



## Pre-lecture brain teaser

Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say  $k$  arrays of size  $n/k$  each?

# ECE-374-B: Lecture 10 - Divide and Conquer Algorithms

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## Quick Sort

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1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
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3. Recursively sort the subarrays, and concatenate them.

## Quick Sort: Example

- example array: 16, 12, 14, 20, 5, 3, 18, 19, 1
- worst case array: 3, 7, 5, 1, 2, 4, 6, 8

See visualizer:

<https://www.hackerearth.com/practice/algorithms/sorting/quick-sort/visualize/>

## Time Analysis

- Let  $k$  be the rank of the chosen pivot. Then,  $T(n) = T(k - 1) + T(n - k) + O(n)$

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Then,  $T(n) = O(n \log n)$ .

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Then,  $T(n) = O(n \log n)$ .
- Typically, pivot is the first or last element of array. Then,

$$T(n) = \max_{1 \leq k \leq n} (T(k - 1) + T(n - k) + O(n))$$

In the worst case  $T(n) = T(n - 1) + O(n)$ , which means  $T(n) = O(n^2)$ .  
Happens if array is already sorted and pivot is always first element.

## Selecting in Unsorted Lists

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## The Selection Problem

Big problem with QuickSort is that the pivot might not be the median.

How long would it take us to find the median of an unsorted list?



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How long would it take us to find the median of an unsorted list?

Sort, then  $A[n/2]$ . **Is this the optimal way?**

## Rank of element in an array

A: an unsorted array of  $n$  integers

For  $0 \leq j \leq n - 1$ , element of rank  $j$  is the  $j + 1$ -th smallest element in A.

0	1	2	3	4	5	6	7	8	Unsorted array
16	14	34	20	12	5	3	19	11	

5	4	8	7	3	1	0	6	2	Rank
---	---	---	---	---	---	---	---	---	------

## Problem - Selection

**Input** Unsorted array  $A$  of  $n$  integers **and** integer  $j$

**Goal** Find the  $j$ -th smallest number in  $A$  (rank  $j$  number)

Median:  $j = \lfloor n/2 \rfloor$

## Problem - Selection

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**Goal** Find the  $j$ -th smallest number in  $A$  (rank  $j$  number)

Median:  $j = \lfloor n/2 \rfloor$

Simplifying assumption for sake of notation: elements of  $A$  are distinct

# Algorithm I

- Sort the elements in  $A$
- Pick  $j$ th element in sorted order

Time taken =  $O(n \log n)$

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- Sort the elements in  $A$
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Do we need to sort? Is there an  $O(n)$  time algorithm?

## Algorithm II

If  $j$  is small or  $n - j$  is small then

- Find  $j$  smallest/largest elements in  $A$  in  $O(jn)$  time. (How?)
- Time to find median is  $O(n^2)$ .

## Quick select

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# QuickSelect

- Pick a pivot element  $a$  from  $A$
- Partition  $A$  based on  $a$ .  
 $A_{\text{less}} = \{x \in A \mid x \leq a\}$  and  $A_{\text{greater}} = \{x \in A \mid x > a\}$
- $|A_{\text{less}}| = j$ : return  $a$
- $|A_{\text{less}}| > j$ : recursively find  $j$ th smallest element in  $A_{\text{less}}$
- $|A_{\text{less}}| < j$ : recursively find  $k$ th smallest element in  $A_{\text{greater}}$  where  $k = j - |A_{\text{less}}|$ .

## Example

16	14	34	20	12	5	3	19	11
----	----	----	----	----	---	---	----	----

# Time Analysis

- Partitioning step:  $O(n)$  time to scan  $A$
- How do we choose pivot? Recursive running time?

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- Partitioning step:  $O(n)$  time to scan  $A$
- How do we choose pivot? Recursive running time?

Suppose we always choose pivot to be  $A[1]$ .

Say  $A$  is sorted in increasing order and  $j = n$ .

How long does this new algorithm take?

## Does this help with QuickSort?

Should we combine this with QuickSort

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Of course not! It takes  $O(n^2)$  which is already the worse case of QuickSort. Need another method....

