Introduction to Linear Algebra

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Vectors

1.1 Real Numbers $\mathbb{R}$

There are many sets of numbers, however, we will only consider numbers that fall within the real number line.

Integers: $\mathbb{Z} = \{..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\}$

Natural numbers (Positive integers): $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ (Sometimes the natural numbers include 0, denoted $\mathbb{N}_0$)

Rational numbers (or fractions): $\mathbb{Q} = \frac{p}{q}$, where p, $q \in \mathbb{Z}$, and $q \neq 0$

Example 1.1.1. $\frac{1}{2}$, $-\frac{5}{2}$, $\frac{47}{431}$

Irrational numbers: Numbers that are not rational

Example 1.1.2. $\pi \approx 3.14159...$, $\sqrt{2} \approx 1.4142...$, $e \approx 2.7182...$

Real numbers: $\mathbb{R}$, contain all of the sets of numbers above, rational numbers, and irrational numbers. The real numbers can be represented by the real number line (Figure 1).

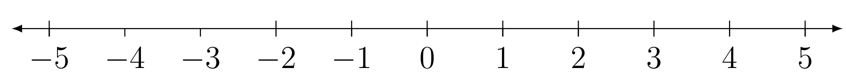


Figure 1.1: Real number line

<u>What operations can we perform using the real numbers $\mathbb{R}$?</u>

If $a, b \in \mathbb{R}$,

Addition: $a + b$
Subtraction: $a - b$
Multiplication: $a \times b$
Division: $\frac{a}{b}$, if $b \neq 0$.

1.1.1 Absolute Value

The absolute value of a number $|a|$ gives the distance from 0 along the number line. This gives us the magnitude (the size) of a number.

Definition 1.1.1. Given $a \in \mathbb{N}_0$ the absolute value of a is defined by:

$$|a| = a$$

$$|-a| = a.$$

Example 1.1.3.

$$|12| = 12$$

$$|-12| = 12$$

$$|0| = 0$$

$$|-\pi| = \pi$$

1.2 Vectors in $\mathbb{R}^n$

Definition 1.2.1. $\mathbb{R}^n$ is the ordered set of a sequence of n real numbers, where $n \geq 0$. E.g. $\mathbb{R}^2$ is the set of sequences with 2 numbers, which can be thought of as the xy-plane (x, y) where $x, y \in \mathbb{R}$.

A vector is an object with both a magnitude (the length) and a direction. We will use the underline notation for a vector $\underline{a}$, it is also common to see a vector denoted with bold text **a**. Firstly, we consider vectors in two-dimensional space, with real number entries. Which can be thought of as existing in the xy-plane. A vector has the same number of entries as it does dimensions, therefore, the two-dimensional vector $\underline{a} = \begin{bmatrix} x \\ y \end{bmatrix}$ has two entries. We can see a plot of the vectors $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ in figure 1.2. Graphically we can display the vectors by arrows, indicating both their magnitude and direction.

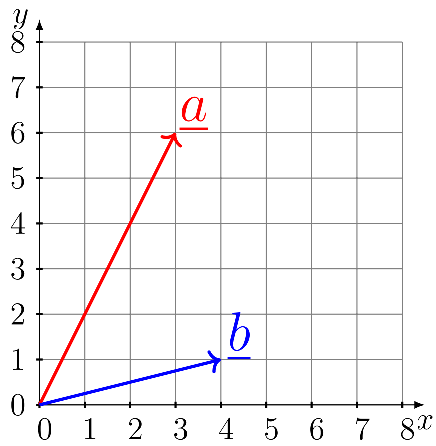


Figure 1.2: Two dimensional vector example, the vector $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is given by the red arrow, and the vector $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is given by the blue arrow.

1.2.1 Vector addition

As we can with real numbers we are also able to add (or subtract) two vectors, as long as they are of the same dimension. When we add two vectors we create a new vector of the same dimension. To add two vectors together we add the individual components in each direction.

Example 1.2.1. Given $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, what is the value of the new vector $\underline{c}$ where $\underline{a} + \underline{b} = \underline{c}$?

$$\begin{aligned} \underline{c} &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3+4 \\ 6+1 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 7 \end{bmatrix} \end{aligned}$$

Vector addition is like travelling along one vector, and then travelling along the second. We can also subtract a vector from another vector to create a new vector. When we add two vectors $\underline{a} + \underline{b}$, we add the individual components of $\underline{b}$ to $\underline{a}$. When we subtract a vector from another $\underline{a} - \underline{b}$, we subtract the individual components of $\underline{b}$ from $\underline{a}$.

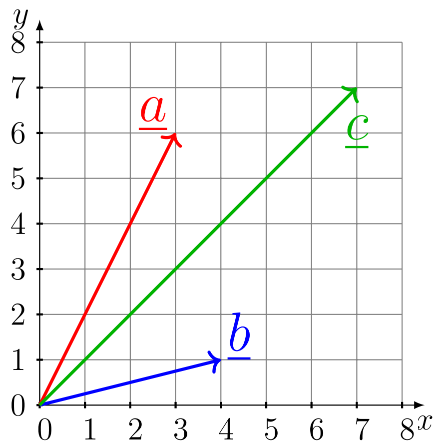


Figure 1.3: Two dimensional vector addition of $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, where $\underline{a} + \underline{b} = \underline{c} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$.

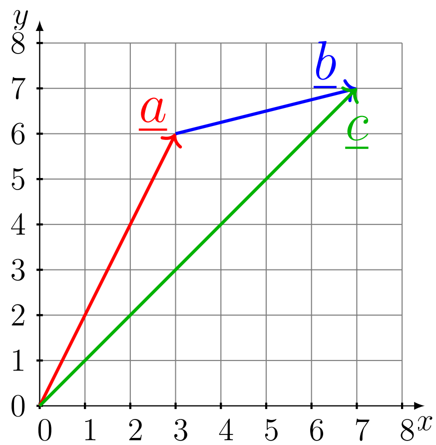


Figure 1.4: Two dimensional vector addition of $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, where $\underline{a} + \underline{b} = \underline{c} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$.

Example 1.2.2. Given $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, what is the value of the new vector $\underline{d}$ where $\underline{a} - \underline{b} = \underline{d}$?

$$\begin{aligned} \underline{d} &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3-4 \\ 6-1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \end{aligned}$$

We also note that $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$, where $-\underline{b} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$.

