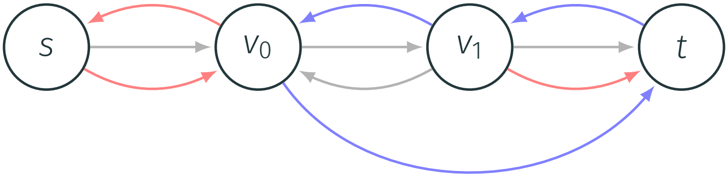
Pre-lecture brain teaser

You have a graph G(V,E). Some of the edges are red, some are white and some are blue. You are given two distinct vertices u and v and want to find a walk $[u \to v]$ such that:

- a white edge must be taken after a red edge only.
- a blue edge must be taken after a white edge only.
- and a red edge may be taken after a blue edge only.
- must start on red edge



ECE-374-B: Lecture 17 - Bellman-Ford and Dynamic Programming on Graphs

Instructor: Nickvash Kani

October 28, 2025

University of Illinois Urbana-Champaign

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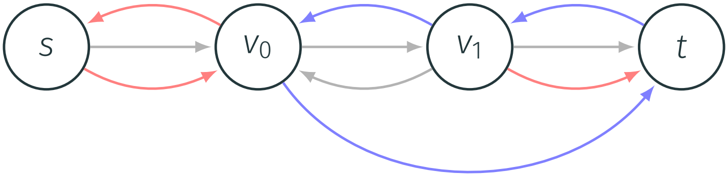

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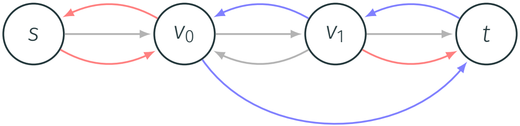

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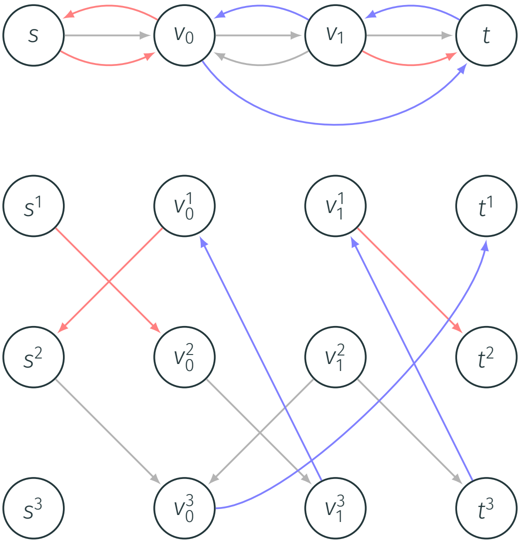


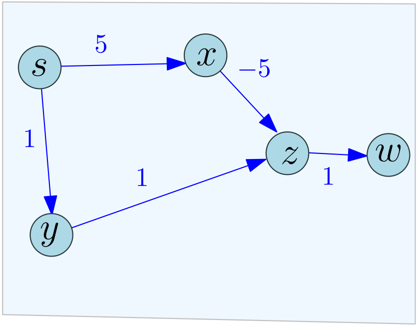
Shortest Paths with Negative Length Edges

Why Dijkstra’s algorithm fails with negative edges

Single-Source Shortest Path Problems
Input: A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths.
For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

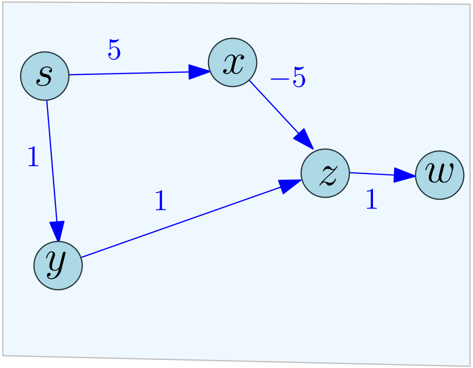
What are the distances computed by Dijkstra's algorithm?



The distance as computed by Dijkstra algorithm starting from s:

1. $s = 0$, $x = 5$, $y = 1$, $z = 0$.
2. $s = 0$, $x = 1$, $y = 2$, $z = 5$.
3. $s = 0$, $x = 5$, $y = 1$, $z = 2$.
4. IDK.

With negative length edges, Dijkstra's algorithm can fail



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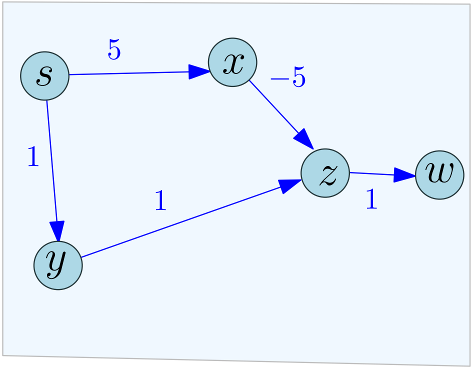

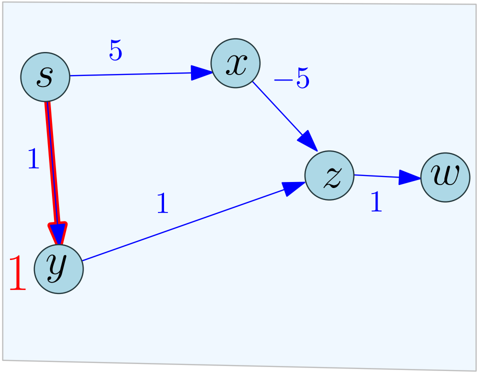

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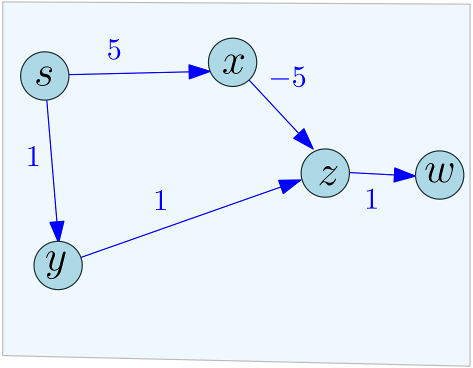

