}T(1)+2^{n-2}+\ldots+1 \\
&= 2^{n-1}+2^{n-2}+\ldots+1 \\
&= (2^n-1)/(2-1)=2^n-1
\end{aligned}
$$

Merge Sort

Sorting

Input Given an array of n elements

Goal Rearrange them in ascending order

MergeSort

1. Input: Array $A[1 \ldots n]$

A L G O R I T H M S

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$$A\ L\ G\ O\ R\ I\ T\ H\ M\ S$$

2. Divide into subarrays $A[1 \ldots m]$ and $A[m+1 \ldots n]$, where $m = \lfloor n/2 \rfloor$

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4. Merge the sorted arrays

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Merging Sorted Arrays

- Use a new array C to store the merged array
- Scan A and B from left-to-right, storing elements in C in order

$$A\ G\ L\ O\ R \qquad H\ I\ M\ S\ T$$

$$A$$

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- Merge two arrays using only constantly more extra space (in-place merge sort): doable but complicated and typically impractical.

Formal Code

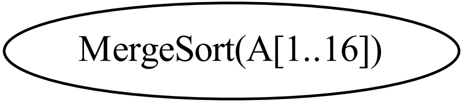
```
MERGESORT(A[1..n]):
  if n > 1
      m ← ⌊n/2⌋
      MERGESORT(A[1..m])
      MERGESORT(A[m+1..n])
      MERGE(A[1..n], m)
```

```
MERGE(A[1..n], m):
  i ← 1;  j ← m+1
  for k ← 1 to n
      if j > n
          B[k] ← A[i];  i ← i+1
      else if i > m
          B[k] ← A[j];  j ← j+1
      else if A[i] < A[j]
          B[k] ← A[i];  i ← i+1
      else
          B[k] ← A[j];  j ← j+1

  for k ← 1 to n
      A[k] ← B[k]
```

Running time analysis of merge-sort: Recursion tree method

Recursion tree



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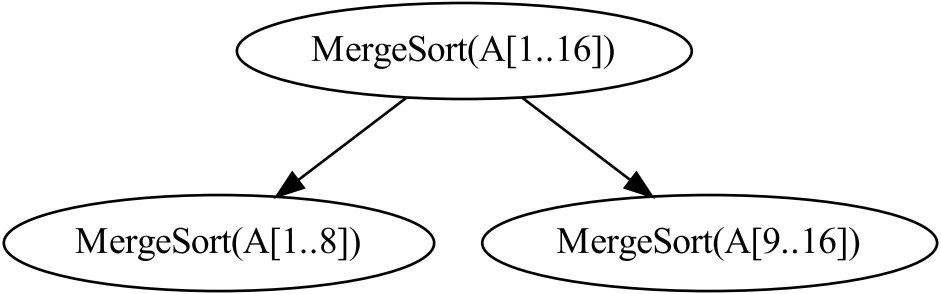

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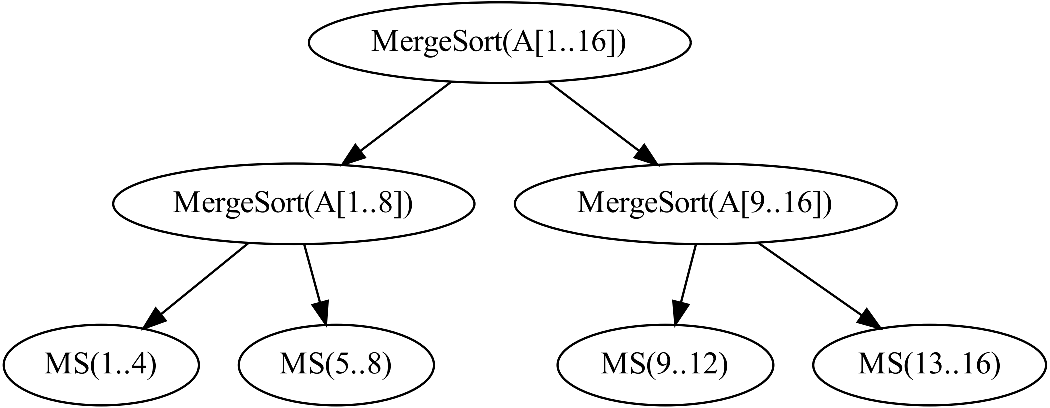

