Pre-lecture brain teaser

Given a directed graph (G), propose an algorithm that finds a vertex that is contained within the source SCC of the meta-graph of G.

ECE-374-B: Lecture 16 - Shortest Paths [BFS, Djikstra]

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University of Illinois Urbana-Champaign

Pre-lecture brain teaser

Given a directed graph (G), propose an algorithm that finds a vertex that is contained within the source SCC of the meta-graph of G.

Breadth First Search

Breadth First Search (BFS)

Overview

(A) BFS is obtained from BasicSearch by processing edges using a queue data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

- DFS good for exploring graph structure
- BFS good for exploring distances

Queue Data Structure

Queues
A queue is a list of elements which supports the operations:

- **enqueue**: Adds an element to the end of the list
- **dequeue**: Removes an element from the front of the list

Elements are extracted in first-in first-out (FIFO) order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

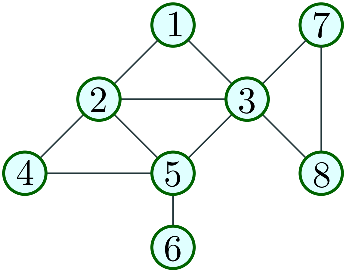
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

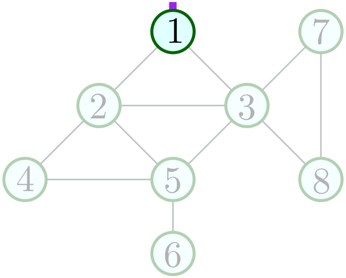
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enqueue(Q, s)
    while Q is nonempty do
        u = dequeue(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enqueue(v)
```

Proposition

BFS(s) *runs in* $O(n + m)$ *time.*

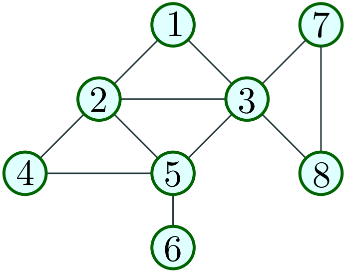
BFS: An Example in Undirected Graphs

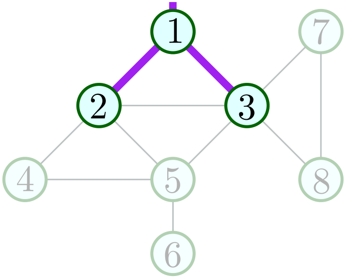




T1. [1]

BFS: An Example in Undirected Graphs

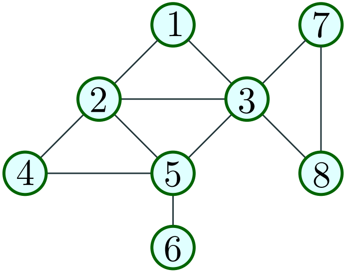


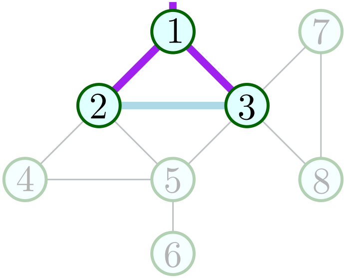


T1. [1]

T2. [2,3]

BFS: An Example in Undirected Graphs

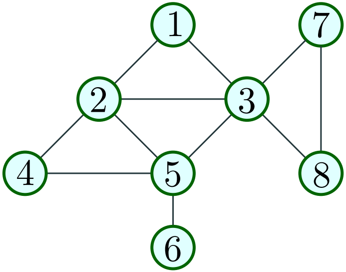


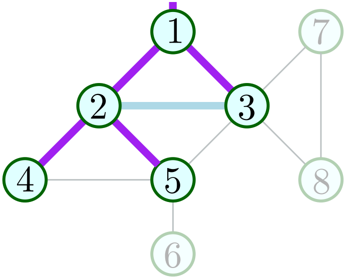


T1. [1]

T2. [2,3]

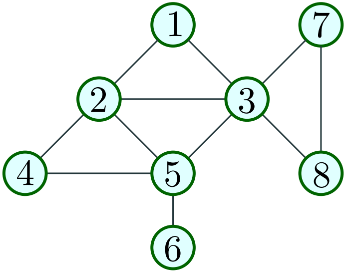
BFS: An Example in Undirected Graphs

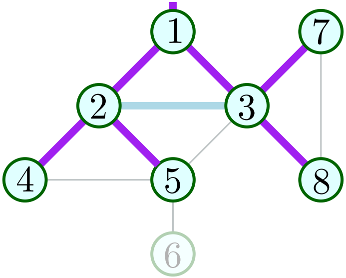




T1. [1]
T2. [2,3]
T3. [3,4,5]

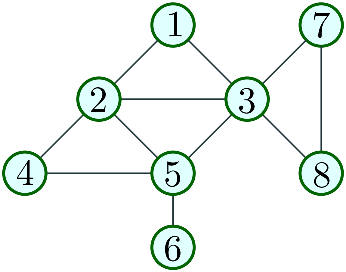
BFS: An Example in Undirected Graphs

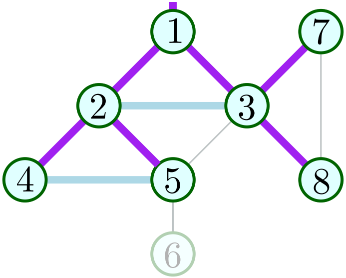




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]

BFS: An Example in Undirected Graphs





6

T1. [1]

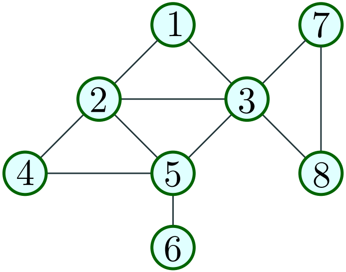
T2. [2,3]

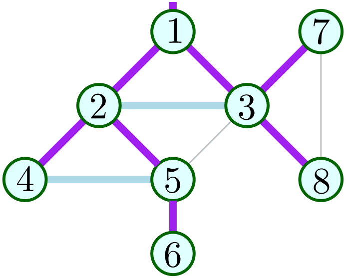
T3. [3,4,5]

T4. [4,5,7,8]

T5. [5,7,8]

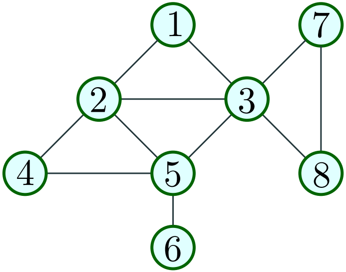
BFS: An Example in Undirected Graphs

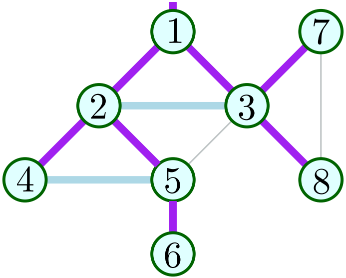




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]

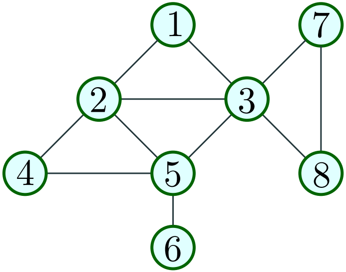
BFS: An Example in Undirected Graphs

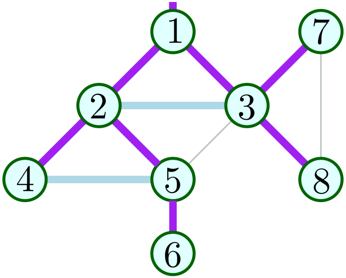




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]