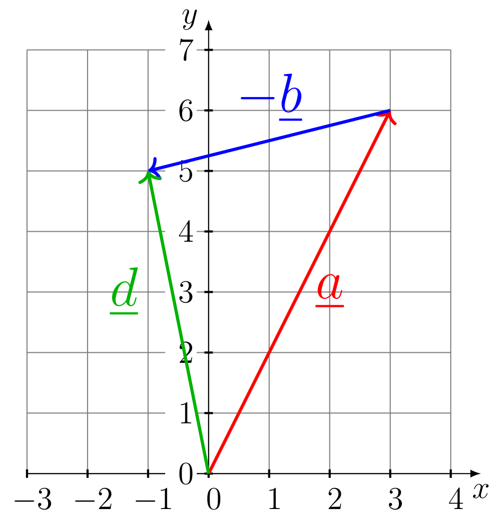


Figure 1.5: Two dimensional vector addition of $\underline{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, where $\underline{a} - \underline{b} = \underline{a} + (-\underline{b}) = \underline{d} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

Vector addition can be treated the same as normal addition. When two vectors are added this can be performed in any order (commutative law),

$$\underline{a} + \underline{b} = \underline{b} + \underline{a},$$

brackets can also be treated in the usual way (associative law),

$$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}.$$

We are also able to rearrange equations,

$$\underline{a} + \underline{b} = \underline{c},$$

is the same as ($-\underline{b}$ from both sides),

$$\underline{a} = \underline{c} - \underline{b}.$$

Definition 1.2.2. Additive Identity (zero vector): For a vector $\underline{a}$ of any dimension n, there exists a zero vector of dimension n $\underline{0}_n$ that consists of only zero entries such that

$$\underline{a} + \underline{0} = \underline{a} = \underline{0} + \underline{a}.$$

Example 1.2.3. Given the two-dimensional vector $\underline{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and the two-dimensional zero vector $\underline{0}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ find $\underline{a} + \underline{0}_2$,

$$\begin{aligned} \underline{a} + \underline{0}_2 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2+0 \\ 3+0 \end{bmatrix} \\ &= \underline{a} \end{aligned}$$

1.2.2 Scalar Multiplication

We cannot multiply two vectors together, however, we can multiply a vector by a real number. This is known as scalar multiplication. To multiply a vector by a real number, we multiply each individual component of the vector by the real number.

Example 1.2.4. If we have the vector $\underline{e} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and the real number 2, what is the vector $2\underline{e} =$?

$$\begin{aligned} 2\underline{e} &= 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 1 \\ 2 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{aligned}$$

How about $-\frac{3}{2}\underline{e}$?

$$\begin{aligned} -\frac{3}{2}\underline{e} &= -\frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2} \times 1 \\ -\frac{3}{2} \times 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2} \\ -3 \end{bmatrix} \end{aligned}$$

When we multiply a vector by a positive real number, the direction that the vector (the arrow) points is the same, but the size of the vector (the length of the arrow) is adjusted. If we multiply by a negative real number, the direction of the vector (the arrow) is in the opposite direction of the original vector, and the size (length of the arrow) is also adjusted. If we multiplied any vector by 0, then we will always end up with the zero vector $\underline{0}$, with the same number of 0 entries as there are dimensions in the original vector.

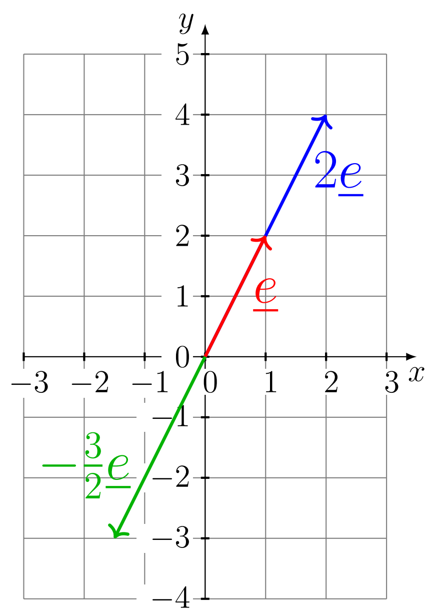


Figure 1.6: Two dimensional vector scalar multiplication of $\underline{e} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (red arrow), $2\underline{e}$ (blue arrow), and $-\frac{3}{2}\underline{e}$ (green arrow).

2

Vector Space and Properties of Vectors

So far we have introduced the real numbers, the notion of what a vector is, and the concepts of vector addition, and scalar multiplication. We will now formally define a vector space.

Definition 2.0.1. A vector space is a set lets say V with the operation of addition and scalar multiplication, such that the following properties hold (we will only be considering the real numbers $\mathbb{R}$ with the standard addition and multiplication):

I $\underline{u} + \underline{v} = \underline{v} + \underline{u}$, for all $\underline{u}, \underline{v}$ in V (commutative),

II $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$, for all $\underline{u}, \underline{v}, \underline{w}$, in V (associative),

III $(ab)\underline{v} = a(b\underline{v})$, for all $\underline{v}$ in V and a, b in $\mathbb{R}$ (associative),

IV there exists an element 0 in V such that $\underline{v} + 0 = \underline{v}$, for all $\underline{v} \in V$ (addition identity),

V for all $\underline{v}$ in V, there exists $\underline{w}$ in V, such that $\underline{v} + \underline{w} = 0$ (addition inverse),

VI $1\underline{v} = \underline{v}$ for all $\underline{v}$ in V (multiplication identity),

VII $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$, and $(a + b)\underline{v} = a\underline{v} + b\underline{v}$ for all a, b in $\mathbb{R}$, and for all $\underline{u}, \underline{v}$ in V (distributive),

If you are working in a different number field, replace $\mathbb{R}$ with your field $\mathbb{F}$ of choice.

Exercise 2.0.1. As an exercise verify $\mathbb{R}^2$ that is the set of two-dimensional vectors with real number entries is a valid vector space over the field of real numbers $\mathbb{R}$. This can be done by showing the conditions $I - VII$ hold.

2.1 Linear Combination

Definition 2.1.1. We say that a vector $\underline{\text{w}}$ is a linear combination of the vectors $\underline{v}_1, ..., \underline{v}_n$ if we choose numbers

$$a_1, \ldots, a_n \in \mathbb{R},$$

such that

$$\underline{w} = a_1\underline{v}_1 + a_2\underline{v}_2 + ... + a_n\underline{v}_n.$$

Example 2.1.1. In $\mathbb{R}^3$ the vector $\begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$,

$$\begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Exercise 2.1.1. Show that in $\mathbb{R}^3$ the vector $\begin{bmatrix} 8 \\ 6 \\ 7 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. To do so you need to show the system of equations

$$\begin{aligned} 8 &= 3a + 1b \\ 6 &= 1a + 2b \\ 7 &= 3a - 1b \end{aligned}$$

has no solutions.