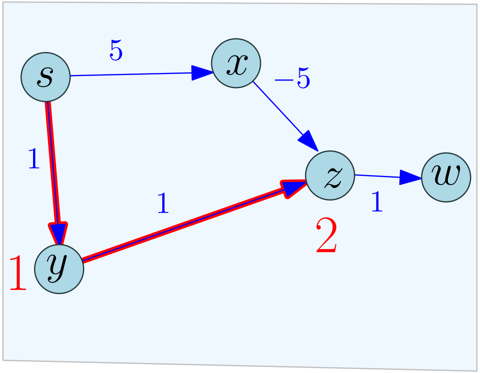

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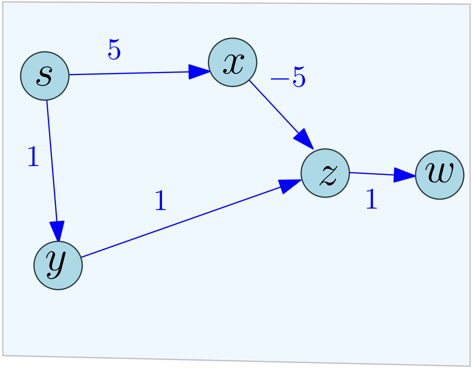

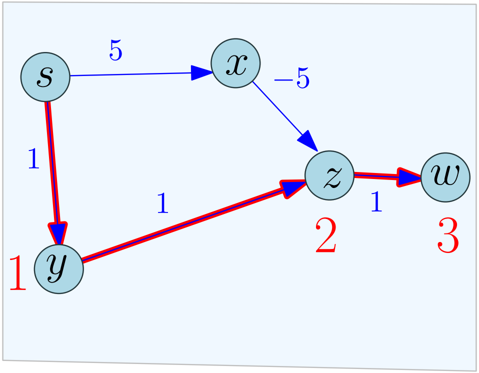

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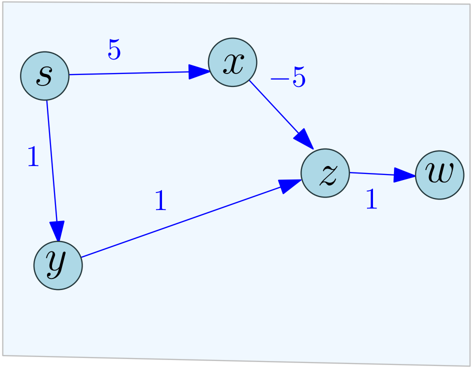

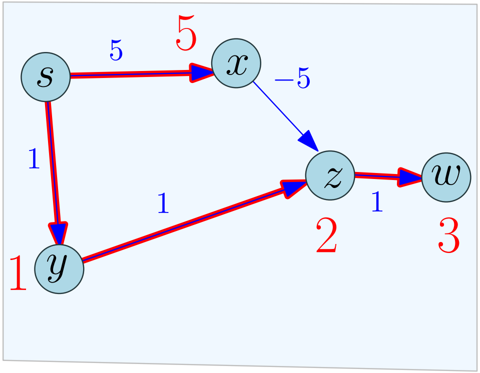

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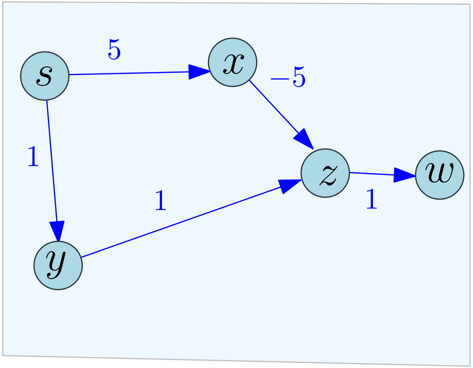

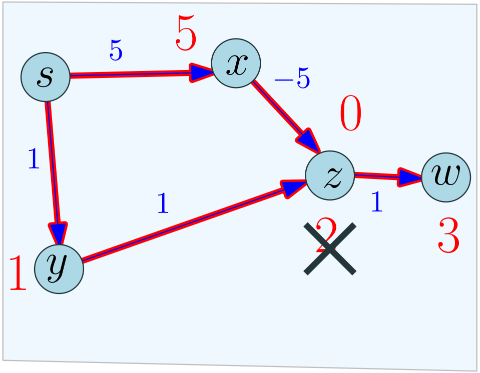

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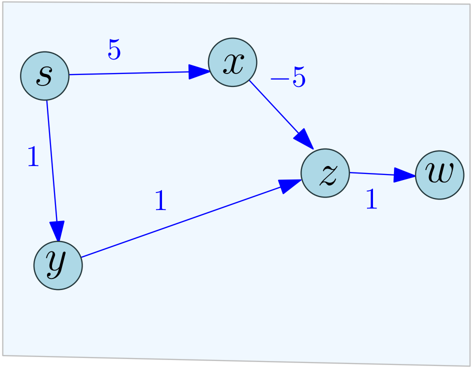
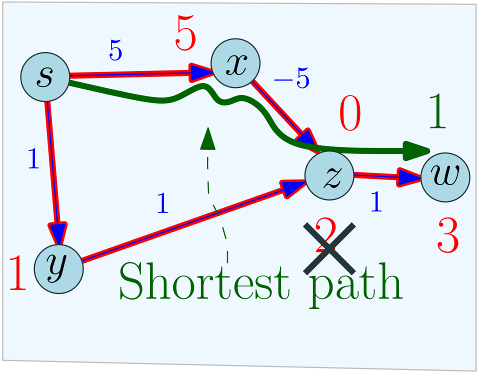
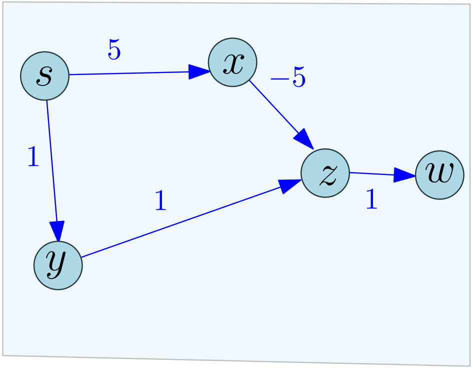
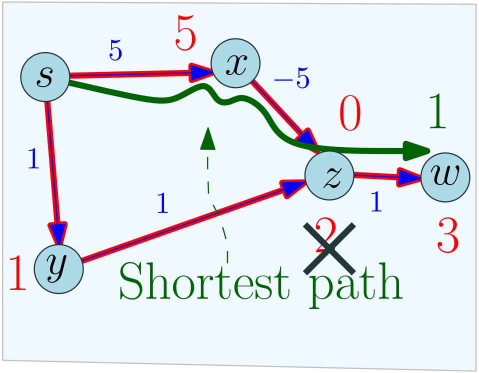




Dijkstra's Algorithm and Negative Lengths

With negative length edges, Dijkstra's algorithm can fail





False assumption: Dijkstra's algorithm assumes that if $s \to v_0 \to v_1 \to v_2 \ldots \to v_k$ is a shortest path from s to v_k then $dist(s, v_i) \leq dist(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.

