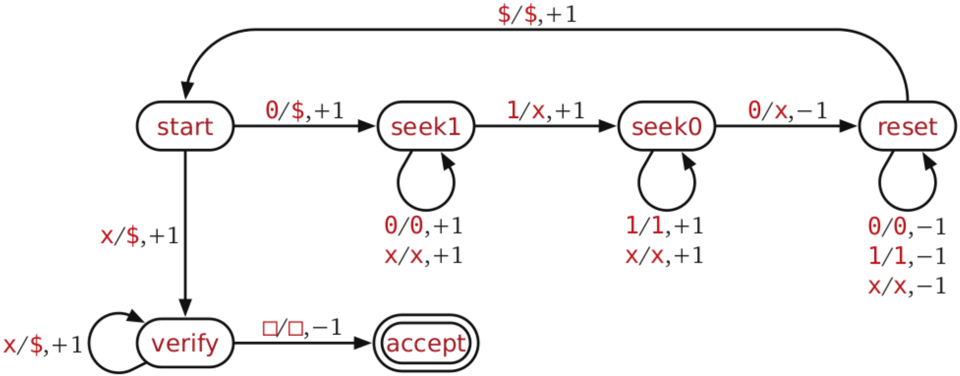
Pre-lecture brain teaser

You have the following Turing machine diagram that accepts a particular language whose alphabet $\Sigma = \{0, 1\}$. Please describe the language.



ECE-374-B: Lecture 8 - Universal Turing Machines

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University of Illinois Urbana-Champaign

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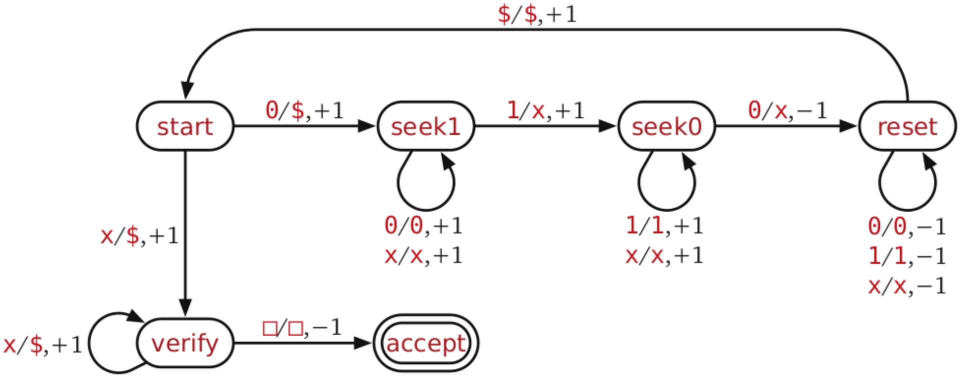


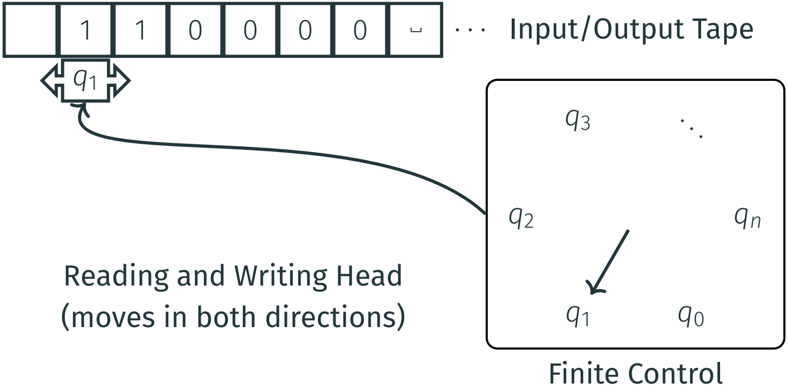
Pre-lecture brain teaser - code

Can simulate TM on `turingmachine.io` using the following code:

```
start state: start
table:
start:
    # Inductive case: start with the same symbol.
    0: {write: '$', R: seek1}
    # Base case: empty string.
    'x': {write: '$', R: verify}
seek1:
    [0,'x']: R
    1: {write: 'x', R: seek0}
seek0:
    [1,'x']: R
    0: {write: 'x', L: reset}
reset:
    [0,1,'x']: L
    '$': {R: start}
verify:
    x: {write: '$', R}
    ' ': {L: accept}
accept:
```

Turing machine recap

Turing machine



- Input written on (infinite) one sided tape.
- Special blank characters.
- Finite state control (similar to DFA).
- Ever step: Read character under head, write character out, move the head

