Arithmetico–Geometric Series

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1. Definition

A (finite) AGS has the form

$$S_n = \sum_{k=1}^{n} (a + (k-1)d)\, r^{\,k-1},$$

i.e. an arithmetic progression $a, (a+d), (a+2d), \dots$ times a geometric progression $1, r, r^2, \dots$.

2. Main Formulae to Remember

For $r \neq 1$:

$$\boxed{\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r}} \qquad \boxed{\sum_{k=1}^{n} k\, r^{k-1} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}}$$

Hence any AGS splits as

$$\begin{aligned} S_n &= a\sum_{k=1}^{n} r^{k-1} + d\sum_{k=1}^{n}(k-1)r^{k-1} \\ &= (a-d)\frac{1-r^n}{1-r} + d\,\frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}. \end{aligned}$$

Special cases:

$$\boxed{r = 1{:}\quad S_n = \frac{n}{2}(2a + (n-1)d)} \qquad \boxed{|r| < 1{:}\quad S_\infty = \frac{a-d}{1-r} + \frac{d}{(1-r)^2}}$$

3. How it appears in recurrences (the classic use)

Consider

$$T(n) = 2T(n-1) + c\,n, \qquad T(0) = 0.$$

Unroll:

$$T(n) = cn + 2c(n-1) + 4c(n-2) + \cdots + 2^{n-1}c = \sum_{i=1}^{n} 2^{\,i-1}\, c\,(n-i+1).$$

This is an AGS with $a = cn$, $d = -c$, $r = 2$. Apply the formulas:

$$\begin{aligned} T(n) &= c\left[n\sum_{i=1}^{n} 2^{i-1} - \sum_{i=1}^{n}(i-1)2^{i-1}\right] \\ &= c\left[n(2^n - 1) - \frac{1-(n+1)2^n + n2^{n+1}}{(1-2)^2} + (2^n - 1)\right] \\ &= c\,(2^{n+1} - n - 2) = \Theta(2^n). \end{aligned}$$

Takeaway. When a linear recurrence $T(n) = \alpha T(n-1) +$ (linear in n) is unrolled, you get an AGS $\sum \alpha^{\,i-1}(n-i+1)$.

4. Other examples for you to try

(A) $r = \frac{1}{2}$**.**

$$S_n = 3 + 5\left(\tfrac{1}{2}\right) + 7\left(\tfrac{1}{2}\right)^2 + \cdots \;\; (n \text{ terms})$$

Here $a = 3,\ d = 2,\ r = \frac{1}{2}$. Using the finite formula gives an explicit S_n; as $n \to \infty$ (since $|r| < 1$),

$$\boxed{S_\infty = \frac{a-d}{1-r} + \frac{d}{(1-r)^2} = \frac{1}{1/2} + \frac{2}{(1/2)^2} = 10.}$$

(B)

$$\boxed{\sum_{k=1}^{n} k\,3^{k-1} = \frac{1-(n+1)3^n + n3^{n+1}}{(1-3)^2} = \frac{1+(2n-1)3^n}{4}}$$

(C) Another recurrence. Solve $T(n) = 3T(n-1) + n$, $T(0) = T_0$.

$$T(n) = 3^n T_0 + \sum_{i=1}^{n} 3^{\,i-1}(n-i+1) = 3^n\left(T_0 + \tfrac{3}{4}\right) - \tfrac{1}{2}n - \tfrac{3}{4}.$$

So $T(n) = \Theta(3^n)$.