2.2 Linear Independence/Dependence of Vectors

Definition 2.2.1. Linear independence: The list of vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$ in V is linearly independent if

$$a_1\underline{v}_1 + a_2\underline{v}_2 + ... + a_n\underline{v}_n = \underline{0},$$

only when $a_1 = a_2 = ... = a_n = 0$ for $a_1, a_2, ..., a_n$ in $\mathbb{R}$.

Example 2.2.1. The vectors in $\mathbb{R}^3$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, are linearly independent. This is because

$$a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

only when $a_1 = a_2 = a_3 = 0$.

Definition 2.2.2. A list of vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$ in V are linearly dependent if they are not linearly independent. That is, we can find $\underline{a_1}, \ldots, \underline{a_n} \in \mathbb{R}$ such that

$$a_1\underline{v}_1 + ... + a_n\underline{v}_n = \underline{0},$$

where at least one of $a_1, ..., a_n \neq 0$.

Example 2.2.2. The vectors in $\mathbb{R}^3$ $\begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, are linearly dependent, we can choose the numbers $1, -2, -2 \in \mathbb{R}$ such that

$$1\begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Exercise 2.2.1. Are the following list of vectors in $\mathbb{R}^2$ linearly independent or dependant?

I $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

II $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$

III $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ a \end{bmatrix}$ for which values of a is the list of vectors linearly independent, and for which values is it linearly dependent?

2.3 Span of Vectors

Definition 2.3.1. Span: The set of all linear combinations of a list of vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$ in V, is called the span of the list of vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$, $\text{span}(\underline{v}_1, \underline{v}_2, ..., \underline{v}_n)$.

Example 2.3.1. If we look at the space of three dimensional vectors with real number entries $V = \mathbb{R}^3$. Firstly with a list containing one vector $\underline{u}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then what is the span of this list, and what shape does it form in the three-dimensional xyz-plane? All linear combinations of the one vector are precisely the scalar multiples of $\underline{u}_1$, given $\lambda \in \mathbb{R}$

then,

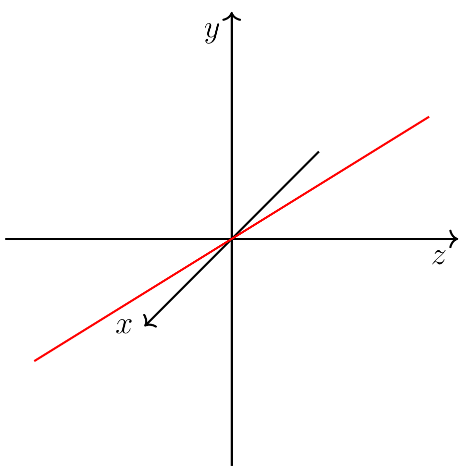
$$\text{span}(\underline{u}_1) = \lambda \underline{u}_1,$$

which forms a line in the xyz-plane. If for example we have the vector $\underline{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,

$$\text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

which gives the line described by

$$\begin{aligned} x &= \lambda \\ y &= 2\lambda \\ z &= 3\lambda \text{ for } \lambda \in \mathbb{R}. \end{aligned}$$



Next we will consider two linearly independent vectors $\underline{u}_1, \underline{u}_2 \in \mathbb{R}^2$, where

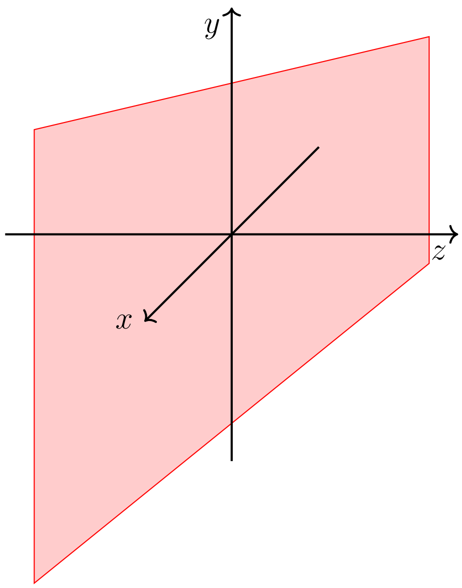
$$\text{span}(\underline{u}_1, \underline{u}_2) = \lambda_1 \underline{u_1} + \lambda_2 \underline{u_2},$$

which forms a two-dimensional plane in the xyz-plane. If for example we consider $\underline{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ then

$$\text{span}\left(\lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \lambda_1 \\ 2\lambda_1 + \lambda_2 \\ 3\lambda_1 \end{bmatrix},$$

which gives a plane described by

$$x = \lambda_1$$
$$y = 2\lambda_1 + \lambda_2$$
$$z = 3\lambda_1.$$



Lastly if we consider three linearly independent vectors $\underline{u_1}, \underline{u_2}, \underline{u_3} \in \mathbb{R}^3$, and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

$$\text{span}(\underline{u_1}, \underline{u_2}, \underline{u_3}) = \lambda_1\underline{u_1} + \lambda_2\underline{u_2} + \lambda_3\underline{u_3}, \tag{2.1}$$

which is equal to the whole xyz-plan. This is because if the vectors are of dimension n then the span of n linearly independent vectors always describes the whole n-dimensional space.

Definition 2.3.2. Spans: If the span of $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$ is equal to V, we say that $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$ spans V.

Example 2.3.2. Given we have the vector space $\mathbb{R}^3$ with $\mathbb{R}$ entries, we want to find a list of vectors that spans all of $\mathbb{R}^3$. We can think of $\mathbb{R}^3$ as three-dimensional space, or equivalently as the xyz-plane. This means a linear combination of vectors that spans all of the xyz-plane is able to reach every real point in the xyz-plane. For example, if we consider the list of vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

does this span $\mathbb{R}^3$? Yes this does span $\mathbb{R}^3$, for example, if we want to create the new vector $\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} \in \mathbb{R}$,

$$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} \in \mathbb{R}.$$

Exercise 2.3.1. Show $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ spans $\mathbb{R}^3$ with $\mathbb{R}$ entries.

2.4 Basis

Definition 2.4.1. Basis: A basis of V is a list of vectors in V that is both linearly independent and spans V.

Example 2.4.1. For a basis of $\mathbb{R}^3$, we know the list of vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

spans $\mathbb{R}^3$. They are also linearly independent, and therefore, are a basis of $\mathbb{R}^3$ (this list of vectors are known as the standard basis for $\mathbb{R}^3$).

Theorem 2.4.1. *If we have a list of n linearly independent vectors in $\mathbb{R}^n$, then this list of vectors spans all of $\mathbb{R}^n$.*

From this theorem, this means if we have a list of n vectors in $\mathbb{R}^n$ that are linearly independent, then they form a basis for $\mathbb{R}^n$.

Example 2.4.2. Another example of a basis of $\mathbb{R}^3$ is given by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Exercise 2.4.1. Can you find another basis of $\mathbb{R}^3$?

