Mathematical and Logical Foundations of Computer Science

Lecture 5b - Propositional Logic (Natural Deduction & Sequent Calculus)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

Today

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

Further reading

- Section 5 in “Proof and Types”
  https://www.paultaylor.eu/stable/prot.pdf
- Chapter 3 of
  http://leanprover.github.io/logic_and_proof/

Recap: Propositional logic syntax

Syntax:

$$P ::= a \mid P \land P \mid P \lor P \mid P \to P \mid \neg P$$

Recap: Propositional logic syntax

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$$P ::= a \mid P \wedge P \mid P \vee P \mid P \to P \mid \neg P$$

Lower-case letters are atoms: p, q, r, etc.

Upper-case letters stand for any proposition: P, Q, R, etc.

Recap: Propositional logic syntax

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Upper-case letters stand for any proposition: P, Q, R, etc.

Two special atoms:

- $\top$ which stands for True
- $\bot$ which stands for False

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Lower-case letters are atoms: p, q, r, etc.

Upper-case letters stand for any proposition: P, Q, R, etc.

Two special atoms:

- $\top$ which stands for True
- $\bot$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \to Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \to \bot$

Recap: Propositional logic syntax

How would you express these sentences in propositional logic?

- "if $x > 2$ then $x > 1$"

- "if $x > 2$ and x is even then $x > 3$"

Recap: Propositional logic syntax

How would you express these sentences in propositional logic?

- "if $x > 2$ then $x > 1$"
  - atom p: "$x > 2$"
  - atom q: "$x > 1$"
  - proposition: $p \rightarrow q$
- "if $x > 2$ and x is even then $x > 3$"

Recap: Propositional logic syntax

How would you express these sentences in propositional logic?

- "if $x > 2$ then $x > 1$"
  - atom p: "$x > 2$"
  - atom q: "$x > 1$"
  - proposition: $p \to q$

- "if $x > 2$ and x is even then $x > 3$"
  - atom p: "$x > 2$"
  - atom q: "x is even"
  - atom r: "$x > 3$
  - proposition: $(p \wedge q) \to r$
  - we don't need parentheses, and can just write: $p \wedge q \to r$

Recap: Natural deduction vs. Sequent Calculus

2 deduction systems for propositional logic (don't mix their rules!)

Recap: Natural deduction vs. Sequent Calculus

2 deduction systems for propositional logic (don't mix their rules!)

Natural Deduction

- "natural" style of constructing a proof
- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion
- Can work both forwards and backwards
- "natural" doesn't mean there is a unique proof

Recap: Natural deduction vs. Sequent Calculus

2 deduction systems for propositional logic (don't mix their rules!)

Natural Deduction

- "natural" style of constructing a proof
- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion
- Can work both forwards and backwards
- "natural" doesn't mean there is a unique proof

Sequent Calculus

- hypotheses are made explicit in a **context**
- instead of deriving proposition, we derive **sequents**
  - a sequent is of the form $\Gamma \vdash P$
  - where the environment/context Γ is a list of propositions
  - and P is a proposition
  - intuitively: P is true assuming that the formulas in Γ are true
- we typically go backward

Recap: Natural Deduction

Natural Deduction rules:

$$\dfrac{\bot}{A}\;[\bot E] \qquad \dfrac{}{\top}\;[\top I] \qquad \dfrac{\begin{array}{c}\overline{A}^{\;1}\\ \vdots\\ B\end{array}}{A \to B}\;1\;[\to I] \qquad \dfrac{A \to B \quad A}{B}\;[\to E]$$

$$\dfrac{\begin{array}{c}\overline{A}^{\;1}\\ \vdots\\ \bot\end{array}}{\neg A}\;1\;[\neg I] \qquad \dfrac{\neg A \quad A}{\bot}\;[\neg E]$$

$$\dfrac{A}{A \vee B}\;[\vee I_L] \qquad \dfrac{A}{B \vee A}\;[\vee I_R] \qquad \dfrac{A \vee B \quad A \to C \quad B \to C}{C}\;[\vee E]$$

$$\dfrac{A \quad B}{A \wedge B}\;[\wedge I] \qquad \dfrac{A \wedge B}{B}\;[\wedge E_R] \qquad \dfrac{A \wedge B}{A}\;[\wedge E_L]$$