Transition function

Transition Function

$$\delta : Q \times \Gamma = Q \times \Gamma \times \{L, R\}$$

Current state

Scanned symbol

New State

Symbol to write

Direction to move on tape

$\delta(q, a) = (p, b, L)$ means

from state q, on reading a:

- go to state p
- write b
- move head Left

Turing machine varients

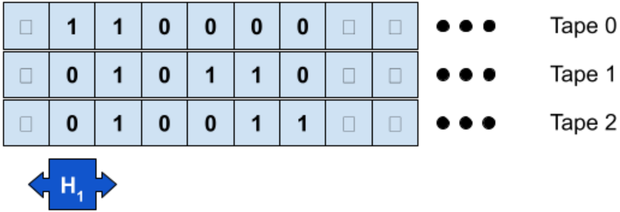
Equivalent Turing Machines

Several variations of a Turing machine:

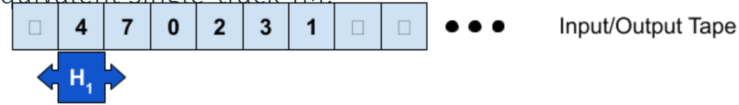
- Standard Turing machine (single infinite tape)
- Multi-track tapes
- Doubly-Infinite Tape
- Multiple heads
- Multiple heads and tapes

Multi-track Tapes

Suppose we have a TM with multiple tracks:



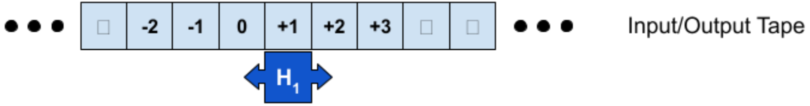
Is there an equivalent single-track TM?



New transition function: $\delta : Q \times \Gamma_1 \times \Gamma_2 \times \Gamma_3 \to Q \times \Gamma_1 \times \Gamma_2 \times \Gamma_3 \times \{-1, +1\}$

Infinite Bi-directional Tape

Suppose we have a TM with multiple tracks:



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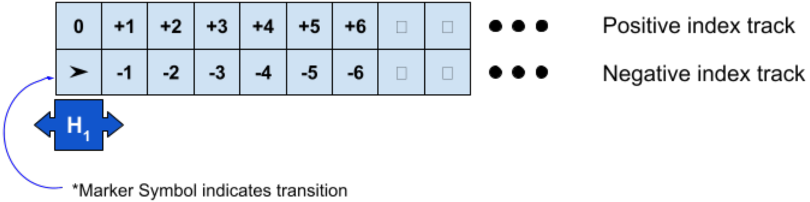
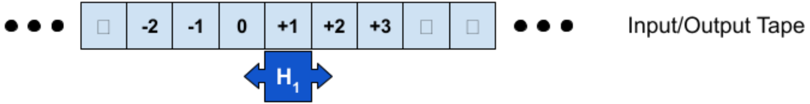


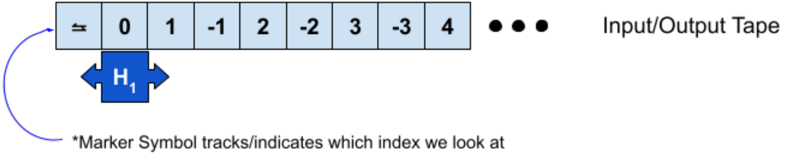
Can model as multiple tapes.

Infinite Bi-directional Tape

Suppose we have a TM with a bidirectional tape:



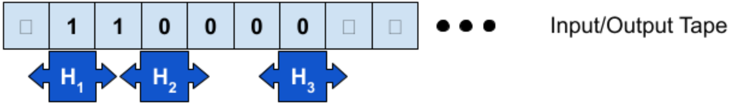
Is there an equivalent single-track TM?



Or as single tape interleaved with positive and negative indexes.

Multiple Read/Write Heads

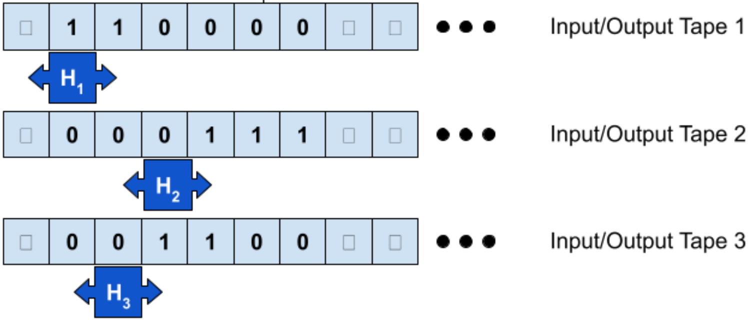
Suppose we have a TM with multiple heads:



What does the transition function for the equivalent nominal TM look like?

