

# Colouring Knots with Primes

Ed Hawkins (any/all)

13 March 2025

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- 4 Saying "Knot" and "Not" is confusing...





## Definition: Knot

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## New (Better) "Definition"

A knot is a line (piece of string) we move around in 3D space, and then glue its ends together. We do not care about the length of said line, only the way in which it becomes "knotted" (I'm not defining what I mean by "knotted").

# Knot Projections

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A **Knot Projection** is a projection of the embedded knot onto the plane. At any crossing of the knot, we indicate an over strand.

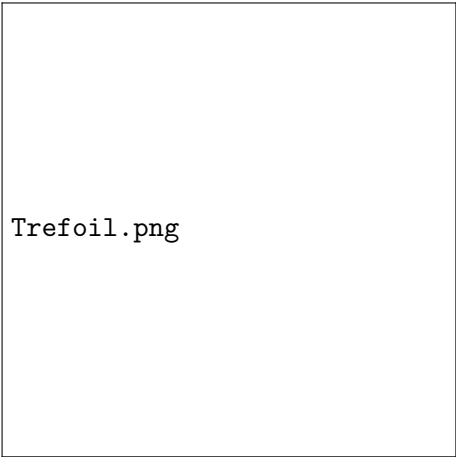
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This gives us a convenient way to represent knots, without a mess of (possibly unclear) instructions as to how to tie them.

# Examples of Knots

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Trefoil.png

Figure: Knot  $3_1$  ; the Trefoil



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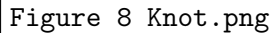
A square frame containing the text "Figure 8 Knot.png". The actual knot diagram is not visible, only the placeholder text.

Figure 8 Knot.png

Figure: Knot  $4_1$  ; the Figure-Eight knot

# Examples of Knots

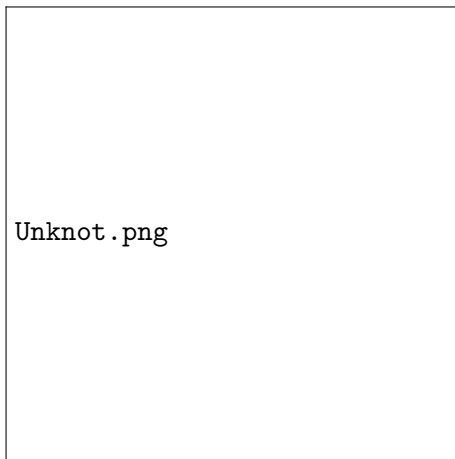


Figure: Knot  $0_1$  ; the Unknot

# Knot Examples

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But we need to introduce a few more tools and ideas before we can.

# Knot Equivalence

## Definition: Ambient Isotopy

We say two knots are equivalent if we can continuously transform one into the other without cutting the line, or passing through itself; i.e. can we "untie" them until they look the same? If so, we then say they are **Ambient-Isotopic** to one another.

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# Reidemeister Moves

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# Reidemeister moves example

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Figure 8 Knot transformation.jpeg

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# Knot Invariants



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## Definition: Knot Invariant

A **Knot Invariant**  $\varphi(K)$  is some function defined for all knots such that if two knots  $K_1$  and  $K_2$  are equivalent, then  $\varphi(K_1) = \varphi(K_2)$ .

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We importantly care about the contrapositive of this statement, being that if  $\varphi(K_1) \neq \varphi(K_2)$  then  $K_1$  and  $K_2$  are different knots.

# Knot Colourability

## Colourability

We say a knot diagram is colourable if we can assign each strand one of three labels (colours) such that the following hold:

- ① We use at least two distinct colours, and
- ② At any crossing, if two colours appear, then all three colours appear.

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We now prove that this is infact a knot invariant.

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Coloured Reidemeister moves.png

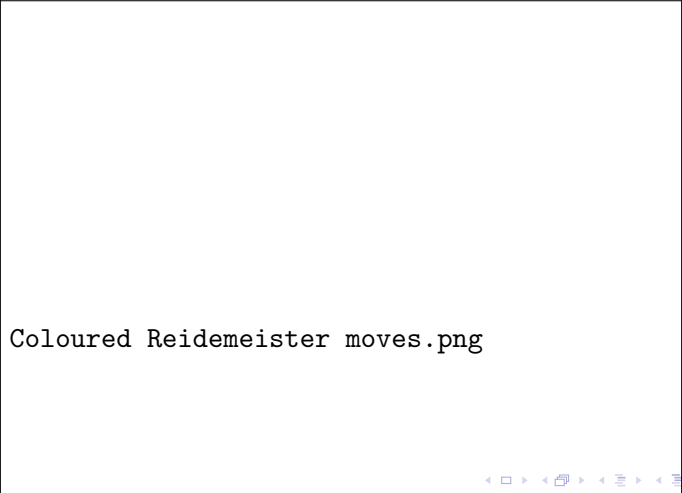
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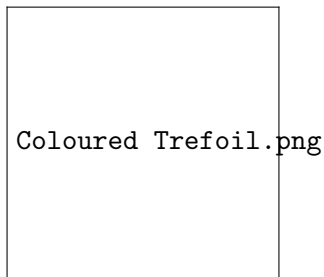


Figure: Coloured Trefoil

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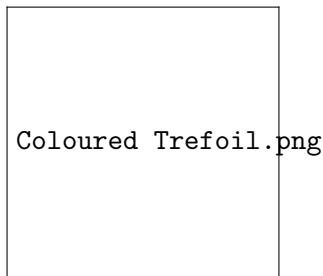


Figure: Coloured Trefoil

So since we've shown that colourability is a knot invariant, we now know the trefoil and the unknot are actually different knots.

