

Exercise Sheet 7 - Mathematics

Unassessed exercises

Write out your answers to all exercises and submit via Canvas by next week, Tuesday, 11am. (We will review a sample of answers but not be able to give feedback to everyone.)

Exercise 7.1

Let X be the smallest set of natural numbers such that the following two conditions are satisfied:

- $2 \in X$;
- if $x \in X$ then $x^2 \in X$.

- Apply the rules repeatedly to demonstrate that 256 is an element of X .
- Use structural induction to show that every element of X is of the form 2^{2^n} .
- Give the (inductive) definition of a function that returns for each element $x = 2^{2^n}$ the number n .

Exercise 7.2

Let $X \subseteq \mathbb{Z}$ be the smallest set of integers such that the following three conditions are satisfied:

- $9 \in X$
- $15 \in X$
- If a and b are in X , then so is $a - b$.

- Show that 33 belongs to X .
- Prove a property about the elements of X which allows you to conclude that 32 is not a member of X .
- Consider the following inductive definition of a relation R from X to \mathbb{Z} :
 - $(9, 1) \in R$
 - $(15, 0) \in R$
 - If (a, n) and (b, m) are in R , then so is $(a - b, n - m)$.

The relation R is **not** a function. Why?

Exercise 7.3

This question is about the inductively defined set wbb of “well-balanced (square) brackets,” considered in the lecture.

For two elements s and t of wbb say that they are in relation R if the level of nesting of s is greater or equal than that of t .

- Obviously, R is reflexive and transitive, but it is not anti-symmetric. Give an example that illustrates the lack of anti-symmetry.
- Show that the relation R (as a subset of $wbb \times wbb$) can itself be defined inductively.

Exercise 7.4

Find an inductive definition for the set of strings over the alphabet $\Sigma = \{a, b\}$ that contain as many a 's as b 's. So the strings ϵ , “abba”, and “bababa” are members but “bab” or “aabba” are not. Sketch a *valid argument* that shows that all such strings can indeed be derived from your inductive definition.