T7. [8,6]

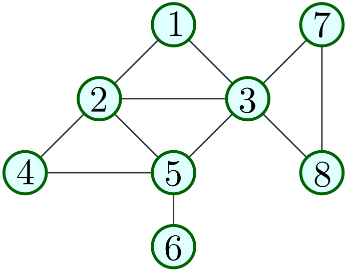
BFS: An Example in Undirected Graphs

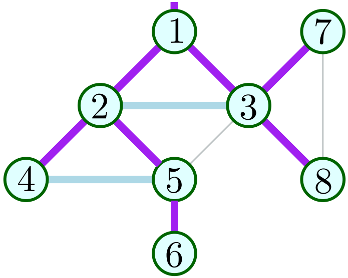




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]
T7. [8,6]
T8. [6]

BFS: An Example in Undirected Graphs

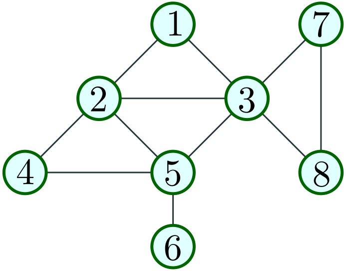


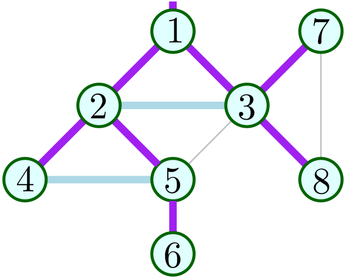


T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]
T7. [8,6]
T8. [6]
T9. []

BFS tree is the set of purple edges.

BFS: An Example in Undirected Graphs

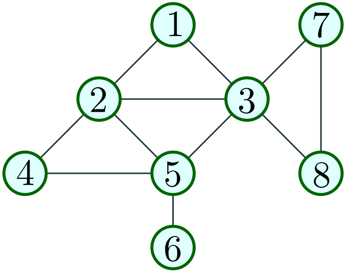


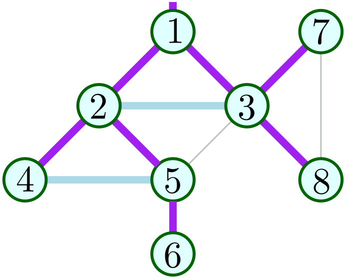


T1.	[1]	T4.	[4,5,7,8]	T7.	[8,6]
T2.	[2,3]	T5.	[5,7,8]	T8.	[6]
T3.	[3,4,5]	T6.	[7,8,6]	T9.	[]

BFS tree is the set of purple edges.

BFS: An Example in Undirected Graphs





T1.	[1]	T4.	[4,5,7,8]	T7.	[8,6]
T2.	[2,3]	T5.	[5,7,8]	T8.	[6]
T3.	[3,4,5]	T6.	[7,8,6]	T9.	[]

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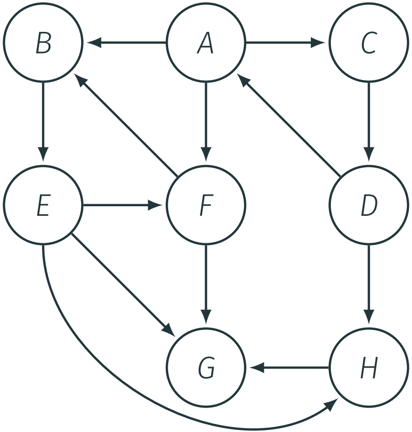
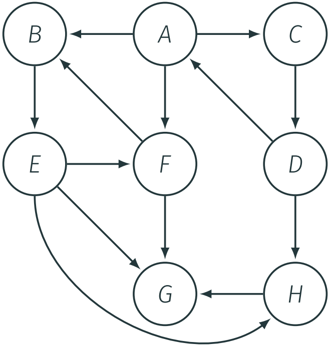
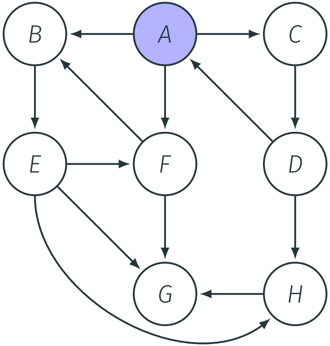
BFS: An Example in Directed Graphs



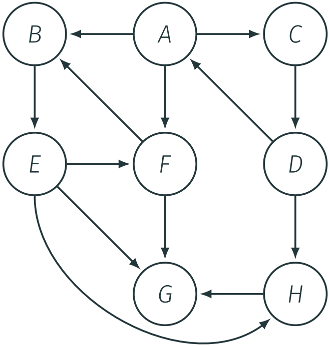
BFS: An Example in Directed Graphs

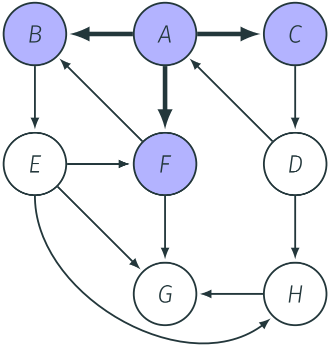




T1. [A]

BFS: An Example in Directed Graphs

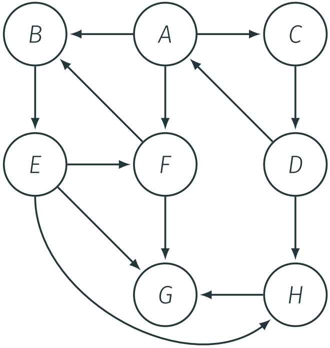


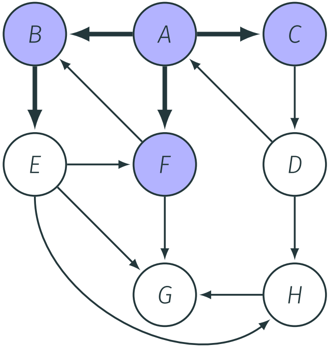


T1. [A]

T2. [B,C,F]

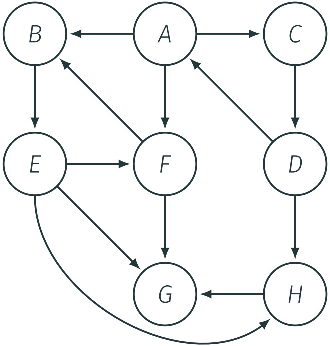
BFS: An Example in Directed Graphs

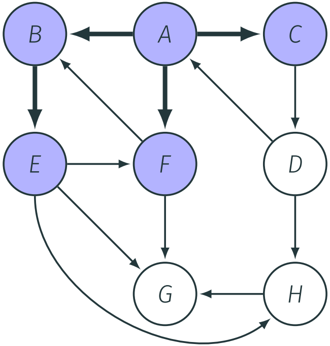




T1. [A]
T2. [B,C,F]

BFS: An Example in Directed Graphs



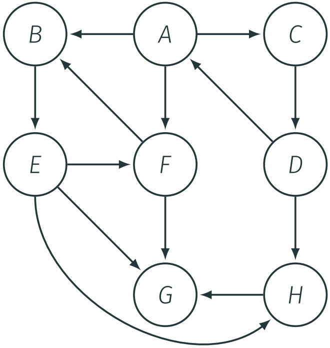


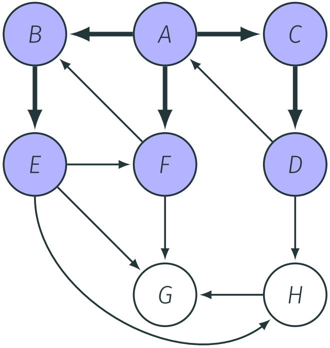
T1. [A]

T2. [B,C,F]

T3. [C,F,E]

BFS: An Example in Directed Graphs





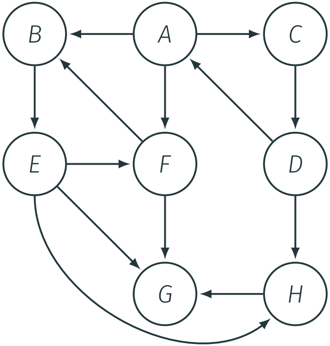
T1. [A]

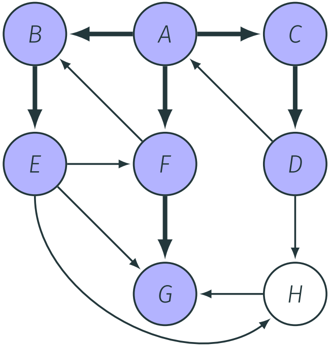
T2. [B,C,F]

T3. [C,F,E]

T4. [F,E,D]

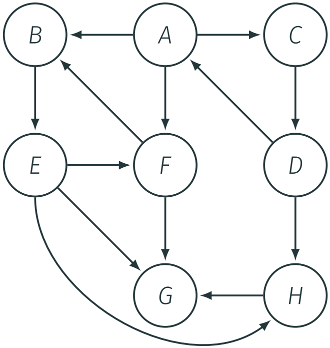
BFS: An Example in Directed Graphs

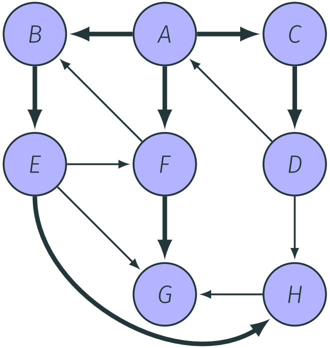




T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]

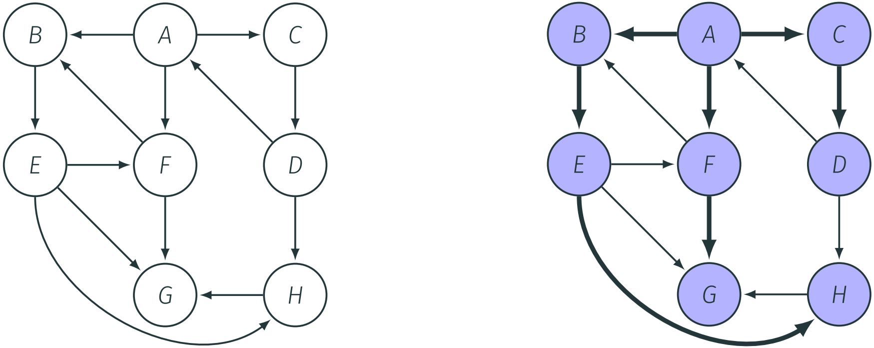
BFS: An Example in Directed Graphs





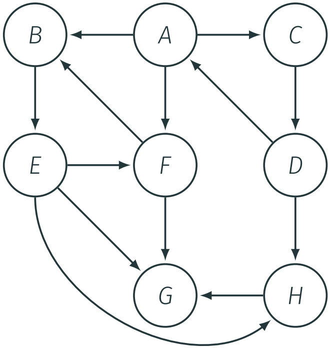
T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]

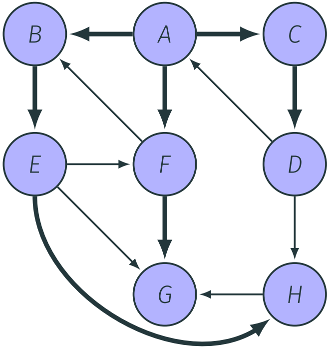
BFS: An Example in Directed Graphs



T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]
T7. [G,H]

BFS: An Example in Directed Graphs





T1. [A]

T2. [B,C,F]

T3. [C,F,E]

T4. [F,E,D]

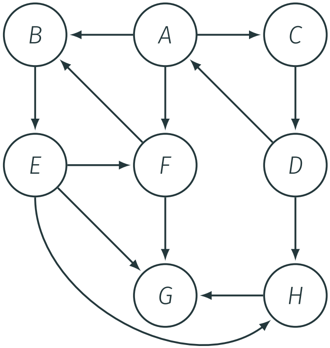
T5. [E,D,G]

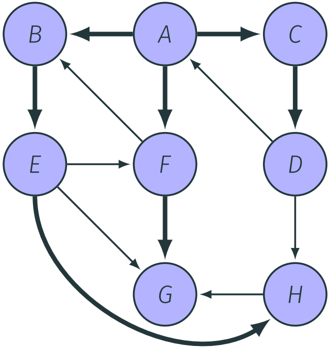
T6. [D,G,H]

T7. [G,H]

T8. [H]

BFS: An Example in Directed Graphs





T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]
T7. [G,H]
T8. [H]
T9. []

BFS with distances and layers

BFS with distances