## Does this help with QuickSort?

Looking at the quicksort recurrence again:

$$T(n) = T(k - 1) + T(n - k) + O(n)$$

Does  $k$  need to be  $n/2$ ?



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What if  $k = \frac{3}{5}n$ ?

What if  $k = \frac{7}{10}n$ ?

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Looking at the quicksort recurrence again:

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What if  $k = \frac{3}{5}n$ ?

What if  $k = \frac{7}{10}n$ ?

we only need to be able to find a rough median! .... How do we do that?

## Median of Medians

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# Divide and Conquer Approach

## Idea

- Break input  $A$  into many subarrays:  $L_1, \dots, L_k$ .
- Find median  $m_i$  in each subarray  $L_i$ .
- Find the median  $x$  of the medians  $m_1, \dots, m_k$ .
- Intuition: The median  $x$  should be close to being a good median of all the numbers in  $A$ .
- Use  $x$  as pivot in previous algorithm.

## Deterministic selection - example

Given an array  $A = [0, \dots, n - 1]$  of  $n$  numbers and an index  $i$ , where  $0 \leq i \leq n - 1$ , find the  $i^{\text{th}}$  smallest element of  $A$ .

For instance, assume  $n = 20$  and  $i = 10$ .

3	2	14	6	0	16	8	9	13	12	7	17	10	1	11	15	5	18	4	19
---	---	----	---	---	----	---	---	----	----	---	----	----	---	----	----	---	----	---	----

The smallest element of rank 10 would be 10. But how do we figure that out?

Do median of medians.....

Call **Median-of-Medians**( $A$ , 10)

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## Deterministic selection - example

First we reorganize:

3	16	7	15
2	8	17	5
14	0	10	18
6	13	1	4
0	12	11	19

## Deterministic selection - example

First we reorganize:

3	16	7	15
2	8	17	5
14	0	10	18
6	13	1	4
0	12	11	19

Then we sort each column:

0	8	1	4
2	9	7	5
3	12	10	15
6	13	11	18
14	16	17	19

## Deterministic selection - example

First we reorganize:

3	16	7	15
2	8	17	5
14	0	10	18
6	13	1	4
0	12	11	19

Then we sort each column:

0	8	1	4
2	9	7	5
3	12	10	15
6	13	11	18
14	16	17	19

Still need the pivot. Find median of medians

## Deterministic selection - example

0	8	1	4
2	9	7	5
3	12	10	15
6	13	11	18
14	16	17	19

## Deterministic selection - example

0	8	1	4
2	9	7	5
3	12	10	15
6	13	11	18
14	16	17	19

- Call **Median-of-Medians**([3,12,10,15],  $\text{floor}(\text{len}/2) = 2$ )
- Can sort this in linear time.
- Get back 12.
- **12** is our new pivot!

## Deterministic selection - example

Back to our original array! Use the pivot (=12) to break it up into two.

3	2	14	6	0	16	8	9	13	12	7	17	10	1	11	15	5	18	4	19
---	---	----	---	---	----	---	---	----	----	---	----	----	---	----	----	---	----	---	----

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---

12
----

14	16	13	17	15	18	19
----	----	----	----	----	----	----

We know the following:

- $\text{len}(A_{\text{Lower}}) = 12$
- $\text{len}(A_{\text{Upper}}) = 7$
- Want  $k = 10$

## Deterministic selection - example

Back to our original array! Use the pivot (=12) to break it up into two.

3	2	14	6	0	16	8	9	13	12	7	17	10	1	11	15	5	18	4	19
---	---	----	---	---	----	---	---	----	----	---	----	----	---	----	----	---	----	---	----

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---

12
----

14	16	13	17	15	18	19
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We know the following:

- $\text{len}(A_{\text{Lower}}) = 12$
- $\text{len}(A_{\text{Upper}}) = 7$
- Want  $k = 10$

Call **Median-of-Medians**( $A_{\text{Lower}}, 10$ )

## Deterministic selection - example

Then we do this again:

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---



## Deterministic selection - example

Then we do this again:

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---

First we reorganize:

3	9	
2	7	5
6	10	4
0	1	
8	11	

## Deterministic selection - example

Then we do this again:

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---

First we reorganize:

3	9	
2	7	5
6	10	4
0	1	
8	11	

Then we sort each column:

0	1	
2	7	4
3	9	5
6	10	
8	11	

## Deterministic selection - example

0	1	
2	7	4
3	9	5
6	10	
8	11	

## Deterministic selection - example

0	1	
2	7	4
3	9	5
6	10	
8	11	

- Call **Median-of-Medians**([3,9,5],  $\text{floor}(n/2) = 1$ )
- Can sort this in linear time.
- Get back 5.
- **5** is our new pivot!