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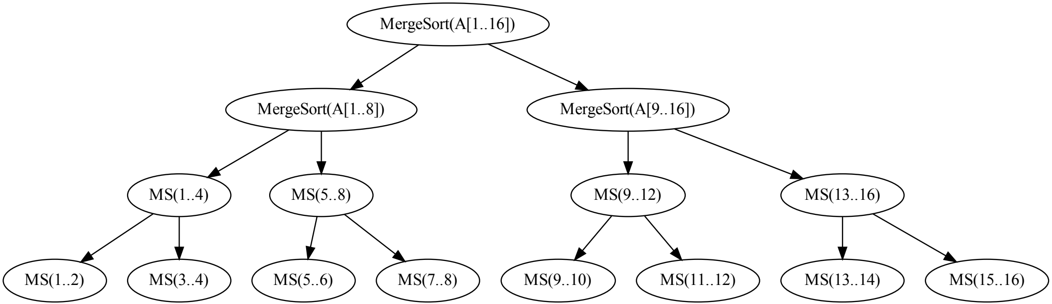

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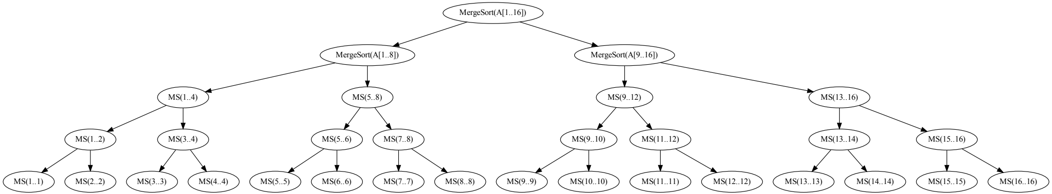
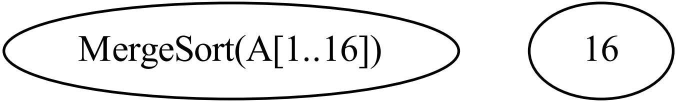


Recursion tree: subproblem sizes



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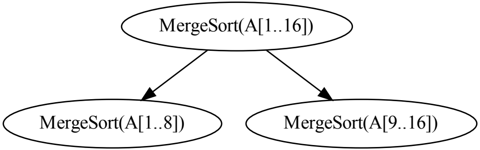

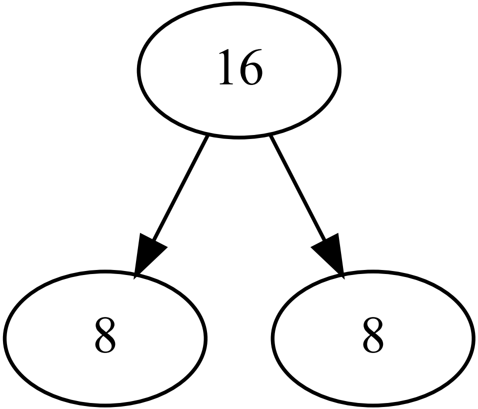

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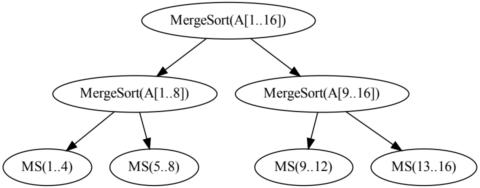

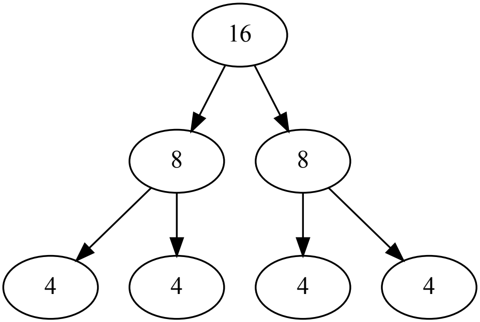

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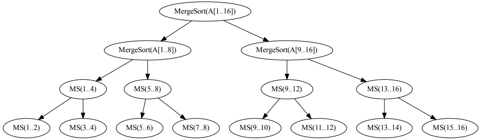