Lemma
Let G be a directed graph with arbitrary edge lengths. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

- $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is a shortest path from s to v_i*

Shortest Paths with Negative Lengths

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Let G be a directed graph with arbitrary edge lengths. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from s to v_k then for $1 \le i < k$:

- $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is a shortest path from s to v_i*
- *False:* $dist(s, v_i) \le dist(s, v_k)$ *for $1 \le i < k$. Holds true only for non-negative edge lengths.*

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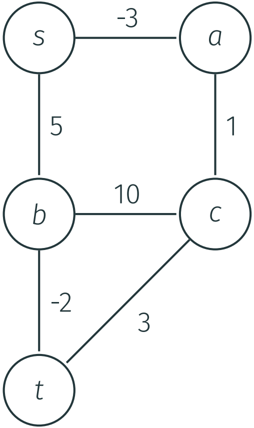
- $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ *is a shortest path from s to v_i*
- *False:* $\mathit{dist}(s, v_i) \leq \mathit{dist}(s, v_k)$ *for $1 \leq i < k$.* *Holds true only for non-negative edge lengths.*

Cannot explore nodes in increasing order of distance! We need other strategies.

Why can’t we just re-normalize the edge lengths!?

Instinctual thought

Why can't we simply add a weight to each edge so that the shortest length is 0 (or positive).



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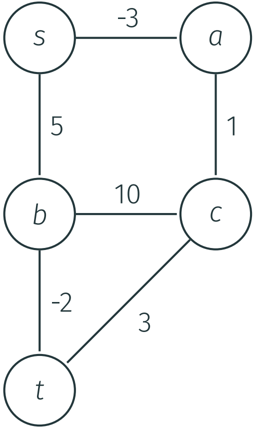

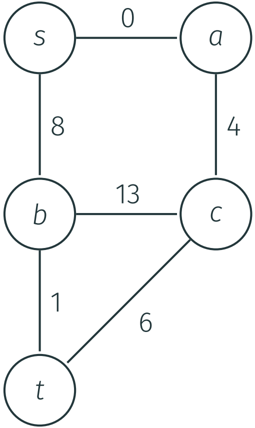

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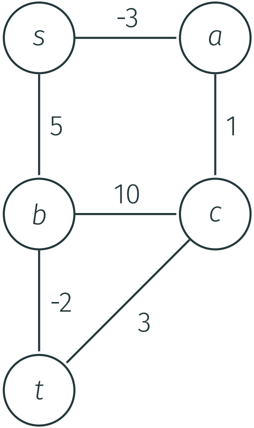
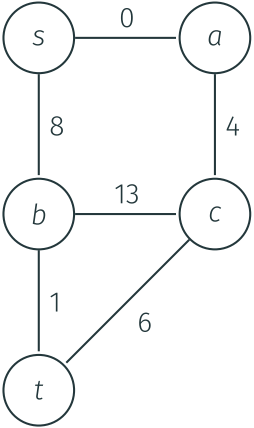


Shortest Path: $s \to a \to c \to t$



Shortest Path: $s \to b \to t$

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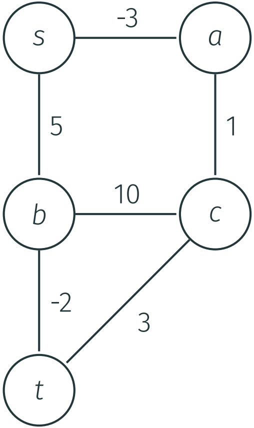

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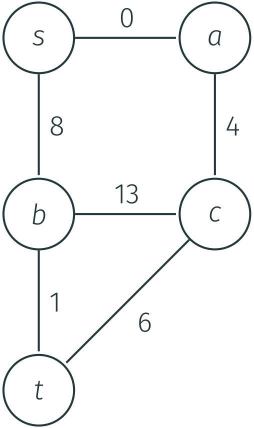
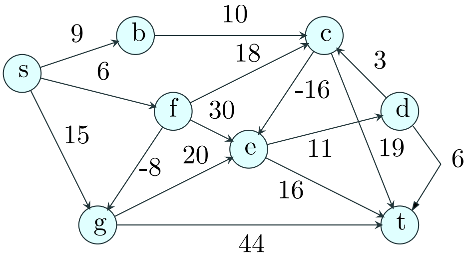


Shortest Path: $s \to b \to t$

But wait! Things get worse: Negative cycles

Definition
A cycle C is a negative length cycle if the sum of the edge lengths of C is negative.



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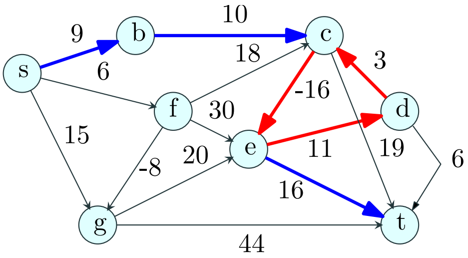

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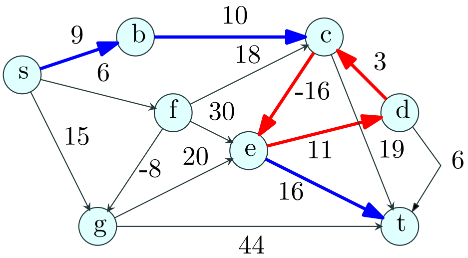

What is the shortest path distance between s and t?

Reminder: Paths have to be simple...

Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and s, t. Suppose

- G has a negative length cycle C, and
- s can reach C and C can reach t.

Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and s, t. Suppose

- G has a negative length cycle C, and
- s can reach C and C can reach t.

Question: What is the shortest distance from s to t?