Yippee.png





Recall our definition of colourability:

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- 1 We use at least two distinct colours, and
- 2 At any crossing, if two colours appear, then all three colours appear.

Observe that if we colour our strands with elements from  $\{0, 1, 2\} = \mathbb{Z}_3$ , condition (2) is equivalent to:

For  $x, y, z \in \mathbb{Z}_3$ , at any crossing with over strand coloured  $z$ , and under-strands coloured  $x$  and  $y$ ,

$$x + y \equiv 2z \pmod{3}$$

## p-Colourability

We say a knot diagram is  $p$ -colourable for some prime  $p \geq 3$  if we can label (colour) it using elements from  $\mathbb{Z}_p$  such that:

- 1 At least two distinct elements are used, and
- 2 At any crossing with over-strand coloured  $z$  and under-strands coloured  $x$  and  $y$ ,

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This is a knot invariant  $\forall$  primes  $p \geq 3$  (trust)

# p-Colourable Example

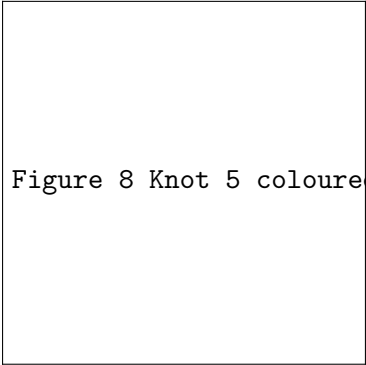


Figure 8 Knot 5 coloured.png

Figure: The Figure-8 Knot is 5-colourable

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If we then label each of our  $n$  crossings  $(1, 2, 3, \dots, n)$ , and each of our  $n$  strands  $(x_1, x_2, x_3, \dots, x_n)$ , we obtain a system of  $n$  linear equations, corresponding to each of the  $n$  crossings.



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We construct such a matrix carefully because of the unique case where the over strand is one of the under strands.

# Constructing Matrices

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- 3 Repeat  $\forall i \in \{1, 2, 3, \dots, n\}$
- 4 Remove one arbitrary row and column

This actually **isn't** a knot invariant, but the following is...

# Knot Determinants



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Given a knot  $K$  and a corresponding matrix constructed as above  $M_K$ , we define the Knot Determinant of  $K$  as

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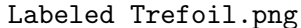
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But first, lets do an example.

# Knot Determinate Example




Labeled Trefoil.png

- 1 We label a knot and its crossings, and create a  $3 \times 3$  matrix of all 0's:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure:  $3_1$ ; the Trefoil

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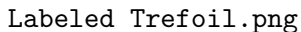
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② We add 2 to each over-strand entry:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Figure:  $3_1$ ; the Trefoil

# Knot Determinate Example




Labeled Trefoil.png

- 3 And add -1 to each under-strand entry:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Figure:  $3_1$ ; the Trefoil

# Knot Determinate Example



Labeled Trefoil.png

- ④ We remove an arbitrary row and column, and compute the determinate:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$\therefore \det(3_1) = 3$$

Figure:  $3_1$ ; the Trefoil



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Which isn't immediately obvious :(

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We are looking for the null-space of this matrix ( $M_K$ ), as that is the solution to the system of equations we generated before, and we want a non-trivial NS.

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# (At least) Countably infinite knots

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We will now prove this.

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Consider a knot with  $n \geq 3$  crossings s.t.  $n$  is odd, and each strand is the over-strand of **exactly** one crossing that is not said strands end.

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Consider a knot with  $n \geq 3$  crossings s.t.  $n$  is odd, and each strand is the over-strand of **exactly** one crossing that is not said strands end.

The  $n \times n$  matrix corresponding to such a knot is given by:

$$\begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}$$

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We choose to remove the  $n^{\text{th}}$  row and column to make our lives easier.

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This leaves us with the  $n - 1 \times n - 1$  matrix:

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Now we need only show that such a knot exists for each odd  $n \geq 3$ , since the determinate of our knots is an invariant.

# Intermediate Definition



## Definition: Braid

A **Braid** is a set of  $n$  strings which can be interpreted as attached to a horizontal bar at each end. Each string is positioned such that it intersects a horizontal plane exactly once.

By connecting each of the strands on the top bar with corresponding strands on the bottom bar, we obtain a knot or link.

For a knot  $K$  obtained this way, we say the braid is the *braid representation* of  $K$ .

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Ok now we can get back to our proof.

# Back to Proving (at least) countably infinite knots

We show such a knot exists using the following braid, which is just a fancy way of representing a knot.

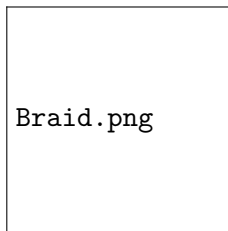


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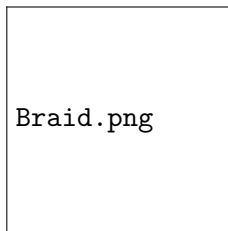


Figure: Braid 1

We attach  $n$  of this braid together "on-top" of each other.

# Proving (at least) countably infinite knots

n-braid.png

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So now you know:

- 1 What a knot is, and what it means for two of them to be the same
- 2 What it means for two knots to be different, and some tools to help determine this
- 3 That there are (at least) a countably infinite number of knots

- ① C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, W.H. Freeman and Company, New York (1994).
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