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COL726 Assignment 2

Raval Vedant Sanjay 2017CS10366

Ans 1:

$\Rightarrow$ We have $\boxed{C = AB}$ with all of them being $m \times m$ matrices.

$\Rightarrow$ Now, we can look at the Invertibility of matrix A, B

→ Both A & B are invertible:

→ We can use: $\|C\|_2 = \|AB\|_2 \leq \|A\|_2 \cdot \|B\|_2$

$$\Rightarrow \boxed{c_1 \leq a_1 b_1} \leftarrow (1)$$

→ Also taking inverse on both sides, we have: $C^{-1} = B^{-1}A^{-1} \Rightarrow \|C^{-1}\|_2 \leq \|B^{-1}\|_2 \|A^{-1}\|_2$

$$\Rightarrow \boxed{\frac{1}{c_m} \leq \frac{1}{b_m} \cdot \frac{1}{a_m}}$$

($\because$ A, B invertible → C invertible. So $c_m, a_m, b_m \neq 0$
And c_m smallest singular value for C.
So $1/c_m$ largest singular value for C)

$$\Rightarrow \boxed{c_m \geq a_m b_m} \leftarrow (2)$$

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* ~~Atleast one of A & B is not invertible~~

→ ~~So here we can say that C won't be invertible~~

→ We can also write : $\boxed{CB^{-1} = A}$ $\Rightarrow \|A\|_2 \le \|C\|_2 \|B^{-1}\|_2$
(From $C = AB$)

$\Rightarrow \boxed{c_1 \ge a_1 b_m}$ ← (3)

→ Similarly, from $\boxed{A^{-1}C = B}$ $\Rightarrow \boxed{c_1 \ge b_1 a_m}$ ← (4)

→ Combining the eqⁿ (1), (3), (4) : $\boxed{a_1 b_1 \ge c_1 \ge \max(a_1 b_m, b_1 a_m)}$

→ Also, if we consider : $\boxed{C^{-1} = B^{-1}A^{-1}}$

$\boxed{c_m \le b_1 a_m} \Leftarrow \boxed{BC^{-1} = A^{-1}}$

$\boxed{C^{-1}A = B^{-1}} \Rightarrow \boxed{c_m \le a_1 b_m}$

→ The above inequalities we obtained in a similar manner as done for the previous case & so by combining them along with eqⁿ (2) :

$\boxed{a_m b_m \le c_m \le \min(a_1 b_m, b_1 a_m)}$

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* If atleast one of A & B are not invertible

→ We can say that C won't be invertible. So $\boxed{c_m = 0}$

→ Now: $a_m b_m = 0$ ($\because$ atleast one of them (a_m, b_m) will be zero)

& $\min(a_1 b_m, b_1 a_m) = 0$ ($\because$ atleast one of (a_m, b_m) is zero)

→ So we can trivially say: $\boxed{a_m b_m \le c_m \le \min(a_1 b_m, b_1 a_m)}$

→ Now, we know that $\boxed{c_1 \le a_1 b_1}$ will hold regardless of the case that A & B are invertible or not.

→ Now, for proving $\boxed{c_1 \ge \max(a_1 b_m, b_1 a_m)}$, we consider the further two cases

(1) Both A & B are not invertible $\Rightarrow$ $a_m = 0$ & $b_m = 0$ $\Rightarrow$ $\max(a_1 b_m, b_1 a_m) = 0$

$\Rightarrow \boxed{c_1 \ge \max(a_1 b_m, b_1 a_m)}$ Trivially

(2) One of A & B is not invertible $\Rightarrow$ WLOG assume that A isn't invertible. But B is. So $C = AB \Rightarrow \boxed{CB^{-1} = A}$

$\boxed{c_1 \ge a_1 b_m}$

$\boxed{c_1 \ge \max(a_1 b_m, b_1 a_m)}$ (as $a_m = 0$)

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$\Rightarrow$ So, here also we can say: $\boxed{a_1 b_1 \geq c_1 \geq \max\left(a_1 b_m, b_1 a_m\right)}$

$\Rightarrow$ So for any two matrix A, B & a matrix C s.t. $C = AB$, we proved the asked inequalities.

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Ans 2:

$\Rightarrow$ Let us define $T' = \{ v \in \mathbb{C}^m \mid u^* v \ \forall u \in S \}$, we need to prove that $T = T'$.

