Pre-lecture brain teaser

We know that SAT is NP-complete which means that it is in NP-Hard. HALT is also in NP-Hard. Is SAT reducible to HALT? How?

ECE-374-B: Lecture 23 - Decidability II

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University of Illinois Urbana-Champaign

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Reductions

Reduction

Meta definition: Problem X reduces to problem Y, if given a solution to Y, then it implies a solution for X. Namely, we can solve Y then we can solve X. We will done this by $X \implies Y$.

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Definition
<u>oracle</u> ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

Lemma
A language X <u>reduces</u> to a language Y, if one can construct a TM decider for X using a given oracle ORAC_Y *for Y.*

We will denote this fact by $X \implies Y$.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.

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- L: language of Y.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem X using M.
- Result in decider for X (i.e., A_{TM}).
- Contradiction X is not decidable.
- Thus, L must be not decidable.

Reduction implies decidability

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If Y is decidable then X is decidable.

Proof.
Let T be a decider for Y (i.e., a program or a TM). Since X reduces to Y, it follows that there is a procedure $\mathsf{T}_{X|Y}$ (i.e., decider) for X that uses an oracle for Y as a subroutine. We replace the calls to this oracle in $\mathsf{T}_{X|Y}$ by calls to T. The resulting program T_X is a decider and its language is X. Thus X is decidable (or more formally TM decidable). $\square$

The countrapositive...

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If X is undecidable then Y is undecidable.

Halting

The halting problem

Language of all pairs $\langle M, w\rangle$ such that M halts on w:

$$A_{\mathrm{Halt}} = \left\{ \langle M, w\rangle \,\middle|\, M \text{ is a } TM \text{ and } M \text{ stops on } w \right\}.$$

Similar to language already known to be undecidable:

$$\mathrm{A}_{TM} = \left\{ \langle M, w\rangle \,\middle|\, M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

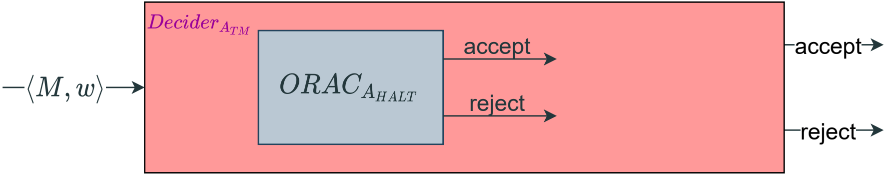
One way to proving that Halting is undecidable...

Lemma
The language A_{TM} *reduces to* A_{Halt}*. Namely, given an oracle for* A_{Halt} *one can build a decider (that uses this oracle) for* A_{TM}*.*

One way to proving that Halting is undecidable...

Lemma

The language A_{TM} reduces to A_{Halt}. Namely, given an oracle for A_{Halt} one can build a decider (that uses this oracle) for A_{TM}.



One way to proving that Halting is undecidable...

Proof.
Let ORAC_{Halt} be the given oracle for A_{Halt}. We build the following decider for A_{TM}.

AnotherDecider-A_{TM} $\left(\langle M, w\rangle\right)$

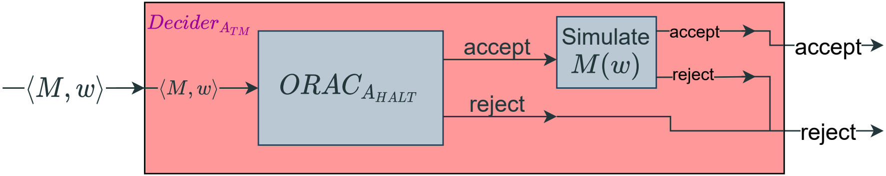
```
    res ← ORAC_Halt(⟨M, w⟩)
    // if M does not halt on w then reject.
    if res = reject then
            halt and reject.
    // M halts on w since res =accept.
    // Simulating M on w terminates in finite time.
    res_2 ←Simulate M on w.
    return res_2.
```

This procedure always return and as such its a decider for A_{TM}. $\square$

Theorem
The language A_{Halt} is not decidable.

Proof.
Assume, for the sake of contradiction, that A_{Halt} is decidable. As such, there is a TM, denoted by TM_{Halt}, that is a decider for A_{Halt}. We can use TM_{Halt} as an implementation of an oracle for A_{Halt}, which would imply that one can build a decider for A_{TM}. However, A_{TM} is undecidable. A contradiction. It must be that A_{Halt} is undecidable. $\square$

The same proof by figure...