2.5 The Norm and Dot Product of a Vector

In section 1.1.1 we saw that the absolute value of a real number is the distance along the number line from the origin, and is always a positive number. For vectors we have a similar concept known as the norm of a vector. The norm of a vector gives us the length of the vector from the origin.

Definition 2.5.1. Norm: The norm of a vector $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$ is denoted as $||\underline{a}||$, and is defined by

$$||\underline{a}|| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}.$$

(This might be familiar as the calculation necessary to calculate the hypotenuse in the Pythagoras theorem.)

Example 2.5.1. Given the vector $\underline{a} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$ find $||\underline{a}||$,

$$\begin{aligned} ||\underline{a}|| &= \sqrt{3^2 + (-2)^2 + 5^2} \\ &= \sqrt{9 + 4 + 25} \\ &= \sqrt{38}. \end{aligned}$$

Exercise 2.5.1. Given,

$$\underline{a} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} \in \mathbb{R}^3,$$
$$\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^4,$$

find $||\underline{a}||$ and $||\underline{b}||$.

Definition 2.5.2. If the norm of a vector equals to 1, $||a|| = 1$, then we call this a unit vector.

Example 2.5.2. For example, the vector,

$$\underline{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

is a unit vector

$$\begin{aligned} ||\underline{a}|| &= \sqrt{1^2 + 0^2} \\ &= 1. \end{aligned}$$

We can also find something called the dot product of two vectors. Algebraically the dot product can be thought of as the sum of the product of the individual entries of the vectors.

Definition 2.5.3. Dot Product: The dot product of two vectors of the same length $\underline{a}, \underline{b} \in \mathbb{R}^n$ is defined by,

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + ... + a_n b_n.$$

Example 2.5.3. Given the vectors $\underline{a} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, in $\mathbb{R}^3$, find $\underline{a} \cdot \underline{b}$,

$$\begin{aligned} \underline{a} \cdot \underline{b} &= 2 \times 1 + (-2) \times 4 + 2 \times 5 \\ &= 2 - 8 + 10 \\ &= 4. \end{aligned}$$

Exercise 2.5.2. Find the dot product of the two vectors $\underline{a} = \begin{bmatrix} -1 \\ 8 \\ 2 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

We now know how to calculate the dot product, however, why is the dot product useful and what can it be used to calculate?

Definition 2.5.4. The dot product of a vector $\underline{a}$ with a unit vector $\underline{u}$ is the projection of the length of $\underline{a}$ in the direction of the vector $\underline{u}$. This is easiest to see geometrically in figure 2.1, the dot product of $\underline{a} \cdot \underline{u}$ is the length of the line segment from A to B. We can also find the length of this line segment given the angle θ between $\underline{a}$ and $\underline{u}$,

$$\underline{a} \cdot \underline{u} = ||\underline{a}|| \cos \theta.$$

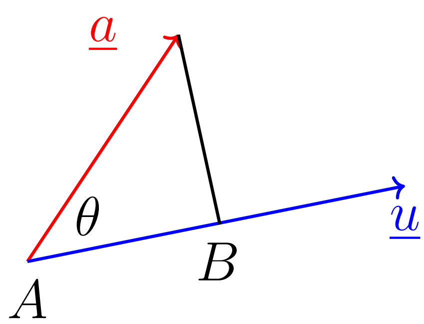


Figure 2.1: The dot product of a vector with a unit vector is the projection of the length of the vector.

Therefore, the angle between two vectors can also be calculated using the dot product. However, what happens when the vector we are calculating the dot product with is not a unit vector? In this case the dot product is just scaled by the length of the second vector and we can thus find the angle between two vectors in general.

Definition 2.5.5. The angle θ between two vectors $\underline{a}$, and $\underline{b}$ can be found using the dot product by the following formula

$$\frac{\underline{a}\cdot\underline{b}}{||\underline{b}||} = ||\underline{a}||\,\cos\theta,$$

this is the similar as we saw for a unit vector but just scaled by the length of the vector $\underline{b}$. If $||\underline{b}|| = 1$, we would have our previous formula,

$$\underline{a}\cdot\underline{b} = ||\underline{a}||\cos\theta.$$

We can now rearrange the formula to find the angle,

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{||\underline{a}||\,||\underline{b}||}.$$

Example 2.5.4. Given the vectors $\underline{a} = \begin{bmatrix}5\\2\end{bmatrix}$, and $\underline{b} = \begin{bmatrix}4\\5\end{bmatrix}$, find the angle θ between them (see figure 2.2). Firstly we will compute the norms of $\underline{a}$, and $\underline{b}$,

$$\begin{aligned}||\underline{a}|| &= \sqrt{5^2+2^2}\\ &= \sqrt{29},\\ ||\underline{b}|| &= \sqrt{4^2+5^2}\\ &= \sqrt{41}.\end{aligned}$$

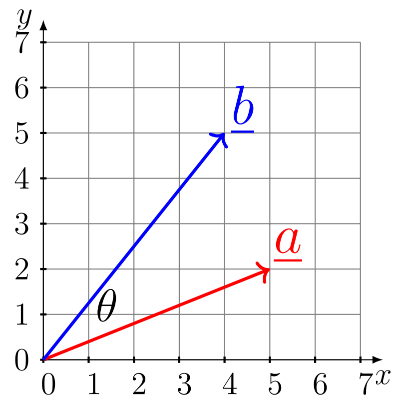


Figure 2.2: Two dimensional vectors $\underline{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, where θ is the angle between then two vectors.

Next, we will compute the dot product of $\underline{a}$, and $\underline{b}$,

$$\begin{aligned} \underline{a} \cdot \underline{b} &= 5 \times 4 + 2 \times 5 \\ &= 30. \end{aligned}$$

Thus we now have from the formula,

$$\cos\theta = \frac{30}{\sqrt{29}\sqrt{41}},$$

taking the inverse of cos on both sides,

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{30}{\sqrt{29}\sqrt{41}}\right) \\ &\approx 29.54^\circ. \end{aligned}$$

Definition 2.5.6. If the angle between two non-zero vectors $\underline{a}$, $\underline{b}$ is 90°, then the dot product

$$\underline{a} \cdot \underline{b} = 0.$$

This is the case because $\cos 90 = 0$.

Exercise 2.5.3. Find the angle θ between the two vectors $\underline{a} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$.

