Folding Spaces to Identify Holes

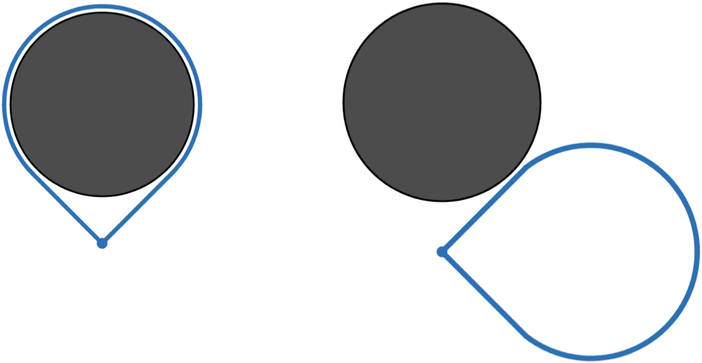
An Overview of the Fundamental Group and Covering Spaces

Simon Brims

How to Detect a Hole

Aim: Find a way to detect and identify 'holes' in a space.

Idea: A hole is something that loops get caught on.



How to Detect a Hole

Definition (Path Homotopy)

Let $p_0, p_1 : [0,1] \to X$ be continuous paths in X. Then p_0 is homotopic to p_1, denoted $p_0 \simeq p_1$, if there exists a continuous function $F : [0,1] \times [0,1] \to X$ such that $F(x,0) = p_0(x)$ and $F(x,1) = p_1(x)$.

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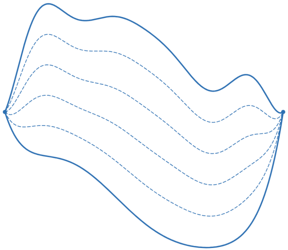

- Continuous deformation from p_0 to p_1.

How to Detect a Hole

Aim:

- Pick a base-point $b \in X$.
- Want to classify all loops in X with base-point b, up to homotopy.

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- Pick a base-point $b \in X$.
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Key Observations:

- Composition of loops preserves homotopy.
- Homotopy classes of loops forms a group under composition.

The Fundamental Group

Definition (The Fundamental Group)

The group of homotopy classes of loops in X is called the fundamental group of X, denoted $\pi_1(X)$.

The Fundamental Group

Example ($\mathbb{R}^n$)

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$$\pi_1(\mathbb{R}^n) = \{0\}$$

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Example (Sphere)

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$$\pi_1(\mathbb{R}^n) = \{0\}$$

Example (Sphere)

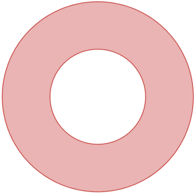
$$\pi_1(\mathbb{S}^2) = \{0\}$$

Definition

If $\pi_1(X) = \{0\}$, then we call it **simply connected**.

The Fundamental Group

Example (The Annulus)



The Fundamental Group

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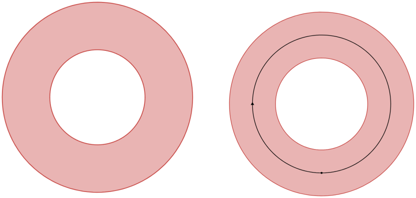

The Fundamental Group

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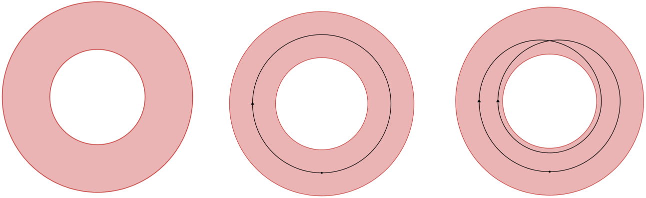