... if A_{Halt} is decidable, then A_{TM} is decidable, which is impossible.

Emptiness

The language of empty languages

- $E_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) = \emptyset \right\}.$
- TM_{ETM}: Assume we are given this decider for E_{TM}.
- Need to use TM_{ETM} to build a decider for A_{TM}.
- Decider for A_{TM} is given M and w and must decide whether M accepts w.
- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.

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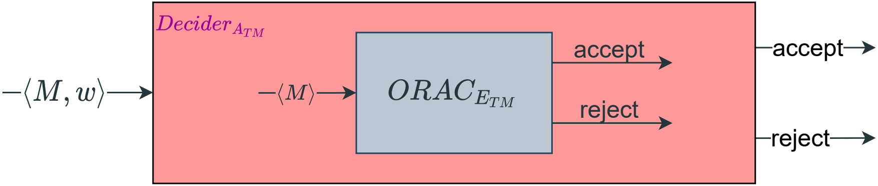

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- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.
- Idea: hard-code w into M, creating a TM M_w which runs M on the fixed string w.
- TM $M_w(x)$:
    1. Input = x (which will be ignored)
    2. Simulate M on w.
    3. If the simulation accepts, accept. Else, reject.

Embedding strings...

- Given program $\langle M \rangle$ and input w...
- ...can output a program $\langle M_w \rangle$.
- The program M_w simulates M on w. And accepts/rejects accordingly.
- EmbedString($\langle M, w \rangle$) input two strings $\langle M \rangle$ and w, and output a string encoding (TM) $\langle M_w \rangle$.

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- What is $L(M_w)$?
- Since M_w ignores input x.. language M_w is either Σ^* or $\emptyset$.
  It is Σ^* if M accepts w, and it is $\emptyset$ if M does not accept w.

Emptiness is undecidable

Theorem
The language E_{TM} is undecidable.

- Assume (for contradiction), that E_{TM} is decidable.
- TM_{ETM} be its decider.
- Build decider **AnotherDecider-A_{TM}** for A_{TM}:

AnotherDecider-A_{TM} $(\langle M, w\rangle)$

  $\langle M_w\rangle \leftarrow$ **EmbedString** $(\langle M, w\rangle)$
  $r \leftarrow TM_{ETM}(\langle M_w\rangle)$.
  if $r =$ accept **then**
    return reject
  `//` $TM_{ETM}(\langle M_w\rangle)$ `rejected its input`
  return accept

Emptiness is undecidable...

Consider the possible behavior of **AnotherDecider-**A_{TM} on the input $\langle M, w\rangle$.

- If TM_{ETM} accepts $\langle M_w\rangle$, then $L(M_w)$ is empty. This implies that M does not accept w. As such, **AnotherDecider-**A_{TM} rejects its input $\langle M, w\rangle$.
- If TM_{ETM} accepts $\langle M_w\rangle$, then $L(M_w)$ is not empty. This implies that M accepts w. So **AnotherDecider-**A_{TM} accepts $\langle M, w\rangle$.

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$\implies$ **AnotherDecider-A_{TM}** is decider for A_{TM}.

But A_{TM} is undecidable...

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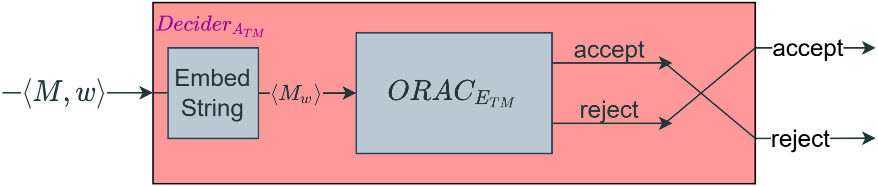
But A_{TM} is undecidable...

...must be assumption that E_{TM} is decidable is false.

Emptiness is undecidable via diagram



AnotherDecider-A_{TM} never actually runs the code for M_W. It hands the code to a function TM_{ETM} which analyzes what the code would do if run it. So it does not matter that M_W might go into an infinite loop.