## Deterministic selection - example

Back to our original array! Use the pivot (=5) to break it up into two (well three).

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---

3	2	0	1	4
---	---	---	---	---

5
---

6	8	9	7	10	11
---	---	---	---	----	----

We know the following:

- $\text{len}(A_{\text{Lower}}) = 5$
- $\text{len}(A_{\text{Upper}}) = 6$
- Want  $k = 10$  (pivot is of rank 5)

## Deterministic selection - example

Back to our original array! Use the pivot (=5) to break it up into two (well three).

3	2	6	0	8	9	7	10	1	11	5	4
---	---	---	---	---	---	---	----	---	----	---	---

3	2	0	1	4
---	---	---	---	---

5
---

6	8	9	7	10	11
---	---	---	---	----	----

We know the following:

- $\text{len}(A_{\text{Lower}}) = 5$
- $\text{len}(A_{\text{Upper}}) = 6$
- Want  $k = 10$  (pivot is of rank 5)

Call **Median-of-Medians**( $A_{\text{Upper}}$ ,  $10 - (5 + 1) = 4$ )

## Deterministic selection - example

Then we do this again:

6	8	9	7	10	11
---	---	---	---	----	----

## Deterministic selection - example

Then we do this again:

6	8	9	7	10	11
---	---	---	---	----	----

First we reorganize:

6	
8	
9	11
7	
10	



## Deterministic selection - example

Then we do this again:

6	8	9	7	10	11
---	---	---	---	----	----

First we reorganize:

6	
8	
9	11
7	
10	

Then we sort each column:

6	
7	
8	11
9	
10	

## Deterministic selection - example

6	
7	
8	11
9	
10	

## Deterministic selection - example

6	
7	
8	11
9	
10	

- Call **Median-of-Medians**([8,11],  $\text{floor}(\text{len}/2) = 1$ )
- Can sort this in linear time.
- Get back 11.
- **11** is our new pivot!

## Deterministic selection - example

Back to our original array! Use the pivot (=11) to break it up into partitions.

6	8	9	7	10	11
---	---	---	---	----	----

6	8	9	7	10
---	---	---	---	----

11
----

We know the following:

- $\text{len}(A_{\text{Lower}}) = 5$
- $\text{len}(A_{\text{Upper}}) = 0$
- Want  $k = 4$  (pivot is of rank 5)

## Deterministic selection - example

Back to our original array! Use the pivot (=11) to break it up into partitions.

6	8	9	7	10	11
---	---	---	---	----	----

6	8	9	7	10
---	---	---	---	----

11
----

We know the following:

- $\text{len}(A_{\text{Lower}}) = 5$
- $\text{len}(A_{\text{Upper}}) = 0$
- Want  $k = 4$  (pivot is of rank 5)

## Deterministic selection - example

Final Step!

6	8	9	7	10
---	---	---	---	----

Can sort in linear time!

6	7	8	9	10
---	---	---	---	----

Return  $\text{Sorted}(A[k] = A[4]) = 10$

# Algorithm for Selection

`select(A, j):`

Form lists  $L_1, L_2, \dots, L_{\lceil n/5 \rceil}$  where  $L_i = \{A[5i-4], \dots, A[5i]\}$

Find median  $b_i$  of each  $L_i$  using brute-force

Find median  $b$  of  $B = \{b_1, b_2, \dots, b_{\lceil n/5 \rceil}\}$

Partition  $A$  into  $A_{\text{less}}$  and  $A_{\text{greater}}$  using  $b$  as pivot

if  $(|A_{\text{less}}|) = j$  return  $b$

else if  $(|A_{\text{less}}|) > j$

return `select`( $A_{\text{less}}$ ,  $j$ )

else

return `select`( $A_{\text{greater}}$ ,  $j - |A_{\text{less}}|$ )

