પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question
Sub-Question No.

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COL726 Assignment 1

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Ans 1 (a) :⇒ We know that by the Supremum definition:

$$K_f(x) = \frac{\|x\|}{\|f(x)\|} \cdot \sup_{\delta x} \frac{\|\delta(f(x))\|}{\|\delta x\|} \quad \leftarrow (1)$$

$$K_g(y) = \frac{\|y\|}{\|g(y)\|} \sup_{\delta y} \frac{\|\delta(g(y))\|}{\|\delta y\|} \quad \leftarrow (2)$$

$$K_h(x) = \frac{\|x\|}{\|h(x)\|} \cdot \sup_{\delta x} \frac{\|\delta h(x)\|}{\|\delta x\|} \quad \leftarrow (3)$$

⇒ Now, using the fact that $h(x) = g(y)$, where $y = f(x)$, we can get:

$$K_h(x) = \frac{\|x\|}{\|g(y)\|} \cdot \sup_{\delta x} \frac{\|\delta g(y)\|}{\|\delta x\|} \quad \leftarrow (4)$$

where $y = f(x)$

Question / Sub-Question No.

$\square + \square + \square + \square = \square$

Q. ____

$\Rightarrow$ From Equation (4), we can have:

*
$$\frac{\|x\|}{\|g(y)\|} = \frac{\|x\|}{\|g(y)\|} \cdot \frac{\|y\|}{\|y\|} = \left(\frac{\|x\|}{\|f(x)\|} \cdot \frac{\|y\|}{\|g(y)\|}\right)$$

$(y = f(x))$ $\leftarrow$ (5)

* Similarly, we can get:

$$\sup_{\delta x}\left(\frac{\|\delta g(y)\|}{\|\delta x\|}\right) = \sup_{\delta x}\left(\frac{\|\delta g(y)\|}{\|\delta y\|} \cdot \frac{\|\delta f(x)\|}{\|\delta x\|}\right) \leftarrow (6)$$

$\Rightarrow$ Thus, using (5) & (6) in Eq^n (4) gives:

$$K_h(x) = \frac{\|x\|}{\|f(x)\|} \cdot \frac{\|y\|}{\|g(y)\|}\left(\sup_{\delta x}\left(\frac{\|\delta g(y)\|}{\|\delta y\|} \cdot \frac{\|\delta f(x)\|}{\|\delta x\|}\right)\right)$$

$\Downarrow$ Using $\frac{\|\delta g(y)\|}{\|\delta y\|} \leq \sup_{\delta y}\left(\frac{\|\delta g(y)\|}{\|\delta y\|}\right)$

$$K_h(x) \leq \left(\frac{\|x\|}{\|f(x)\|} \cdot \frac{\|y\|}{\|g(y)\|}\right)\left(\sup_{\delta x}\left(\frac{\|\delta f(x)\|}{\|\delta x\|} \cdot \left(\sup_{\delta y} \frac{\|\delta g(y)\|}{\|\delta y\|}\right)\right)\right)$$

$\leftarrow$ (7)

પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question Sub-Question No.

Q. ______

$\Rightarrow$ From eqⁿ ⑦, We can write :

$$K_h(x) \leq \left(\frac{\|x\|}{\|f(x)\|} \cdot \sup_{\delta x} \frac{\|\delta f(x)\|}{\|\delta x\|} \right) \cdot \left(\frac{\|y\|}{\|g(y)\|} \cdot \sup_{\delta y} \frac{\|\delta g(y)\|}{\|\delta y\|} \right)$$

$$\therefore \boxed{K_h(x) \leq K_f(x) \cdot K_g(y)}$$

પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question Sub-Question No.