3

Linear Mapping

To start with if we a map that takes a number $x \in \mathbb{R}$ and multiplies x by 2. This is the map Φ takes the real numbers to the reals,

$$\Phi : \mathbb{R} \to \mathbb{R},$$

where it takes an input x and outputs $2x$,

$$\Phi(x) = 2x.$$

Example 3.0.1. Given the map,

$$\Phi : \mathbb{R} \to \mathbb{R},$$
$$\Phi(x) = 2x.$$

Then we have for example,

$$\Phi(2) = 4,$$
$$\Phi(-3) = -6.$$

Example 3.0.2. Another example of a map is say we have a point $(x, y) \in \mathbb{R}^2$, can we find the map that rotates this point in the xy-plane by 90 degrees? We have the corresponding map,

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2,$$
$$\Phi((x, y)) = (-y, x).$$

If for example we had the point $(2, 3) \in \mathbb{R}^2$, then $\Phi((2, 3)) = (-3, 2)$.

Here we explore mappings on vector spaces that preserve the structure of the vector spaces.

Definition 3.0.1. Mapping: Consider two vector spaces V, and W, a mapping

$$\Phi : V \to W,$$

preserves the structure of the vector space if,

$$\begin{aligned} \Phi(\underline{x} + \underline{y}) &= \Phi(\underline{x}) + \Phi(\underline{y}), \\ \Phi(\lambda \underline{x}) &= \lambda \Phi(\underline{x}), \\ \forall \underline{x}, \underline{y} &\in V,\ \lambda \in \mathbb{R}. \end{aligned}$$

Definition 3.0.2. Linear Mapping: Given V, W, two vector spaces and a mapping $\Phi : V \to W$, the mapping is linear if,

$$\forall \underline{x}, \underline{y} \in V,\ \forall \lambda, \phi \in \mathbb{R} :\ \Phi(\lambda \underline{x} + \phi \underline{y}) = \lambda \Phi(\underline{x}) + \phi \Phi(\underline{y}).$$

Theorem 3.0.1. *Unique Linear Map: Let $(\underline{v_1}, \ldots, \underline{v_n})$ be a basis of V, and $(\underline{w_1}, \ldots \underline{w_m})$ be an arbitrary list of vectors in W. Then there exists a unique linear map,*

$$T : V \to W \text{ such that } T(\underline{v_i}) = \underline{w_i}.$$

We will now introduce the concept of a matrix. Matrices provide an intuitive way to compute and work with linear maps.

Definition 3.0.3. Every linear map can be represented by a matrix.

Definition 3.0.4. Matrix: A matrix is nothing more than a rectangular array of numbers. We will always be treating these as real numbers. The vertical entries of a matrix are the **rows** and the horizontal entries of a matrix are the **columns**. A matrix A, the i-th row and the j-th column entry is denoted by $A_{i,j}$. We write an $m \times n$ matrix, this is a matrix with m rows and n columns by,

$$A = \begin{bmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \ldots & A_{m,n} \end{bmatrix}$$

Definition 3.0.5. Every linear map can be represented by a matrix and vice versa, every matrix can define a linear map. Let V, W be two vector spaces with the linear map

$$T : V \to W.$$

If we have the basis for V $(\underline{v_1}, \ldots \underline{v_n})$, and a basis for W $(\underline{w_1}, \ldots \underline{w_m})$. From theorem 3.0.1 T is uniquely determined if we specify,

$$T\underline{v_1}, \ldots, T\underline{v_n} \in W,$$

because for W there exists unique scalars $a_{ij} \in \mathbb{R}$ s.t.

$$\begin{aligned} T\underline{v_j} &= a_{1j}\underline{w_1} + \ldots + a_{mj}\underline{w_m} \text{ for } 1 \leq j \leq m \\ &= \sum_{i=1}^{m} a_{ij}\underline{w_i}. \end{aligned}$$

The scalars can then be arranged to form a matrix A of m rows and n columns, where $A \in \mathbb{R}^{m\times n}$.

We will now look at some examples of matrices from the point of view from linear maps.

Example 3.0.3. Let the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ for $a, b, c, d \in \mathbb{R}$, then for the standard basis $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ we have

$$\begin{aligned} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} a \\ c \end{bmatrix} \\ T\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} b \\ d \end{bmatrix} \end{aligned}$$

which forms the matrix,

$$M(T) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Here we saw what happens when we have a linear map from $T : \mathbb{R}^2 \to \mathbb{R}^2$, but what happens when the dimensions are different for example a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$?

Example 3.0.4. Let the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ x-y \\ 2x+y \end{bmatrix}$ with the standard basis $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ for $\mathbb{R}^2$, and $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ for $\mathbb{R}^3$. Then $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ which forms the matrix,

$$M(T) = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

4

Matrices Operations

Example 4.0.1. An example 2×2 matrix A,

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$$

where,

$$A_{1,1} = 3,\ A_{1,2} = 2,\ A_{2,1} = -1,\ \text{and}\ A_{2,2} = 4.$$

4.1 Matrix Addition, Scalar Multiplication, and Matrix Multiplication

In the same way as vectors, matrices of the same size can be added together to form a new matrix.

Definition 4.1.1. Given two $m \times n$ matrices A, and B, then $A + B$ is given by,

$$\begin{aligned} &\begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} + \begin{bmatrix} B_{1,1} & \dots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{m,1} & \dots & B_{m,n} \end{bmatrix} \\ &= \begin{bmatrix} A_{1,1} + B_{1,1} & \dots & A_{1,n} + B_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} + B_{m,2} & \dots & A_{m,n} + B_{m,n} \end{bmatrix}. \end{aligned}$$

Example 4.1.1. Given the 2×3 matrix A, and B, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$
$$B = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix},$$

find the matrix $C = A + B$,

$$\begin{aligned} C &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 2+4 & 3+6 \\ 4+6 & 5+4 & 6+2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 & 9 \\ 10 & 9 & 8 \end{bmatrix}. \end{aligned}$$

Exercise 4.1.1. Given the 3×3 matrices A, and B, find $C = A + B$.

$$A = \begin{bmatrix} 7 & 4 & -2 \\ -1 & 8 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & -2 \\ 4 & 7 & 2 \\ 8 & 0 & 5 \end{bmatrix}.$$

We also have scalar multiplication of matrices in the same way as we have seen with vectors.

Definition 4.1.2. Scalar matrix multiplication: Given an $m \times n$ matrix, and a scalar $\alpha \in \mathbb{R}$, then αA is given by,

$$\begin{aligned} \alpha A &= \alpha \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,2} & \dots & A_{m,n} \end{bmatrix} \\ &= \begin{bmatrix} \alpha A_{1,1} & \dots & \alpha A_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha A_{m,2} & \dots & \alpha A_{m,n} \end{bmatrix}. \end{aligned}$$

Example 4.1.2. Given $\alpha = 3$, and the 2×2 matrix A,

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix},$$

find αA,

$$\begin{aligned}\alpha A &= 3\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3\times 3 & 3\times 2 \\ 3\times -1 & 3\times 4 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 6 \\ -3 & 12 \end{bmatrix}.\end{aligned}$$

Exercise 4.1.2. Given $\alpha = 4$ and the 3×3 matrices A

$$A = \begin{bmatrix} 7 & 4 & -2 \\ -1 & 8 & 0 \\ 0 & 1 & 4 \end{bmatrix},$$

find αA.