```
BFS(s)
    Mark all vertices as unvisited; for each v set dist(v) = ∞
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enqueue(s)
    while Q is nonempty do
        u = dequeue(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enqueue(v)
                and set dist(v) = dist(u) + 1
```

Properties of BFS: Undirected Graphs

Theorem
The following properties hold upon termination of BFS(s)

(A) *Search tree contains exactly the set of vertices in the connected component of s.*

(B) *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then u is visited before v.*

(C) *For every vertex u,* $\mathrm{dist}(u)$ *is the length of a shortest path (in terms of number of edges) from s to u.*

(D) *If u, v are in connected component of s and $e = \{u, v\}$ is an edge of G, then* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Properties of BFS: Directed Graphs

Theorem

The following properties hold upon termination of BFS(s):

(A) *The search tree contains exactly the set of vertices reachable from s*

(B) *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then u is visited before v*

(C) *For every vertex u,* $\mathrm{dist}(u)$ *is indeed the length of shortest path from s to u*

(D) *If u is reachable from s and $e = (u, v)$ is an edge of G, then* $\mathrm{dist}(v) - \mathrm{dist}(u) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(u) - \mathrm{dist}(v) \leq 1$.

BFS with Layers

BFSLayers(s):

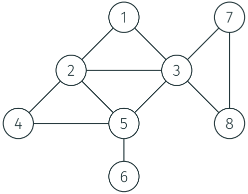
Mark all vertices as unvisited and initialize T to be empty
Mark s as visited and set $L_0 = \{s\}$
$i = 0$
while L_i is not empty **do**
    initialize L_{i+1} to be an empty list
    for each u in L_i **do**
        for each edge $(u, v) \in \mathrm{Adj}(u)$ **do**
        if v is not visited
            mark v as visited
            add (u, v) to tree T
            add v to L_{i+1}
    $i = i + 1$

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
        initialize L_{i+1} to be an empty list
        for each u in L_i do
            for each edge (u,v) ∈ Adj(u) do
            if v is not visited
                mark v as visited
                add (u,v) to tree T
                add v to L_{i+1}
        i = i + 1
```