Possible answers: Define shortest distance to be:

- undefined, that is $-\infty$, OR
- the length of a shortest simple path from s to t.

Really bad new about negative edges, and shortest path...

Lemma
If there is an efficient algorithm to find a shortest simple $s \to t$ path in a graph with negative edge lengths, then there is an efficient algorithm to find the longest simple $s \to t$ path in a graph with positive edge lengths.

Finding the $s \to t$ longest path is difficult. NP-HARD!

Restating problem of Shortest path with negative edges

Alternatively: Finding Shortest Walks

Given a graph $G = (V, E)$:

- A path is a sequence of <u>distinct</u> vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \le i \le k-1$.
- A walk is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \le i \le k-1$. Vertices are allowed to repeat.

Define $dist(u, v)$ to be the length of a shortest walk from u to v.

- If there is a walk from u to v that contains negative length cycle then $dist(u, v) = -\infty$
- Else there is a path with at most $n-1$ edges whose length is equal to the length of a shortest walk and $dist(u, v)$ is finite

Helpful to think about walks

Algorithmic Problems

Input: A directed graph $G = (V, E)$ with edge lengths (could be negative). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

Questions:

- Given nodes s, t, either find a negative length cycle C that s can reach or find a shortest path from s to t.
- Given node s, either find a negative length cycle C that s can reach or find shortest path distances from s to all reachable nodes.
- Check if G has a negative length cycle or not.

Shortest Paths with Negative Edge Lengths - In Undirected Graphs

Note: With negative lengths, shortest path problems and negative cycle detection in undirected graphs cannot be reduced to directed graphs by bi-directing each undirected edge. Why?

Problem can be solved efficiently in undirected graphs but algorithms are different and significantly more involved than those for directed graphs. One need to compute T-joins in the relevant graph. Pretty painful stuff.

Bellman Ford Algorithm

Shortest path via number of hops

Shortest Paths and Recursion

- Compute the shortest path distance from s to t recursively?
- What are the smaller sub-problems?

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Sub-problem idea: paths of fewer hops/edges

Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source s.

Assume that all nodes can be reached by s in G

Assume G has no negative-length cycle (for now).

$d(v, k)$: shortest walk length from s to v using at most k edges.

Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source s.

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$d(v, k)$: shortest walk length from s to v using at most k edges.

Note: $dist(s, v) = d(v, n - 1)$.

Hop-based Recursion: Bellman-Ford Algorithm

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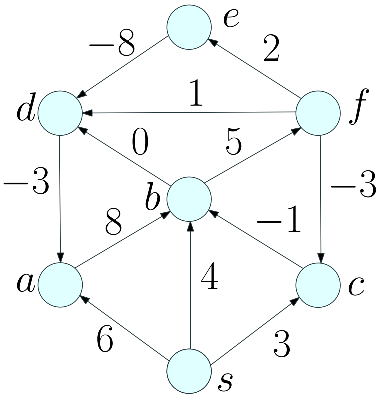
$d(v, k)$: shortest walk length from s to v using at most k edges.

Note: $dist(s, v) = d(v, n-1)$. Recursion for $d(v, k)$:

$$d(v,k) = \min \begin{cases} \min_{u \in V}(d(u, k-1) + \ell(u, v)). \\ d(v, k-1) \end{cases}$$

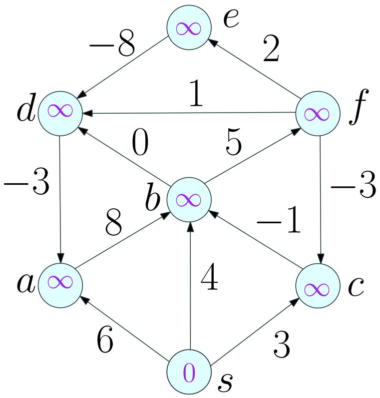
Base case: $d(s, 0) = 0$ and $d(v, 0) = \infty$ for all $v \neq s$.

Example



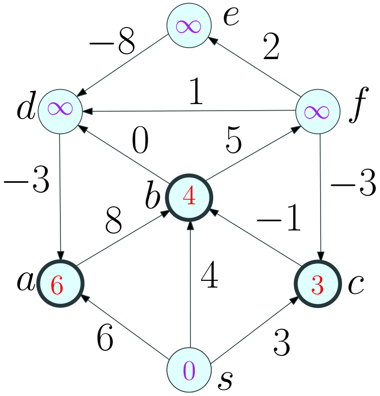
round	s	a	b	c	d	e	f

Example



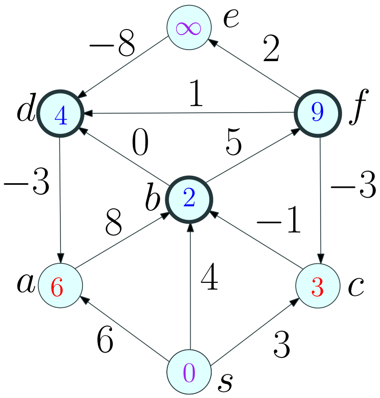
round	s	a	b	c	d	e	f
0	0	∞	∞	∞	∞	∞	∞

Example



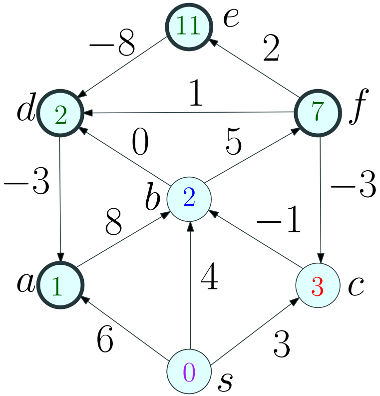
round	s	a	b	c	d	e	f
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞

Example



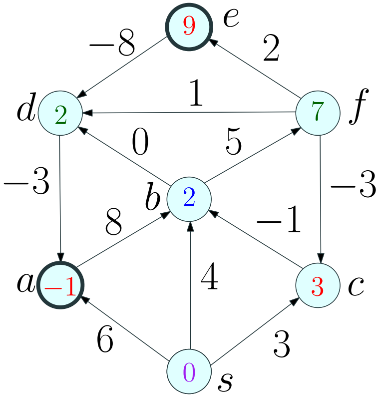
round	s	a	b	c	d	e	f
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9