$\square + \square + \square + \square = \square$

Q. ____

Ans 2 (a) $\Rightarrow$ We have : $\bar{f}(x) = \bar{x} \otimes (1 \ominus \bar{x})$

$$\Downarrow$$

$\left(fl(x) = x(1+\epsilon_1) \right.$ & $\left. |\epsilon_1| \le \epsilon_m \right)$

$$\bar{f}(x) = fl(x) \otimes (1 \ominus fl(x))$$

$\downarrow$ By floating Point axiom

$(1 - fl(x))(1+\epsilon_2)$

where $|\epsilon_2| \le \epsilon_m$

This Operator can also be converted like we did for $\ominus$ using FP axioms

$$\Downarrow$$

$$\bar{f}(x) = (fl(x) \times (1 - fl(x))) \times (1+\epsilon_2) \times (1+\epsilon_3)$$

with $|\epsilon_2|, |\epsilon_3| \le \epsilon_m$

$$\Downarrow$$

$$\boxed{\bar{f}(x) = (x(1+\epsilon_1)) \times (1 - x(1+\epsilon_1)) \times (1+\epsilon_4)}$$

where : $|\epsilon_1| \le \epsilon_m$ & $|\epsilon_4| \le 2\epsilon_m + O(\epsilon_m^2)$

$\Rightarrow$ The above Equation Can be written as :

$$\bar{f}(x) = x(1+\epsilon_1)(1+\epsilon_4) - x^2(1+\epsilon_1)(1+\epsilon_1)(1+\epsilon_4)$$

$$= \boxed{x(1+\epsilon_5) - x^2(1+\epsilon_6)}$$

where : $|\epsilon_5| \le 3\epsilon_m + O(\epsilon_m^2) + O(\epsilon_m^3)$ And $||\epsilon_6|| \le 4\epsilon_m + O(\epsilon_m^2) + O(\epsilon_m^3) + O(\epsilon_m^4)$

પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question Sub-Question No.

$\square + \square + \square + \square = \square$

Q. ____

$\Rightarrow$ Thus, $\bar{f}(x) - f(x) = \left(x(1+\epsilon_5) - x^2(1+\epsilon_6)\right) - (x - x^2)$

$\therefore$ $\boxed{f.e. = \left|x(\epsilon_5) - x^2(\epsilon_6)\right|}$

$\Downarrow$ (using $f.e \le |x\epsilon_5| + |x^2 \epsilon_6|$)

$\boxed{O(f.e) = O(3\epsilon_m x) + O(4\epsilon_m x^2)}$

$O(\epsilon_m x^2)$

(would get to infinity for $x \to \infty$)

b) $\Rightarrow$ we get, $\boxed{r.f.e = \frac{|f.e|}{|f(x)|} = \frac{|\epsilon_5 - x\epsilon_6|}{(1-x)}}$

$\Downarrow$

$\boxed{\text{Bound on r.f.e.} = \frac{|3\epsilon_m|}{|1-x|} + \frac{|4\epsilon_m x|}{|1-x|}}$

→ In the vicinity of $x=1$, rfe becomes very high.
Thus, not accurate at $\boxed{x=1}$

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પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question Sub-Question No.

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⇒ For Backward stability, we need :

$$\boxed{\tilde{f}(x) = f(\tilde{x})} \quad \text{for some } \tilde{x} \text{ s.t. } \boxed{\frac{|\tilde{x} - x|}{|x|} = O(\epsilon_m)}$$

Thus we have : $\boxed{x(1+\epsilon_5) - x^2(1+\epsilon_6) = \tilde{x} - \tilde{x}^2}$

$$\Downarrow$$

$$\tilde{x} = \tilde{x}^2 + x + x\epsilon_5 - x^2(1+\epsilon_6)$$

$$\Rightarrow (\tilde{x} - x) - (\tilde{x}^2 - x^2) = x\epsilon_5 - x^2\epsilon_6$$

$$\Rightarrow (\tilde{x} - x)(1 - (\tilde{x} + x)) = x(\epsilon_5 - x\epsilon_6)$$

$$\Rightarrow \boxed{\left|\frac{\tilde{x} - x}{x}\right| = \left|\frac{\epsilon_5 - x\epsilon_6}{1 - (\tilde{x} + x)}\right|}$$

→ Should be $O(\epsilon_m)$

⇒ Thus, here it can be seen that the above ξ^n fails the Backward Stability Criterion when $\boxed{x + \tilde{x} = 1}$

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Question
...b-Question No.