$\Rightarrow$ Now, from the definition of Orthogonal Subspaces; if S & T are orthogonal then $\forall u \in S$ & $\forall v \in T$, $u^* v = 0$. Now suppose some $t \in T$. Then we can trivially see that $t \in T'$ too. Thus $\boxed{T \subseteq T'}$

$\Rightarrow$ ~~[illegible]~~ Now, let some $t \in T'$. Since $t \in \mathbb{C}^m$ & we have that $S + T = \mathbb{C}^m$, $\exists u \in S$ & $\exists v \in T$ s.t. $t = u + v$. Now, from the definition of T', we have $p^* t = 0$ ~~[illegible]~~ with $\forall p \in S$.

$\Rightarrow$ Thus: $p^* u + p^* v = 0 \Rightarrow \boxed{p^* u = 0 \quad \forall p \in S}$

$\Downarrow$

$\boxed{u = 0}$

$\boxed{t \in T} \Leftarrow \boxed{t = v} \Leftarrow$

$\hookrightarrow$ Thus, we can see that $\boxed{T' \subseteq T}$

$\Rightarrow$ Thus, we get $\boxed{T = T'}$. Note that till now, we only used the condition of $S \perp T$ & $S + T = \mathbb{C}^m$. If we add an additional constraint of $S \cap T = \{0\}$ then the set T still remains the same as it already satisfies this constraint.

$$\left(v \in \mathbb{C}^m \text{ s.t. } v \in T \ \& \ v \in S \Rightarrow v^* v = 0 \Rightarrow \|v\|_2 = 0 \Rightarrow v = 0 \right)$$

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Ans 3: (a) $\Rightarrow$ Using $\|x_i\|_2 = \|y_i\|_2 \; \forall i \Rightarrow \boxed{x_i^* x_i = y_i^* y_i \quad \forall i}$ $\leftarrow$ (1)

(Using $\|x\|_2^2 = x^* x$)

$\Rightarrow$ Using $\|x_i - x_j\|_2 = \|y_i - y_j\|_2 \; \forall i, j$ s.t. $i \neq j$ $\Rightarrow$ $\|x_i\|_2^2 + \|x_j\|_2^2 - 2x_i^* x_j$

$= \|y_i\|_2^2 + \|y_j\|_2^2 - 2y_i^* y_j$

(Using : $\|x-y\|_2^2 = (x^* - y^*)(x-y)$
$= \|x\|_2^2 + \|y\|_2^2 - 2x^* y$)

$\boxed{x_i^* x_j = y_i^* y_j} \; \forall i \neq j$ $\leftarrow$ (2)

$\Rightarrow$ Thus, we get $\boxed{x_i^* x_j = y_i^* y_j}$ for all i, j

$\Rightarrow$ Now, looking at $X^* X$, we can see that $\boxed{(X^* X)_{ij} = x_i^* x_j}$

(Similarly for $Y^* Y$)

$\Rightarrow$ Using this, we get $\boxed{X^* X = Y^* Y}$

$\rightarrow \boxed{X^T X = Y^T Y}$ (Since they are real matrices)

⇒ Now, if we consider the (reduced) QR factorization of X & Y:

$$X = \hat{Q}_x \hat{R}_x \quad \& \quad Y = \hat{Q}_y \hat{R}_y \implies \boxed{\hat{R}_x^T \hat{Q}_x^T \hat{Q}_x R_x = \hat{R}_y^T \hat{Q}_y^T \hat{Q}_y \hat{R}_y}$$

$$\Downarrow$$

$$\boxed{\hat{R}_x^T \hat{R}_x = \hat{R}_y^T \hat{R}_y}$$

$$(\because \hat{Q}_x^T \hat{Q}_x = \hat{Q}_y^T \hat{Q}_y = I)$$

⇒ Now since $\hat{R}_x$ is a $n \times n$ Upper Triangular matrix:

$$\hat{R}_x^T \hat{R}_x = \begin{bmatrix} r_{11} & & & \\ r_{12} & r_{22} & & \\ \vdots & & \ddots & \\ r_{1n} & \cdots & & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & & \ddots & \\ & & & r_{nn} \end{bmatrix} = \begin{bmatrix} r_{11}^2 & r_{11} \cdot r_{12} & \cdots & r_{11} \cdot r_{1n} \\ r_{11} r_{12} & & & \\ \vdots & & & \\ r_{11} \cdot r_{1n} & & & \end{bmatrix}$$

Zero Entries

(n-1 x n-1) Segment

$$\begin{bmatrix} r_{22}^2 + r_{12}^2 & \cdots & r_{12} r_{1n} + r_{22} r_{2n} \\ \vdots & & \\ r_{12} r_{1n} + r_{22} r_{2n} & & \end{bmatrix}$$

(n-1 x n-1)

Similarly for the (n-2 x n-2) sized sub matrix & so on...