When we were considering vectors there does not exist such a thing as multiplication of vectors with other vectors. However, we are able to multiply matrices with other matrices as long as the shape of the matrices allow.

Definition 4.1.3. Matrix Multiplication: In order to multiply two matrices A, and B together AB, the number of columns of the first matrix has to equal the number of rows of the second matrix. This is you can multiply an $m\times n$ matrix by an $n\times l$ matrix, to form a new $m\times l$ matrix $C = AB$. Where the i-th row and j-th column entry of the new matrix C is found by multiplying and then adding the entries of the i-th row of the first matrix A by the j-th column of the second matrix B.

We note that you have to be careful $AB \neq BA$ in general, and often only one is possible to compute.

Example 4.1.3. Multiply the following 3×2 matrix A, by the 2×3 matrix B, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 9 & 3 \end{bmatrix},$$

we have the solution AB given by,

$$\begin{aligned}C &= AB \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 7 & 2 & 1 \\ 0 & 9 & 3 \end{bmatrix},\end{aligned}$$

to find the first row and first column entry of C, $C_{1,1}$, we multiply the first row of A,

$\begin{bmatrix}1 & 2\end{bmatrix}$, by the first column entry of B, $\begin{bmatrix}7\\0\end{bmatrix}$, and then add the numbers together,

$$\begin{aligned}C_{1,1} &= \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}7\\0\end{bmatrix}\\ &= \begin{bmatrix}1\times 7+2\times 0\end{bmatrix}\\ &= (7).\end{aligned}$$

Therefore, for the matrix AB we have

$$\begin{aligned}C &= AB\\ &= \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}7 & 2 & 1\\0 & 9 & 3\end{bmatrix}\\ &= \begin{bmatrix}1\times 7+2\times 0 & 1\times 2+2\times 9 & 1\times 1+2\times 3\\3\times 7+4\times 0 & 3\times 2+4\times 9 & 3\times 1+4\times 3\\5\times 7+6\times 0 & 5\times 2+6\times 9 & 5\times 1+6\times 3\end{bmatrix}\\ &= \begin{bmatrix}7+0 & 2+18 & 1+6\\21+0 & 6+36 & 3+12\\35+0 & 10+54 & 5+18\end{bmatrix}\\ &= \begin{bmatrix}7 & 20 & 7\\21 & 42 & 15\\35 & 64 & 23\end{bmatrix}.\end{aligned}$$

Definition 4.1.4. A square matrix is a matrix with the same number of rows and column, this is an $n\times n$ matrix. Only a square matrix can be multiplied by itself. The matrix multiplication AB, and BA both only exist when A is an $n\times m$ matrix, and B is an $m\times n$ matrix. however, $AB\neq BA$ in general.

Example 4.1.4. If we have the 2×2 matrix A, the $2\times$ matrix B, and the 2×3 matrix C where

$$A=\begin{bmatrix}5 & 1\\7 & 2\end{bmatrix}, B=\begin{bmatrix}2 & 0\\1 & 3\end{bmatrix}, C=\begin{bmatrix}1 & 2\\0 & 5\\9 & 2\end{bmatrix},$$

if possible find AB, BA, AC, BC, CA, and CB. Firstly we calculate AB this is possible as A, and B, have the same size,

$$AB=\begin{bmatrix}11 & 3\\16 & 6\end{bmatrix},$$

next we calculate BA

$$BA = \begin{bmatrix} 10 & 2 \\ 26 & 7 \end{bmatrix},$$

we note that $AB \neq BA$. The matrix AC can not be calculated because the number of columns of A does not equal the number of rows of C, $2 \neq 3$ this is the same for BC. However, because the number of columns of C does equal the number of rows of A. Thus we can calculate CA

$$CA = \begin{bmatrix} 19 & 5 \\ 35 & 10 \\ 59 & 13 \end{bmatrix},$$

as an exercise calculate CB.

Definition 4.1.5. Give a square $m \times m$ matrix A and a positive integer $n \in \mathbb{Z}^+$, we can calculate

$$A^n = \underbrace{A \times \cdots \times A}_{n}.$$

.

Exercise 4.1.3. Given the 2×2 square matrix A,

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix},$$

calculate $A^2 = AA$.

When we are thinking out just real numbers $\mathbb{R}$, the identity element is $1 \in \mathbb{R}$. Explicitly this means given any real number $a \in \mathbb{R}$, $1 \times a = a = a \times 1$. When we consider scalar multiplication for both a vector or a matrix, we are simply multiplying be a real number. The identity element for scalar multiplication is also the identity element of the real numbers 1. However, for matrices we also have multiplication of a matrix by a matrix, and one question we might have is what is the identity element for matrix multiplication?

4.2 Identity Matrix, Transpose of a Matrix, Orthogonal Matrix and Diagonal Matrix

Definition 4.2.1. Identity Matrix: The set of identity matrices are square matrices with value 1 along the leading diagonal entries (these are the entries where the row number matches the column number), and 0 entries everywhere else. The size of the identity matrix is indicated by a subscript denoting the number of rows and columns, for example,

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Definition 4.2.2. Choosing The Correct Identity Matrix: Given an $m \times n$ matrix A, we choose the following identity matrix,

$$AI_n = A = I_m A.$$

Example 4.2.1. Given the 3×2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, show $AI_2 = A = I_3 A$

$$\begin{aligned} AI_2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 1 + 2 \times 0 & 1 \times 0 + 2 \times 1 \\ 3 \times 1 + 4 \times 0 & 3 \times 0 + 4 \times 1 \\ 5 \times 1 + 6 \times 0 & 5 \times 0 + 6 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \\ &= A, \end{aligned}$$

$$\begin{aligned} I_3 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 1 + 0 \times 3 + 0 \times 5 & 1 \times 2 + 0 \times 4 + 0 \times 6 \\ 0 \times 1 + 0 \times 3 + 1 \times 5 & 0 \times 2 + 1 \times 4 + 0 \times 6 \\ 0 \times 1 + 0 \times 3 + 1 \times 5 & 0 \times 2 + 0 \times 4 + 1 \times 6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \\ &= A \end{aligned}$$

Exercise 4.2.1. Given the matrix $B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 9 & 3 \end{bmatrix}$, choose the appropriate identity matrix $I_?$ to calculate $BI_?$, and $I_?B$ and confirm $BI_? = B = I_?B$.