Example



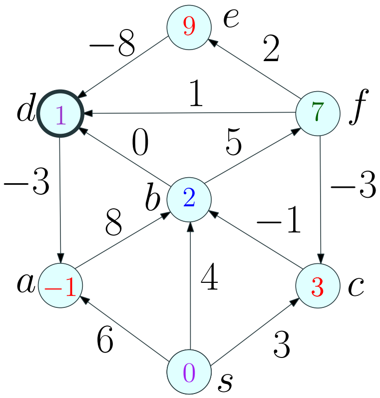
round	s	a	b	c	d	e	f
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7

Example



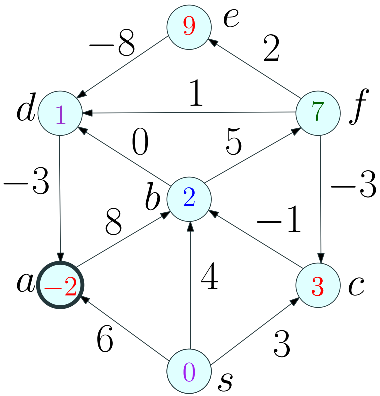
round	s	a	b	c	d	e	f
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7

Example



round	s	a	b	c	d	e	f
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7
5	0	-1	2	3	1	9	7

Example



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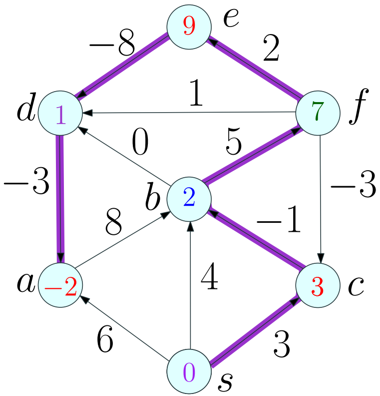
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7
5	0	-1	2	3	1	9	7
6	0	-2	2	3	1	9	7

Example



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0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7
5	0	-1	2	3	1	9	7
6	0	-2	2	3	1	9	7

The Bellman-Ford Algorithm

Bellman-Ford Algorithm

```
Create in(G) list from adj(G)

for each u ∈ V do
    d(u, 0) ← ∞
d(s, 0) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        d(v, k) ← d(v, k − 1)
        for each edge (u, v) ∈ in(v) do
            d(v, k) = min{d(v, k), d(u, k − 1) + ℓ(u, v)}

for each v ∈ V do
    dist(s, v) ← d(v, n − 1)
```

Bellman-Ford Algorithm

```
Create in(G) list from adj(G)

for each u ∈ V do
    d(u,0) ← ∞
d(s,0) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        d(v,k) ← d(v,k − 1)
        for each edge (u,v) ∈ in(v) do
            d(v,k) = min{d(v,k), d(u,k − 1) + ℓ(u,v)}

for each v ∈ V do
    dist(s,v) ← d(v,n − 1)
```

Running time:

Bellman-Ford Algorithm

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for each v ∈ V do
    dist(s, v) ← d(v, n − 1)
```

Running time: $O(n(n+m))$

Bellman-Ford Algorithm

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for each v ∈ V do
    dist(s, v) ← d(v, n − 1)
```

Running time: $O(n(n+m))$ Space:

Bellman-Ford Algorithm

```
Create in(G) list from adj(G)

for each u ∈ V do
    d(u, 0) ← ∞
d(s, 0) ← 0

for k = 1 to n − 1 do
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            d(v, k) = min{d(v, k), d(u, k − 1) + ℓ(u, v)}

for each v ∈ V do
    dist(s, v) ← d(v, n − 1)
```

Running time: $O(n(n+m))$ Space: $O(m+n^2)$ (Space can be reduced to $O(m+n)$).

Bellman-Ford Algorithm: Cleaner version

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ in(v) do
            d(v) = min{d(v), d(u) + ℓ(u, v)}

for each v ∈ V do
        dist(s, v) ← d(v)
```

Running time: $O(mn)$ Space: $O(m + n)$

Bellman-Ford Algorithm: Cleaner version

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ in(v) do
            d(v) = min{d(v), d(u) + ℓ(u, v)}

for each v ∈ V do
        dist(s, v) ← d(v)
```

Running time: $O(mn)$ Space: $O(m+n)$ Do we need the in(V) list?

Bellman-Ford Algorithm: Cleaner version

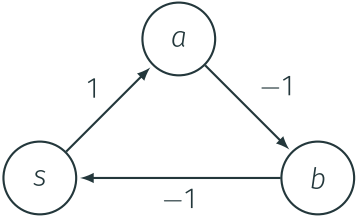
```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each edge (u, v) ∈ G do
        d(v) = min{d(v), d(u) + ℓ(u, v)}

for each v ∈ V do
        dist(s, v) ← d(v)
```

Running time: $O(mn)$ Space: $O(n)$

Bellman-Ford: Detecting negative cycles

Negative cycles

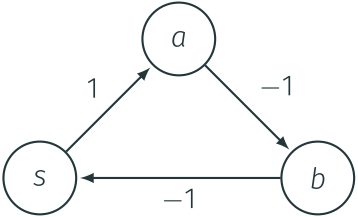
What happens if we run this on a graph with negative cycles?



round	s	a	b

Negative cycles

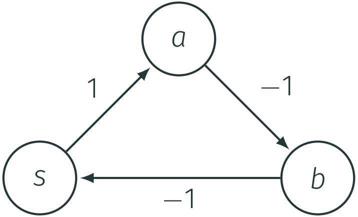
What happens if we run this on a graph with negative cycles?



round	s	a	b
0	0	∞	∞

Negative cycles

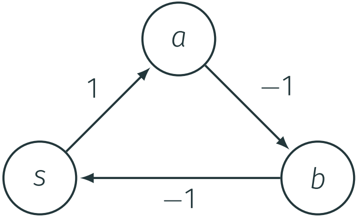
What happens if we run this on a graph with negative cycles?



round	s	a	b
0	0	∞	∞
1	0	1	∞

Negative cycles

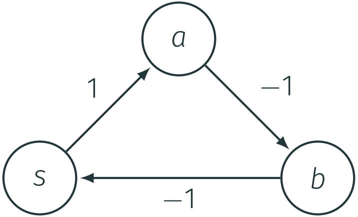
What happens if we run this on a graph with negative cycles?



