Relations

Chapter 9

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Chapter Summary

- 9.1: Relations and Their Properties
- *9.2: N-ary* Relations and Their Applications
- 9.3: Representing Relations
- 9.4: Closures of Relations
- 9.5: Equivalence Relations

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Relations and Their Properties

Section 9.1

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Section Summary

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

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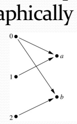
Binary Relations

Definition: A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.

Example:
- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a), (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

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R

Binary Relation on a Set

Definition: A binary relation R *on a set A* is a subset of $A \times A$ or a relation from A to A.

Example:
- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are $(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

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Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when $|A| = n$ elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

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Binary Relations on a Set (*cont*.)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and therefore each of these relations is an infinite set. These examples will be used to illustrate the various properties of relations which are of particular interest in Mathematics and Computing.

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Consider the following relations on $\mathbb{Z}$

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

These are relations on an infinite set ($\mathbb{Z}$) and therefore each of these relations is an infinite set.

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Question:

Which of these relations contain the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution:

(1,1) is in R_1, R_3, R_4 , and R_6;
(1,2) is in R_1 and R_6;
(2,1) is in R_2, R_5, and R_6;
(1, −1) is in R_2, R_3, and R_6 ;
(2,2) is in R_1, R_3, and R_4.

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Reflexive Relations

Definition: R is *reflexive* iff' $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \longrightarrow (x,x) \in R]$$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$.

If $A = \emptyset$ then the empty relation is reflexive *vacuously*; i.e., ***the empty relation on an empty set is reflexive!***

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that, e.g., $3 \ngtr 3$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that, e.g., $3 \neq 3 + 1$),
$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that, e.g., $4 + 4 \nleq 3$).

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Symmetric Relations

Definition: R is *symmetric* iff' $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$. *What property of arithmetic guarantees that this relation is symmetric??*

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),
$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Commutative ?

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Antisymmetric Relations

Definition: A relation R on a set A such that $\forall\ a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y\ [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$$

- **Example**: The following relations on the integers are antisymmetric:
  - $R_1 = \{(a,b) \mid a \leq b\}$,
  - $R_2 = \{(a,b) \mid a > b\}$,
  - $R_4 = \{(a,b) \mid a = b\}$,
  - $R_5 = \{(a,b) \mid a = b + 1\}$.

  For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.
- The following relations are not antisymmetric:
  - $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$ (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),
  - $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

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Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y\ \forall z[(x,y) \in R \wedge (y,z) \in R \longrightarrow (x,z) \in R\ ]$$

- **Example**: The following relations on the integers are transitive:
  - $R_1 = \{(a,b) \mid a \leq b\}$,
  - $R_2 = \{(a,b) \mid a > b\}$,
  - $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  - $R_4 = \{(a,b) \mid a = b\}$.

  For every integer, $a \leq b$ and $b \leq c$, then $a \leq c$.

  The following are not transitive:
  - $R_5 = \{(a,b) \mid a = b + 1\}$ (note that both $(4,3)$ and $(3,2)$ belong to R_5, but not $(4,2)$)
  - $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(2,1)$ and $(1,2)$ belong to R_6, but not $(2,2)$).

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Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$, then: AxB=$\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2)$ $(3,3),(3,4)\}$; let $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$. R_1 & R_2 can be combined:

  $R_1 \cap R_2 = \{(1,1)\}$ $\quad$ $R_1 - R_2 = \{(2,2),(3,3)\}$

  $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

  $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

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Composition

Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

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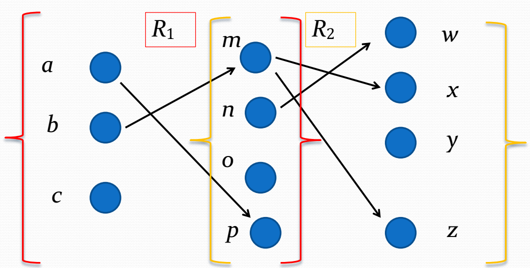
Representing the Composition of a Relation



$R_1 \circ R_2 = \{(b,x),(b,z)\}$

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Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined by:

- $R^1 = R$
- $R^{n+1} = R^n \circ R$

NB: *The powers of a* ***transitive*** *relation are* ***subsets*** *of the relation*. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots.$

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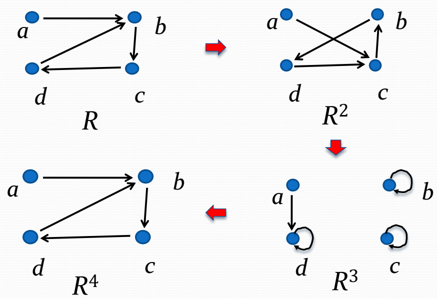
Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