# Algorithm for Selection

**select**( $A, j$ ):

Form lists  $L_1, L_2, \dots, L_{\lceil n/5 \rceil}$  where  $L_i = \{A[5i-4], \dots, A[5i]\}$

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**Find median  $b$  of  $B = \{b_1, b_2, \dots, b_{\lceil n/5 \rceil}\}$**

Partition  $A$  into  $A_{\text{less}}$  and  $A_{\text{greater}}$  using  $b$  as pivot

if ( $|A_{\text{less}}| = j$ ) return  $b$

else if ( $|A_{\text{less}}| > j$ )

    return **select**( $A_{\text{less}}, j$ )

else

    return **select**( $A_{\text{greater}}, j - |A_{\text{less}}|$ )

How do we find median of  $B$ ?



# Algorithm for Selection

**select**( $A, j$ ):

Form lists  $L_1, L_2, \dots, L_{\lceil n/5 \rceil}$  where  $L_i = \{A[5i-4], \dots, A[5i]\}$

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**Find median  $b$  of  $B = \{b_1, b_2, \dots, b_{\lceil n/5 \rceil}\}$**

Partition  $A$  into  $A_{\text{less}}$  and  $A_{\text{greater}}$  using  $b$  as pivot

if ( $|A_{\text{less}}| = j$ ) return  $b$

else if ( $|A_{\text{less}}| > j$ )

    return **select**( $A_{\text{less}}, j$ )

else

    return **select**( $A_{\text{greater}}, j - |A_{\text{less}}|$ )

How do we find median of  $B$ ? Recursively!

## Running time of deterministic median selection

---

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$$T(n) \leq T(\lceil n/5 \rceil) + \max\{T(|A_{\text{less}}|), T(|A_{\text{greater}}|)\} + O(n)$$

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From Lemma,

$$T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 \rfloor) + O(n)$$

and

$$T(n) = O(1) \quad n < 10$$

## Running time of deterministic median selection

$$T(n) \leq T(\lceil n/5 \rceil) + \max\{T(|A_{\text{less}}|), T(|A_{\text{greater}}|)\} + O(n)$$

From Lemma,

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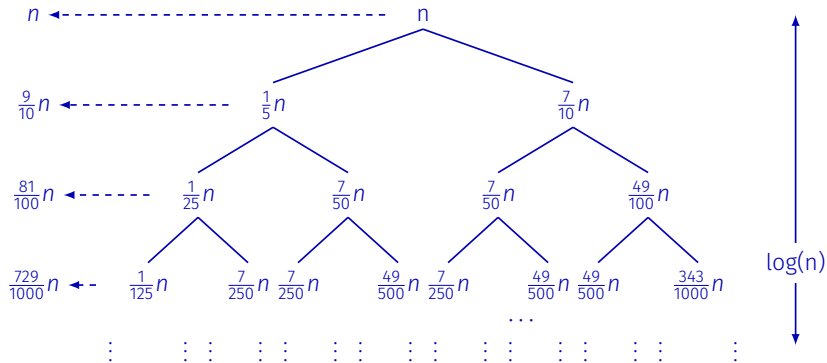
and

$$T(n) = O(1) \quad n < 10$$

**Exercise:** show that  $T(n) = O(n)$

## Recursion tree fill-in

If the workload is decreasing at every level, then total work is dominated by the root.



$$T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 \rfloor) + O(n) = O(n)$$

## What about QuickSort?

How would we use the median of medians approach for quicksort?

## What about QuickSort?

How would we use the median of medians approach for quicksort?

Just use MoM to find pivot!

- Original recurrence:  $T(n) = T(k - 1) + T(n - k) + O(n)$
- With MoM:  $T(n) = T(\frac{3}{10}n) + T(\frac{7}{10}n) + O(n) + O(n)$



# Median of Medians Algorithm

Due to: M. Blum, R. Floyd, D. Knuth, V. Pratt, R. Rivest, and R. Tarjan.

“Time bounds for selection”.