The Fundamental Group

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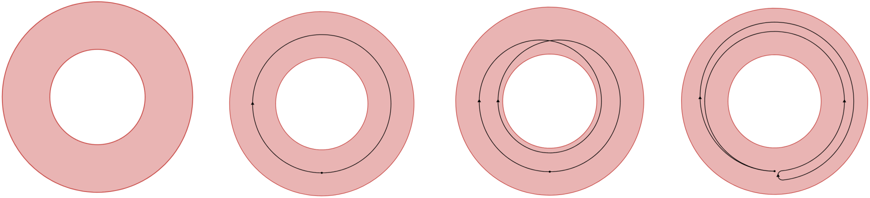

The Fundamental Group

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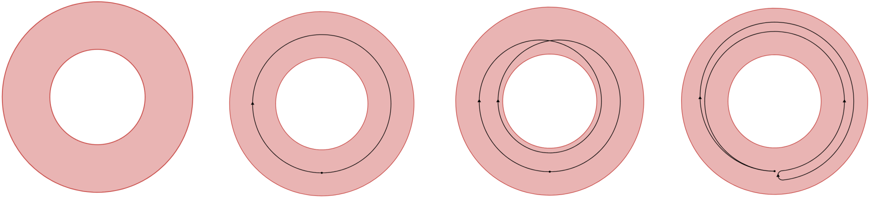


$$\pi_1(Annulus) = \mathbb{Z}$$

- Counts the number of clockwise windings around the hole.

The Fundamental Group

Example (The Cylinder)

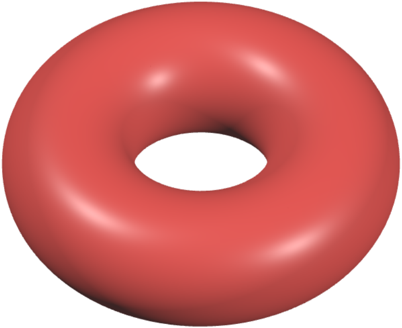
The Fundamental Group

Example (The Cylinder)

$$\pi_1(\mathbb{S}^1 \times \mathbb{R}) = \mathbb{Z}$$

The Fundamental Group

Example (The Torus)



The Fundamental Group

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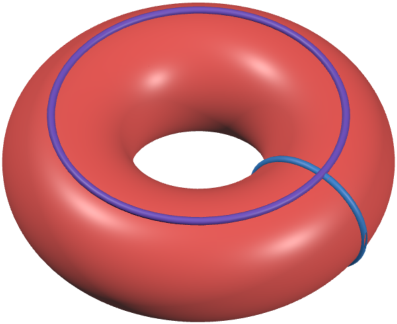

The Fundamental Group

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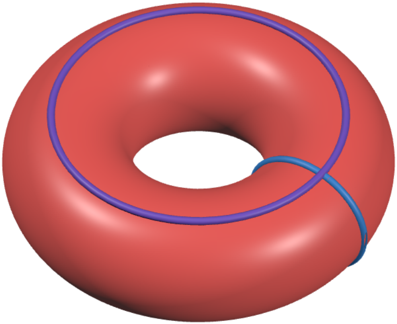

$$\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$$

Issues with finding the Fundamental Group

- How do I find generating loops?
- How do I know if I've found ALL of the loops?
- How do I know if two loops are not homotopic?
- What do I do if I can't picture the space?

The Fundamental Group

Example (Projective Plane)

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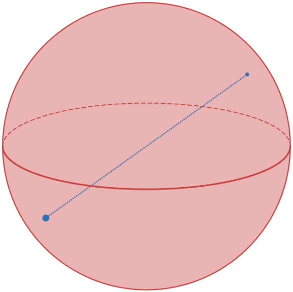
The Fundamental Group

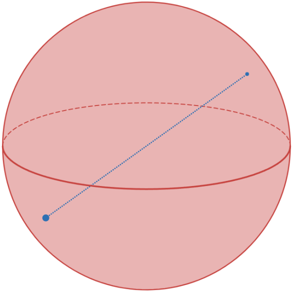
Example (Projective Plane)



$$\xrightarrow{\pi} \mathbb{RP}^2$$

The Fundamental Group

Example (Projective Plane)