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N-ary Relations

Section 9.2

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N-ary Relations

- Section Summary
  - What is an *n-ary* relation?
  - Operations on *n-ary relations*
  - *N-ary* relations and relational databases

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N-ary Relations

- ***Definition:*** Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is its degree.
- ***Example:*** Let R be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consisting of triples (a, b, c) in which a,b,c form an arithmetic progression. That is $(a,b,c) \in R$ iff' there exist some k integer such that b=a+k and c=a+2k=b+k.
  - What is the degree of this relation?
  - What are the domains of this relation?
  - Are the following tuples in this relation? (1, 3, 5); (2, 5, 9)
- Answers: $n = 3$; $\mathbb{Z}$; (1, 3, 5), YES; (2, 5, 9) NO

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N-ary relations are the basis of relational database management systems

- Data is stored in relations (a.k.a., tables), e.g., ***Students** and **Enrollment*** (note the common field containing the student ID number — this is the *primary key)*

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Enrollment	
Stud_ID	Course
334322	CS 441
334322	Math 336
546346	Math 422
964389	Art 707

- Columns of a table represent the attributes of a relation Rows, or records, contain the actual data defining the relation

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Operations on an RDBMS are formally defined in terms of a relational algebra

- Relational algebra gives a formal semantics to the operations performed on a database by rigorously defining these operations in terms of manipulations on sets of tuples (i.e., records)
- Operators in relational algebra include (*= covered here):
  - Selection *
  - Projection *
  - Rename
  - Join
    - Equijoin*, Left outer join & Right outer join
  - Aggregation

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The selection operator allows us to filter the rows in a table

- ***Definition:*** Let R be an n-ary relation and let C be a condition that elements in R must satisfy. The selection s_c maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.
- ***Example:*** Consider the Students relation from an earlier slide. Let the condition C_1 be Major="CS" and let C_2 be GPA > 2.5. What is the result of $s_{C_1 \wedge C_2}$(Students)?

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Answer:
- (Alice, 334322, CS, 3.45)
- (Charlie, 045628, CS, 2.75)

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The projection operator allows us to consider only a subset of the columns of a table

- ***Definition:*** The projection $P_{i_1,\ldots,i_n}$ maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, \ldots, a_{i_m})$ where $m \le n$
- ***Example:*** What is the result of applying the projection $P_{1,3}$ to the Students table?

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Name	Major
Alice	CS
Bob	Math
Charlie	CS
Denise	Art

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The equijoin operator allows us to create a new table based on data from two or more related tables

- ***Definition:*** Let R be a relation of degree m and S be a relation of degree n. The equijoin $J_{i_1=j_1,\ldots,i_k=j_k}$, where $j \le m$ and $k \le n$, creates a new relation of degree m+n-k containing the subset of $S \times R$ in which $s_{i_1} = r_{j_1}, \ldots, s_{i_k}=r_{j_k}$ and duplicate columns are removed (via projection).

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Equijoin

- ***Example:*** What is the result of the equijoin $J_{2=1}$ on the Students and Enrollment tables?

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Enrollment	
Stud_ID	Course
334322	CS 441
334322	Math 336
546346	Math 422
964389	Art 707

Name	ID	Major	GPA	Course
Alice	334322	CS	3.45	CS 441
Alice	334322	CS	3.45	Math 336
Bob	546346	Math	3.23	Math 422
Denise	964389	Art	4.0	Art 707

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SQL queries correspond to statements in relational algebra

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Name	ID
Alice	334322
Charlie	045628

SELECT Name, ID FROM Students WHERE Major = “CS” AND GPA > 2.5

SELECT is actually a projection (in this case, $P_{1,2}$)

The WHERE clause lets us filter (i.e., $S_{major=\text{“CS”} \wedge GPA>2.5}$)

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SQL: An Equijoin Example

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Enrollment	
Stud_ID	Course
334322	CS 441
334322	Math 336
546346	Math 422
964389	Art 707

SELECT Name, ID, Major, GPA, Course FROM Students, Enrollment WHERE ID = Stud_ID

Name	ID	Major	GPA	Course
Alice	334322	CS	3.45	CS 441
Alice	334322	CS	3.45	Math 336
Bob	546346	Math	3.23	Math 422
Denise	964389	Art	4.0	Art 707

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Your Turn!

