Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size k.

ECE-374-B: Lecture 11 - Backtracking and memorization

Instructor: Nickvash Kani

October 07, 2025

University of Illinois Urbana-Champaign

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Median of medians time analysis

```
Median-of-medians(A, i):
    sublists = [A[j:j+5] for j ∈range(0, len(A), 5)]
    medians = [sorted (sublist)[len (sublist)/2] for sublist ∈sublists]

    // Base Case
    if len (A) ≤ 5 return sorted (a)[i]

    // Find median of medians
    if len (medians) ≤ 5
        pivot = sorted (medians)[len (medians)/2]
    else
        pivot = Median-of-medians (medians, len/2)

    // Partitioning Step
    low = [j for j ∈A if j < pivot]
    high = [j for j ∈A if j > pivot]

    k = len (low)
    if i < k
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    elseif i > k
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    else
    return pivot
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What about $k = 7$?

$$T(n) = T(\frac{1}{7}n) + T(\frac{10}{14}n) + cn$$

On different techniques for recursive algorithms

Recursion

Reduction: Reduce one problem to another

Recursion
A special case of reduction

- reduce problem to a smaller instance of itself
- self-reduction

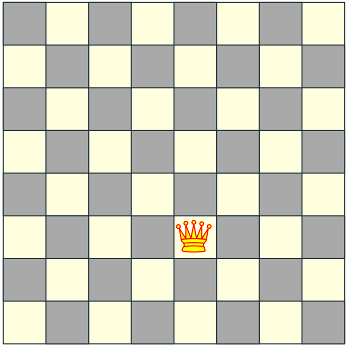
- Problem instance of size n is reduced to one or more instances of size $n - 1$ or less.
- For termination, problem instances of small size are solved by some other method as base cases.

Recursion in Algorithm Design

- Tail Recursion: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.
  Examples: Interval scheduling, MST algorithms....
- Divide and Conquer: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
  Examples: Closest pair, median selection, quick sort.
- Backtracking: Refinement of brute force search. Build solution incrementally by invoking recursion to try all possibilities for the decision in each step.