round	s	a	b
0	0	∞	∞
1	0	1	∞
2	0	1	0

Negative cycles

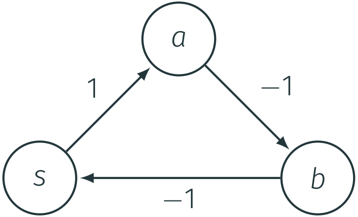
What happens if we run this on a graph with negative cycles?



round	s	a	b
0	0	∞	∞
1	0	1	∞
2	0	1	0
3	-1	1	0

Negative cycles

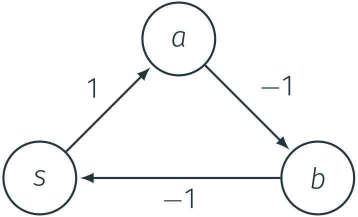
What happens if we run this on a graph with negative cycles?



round	s	a	b
0	0	∞	∞
1	0	1	∞
2	0	1	0
3	-1	1	0
4	-1	0	0

Negative cycles

What happens if we run this on a graph with negative cycles?



round	s	a	b
0	0	∞	∞
1	0	1	∞
2	0	1	0
3	-1	1	0
4	-1	0	0
5	-1	0	-1

Negative cycles can not hide

Lemma restated
If G does not has a negative length cycle reachable from $s \implies \forall v$: $d(v, n) = d(v, n-1)$.

Also, $d(v, n-1)$ is the length of the shortest path between s and v.

Put together are the following:

Lemma
G has a negative length cycle reachable from $s \iff$ there is some node v such that $d(v, n) < d(v, n-1)$.

Bellman-Ford: Negative Cycle Detection - final version

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ in(v) do
            d(v) = min{d(v), d(u) + ℓ(u, v)}
(* One more iteration to check if distances change *)
for each v ∈ V do
    for each edge (u, v) ∈ in(v) do
        if (d(v) > d(u) + ℓ(u, v))
            Output ``Negative Cycle''

for each v ∈ V do
    dist(s, v) ← d(v)
```

Variants on Bellman-Ford

Finding the Paths and a Shortest Path Tree

How do we find a shortest path tree in addition to distances?

- For each v the $d(v)$ can only get smaller as algorithm proceeds.
- If $d(v)$ becomes smaller it is because we found a vertex u such that $d(v) > d(u) + \ell(u, v)$ and we update $d(v) = d(u) + \ell(u, v)$. That is, we found a shorter path to v through u.
- For each v have a $prev(v)$ pointer and update it to point to u if v finds a shorter path via u.
- At end of algorithm $prev(v)$ pointers give a shortest path tree oriented towards the source s.

Negative Cycle Detection

Given directed graph G with arbitrary edge lengths, does it have a negative length cycle?

Negative Cycle Detection

Given directed graph G with arbitrary edge lengths, does it have a negative length cycle?

- Bellman-Ford checks whether there is a negative cycle C that is reachable from a specific vertex s. There may negative cycles not reachable from s.
- Run Bellman-Ford $|V|$ times, once from each node u?

Negative Cycle Detection

- Add a new node s' and connect it to all nodes of G with zero length edges. Bellman-Ford from s' will fill find a negative length cycle if there is one.
- Negative cycle detection can be done with one Bellman-Ford invocation.

Shortest Paths in DAGs

Single-Source Shortest Path Problems

Input A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

Shortest Paths in a DAG

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- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

Simplification of algorithms for DAGs

- No cycles and hence no negative length cycles! Hence can find shortest paths even for negative length edges
- Can order nodes using topological sort

Algorithm for DAGs

- Want to find shortest paths from s. Ignore nodes not reachable from s.
- Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G

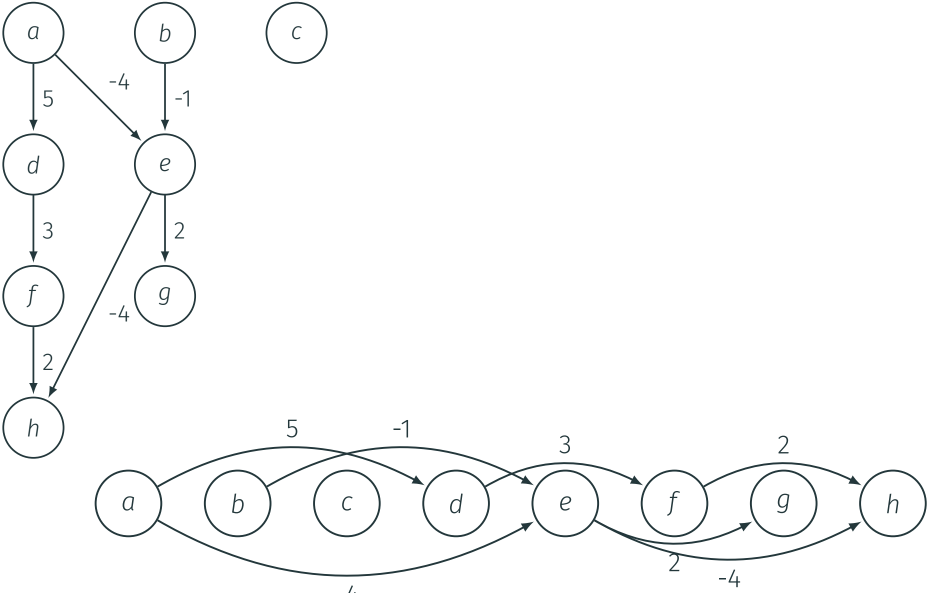
Algorithm for DAGs

- Want to find shortest paths from s. Ignore nodes not reachable from s.
- Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G

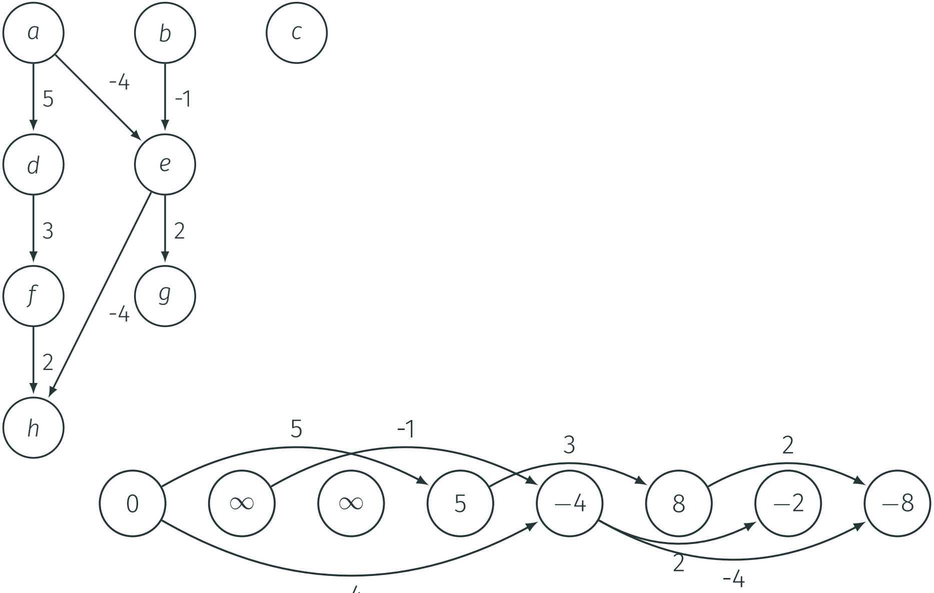
Observation:

- shortest path from s to v_i cannot use any node from $v_{i+1}, \ldots, v_n$
- can find shortest paths in topological sort order.

Shortest Paths for DAGs - Example



Shortest Paths for DAGs - Example



Algorithm for DAGs

