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COL726 Assignment 3

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Ans 1:

(a) $\Rightarrow$ Given a square matrix A, we know it's decomposition as: $\boxed{A = PVQ}$ ($V \rightarrow$ Upper Triangular; $P, Q \rightarrow$ Permutation matrices)

$\Rightarrow$ Now, let us take a upper triangular matrix of size 2×2 as: $V_{(2)} = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$. Let a Permutation matrix $P'_{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow$ Now: $V_{(2)} P'_{(2)} = \begin{bmatrix} a_2 & a_1 \\ a_3 & 0 \end{bmatrix}$ (Columns being interchanged)

$\Rightarrow$ And: $P'_{(2)} V_{(2)} P'_{(2)} = \begin{bmatrix} a_3 & 0 \\ a_2 & a_1 \end{bmatrix}$ (Columns being inverted) Lower Triangular

$\Rightarrow$ Thus, from this intuition if we do the same exercise for 3×3, 4×4 etc. matrices, we will get the same result.

$\Rightarrow$ Thus given $U = \begin{bmatrix} \times & \times & \cdots & \times \\ & \times & & \times \\ & & \ddots & \vdots \\ & & & \times \end{bmatrix}$ & $P' = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$;

We have: $UP' = \begin{bmatrix} \times & \cdots & \times & \times \\ \times & \cdots & \times & \\ \vdots & \iddots & & \\ \times & & & \end{bmatrix}$ & $P'UP' = \begin{bmatrix} \times & & & \\ \times & \times & & \\ \vdots & & \ddots & \\ \times & \cdots & \times & \times \end{bmatrix}$

Lower Triangular

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~~$\Rightarrow$ Thus we have $A = PUQ = P($~~

$\Rightarrow$ Thus our upper triangular matrix U can be converted to a lower triangular matrix L by the virtue of P' defined earlier as : $\boxed{L = P'UP'}$

$\Downarrow$

$\boxed{U = P''LP''} \Longleftarrow \boxed{U = (P')^{-1} L (P')^{-1}}$

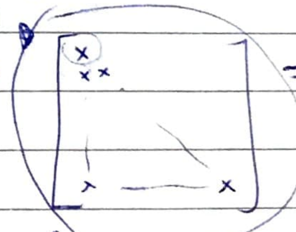
(Inverse of a Perm. matrix is also a Perm. matrix)

$\Rightarrow$ Thus : $A = PUQ = P(P''LP'')Q$

$\therefore A = (PP'')L(P''Q) = \boxed{P_0 L Q_0}$ Hence Proved

(Multiplication of Perm. matrices is also Perm. matrix)

(b) $\Rightarrow$ Given : $A = PLQ$ with L having no extra zeros



$\Rightarrow$ i^{th} Col has i Zeros

$\hookrightarrow$ i^{th} row has $m-(i+1)$ Zeros

Indices start from 0

* If a row r_i has c_{r_i} Zeros then it would be obtained from the row $r_{m-(c_{r_i}+1)}$ of matrix L

* If a Col c_i has c_{c_i} Zeros then it would be obtained from the Col $c_{(c_{c_i})}$ of matrix L

$\hookrightarrow$ Intuition is that the Permutation matrices just Swap the rows & Col and so the number of Zeros remains the Same.

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$\Rightarrow$ From the above intuition, to get $A^{(\text{from } L)}$ we need to:

- Swap rows i & $m - (c_{r_i}+1)$ $\Rightarrow$ Done by P
- Swap col i & (c_{c_i}) $\Rightarrow$ Done by Q

$\Rightarrow$ Thus to get P, we have: i^{th} row of P

If the i^{th} row of A has c_{r_i} zeros then: $P_i^T = [0 \text{ --- } 1 \text{ --- } 0]$

index of $(m-(c_{r_i}+1))$ (Starting from 0)

$O(m^2)$ Time (Going through all entries of A once)

$\Rightarrow$ Similarly to get Q: If i^{th} col of A has c_{c_i} zeros then: $Q_i = \begin{bmatrix} 0 \\ | \\ 1 \\ | \\ 0 \end{bmatrix}$ $\rightarrow$ index of (c_{c_i}) (Starting from 0)

i^{th} Column of Q

$O(m^2)$ Time (Going through all entries of A once)

$\Rightarrow$ Thus, we would get the Permutation matrices P & Q in $O(m^2)$ time.

