

A **subsequence** of a sequence (for example, an array, linked list, or string), obtained by removing zero or more elements and keeping the rest in the same sequence order. A subsequence is called a **substring** if its elements are contiguous in the original sequence. For example:

- **SUBSEQUENCE**, **UBSEQU**, and the empty string  $\varepsilon$  are all substrings (and therefore subsequences) of the string **SUBSEQUENCE**;
- **SBSQNC**, **SQUEE**, and **EEE** are all subsequences of **SUBSEQUENCE** but not substrings;
- **QUEUE**, **EQUUS**, and **DIMAGGIO** are not subsequences (and therefore not substrings) of **SUBSEQUENCE**.

Describe recursive backtracking algorithms for the following problems. *Don't worry about running times.*

1. Given an array  $A[1..n]$  of integers, compute the length of a **longest increasing subsequence**. A sequence  $B[1..\ell]$  is *increasing* if  $B[i] > B[i-1]$  for every index  $i \geq 2$ .

For example, given the array

$\langle 3, \underline{1}, \underline{4}, 1, \underline{5}, 9, 2, \underline{6}, 5, 3, 5, \underline{8}, \underline{9}, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

your algorithm should return the integer 6, because  $\langle 1, 4, 5, 6, 8, 9 \rangle$  is a longest increasing subsequence (one of many).

2. Given an array  $A[1..n]$  of integers, compute the length of a **longest decreasing subsequence**. A sequence  $B[1..\ell]$  is *decreasing* if  $B[i] < B[i-1]$  for every index  $i \geq 2$ .

For example, given the array

$\langle 3, 1, 4, 1, 5, \underline{9}, 2, \underline{6}, 5, 3, \underline{5}, 8, 9, 7, 9, 3, 2, 3, 8, \underline{4}, 6, \underline{2}, 7 \rangle$

your algorithm should return the integer 5, because  $\langle 9, 6, 5, 4, 2 \rangle$  is a longest decreasing subsequence (one of many).

3. Given an array  $A[1..n]$  of integers, compute the length of a **longest alternating subsequence**. A sequence  $B[1..\ell]$  is *alternating* if  $B[i] < B[i-1]$  for every even index  $i \geq 2$ , and  $B[i] > B[i-1]$  for every odd index  $i \geq 3$ .

For example, given the array

$\langle \underline{3}, \underline{1}, \underline{4}, \underline{1}, \underline{5}, 9, \underline{2}, \underline{6}, \underline{5}, 3, 5, \underline{8}, 9, \underline{7}, \underline{9}, \underline{3}, 2, 3, \underline{8}, \underline{4}, \underline{6}, \underline{2}, \underline{7} \rangle$

your algorithm should return the integer 17, because  $\langle 3, 1, 4, 1, 5, 2, 6, 5, 8, 7, 9, 3, 8, 4, 6, 2, 7 \rangle$  is a longest alternating subsequence (one of many).

**To think about later:**

4. Given an array  $A[1..n]$  of integers, compute the length of a longest **convex** subsequence of  $A$ . A sequence  $B[1..l]$  is *convex* if  $B[i] - B[i-1] > B[i-1] - B[i-2]$  for every index  $i \geq 3$ .

For example, given the array

$\langle \underline{3}, \underline{1}, 4, \underline{1}, 5, 9, \underline{2}, 6, 5, 3, \underline{5}, 8, \underline{9}, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

your algorithm should return the integer 6, because  $\langle 3, 1, 1, 2, 5, 9 \rangle$  is a longest convex subsequence (one of many).

5. Given an array  $A[1..n]$  of integers, compute the length of a longest **weakly-increasing** subsequence of  $A$ . A sequence  $B[1..l]$  is *weakly-increasing* if each element is larger than the average of its two previous elements (i.e.,  $2 \cdot B[i] > B[i-1] + B[i-2]$  for all  $i > 2$ ).

For example, given the array

$\langle 3, \underline{1}, 4, \underline{1}, 5, 9, \underline{2}, 6, 5, \underline{3}, 5, 8, 9, 7, 9, \underline{3}, 2, 3, 8, \underline{4}, \underline{6}, 2, \underline{7} \rangle$

your algorithm should return the integer 8, because  $\langle 1, 1, 2, 3, 3, 4, 6, 7 \rangle$  is a longest weakly increasing subsequence (one of many).

6. Given an array  $A[1..n]$ , compute the length of a longest **palindrome** subsequence of  $A$ . Recall that a sequence  $B[1..l]$  is a *palindrome* if  $B[i] = B[l - i + 1]$  for every index  $i$ .

For example, given the array

$\langle \underline{3}, 1, 4, 1, 5, 9, \underline{2}, 6, 5, \underline{3}, 5, 8, \underline{9}, \underline{7}, \underline{9}, \underline{3}, \underline{2}, \underline{3}, 8, 4, 6, 2, 7 \rangle$

your algorithm should return the integer 9, because  $\langle 3, 2, 3, 9, 7, 9, 3, 2, 3 \rangle$  is a longest palindrome subsequence (there may be others).