Running time: $O(n+m)$

Example



Layer 0: 1

Layer 1: 2, 3

Layer 2: 4, 5, 7, 8

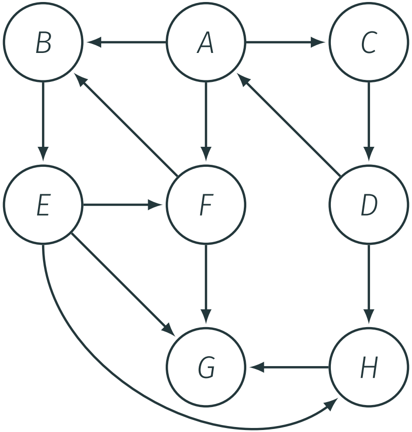
Layer 3: 6

BFS with Layers: Properties

Proposition
The following properties hold on termination of **BFSLayers**(s).

- **BFSLayers**(s) *outputs a* **BFS** *tree*
- L_i *is the set of vertices at distance exactly* i *from* s
- *If* G *is undirected, each edge* $e = \{u, v\}$ *is one of three types:*
  - *tree edge between two consecutive layers*
  - *non-tree forward/backward edge between two consecutive layers*
  - *non-tree cross-edge with both* u, v *in same layer*
  - $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Example



Layer 0: A

Layer 1: B, F, C

Layer 2: E, G, D

Layer 3: H

BFS with Layers: Properties for directed graphs

Proposition
The following properties hold on termination of BFSLayers(s)*, if* G *is directed.*

For each edge $e = (u, v)$ *is one of four types:*

- *a tree edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
- *a non-tree forward edge between consecutive layers*
- *a non-tree backward edge*
- *a cross-edge with both* u, v *in same layer*

Shortest Paths and Dijkstra's Algorithm

Problem definition

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
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Shortest Path Problems

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Many applications!

Single-Source Shortest Paths: Non-Negative Edge Lengths

- Single-Source Shortest Path Problems
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  - Restrict attention to directed graphs
  - Undirected graph problem can be reduced to directed graph problem - how?
    - Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by (u, v) and (v, u) in G'.
    - set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
    - Exercise: show reduction works. Relies on non-negativity!

Shortest path in the weighted case using BFS

Single-Source Shortest Paths via BFS

- **Special case:** All edge lengths are 1.

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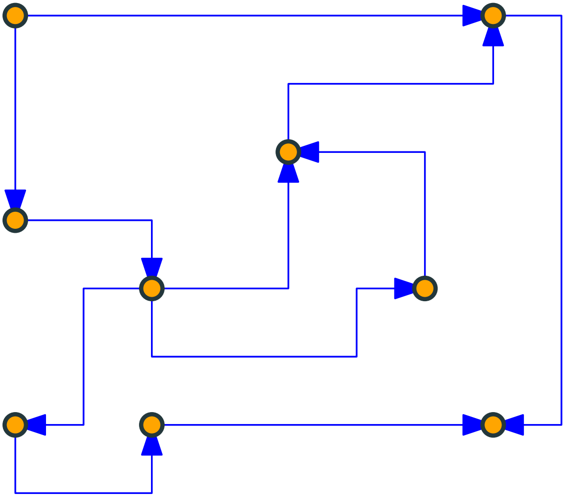
  Can we use BFS? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e.

Example of edge refinement



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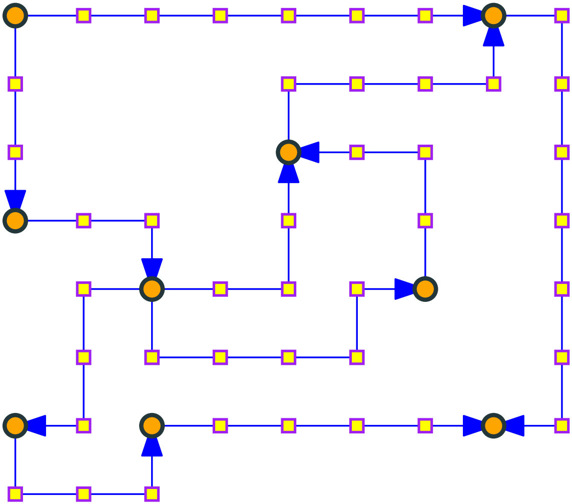

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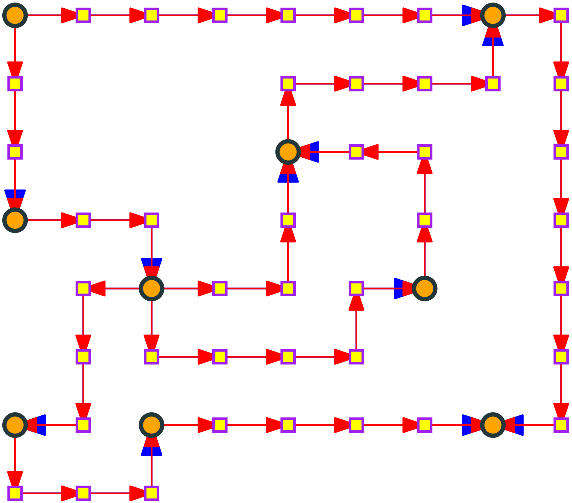


Shortest path using BFS

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. BFS takes $O(mL + n)$ time. Not efficient if L is large.