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A. 2 : (a) ⇒ To get the Cholesky factorization normally, we'd have

$$A = \left[\begin{array}{c|c} a & \vec{b}^* \\ \hline \vec{b} & c \end{array}\right] = \underbrace{\left[\begin{array}{c|c} \sqrt{a} & 0 \\ \hline \frac{\vec{b}}{\sqrt{a}} & I \end{array}\right]}_{L_1} \underbrace{\left[\begin{array}{c|c} \boxed{1} & 0 \\ \hline 0 & c - \frac{\vec{b}\vec{b}^*}{a} \end{array}\right]}_{A_1} \underbrace{\left[\begin{array}{c|c} \sqrt{a} & \vec{b}^*/\sqrt{a} \\ \hline 0 & I \end{array}\right]}_{L_1^*}$$

$$A_1 \to L_2 A_2 L_2^* \; \vdots$$

$$\boxed{A = LL^*} \Leftarrow \boxed{A = (L_1 \cdots L_m)\, \underbrace{A_m}_{I}\, (L_1 - L_m)^*}$$

⇒ But now we don't want an Identity matrix. So the top-left element of A_1 need not be 1. So now we get:

$$A = \left[\begin{array}{c|c} a & \vec{b}^* \\ \hline \vec{b} & c \end{array}\right] = \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline \frac{\vec{b}}{a} & I \end{array}\right]}_{L_1} \underbrace{\left[\begin{array}{c|c} a & 0 \\ \hline 0 & c - \frac{bb^*}{a} \end{array}\right]}_{A_1} \underbrace{\left[\begin{array}{c|c} 1 & \vec{b}^*/a \\ \hline 0 & I \end{array}\right]}_{L_1^*}$$

$$\Rightarrow A = (L_1 - L_m)\, A_m\, (L_1 - L_m)^* \Rightarrow \boxed{A = LDL^*}$$

$$A_m \to D \;\Leftarrow\; \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nm} \end{bmatrix}$$

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Pseudo Code

⇒ The (Similar to what was defined in the class):

$D = 0$

$L = A$,

For cols $k = 1, \dots, m$:

let $\vec{b} = L_{k+1:m, k}$

$D_{kk} = l_{kk}$

$L_{k+1:m, k+1:m} = L_{k+1:m, k+1:m} - \frac{\vec{b}\vec{b}^*}{l_{kk}}$

$l_k = l_k / l_{kk}$

→ k^{th} column of matrix L

Only acts on the lower triangle of L

(b) ⇒ No. It won't work for all non singular Hermitian matrices. The counter example being $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

→ Division by l_{kk} won't be possible.

Ans 3: (a) ⇒ Let A & B be two c-sparse matrices of $m \times m$. To get AB, let us consider column B_i of B: AB_i $\boxed{i^{th} \text{ col. of } AB}$

⇒ Now if the no. of non zero elements in B_i is c. Then the columns of A that correspond to the zero elements of B_i can be ignored to get AB_i

⇒ Now this is equivalent to multiplying a submatrix of A of the dimension $m \times c$ ($A_{S.S}$), & the vector of dim. $c \times 1$ ($b_{S.S}$) corresponding to the c non-zeros of B_i

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$\Rightarrow$ So the i^{th} col. of AB = $A_{Sub}\ B_{Sub}$

$$= \boxed{\begin{bmatrix} & \\ & \end{bmatrix}_{m\times c} \begin{bmatrix} \\ \end{bmatrix}_{c\times 1}}$$

$\Rightarrow$ Then as the sparsity of cols = c, so the max. number of non zero elts in A_{sub} will be $c \times c$ = c^2

$\Rightarrow$ Now these c^2 entries can be spread such that each row of A_{Sub} has one of these non-zero elets. So the max. no. of non-zero elets in the Product $A_{Sub}\ b_{Sub}$ = $\boxed{m^2}$

$\Rightarrow$ Thus there are atmost m^2 elets in each column of AB. We can similarly prove for the rows of AB.