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$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}$$

The Fundamental Group

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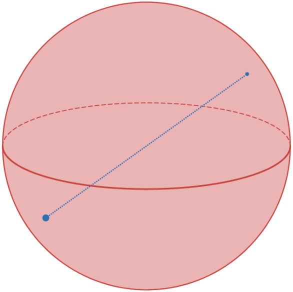


$$\xrightarrow{\pi} \mathbb{RP}^2$$

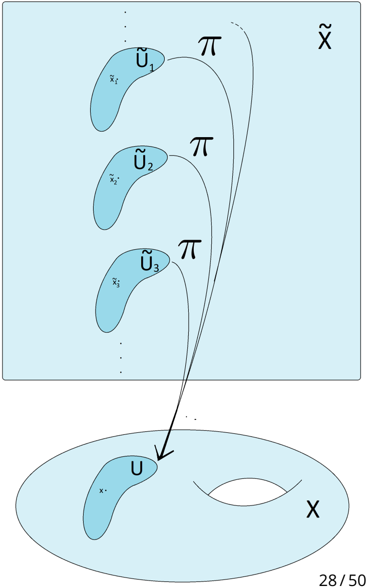
$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$$

Covering Space

Definition (Covering Space)

A covering space of X is a space $\tilde{X}$ paired with a map $\pi : \tilde{X} \to X$ such that for each point $x \in X$, there exists a neighbourhood U such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, each of which maps homeomorphically onto U via p.

- For each point, there exists a local region such that its preimage under π looks like a bunch of disjoint copies of the region.
- Call the disjoint copies the **Decks**.



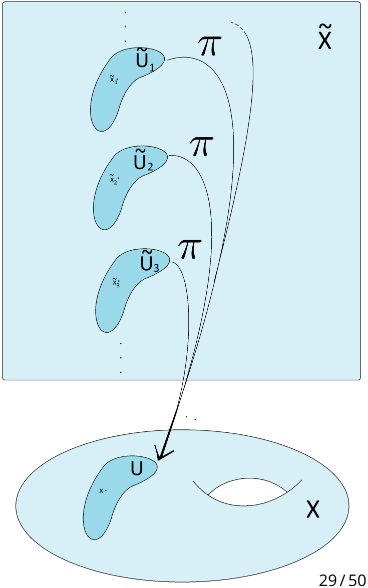
The Universal Cover

Definition (Universal Cover)

If $\tilde{X}$ is simply connected, it's called the universal cover.

Example (The Cylinder)

LIVE DEMONSTRATION



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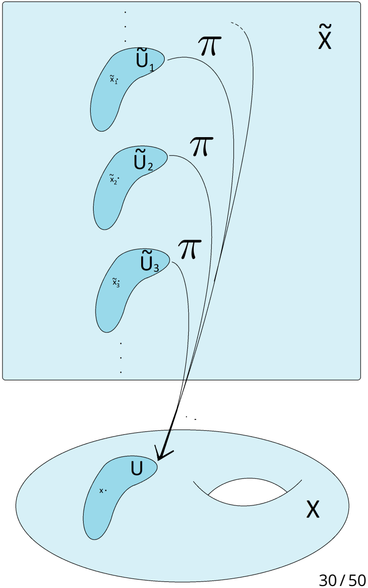
Example (The Cylinder)

LIVE DEMONSTRATION

- The map introduces holes into the space.
- Layout of decks describes how holes are introduced.