Recap: Sequent Calculus

Sequence Calculus rules:

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ [\to L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ [\neg L] \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ [\neg R]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ [\vee L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ [\wedge L] \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{}{A \vdash A}\ [Id] \qquad \frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ [Cut]$$

$$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X] \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W] \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ [C]$$

Recap: Sequent Calculus

In addition we allow using the following **derived rules**:

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \to B, \Gamma_2 \vdash C} [\to L] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B} [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C} [\vee L] \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C} [\wedge L]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B} [W] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B} [C]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A} [Id]$$

Recap: Sequent Calculus

In addition we allow using the following **derived rules**:

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \rightarrow B, \Gamma_2 \vdash C} \ [\rightarrow L] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B} \ [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C} \ [\vee L] \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C} \ [\wedge L]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B} \ [W] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B} \ [C]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A} \ [Id]$$

All these **derived rules** can be proved/derived using the rules on the previous slide

Recap: Proofs

Natural Deduction

Sequent Calculus

Recap: Proofs

Natural Deduction

introduction/elimination rules

Sequent Calculus

right/left rules

Recap: Proofs

Natural Deduction	**Sequent Calculus**
introduction/elimination rules	right/left rules
natural proofs	amenable to automation

Recap: Proofs

Natural Deduction

introduction/elimination rules

natural proofs

$$\dfrac{\dfrac{}{A}\,1 \\ \vdots \\ B}{A \to B}\ 1\ [\to I]$$

Sequent Calculus

right/left rules

amenable to automation

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

Recap: Proofs

Natural Deduction

introduction/elimination rules

natural proofs

$$\dfrac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ B\end{array}}{A \to B}\;1\;[\to I]$$

Sequent Calculus

right/left rules

amenable to automation

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\;[\to R]$$

- in the Sequent Calculus the discharged hypothesis A is kept in the context!

Recap: Proofs

Natural Deduction

introduction/elimination rules

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$$\dfrac{\begin{array}{c}\overline{A}^{\;1} \\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I]$$

Sequent Calculus

right/left rules

amenable to automation

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

- in the Sequent Calculus the discharged hypothesis A is kept in the context!
- all the available hypotheses are always kept in the context part of sequents

Recap: Proofs

Natural Deduction

introduction/elimination rules

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$$\dfrac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ B\end{array}}{A \to B}\ 1\ [\to I]$$

Sequent Calculus

right/left rules

amenable to automation

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

- in the Sequent Calculus the discharged hypothesis A is kept in the context!
- all the available hypotheses are always kept in the context part of sequents
- a proposition provable in one system is provable in the other

Example 1

Provide a Natural Deduction proof of $(A \land B) \to (B \land A)$

Example 1

Provide a Natural Deduction proof of $(A \wedge B) \to (B \wedge A)$

Here is an example of a backward proof:

$$\dfrac{\dfrac{\dfrac{\quad}{\quad} \qquad \dfrac{\quad}{\quad}}{\quad}}{(A \wedge B) \to (B \wedge A)}$$

Example 1

Provide a Natural Deduction proof of $(A \wedge B) \to (B \wedge A)$

Here is an example of a backward proof:

$$\underline{\overline{A \wedge B}^{\,1}} \qquad\qquad \underline{\overline{A \wedge B}^{\,1}}$$

$$\dfrac{\overline{\quad B \wedge A \quad}}{(A \wedge B) \to (B \wedge A)}\ 1\ [\to I]$$

Example 1

Provide a Natural Deduction proof of $(A \wedge B) \rightarrow (B \wedge A)$

Here is an example of a backward proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,1}{B} \qquad \dfrac{\dfrac{}{A \wedge B}\,1}{A}}{B \wedge A}\,[\wedge I]}{(A \wedge B) \rightarrow (B \wedge A)}\,1\ [\rightarrow I]$$

Example 1

Provide a Natural Deduction proof of $(A \wedge B) \to (B \wedge A)$

Here is an example of a backward proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,{}^{1}}{B}\,[\wedge E_R] \qquad \dfrac{\dfrac{}{A \wedge B}\,{}^{1}}{A}}{B \wedge A}\,[\wedge I]}{(A \wedge B) \to (B \wedge A)}\;1\;[\to I]$$

Example 1

Provide a Natural Deduction proof of $(A \wedge B) \rightarrow (B \wedge A)$

Here is an example of a backward proof:

$$
\dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{\,1}}{B}\,[\wedge E_R] \qquad \dfrac{\overline{A \wedge B}^{\,1}}{A}\,[\wedge E_L]}{B \wedge A}\,[\wedge I]}{(A \wedge B) \rightarrow (B \wedge A)}\;1\;[\rightarrow I]
$$

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Here is an example of a backward proof:

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How do we know where the introduced hypotheses ($A \wedge B$ above) will be used in the proof?