Students			
Name	ID	Major	GPA
Alice	334322	CS	3.45
Bob	546346	Math	3.23
Charlie	045628	CS	2.75
Denise	964389	Art	4.0

Enrollment	
Stud_ID	Course
334322	CS 441
334322	Math 336
546346	Math 422
964389	Art 707

Problem 1: What is $P_{1,4}$(Students)?
Problem 2: What relational operators would you use to generate a table containing only the names of Math and CS majors with a GPA > 3.0?
Problem 3: Write an SQL statement corresponding to the solution to problem 2.

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Representing Relations

Section 9.3

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Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

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Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

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Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

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Examples of Representing Relations Using Matrices (*cont.*)

Example 2: Let $A = \{a_1,a_2, a_3\}$ and $B = \{b_1,b_2, b_3,b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i,b_j) with $m_{ij} = 1$, it follows that:

$R = \{(a_1, b_2), (a_2, b_1),(a_2, b_3), (a_2, b_4),(a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

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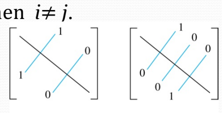
Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1. $\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$
- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

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Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

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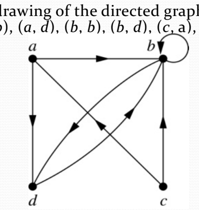
Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

- An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



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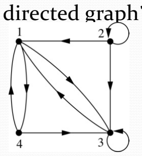
Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)

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Determining which Properties a Relation has from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

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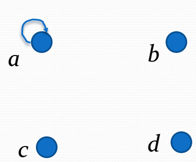
Determining which Properties a Relation has from its Digraph – Example 1



- *Reflexive?* No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from one vertex to another

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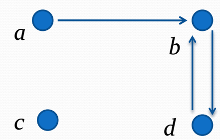
Determining which Properties a Relation has from its Digraph – Example 2



- *Reflexive?* No, there are no loops
- *Symmetric?* No, there is an edge from *a* to *b*, but not from *b* to *a*
- *Antisymmetric?* No, there is an edge from *d* to *b* and *b* to *d*
- *Transitive?* No, there are edges from *a* to *b* and from *b* to *d*, but there is no edge from *a* to *d*

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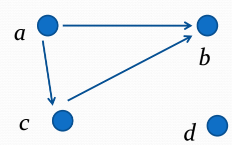
Determining which Properties a Relation has from its Digraph – Example 3



Reflexive? No, there are no loops
Symmetric? No, for example, there is no edge from *c* to *a*
Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back
Transitive? Yes, there is an edge from *a* to *b*

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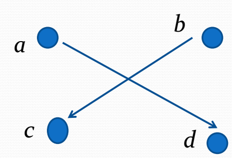
Determining which Properties a Relation has from its Digraph – Example 4



- *Reflexive?* No, there are no loops
- *Symmetric?* No, for example, there is no edge from *d* to *a*
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

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Closures of Relations

Section 9.4

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Definition: Closure of a Relation

- Let R be a relation on a set A. The relation R may or may not have some property **P** such as reflexivity, symmetry or transitivity.
- Given a relation R on a set A and a property P of relations, the closure of R with respect to property P is smallest relation on A that contains R and has property P. That is, ***closure of R with respect to P is the relation obtained by adding the minimum number of ordered pairs to R necessary to obtain property P.***
- In simpler terms: by "adding" some relation **S which has the desired property P** to the relation **R**, the property **P** can be conferred on **R.**

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Reflexive Closure

- Let A be the set {1,2,3,4} and R be the relation R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)} on A.
- $M_R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$
- Is this relation reflexive? If not, can it be made reflexive? (i.e., what is the reflexive closure of this relation?) ?). Not reflexive, because not all possible pairs of the form (a, a) are in R.
- **Solution:** Add the pair (3,3) to the relation.

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Reflexive Closure

- Let R be a relation on a set A. The *reflexive closure* of $\boldsymbol{R}$ is $\boldsymbol{R} \cup \Delta$ where $\Delta = \{(a, a) | a \in \boldsymbol{A}\}$.
- Δ is called the *diagonal relation* on $\boldsymbol{A}$.
- If **A** is {1,2,3,4} then Δ={(1,1),(2,2),(3,3), (4,4)}
- $\boldsymbol{R}$ = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
- **R** ∪ Δ = {(1, 1), (1, 2), (2, 1), (2, 2), (3,3), (3, 4), (4, 1), (4, 4)}
- **R** ∪ Δ is reflexive.