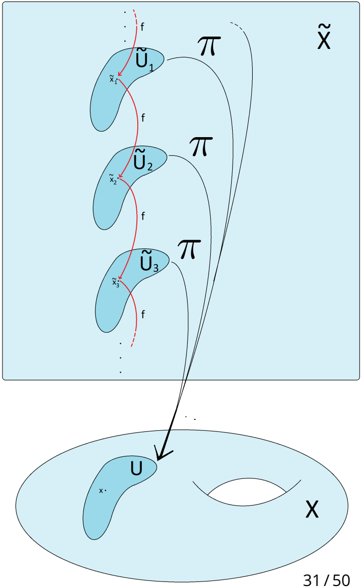
Deck Transformations

Definition (Deck Transformation)

Let $f : \tilde{X} \to \tilde{X}$ be a continuous function such that $\pi \circ f = \pi$. Then f is called a deck transformation

Deck transformations form a group $G(\tilde{X})$

- The group of transformations that preserve π
- Each transformation is determined by their action on a single element



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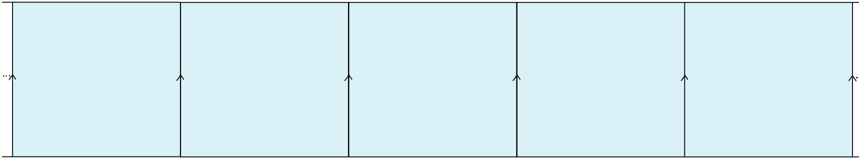

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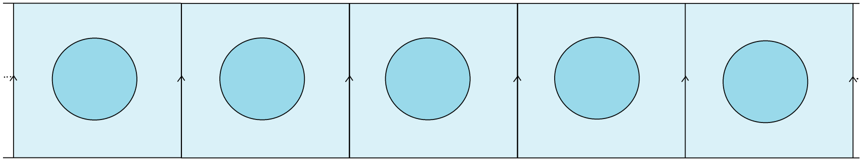

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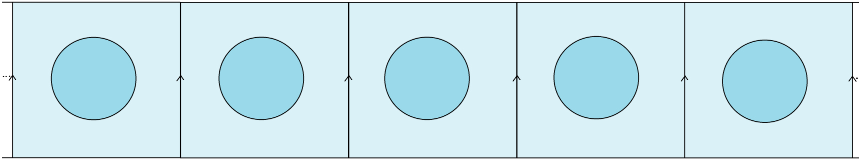
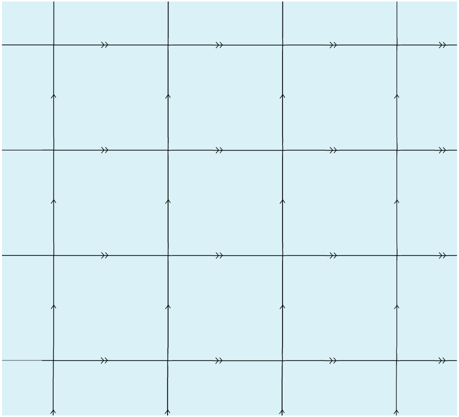


$$G(\tilde{X}) = \mathbb{Z}$$

Deck Transformations

Example (The Torus)



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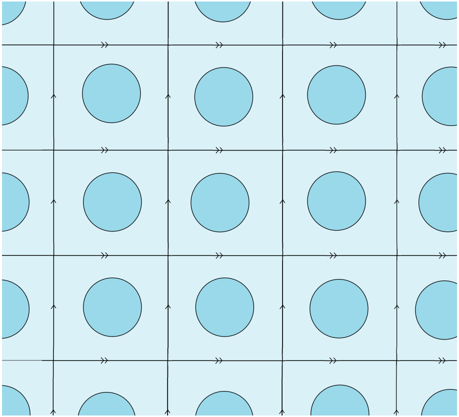