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Provide a Natural Deduction proof of $(A \wedge B) \rightarrow (B \wedge A)$

Here is an example of a backward proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,1}{B}\,[\wedge E_R] \quad \dfrac{\dfrac{}{A \wedge B}\,1}{A}\,[\wedge E_L]}{B \wedge A}\,[\wedge I]}{(A \wedge B) \rightarrow (B \wedge A)}\,1\ [\rightarrow I]$$

How do we know where the introduced hypotheses ($A \wedge B$ above) will be used in the proof?

We typically don't so we can keep track of them on the side while doing the proof as follows:

Example 1

Let us prove $(A \wedge B) \rightarrow (B \wedge A)$ again:

Example 1

Let us prove $(A \wedge B) \to (B \wedge A)$ again:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\quad}}{\quad} \qquad \dfrac{\dfrac{}{\quad}}{\quad}}{\quad}}{(A \wedge B) \to (B \wedge A)}$$

Hypotheses:

Example 1

Let us prove $(A \wedge B) \to (B \wedge A)$ again:

$$\dfrac{\dfrac{\dfrac{\dfrac{\quad}{\quad}}{\quad} \qquad \dfrac{\dfrac{\quad}{\quad}}{\quad}}{B \wedge A}}{(A \wedge B) \to (B \wedge A)}\ 1\ [\to I]$$

Hypotheses:

- hypothesis 1: $A \wedge B$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\quad}}{B} \qquad \dfrac{\dfrac{}{\quad}}{A}}{B \wedge A}\,[\wedge I]}{(A \wedge B) \to (B \wedge A)}\; 1\ [\to I]$$

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Let us prove $(A \wedge B) \rightarrow (B \wedge A)$ again:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}}{B}\,[\wedge E_R] \qquad \dfrac{\dfrac{}{}}{A}}{B \wedge A}\,[\wedge I]}{(A \wedge B) \rightarrow (B \wedge A)}\,1\ [\rightarrow I]$$

Hypotheses:

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Let us prove $(A \wedge B) \to (B \wedge A)$ again:

$$
\dfrac{
  \dfrac{\dfrac{}{A \wedge B}\,1}{B}\,[\wedge E_R]
  \qquad
  \dfrac{\dfrac{}{\quad}}{A}
}{
  \dfrac{B \wedge A}{(A \wedge B) \to (B \wedge A)}\,1\ [\to I]
}\,[\wedge I]
$$

Hypotheses:

- hypothesis 1: $A \wedge B$

Example 1

Let us prove $(A \land B) \to (B \land A)$ again:

$$\dfrac{\dfrac{\dfrac{}{A \land B}\,1}{B}\,[\land E_R] \qquad \dfrac{\dfrac{}{A \land B}}{A}\,[\land E_L]}{\dfrac{B \land A}{(A \land B) \to (B \land A)}\,1\ [\to I]}\,[\land I]$$

Hypotheses:

- hypothesis 1: $A \land B$

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Let us prove $(A \wedge B) \to (B \wedge A)$ again:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,{}^{1}}{B}\,[\wedge E_R] \qquad \dfrac{\dfrac{}{A \wedge B}\,{}^{1}}{A}\,[\wedge E_L]}{B \wedge A}\,[\wedge I]}{(A \wedge B) \to (B \wedge A)}\,1\ [\to I]$$

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Let us prove $(A \wedge B) \rightarrow (B \wedge A)$ again:

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Hypotheses:

- hypothesis 1: $A \wedge B$

This can be achieved using the Sequent Calculus!