⇒ Having This, if we Try to get $\boxed{\hat{R}_x^T \hat{R}_x = \hat{R}_y^T \hat{R}_y}$, we can

See that $\boxed{(r_x)_{11}^2 = (r_y)_{11}^2}$ ⇒ $\boxed{(r_x)_{11} = (r_y)_{11}}$

Comparing The 1st Diagonal entry

Then we Compare the first row / Column & get $\boxed{(r_x)_{11} (r_x)_{1j} = (r_y)_{11} (r_y)_{1j}}$

⇓

$\boxed{(r_x)_{1j} = (r_y)_{1j} \quad \forall j > 1}$

Then we Compare the Second diagonal element and get $\boxed{(r_x)_{22} = (r_y)_{22}}$

Then, we See the Second row / Column & on Comparing & using the Previous results for r_x & r_y, we get : $\boxed{(r_x)_{2j} = (r_y)_{2j} \quad \forall j > 2}$

And So on

⇒ Then by ~~Comparing~~ Comparing like this, we get : $\boxed{\hat{R}_x = \hat{R}_y}$

Hence Proved.

(b) ⇒ By Taking all $Qx_1 = y_1$, $Qx_2 = y_2$, ——— $Qx_n = y_n$; we can see that $\boxed{QX = Y}$

⇒ Now Since $\boxed{\hat{R}_x = \hat{R}_y}$ ⇒ $\boxed{R_x = R_y}$ (the last rows are just extended by 0)

Full QR

⇒ Now, Using $\boxed{X = Q_x R_x}$ & $\boxed{Y = Q_y R_y}$, we can see

$$Q\,(Q_x R_x) = Q_y R_y \implies \boxed{Q\,Q_x = Q_y}$$

(Usg $R_x = R_y$ & the fact that they are Upp Trgl)

$$\Downarrow$$

$$\boxed{Q = Q_y\,Q_x^T}$$

($\because$ Q_x is a Square matrix with Orthonorml Columns)

$$\boxed{Q \text{ is orthgl as } Q^T Q = I}$$

$$\left(Q_x Q_y^T Q_y Q_x^T = Q_x Q_x^T = I \right)$$

Ans 4: (a) $\Rightarrow$ we have : $FA = A - \dfrac{2\, v v^*}{v^* v} A$

$$\boxed{FA = A + v \left(\frac{-2 v^* A}{v^* v} \right) = A + v w^*}$$

$\frac{-2 v^* A}{v^* v} \to w^*$

(i) Computing F first & then multiplying with A

$$F = I - \frac{2\,(v v^*)}{(v^* v)}$$

4. $I - \frac{2 v v^*}{v^* v}$ — Needs m^2 Subtractions

3. $\frac{2 v v^*}{v^* v}$ — Needs $2m^2$ multiplications $\left(2 \times (v^* v^{-1}) \times \alpha_{ij}\right.$ with $\left.\alpha = v v^*\right)$

1. $v v^*$ — Needs m^2 multiplications

2. $v^* v$ — Needs m multiplications & $(m-1)$ additions

Note : 1., 2., 3., 4. are the order of Computation of these Terms.
Thus we get vv^* first, then v^*v, then $\frac{2(vv^*)}{(v^*v)}$, & then $I - \beta$

$\beta = \frac{2(vv^*)}{(v^*v)}$

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→ Thus, we needed : $\boxed{\sim 4m^2 \text{ flops}}$ to Compute F

→ Then to get FA, We will need m multiplications & (m-1) additions to Compute each element of FA

→ $\boxed{\sim (2m)(mn) \text{ flops}}$

→ Thus the asymptotic Operation Count here : $\boxed{\sim 2m^2 n \text{ flops}}$ ✓

(ii) Computing W^* first, then VW^*, & then matrix addition

* $W^* = \dfrac{-2\, V^*A}{V^*V}$

3. 1. V^*A → $(2m-1)(n)$ flops .

2. V^*V → m multiplications + (m-1) additions

→ 2n flops

→ Needs $\boxed{\sim 2mn \text{ flops}}$ to be Computed

→ VW^* → Needs $\boxed{\sim mn \text{ flops}}$ To be Computed

→ $A + VW^*$ → Needs $\boxed{\sim mn \text{ flops}}$ to be Computed

→ Thus the asymptotic Operation Count here : $\boxed{\sim 4mn \text{ flops}}$ ✓

⇒ Thus, the Second way is much more efficient than the first.