We have seen that there always exists a multiplicative identity element, also similar to vectors there exists an additive identity.

Definition 4.2.3. For a matrix A of dimension $n \times m$ then there exists a zero matrix of the same dimension $n \times m$ with all zero entries $0_{n\times m}$ such that,

$$A + 0_{n\times m} = A = 0_{n\times m} + A.$$

Example 4.2.2. Given the matrix 3×2 dimensional matrix A and the 3×2 dimensional zero matrix $0_{3\times 2}$ where

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 6 \\ 1 & 0 \end{bmatrix}, \quad 0_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

show $A + 0_{3\times 2} = A$,

$$\begin{aligned} A + 0_{3\times 2} &= \begin{bmatrix} 3 & 4 \\ -2 & 6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3+0 & 4+0 \\ -2+0 & 6+0 \\ 1+0 & 0+0 \end{bmatrix} \\ &= A. \end{aligned}$$

When considering a real number $a \in \mathbb{R}$ its inverse is $\frac{1}{a}$, this means $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$. The real number a times its identity element 1, equals the real number a. We can do the same for a square matrix.

Definition 4.2.4. Inverse Matrix: Given an $m \times m$ matrix A, we denote its inverse matrix by A^{-1} such that,

$$AA^{-1} = I_m = A^{-1}A.$$

Definition 4.2.5. 2×2 Matrix Inverse: For a 2×2 square matrix A there exists a formula to find its inverse matrix A^{-1}. The formula is as follows,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we have

$$\begin{aligned} A^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \end{aligned}$$

Example 4.2.3. Given the 2×2 matrix A,

$$A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix},$$

find its inverse A^{-1}. Using the formula above we have,

$$\begin{aligned} A^{-1} &= \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}^{-1} \\ &= \frac{1}{1 \times 4 - 6 \times 2} \begin{bmatrix} 4 & -6 \\ -2 & 1 \end{bmatrix} \\ &= -\frac{1}{8} \begin{bmatrix} 4 & -6 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \end{aligned}$$

Exercise 4.2.2. Verify that $AA^{-1} = I_2 = A^{-1}A$, given

$$A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}.$$

Definition 4.2.6. Transpose of a Matrix: Given the matrix A, then A^T is the transpose matrix of A. The transpose matrix is where the rows and columns of a matrix are switched. Therefore, an $m \times n$ matrix, its transpose is an $n \times m$ matrix.

Example 4.2.4. Given the 2×3 matrix A,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

its transpose is given by,

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Exercise 4.2.3. Given the matrix $B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 9 & 3 \end{bmatrix}$, find B^T.

So far we have explored various computations with matrices, however, matrices with certain structures can have very interesting and useful properties. One such class of matrices are diagonal matrices.

Definition 4.2.7. A diagonal matrix is a square matrix where all entries outside the main diagonal are zero. The main diagonal is from the top left to the bottom right entries of the matrix. The diagonal entries can be any real number $\mathbb{R}$ including zero.

Example 4.2.5. We have already come across two examples of a diagonal matrix, the first is the identity matrix, second is any square $n \times n$ zero matrix. Although there are no non-zero entries in a zero square matrix, the matrix still follows the rule that all entries outside of the main diagonal are zero.

Example 4.2.6. Here we have an example of a 3×3 diagonal matrix A where

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Definition 4.2.8. Orthogonal Matrix: If we have a square matrix A then it is orthogonal if $AA^T = I$.

Example 4.2.7. Show the 2×2 matrix A is orthogonal where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

thus, we have

$$\begin{aligned} AA^T &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2. \end{aligned}$$

Diagonal matrices have several interesting when added or multiplied to other matrices.

Definition 4.2.9. Given two $n \times n$ diagonal matrices A, B then the new matrix formed by addition $A + B = C$ is also a diagonal matrix.

Example 4.2.8. Given the 2×2 diagonal matrix A, and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

show $A + B$ is diagonal. Thus, we have

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2+1 & 0 \\ 0 & 3+2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = C, \end{aligned}$$

which is also a diagonal matrix.

Definition 4.2.10. Diagonal matrices are always commutative, that is given two $n \times n$ diagonal matrices $AB = BA$.

Example 4.2.9. Given the 2×2 diagonal matrix A, and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

show $AB = BA$. Thus we have,

$$\begin{aligned}
AB &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 1 + 0 \times 0 & 2 \times 0 + 0 \times 2 \\ 0 \times 1 + 3 \times 0 & 0 \times 0 + 3 \times 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \\
BA &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 0 + 2 \times 1 & 0 \times 2 + 2 \times 0 \\ 3 \times 0 + 0 \times 1 & 3 \times 2 + 0 \times 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},
\end{aligned}$$

and hence we have $AB = BA$.

4.3 Matrices are a Vector Space

Now that we have explored all of the necessary rules of computations with matrices we can show that by verifying each of the seven requirements necessary to be a vector space as seen in chapter 2, and repeated here

I $\underline{u} + \underline{v} = \underline{v} + \underline{u}$, for all $\underline{u}, \underline{v}$ in V (commutative),

II $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$, for all $\underline{u}, \underline{v}, \underline{w}$, in V (associative),

III $(ab)\underline{v} = a(b\underline{v})$, for all $\underline{v}$ in V and a, b in $\mathbb{R}$ (associative),

IV there exists an element 0 in V such that $\underline{v} + 0 = \underline{v}$, for all $\underline{v} \in V$ (addition identity),

V for all $\underline{v}$ in V, there exists $\underline{w}$ in V, such that $\underline{v} + \underline{w} = 0$ (addition inverse),

VI $1\underline{v} = \underline{v}$ for all $\underline{v}$ in V (multiplication identity),

VII $a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}$, and $(a + b)\underline{v} = a\underline{v} + b\underline{v}$ for all a, b in $\mathbb{R}$, and for all $\underline{u}, \underline{v}$ in V (distributive).

Firstly, for I we have already seen,

$$\begin{aligned}
A + B &= \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} + \begin{bmatrix} B_{1,1} & \dots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{m,1} & \dots & B_{m,n} \end{bmatrix} \\
&= \begin{bmatrix} A_{1,1} + B_{1,1} & \dots & A_{1,n} + B_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} + B_{m,2} & \dots & A_{m,n} + B_{m,n} \end{bmatrix},
\end{aligned}$$

however, we now that real numbers are commutative

$$\begin{bmatrix} A_{1,1}+B_{1,1} & \dots & A_{1,n}+B_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1}+B_{m,2} & \dots & A_{m,n}+B_{m,n} \end{bmatrix} = \begin{bmatrix} B_{1,1}+A_{1,1} & \dots & B_{1,n}+A_{1,n} \\ \vdots & \ddots & \vdots \\ B_{m,2}+A_{m,1} & \dots & B_{m,n}+A_{m,n} \end{bmatrix}$$
$$= B + A.$$

Exercise 4.3.1. For II we have a very similar process but with the matrices A, B, and C. For III we know that multiplication of a matrix by a scalar is element wise multiplication where the entries of the matrices themselves are scalars. It is left to the reader to verify these conditions.