- Dynamic Programming: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use memorization to avoid recomputation of common solutions leading to iterative bottom-up algorithm.

Search trees and backtracking

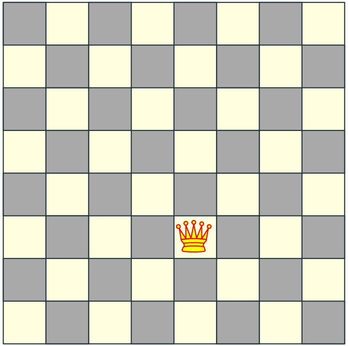
The queens problem



Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board?

The queens problem



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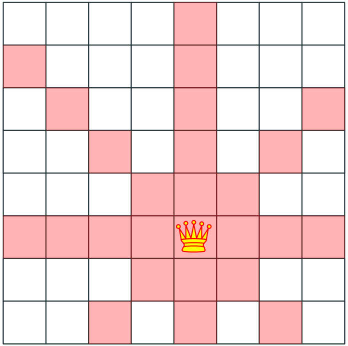

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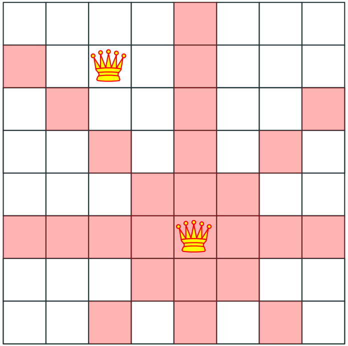

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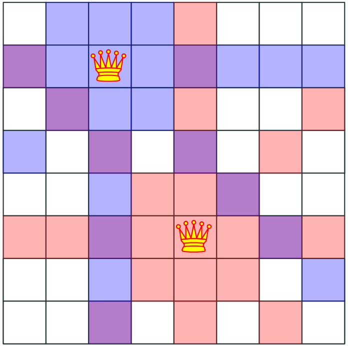

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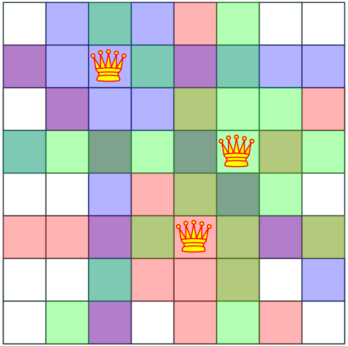

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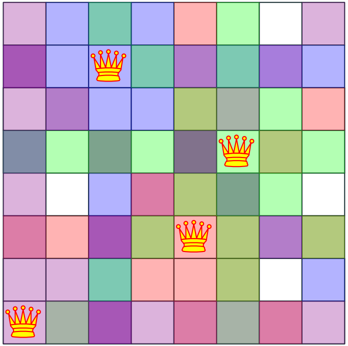

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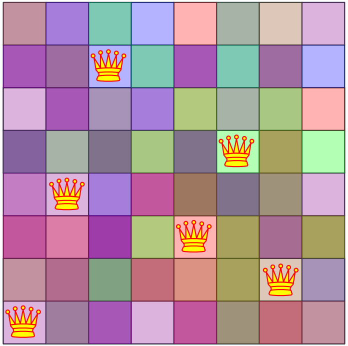

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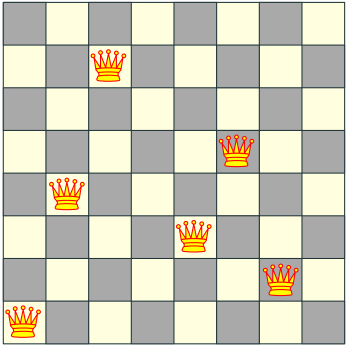
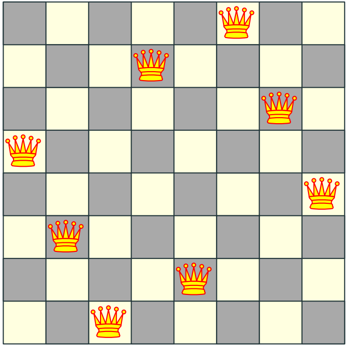


Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board? How many permutations?

The eight queens puzzle

Problem published in 1848, solved in 1850.



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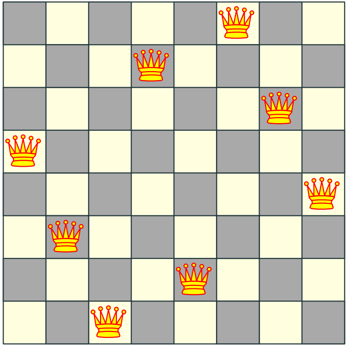
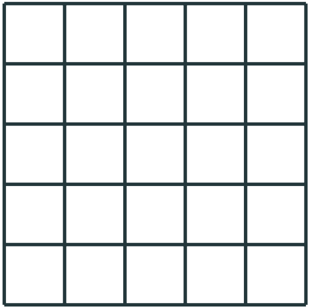


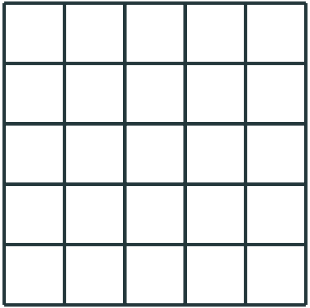
Q: How to solve problem for general *n*?

Introducing concept of state tree



What if we attempt to find all the possible permutations and then check?

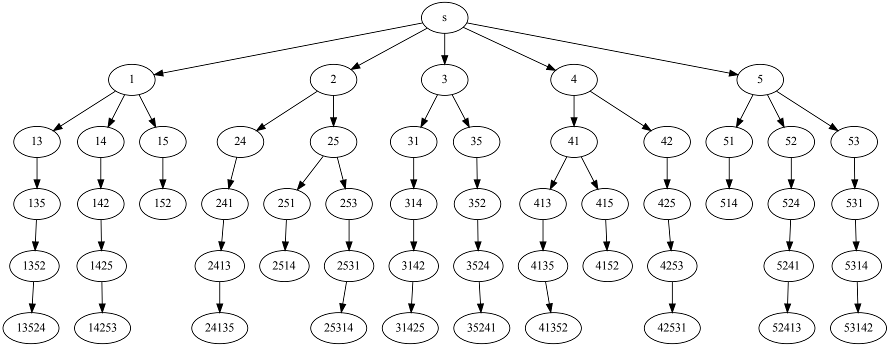
Search tree for 5 queens



Let’s be a bit smarter and recognize that:

- Queens can’t be on the same row, column or diagonal
- Can have *n* queens max.

Search tree for 5 queens



Backtracking: Informal definition

Recursive search over an implicit tree, where we "backtrack" if certain possibilities do not work.

n queens C++ code