$\Rightarrow$ Thus the Product of two c-sparse matrices is only c^2-sparse.

(b) ⇒ Let us Take a matrix :

$$A = \begin{bmatrix} a & \times & & & & \times \\ \times & \times & \times & & & \\ & \times & \times & \times & & \\ & & \times & \times & \times & \\ & & & \ddots & \ddots & \ddots \\ & & & \times & \times & \times \\ & & & & \times & \times \\ \times & & & & \times & \times \end{bmatrix}_{m \times m} \quad \to \vec{b}^T, \; \to C, \; \vec{b} \leftarrow$$

3-sparse matrix ⇐ A Tridiagonal matrix with non-zero elts at $(0, m-1)$ & $(m-1, 0)$

⇒ To get the first row of A zeroed out (except for the first elt as it'll be a lower trgl matrix), we run one iteration of cholesky. Now: $\vec{b}\vec{b}^T = \begin{bmatrix} \times & & \times \\ & & \\ \times & & \times \end{bmatrix}$

⇒ So: $C - \frac{bb^T}{a} = \begin{bmatrix} \times & \times & & & \times \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \ddots & \ddots & \ddots \\ & & \times & \times & \times \\ \times & & & \times & \times \end{bmatrix}_{(m-1)\times(m-1)}$ → Same structure as A, just one row & column less.

$\Rightarrow$ So we got :

$$A \rightarrow \left[\begin{array}{c|c} \times & 0 \\ \hline \times & \\ & A_1 \\ \times & \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} \times & & 0 \\ \hline \times & \times & \\ \hline & \times & A_2 \\ \times & \times & \end{array}\right]$$

$\downarrow$ after m steps

$$\begin{bmatrix} \times & & & & \\ \times & \times & & & \\ & \times & \times & & \\ & & & \ddots & \\ \times & \times & \cdots & & \times \end{bmatrix}$$

$\Longleftarrow$ Not Sparse as the last row has all the entries non-zero

$\Longleftarrow$ Thus, for any m $\exists A \in \mathbb{C}^{m \times m}$ s.t. A is a 3-sparse HPD whose cholesky factorization is not sparse. ✓

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A 4 (a) ⇒ Let us have after n^{th} iterate; the value of $\vec{x}$ as $\vec{x}_n$. Now we have to minimize $\|A\vec{x}_n - b\|_2$

⇒ Taking $x_n = K_n y$ ⇒ $\min \|(AK_n) y - b\|_2$

$$AK_n \xrightarrow{QR} Q_n R_n$$

$$\Downarrow$$

$$\boxed{\min \|R_n y - Q_n^* b\|_2} \Leftarrow \boxed{\min \|Q_n R_n y - b\|_2}$$

⇒ Thus the algorithm we'd get will be:

for $n = 1, 2, \ldots$

$K_n = [K_{n-1} \mid A^{n-1} b]$ → Apply $A^{n-1}b$: $O(cm)$ flops

$AK_n = Q_n R_n$ (QR factorization) → To get $Q_n R_n$: $O\left(2\left(m - \frac{n}{3}\right)n^2\right)$ flops

Solve $\min_y \|R_n y - Q_n^* b\|_2$ → $O(n^3)$ GMRES

$x_n = K_n y$ → $O(mn)$ Flops (Matrix multiply)

To compute $AK_n = [AK_{n-1} \mid A^n b]$ → $\boxed{O(cm) \text{ flops}}$

AK_{n-1}: known (from last iteration)

Householder algo. for QR

⇒ This is the algorithm & the corresponding Operation Count.