Multiple Read/Write Heads

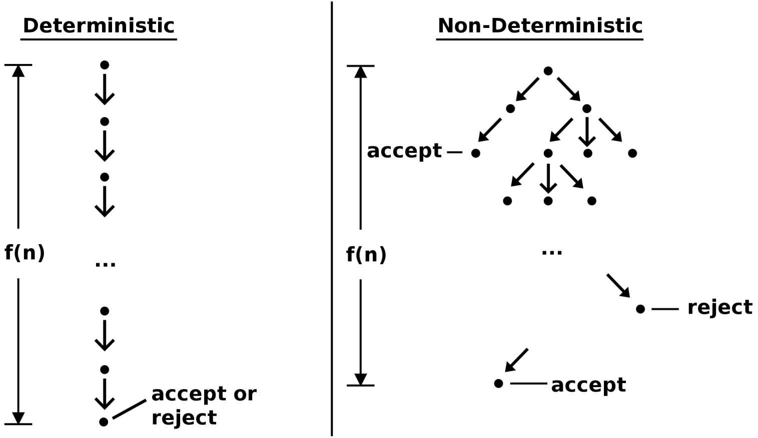
Suppose we have a TM with multiple heads and tracks:



What does the transition function for the equivalent nominal TM look like?

Determinism in Turing machines

Remember Non-determinism?



Non-deterministic Turing machine?

What does a non-deterministic Turing machine look like?

Non-deterministic Turing machine?

What does a non-deterministic Turing machine look like?

Is a NTM more powerful than a DTM?

Power of NTM

No. A DTM can simulate a NTM in the following ways:

- **Multiplicity of configuration of states**
  1. Have the store multiple configurations of the NTM.
  2. At every timestep, process each configuration. Add configurations to the set if multiple paths exist.
- **Multiple Tapes** - Can simulate NTM with 3-tape DTM:
  1. First tape holds original input
  2. Second used to simulate a particular computation of NTM
  3. Third tape encodes path in NTM computation tree.

  Effectively this is a breadth-first search of non-deterministic computation tree.

Savitch's Theorem

Proved by Walter Savitch in 1970, states that for any function $f \in \Omega(\log(n))$:

$$\text{NSPACE}(f(n)) \subseteq \text{DSPACE}(f(n)^2)$$

Lemma

If a NTM can solve a problem using $f(n)$ space, a DTM can solve the same problem int he square of that space bound.

$\implies$ Even though non-determinism significantly reduces time to solve problem, it reduces space requirements far less!

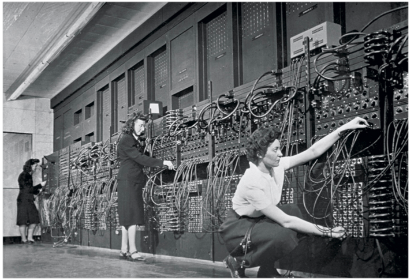
Universal Turing Machine

Special Purpose Machines?

We've seen that you need different DFAs for different languages.

We've seen that you need different TMs for different languages.

Early computers were no different.



Universal Turing Machine

A single TM M_U that can compute anything computable!

Takes as input:

- the description of some other TM M
- data w for M to run on

Outputs:

- results of running $M(w)$

Coding of TMs

Show how to represent every *TM* as a natural number

Lemma

If L over alphabet $\{0, 1\}$ is accepted by some TM M, then there is a one-tape TM M that accepts L, such that

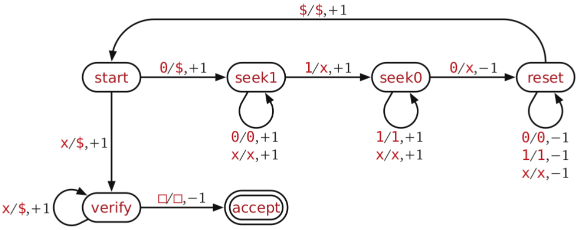
- $\Gamma = \{0, 1, B\}$
- *states numbered* $1, \ldots, k$
- q_1 *is a unique start state*
- q_2 *is a unique halt/accept state*
- q_3 *is a unique halt/reject state*

So to represent a TM, we need only list its set of transitions - everything else is implicit by the above.

