Why Knot?

Turning & Winding Through Knots via Grid Diagrams

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MSS Maths Talks

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Supervisors

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Definition & Convention

Definition

A *knot*, K, is a smooth embedding of S^1 into S^3. All knots considered will have the typical orientation [5].

Remark

Grid diagrams exist and generally work nicely for links as well, but were not considered for reasons that will become very evident shortly.

Torus Knots

Definition

Let p, q be relatively prime integers. Then,

$$T_{p,q} = \left\{(z_1, z_2) \in \mathbb{C}^2 : z_1\overline{z_1} + z_2\overline{z_2} = 1, z_1^p z_2^q = 0\right\}$$

is the *torus knot* $T_{p,q}$ (or the (p, q) torus knot).

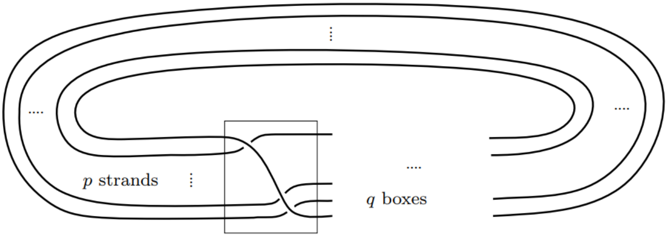


Figure 3: Reproduced from [5].

Torus Knot Examples

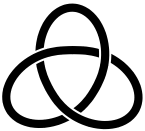


Figure 4: $T_{2,3} = 3_1$. Made with KnotPlot [6].

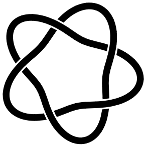


Figure 5: $T_{2,5} = 5_1$. Made with KnotPlot [6].

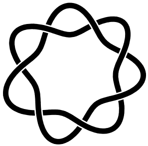
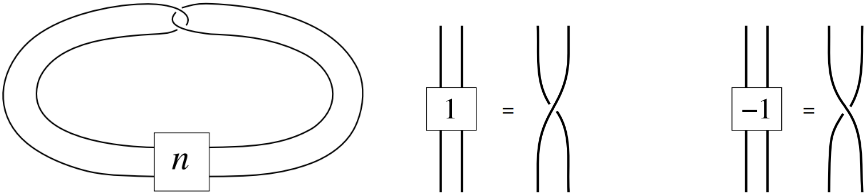


Figure 6: $T_{2,7} = 7_1$. Made with KnotPlot [6].

Twist Knots

Definition

Let $n \in \mathbb{Z}$. The *twist knot*, W_n, is the knot created by n many half-twists of a closed loop that is then linked at the ends [5].



(a) General configuration of W_n. **(b)** Convention for orientation of twists.

Figure 7: Reproduced from [5].

Twist Knot Examples

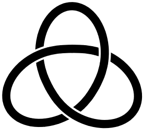


Figure 8: $W_1 = 3_1$. Made with KnotPlot [6].

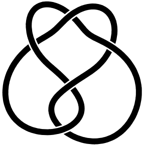


Figure 9: $W_3 = 5_2$. Made with KnotPlot [6].

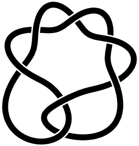


Figure 10: $W_5 = 7_2$. Made with KnotPlot [6].

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Planar Grid Diagrams

Definition

A *planar grid diagram* (or simply just a grid), $\mathbb{G}$, is an $n \times n$ grid on the plane with two ordered tuples of markings each of size n, commonly denoted $\mathbb{X}$ and $\mathbb{O}$ [2].

"Sudoku" Rule

Each row and column must have exactly one X marking and one O marking.

Conventions

- Vertical lines are always overpasses.
- Indexing starts at 0, i.e. $i \in \{0, 1, \ldots, n-1\}$.
- Orientation is $X \to O$ horizontally and $O \to X$ vertically.
- Grids are read left to right, bottom to top.

Remark

Item 1 is actually more than a convention.

Note

These are typically open to alteration and often dependent on the current use case.

Examples of Grids

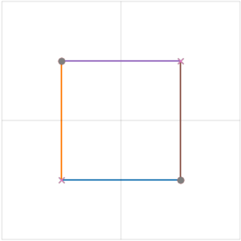

Figure 11: Grid diagram of the unknot. Created with GridPythonModule [2].

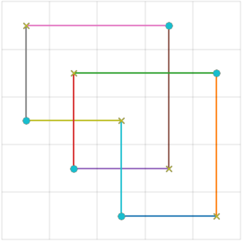

Figure 12: Grid diagram of the trefoil knot. Created with GridPythonModule [2].

Examples of Grids Continued

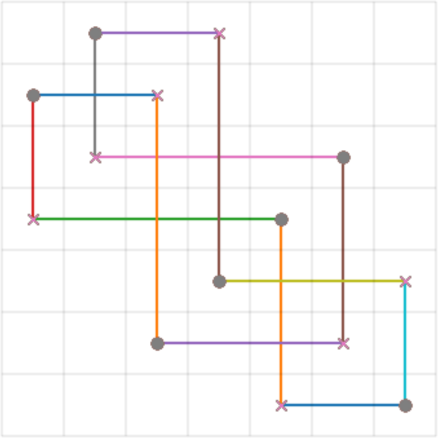

Figure 13: Grid diagram of the 5_1 knot. Created with GridPythonModule [2].