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All except Vaughan Pratt! **Favorite Knuth quote:** He once warned a correspondent, “Beware of bugs in the above code; I have only proved it correct, not tried it.”

## Takeaway Points

- Recursion tree method and guess and verify are the most reliable methods to analyze recursions in algorithms.
- Recursive algorithms naturally lead to recurrences.
- Some times one can look for certain type of recursive algorithms (reverse engineering) by understanding recurrences and their behavior.

Problem statement: Multiplying  
numbers + a slow algorithm

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## The Problem: Multiplying numbers

Given two large positive integer numbers  $b$  and  $c$ , with  $n$  digits, compute the number  $b * c$ .

## Egyptian multiplication: 1850BC (3870 years ago?)

76 | 35 |



## Egyptian multiplication: 1850BC (3870 years ago?)

$$\begin{array}{c|c|c} 76 & 35 & \\ 76 & 34 + 1 & 76 \end{array}$$

## Egyptian multiplication: 1850BC (3870 years ago?)

$$\begin{array}{c|c|c} 76 & 35 & \\ 76 & 34 + 1 & 76 \\ 76 & 34 & \end{array}$$

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152
152	16	

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152
152	16	
304	8	

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152
152	16	
304	8	
608	4	

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152
152	16	
304	8	
608	4	
1216	2	



## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	34 + 1	76
76	34	
152	17	
152	16 + 1	152
152	16	
304	8	
608	4	
1216	2	
2432	1	2432

## Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152
152	16	
304	8	
608	4	
1216	2	
2432	1	2432
		2660

# The problem: Multiplying Numbers

**Problem** Given two  $n$ -digit numbers  $x$  and  $y$ , compute their product.

## Grade School Multiplication

Compute “partial product” by multiplying each digit of  $y$  with  $x$  and adding the partial products.

$$\begin{array}{r} 3141 \\ \times 2718 \\ \hline 25128 \\ 3141 \phantom{0} \\ 21987 \phantom{00} \\ 6282 \phantom{000} \\ \hline 8537238 \end{array}$$

## Time Analysis of Grade School Multiplication

- Each partial product:  $\Theta(n)$
- Number of partial products:  $\Theta(n)$
- Addition of partial products:  $\Theta(n^2)$
- Total time:  $\Theta(n^2)$

## Multiplication using Divide and Conquer

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# Divide and Conquer

Assume  $n$  is a power of 2 for simplicity and numbers are in decimal.

Split each number into two numbers with equal number of digits

- $b = b_{n-1}b_{n-2} \dots b_0$  and  $c = c_{n-1}c_{n-2} \dots c_0$
- $b = b_{n-1} \dots b_{n/2} 0 \dots 0 + b_{n/2-1} \dots b_0$
- $b(x) = b_L x + b_R$ , where  $x = 10^{n/2}$ ,  $b_L = b_{n-1} \dots b_{n/2}$  and  $b_R = b_{n/2-1} \dots b_0$
- Similarly  $c(x) = c_L x + c_R$  where  $c_L = c_{n-1} \dots c_{n/2}$  and  $c_R = c_{n/2-1} \dots c_0$

## Example

$$\begin{aligned} 1234 \times 5678 &= (12x + 34) \times (56x + 78) && \text{for } x = 100. \\ &= 12 \cdot 56 \cdot x^2 + (12 \cdot 78 + 34 \cdot 56)x + 34 \cdot 78. \end{aligned}$$

$$\begin{aligned} 1234 \times 5678 &= (100 \times 12 + 34) \times (100 \times 56 + 78) \\ &= 10000 \times 12 \times 56 \\ &\quad + 100 \times (12 \times 78 + 34 \times 56) \\ &\quad + 34 \times 78 \end{aligned}$$

# Divide and Conquer for multiplication

Assume  $n$  is a power of 2 for simplicity and numbers are in decimal.