Equality

Equality is undecidable

$$EQ_{TM} = \left\{ \langle M, N \rangle \,\middle|\, M \text{ and } N \text{ are TM's and } L(M) = L(N) \right\}.$$

Lemma
The language EQ_{TM} is undecidable.

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Let's try something different. We know E_{TM} is undecidable. Let's use that:

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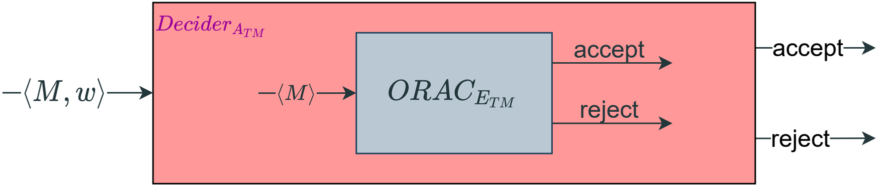
$$E_{TM} \implies EQ_{TM}$$

Equality diagram



Proof

Proof.
Suppose that we had a decider **DeciderEqual** for EQ_{TM}. Then we can build a decider for E_{TM} as follows:

TM R:

1. Input = $\langle M \rangle$
2. Include the (constant) code for a TM T that rejects all its input. We denote the string encoding T by $\langle T \rangle$.
3. Run **DeciderEqual** on $\langle M, T \rangle$.
4. If **DeciderEqual** accepts, then accept.
5. If **DeciderEqual** rejects, then reject.

$\square$

DFAs

DFAs are empty?

$$E_{DFA} = \left\{ \langle A \rangle \,\middle|\, A \text{ is a DFA and } L(A) = \emptyset \right\}.$$

What does the above language describe?

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Lemma
The language E_{DFA} is decidable:

Scratch

Proof

Proof.
Unlike in the previous cases, we can directly build a decider (DeciderEmptyDFA) for E_{DFA}

TM R:

1. Input = $\langle A \rangle$
2. Mark start state of A as visited.
3. Repeat until no new states get marked:
    - Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, then accept.
5. Otherwise, then reject.

$\square$

Equal DFAs

DFAs are equal?

$$EQ_{DFA} = \left\{ \langle A, b \rangle \;\middle|\; A \text{ and } B \text{ are DFAs and } L(A) = L(B) \right\}.$$

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Lemma
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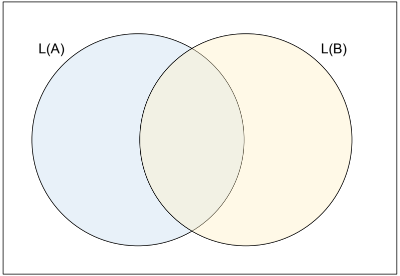
Can we show this using reductions?

Equal DFA trick I

Need a way to determine if there any strings in one language and not the other....



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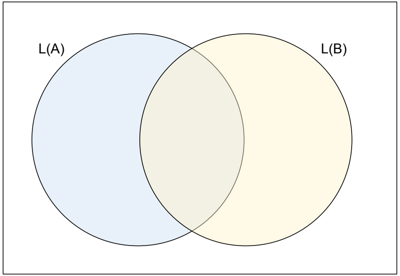

This is known as the symmetric difference. Can create a new DFA (C) which represents the symmetric difference of L_A and L_B

Equal DFA trick II

Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \emptyset$
- If $L(A) \neq L(B)$ then $L(C)$ is not empty

Good time to use E_{DFA} proof from before.....How do we show EQ_{DFA} is decidable using a reduction?

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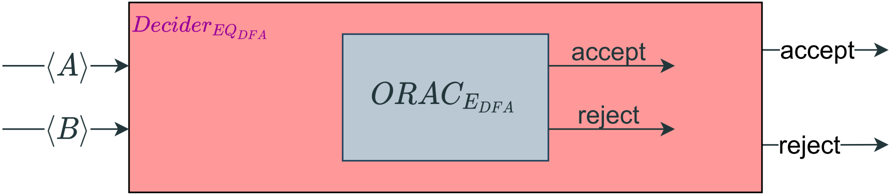

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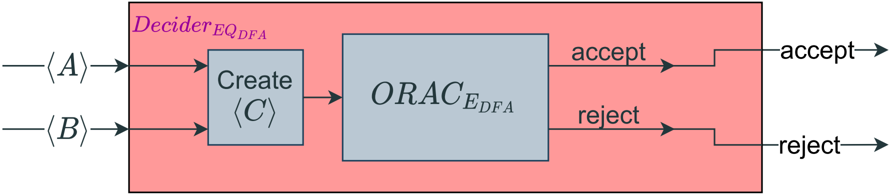