```
for i = 1 to n do
        d(s, v_i) = ∞
d(s, s) = 0

for i = 1 to n − 1 do
        for each edge (v_i, v_j) in Adj(v_i) do
                d(s, v_j) = min{d(s, v_j), d(s, v_i) + ℓ(v_i, v_j)}

return d(s, ·) values computed
```

Correctness: induction on i and observation in previous slide.

Running time: $O(m + n)$ time algorithm! Works for negative edge lengths and hence can find longest paths in a DAG.

All Pairs Shortest Paths

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.
- Find shortest paths for all pairs of nodes.

Single-Source Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

SSSP: Single-Source Shortest Paths

Single-Source Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

Dijkstra's algorithm for non-negative edge lengths. Running time: $O((m + n) \log n)$ with heaps and $O(m + n \log n)$ with advanced priority queues.

Bellman-Ford algorithm for arbitrary edge lengths. Running time: $O(nm)$.

All-Pairs Shortest Paths - Using known algorithms...

All-Pairs Shortest Path Problem

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Find shortest paths for all pairs of nodes.

All-Pairs Shortest Paths - Using known algorithms...

All-Pairs Shortest Path Problem

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Find shortest paths for all pairs of nodes.

Apply single-source algorithms n times, once for each vertex.

- Non-negative lengths. $O(nm \log n)$ with heaps and $O(nm + n^2 \log n)$ using advanced priority queues.
- Arbitrary edge lengths: $O(n^2 m)$.
  $\Theta(n^4)$ if $m = \Omega(n^2)$.

All-Pairs Shortest Path Problem

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Find shortest paths for all pairs of nodes.

Apply single-source algorithms n times, once for each vertex.

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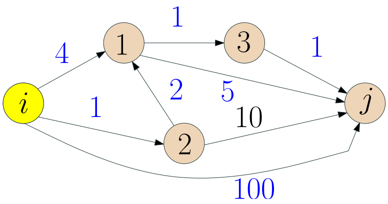
- Arbitrary edge lengths: $O(n^2 m)$.
  $\Theta(n^4)$ if $m = \Omega(n^2)$.

Can we do better?

All Pairs Shortest Paths: A recursive solution

All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $dist(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



$dist(i, j, 0) =$

$dist(i, j, 1) =$

$dist(i, j, 2) =$

$dist(i, j, 3) =$

All-Pairs: Recursion on index of intermediate nodes

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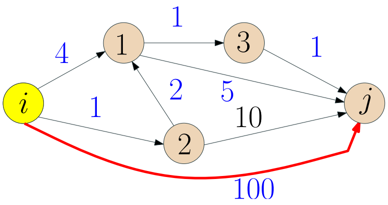
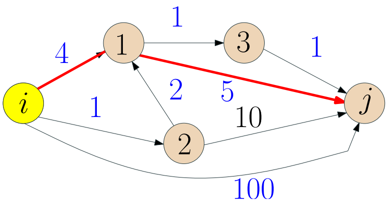
- $dist(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



$$dist(i, j, 0) = \quad 100$$
$$dist(i, j, 1) =$$
$$dist(i, j, 2) =$$
$$dist(i, j, 3) =$$

All-Pairs: Recursion on index of intermediate nodes

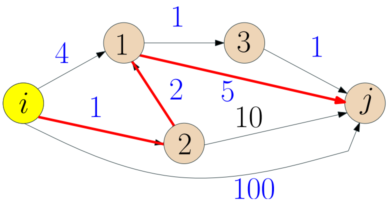
- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $dist(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



$$dist(i, j, 0) = \quad 100$$
$$dist(i, j, 1) = \quad 9$$
$$dist(i, j, 2) =$$
$$dist(i, j, 3) =$$

All-Pairs: Recursion on index of intermediate nodes

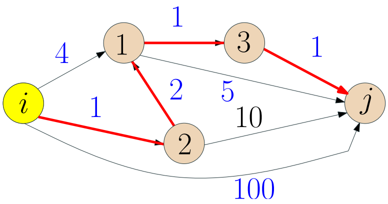
- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $dist(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



$$dist(i, j, 0) = 100$$
$$dist(i, j, 1) = 9$$
$$dist(i, j, 2) = 8$$
$$dist(i, j, 3) =$$

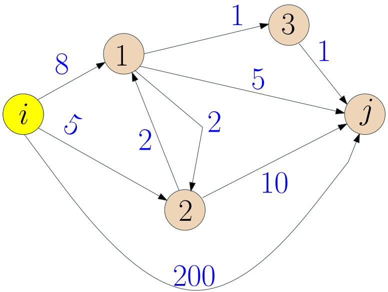
All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $dist(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an <u>intermediate node</u> is at most k (could be $-\infty$ if there is a negative length cycle).



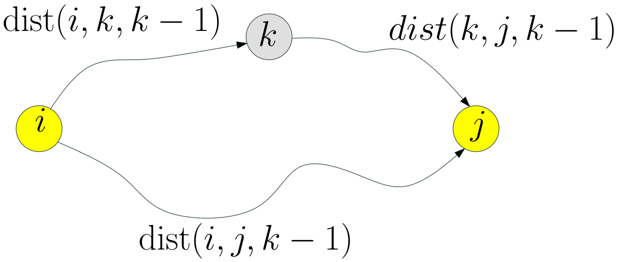
$$dist(i, j, 0) = 100$$
$$dist(i, j, 1) = 9$$
$$dist(i, j, 2) = 8$$
$$dist(i, j, 3) = 5$$

For the following graph, $\text{dist}(i, j, 2)$ is...



1. 9
2. 10
3. 11
4. 12
5. 15

All-Pairs: Recursion on index of intermediate nodes



$$dist(i,j,k) = \min \begin{cases} dist(i,j,k-1) \\ dist(i,k,k-1) + dist(k,j,k-1) \end{cases}$$

Base case: $dist(i,j,0) = \ell(i,j)$ if $(i,j) \in E$, otherwise ∞

Correctness: If $i \to j$ shortest walk goes through k then k occurs only once on the

All-Pairs: Recursion on index of intermediate nodes

If i can reach k <u>and</u> k can reach j <u>and</u> $dist(k, k, k-1) < 0$ then G has a negative length cycle containing k and $dist(i, j, k) = -\infty$.

Recursion below is valid only if $dist(k, k, k-1) \geq 0$. We can detect this during the algorithm or wait till the end.

$$dist(i, j, k) = \min \begin{cases} dist(i, j, k-1) \\ dist(i, k, k-1) + dist(k, j, k-1) \end{cases}$$

Floyd-Warshall algorithm

Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

$$d(i,j,k) = \min \begin{cases} d(i,j,k-1) \\ d(i,k,k-1) + d(k,j,k-1) \end{cases}$$