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(b) ⇒ Given $\dfrac{\|\hat{v} - v\|_2}{\|v\|_2} = O(\epsilon_m)$, we have $\boxed{\hat{v} = v(1+\epsilon_1)}$

ϵ_1 is $O(\epsilon_m)$

⇒ Now, if we consider the Errors Part in the different Operators involved here; Then we get:

$$\hat{F} = \left(I - 2\left(\frac{vv^*(1+\epsilon_1)^2(1+\epsilon_3)}{v^*v(1+\epsilon_1)^2(1+\epsilon_3)}\right)(1+\epsilon_4)\right)(1+\epsilon_5)$$

All ϵ_i's are $O(\epsilon_m)$

Can be reduced to

⇓

(Along with $1+\epsilon_5$)

Ignored

$$\hat{F} = I(1+\epsilon_5) - 2\left(\frac{vv^*}{v^*v}\right)\left(1+\epsilon_6 + O(\epsilon_m^2) + \dots\right)$$

with ϵ_5, ϵ_6 in $O(\epsilon_m)$

⇓

$$\|\hat{F} - F\|_2 = \left\| I\epsilon_5 - 2\epsilon_6\left(\frac{vv^*}{v^*v}\right)\right\|_2 \le \epsilon_5 \overbrace{\|I\|_2}^{1} + 2\epsilon_6 \overbrace{\left\|\frac{vv^*}{v^*v}\right\|_2}^{1}$$

$$\boxed{\|\hat{F} - F\|_2 \le \epsilon_5 + 2\epsilon_6 \le k\epsilon_m}$$

(as ϵ_5, ϵ_6 in $O(\epsilon_m)$)

⇓

Thus, $\boxed{\|\hat{F} - F\|_2 = O(\epsilon_m)}$ H.P.

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⇒ If we take Error gr by (fl) operator as ϵ_7 in $O(\epsilon_m)$, we get

⇒ $$fl(\hat{F}A) = \left(A(1+\epsilon_5) - 2\left(\frac{vv^*A}{v^*v}\right)(1+\epsilon_6)\right)(1+\epsilon_7)$$

$$= A(1+\epsilon_8) - 2\left(\frac{vv^*A}{v^*v}\right)(1+\epsilon_9)$$ with ϵ_8, ϵ_9 in $O(\epsilon_m)$

⇒ Now, $$F^2 = \left(I - \frac{2vv^*}{v^*v}\right)\left(I - \frac{2vv^*}{v^*v}\right) = I - \frac{4vv^*}{v^*v} + 4\left(\frac{vv^*}{v^*v}\right)^2 = I$$

$$\left(\frac{vv^*}{v^*v}\right)^2 = \frac{v(v^*v)v^*}{(v^*v)(v^*v)} = \frac{vv^*}{v^*v}$$

& $$\|F\|_2 \le \|I\|_2 + 2\left\|\frac{vv^*}{v^*v}\right\|_2 = 3$$ $\rightarrow 1$

& $$FA = A - \frac{2vv^*}{v^*v}A$$

⇒ Using the above eq^ns ⇒ $$fl(\hat{F}A) = \left(A - \frac{2vv^*A}{v^*v}\right) + \left(A\epsilon_8 - \frac{2\epsilon_9 vv^*A}{v^*v}\right)$$

$$fl(\hat{F}A) = FA + IA\left(\epsilon_8 - 2\epsilon_9\left(\frac{vv^*}{v^*v}\right)\right)$$

$$\boxed{fl(\hat{F}A) = F\left(A + FA\left(\epsilon_8 - 2\epsilon_9\left(\frac{vv^*}{v^*v}\right)\right)\right)}$$ (using $I = F.F$)

⇓

$$\boxed{\delta A = FA\left(\epsilon_8 - 2\epsilon_9\left(\frac{vv^*}{v^*v}\right)\right)}$$

$$\boxed{\|\delta A\|_2 \le \epsilon_8\|F\|_2\|A\|_2 + 2\epsilon_9\|F\|_2\|A\|_2\left\|\frac{vv^*}{v^*v}\right\|_2}$$

⇒ $$\frac{\|\delta A\|_2}{\|A\|_2} \le (3\epsilon_8 + 6\epsilon_9)$$ in $O(\epsilon_m)$ ⇒ $$\boxed{\frac{\|\delta A\|_2}{\|A\|_2} = O(\epsilon_m)}$$

H.P.

