Chapter 1—Test 1

1. What is the truth value of $(p \vee q) \rightarrow (p \wedge q)$ when both p and q are false?

2. What are the converse and contrapositive of the statement "If it is sunny, then I will go swimming"?

3. Show that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are equivalent

(a) using a truth table.

(b) using logical equivalences.

4. Suppose that $Q(x)$ is the statement "$x + 1 = 2x$." What are the truth values of $\forall x\, Q(x)$ and $\exists x\, Q(x)$?

5. Prove each of the following statements.

(a) The sum of two even integers is always even.

(b) The sum of an even integer and an odd integer is always odd.

6. Prove that there are no solutions in positive integers to the equation $x^4 + y^4 = 100$.

Chapter 1—Test 1 Solutions

1. When p and q are both false, so are $(p \vee q)$ and $(p \wedge q)$. Hence $(p \vee q) \rightarrow (p \wedge q)$ is true.

2. The converse of the statement is "If I go swimming, then it is sunny." The contrapositive of the statement is "If I do not go swimming, then it is not sunny."

3. (a) We have the following truth table.

p	q	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p$	$q \wedge \neg p$
T	T	F	T	F	F	F
T	F	T	T	F	F	F
F	T	F	F	T	T	T
F	F	T	T	F	T	F

Since the fifth and seventh columns agree, we conclude that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are equivalent.

(b) By De Morgan's law $\neg(p \vee \neg q)$ and $\neg p \wedge \neg\neg q$ are equivalent. By the double negation law, this is equivalent to $\neg p \wedge q$, which is equivalent to $q \wedge \neg p$ by the commutative law. We conclude that $\neg(p \vee \neg q)$ and $q \wedge \neg p$ are equivalent.

4. Since $x + 1 = 2x$ is true if and only if $x = 1$, we see that $Q(x)$ is true if and only if $x = 1$. It follows that $\forall x\, Q(x)$ is false and $\exists x\, Q(x)$ is true.

5. (a) Suppose that m and n are even integers. Then there are integers j and k such that $m = 2j$ and $n = 2k$. It follows that $m + n = 2j + 2k = 2(j + k) = 2l$, where $l = j + k$. Hence $m + n$ is even.

(b) Suppose that m is even and n is odd. Then there are integers j and k such that such that $m = 2j$ and $n = 2k + 1$. It follows that $m + n = 2j + (2k + 1) = 2(j + k) + 1 = 2l + 1$, where $l = j + k$. Hence $m + n$ is odd.

6. If $x^4 + y^4 = 100$, then both x and y must be less than 4, since $4^4 = 256$. Therefore the only possible values for x and y are 1, 2, and 3, and the fourth powers of these are 1, 16, and 81. Since none of $1 + 1$, $1 + 16$, $1 + 81$, $16 + 16$, $16 + 81$, and $81 + 81$ is 100, there can be no solution.

Chapter 1—Test 2

1. Prove or disprove that $(p \to q) \to r$ and $p \to (q \to r)$ are equivalent.

2. Let $P(m,n)$ be "n is greater than or equal to m" where the domain (universe of discourse) is the set of nonnegative integers. What are the truth values of $\exists n \forall m\, P(m,n)$ and $\forall m \exists n\, P(m,n)$?

3. Prove that all the solutions to the equation $x^2 = x + 1$ are irrational.

4. (a) Prove or disprove that a 6×6 checkerboard can be covered with straight triominoes.

(b) Prove or disprove that an 8×8 checkerboard can be covered with straight triominoes.

5. A stamp collector wants to include in her collection exactly one stamp from each country of Africa. If $I(s)$ means that she has stamp s in her collection, $F(s,c)$ means that stamp s was issued by country c, the domain for s is all stamps, and the domain for c is all countries of Africa, express the statement that her collection satisfies her requirement. Do not use the $\exists!$ symbol.

Chapter 1—Test 2 Solutions

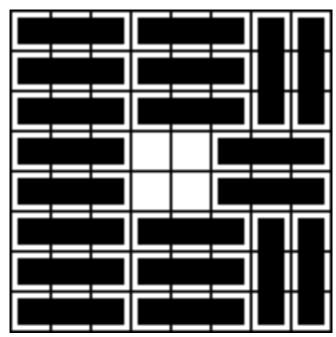
1. Suppose that p is false, q is true, and r is false. Then $(p \to q) \to r$ is false since its premise $p \to q$ is true while its conclusion r is false. On the other hand, $p \to (q \to r)$ is true in this situation since its premise p is false. Therefore $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent.

2. For every positive integer n there is an integer m such that $n < m$ (take $m = n+1$ for instance). Hence $\exists n \forall m\, P(m,n)$ is false. For every integer m there is an integer n such that $n \geq m$ (take $n = m+1$ for instance). Hence $\forall m \exists n\, P(m,n)$ is true.

3. This equation is equivalent to (and therefore has the same solutions as) $x^2 - x - 1 = 0$. By the quadratic formula, the solutions are exactly $(1 \pm \sqrt{5})/2$. If either of these were a rational number r, then we would have $\sqrt{5} = \pm(2r-1)$. Since the rational numbers are closed under the arithmetic operations, this would tell us that $\sqrt{5}$ was rational, which we know from this chapter it is not.

4. (a) The 6×6 board with four squares removed has $36 - 4 = 32$ squares. Since 32 is not a multiple of 3, it cannot be covered by pieces that cover 3 squares each.

(b) The following picture shows that it is possible.



5. The simplest formula is $\forall c\, \exists s\, \forall x((I(x) \wedge F(x,c)) \leftrightarrow x = s)$.