$\Rightarrow$ But we also know that $f(x) = f(1-x)$.
So if for some $\bar{x}$ not close to x, we get $x + \bar{x} = 1$ & $\tilde{f}(x) = f(\bar{x})$. Then we can have $1-\bar{x}$ close to x & $\tilde{f}(x) = f(1-\bar{x})$
So for $\bar{x} \neq x$ (ie $x \neq 0.5$), the System is still Backward Stable.

$\Rightarrow$ And when we look for $x = 0.5$, then it can be seen that $\bar{x}$ is already in the vicinity of x & satisfying $\tilde{f}(x) = f(\bar{x})$. And so for a difference between $|\bar{x} - x|$ as δ when $x = 0.5$, we can see that the R.H.S. is bounded as $O(1/\delta)$ (which, inspite of being very large, is still finite).

$\Rightarrow$ Thus the algorithm is backward stable at all values.

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Question
Sub-Question No.

Columns of X

Columns of the resulting matrix

Ans 3: ⇒ We can write $UX = \left[UX_1 \mid UX_2 \mid \cdots \mid UX_m \right]_{m \times m}$

⇒ Since X, L, Y can be broken down into columns to give us:

$$\left[(UX_1 + XL_1) \mid (UX_2 + XL_2) \mid \cdots \mid UX_m + XL_m \right] = \left[Y_1 \mid Y_2 \mid \cdots \mid Y_m \right]$$

⇒ If we start from the last column (ie m); we get:

$\boxed{UX_m + XL_m = Y_m}$ → Now using that L is a lower triangular matrix, we get: $\boxed{XL_m = l_{mm} X_m}$

Entry of m^{th} column of m^{th} row

(Only non-zero entry in the column L_m)

⇒ Thus, we get: $UX_m + l_{mm} X_m = Y_m$

$$\Downarrow$$

$$\boxed{(U + l_{mm} I) X_m = Y_m}$$

Upper triangular Matrix

↓

We solve this by Back substitution in $O(m^2)$ time & get the column X_m.

પ્રશ્ન /
પેટા પ્રશ્ન ક્રમાંક
Question
Sub-Question No.

$\square + \square + \square + \square = \square$

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$\Rightarrow$ Then to find X_{m-1}, we look at the $(m-1)^{th}$ column:

$$U X_{m-1} + X L_{m-1} = Y_{m-1}$$

$\downarrow$ L_{m-1} has only 2 non-zero elements ($\because$ Lower triangular)

$$(l_{(m-1),(m-1)} X_{m-1}) + (l_{m,m-1} X_m)$$

$\Downarrow$

Upper triangular

$$\left(U + l_{(m-1),(m-1)} I\right) X_{m-1} = \left(Y_{m-1} - l_{m,m-1} X_m\right).$$

$\hookrightarrow$ Gives X_{m-1} by back substitution in $O(m^2)$.

$\Rightarrow$ Similarly, we can find $X_{m-2}, X_{m-3}, \ldots, X_1$. Thus we get X in $O(m^3)$ time.

$\hookrightarrow$ To find X_{m-j}:

$$\left(U + l_{(m-j),(m-j)} I\right) X_{m-j} = \left(Y_{m-j} - \sum_{i=0}^{j-1} l_{m-i,m-j} X_{m-i}\right)$$

પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question Sub-Question No.

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Q. ______

Ans 4 i

⇒ Given $\|Ax\|_2 = \|x\|_2$; we know that: $\boxed{|(Ax)^*(Ax)| = |x^* x|}$

Say matrix B

⇒ So, $\boxed{|x^* (A^* A) x| = |x^* x|}$

⇒ By making use of the expression for $x^* B x$; we can see that: $\boxed{B = I}$ ⇒ $\boxed{A^* A = I}$ → $\boxed{m \times m}$

⇒ Thus; the matrix A is Unitary ← (1)

⇒ Now, if we try to break this into cases:

* If $m \leq n$: Using $\boxed{\text{rank}(A^* A) = \text{rank}(A)}$
 $\text{rank}(A^*A) \to m$, $\text{rank}(A) \to \leq m$

A is Unitary iff the colns are Orthonormal & so here $\boxed{\text{rank}(A) = m}$ ← (2)

* If $n \leq m$: Using $\boxed{\text{rank}(A^* A) = \text{rank}(A)}$ → $\leq n$
 $\text{rank}(A^*A) \to m$

Here, we can see that the ranks of these matrices can never be equal. And so, A can't be a Unitary matrix here if $\boxed{m > n}$

⇒ So, the Conditions that we obtained here are:

- $\boxed{\text{Necessary}}$: The matrix $A_{m\times n}$ is unitary
- $\boxed{\text{Sufficient}}$: $\boxed{n > m}$ & $\boxed{\text{rank}(A) = m}$

પ્રશ્ન / પેટા પ્રશ્ન ક્રમાંક
Question
Sub-Question No.