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Ans 5 : (a) $\Rightarrow$ We desire to minimize the residual norm $\|r\|_2 = \|b - Ax\|_2$. We claim that this is possible iff $r \perp$ range $(A) \Rightarrow \boxed{A^* r = 0}$

* Then, from the definition of Orthogonal Projectors; the above statement would imply:

$\boxed{y = Ax = Pb}$ would be the unique point minimizing $\|b - y\|_2$

$\rightarrow$ Orthogonal Projector onto Range (A)

* The Proof of this claim can be proceeded by taking some $z \neq y$ to be another point in range (A).

* Now since $(z-y)$ is orthogonal to $(b-y)$, we can write

$$\|b - z\|_2^2 = \|(b-y) + (y-z)\|_2^2 = \|b-y\|_2^2 + \|y-z\|_2^2$$

($\because$ $z-y$ & $b-y$ are orthogonal)

H.P. $\boxed{\|b-z\|^2 > \|b-y\|^2}$

$\Rightarrow$ Thus we are needed to obtain the orthogonal Projector matrix P. As obtained in Eqn (6.6) of Trefethen & Bau, this orthogonal Projector P can be given as: $\boxed{P = \hat{U}\hat{U}^*}$ where $\boxed{A = \hat{U}\hat{\Sigma}\hat{V}^*}$ (reduced SVD)

$\Rightarrow$ We can obtain $\hat{U}$ for the U corresponding to the full SVD of A ($A = U \Sigma V^*$) by taking the first r columns of U.

$\Rightarrow$ Thus; we get the unique vector $\boxed{y = \hat{U}\hat{U}^* b}$

(b) $\Rightarrow$ Taking the full SVD of A again as $A = U\Sigma V^*$; we can make use of the fact that $\boxed{VV^* = I}$ (as V is unitary)

$$\underbrace{\min_x \| b - Ax \|_2^2 = \min_x \| U^* (Ax - b) \|_2^2}$$

$\| x \|_2^2 = \| U^* x \|_2^2$ if U^* is Unitary

$$= \min_x \| U^* (AVV^* x - b) \|_2^2$$

$$= \min_x \| U^* (U \Sigma V^* VV^* x - b) \|_2^2$$

$$= \min_x \| \Sigma V^* x - U^* b \|_2^2$$

$\Rightarrow$ Now taking $V^* x$ as z; we can obtain:

$$\min_x \| b - Ax \|^2 = \min_z \| \Sigma z - U^* b \|_2^2$$

$$= \boxed{\sum_{i=1}^{r} (\sigma_i z_i - u_i^* b)^2 + \sum_{i=r+1}^{m} (u_i^* b)^2}$$

(Because Rank of $A = r \Rightarrow \sigma_j = 0$ for $\forall j \geq r+1$)

$\Rightarrow$ So, to get the minimum value, we can see:

$$\boxed{\sigma_i z_i = u_i^* b \quad \forall i = 1 \dots r}$$

$$\Downarrow$$

$$z_i = \begin{cases} \dfrac{u_i^* b}{\sigma_i} & 1 \le i \le r \ \& \ i \in \mathbb{N} \\ \text{Arbitrary} & r+1 \le i \le n \ \& \ i \in \mathbb{N} \end{cases}$$

$\Rightarrow$ We have $z = V^* x \Rightarrow \boxed{x = Vz}$ ($\because V$ is unitary).

$\hookrightarrow$ General Solution of The least-Squares Problem

$\Rightarrow$ Now, if we want to minimize $\|x\|_2$, we know that

$\|x\|_2 = \|Vx\|_2 = \|z\|_2 \longrightarrow$ will be minimal if $|z_j| = 0 \quad \forall j \ge r+1$

($\because V$ is unitary)

$$\boxed{x' = Vz'} \Leftarrow \boxed{z'_i = \begin{cases} \dfrac{u_i^* b}{\sigma_i} & i = 1, 2, \dots r \\ 0 & i = r+1, \dots n \end{cases}}$$

$\boxed{\text{Minimizes } \|x\|_2}$

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Ans 6: (a) $\Rightarrow$ Suppose $[p] = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$ & $[q] = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$ $(p, q \in P_n)$

$\Rightarrow$ Now: $(p,q) = \int_{-1}^{1} \overline{p(x)}\, q(x)\, dx$

$\therefore (p,q) = \int_{-1}^{1} (\bar{a}_0 + \bar{a}_1 x + \_\_\_ + \bar{a}_n x^n)(b_0 + b_1 x + \_\_\_ + b_n x^n)\, dx$