Consider the TM that recognizes the language $L = \{0^n 1^n 0^n | n \geq 0\}$ with the state diagram shown below:

Input encoding:

- $\langle 0 \rangle = 001$
- $\langle 1 \rangle = 010$
- $\langle \$ \rangle = 011$
- $\langle x \rangle = 100$
- $\langle \sqcup \rangle = 000$



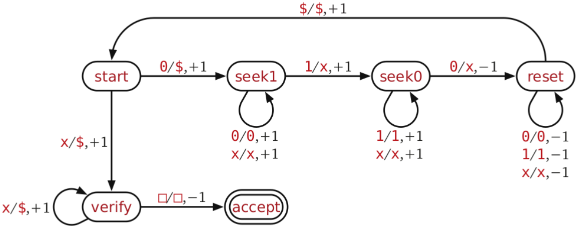
Example: $\langle 001100 \rangle = [001 \cdot 001 \cdot 010 \cdot 010 \cdot 001 \cdot 001]$
(Putting · separators for the sake of legibility)

Encoding states

Consider the TM that recognizes the language $L = \{0^n 1^n 0^n | n \geq 0\}$ with the state diagram shown below:

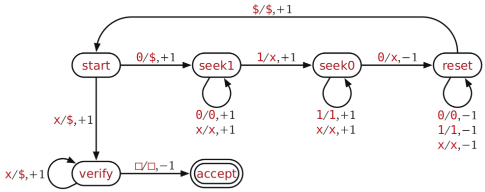
State encoding:

- $\langle \text{start} \rangle = 001$
- $\langle \text{seek1} \rangle = 010$
- $\langle \text{seek0} \rangle = 011$
- $\langle \text{reset} \rangle = 100$
- $\langle \text{verify} \rangle = 101$
- $\langle \text{accept} \rangle = 110$
- $\langle \text{reject} \rangle = 000$



Encoding States and Alphabet

Consider the TM that recognizes the language $L = \{0^n 1^n 0^n | n \geq 0\}$ with the state diagram shown below:

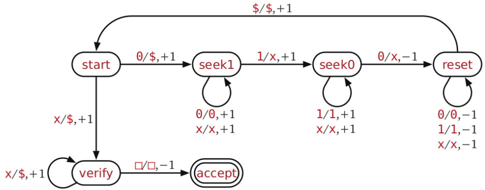


Now we need to encode a transition. Last thing we'll need is to encode the movement of the head whihc we'll describe as: $[\text{left}, \text{right}] = [0, 1]$.

Example: How do we encode: $\delta(\text{reset}, \$) = (\text{start}, \$, \text{right})$

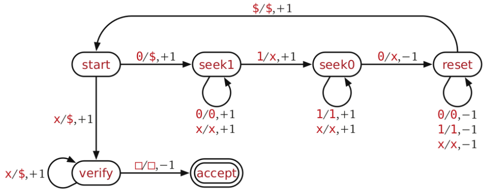
Answer: $[100 \cdot 011|001 \cdot 011 \cdot 1]$

Encoding machine through transitions



$$
\begin{aligned}
\delta^{\mathsf{M}} = [&[\mathtt{001}\cdot\mathtt{001}|\mathtt{010}\cdot\mathtt{011}\cdot\mathtt{1}][\mathtt{001}\cdot\mathtt{100}|\mathtt{101}\cdot\mathtt{011}\cdot\mathtt{1}]\\
&[\mathtt{010}\cdot\mathtt{001}|\mathtt{010}\cdot\mathtt{001}\cdot\mathtt{1}][\mathtt{010}\cdot\mathtt{100}|\mathtt{010}\cdot\mathtt{100}\cdot\mathtt{1}]\\
&[\mathtt{010}\cdot\mathtt{010}|\mathtt{011}\cdot\mathtt{100}\cdot\mathtt{1}][\mathtt{011}\cdot\mathtt{010}|\mathtt{011}\cdot\mathtt{010}\cdot\mathtt{1}]\\
&[\mathtt{011}\cdot\mathtt{100}|\mathtt{011}\cdot\mathtt{100}\cdot\mathtt{1}][\mathtt{011}\cdot\mathtt{001}|\mathtt{100}\cdot\mathtt{100}\cdot\mathtt{1}]\\
&[\mathtt{100}\cdot\mathtt{001}|\mathtt{100}\cdot\mathtt{001}\cdot\mathtt{0}][\mathtt{100}\cdot\mathtt{010}|\mathtt{100}\cdot\mathtt{010}\cdot\mathtt{0}]\\
&[\mathtt{100}\cdot\mathtt{100}|\mathtt{100}\cdot\mathtt{100}\cdot\mathtt{0}][\mathtt{100}\cdot\mathtt{011}|\mathtt{001}\cdot\mathtt{011}\cdot\mathtt{1}]\\
&[\mathtt{101}\cdot\mathtt{100}|\mathtt{101}\cdot\mathtt{011}\cdot\mathtt{1}][\mathtt{101}\cdot\mathtt{000}|\mathtt{110}\cdot\mathtt{000}\cdot\mathtt{0}]]
\end{aligned}
$$

