2.7 Joint Distributions

Previously we have looked at discrete random variables, which essentially take values that depend on the outcome of some random experiment. In this section we want to study pairs of random variables which may or may not depend on the same random experiment. One of main ideas here is for us to have a similar notion of conditional events but applied instead to random variables.

In an analogous way to the single random variable case, we can define a distribution function for pairs of random variables. In the single variable case, the distribution function tells us the likelihood of each possible outcome. Similarly in the two variable case, the distribution function tells us how likely we are to observe every possible pair of outcomes.

Definition 2.7.1. Suppose X and Y are discrete random variables. Then we define the *joint distribution function* of X and Y as follows:

$$f_{X,Y}(i,j) = \mathbb{P}(\{X = i\} \cap \{Y = j\}),$$

for each $i, j \in \mathbb{N}_0$.

Essentially the distribution function just tells us likelihood of X taking one values and Y simultaneously taking another value. This is made more apparent in the following example:

Example 2.7.1. Suppose we play the following game: We flip a coin, if the coin shows heads then we roll a five-sided fair die, if the coin shows tail then we roll a four-sided fair die. Let X be a random variable which is one if the coin shows heads, and zero if the coin shows tails. Let Y be the outcome of the die which is rolled. What is the joint distribution function of X and Y?

We should first note that the support of X is $\{0, 1\}$, while the support of Y is $\{1, 2, 3, 4, 5\}$. So we need to consider each pair of values across both supports. Firstly let us consider, $f_{X,Y}(0, 1) = \mathbb{P}(X = 0 \cap Y = 1)$, by using the definition of conditional probability (Definition 1.5.1) we have that:

$$\mathbb{P}(X = 0 \cap Y = 1) = \mathbb{P}(Y = 1|X = 0)\mathbb{P}(X = 0).$$

We note that $X = 0$ whenever the coin shows tails, therefore $\mathbb{P}(X = 0) = 1/2$. For $\mathbb{P}(Y = 1|X = 0)$, we note that as it is given that $X = 0$, then the coin must have shown tails. This implies that we must have rolled the four-sided die, therefore the probability of seeing a one is $1/4$. Hence we have that:

$$\mathbb{P}(X = 0 \cap Y = 1) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$$

Now we note there was nothing special about choosing $Y = 1$ here, as each face of the die is equally likely, therefore for each $j \in \{1, 2, 3, 4\}$ we have that $f_{X,Y}(0, j) = 1/8$. We follow a similar argument for when $X = 1$, in this case we end up rolling a five-sided die thus for each $k \in \{1, 2, 3, 4, 5\}$ we have that:

$$f_{X,Y}(1,k) = \mathbb{P}(Y = k \cap X = 1) = \mathbb{P}(Y = k|X = 1)\mathbb{P}(X = 1) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}.$$

We may note that there is one pair missing $f_{X,Y}(0,5)$. We note that if $X = 0$, we roll a four-sided die, therefore it is impossible for it show a five, hence $f_{X,Y}(0,5) = 0$. Therefore we can list the complete joint distribution as follows:

$f_{X,Y}$		Y				
		1	2	3	4	5
X	0	1/8	1/8	1/8	1/8	0
	1	1/10	1/10	1/10	1/10	1/10