Mathematical and Logical Foundations of Computer Science

Lecture 7 - Propositional Logic (Semantics)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- semantics of propositional logic
- satisfiability & validity
- truth tables
- soundness & completeness

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Further reading:

- Chapter 6 of
  http://leanprover.github.io/logic_and_proof/

Recap: Propositional logic syntax

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

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- $P \land Q$: we have a proof of both P and Q
- $P \lor Q$: we have a proof of at least one of P and Q
- $P \to Q$: if we have a proof of P then we have a proof of Q

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- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \bot$

Syntax vs. Semantics

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Syntax and Semantics for the English language?

- Syntax: alphabet and grammar
- Semantics: meanings for words

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We will see others towards the end of the module.

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- $\phi(p) = \mathbf{T}, \phi(q) = \mathbf{T}$ and $\phi(p) = \mathbf{T}, \phi(q) = \mathbf{F}$ and $\phi(p) = \mathbf{F}, \phi(q) = \mathbf{T}$ and $\phi(p) = \mathbf{F}, \phi(q) = \mathbf{F}$

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Conventions:

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Conventions:

- The atoms $\top, \bot$ have the interpretations $\mathbf{T}, \mathbf{F}$ respectively

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Conventions:

- The atoms $\top, \bot$ have the interpretations $\mathbf{T}, \mathbf{F}$ respectively
- $\phi(\top) = \mathbf{T}$ and $\phi(\bot) = \mathbf{F}$

Semantics of logical connectives

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For example given a conjunction $A \wedge B$, we first have to evaluate the truth-values of A and B to compute the truth-value of $A \wedge B$.

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For example given a conjunction $A \wedge B$, we first have to evaluate the truth-values of A and B to compute the truth-value of $A \wedge B$.

I.e., $\phi(A \wedge B) = \mathbf{T}$ iff both $\phi(A) = \mathbf{T}$ and $\phi(B) = \mathbf{T}$.

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- $\phi(A \wedge B) = \mathbf{T}$ iff both $\phi(A) = \mathbf{T}$ and $\phi(B) = \mathbf{T}$
- $\phi(A \rightarrow B) = \mathbf{T}$ iff $\phi(B) = \mathbf{T}$ whenever $\phi(A) = \mathbf{T}$

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- $\phi(A \rightarrow B) = \mathbf{T}$ iff $\phi(B) = \mathbf{T}$ whenever $\phi(A) = \mathbf{T}$
- $\phi(\neg A) = \mathbf{T}$ iff $\phi(A) = \mathbf{F}$

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What is $\phi(2 > 1 \wedge 1 > 0)$? (inequalities are atomic propositions)

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$\phi(2 > 1 \wedge 1 > 0) = \mathbf{T}$ because $\phi(2 > 1) = \mathbf{T}$ and $\phi(1 > 0) = \mathbf{T}$

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What is $\phi(2 > 1 \wedge 0 > 1)$?

What is $\phi(x > 1 \wedge 3 > x)$?

What is $\phi(x > 1 \vee 2 > x)$?

Semantics of logical connectives

What is $\phi(2 > 1 \land 1 > 0)$? (inequalities are atomic propositions)

$\phi(2 > 1 \land 1 > 0) =$ **T** because $\phi(2 > 1) =$ **T** and $\phi(1 > 0) =$ **T**

What is $\phi(2 > 1 \land 0 > 1)$?

$\phi(2 > 1 \land 0 > 1) =$ **F** because $\phi(0 > 1) =$ **F**

What is $\phi(x > 1 \land 3 > x)$?

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What is $\phi(2 > 1 \wedge 0 > 1)$?

$\phi(2 > 1 \wedge 0 > 1) = \mathbf{F}$ because $\phi(0 > 1) = \mathbf{F}$

What is $\phi(x > 1 \wedge 3 > x)$?

we don't know: it depends on $\phi(x > 1)$ and $\phi(3 > x)$

What is $\phi(x > 1 \vee 2 > x)$?

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$\phi(2 > 1 \wedge 1 > 0) =$ **T** because $\phi(2 > 1) =$ **T** and $\phi(1 > 0) =$ **T**

What is $\phi(2 > 1 \wedge 0 > 1)$?