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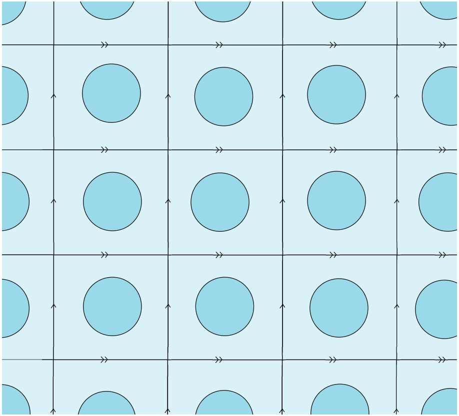

$$G(\tilde{X}) = \mathbb{Z} \times \mathbb{Z}$$

Deck Transformations and the Fundamental Group

Theorem

$$G(\tilde{X}) \cong \pi_1(X)$$

Idea: Loops lift to paths that connect
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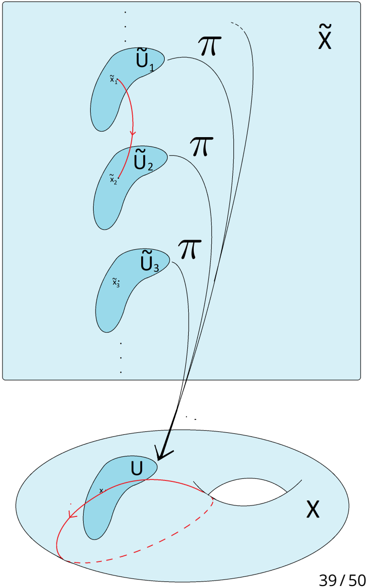

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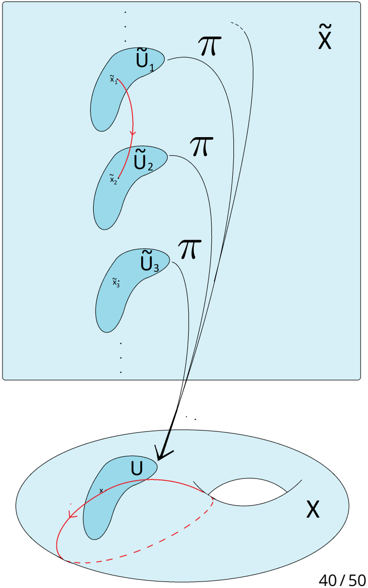
- Non-trivial loops correspond with paths between distinct decks.



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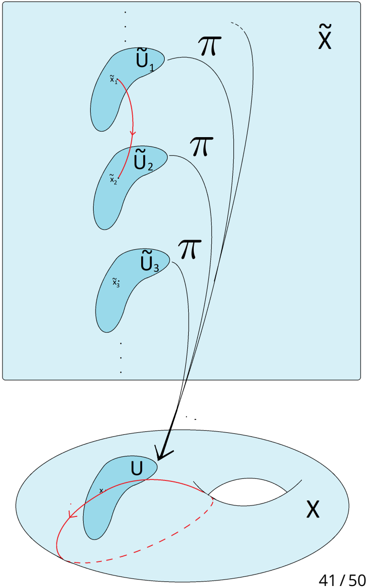
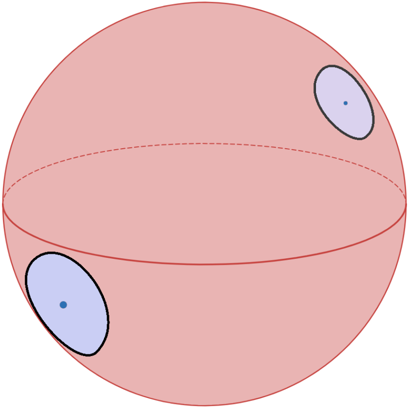
- Non-trivial loops correspond with paths between distinct decks.
- Every *nice enough* space has a unique universal cover.