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Provide a Sequent Calculus proof of $(A \wedge B) \rightarrow (B \wedge A)$

Example 1

Provide a Sequent Calculus proof of $(A \land B) \to (B \land A)$

$$\dfrac{\dfrac{\dfrac{\qquad}{\qquad}}{\qquad}}{\vdash (A \land B) \to (B \land A)}$$

Example 1

Provide a Sequent Calculus proof of $(A \wedge B) \rightarrow (B \wedge A)$

$$\dfrac{\dfrac{\dfrac{}{}}{A \wedge B \vdash B \wedge A}}{\vdash (A \wedge B) \rightarrow (B \wedge A)}\ [\rightarrow R]$$

Example 1

Provide a Sequent Calculus proof of $(A \land B) \to (B \land A)$

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash B \land A}}{A \land B \vdash B \land A}\;[\land L]}{\vdash (A \land B) \to (B \land A)}\;[\to R]$$

Example 1

Provide a Sequent Calculus proof of $(A \wedge B) \rightarrow (B \wedge A)$

$$
\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,[Id] \quad \dfrac{}{A, B \vdash A}\,[Id]}{A, B \vdash B \wedge A}\,[\wedge R]}{\dfrac{A \wedge B \vdash B \wedge A}{\vdash (A \wedge B) \rightarrow (B \wedge A)}\,[\rightarrow R]}\,[\wedge L]
$$

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

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Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

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$$\dfrac{}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}$$

Hypotheses:

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{}{(C \to D) \to (A \lor C) \to (B \lor D)}}{(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\ 1\ [\to I]$$

Hypotheses:

- hyp. 1: $A \to B$

Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$
\dfrac{\dfrac{\overline{(A \vee C) \to (B \vee D)}}{(C \to D) \to (A \vee C) \to (B \vee D)}\ 2\ [\to I]}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\ 1\ [\to I]
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{B \vee D}{(A \vee C) \to (B \vee D)}\ 3\ [\to I]}{(C \to D) \to (A \vee C) \to (B \vee D)}\ 2\ [\to I]}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\ 1\ [\to I]$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{A \lor C \qquad A \to (B \lor D) \qquad C \to (B \lor D)}{B \lor D}\ [\lor E]}{(A \lor C) \to (B \lor D)}\ 3\ [\to I]}{(C \to D) \to (A \lor C) \to (B \lor D)}\ 2\ [\to I]}{(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\ 1\ [\to I]$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \lor C$

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vee C}\,3 \qquad \dfrac{}{A \to (B \vee D)} \qquad \dfrac{}{C \to (B \vee D)}}{B \vee D}\,[\vee E]}{(A \vee C) \to (B \vee D)}\,3\,[\to I]}{(C \to D) \to (A \vee C) \to (B \vee D)}\,2\,[\to I]}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,1\,[\to I]$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$

Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vee C}\ 3 \qquad \dfrac{B \vee D}{A \to (B \vee D)}\ 4\ [\to I] \qquad \dfrac{}{C \to (B \vee D)}}{B \vee D}\ [\vee E]}{(A \vee C) \to (B \vee D)}\ 3\ [\to I]}{(C \to D) \to (A \vee C) \to (B \vee D)}\ 2\ [\to I]}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\ 1\ [\to I]$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$
- hyp. 4: A

Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{A \lor C}\;3
      \qquad
      \dfrac{
        \dfrac{
          \dfrac{\dfrac{\dfrac{}{\qquad} \quad \dfrac{}{\,}}{}}{B}
        }{B \lor D}\;[\lor I_L]
      }{A \to (B \lor D)}\;4\;[\to I]
      \qquad
      \dfrac{
        \dfrac{\dfrac{\dfrac{\dfrac{}{\qquad} \quad \dfrac{}{\,}}{}}{}}{}
      }{C \to (B \lor D)}
    }{B \lor D}\;[\lor E]
  }{(A \lor C) \to (B \lor D)}\;3\;[\to I]
}{
  \dfrac{}{(C \to D) \to (A \lor C) \to (B \lor D)}\;2\;[\to I]
}
$$

$$
\dfrac{(C \to D) \to (A \lor C) \to (B \lor D)}{(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\;1\;[\to I]
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
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Provide a Natural Deduction proof of