- $b = b_{n-1}b_{n-2} \dots b_0$  and  $c = c_{n-1}c_{n-2} \dots c_0$
- $b \equiv b(x) = b_Lx + b_R$   
where  $x = 10^{n/2}$ ,  $b_L = b_{n-1} \dots b_{n/2}$  and  $b_R = b_{n/2-1} \dots b_0$
- $c \equiv c(x) = c_Lx + c_R$  where  $c_L = c_{n-1} \dots c_{n/2}$  and  $c_R = c_{n/2-1} \dots c_0$



# Divide and Conquer for multiplication

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- $c \equiv c(x) = c_Lx + c_R$  where  $c_L = c_{n-1} \dots c_{n/2}$  and  $c_R = c_{n/2-1} \dots c_0$

Therefore, for  $x = 10^{n/2}$ , we have

$$\begin{aligned}bc &= b(x)c(x) = (b_Lx + b_R)(c_Lx + c_R) \\&= b_Lc_Lx^2 + (b_Lc_R + b_Rc_L)x + b_Rc_R \\&= 10^n b_Lc_L + 10^{n/2}(b_Lc_R + b_Rc_L) + b_Rc_R\end{aligned}$$

$$bc = 10^n b_L c_L + 10^{n/2} (b_L c_R + b_R c_L) + b_R c_R$$

4 recursive multiplications of number of size  $n/2$  each plus 4 additions and left shifts (adding enough 0's to the right)

$$bc = 10^n b_L c_L + 10^{n/2} (b_L c_R + b_R c_L) + b_R c_R$$

4 recursive multiplications of number of size  $n/2$  each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \quad T(1) = O(1)$$

$$bc = 10^n b_L c_L + 10^{n/2} (b_L c_R + b_R c_L) + b_R c_R$$

4 recursive multiplications of number of size  $n/2$  each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \quad T(1) = O(1)$$

$T(n) = \Theta(n^2)$ . No better than grade school multiplication!

## Faster multiplication: Karatsuba's Algorithm

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# A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers:  $(a + bi)$  and  $(c + di)$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

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How many multiplications do we need?

Only 3! If we do extra additions and subtractions.

Compute  $ac$ ,  $bd$ ,  $(a + b)(c + d)$ . Then



## Gauss technique for polynomials

$$p(x) = ax + b \quad \text{and} \quad q(x) = cx + d.$$

$$p(x)q(x) = acx^2 + (ad + bc)x + bd.$$

## Gauss technique for polynomials

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$$p(x)q(x) = acx^2 + ((a + b)(c + d) - ac - bd)x + bd.$$

$$bc = b(x)c(x) = (b_Lx + b_R)(c_Lx + c_R)$$

## Improving the Running Time

$$\begin{aligned}bc &= b(x)c(x) = (b_Lx + b_R)(c_Lx + c_R) \\ &= b_Lc_Lx^2 + (b_Lc_R + b_Rc_L)x + b_Rc_R\end{aligned}$$

## Improving the Running Time

$$\begin{aligned}bc &= b(x)c(x) = (b_Lx + b_R)(c_Lx + c_R) \\&= b_Lc_Lx^2 + (b_Lc_R + b_Rc_L)x + b_Rc_R \\&= (b_L * c_L)x^2 + \left( (b_L + b_R) * (c_L + c_R) - b_L * c_L - b_R * c_R \right)x \\&\quad + b_R * c_R\end{aligned}$$

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Recursively compute only  $b_Lc_L, b_Rc_R, (b_L + b_R)(c_L + c_R)$ .

# Improving the Running Time

$$\begin{aligned}bc &= b(x)c(x) = (b_Lx + b_R)(c_Lx + c_R) \\&= b_Lc_Lx^2 + (b_Lc_R + b_Rc_L)x + b_Rc_R \\&= (b_L * c_L)x^2 + \left( (b_L + b_R) * (c_L + c_R) - b_L * c_L - b_R * c_R \right)x \\&\quad + b_R * c_R\end{aligned}$$

Recursively compute only  $b_Lc_L$ ,  $b_Rc_R$ ,  $(b_L + b_R)(c_L + c_R)$ .

## Time Analysis

Running time is given by

$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means  $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

# State of the Art

Schönhage-Strassen 1971:  $O(n \log n \log \log n)$  time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007:  $O(n \log n 2^{O(\log^* n)})$  time

**Conjecture:** There is an  $O(n \log n)$  time algorithm.