1 History & Context

- Based on a YouTube video by Michael Penn
- Fractional derivatives have many different and not necessarily equivalent definitions
- This construction is "fairly easy" to follow
  - It is equivalent to the Riemann-Liouville construction
  - They define a fractional integral first and then reverse it
- Important work on this is also done by Fourier
- There is recent work by Caputo-Fabrizio (2015) and Atangana–Baleanu (2016) both introducing new constructions of fractional derivatives

2 Laplace Transform

Definition (Laplace Transform)**.** For "nice enough" functions f, define

$$\mathcal{L}(f) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t, \quad s \in \mathbb{C}$$

- Linear operator
  - Integrals are linear over
  - Choose f "nice enough" so that integral converges
- Technically, bounded as an operator over $L^2(\mathbb{R}_+)$
  - $\|\mathcal{L}\|_{op} = \sqrt{\pi}$
  - This is hard to do

Definition (Gamma Function)**.** For $z \in \mathbb{C}$, define

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$$

- This is an analytic continuation of factorial: $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$

2.1 Laplace E.g.s

2.1.1 $\mathcal{L}(1)$:

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

- We can clearly then do constants by $\mathcal{L}(c) = \frac{c}{s}, c \in \mathbb{R}$.

2.1.2 $\mathcal{L}(t)$:

$$\mathcal{L}(t) = \underbrace{\int_0^\infty te^{-st}\,\mathrm{d}t = -\frac{t}{s}e^{-st}\Big|_0^\infty - \frac{1}{s^2}e^{-st}\Big|_0^\infty}_{\text{By Parts: DI Method}} = \frac{1}{s^2}$$

- This can generalise to $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, n \in \mathbb{N}$
- Proof is obvious by induction but also probably can just be brute forced.
- What does the inverse tells us here?

2.2 Properties of $\mathcal{L}$:

Claim 1.

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Proof.

$$\begin{aligned}
\mathcal{L}(t^\alpha) &= \int_0^\infty t^\alpha e^{-st}\,\mathrm{d}t \\
\text{u-sub.}\ & \boxed{\begin{aligned} u = st &\implies t = \frac{u}{s} \\ \implies \mathrm{d}t &= \frac{\mathrm{d}u}{s} \\ \implies &\begin{cases} t = 0 \rightsquigarrow u = 0 \\ t \to \infty \rightsquigarrow u \to \infty \end{cases} \end{aligned}} \\
&= \int_0^\infty \frac{u^\alpha}{s^\alpha} e^{-u}\frac{\mathrm{d}u}{s} \\
&= \frac{1}{s^{\alpha+1}} \underbrace{\int_0^\infty u^{(\alpha+1)-1}e^{-u}\,\mathrm{d}u}_{=\Gamma(\alpha+1)} \\
&= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}
\end{aligned}$$

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- Again, what does this imply for the inverse?

Claim 2.

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

Proof.

$$\begin{aligned}
\mathcal{L}(f') &= \int_0^\infty f'(t)e^{-st}\,\mathrm{d}t \\
&= \int_0^\infty \underbrace{e^{-st}}_{u}\underbrace{f'(t)}_{\mathrm{d}v} \\
&= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\
&= -f(0) + s\mathcal{L}(f)
\end{aligned}$$

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Claim 3.

$$\mathcal{L}(\delta) = 1$$

Proof.

$$\mathcal{L}(\delta) = \int_0^\infty \delta(t)e^{-st}\,\mathrm{d}t = e^{-s\cdot 0} = 1$$

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- Once more, what does this imply for the inverse?

3 The Laplace Transform Operator:

Definition (Differential Operator)**.**

$$D = \frac{\mathrm{d}}{\mathrm{d}x}, D^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \text{etc.}$$

Motivation.

$$Df = \mathcal{L}^{-1}\left(s\mathcal{L}(f)\right)$$

Derivation.

$$\begin{aligned}
Df &= f' \\
&= \mathcal{L}^{-1}\left(\mathcal{L}(f')\right) \\
&= \mathcal{L}^{-1}\left(s\mathcal{L}(f) - f(0)\right) \\
&= \mathcal{L}^{-1}\left(s\mathcal{L}(f)\right) - \underbrace{\mathcal{L}^{-1}(f(0))}_{f(0)\delta(t)=0 \because t>0} \\
&= \mathcal{L}^{-1}\left(s\mathcal{L}(f)\right)
\end{aligned}$$

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Definition (Fractional Derivative Operator)**.** For $\alpha \in \mathbb{R}$, define D^α by

$$D^\alpha f = \mathcal{L}^{-1}(s^\alpha \mathcal{L}(f)).$$

Remark. Note that this is also a linear operator. Boundedness is ... up for debate.

3.1 Example

3.1.1 $D^{1/2}(t)$:

$$\begin{aligned} D^{1/2}(t) &= \mathcal{L}^{-1}(s^{1/2}\mathcal{L}(t)) \\ &= \mathcal{L}^{-1}(s^{1/2} \cdot s^{-2}) \\ &= \mathcal{L}^{-1}(s^{-3/2}) \\ &= \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} \\ &= \frac{2t^{1/2}}{\sqrt{\pi}} \end{aligned}$$

Claim 4.

$$D^{1/2}D^{1/2}(t) = D(t)$$

Proof.

$$\begin{aligned} D^{1/2}D^{1/2}(t) &= \frac{2}{\sqrt{\pi}}D^{1/2}(t^{1/2}) \\ &= \frac{2}{\sqrt{\pi}}\mathcal{L}^{-1}\left(s^{1/2}\mathcal{L}\left(t^{1/2}\right)\right) \\ &= \frac{2}{\sqrt{\pi}}\mathcal{L}^{-1}\left(s^{1/2}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}\right) \\ &= \frac{2}{\sqrt{\pi}}\mathcal{L}^{-1}\left(\frac{\sqrt{\pi}}{2} \cdot \frac{1}{s}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s}\right) \\ &= 1 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}t \\ &= D(t) \end{aligned}$$

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Claim 5.

$$D^{1/2}(t^\alpha) = \Gamma(\alpha+1)\frac{t^{\alpha-1/2}}{\Gamma\left(\alpha+\frac{1}{2}\right)}$$

Proof.

$$\begin{aligned} D^{1/2}(t^\alpha) &= \mathcal{L}^{-1}\left(s^{1/2}\mathcal{L}\left(t^\alpha\right)\right) \\ &= \mathcal{L}^{-1}\left(s^{1/2}\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right) \\ &= \Gamma(\alpha+1)\mathcal{L}^{-1}\left(\frac{1}{s^{\alpha-1/2+1}}\right) \\ &= \Gamma(\alpha+1)\frac{t^{\alpha-1/2}}{\Gamma\left(\alpha+\frac{1}{2}\right)} \end{aligned}$$

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4 Problem At Hand

4.1 $f = \cos$

Remark. Recall that the Taylor series of cosine is:

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^2n}{(2n)!}$$

Query.

$$D^{1/2}\cos(x) = ?$$

- The problem now arises as to whether we can interchange the operator and the sum
- Boundedness of the operator (and equivalently continuity) becomes important
- Two conjectures:
  - D^α is probably unbounded but would be bounded on a "nice enough" domain, likely the Sobolev space $H_0^1(\Omega)$ =the closure of the infinitely differential functions that are compactly supported in $\Omega \subset \mathbb{R}^n$ (open in $\mathbb{R}^n$) in $W^{1,2}(\Omega)$.
  - $D^{1/2}$ as is currently defined likely does not exist/work for a function like cosine but could probably be tweaked into working by utilising a variation of the Laplace transform.