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$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\overline{A \lor C}\;3
\qquad
\dfrac{
\dfrac{
\dfrac{\overline{A \to B} \qquad \overline{A}}{B}\,[\to E]
}{B \lor D}\,[\lor I_L]
}{A \to (B \lor D)}\;4\,[\to I]
\qquad
\dfrac{
\dfrac{
\dfrac{\overline{\qquad} \qquad \overline{\quad}}{\qquad\qquad}
}{\qquad}
}{C \to (B \lor D)}
}{B \lor D}\,[\lor E]
}{(A \lor C) \to (B \lor D)}\;3\,[\to I]
}{(C \to D) \to (A \lor C) \to (B \lor D)}\;2\,[\to I]
}{(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\;1\,[\to I]
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
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Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$A \vee C$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{}
\UnaryInfC{A}
\RightLabel{$[\to E]$}
\BinaryInfC{B}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$B \vee D$}
\RightLabel{$4\ [\to I]$}
\UnaryInfC{$A \to (B \vee D)$}
\AxiomC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\BinaryInfC{}
\UnaryInfC{}
\UnaryInfC{$C \to (B \vee D)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$B \vee D$}
\RightLabel{$3\ [\to I]$}
\UnaryInfC{$(A \vee C) \to (B \vee D)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$(C \to D) \to (A \vee C) \to (B \vee D)$}
\RightLabel{$1\ [\to I]$}
\UnaryInfC{$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$
- hyp. 4: A

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$A \vee C$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{A}
\RightLabel{$[\to E]$}
\BinaryInfC{B}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$B \vee D$}
\RightLabel{4 $[\to I]$}
\UnaryInfC{$A \to (B \vee D)$}
\AxiomC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\BinaryInfC{}
\UnaryInfC{}
\UnaryInfC{$C \to (B \vee D)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$B \vee D$}
\RightLabel{3 $[\to I]$}
\UnaryInfC{$(A \vee C) \to (B \vee D)$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$(C \to D) \to (A \vee C) \to (B \vee D)$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$
- hyp. 4: A

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$A \lor C$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{A}
\RightLabel{$[\to E]$}
\BinaryInfC{B}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$B \lor D$}
\RightLabel{4 $[\to I]$}
\UnaryInfC{$A \to (B \lor D)$}
\AxiomC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\BinaryInfC{}
\UnaryInfC{$B \lor D$}
\RightLabel{5 $[\to I]$}
\UnaryInfC{$C \to (B \lor D)$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$B \lor D$}
\RightLabel{3 $[\to I]$}
\UnaryInfC{$(A \lor C) \to (B \lor D)$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$(C \to D) \to (A \lor C) \to (B \lor D)$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \lor C$
- hyp. 4: A
- hyp. 5: C

Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vee C}\;3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\;1 \quad \dfrac{}{A}\;4}{B}\;[\to E]}{B \vee D}\;[\vee I_L]}{A \to (B \vee D)}\;4\;[\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\qquad} \quad \dfrac{}{\;}}{D}}{B \vee D}\;[\vee I_R]}{C \to (B \vee D)}\;5\;[\to I]}{B \vee D}\;[\vee E]}{(A \vee C) \to (B \vee D)}\;3\;[\to I]}{(C \to D) \to (A \vee C) \to (B \vee D)}\;2\;[\to I]}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\;1\;[\to I]$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$
- hyp. 4: A
- hyp. 5: C

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$
\dfrac{\dfrac{\dfrac{\overline{A \lor C}^{\;3} \qquad \dfrac{\dfrac{\dfrac{\overline{A \to B}^{\;1} \quad \overline{A}^{\;4}}{B}\,{\scriptstyle [\to E]}}{B \lor D}\,{\scriptstyle [\lor I_L]}}{A \to (B \lor D)}\,{\scriptstyle 4\,[\to I]} \qquad \dfrac{\dfrac{\dfrac{\overline{C \to D} \quad \overline{C}}{D}\,{\scriptstyle [\to E]}}{B \lor D}\,{\scriptstyle [\lor I_R]}}{C \to (B \lor D)}\,{\scriptstyle 5\,[\to I]}}{\dfrac{B \lor D}{(A \lor C) \to (B \lor D)}\,{\scriptstyle 3\,[\to I]}}\,{\scriptstyle [\lor E]}}{(C \to D) \to (A \lor C) \to (B \lor D)}\,{\scriptstyle 2\,[\to I]}}{(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\,{\scriptstyle 1\,[\to I]}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \lor C$
- hyp. 4: A
- hyp. 5: C

