Wheels.

Rhea Wolski

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Why was this talk (mostly) written ~~last night~~ an hour ago?

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Incredibly cruel and mean criticism

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"I wouldn't say it was perfect... A perfect talk involves insulting much more than 2 people" - Luke

"You weren't ... accurate... that wasn't ... better" - Michael

"It was so boring, I walked out in 5 minutes" - Max Orchard

"Did you know that integration bee is next wednesday and Rhea keeps forgetting to post an annoucement" - Gabe

"Only slightly worse than watching paint dry for 25 minutes" - unknown incredibly mean student.

Examples of Algebraic Structures

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The big important well-known algebraic structures:

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The obscure, unimportant algebraic structures:

- groups
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- fields

Build your own Algebraic Structure

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For example

- Binary operations take in two elements and output a single element. E.g. group multiplication.
- Unary operations take a single element and output a single element. E.g. a function that sends a group element to its inverse.
- Nullary operations take no elements and output a single element. E.g. The identity of a group.

Entering the world of Universal Algebra

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Legal Axioms

$$\forall x, y, z \in A,$$

$$(x * y) * z = x * (y * z)$$

$$x * y = y * x$$

$$x \times (y + z) = x \times y + x \times z$$

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$$x \times (y + z) = x \times y + x \times z$$

Evil Crime Axioms

$$\exists e \text{ st } x * e = e * x = x \ \forall x \in x$$
$$\forall x \in A, \exists x^{-1} \text{ s.t. } x \times x^{-1} = 1$$
$$\forall x \in A \setminus \{0\}, \exists x^{-1} \text{ such that:}$$
$$x \times x^{-1} = 1$$

Are Groups a variety?

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A group is a set G along with a binary operation $*$ satisfying the following three axioms:

1. $\forall\, x, y, z \in G$, $(x * y) * z = x * (y * z)$
2. $\exists\, e \in G$ such that $\forall x \in G$, $x * e = e * x = x$
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A Group is a set G along with three operations, a binary operation $*$, a unary operation Inv and a nullary operation e, satisfying the following three axioms:

1. $\forall\, x, y, z \in G,\ (x * y) * z = x * (y * z)$
2. $\forall\, x \in G,\ x * e = e * x = x$
3. $\forall\, x \in G,\ x * \mathrm{Inv}(x) = \mathrm{Inv}(x) * x = e$

So Groups are a variety!

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Adapts well into category theory (for instance, defining a group object on a non-set).

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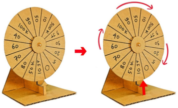
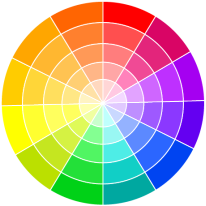
We cannot possibly rewrite this with exclusively equality, so Fields fail to form a Variety unless we can invert 0.

Why can't we invert 0 in a field?

- $0 \cdot b = (a - a) \cdot b = ab - ab = 0$
- $1 = \frac{1}{0} \cdot 0 = 0$, but 1 and 0 must be distinct.

Introducing the Wheel





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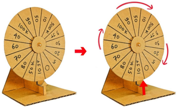

Introducing the Wheel

A Wheel is a set W alongside two binary operations $+$ and $\cdot$, one unary operation $/$ and two nullary operations 1 and 0 satisfying the following axioms:

- $a + b = b + a$
- $a \cdot b = b \cdot a$
- $(a + b) + c = a + (b + c)$
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $a + 0 = 0 + a = a$
- $a \cdot 1 = 1 \cdot a = a$
- $//a = a$
- $/(x \cdot y) = /x \cdot /y$
- $(x + y) \cdot z + 0 \cdot z = x \cdot y + y \cdot z$
- $(x + y \cdot z) \cdot /y = x \cdot /y + z + 0 \cdot y$
- $0 \cdot 0 = 0$
- $(x + 0 \cdot y) \cdot z = x \cdot z + 0 \cdot y$
- $/(x + 0 \cdot y) = /x + 0 \cdot y$
- $0 \cdot /0 + x = 0 \cdot /0$

Fun(?) facts about wheels

We do not necessarily have $0 \cdot x = 0$

It is not necessarily true that $x \cdot /x = 1$

We do not necessarily have $x - x = 0$, in fact in general, $x - x = 0 \cdot x^2$

Is wheel theory useful?

Is wheel theory useful?

No

Examples of wheels

Wheel of fractions [edit]

Let A be a commutative ring, and let S be a multiplicative submonoid of A. Define the congruence relation $\sim_S$ on $A \times A$ via

$(x_1, x_2) \sim_S (y_1, y_2)$ means that there exist $s_x, s_y \in S$ such that $(s_x x_1, s_x x_2) = (s_y y_1, s_y y_2)$.

Define the *wheel of fractions* of A with respect to S as the quotient $A \times A \,/\!\sim_S$ (and denoting the equivalence class containing (x_1, x_2) as $[x_1, x_2]$) with the operations

$0 = [0_A, 1_A]$ (additive identity)

$1 = [1_A, 1_A]$ (multiplicative identity)

$/[x_1, x_2] = [x_2, x_1]$ (reciprocal operation)

$[x_1, x_2] + [y_1, y_2] = [x_1 y_2 + x_2 y_1, x_2 y_2]$ (addition operation)

$[x_1, x_2] \cdot [y_1, y_2] = [x_1 y_1, x_2 y_2]$ (multiplication operation)

In general, this structure is not a ring unless it is trivial, as $0x \neq 0$ in the usual sense – here with $x = [0, 0]$ we get $0x = [0, 0]$, although that implies that $\sim_S$ is an improper relation on our wheel W.

This follows from the fact that $[0, 0] = [0, 1] \implies 0 \in S$, which is also not true in general.[1]

Projective line and Riemann sphere [edit]

The special case of the above starting with a field produces a projective line extended to a wheel by adjoining a bottom element noted $\bot$, where $0/0 = \bot$. The projective line is itself an extension of the original field by an element ∞, where $z/0 = \infty$ for any element $z \neq 0$ in the field. However, $0/0$ is still undefined on the projective line, but is defined in its extension to a wheel.

Starting with the real numbers, the corresponding projective "line" is geometrically a circle, and then the extra point $0/0$ gives the shape that is the source of the term "wheel". Or starting with the complex numbers instead, the corresponding projective "line" is a sphere (the Riemann sphere), and then the extra point gives a 3-dimensional version of a wheel.

Figure: Stolen from wikipedia, CC-BY-SA 4.0

The talk is over now.

Figure: A couple of wheel enjoyers