```
void  generate_permutations( int  * permut, int  row, int  n )
{
   if  ( row == n ) {
      print_board( permut, n );
      return;
   }

   for  ( int  val = 1; val <= n; val++ )
      if  ( isValid( permut, row, val ) ) {
         permut[ row ] = val;
         generate_permutations( permut, row + 1, n );
      }
}

generate_permutations( permut, 0, 8 );
```

Quick note: n queens - number of solutions

N	Number of Solutions	Number of Unique Solutions
1	1	1
2	0	0
3	0	0
4	2	1
5	10	2
6	4	1
7	40	6
8	92	12
9	352	46
10	724	92
11	2,680	341
12	14,200	1,787
13	73,712	9,233
14	365,596	45,752
15	2,279,184	285,053

Sudoku

Sudoku problem

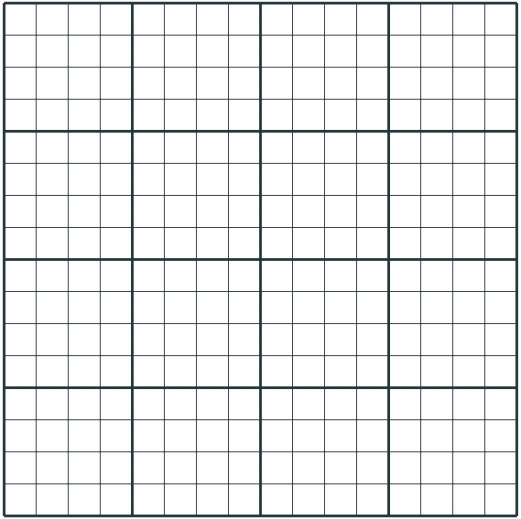
	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

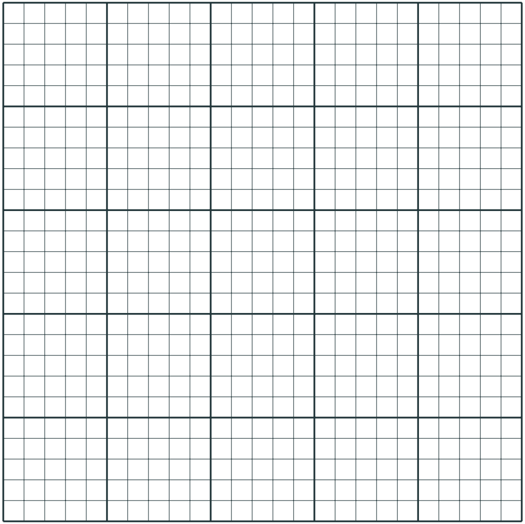
Unsolved Sudoku

4	2	*6*	5	*7*	1	*3*	9	*8*
8	*5*	*7*	2	*9*	3	*1*	*4*	6
1	3	*9*	*4*	6	*8*	*2*	7	*5*
9	*7*	1	*3*	*8*	*5*	6	*2*	*4*
5	4	*3*	*7*	*2*	*6*	*8*	1	9
6	*8*	2	*1*	*4*	*9*	7	*5*	*3*
7	9	*4*	*6*	3	*2*	*5*	8	*1*
2	*6*	*5*	8	*1*	4	*9*	*3*	7
3	1	*8*	9	*5*	7	*4*	6	*2*

Solved Sudoku

Variable Sized Sudoku





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```
algSudokuNaive(S[0..n − 1, 0..n − 1]):
    for possible value (X) in empty space do
        if SudokuValid? == True then
            return X

    return NULL
```