Encoding machine through transitions



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&[\mathtt{100}\cdot\mathtt{001}|\mathtt{100}\cdot\mathtt{001}\cdot\mathtt{0}][\mathtt{100}\cdot\mathtt{010}|\mathtt{100}\cdot\mathtt{010}\cdot\mathtt{0}]\\
&[\mathtt{100}\cdot\mathtt{100}|\mathtt{100}\cdot\mathtt{100}\cdot\mathtt{0}][\mathtt{100}\cdot\mathtt{011}|\mathtt{001}\cdot\mathtt{011}\cdot\mathtt{1}]\\
&[\mathtt{101}\cdot\mathtt{100}|\mathtt{101}\cdot\mathtt{011}\cdot\mathtt{1}][\mathtt{101}\cdot\mathtt{000}|\mathtt{110}\cdot\mathtt{000}\cdot\mathtt{0}]]
\end{aligned}
$$

Encoding initial state

Ok so now we've encoded the Turing machine (M) into a string, how do we make a machine $M_u(M, w)$ which accepts if $M(w)$ accepts, and rejects if $M(w)$ rejects?

Encoding initial state

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Let's start with the encoding of w (let's say $w = 001100$):
$\langle 001100 \rangle = [001 \cdot 001 \cdot 010 \cdot 010 \cdot 001 \cdot 001]$

Encoding initial state

Ok so now we've encoded the Turing machine (M) into a string, how do we make a machine $M_u(M, w)$ which accepts if $M(w)$ accepts, and rejects if $M(w)$ rejects?

Let's start with the encoding of w (let's say $w = 001100$):
$\langle 001100 \rangle = [001 \cdot 001 \cdot 010 \cdot 010 \cdot 001 \cdot 001]$

Now let's add spaces next to each character so we can mark where M's head is:
$[[000 \cdot 001][000 \cdot 001][000 \cdot 010][000 \cdot 010][000 \cdot 001][000 \cdot 001]]$

Encoding states

Padding used to mark state.

In the beginning, $q = \langle \text{start} \rangle = 001$ so our machine tapes initial string is:
$[[\underline{001} \cdot 001][000 \cdot 001][000 \cdot 010][000 \cdot 010][000 \cdot 001][000 \cdot 001]]$

Similarly intermediate configuration
$M = \langle \text{state}, \text{tape string}, \text{head position} \rangle = (\text{seek1}, \$0x1x0, 3)$ would be marked as:

$$[\underbrace{[000 \cdot 011]}_{\text{reject }\$}\ \underbrace{[000 \cdot 001]}_{\text{reject }0}\ \underbrace{[000 \cdot 100]}_{\text{reject x}}\ \underbrace{[010 \cdot 010]}_{\text{seek1 }1}\ \underbrace{[000 \cdot 100]}_{\text{reject x}}\ \underbrace{[000 \cdot 001]}_{\text{reject }0}]$$

The universal Turing machine

UTM introduction

Now that we are able to encode Turing machines, we want to construct a Turing machine such that:

$L(M_u) = \{\langle M \rangle \# w | M \text{ accepts } w\}$

M_u is a stored-program computer. It reads $< M >$ and executes it on data w.

M_u simulates the run of M on w.

Encodings

M: Turing machine

$\langle M \rangle$: a string uniquely describing M (i.e., it is a number.

w: An input string.

$\langle M, w \rangle$: A unique string encoding both M and input w.

$$L(M_u) = \{\langle M, w \rangle \; M \text{ is a TM } \text{ and } M \text{ accepts } w\}.$$

M_u Operational concept

We assume without a loss of generality that our universal turing machine (M_u) has two tapes and two heads:

- **Input tape:** which stores the encoding of $\langle M \rangle = \langle \text{state}, \text{tape input}, \text{head position} \rangle$
- **Machine tape:**Encoding tape which stores M's encoding

General Idea: For any given configuration of M, our M_u will.