For the condition IV we have already in definition 4.2.3 that the zero element is the zero matrix, and there always exists such a zero matrix for and matrix. For condition V, we require a $n \times m$ dimensional matrix A with a corresponding matrix B such that $A + B = 0_{m \times n}$.

Definition 4.3.1. Given an $n \times m$ matrix A because the entries are scalars and matrices have element wise addition we have the matrix $-1 \times A = -A$, such that

$$A + (-A) = 0_{n \times m}$$

.

For the condition VI we have already shown that there exists an identity matrix I_n for any matrix. Finally for the distributive properly, directly from the distributive property of real numbers where our matrix entries are real numbers with element wise addition this property holds. Therefore, matrices satisfy all of the necessary conditions and matrices are also a vector space.

5

Matrices and Vectors

5.1 Multiplication of a Matrix by a Vector and Rotation Matrices

We can think of a column vector as a matrix with only one column, an $m \times 1$ matrix.

Definition 5.1.1. Given an $m \times m$ matrix A, we can multiply this by an m dimensional vector $\underline{a}$ (which is a vector with m rows or an $m \times 1$ matrix), the result is a new m dimensional vector $\underline{b} = A\underline{a}$,

$$\begin{aligned} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} &= \begin{bmatrix} A_{1,1} & \dots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,m} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \\ &= \begin{bmatrix} A_{1,1}a_1 + \ldots + A_{1,m}a_m \\ \vdots \\ A_{m,1}a_1 + \ldots + A_{m,m}a_m \end{bmatrix}. \end{aligned}$$

Definition 5.1.2. From the point of view of linear maps, multiplying by a vector is taking the image of the vector under the map.

Example 5.1.1. Given the matrix 2×2 A, and the 2 dimensional vector $\underline{\text{a}}$ where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

find the new 2 dimensional vector $\underline{b} = A\underline{a}$,

$$\begin{aligned}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} &= \begin{bmatrix}1\times 1+2\times 2\\3\times 1+4\times 2\end{bmatrix}\\&= \begin{bmatrix}5\\11\end{bmatrix}\\&= \underline{b}.\end{aligned}$$

We have one more interesting result of diagonal matrices. If we multiply a diagonal matrix by a vector, then the entries of the vector are multiplied by the corresponding non-zero row value of the diagonal matrix.

Example 5.1.2. Given the 2×2 dimensional diagonal matrix A, and the 2 dimensional vector $\underline{v}$ where, A =

$$\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}, \quad \underline{v} = \begin{bmatrix}3\\-1\end{bmatrix},$$

find $A\underline{v}$. We have,

$$\begin{aligned}A\underline{v} &= \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}\\&= \begin{bmatrix}2\times 3+0\times -1\\0\times 3+1\times(-1)\end{bmatrix}\\&= \begin{bmatrix}6\\-1\end{bmatrix}.\end{aligned}$$

In section 1.2.2 we saw that we can multiply a vector by a scalar, this can change the length of the vector, act as a reflection through the origin, or both. However, now we have seen that we can multiply a matrix by a vector to compute a new vector, this will allow us to rotate the vector by a given angle.

Definition 5.1.3. In two dimensions we can rotate a vector $\underline{a} = \begin{bmatrix}x\\y\end{bmatrix}$, by a given angle θ counterclockwise by multiplying the 2×2 rotation matrix

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix},$$

therefore, we have

$$\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\cos\theta\times x-\sin\theta\times y\\\sin\theta\times x+\cos\theta\times y\end{bmatrix}.$$

Example 5.1.3. Given the vector $\underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, rotate $\underline{a}$ by 90 degrees counterclockwise,

$$\begin{aligned} \begin{bmatrix} \cos(90) & -\sin(90) \\ \sin(90) & \cos(90) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} \cos(90) \times 1 - \sin(90) \times 2 \\ \sin(90) \times 1 + \cos(90) \times 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \end{aligned}$$

rotating the vector $\underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ counterclockwise by 90 degrees gives us the new vector $\underline{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, see figure 5.1.

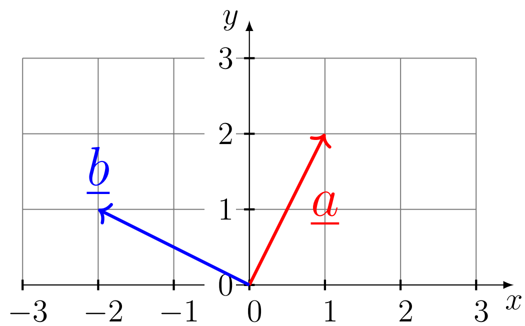


Figure 5.1: The vectors $\underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ rotated counterclockwise by 90 degrees to create the new vector $\underline{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

5.2 Eigenvectors and Eigenvalues

Definition 5.2.1. An Eigenvector $\underline{v}$ of a linear map T is a non-zero vector, such that when T is applied to $\underline{v}$, $\underline{v}$ does not change direction. This is applying T to $\underline{v}$ scales $\underline{v}$ by a scalar λ know as an eigenvalue. Mathematically this is,

$$T(\underline{v}) = \lambda \underline{v}.$$

We have seen how how linear maps relate to matrices and therefore we have an equivalent definition for matrices. We have seen that if we multiply a square $n \times n$ matrix A by a n dimensional vector $\underline{a}$ the output is a n dimensional vector $\underline{b}$, $A\underline{a} = \underline{b}$.

Definition 5.2.2. Given an $n \times n$ square matrix A, and the n dimensional vector $\underline{a}$, if

$$A\underline{a} = \lambda \underline{a},$$

where $\lambda \in \mathbb{R}$, we say that λ is an eigenvalue of the matrix A. That is multiplying the matrix A by a vector $\underline{a}$, returns a scalar multiply of the vector $\underline{a}$.

Example 5.2.1. Given the 2×2 dimensional matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, show the following vectors $\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors for the matrix A and find the corresponding eigenvalue. Firstly, we consider the vector $\underline{u}$,

$$\begin{aligned} A\underline{u} &= \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned}$$

therefore, $\underline{u}$ is an eigenvector of A with eigenvalue $\lambda = 1$. Secondly, we consider the vector $\underline{u}$,

$$\begin{aligned} A\underline{v} &= \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ &= 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

therefore, $\underline{v}$ is an eigenvector of A with eigenvalue $\lambda = 4$.