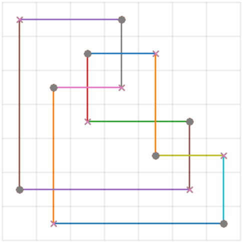

Figure 14: Grid diagram of the 5_2 knot. Created with GridPythonModule [2].

Commutation

Definition

A *row commutation* (resp. *column commutation*) is the swapping of two rows (resp. columns) that are either disjoint or strictly contained in one another.

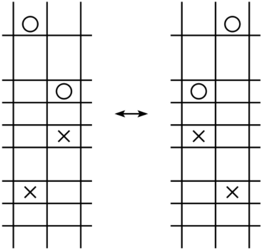

Figure 15: Example of column commutation of strictly contained columns. Reproduced from [5].

Stabilisation

Definition

A *stabilisation* is a move from a $n \times n$ grid to a $(n+1) \times (n+1)$ grid by separating a row and column in two according to the rule below. A *destabilisation* is the inverse of a stabilisation.

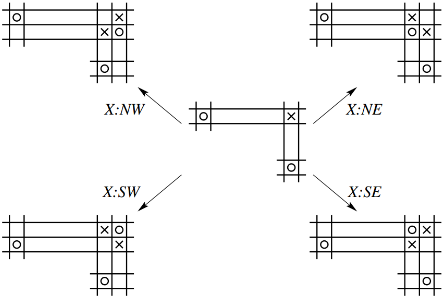

Figure 16: Example of stabilisation at X marking. Reproduced from [5].

Cromwell's Theorem

Theorem (Cromwell)

Two grid diagrams represent equivalent links if and only if there exists a finite sequence of commutations, stabilisations, and destabilisations (sometimes called grid moves and/or a subset of Cromwell moves) that transform one into the other.

Existence

Theorem

Every oriented link in S^3 can be represented by a grid diagram [5].

Existence

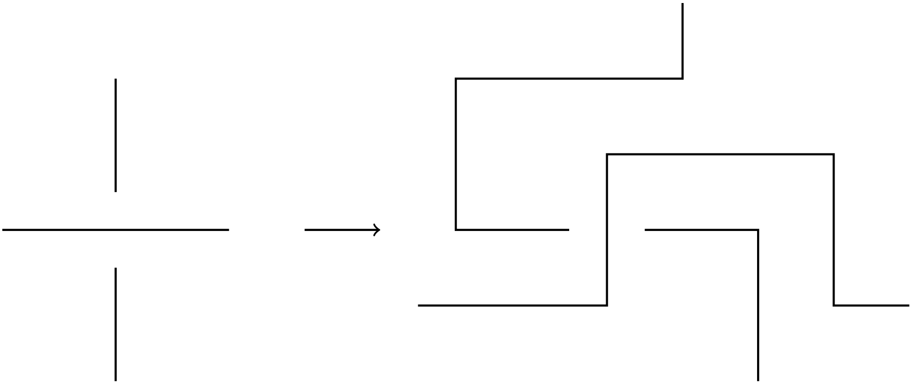

Figure 17: Reproduced from [5] with Tikz.

Existence

Theorem

Every oriented link in S^3 can be represented by a grid diagram [5].

Proof Sketch.

Begin by approximating the link with a PL-embedding such that the projection consists only of vertical and horizontal segments. If a crossing has the vertical segment as an overpass (as prescribed), then leave it unchanged. If the horizontal segment is the overpass, then modify the diagram as per fig. 17. Repeat as necessary. Finally, adjust the position of the segments such that no vertical (or horizontal) segments are collinear. Then, fill in with the X and O markings as per the "Sudoku" rule and following the convention for orientation. The result is a grid diagram representing the link. ∎

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Intuition

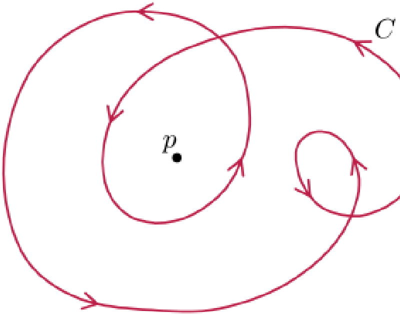

Figure 18: Created by [3]. Example of a curve with various winding numbers depending on the point chosen, and specifically winding number 2 at the point p.

Details

Definition

Let C be a closed, piecewise linear, oriented curve in the plane and p a point such that $p \in \mathbb{R}^2 \setminus C$. The *winding number*, $w(p, C)$, of a C around p is found by drawing a ray from p to ∞ and counting the number of algebraic intersections of the ray with C [5]. This is independent of the choice of ray.