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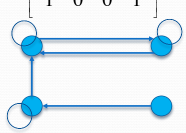
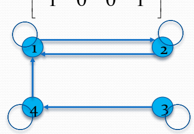
Reflexive Closure

- **The zero-one and digraph representations of $\mathbf{M}_\mathbf{R}$ and $\mathbf{M}_{R\cup\Delta}$**

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{R\cup\Delta} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$





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Symmetric Closure

- With R as previously given:
  R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
  the matrix representation of R is:

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- Given that the matrix of a symmetric relation is symmetric, is R symmetric and, if not, what must be "added" to R to make it symmetric?
- **Solution:** M_R would be symmetric if there were 1's in locations $m_{1,4}$ (to match the 1 at $m_{4,1}$) and at $m_{4,3}$ (to match the 1 at $m_{3,4}$).
- This is equivalent to adding edges from vertex 1 to vertex 4 and from vertex 4 to vertex 3.

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Symmetric Closure

- Let R be a relation on a set A. The symmetric closure of R is
  $$R \cup R^{-1}$$
- Where:
  $$R^{-1} = \{(b, a) | (a, b) \in R\}$$
  is the *inverse relation* of R
- e.g.: Let A be the set {1,2,3,4} and R be the relation
  R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}.

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Symmetric Closure

- Given $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
  then $R^{-1} = \{(1,1),(2,1),(1,2),(2,2),(4,3),(1,4),(4,4)\}$
- So:
  $R \cup R^{-1} = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,4),(4,1),(4,3),(4,4)\}$
- And
  $$M_{R \cup R^{-1}} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$
  which is symmetric.
- Exercise—draw the digraphs for both R and $R \cup R^{-1}$ and examine them, too.

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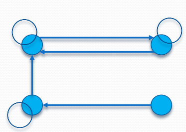
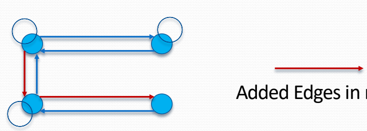
Symmetric Closure

This is the digraph of the original relation:



And this is the digraph of $R \cup R^{-1}$:



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Anti-symmetric Closure

- To make an antisymmetric closure, assuming that ***R*** is not already antisymmetric, edges would have to be *removed* from ***R*** which violates the requirement that ***R*** ⊆ any *closure of* ***R; therefore, there is no such construct as the "anti-symmetric closure" of a relation.***

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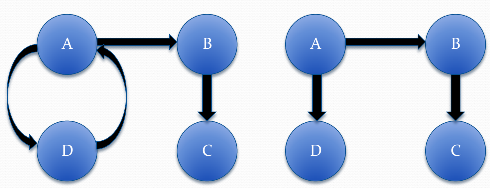
Why Antisymmetric Closure Does Not Exist

- Not Antisymmetric Antisymmetric



- R={(A,D),(D,A),(A,B),(B,C)} R*={(A,D),(A,B),(B,C)}
- Note that $\boldsymbol{R} \nsubseteq \boldsymbol{R}^*$

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Transitive Closure

- This is a bit more complicated; it involves several previously covered operations such as
  - **Composition of relations**: Let R be a relation from a set A to a set B, and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by S ∘ R.
  - **Powers of a Relation**: Let R be a relation on the set A. The powers, R^n, n = 1,2,..., are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

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Transitive Closure

- A **path** from a to b in a directed graph G is a sequence of edges (x_0,x_1), (x_1,x_2), …, (x_{n-1}, x_n) in G, where n is a non-negative integer, and x_0 = a and x_n = b, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex of the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n. ***We view the empty set of edges as a path from a to a.***
- A path of length n ≥ 1 that begins and ends at the same vertex is called a *circuit* or *cycle*.
- **Definition**: There is a path from a to b in a relation R if there is a sequence of elements a, $x_1, x_2, \ldots, x_{n-1}$, b with $(a, x_1) \in R$, $(x_1, x_2) \in R$, …, $(x_{n-1}, b) \in R$. This path is of length **n**.

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Transitive Closure

- **The Join Operation**: Let A = $[a_{ij}]$ and B = $[b_{ij}]$ be m × n zero-one matrices. Then, the join of A and B, denoted by A∨B, is the m×n zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$.
- **The Boolean Product**: LetA=$[a_{ij}]$be an m×k zero-one matrix and B=$[b_{ij}]$ be a k×n zero-one matrix. Then, the Boolean product of A and B, denoted by A⊙B, is the m×n matrix with (i,j)th entry $[c_{ij}]$, where
- $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$.