Equal DFA decider

TM F:

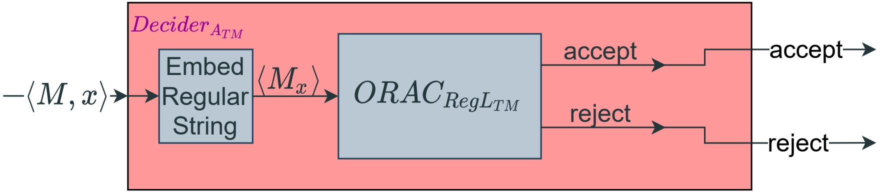
1. Input = $\langle A, B \rangle$ where A and B are DFAs
2. Construct DFA C as described before
3. Run **DeciderEmptyDFA** from previous slide on C
4. If accepts, then accept.
5. If rejects, then reject.

Regularity

Many undecidable languages

- Almost any property defining a TM language induces a language which is undecidable.
- proofs all have the same basic pattern.
- Regularity language: $\mathrm{Regular}_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) \text{ is regular} \right\}$.
- **DeciderRegL**: Assume TM decider for $\mathrm{Regular}_{TM}$.
- Reduction from halting requires to turn problem about deciding whether a TM M accepts w (i.e., is $w \in \mathrm{A}_{TM}$) into a problem about whether some TM accepts a regular set of strings.

Outline of IsRegular? reductionr



Proof continued...

- Given M and w, consider the following TM M'_w:
  TM M'_w:
  (i) Input = x
  (ii) If x has the form $a^n b^n$, halt and accept.
  (iii) Otherwise, simulate M on w.
  (iv) If the simulation accepts, then accept.
  (v) If the simulation rejects, then reject.
- not executing M'_w!
- feed string $\langle M'_w \rangle$ into **DeciderRegL**
- **EmbedRegularString**: program with input $\langle M \rangle$ and w, and outputs $\langle M'_w \rangle$, encoding the program M'_w.
- If M accepts w, then any x accepted by M'_w: $L(M'_w) = \Sigma^*$.
- If M does not accept w, then $L(M'_w) = \{a^n b^n \mid n \geq 0\}$.

Proof continued...

- $a^n b^n$ is not regular...
- Use **DeciderRegL** on M'_w to distinguish these two cases.
- Note - cooked M'_w to the decider at hand.
- A decider for A_{TM} as follows.

AnotherDecider-A_{TM} $(\langle M, w\rangle)$
  $\langle M'_w\rangle \leftarrow$ **EmbedRegularString** $(\langle M, w\rangle)$
  $r \leftarrow$ **DeciderRegL**$(\langle M'_w\rangle)$.
  return r

- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*)

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- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*) $\implies M$ accepts w. So **AnotherDecider-A_{TM}** should accept $\langle M, w\rangle$.

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- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*) $\implies M$ accepts w. So **AnotherDecider-**A_{TM} should accept $\langle M, w\rangle$.
- If **DeciderRegL** rejects $\implies L(M'_w)$ is not regular $\implies L(M'_w) = a^n b^n$

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- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*) $\implies M$ accepts w. So **AnotherDecider-A_{TM}** should accept $\langle M, w\rangle$.
- If **DeciderRegL** rejects $\implies L(M'_w)$ is not regular $\implies L(M'_w) = a^n b^n \implies M$ does not accept $w \implies$ **AnotherDecider-A_{TM}** should reject $\langle M, w\rangle$.

The above proofs were somewhat repetitious...

...they imply a more general result.

Theorem (Rice's Theorem.)
Suppose that L *is a language of Turing machines; that is, each word in* L *encodes a* TM*. Furthermore, assume that the following two properties hold.*

(a) *Membership in* L *depends only on the Turing machine's language, i.e. if* $L(M) = L(N)$ *then* $\langle M \rangle \in \mathrm{L} \Leftrightarrow \langle N \rangle \in \mathrm{L}$.

(b) *The set* L *is "non-trivial," i.e.* $\mathrm{L} \neq \emptyset$ *and* L *does not contain all Turing machines.*

Then L *is a undecidable.*