Note

There are quite a few equivalent definitions. Examples include covering maps, a combinatorial rule (see [4] & [1]) and line integrals.

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Turning Number

Definition

The *turning number Tn* of a curve is the winding number with respect to the tangent vector of the path itself. Orientation is given in the usual way.

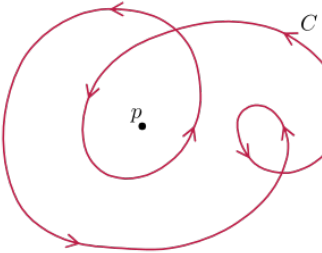
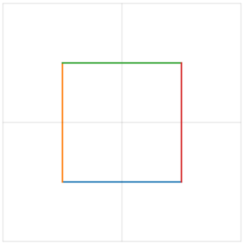


Figure 19: The turning number is a global parameter of the curve. Here C has $Tn = 3$. Diagram by [3].

Polygons

Idea

Treat the grid as a closed polygon.



Remark

For a simple polygon, the Jordan curve theorem implies that the turning number must be 1*. However, we rarely deal with a simple polygon when it comes to grid diagrams.*

Algorithm

Basic Idea

Walk along the grid, noting the sign of each turn.

Details

1. Grids are stored as lists of lists (recall $\mathbb{X}$ and $\mathbb{O}$) so we already have the vertices of the polygon.
2. Run over $i \in \{0, 1, \ldots, n-1\}$ where n is the grid size.
3. Check "x" value (in the Cartesian sense) of the ith element in $\mathbb{X}$ and $\mathbb{O}$.
4. If element arises first in $\mathbb{O}$, then it is to the left of the ith value, and vice versa.
5. Simply count left and right turns and sum up (assigning orientation per convention).

Brief Summary

Two Approaches

1. Sample randomly generated knots
2. Randomly shuffle predetermined knots

Randomly Generated Knots

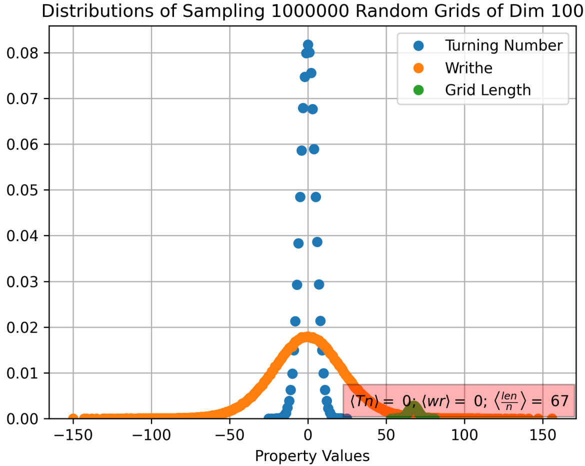


Figure 20: Checking distribution in comparison to other known quantities.

Random Shuffling of Predetermined Knots

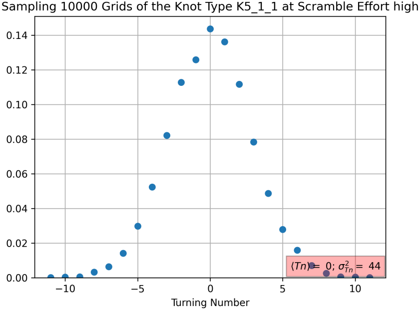


Figure 21: Sampling ten thousand random scrambles of 5_1.

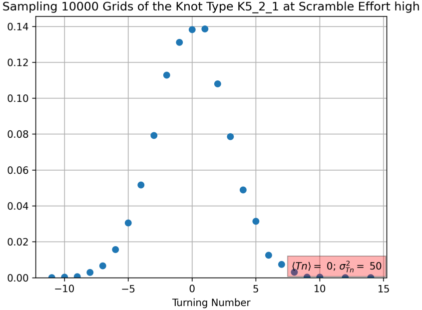


Figure 22: Sampling ten thousand random scrambles of 5_2.

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[2] Agnese Barbensi and Daniele Celoria. "GridPyM: A Python module to handle grid diagrams". In: *Journal of Software for Algebra and Geometry* 14.1 (2024), pp. 31–39. ISSN: 1948-7916. DOI: 10.2140/jsag.2024.14.31. URL: http://dx.doi.org/10.2140/jsag.2024.14.31.

[3] Jim.belk. *Winding Number Around Point.svg*. Figure. 2007. URL: https://commons.wikimedia.org/wiki/File:Winding_Number_Around_Point.svg.

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[4] August Ferdinand Möbius. "Ueber die bestimmung des inhaltes eines polyëders". In: *Gesammelte Werke* 2 (1865), pp. 473–512.

[5] Peter S. Ozsvath, Andras I. Stipsicz, and Zoltan Szabo. *Grid Homology for Knots and Links*. Vol. 208. Mathematical Surveys and Monographs. American Mathematical Society, 2015.

[6] Robert Scharein. *KnotPlot*. Computer Program. 2022. URL: https://knotplot.com/.