# A Rapid Introduction to Analytic Number Theory

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M<sub>∏</sub> Talk

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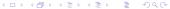
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Riemann zeta function:

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#### Definition

We say that a positive integer is squarefree if it is not divisible by the square of any prime. I.e, the squarefree integers up to 20 are

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exist; and if so, what is its value (which we call the asymptotic density of Q). Exercise:  $P(x) := \pi(x) \log x$ ?



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$$\lim_{x \to \infty} \frac{Q(x)}{x} = \lim_{k \to \infty} \mathbb{P}(\{4 \nmid n\} \cap \dots \cap \{p_k^2 \nmid n\})$$

$$= \lim_{k \to \infty} \mathbb{P}(4 \nmid n) \dots \mathbb{P}(p_k^2 \nmid n) = \prod_{p} \left(1 - \frac{1}{p^2}\right).$$

### The zeta function

#### Definition

Let  $\zeta: \mathbb{R}_{>1} \to \mathbb{R}$  such that

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

We call  $\zeta$  the zeta function on the real line s > 1.

Note: we know that  $\zeta$  is defined on its domain (p-test) and uniformally convergent on any interval  $[\alpha, \infty)$ ,  $\alpha > 1$ , (Weierstrass' M-test).

### Euler products

The zeta function is a "suitably nice" Dirichlet series and so it has an Euler product; namely,

#### Zeta as a product indexed by the primes

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This is well-known but non-trivial. Euler did it without Dirichlet's methods!



### The Proportion of Squarefree Integers

$$\lim_{x \to \infty} \frac{Q(x)}{x} = \frac{6}{\pi^2} \approx 0.608.$$

**Proof [BD]:** We want to make precise the notion of independence of  $\{p^2 \nmid n\}$  from  $\{q^2 \nmid n\}$ .

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There are  $(2^2-1)(3^2-1)\cdots(p_r^2-1)$  such systems of congruences where  $0 < a_i \le p_i^2-1$ . Thus, if  $y = k \cdot 2^2 3^2 \cdots p_r^2$  then

$$Q^{r}(y) = k(2^{2} - 1) \cdots (p_{r}^{2} - 1) = y(1 - 2^{-2}) \cdots (1 - p_{r}^{-2}).$$

### ...the proof continues...

Hence

$$Q^{r}(y) = y(1-2^{-2})\cdots(1-p_{r}^{-2}) \implies \frac{Q^{r}(y)}{y} = (1-2^{-2})\cdots(1-p_{r}^{-2})$$

and it is in this sense that we can consider  $\{p^2 \nmid n\}$  and  $\{q^2 \nmid n\}$  "independent events".

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$$0 \le Q^r(x) - Q^r(y) \le x - y < 2^2 \cdots p_r^2$$

and

$$0 \leq (x-y) \prod_{i=1}^{r} (1-p_i^{-2}) < x-y < 2^2 \cdots p_r^2.$$

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Alas, we write

$$Q^{r}(x) = x \prod_{i=1}^{r} (1 - p_{i}^{-2}) - (x - y) \prod_{i=1}^{r} (1 - p_{i}^{-2}) + Q^{r}(x) - Q^{r}(y)$$
$$= x \prod_{i=1}^{r} (1 - p_{i}^{-2}) + \theta \cdot 2^{2} \cdots p_{r}^{2}, \qquad |\theta| \le 1.$$

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We now need to estimate from below. Observe that

$$Q^{r}(x) - Q(x) \le \operatorname{card}\{n \le x : \exists k > r, p_{k}^{2} \mid n\} \le \sum_{k=r+1}^{\infty} \operatorname{card}\{n \le x : p_{k}^{2} \mid n\}$$
$$= \sum_{k=r+1}^{\infty} \left[\frac{x}{p_{k}^{2}}\right] < \sum_{k=r+1}^{\infty} \frac{x}{k^{2}} < \int_{r}^{\infty} \frac{x}{t^{2}} dt = \frac{x}{r}.$$

Hence

$$\liminf_{x\to\infty}\frac{Q(x)}{x}\geq \liminf_{x\to\infty}\frac{Q^r(x)}{x}-\frac{1}{r}=\prod_{k=1}^r(1-p_k^{-2})-\frac{1}{r}.$$

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Letting  $r \to \infty$ , we obtain

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Therefore

$$\lim_{x \to \infty} \frac{Q(x)}{x} = \prod_{p} (1 - p^{-2}) = \frac{6}{\pi^2} \approx 0.608.$$



### References, Recommendations and QnA

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