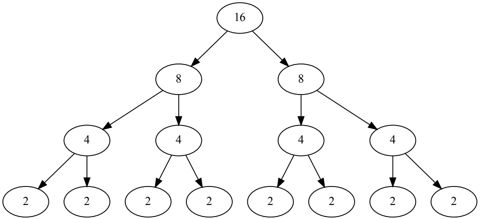

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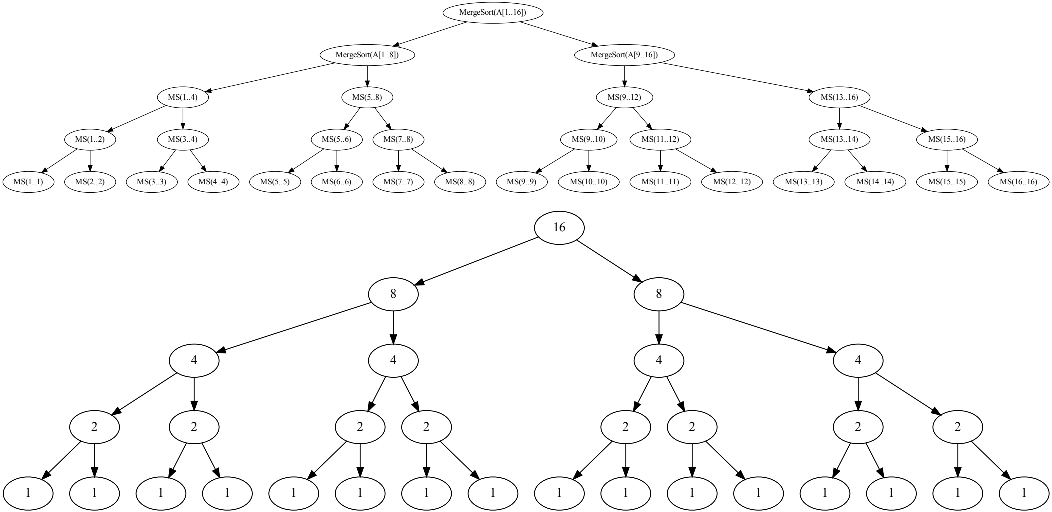
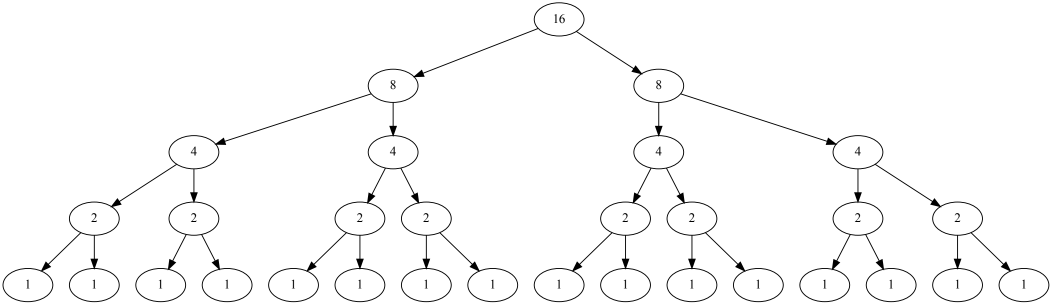


Recursion tree: Total work?



Running Time

$T(n)$: time for merge sort to sort an n element array

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$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

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What do we want as a solution to the recurrence?

Almost always only an asymptotically tight bound. That is we want to know $f(n)$ such that $T(n) = \Theta(f(n))$.