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Transitive Closure

- NOTES: These two theorems (which we will not prove) make the computation of compositions of relations easier if you have the relations in the form of zero-one matrices (⊙ is the Boolean Product):
  - 1: $M_{S \circ R} = M_R \odot M_S$.
  - 2: $M_{R \circ R} = M_R \odot M_R = M_R^{[2]}$
  - Corollary: $M_R^{[n]} = M_R^{[n-1]} \odot M_R$
  - NB!: be careful to keep the matrices in the correct order—remember, matrix multiplication in general DOES NOT COMMUTE

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Transitive Closure

- Let R be a relation on the set $\boldsymbol{A}$. The connectivity relation R^* consists of pairs (a, b) such that there is a path of length at least one from a to b in R.
- **Theorem**: The transitive closure of a relation R equals the connectivity relation $R*$.
- **Theorem**: Let M_R be the zero-one matrix of the relation R on a set with n elements. If $n = |\boldsymbol{A}|$, then the zero-one matrix of the transitive closure $R*$ is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]}$$

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Transitive Closure

- Procedure for Computing the Transitive Closure

```
procedure transitive closure(M_R: zero-one n × n matrix)
{P will store the powers of M_R}
P := M_R
{J will store the join of the powers of M_R}
J := M_R
for i := 2 to n
begin
    P := P ⊙ M_R
    J := J ∨ P
end
{J is the zero-one matrix for R*}
```

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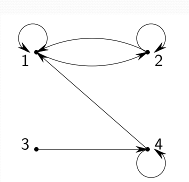
Transitive Closure

- How to do it—Step 1: build M_R
- Let A be the set {1, 2, 3, 4} and R be the relation
  - R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}



$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

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Transitive Closure

- How to do it—Step 2: Compute the powers of M_R through the 4th power (since $|A| = 4$)

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R^{[4]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

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Transitive Closure

- How to do it—Step 3 compute the join of the powers:

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \mathbf{M}_R^{[4]}$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

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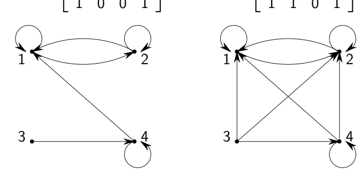
Transitive Closure

- How to do it—Step 4, the result is R^* {compare the matrices and digraphs shown here}

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{R^*} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$



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Equivalence Relations

Section 9.5

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Section Summary

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

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Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

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Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc.

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Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation
$R = \{(a,b) \mid a \equiv b \pmod m\}$
is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod m$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod m$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod m$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\,m$, so $b \equiv a \pmod m$.
- *Transitivity*: Suppose that $a \equiv b \pmod m$ and $b \equiv c \pmod m$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

  $$a - c = (a - b) + (b - c) = km + lm = (k + l)\,m.$$

  Therefore, $a \equiv c \pmod m$.

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Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but the relation is not symmetric. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Even though the "divides" relation is reflexive and transitive, the failure of symmetry means that the relation is not an equivalence…*almost doesn't count*!

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Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.
When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s|(a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. **Any** element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

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Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb
(*ii*) $[a] = [b]$
(*iii*) $[a] \cap [b] \neq \emptyset$

Proof: We show that (*i*) implies (*ii*). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

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Equivalence Classes and Partitions

- **Show that (ii) implies (iii).** Assume that [a] = [b]. It follows that [a] ∩ [b] ≠ ∅ because [a] is nonempty (as a minimum a ∈ [a] because R is reflexive).
- **Show that (iii) implies (i).** Suppose that [a] ∩ [b] ≠ ∅. Then there is an element c with c ∈ [a] and c ∈ [b]. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

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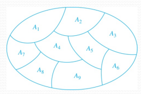
Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



A Partition of a Set

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An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

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An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem. For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

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Partial Orderings

Section 9.6

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Section Summary

- Partial Orderings and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams
- Lattices
- Topological Sorting

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Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering,* or *partial order,* if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

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Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

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Partial Orderings (*continued*)

Example 2: Show that the divisibility relation ($\mid$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- (Z^+, $\mid$) is a poset *[**p**artially **o**rdered **set**].*

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Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

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Comparability

Definition 2: The elements a and b of a poset $(S,\preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.

Definition 4: $(S,\preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

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Lexicographic Order

Definition: Given two posets $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1,b_2), that is,

$$(a_1, a_2) \prec (b_1,b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

- This definition can be easily extended to a lexicographic ordering on strings (*see text*).

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

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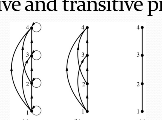
Hasse Diagrams

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

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Procedure for Constructing a Hasse Diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

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