On the hereditary nature of shortest paths

You can not shortcut a shortest path

Lemma

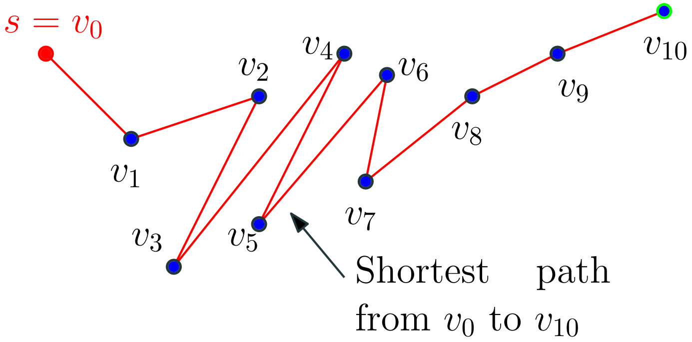
G: directed graph with non-negative edge lengths.

$\mathrm{dist}(s, v)$: *shortest path length from s to v.*

If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ *shortest path from s to* v_k *then for any* $0 \le i < j \le k$:

$v_i \to v_{i+1} \to \ldots \to v_j$ *is shortest path from* v_i *to* v_j

A proof by picture



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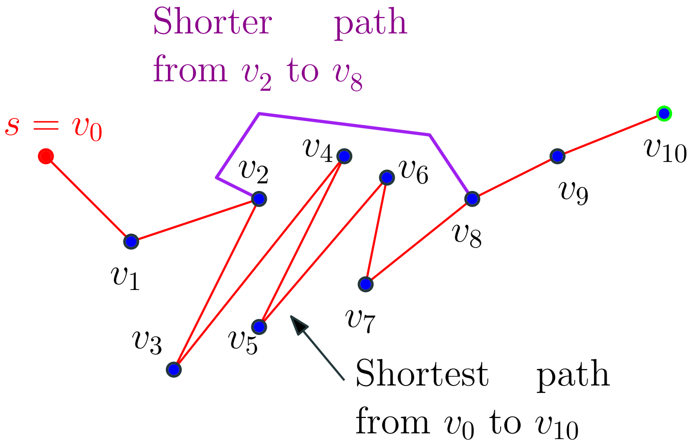
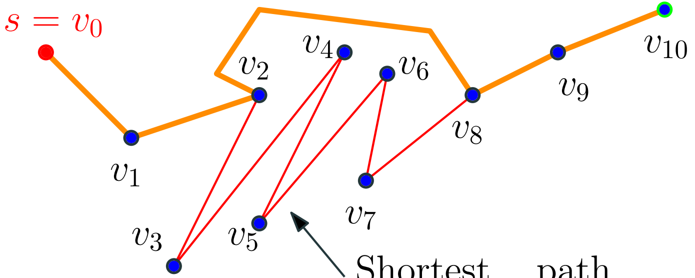


A proof by picture

A shorter path from v_0 to v_{10}. A contradiction.



Shortest path from v_0 to v_{10}

What we really need...

Corollary
G: directed graph with non-negative edge lengths.

$\mathrm{dist}(s, v)$: *shortest path length from s to v.*

If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ *shortest path from s to* v_k *then for any* $0 \leq i \leq k$:

- $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is shortest path from s to* v_i
- $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$. *Relies on non-neg edge lengths.*

The basic algorithm: Find the i^{th} closest vertex

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

Initialize for each node v, $\mathrm{dist}(s, v) = \infty$
Initialize $X = \{s\}$,
for $i = 2$ to $|V|$ **do**
    (* Invariant: X contains the $i-1$ closest nodes to s *)
    Among nodes in $V - X$, find the node v that is the
        $i$þclosest to s
    Update $\mathrm{dist}(s, v)$
    $X = X \cup \{v\}$