- $T(n) = O(f(n))$ - upper bound
- $T(n) = \Omega(f(n))$ - lower bound

Solving Recurrences: Some Techniques

- Know some basic math: geometric series, logarithms, exponentials, elementary calculus
- Expand the recurrence and spot a pattern and use simple math
- Recursion tree method — imagine the computation as a tree
- Guess and verify — useful for proving upper and lower bounds even if not tight bounds

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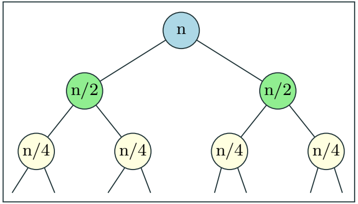
Albert Einstein: “Everything should be made as simple as possible, but not simpler.”

Know where to be loose in analysis and where to be tight. Comes with practice, practice, practice!

Recursion Trees : MergeSort: n is a power of 2



- Unroll the recurrence. $T(n) = 2T(n/2) + cn$

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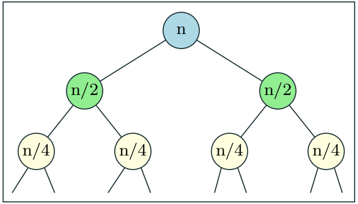
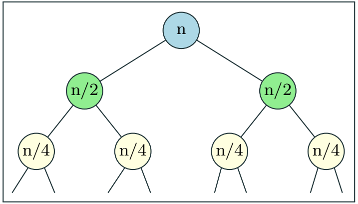


- Unroll the recurrence. $T(n) = 2T(n/2) + cn$
- Identify a pattern. .



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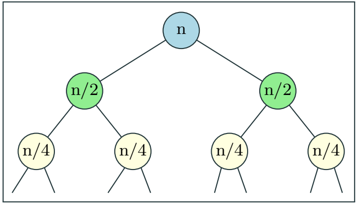

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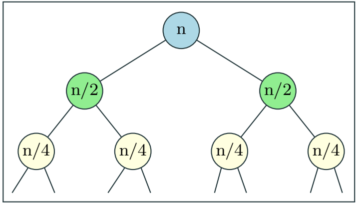
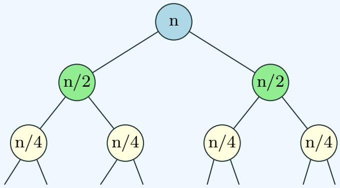


- Unroll the recurrence. $T(n) = 2T(n/2) + cn$
- Identify a pattern. At the $i$þlevel total work is cn.
- Sum over all levels. The number of levels is $\log n$. So total is $cn \log n = O(n \log n)$.

Recursion Trees



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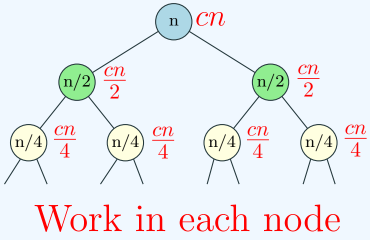
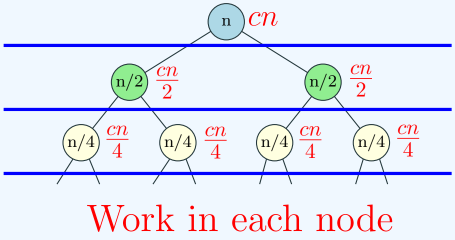


Recursion Trees



Recursion Trees

$$\log n \left\{ \begin{array}{ccc} cn & & = cn \\ \frac{cn}{2} + \frac{cn}{2} & & = cn \\ \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} & & = cn \\ \vdots & & \vdots \\ & & = cn \end{array} \right.$$

Recursion Trees

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$$= cn \log n = O(n \log n)$$

Merge Sort Variant

Question: Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say k arrays of size n/k each?

Binary Search

Binary Search in Sorted Arrays

Input Sorted array A of n numbers and number x

Goal Is x in A?

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```
BinarySearch (A[a..b], x):
        if (b − a < 0) return NO
        mid = A[⌊(a + b)/2⌋]
        if (x = mid) return YES
        if (x < mid)
            return BinarySearch (A[a..⌊(a + b)/2⌋ − 1], x)
        else
            return BinarySearch (A[⌊(a + b)/2⌋ + 1..b],x)
```

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```

Analysis: $T(n) = T(\lfloor n/2 \rfloor) + O(1)$. $T(n) = O(\log n)$.

Observation: After k steps, size of array left is $n/2^k$