Mathematical and Logical Foundations of Computer Science

Lecture 12 - Predicate Logic (Natural Deduction Proofs)

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(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**
- Constructive vs. Classical logic
- Type theory

Today

- Natural Deduction proofs for Predicate Logic
- $\forall/\exists$ rules
- substitution

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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

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- All men are mortal
- Socrates is a man
- Therefore, Socrates is mortal

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- **Constant**: s which stands for Socrates

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$\begin{array}{lcl} t & ::= & x \mid f(t, \ldots, t) \\ P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P \end{array}$$

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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
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The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

Recap: Examples

Consider the following domain and signature:

- Domain: $\mathbb{N}$
- Functions: $0, 1, 2, \ldots$ (arity 0); $+$ (arity 2)
- Predicates: $\texttt{prime}$, $\texttt{even}$, $\texttt{odd}$ (arity 1); $=$, $>$, $\geqslant$ (arity 2)

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Express the following sentences in predicate logic

- All prime numbers are either 2 or odd.
  $\forall x.\texttt{prime}(x) \rightarrow x = 2 \vee \texttt{odd}(x)$
- Every even number is equal to the sum of two primes.
  $\forall x.\texttt{even}(x) \rightarrow \exists y.\exists z.\texttt{prime}(y) \wedge \texttt{prime}(z) \wedge x = y + z$
- There is no number greater than all numbers.
  $\neg\exists x.\forall y.x \geqslant y$
- All numbers have a number greater than them.
  $\forall x.\exists y.y > x$

One more example (from the book – section 7.6.2)

Domain is people, and we have 6 predicates

politician(x) rich(x) crazy(x) trusts(x, y) knows(x, y) related-to(x, y)

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Express the following sentences in predicate logic

- Nobody trusts a politician.
- Anyone who trusts a politician is crazy.
- Everyone knows someone who is related to a politician.
- Everyone who is rich is either a politician or knows a politician.

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The set of variables occurring free/bound in a terms and formulas is recursively computed as follows:

$$
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\mathtt{fv}(x) & = & \{x\} \\
\mathtt{fv}(f(t_1, \ldots, t_n)) & = & \mathtt{fv}(t_1) \cup \ldots \cup \mathtt{fv}(t_n) \\
\mathtt{fv}(p(t_1, \ldots, t_n)) & = & \mathtt{fv}(t_1) \cup \ldots \cup \mathtt{fv}(t_n) \\
\hline
\mathtt{fv}(\neg P) & = & \mathtt{fv}(P) \\
\mathtt{fv}(P_1 \wedge P_2) & = & \mathtt{fv}(P_1) \cup \mathtt{fv}(P_2) \\
\mathtt{fv}(P_1 \vee P_2) & = & \mathtt{fv}(P_1) \cup \mathtt{fv}(P_2) \\
\mathtt{fv}(P_1 \rightarrow P_2) & = & \mathtt{fv}(P_1) \cup \mathtt{fv}(P_2) \\
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\mathtt{fv}(\forall x.P) & = & \mathtt{fv}(P) \backslash \{x\} \\
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What are the free variables of the following formulas

- $P_1 = (\texttt{odd}(x) \wedge \exists y. y < x \wedge \texttt{odd}(y))$

- $P_2 = (\texttt{odd}(x) \wedge x > y \wedge \exists y. y < x \wedge \texttt{odd}(y))$

- $P_3 = (\forall x. \texttt{odd}(x) \wedge x > y \wedge \exists y. y < x \wedge \texttt{odd}(y))$

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The formula $(\texttt{odd}(x) \wedge x > y \wedge \exists y. y < x \wedge \texttt{odd}(y))$ is considered the same as $(\texttt{odd}(x) \wedge x > y \wedge \exists z. z < x \wedge \texttt{odd}(z))$

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We need to be careful with free and bound variables!

Inference Rule for "for all elimination" – 1st attempt

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Predicate P is true for all elements x of the domain

- For any element of the domain t, we can deduce that P is true where x is replaced by t is true
- This “replacing” operation is a **substitution** operation as seen in lecture 2.

Inference Rule for "for all elimination" – 1st attempt

$$\frac{\forall x.P}{?}\ [\forall E]$$

What can we conclude from the fact that P is true for all x?

Predicate P is true for all elements x of the domain

- For any element of the domain t, we can deduce that P is true where x is replaced by t is true
- This "replacing" operation is a **substitution** operation as seen in lecture 2.
- However, we now have to be careful with free/bound variables.

Substitution

Substitution is defined recursively on terms and formulas:

$P[x \backslash t]$ substitute all the free occurrences of x in P with t.

Substitution

Substitution is defined recursively on terms and formulas:

$P[x\backslash t]$ substitute all the free occurrences of x in P with t.

1st attempt (WRONG)

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\begin{array}{lcl}
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x[y\backslash t] & = & x \\
(f(t_1,\ldots,t_n))[x\backslash t] & = & f(t_1[x\backslash t],\ldots,t_n[x\backslash t]) \\
(p(t_1,\ldots,t_n))[x\backslash t] & = & p(t_1[x\backslash t],\ldots,t_n[x\backslash t]) \\
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(\neg P)[x\backslash t] & = & \neg P[x\backslash t] \\
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Why is this wrong?