Q.

Ans 5 : $\Rightarrow$ Let us assume $M(\lambda)$ to be Singular. This implies that $null(M(\lambda)) \neq \{0\}$. Let us take a Vector $x \in null(M(\lambda))$ & $x \neq 0$.

Hyp (P)

$\Rightarrow$ So, $M(\lambda)x = 0 \Rightarrow (I + \lambda A)x = 0$

$$\Rightarrow \lambda A x = -x$$

$$\Rightarrow |\lambda| \, \|Ax\| = \|-x\| = \|x\|$$

$$\Rightarrow |\lambda| \, \|A\| \, \|x\| \geq \|x\|$$

(using $\|AB\| \leq \|A\| \cdot \|B\|$)

$\Rightarrow$ Thus, we get : $\boxed{|\lambda| \geq 1/\|A\|}$ (Conclusion : q)

$(\because \|x\| \neq 0)$

$\Rightarrow$ Thus by taking a Contrapositive of the above statement : $\boxed{|\lambda| < \frac{1}{\|A\|}} \longrightarrow \boxed{M(\lambda) \text{ is non Singular}}$

$$\downarrow$$

Thus the Value of $\delta = \dfrac{1}{\|A\|}$

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Ans 6 :

⇒ From the graph, it can be seen that while the actual solution of the recurrence relation decreases with value of k, the computed solution first decreases along with the actual solution & then after a certain k, shows a dramatic increase in values.

⇒ If we consider the difference $|\bar{x}(k) - x(k)|$ where $x(k)$ is the true solution & $\bar{x}(k)$ is the seq. computed solⁿ at index k, it was observed that this difference monotonicly increases with increasing values of k.

⇒ The reason can be attributed to the fact that the floating point operations carried out while computing the different keep on accumulating errors.

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Question Sub-Question No.

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⇒ Thus : $\hat{X}_{k+1} = 2.25\, \hat{X}_k \ominus 0.5\, \hat{X}_{k-1}$

$$\Rightarrow \hat{X}_{k+1} = \Big(\big(2.25\, X_k (1+\epsilon_1)\big) - \big(0.5\, X_{k-1} (1+\epsilon_2)\big)\Big)(1+\epsilon_3)$$

(where $|\epsilon_1|, |\epsilon_2|, |\epsilon_3| < \epsilon_m$)

$$\Rightarrow \hat{x}_{k+1} = (1+\epsilon_3)\Bigg((1+\epsilon_1)(2.25)\overbrace{\Big(\big(2.25\, x_{k-1}(1+\epsilon_1') - 0.5\, x_{k-2}(1+\epsilon_2')\big)(1+\epsilon_3')\Big)}^{x_k}$$

$$- (1+\epsilon_2)(0.5)\underbrace{\Big(\big(2.25\, x_{k-2}(1+\epsilon_1'') - 0.5\, x_{k-3}(1+\epsilon_2'')\big)(1+\epsilon_3'')\Big)}_{x_{k-1}}\Bigg)$$

⇒ Thus, we can see that these errors keep on getting accumulated for higher values of k.

⇒ So initially $|\hat{x}(k) - x(k)|$ would be very small but with increasing k, the error would keep on increasing till it gets too high as compared to the actual value of $x(k)$.