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Ans 6: (a) ⇒ The instability matrix :

$$\begin{bmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & -1 & \cdots & & 1 \end{bmatrix} \Rightarrow \|U\| = O(2^{m-1})$$

(b) ⇒ The Pseudocode, just like that as Algo 21.1 :

$U = A,\ L = I,\ P = I,\ Q = I$

For $k = 1$ to $m-1$

Select i, l Pivot to maximize $|U_{k:m,\,k:m}|$

$u_{k,\,k:m} \leftrightarrow u_{i,\,k:m}$, $u_{:,k} \leftrightarrow u_{:,l}$ (Interchange Two rows & Two columns)

$l_{k,\,1:k-1} \leftrightarrow l_{i,\,1:k-1}$

$p_{k,:} \leftrightarrow p_{i,:}$

$q_{:,k} \leftrightarrow q_{:,l}$

For $j = k+1$ to m

$l_{jk} = u_{jk} / u_{kk}$

$u_{j,\,k:m} = u_{j,\,k:m} - l_{jk}\, u_{k,\,k:m}$

⇒ Colums of Q are exchanged since AQ swaps columns of A. And columns of U are also interchanged, along with it's rows, because of the addition of this matrix Q in the system.

(c) ⇒ (P) increases exponentially with m for Partial-Pivoting. Gets stable in the case of rook-Pivoting. Backward Error also increases exponentially in the case of Partial-Pivoting, always being more than rook-Pivoting. Has some fluctuations.