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\end{array}
$$

Why is this wrong? $(\forall y.y > x)[x\backslash y]$ would return $\forall y.y > y$, where the free y is now bound! The free y got **captured**! The red occurrences of y stand for different variables than the green ones.

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The additional **conditions** ensure that **free variables do not get captured**.

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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Inference Rule for "for all elimination" – 2nd attempt

The correct rule is:

$$\frac{\forall x.P}{P[x\backslash t]} \quad [\forall E]$$

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Example: consider the formula $\forall x.\exists y.y > x$

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Example: consider the formula $\forall x.\exists y.y > x$

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- P is $\exists y.y > x$

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Example: consider the formula $\forall x.\exists y.y > x$

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- P is $\exists y.y > x$
- Let t be y

Inference Rule for "for all elimination" – 2nd attempt

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- This condition guarantees that we can do the substitution

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- Substituting x with y without renaming bound variables would give the wrong answer (see previous slide)
- Therefore, we first rename bound variables that clash with $\texttt{fv}(t)$, i.e., with y: $\exists z.z > x$
- Then, we substitute: $\exists z.z > y$

Inference Rule for "for all introduction"

$$\frac{?}{\forall x . P} \; [\forall I]$$

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$$\frac{?}{\forall x.P} \quad [\forall I]$$

When can we conclude P is true for all x?

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If we have proved P for a “**general/representative/typical**" variable

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If we have proved P for a "**general/representative/typical**" variable

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I]$$

Condition: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$

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What could go wrong without this condition?

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What could go wrong without this condition?

Otherwise, given the assumption $y > 2$, we could derive $\forall x.x > 2$, which is clearly wrong.

Inference Rule for “exists introduction”

$$\frac{?}{\exists x.P}\ [\exists I]$$

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- Without the substitution conditions $P[x\backslash y]$ would be true

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- We could then deduce $\exists x.\forall y.y = x$, i.e., numbers are all equal to each other — obviously incorrect!

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- The substitution conditions prevents such captures

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- Without the substitution conditions $P[x\backslash y]$ would be true
- We could then deduce $\exists x.\forall y.y = x$, i.e., numbers are all equal to each other — obviously incorrect!
- The substitution conditions prevents such captures
- $[\exists I]$'s condition guarantees that the substitution conditions hold

Inference Rule for “exists elimination”

$$\frac{\exists x.P}{?}\;[\exists E]$$

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We know that it holds about some element of the domain,
but we do not know which

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$$\dfrac{\exists x.P \qquad \begin{matrix} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{matrix}}{Q}\ 1\ [\exists E]$$

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Condition: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

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This rule is similar to OR-elimination!

All four inference rules in one slide

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I]$$

Condition: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$

$$\frac{\forall x.P}{P[x\backslash t]}\ [\forall E]$$

Condition: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$

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A simple proof

Prove that $(\forall z.p(z)) \to \forall x.p(x) \vee q(x)$

A simple proof

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We use backward reasoning

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Conditions:

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$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\forall z.p(z)}\,1}{\quad}}{\dfrac{\quad}{\forall x.p(x) \vee q(x)}}}{(\forall z.p(z)) \to \forall x.p(x) \vee q(x)}\;1\;[\to I]
$$

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$$\dfrac{\dfrac{\dfrac{\overline{\quad\forall z.p(z)\quad}^{\,1}}{} \qquad \overline{p(y) \vee q(y)}}{\forall x.p(x) \vee q(x)}\ [\forall I]}{(\forall z.p(z)) \to \forall x.p(x) \vee q(x)}\ 1\ [\to I]$$

Conditions:

- y does not occur free in not-yet-discharged hypotheses or in $\forall x.p(x) \vee q(x)$

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Prove that $(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall z.p(z)}\ 1}{p(y)}}{p(y) \vee q(y)}\ [\vee I_L]}{\forall x.p(x) \vee q(x)}\ [\forall I]}{(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)}\ 1\ [\rightarrow I]
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Conditions:

- y does not occur free in not-yet-discharged hypotheses or in $\forall x.p(x) \vee q(x)$
- y does not clash with bound variables in $p(z)$

A simple proof

More generally, we can prove:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{(\forall z.P) \to \forall x.P \lor Q}$$

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A simple proof

More generally, we can prove:

$$\dfrac{\overline{\forall z.P}^{\;1}}{}$$

$$\dfrac{\dfrac{\overline{P[x\backslash y] \vee Q[x\backslash y]}}{\forall x.P \vee Q}\;[\forall I]}{(\forall z.P) \to \forall x.P \vee Q}\;1\ [\to I]$$

We assume that y does not occur in P or Q

A simple proof

More generally, we can prove:

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We assume that y does not occur in P or Q

Conclusion

What did we cover today?

- Natural Deduction proofs for Predicate Logic
- $\forall/\exists$ rules
- substitution

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What did we cover today?

- Natural Deduction proofs for Predicate Logic
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Next time?

- Natural Deduction proofs for Predicate Logic – continued