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we don't know: it depends on $\phi(x > 1)$ and $\phi(3 > x)$

What is $\phi(x > 1 \vee 2 > x)$?

it depends on $\phi(x > 1)$ and $\phi(2 > x)$
$\phi(x > 1 \vee 2 > x) =$ **T** for all combinations

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$\phi(2 > 1 \wedge 1 > 0) =$ **T** because $\phi(2 > 1) =$ **T** and $\phi(1 > 0) =$ **T**

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$\phi(2 > 1 \wedge 0 > 1) =$ **F** because $\phi(0 > 1) =$ **F**

What is $\phi(x > 1 \wedge 3 > x)$?

we don't know: it depends on $\phi(x > 1)$ and $\phi(3 > x)$

What is $\phi(x > 1 \vee 2 > x)$?

it depends on $\phi(x > 1)$ and $\phi(2 > x)$

$\phi(x > 1 \vee 2 > x) =$ **T** for all combinations

only 2 possible combinations (the atoms are interdependent):
$\phi(x > 1) =$ **T**, $\phi(2 > x) =$ **F** and $\phi(x > 1) =$ **F**, $\phi(2 > x) =$ **T**

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still $\phi(0 > 2 \rightarrow 1 > 0) = \mathbf{T}$ because $\phi(1 > 0) = \mathbf{T}$

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$\phi(2 > 0 \rightarrow 0 > 1) =$ **F** because $\phi(0 > 1) =$ **F** while $\phi(2 > 0) =$ **T**

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What is $\phi(0 > 2 \rightarrow 0 > 1)$?

$\phi(0 > 2 \rightarrow 0 > 1) = \mathbf{T}$ because $\phi(0 > 2) = \mathbf{F}$

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$\phi(0 > 2 \to 0 > 1) =$ **T** because $\phi(0 > 2) =$ **F**

What is $\phi(x > 2 \to x > 1)$? it depends on $\phi(x > 2)$ and $\phi(x > 1)$

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What is $\phi(2 > 0 \rightarrow 1 > 0)$? (inequalities are atomic propositions)

$\phi(2 > 0 \rightarrow 1 > 0) = \mathbf{T}$ because $\phi(1 > 0) = \mathbf{T}$

What is $\phi(0 > 2 \rightarrow 1 > 0)$?

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$\phi(2 > 0 \rightarrow 0 > 1) = \mathbf{F}$ because $\phi(0 > 1) = \mathbf{F}$ while $\phi(2 > 0) = \mathbf{T}$

What is $\phi(0 > 2 \rightarrow 0 > 1)$?

$\phi(0 > 2 \rightarrow 0 > 1) = \mathbf{T}$ because $\phi(0 > 2) = \mathbf{F}$

What is $\phi(x > 2 \rightarrow x > 1)$? it depends on $\phi(x > 2)$ and $\phi(x > 1)$
$\phi(x > 2 \rightarrow x > 1) = \mathbf{T}$ for all possible combinations (the atoms are interdependent): $\phi(x > 2) = \mathbf{T}, \phi(x > 1) = \mathbf{T}$ and $\phi(x > 2) = \mathbf{F}, \phi(x > 1) = \mathbf{T}$ and $\phi(x > 2) = \mathbf{F}, \phi(x > 1) = \mathbf{F}$

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A method to check satisfiability and validity: **truth tables**

Truth tables

Semantics for "or"

$\phi(A \vee B) = \mathbf{T}$ iff either $\phi(A) = \mathbf{T}$ or $\phi(B) = \mathbf{T}$

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Truth table for "or"

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

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T	**F**	**T**
F	**T**	**T**
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- One row for each valuation
- Last column has the truth value for the corresponding valuation

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T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Truth tables

Semantics for "implies"

$$\phi(A \to B) = \mathbf{T} \text{ iff } \phi(B) = \mathbf{T} \text{ whenever } \phi(A) = \mathbf{T}$$

Truth tables

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Truth table for "implies"

A	B	$A \to B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Truth tables

Semantics for "not"

$$\phi(\neg A) = \textbf{T} \text{ iff } \phi(A) = \textbf{F}$$

Truth tables

Semantics for “not”

$$\phi(\neg A) = \textbf{T} \text{ iff } \phi(A) = \textbf{F}$$

Truth table for “not”

A	$\neg A$
T	**F**
F	**T**

Semantics for compound formulas

We can now construct a truth table for any propositional formula

- consider all possible truth assignments for the atoms
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T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**