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$
\dfrac{
\dfrac{
\dfrac{}{A \vee C}\,{\scriptstyle 3}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A \to B}\,{\scriptstyle 1}
\quad
\dfrac{}{A}\,{\scriptstyle 4}
}{B}\,{\scriptstyle [\to E]}
}{B \vee D}\,{\scriptstyle [\vee I_L]}
}{A \to (B \vee D)}\,{\scriptstyle 4\ [\to I]}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{C \to D}\,{\scriptstyle 2}
\quad
\dfrac{}{C}
}{D}\,{\scriptstyle [\to E]}
}{B \vee D}\,{\scriptstyle [\vee I_R]}
}{C \to (B \vee D)}\,{\scriptstyle 5\ [\to I]}
}{
\dfrac{
\dfrac{B \vee D}{(A \vee C) \to (B \vee D)}\,{\scriptstyle 3\ [\to I]}
}{(C \to D) \to (A \vee C) \to (B \vee D)}\,{\scriptstyle 2\ [\to I]}
}\,{\scriptstyle [\vee E]}
}{(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,{\scriptstyle 1\ [\to I]}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \vee C$
- hyp. 4: A
- hyp. 5: C

Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$A \lor C$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{A}
\RightLabel{$[\to E]$}
\BinaryInfC{B}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$B \lor D$}
\RightLabel{$4\ [\to I]$}
\UnaryInfC{$A \to (B \lor D)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$C \to D$}
\AxiomC{}
\RightLabel{5}
\UnaryInfC{C}
\RightLabel{$[\to E]$}
\BinaryInfC{D}
\RightLabel{$[\lor I_R]$}
\UnaryInfC{$B \lor D$}
\RightLabel{$5\ [\to I]$}
\UnaryInfC{$C \to (B \lor D)$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$B \lor D$}
\RightLabel{$3\ [\to I]$}
\UnaryInfC{$(A \lor C) \to (B \lor D)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$(C \to D) \to (A \lor C) \to (B \lor D)$}
\RightLabel{$1\ [\to I]$}
\UnaryInfC{$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \lor C$
- hyp. 4: A
- hyp. 5: C

Example 2

Provide a Natural Deduction proof of
$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$A \lor C$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{A}
\RightLabel{$[\to E]$}
\BinaryInfC{B}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$B \lor D$}
\RightLabel{4 $[\to I]$}
\UnaryInfC{$A \to (B \lor D)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$C \to D$}
\AxiomC{}
\RightLabel{5}
\UnaryInfC{C}
\RightLabel{$[\to E]$}
\BinaryInfC{D}
\RightLabel{$[\lor I_R]$}
\UnaryInfC{$B \lor D$}
\RightLabel{5 $[\to I]$}
\UnaryInfC{$C \to (B \lor D)$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$B \lor D$}
\RightLabel{3 $[\to I]$}
\UnaryInfC{$(A \lor C) \to (B \lor D)$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$(C \to D) \to (A \lor C) \to (B \lor D)$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \lor C$
- hyp. 4: A
- hyp. 5: C

- If an hypothesis is introduced in a branch, make sure you don't use it in another branch (e.g. 4 cannot be used in the far right branch)

Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

We will keep track of our hypotheses on the side

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor C}\ 3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \to B}\ 1 \quad \dfrac{}{A}\ 4}{B}\ [\to E]}{B \lor D}\ [\lor I_L]}{A \to (B \lor D)}\ 4\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{C \to D}\ 2 \quad \dfrac{}{C}\ 5}{D}\ [\to E]}{B \lor D}\ [\lor I_R]}{C \to (B \lor D)}\ 5\ [\to I]}{B \lor D}\ [\lor E]}{(A \lor C) \to (B \lor D)}\ 3\ [\to I]}{(C \to D) \to (A \lor C) \to (B \lor D)}\ 2\ [\to I]}{(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\ 1\ [\to I]
$$

Hypotheses:

- hyp. 1: $A \to B$
- hyp. 2: $C \to D$
- hyp. 3: $A \lor C$
- hyp. 4: A
- hyp. 5: C