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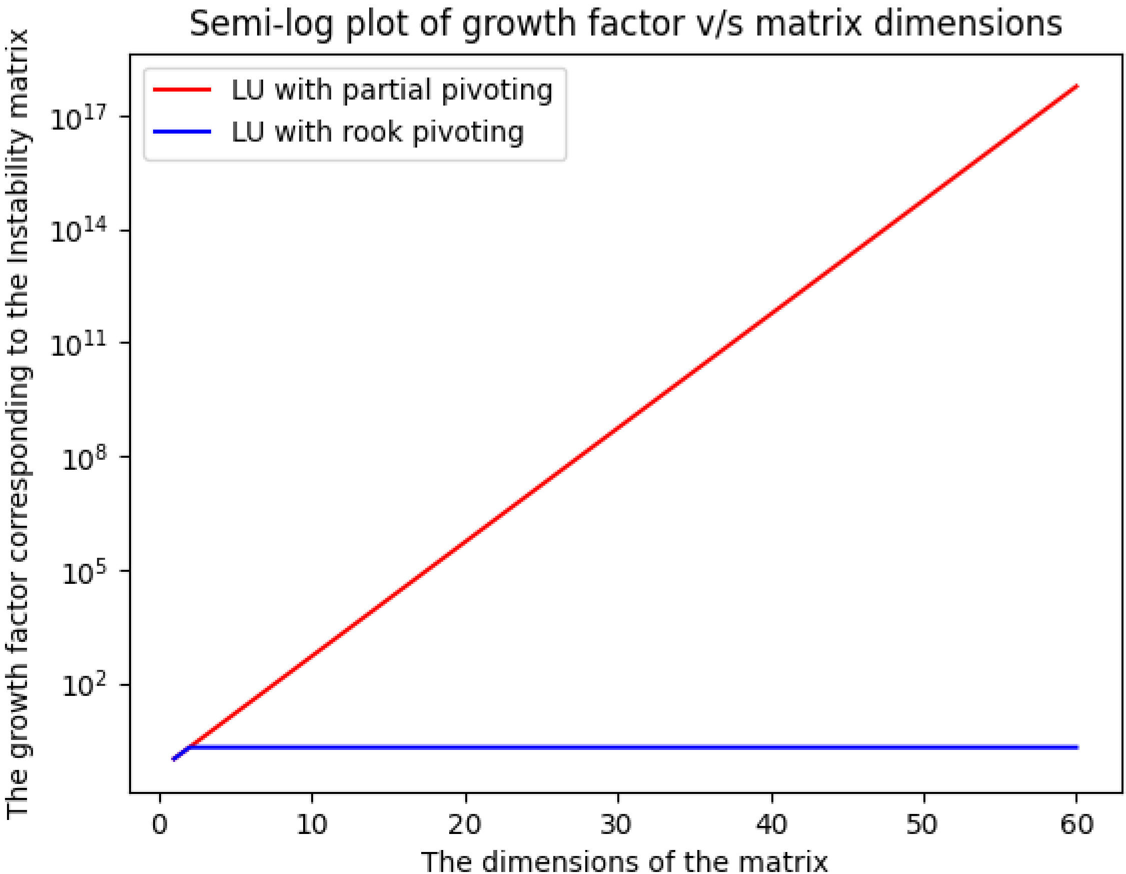
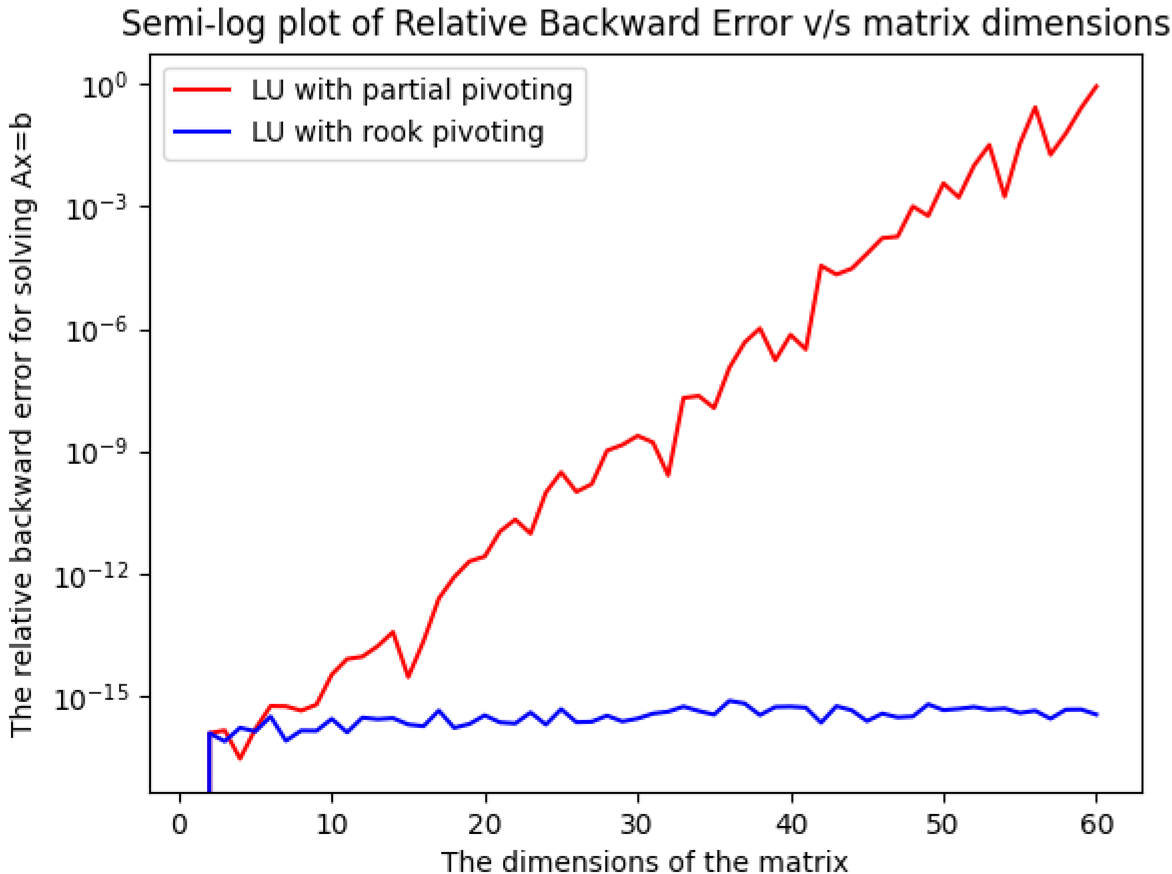


Semi-log plot of growth factor v/s matrix dimensions
LU with partial pivoting
LU with rook pivoting
10^{17}
10^{14}
10^{11}
10^{8}
10^{5}
10^{2}
0
10
20
30
40
50
60
The dimensions of the matrix
The growth factor corresponding to the Instability matrix


Semi-log plot of Relative Backward Error v/s matrix dimensions
LU with partial pivoting
LU with rook pivoting
10^{0}
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
The relative backward error for solving Ax=b
0
10
20
30
40
50
60
The dimensions of the matrix