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T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**

- 2 atoms, and hence $2^2 = 4$ rows (one per interpretation)

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p	q	$p \rightarrow q$	$\neg q$	$(p \rightarrow q) \land \neg q$
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**

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- Rightmost column gives values of the formula

Satisfiability & validity

A formula is **satisfiable** iff there is a valuation that satisfies it

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example: $p \vee \neg p$ **(tautology)**

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Checking validity

- Already seen how to do this using "natural deduction" and "sequent calculus"
- Truth tables is yet another way
- Bonus: yields counterexample if argument is invalid

Checking (semantic) validity

Is $P \to Q, \neg Q \models \neg P$ (semantically) valid?

Checking (semantic) validity

Is $P \rightarrow Q, \neg Q \models \neg P$ (semantically) valid?

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

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T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

Argument is valid: any row where conclusion is **F** then at least one of the premises is also **F**

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P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

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Note that checking $P_1, \ldots, P_n \models C$ is equivalent to checking the validity of $P_1 \to \cdots P_n \to C$

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T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

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Note that checking $P_1, \ldots, P_n \models C$ is equivalent to checking the validity of $P_1 \to \cdots P_n \to C$

i.e., that the cells of the rightmost column of the truth table for $P_1 \to \cdots P_n \to C$ all contain **T**

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Is $\neg P \to \neg R, R \models \neg P$ (semantically) valid?

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P	R	$\neg P$	$\neg R$	$\neg P \to \neg R$	R	$\neg P$
T	**T**	**F**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**

Checking (semantic) validity

Is $\neg P \to \neg R, R \models \neg P$ (semantically) valid?

P	R	$\neg P$	$\neg R$	$\neg P \to \neg R$	R	$\neg P$
T	**T**	**F**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**

Argument is invalid

Checking (semantic) validity

Is $\neg P \to \neg R, R \models \neg P$ (semantically) valid?

P	R	$\neg P$	$\neg R$	$\neg P \to \neg R$	R	$\neg P$
T	**T**	**F**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**

Argument is invalid

- Look at the first row

Checking (semantic) validity

Is $\neg P \to \neg R, R \models \neg P$ (semantically) valid?

P	R	$\neg P$	$\neg R$	$\neg P \to \neg R$	R	$\neg P$
T	**T**	**F**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**

Argument is invalid

- Look at the first row
- Conclusion is **F**, but both premises are **T**

Checking (semantic) validity

Is $\neg P \to \neg R, R \models \neg P$ (semantically) valid?

P	R	$\neg P$	$\neg R$	$\neg P \to \neg R$	R	$\neg P$
T	**T**	**F**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**

Argument is invalid

- Look at the first row
- Conclusion is **F**, but both premises are **T**
- Can we add a premise to make the argument valid?

Checking (semantic) validity

Is $\neg P \rightarrow \neg R, R \models \neg P$ (semantically) valid?

P	R	$\neg P$	$\neg R$	$\neg P \rightarrow \neg R$	R	$\neg P$
T	**T**	**F**	**F**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**

Argument is invalid

- Look at the first row
- Conclusion is **F**, but both premises are **T**
- Can we add a premise to make the argument valid?
  - Yes, we can add $\neg R$, which would be **F** in the first row

Proving anything using contradictions!

Is $P, \neg P \models C$ is (semantically) valid?

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Is $P, \neg P \models C$ is (semantically) valid?

P	C	$\neg P$	C
T	**T**	**F**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**F**	**T**	**F**

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Is $P, \neg P \models C$ is (semantically) valid?

P	C	$\neg P$	C
T	**T**	**F**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**F**	**T**	**F**

Argument is (trivially) valid:

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T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**F**	**T**	**F**

Argument is (trivially) valid:

- Look at any row (we only have to look at rows where the conclusion is **F**)

Proving anything using contradictions!

Is $P, \neg P \models C$ is (semantically) valid?

P	C	$\neg P$	C
T	**T**	**F**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**F**	**T**	**F**

Argument is (trivially) valid:

- Look at any row (we only have to look at rows where the conclusion is **F**)
- One of P and $\neg P$ is **F**

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We will not prove them here

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What did we cover today?

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- satisfiability & validity
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Next time?

- equivalences
- normal forms