The Projective Plane



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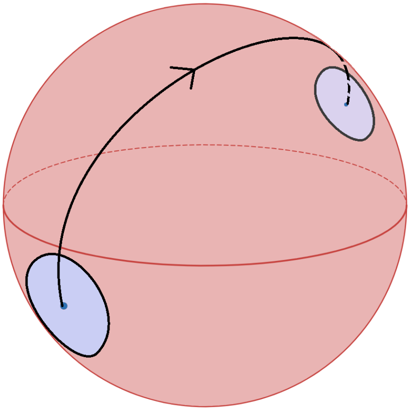

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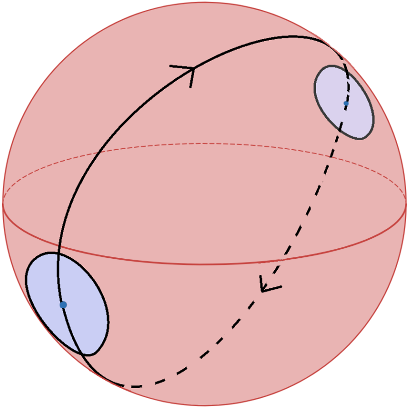

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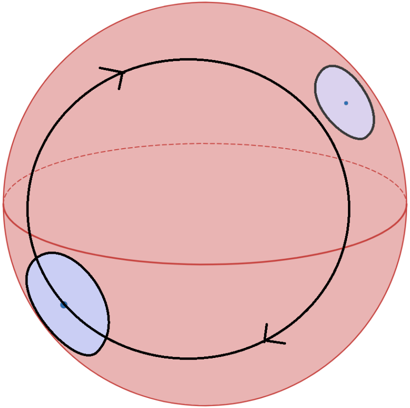

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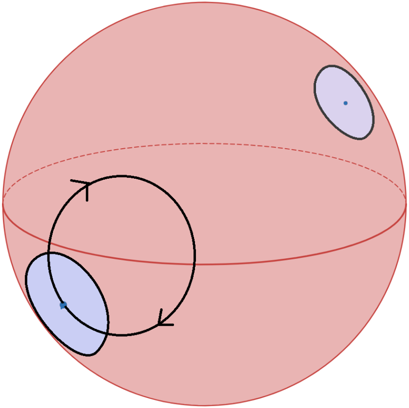

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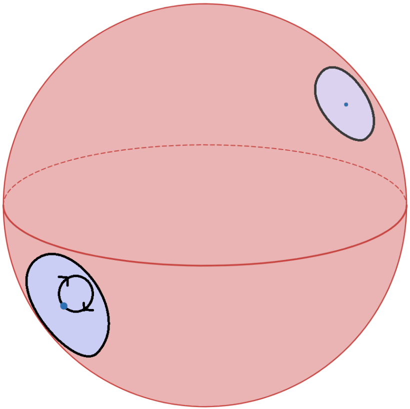

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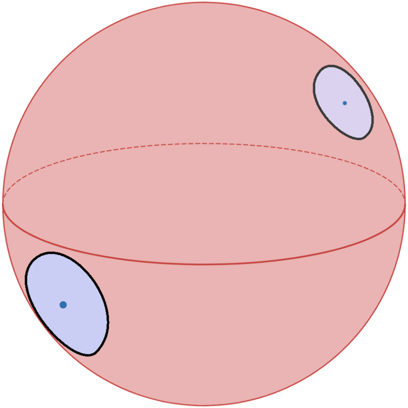

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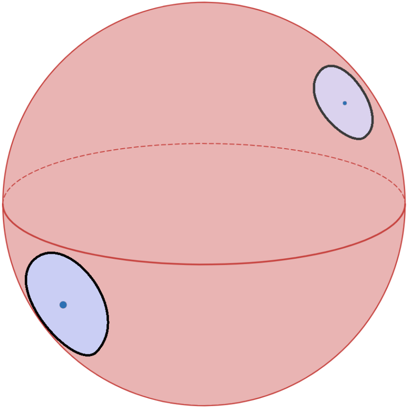

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Thank You

Reference:

A. Hatcher *Algebraic Topology* Cambridge University Press, 2002