- If an hypothesis is introduced in a branch, make sure you don't use it in another branch (e.g. 4 cannot be used in the far right branch)
- This is enforced by sequents in the Sequent Calculus

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$$

Example 2

Provide a Sequent Calculus proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\dfrac{\dfrac{}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\ [\to R]$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\dfrac{\dfrac{\dfrac{}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\;[\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\;[\to R]$$

Example 2

Provide a Sequent Calculus proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\dfrac{\dfrac{\dfrac{A \to B, C \to D, A \vee C \vdash B \vee D}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\ [\to R]}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\ [\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\ [\to R]$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to B, C \to D, A \vdash B \vee D \qquad A \to B, C \to D, C \vdash B \vee D}{A \to B, C \to D, A \vee C \vdash B \vee D}\ [\vee L]}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\ [\to R]}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\ [\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\ [\to R]$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \to D, A \vdash A} \qquad \dfrac{}{B, C \to D, A \vdash B \vee D}}{A \to B, C \to D, A \vdash B \vee D}\,[\to L] \qquad \dfrac{}{A \to B, C \to D, C \vdash B \vee D}}{A \to B, C \to D, A \vee C \vdash B \vee D}\,[\vee L]}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\,[\to R]}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \to D, A \vdash A}\,[Id] \quad \dfrac{\dfrac{}{}}{B, C \to D, A \vdash B \vee D}}{A \to B, C \to D, A \vdash B \vee D}\,[\to L] \quad \dfrac{\dfrac{}{} \quad \dfrac{\dfrac{}{}}{}}{A \to B, C \to D, C \vdash B \vee D}}{A \to B, C \to D, A \vee C \vdash B \vee D}\,[\vee L]}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\,[\to R]}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]
$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$
\dfrac{
  \dfrac{
    \dfrac{}{C \to D, A \vdash A}\,[Id]
    \qquad
    \dfrac{\dfrac{}{B, C \to D, A \vdash B}}{B, C \to D, A \vdash B \vee D}\,[\vee R_1]
  }{A \to B, C \to D, A \vdash B \vee D}\,[\to L]
  \qquad
  \dfrac{\dfrac{}{\qquad} \qquad \dfrac{\dfrac{}{\qquad}}{\qquad}}{A \to B, C \to D, C \vdash B \vee D}
}{
  \dfrac{A \to B, C \to D, A \vee C \vdash B \vee D}{
    \dfrac{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}{
      \dfrac{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]
    }\,[\to R]
  }\,[\to R]
}\,[\vee L]
$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \to D, A \vdash A}\,[Id] \qquad \dfrac{\dfrac{}{B, C \to D, A \vdash B}\,[Id]}{B, C \to D, A \vdash B \vee D}\,[\vee R_1]}{A \to B, C \to D, A \vdash B \vee D}\,[\to L] \qquad \dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{}{}}{}}{A \to B, C \to D, C \vdash B \vee D}}{A \to B, C \to D, A \vee C \vdash B \vee D}\,[\vee L]}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\,[\to R]}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$
\dfrac{
  \dfrac{
    \dfrac{}{C \to D, A \vdash A}\,[Id]
    \qquad
    \dfrac{\dfrac{}{B, C \to D, A \vdash B}\,[Id]}{B, C \to D, A \vdash B \vee D}\,[\vee R_1]
  }{A \to B, C \to D, A \vdash B \vee D}\,[\to L]
  \qquad
  \dfrac{
    \dfrac{}{A \to B, C \vdash C}
    \qquad
    \dfrac{\dfrac{}{}}{A \to B, D, C \vdash B \vee D}
  }{A \to B, C \to D, C \vdash B \vee D}\,[\to L]
}{
  \dfrac{
    \dfrac{
      \dfrac{A \to B, C \to D, A \vee C \vdash B \vee D}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\,[\to R]
    }{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]
  }{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]
}\,[\vee L]
$$