$= \int_{-1}^{1} \left((\bar{a}_0 b_0) + (\bar{a}_1 b_0 + \bar{a}_0 b_1) x + (\bar{a}_2 b_0 + \bar{a}_1 b_1 + \bar{a}_0 b_2) x^2 + \_\_\_ \right) dx$

$$\boxed{\text{Coefficient of } x^j = \sum_{i=0}^{j} (\bar{a}_i\, b_{j-i})}$$

$\boxed{\text{Highest Power Term} : x^{2n}}$

$\Rightarrow$ Doing the above integral, we get:

$$\boxed{(p,q) = 2\left(\bar{a}_0 b_0 + \frac{(\bar{a}_2 b_0 + \bar{a}_1 b_1 + \bar{a}_0 b_2)}{3} + \frac{(\bar{a}_4 b_0 + \bar{a}_3 b_1 + \bar{a}_2 b_2 + \bar{a}_1 b_3 + \bar{a}_0 b_4)}{5} + \dots \right)}$$

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$\Rightarrow$ In the above expression, if we consider the term $\bar{a}_i b_j$ with $0 \le i, j \le n$, the co-efficient would be:

$$C_{ij} = \begin{cases} \dfrac{2}{i+j+1} & \text{if } (i+j) \text{ is even} \\ 0 & \text{Else} \end{cases}$$

$\Rightarrow$ Thus, we have: $\boxed{(p, q) = \sum_j \sum_i \bar{a}_i C_{ij} b_j}$

$\Rightarrow$ Now, we also know that: $\boxed{(p)^* G [q] = \sum_j \sum_i \bar{a}_i G_{ij} b_j}$

$\Rightarrow$ From comparing the above two Eq^n's, we get:

$$\boxed{G_{ij} = \begin{cases} \dfrac{2}{i+j+1} & \text{if } (i+j) \text{ is even} \\ 0 & \text{Else} \end{cases}}$$

$\boxed{0 \le i, j \le n}$

($n+1 \times n+1$ matrix)

$\hookrightarrow$ Satisfies $\boxed{G = G^*}$

Thus, Hermition

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$c[p]$ ~~will be the~~ Coefficient Vector

(b) $\Rightarrow$ The Subspace $\langle p \rangle$ will be defined as : $\{ cp \mid c \in \mathbb{C} \}$

$\Rightarrow$ Projecting q orthogonally onto the Subspace $\langle p \rangle$ wrt the inner (p, q) can be defined as :

Having $\boxed{q = r + (q-r)}$

s.t. $\boxed{r \in \langle p \rangle}$ & $\boxed{(r, (q-r)) = 0}$

$\Rightarrow$ Since $r \in \langle p \rangle$; We can write $\boxed{r = cp \quad \exists c \in \mathbb{C}}$

$\Rightarrow$ So, $(cp, q - cp) = 0 \Rightarrow \boxed{(cp, q) - (cp, cp) = 0}$

$$\left(\text{Since } (a, b-c) = \int_{-1}^{1} \overline{a(x)} \, (b(x) - c(x)) \, dx = \int_{-1}^{1} \overline{a(x)} \, b(x) \, dx - \int_{-1}^{1} \overline{a(x)} \, c(x) \, dx \right)$$

$\Rightarrow$ Then, we can write :

$$(cp, q) = \int_{-1}^{1} \overline{c\,p(x)} \, q(x) \, dx = \bar{c} \int_{-1}^{1} \overline{p(x)} \, q(x) \, dx$$

$$\& \quad (cp, cp) = c \cdot \bar{c} \int_{-1}^{1} \overline{p(x)} \, p(x) \, dx$$

$\Rightarrow$ So, we have : $(cp, q) = (cp, cp) \Rightarrow \boxed{\bar{c}\,(p, q) = c\bar{c}\,(p, p)}$

$$\boxed{c = \frac{(p, q)}{(p, p)}}$$

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$\Rightarrow$ Thus, the Projection Vector will be $\boxed{P \cdot \left(\frac{(P, q)}{(P, P)} \right)}$

$\Rightarrow$ Now using that $(P, q) = [P]^{*} G [q]$, we get:

$$\left[P \left(\frac{(P, q)}{(P, P)} \right) \right] = [P] \left(\frac{[P]^{*} G [q]}{[P]^{*} G [P]} \right) = \boxed{\left(\frac{[P][P]^{*} G}{[P]^{*} G [P]} \right) [q]}$$

Projector matrix acting on $[q]$