(The absolute & relative errors start off from an order of 10^{-17} & exponentially rise to very high values)

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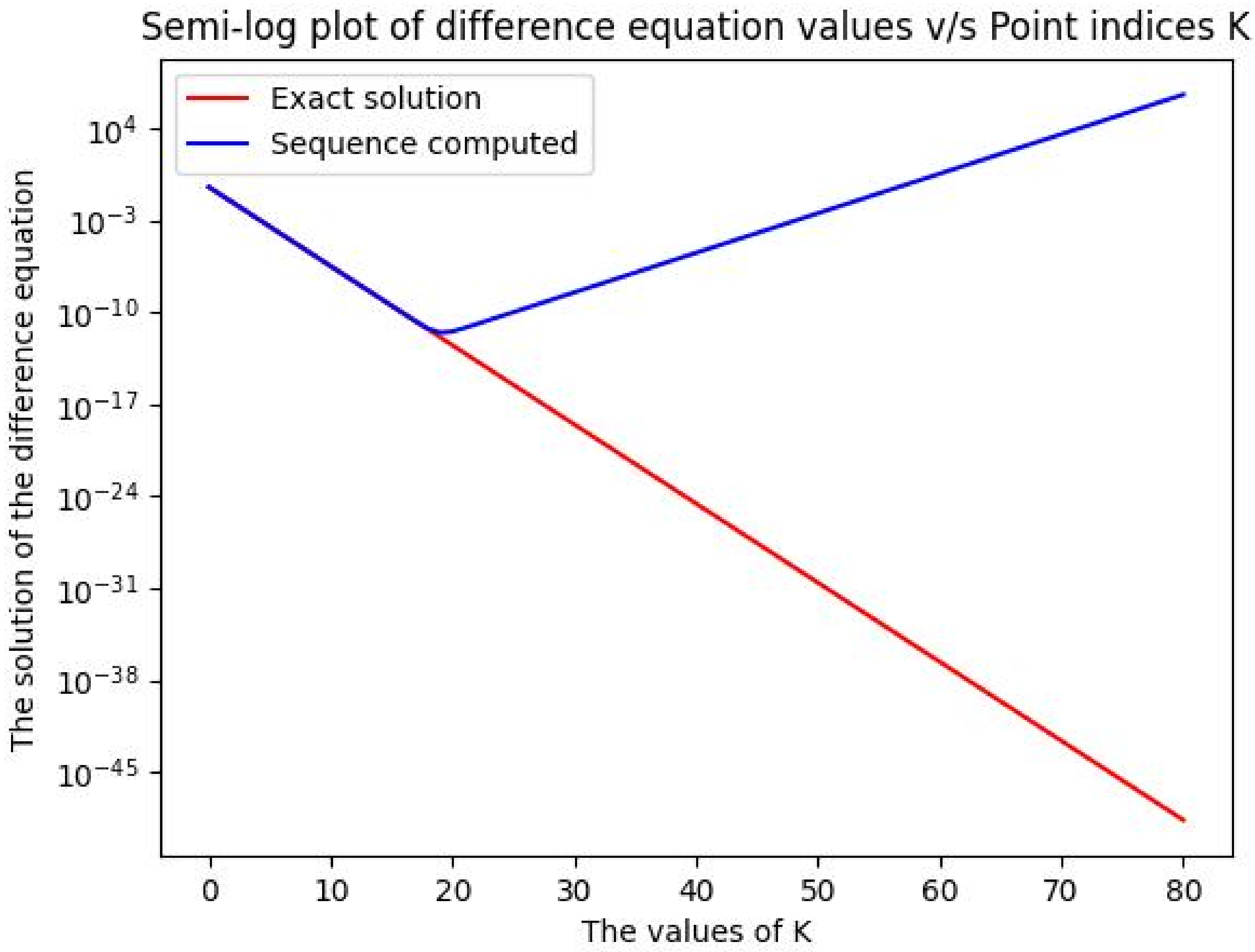
⇒ ~~Another reason can also be attributed to the fact that the relative error for~~

⇒ So becau of lower initial values of relative & absolute errors, the computed sequence solution initially follows the same trend as the actual soln & then because of the exponential growth, the errors dominate the actual answer & so the sequence soln can be seen to be diverging linearly in the log scale plot.



Semi-log plot of difference equation values v/s Point indices K
Exact solution
Sequence computed
10^{4}
10^{-3}
10^{-10}
10^{-17}
10^{-24}
10^{-31}
10^{-38}
10^{-45}
The solution of the difference equation
0
10
20
30
40
50
60
70
80
The values of K