- Starting from leftmost of input tape, scan tape for first state which is not $\langle \text{reject} \rangle$
- M_u scans machine tape for the transition function that matches the substring found in the input tape.
- Based on transition function, M_u writes the right half of this transition function into the current input tape cell.
- Based on head direction of the transition function, M_u moves the current

Simulation example I

Let's start with the configuration: $M = (\mathrm{seek1}, \$\$\mathrm{x1x0}, 3)$:

- Input-Tape = $\underset{\triangle}{[}$ [000 · 011][000 · 011][000 · 100][010 · 010][000 · 100][000 · 001]]
- Machine-Tape = $\delta^M =$
  $\underset{\triangle}{[}$ [001 · 001|010 · 011 · 1][001 · 100|101 · 011 · 1][010 · 001| . . .

First M_u searchers for none reject state:

- Input-Tape = [[000 · 011][000 · 011][000 · 100][010 $\underset{\triangle}{\cdot}$ 010][000 · 100][000 · 001]]
- Machine-Tape = $\delta^M =$
  $\underset{\triangle}{[}$ [001 · 001|010 · 011 · 1][001 · 100|101 · 011 · 1][010 · 001| . . .

Simulation example II

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][010 \underset{\triangle}{\cdot} 010][000 \cdot 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\underset{\triangle}{[}\,[001 \cdot 001|010 \cdot 011 \cdot 1][001 \cdot 100|101 \cdot 011 \cdot 1][010 \cdot 001| \ldots$

Then M_u searches for transition whose left side matches the input cell:

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][010 \underset{\triangle}{\cdot} 010][000 \cdot 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\ldots 100 \cdot 1][010 \underset{\triangle}{\cdot} 010|011 \cdot 100 \cdot 1][011 \cdot 010|011 \cdot 010 \cdot 1] \ldots$

Simulation example III

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][010 \underset{\triangle}{\cdot} 010][000 \cdot 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\ldots 100 \cdot 1][010 \underset{\triangle}{\cdot} 010|011 \cdot 100 \cdot 1][011 \cdot 010|011 \cdot 010 \cdot 1]\ldots$

Then M_u copies the right side of the transition function into the input tape:

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][011 \cdot 100 \underset{\triangle}{]} [000 \cdot 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\ldots 100 \cdot 1][010 \cdot 010|011 \cdot 100 \underset{\triangle}{\cdot} 1][011 \cdot 010|011 \cdot 010 \cdot 1]\ldots$

Simulation example IV

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][011 \cdot 100\underset{\triangle}{]}[000 \cdot 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\ldots 100 \cdot 1][010 \cdot 010|011 \cdot 100 \underset{\triangle}{\cdot} 1][011 \cdot 010|011 \cdot 010 \cdot 1]\ldots$

Then M_u move the state of the configuration according to the transition function:

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][000 \cdot 100][011 \underset{\triangle}{\cdot} 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\ldots 100 \cdot 1][010 \cdot 010|011 \cdot 100 \cdot \underset{\triangle}{1}][011 \cdot 010|011 \cdot 010 \cdot 1]\ldots$

Simulation example V

- Input-Tape = $[[000 \cdot 011][000 \cdot 011][000 \cdot 100][000 \cdot 100][011 \underset{\triangle}{\cdot} 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\ldots 100 \cdot 1][010 \cdot 010|011 \cdot 100 \cdot \underset{\triangle}{1}][011 \cdot 010|011 \cdot 010 \cdot 1]\ldots$

Then we reset:

- Input-Tape = $\underset{\triangle}{[}\,[000 \cdot 011][000 \cdot 011][000 \cdot 100][000 \cdot 100][011 \cdot 100][000 \cdot 001]]$
- Machine-Tape = $\delta^M =$
  $\underset{\triangle}{[}\,[001 \cdot 001|010 \cdot 011 \cdot 1][001 \cdot 100|101 \cdot 011 \cdot 1][010 \cdot 001|\ldots$

What does this show?

- Every TM is encoded by a unique element of N (where N is a natural number)
- **Convention:** elements of N that do not correspond to any TM encoding represent the “null TM” that accepts nothing.
- Thus, every TM is a number, and vice versa
- Let <M> mean the number that encodes M. Conversely, let M_n be the TM with encoding n.

Big Idea: Every TM can be represent by a number (strings of 0’s and 1’s) and there exists a universal TM, M_u, that can simulate any other TM.