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    X = X ∪ {v}
```

How can we implement the step in the for loop?

Finding the i^{th} closest node

- X contains the $i-1$ closest nodes to s
- Want to find the i^{th} closest node from $V - X$.

What do we know about the i^{th} closest node?

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Let P be a shortest path from s to v where v is the i^{th} closest node. Then, all intermediate nodes in P belong to X.

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What do we know about the i^{th} closest node?

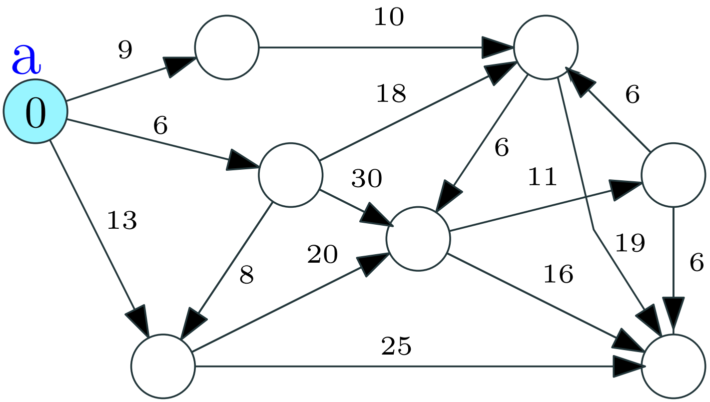
Claim

Let P be a shortest path from s to v where v is the i^{th} closest node. Then, all intermediate nodes in P belong to X.

Proof.

If P had an intermediate node u not in X then u will be closer to s than v. Implies v is not the i^{th} closest node to s - recall that X already has the $i - 1$ closest nodes. $\square$

Finding the i^{th} closest node repeatedly



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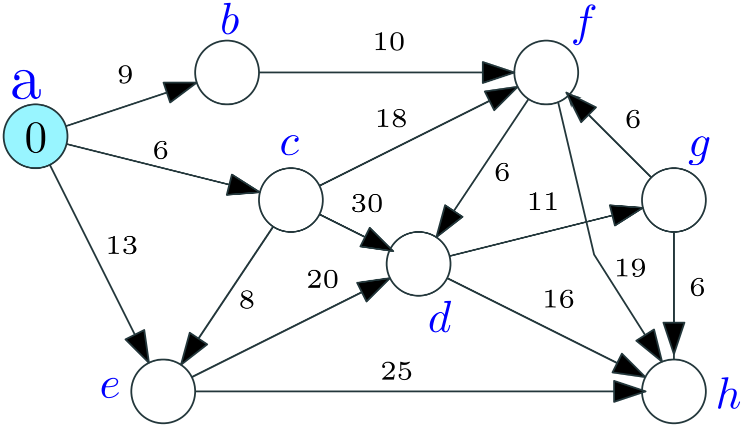

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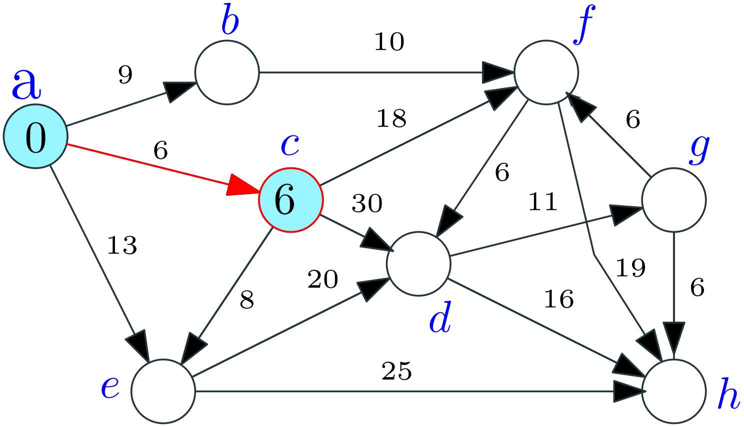

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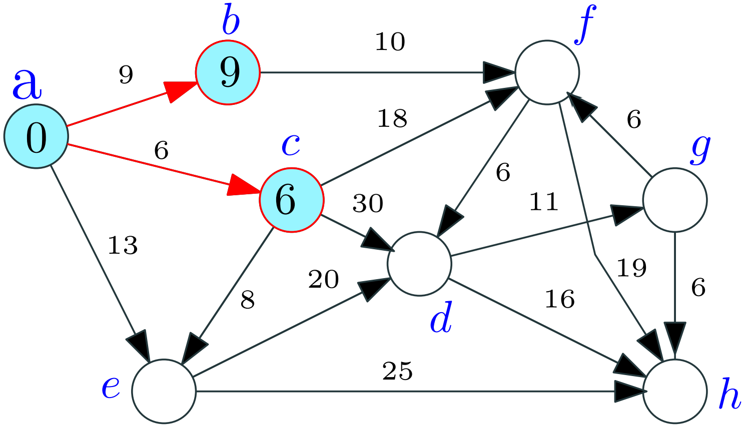

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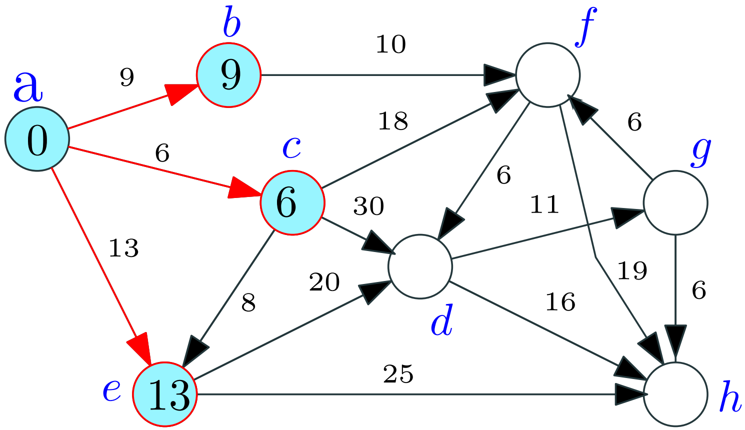

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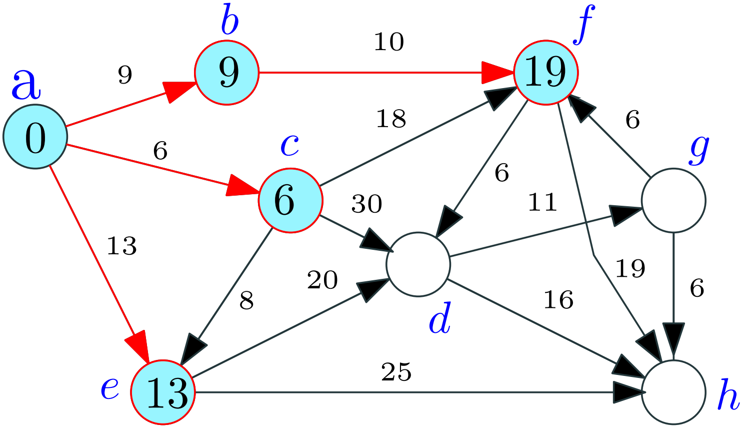

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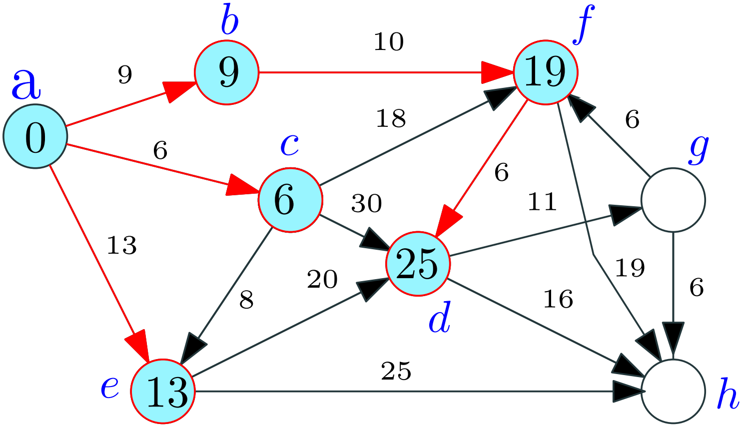

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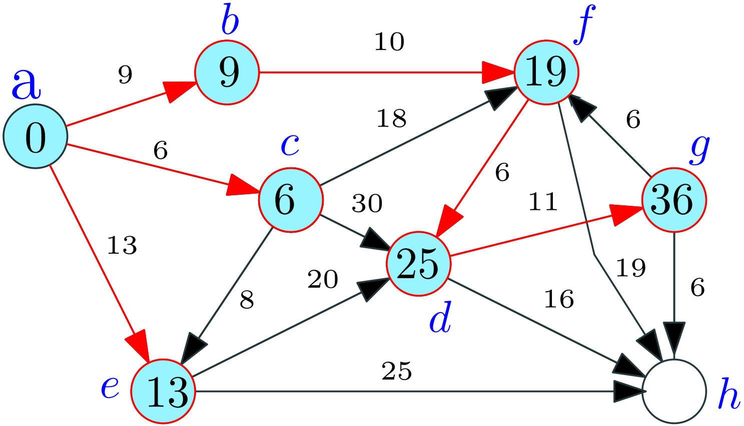

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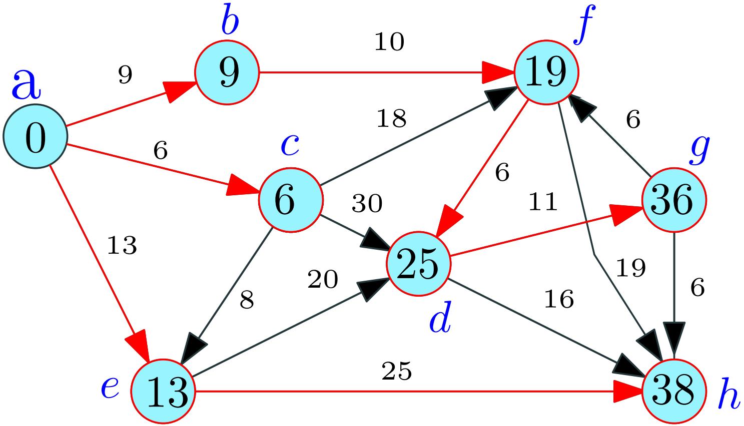

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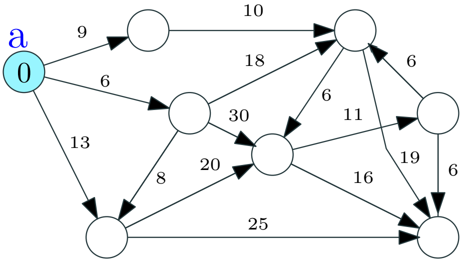

Corollary
The i^{th} closest node is adjacent to X.

Algorithm