Naive Enumeration

	2		5		1		9	
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Running time: $O(n^2 9^{n^2})$.

n^2 time to check all rows/columns/squares contain values 1 through n

9 possibilities per square for n^2 squares

Better Enumeration

	2		5		1		9	
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```
Initialize Bitmap (BM) to contain only
    values available for each square
algSudoku-smaller(S[0..n − 1, 0..n − 1], BM[0..n − 1, 0..n − 1]):
    for each empty space X do
        for each possible value x for X according to BM do
            S-new = S(Assign X = x)
            BM-new = Modify BM removing x from same
                        row/column/square
            if no more empty squares
                return X
            else
                algSudoku-smaller(S, BM)

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Running time: $O(9^{n^2})$.

9 possibilities per square for n^2 squares

Longest Increasing Sub-sequence

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a subsequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Sequences - Example...

Example

- Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing <u>subsequence</u> of the first sequence: $2, 7, 9$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an <u>increasing subsequence</u> $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

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Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

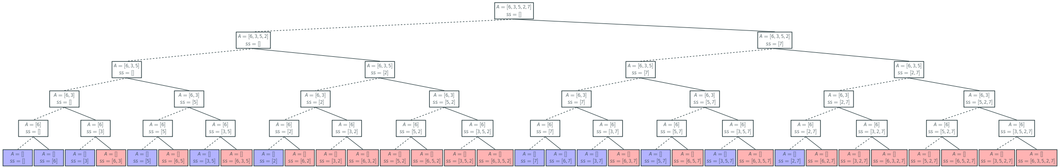
Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Naive Recursion Enumeration - State Tree



- This is just for [6,3,5,2,7]! (Tikz won't print larger trees)
- How many leafs are there for the full [6,3,5,2,7, 8, 1] sequence
- What is the running time?

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Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Recursive Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[1..n]$):

Recursive Approach: LIS: Longest increasing subsequence

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- Case 1: Does not contain $A[n]$ in which case LIS($A[1..n]$) = LIS($A[1..(n-1)]$)
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Recursive Approach: LIS: Longest increasing subsequence

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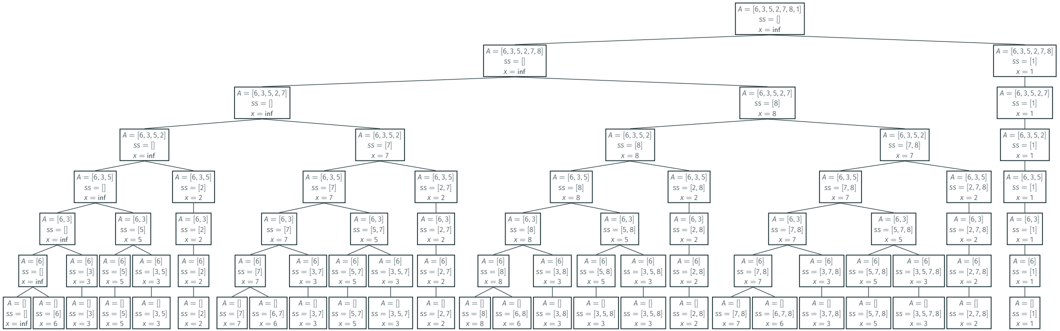
Observation
For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is LIS_smaller($A[1..n], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[0..6] = 6, 3, 5, 2, 7, 8, 1$



Recursive Approach

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

Running time analysis

Running time of LIS([1..n])

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Running time of LIS([1..n])

Lemma

LIS_smaller *runs in* $O(2^n)$ *time.*

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....one can do much better using memorization!