```
for i = 1 to n do
    for j = 1 to n do
        d(i,j,0) = ℓ(i,j)
 (* ℓ(i,j) = ∞ if (i,j) ∉ E, 0 if i = j *)

for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            d(i,j,k) = min { d(i,j,k−1),
                           { d(i,k,k−1) + d(k,j,k−1)
for i = 1 to n do
    if (dist(i,i,n) < 0) then
        Output ∃ negative cycle in G
```

Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

$$d(i,j,k) = \min \begin{cases} d(i,j,k-1) \\ d(i,k,k-1) + d(k,j,k-1) \end{cases}$$

for $i = 1$ to n do
    for $j = 1$ to n do
        $d(i,j,0) = \ell(i,j)$
  (* $\ell(i,j) = \infty$ if $(i,j) \notin E$, 0 if $i = j$ *)

for $k = 1$ to n do
    for $i = 1$ to n do
        for $j = 1$ to n do
            $d(i,j,k) = \min \begin{cases} d(i,j,k-1), \\ d(i,k,k-1) + d(k,j,k-1) \end{cases}$
for $i = 1$ to n do
    if $(dist(i,i,n) < 0)$ then
        Output $\exists$ negative cycle in G

Running Time:

Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

$$d(i,j,k) = \min \begin{cases} d(i,j,k-1) \\ d(i,k,k-1) + d(k,j,k-1) \end{cases}$$

for $i = 1$ `to` n **do**
    for $j = 1$ `to` n **do**
        $d(i,j,0) = \ell(i,j)$
`(*` $\ell(i,j) = \infty$ `if` $(i,j) \notin E$`, 0 if` $i = j$ `*)`

for $k = 1$ `to` n **do**
    for $i = 1$ `to` n **do**
        for $j = 1$ `to` n **do**

$$d(i,j,k) = \min \begin{cases} d(i,j,k-1), \\ d(i,k,k-1) + d(k,j,k-1) \end{cases}$$

for $i = 1$ `to` n **do**
    if $(dist(i,i,n) < 0)$ **then**
        `Output` $\exists$ `negative cycle in` G

Running Time: $\Theta(n^3)$. Space: $\Theta(n^3)$.

Floyd-Warshall Algorithm: Finding the Paths

Question: Can we find the paths in addition to the distances?

Floyd-Warshall Algorithm: Finding the Paths

Question: Can we find the paths in addition to the distances?

- Create a $n \times n$ array Next that stores the next vertex on shortest path for each pair of vertices
- With array Next, for any pair of given vertices i, j can compute a shortest path in $O(n)$ time.

Floyd-Warshall Algorithm - Finding the Paths

```
for i = 1 to n do
    for j = 1 to n do
        d(i, j, 0) = ℓ(i, j)
(* ℓ(i, j) = ∞ if (i, j) not edge, 0 if i = j *)
        Next(i, j) = −1
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            if (d(i, j, k − 1) > d(i, k, k − 1) + d(k, j, k − 1)) then
                d(i, j, k) = d(i, k, k − 1) + d(k, j, k − 1)
                Next(i, j) = k
for i = 1 to n do
    if (d(i, i, n) < 0) then
        Output that there is a negative length cycle in G
```

Exercise: Given *Next* array and any two vertices i, j describe an $O(n)$ algorithm to find a i-j shortest path.

Summary of shortest path algorithms

Summary of results on shortest paths

Single source		
No negative edges	Dijkstra	$O(n \log n + m)$
Edge lengths can be negative	Bellman Ford	$O(nm)$

All Pairs Shortest Paths

No negative edges	n * Dijkstra	$O\left(n^2 \log n + nm\right)$
No negative cycles	n * Bellman Ford	$O\left(n^2 m\right) = O\left(n^4\right)$
No negative cycles	Johnson's [1]	$O\left(nm + n^2 \log n\right)$
No negative cycles	Floyd-Warshall	$O\left(n^3\right)$
Unweighted	Matrix multiplication [2]	$O(n^{2.38})$, $O(n^{2.58})$

Summary of results on shortest paths

(1): The algorithm for the case that there are no negative cycles, and doing all shortest paths, works by computing a potential function using Bellman-Ford and then doing Dijkstra. It is mentioned for the sake of completeness, but it outside the scope of the class.

(2): `https://resources.mpi-inf.mpg.de/departments/d1/teaching/ss12/AdvancedGraphAlgorithms/Slides14.pdf`

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