```
Initialize for each node v: dist(s, v) = ∞
Initialize X = ∅, d'(s, s) = 0
for i = 1 to |V| do
    (* Invariant: X contains the i − 1 closest nodes to s *)
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     using only X as intermediate nodes*)
    Let v be such that d'(s, v) = min_{u∈V−X} d'(s, u)
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Running time:

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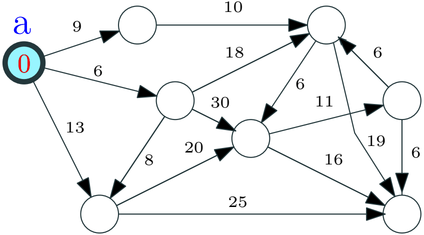
Running time: $O(n \cdot (n+m))$ time.

- n outer iterations. In each iteration, $d'(s,u)$ for each u by scanning all edges out of nodes in X; $O(m+n)$ time/iteration.

Dijkstra’s algorithm

Example: Dijkstra algorithm in action



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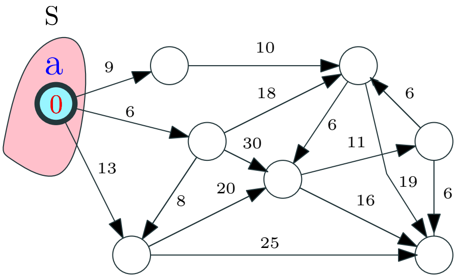

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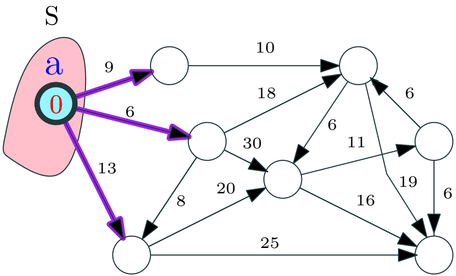

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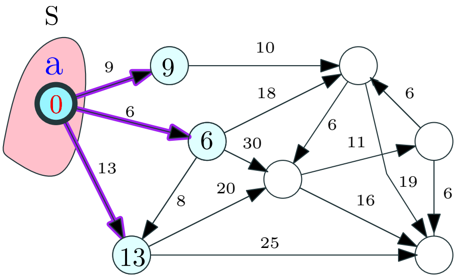

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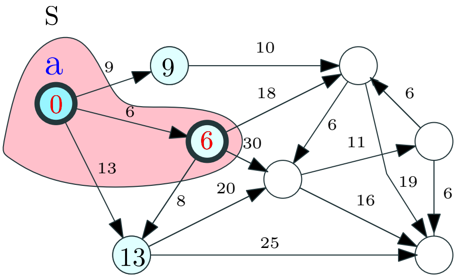

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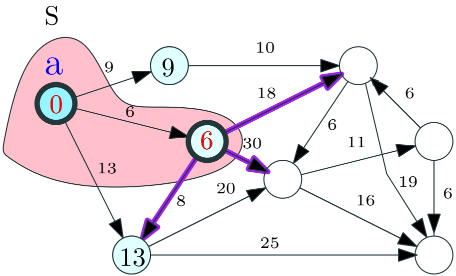

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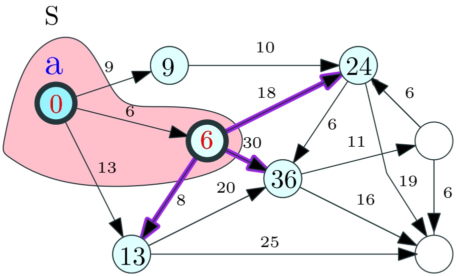

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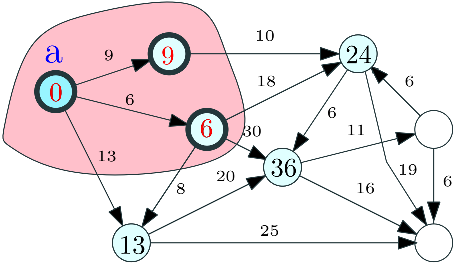

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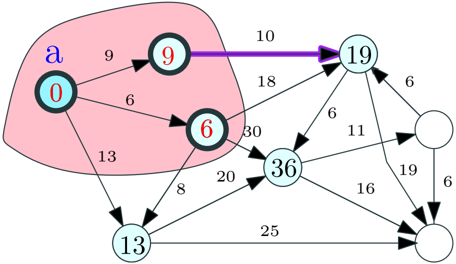

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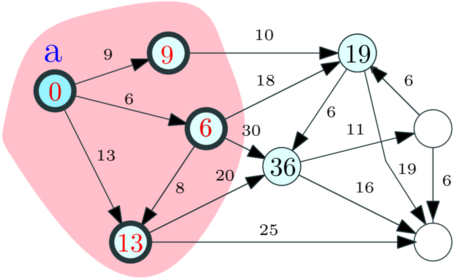

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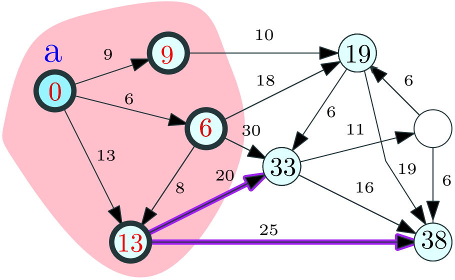

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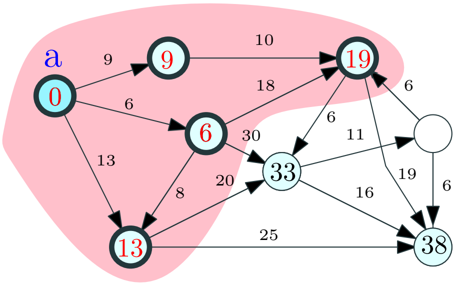

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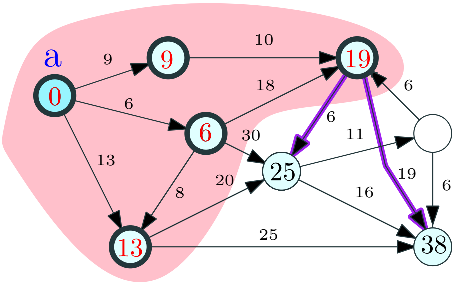

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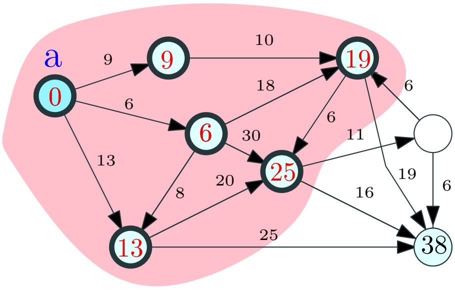

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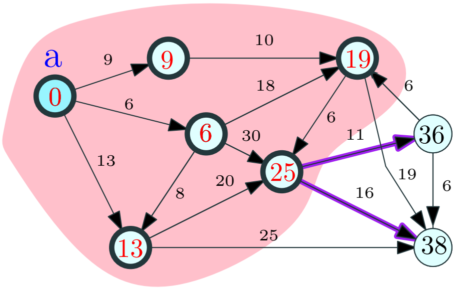

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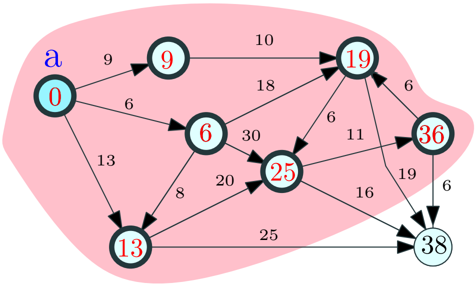

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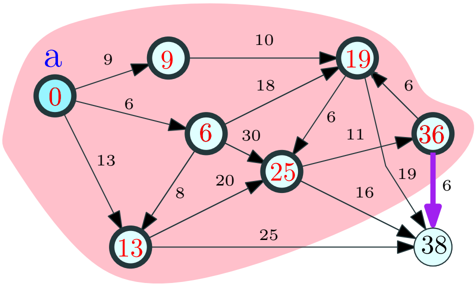

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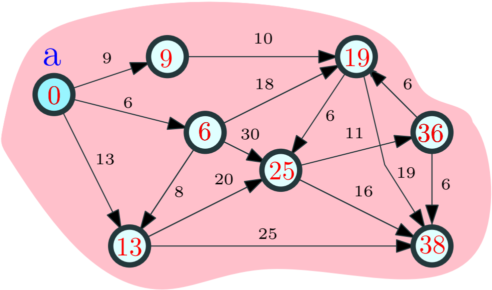

Improved Algorithm

- Main work is to compute the $d'(s, u)$ values in each iteration
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Running time:

Improved Algorithm

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Running time: $O(m + n^2)$ time.

- n outer iterations and in each iteration following steps
- updating $d'(s, u)$ after v is added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters X only once
- Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

- eliminate $d'(s,u)$ and let $\mathrm{dist}(s,u)$ maintain it
- update *dist* values after adding v by scanning edges out of v

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Priority Queues to maintain *dist* values for faster running time

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Priority Queues to maintain *dist* values for faster running time

- Using heaps and standard priority queues: $O((m+n)\log n)$
- Using Fibonacci heaps: $O(m + n\log n)$.

Dijkstra using priority queues

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

- makePQ: create an empty queue.
- findMin: find the minimum key in S.
- extractMin: Remove $v \in S$ with smallest key and return it.
- insert$(v, k(v))$: Add new element v with key $k(v)$ to S.
- delete(v): Remove element v from S.

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- **findMin**: find the minimum key in S.
- **extractMin**: Remove $v \in S$ with smallest key and return it.
- **insert**$(v, k(v))$: Add new element v with key $k(v)$ to S.
- **delete**(v): Remove element v from S.
- **decreaseKey**$(v, k'(v))$: <u>decrease</u> key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
- **meld**: merge two separate priority queues into one.

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

- makePQ: create an empty queue.
- findMin: find the minimum key in S.
- extractMin: Remove $v \in S$ with smallest key and return it.
- insert$(v, k(v))$: Add new element v with key $k(v)$ to S.
- delete(v): Remove element v from S.
- decreaseKey$(v, k'(v))$: decrease key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
- meld: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time.

decreaseKey is implemented via delete and insert.

Dijkstra's Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s,0))
for each node u ≠ s do
    insert(Q, (u,∞))
X ← ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s,u), dist(s,v) + ℓ(v,u)))).
```

Priority Queue operations:

- $O(n)$ **insert** operations
- $O(n)$ **extractMin** operations
- $O(m)$ **decreaseKey** operations

Using Heaps
Store elements in a heap based on the key value

- All operations can be done in $O(\log n)$ time

Implementing Priority Queues via Heaps

Using Heaps
Store elements in a heap based on the key value

- All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n+m)\log n)$ time.

Fibonacci Heaps

- extractMin, insert, delete, meld in $O(\log n)$ time
- decreaseKey in $O(1)$ amortized time:

Fibonacci Heaps

- extractMin, insert, delete, meld in $O(\log n)$ time
- decreaseKey in $O(1)$ amortized time: ℓ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
- Relaxed Heaps: decreaseKey in $O(1)$ worst case time but at the expense of meld (not necessary for Dijkstra's algorithm)

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

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- Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.

Priority Queues: Fibonacci Heaps/Relaxed Heaps

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- extractMin, insert, delete, meld in $O(\log n)$ time
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- Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
- Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps, ...
- Boost library implements both Fibonacci heaps and rank-pairing heaps.

Shortest path trees and variants

Shortest Path Tree

Dijkstra's alg. finds the shortest path distances from s to V.

Question: How do we find the paths themselves?

Shortest Path Tree

Dijkstra's alg. finds the shortest path distances from s to V.

Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s,0))
prev(s) ← null
for each node u ≠ s do
    insert(Q, (u,∞) )
    prev(u) ← null

X = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u)) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)))
            prev(u) = v
```

Lemma

The edge set $(u, \mathrm{prev}(u))$ *is the* <u>*reverse*</u> *of a shortest path tree rooted at* s*. For each* u*, the reverse of the path from* u *to* s *in the tree is a shortest path from* s *to* u*.*

Proof Sketch.

- The edge set $\{(u, \mathrm{prev}(u)) \mid u \in V\}$ induces a directed in-tree rooted at s (Why?)
- Use induction on $|X|$ to argue that the tree is a shortest path tree for nodes in V.

$\square$

Shortest paths to s

Dijkstra's alg. gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

Shortest paths to s

Dijkstra's alg. gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

- In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
- In directed graphs, use Dijkstra's algorithm in G^{rev}!