Example 2

Provide a Sequent Calculus proof of

$$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{}{C \to D, A \vdash A}\,[Id] \quad \dfrac{\dfrac{}{B, C \to D, A \vdash B}\,[Id]}{B, C \to D, A \vdash B \vee D}\,[\vee R_1]}{A \to B, C \to D, A \vdash B \vee D}\,[\to L]
\quad
\dfrac{\dfrac{}{A \to B, C \vdash C}\,[Id] \quad \dfrac{\dfrac{}{}}{A \to B, D, C \vdash B \vee D}}{A \to B, C \to D, C \vdash B \vee D}\,[\to L]
}{A \to B, C \to D, A \vee C \vdash B \vee D}\,[\vee L]}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\,[\to R]}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\,[\to R]
$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{C \to D, A \vdash A}\,[Id]
      \qquad
      \dfrac{\dfrac{}{B, C \to D, A \vdash B}\,[Id]}{B, C \to D, A \vdash B \lor D}\,[\lor R_1]
    }{A \to B, C \to D, A \vdash B \lor D}\,[\to L]
    \qquad
    \dfrac{
      \dfrac{}{A \to B, C \vdash C}\,[Id]
      \qquad
      \dfrac{\dfrac{}{A \to B, D, C \vdash D}\,[Id]}{A \to B, D, C \vdash B \lor D}\,[\lor R_2]
    }{A \to B, C \to D, C \vdash B \lor D}\,[\to L]
  }{
    \dfrac{
      \dfrac{A \to B, C \to D, A \lor C \vdash B \lor D}{A \to B, C \to D \vdash (A \lor C) \to (B \lor D)}\,[\to R]
    }{A \to B \vdash (C \to D) \to (A \lor C) \to (B \lor D)}\,[\to R]
  }
}{\vdash (A \to B) \to (C \to D) \to (A \lor C) \to (B \lor D)}\,[\to R]
$$

Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$C \to D, A \vdash A$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$B, C \to D, A \vdash B$}
\RightLabel{$[\vee R_1]$}
\UnaryInfC{$B, C \to D, A \vdash B \vee D$}
\RightLabel{$[\to L]$}
\BinaryInfC{$A \to B, C \to D, A \vdash B \vee D$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A \to B, C \vdash C$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A \to B, D, C \vdash D$}
\RightLabel{$[\vee R_2]$}
\UnaryInfC{$A \to B, D, C \vdash B \vee D$}
\RightLabel{$[\to L]$}
\BinaryInfC{$A \to B, C \to D, C \vdash B \vee D$}
\RightLabel{$[\vee L]$}
\BinaryInfC{$A \to B, C \to D, A \vee C \vdash B \vee D$}
\RightLabel{$[\to R]$}
\UnaryInfC{$A \to B, C \to D \vdash (A \vee C) \to (B \vee D)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$}
\end{prooftree}
$$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

Hypotheses:

$$\dfrac{}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}$$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{}{A \to \neg(B \wedge C)}}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\ 1\ [\to I]$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A

$$\dfrac{\dfrac{\overline{\neg(B \wedge C)}}{A \to \neg(B \wedge C)}\ 2\ [\to I]}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\ 1\ [\to I]$$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

$$\dfrac{\dfrac{\dfrac{\perp}{\neg(B \wedge C)}\ 3\ [\neg I]}{A \to \neg(B \wedge C)}\ 2\ [\to I]}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\ 1\ [\to I]$$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg A \qquad A}{\bot}\ [\neg E]}{\neg(B \wedge C)}\ 3\ [\neg I]}{A \to \neg(B \wedge C)}\ 2\ [\to I]}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\ 1\ [\to I]$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg A \qquad \overline{A}^{\,2}}{\bot}\,[\neg E]}{\neg(B \wedge C)}\,3\,[\neg I]}{A \to \neg(B \wedge C)}\,2\,[\to I]}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\,1\,[\to I]$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to \neg A \qquad C}{\neg A}\,[\to E] \qquad \dfrac{}{A}\,2}{\bot}\,[\neg E]}{\neg(B \wedge C)}\,3\ [\neg I]}{A \to \neg(B \wedge C)}\,2\ [\to I]}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\,1\ [\to I]$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C \to \neg A} \quad \dfrac{\dfrac{}{}}{B}}{C \to \neg A}\;[\to E] \quad \dfrac{\dfrac{}{}}{C}}{\neg A}\;[\to E] \quad \dfrac{}{A}\;2}{\bot}\;[\neg E]}{\neg(B \land C)}\;3\;[\neg I]}{A \to \neg(B \land C)}\;2\;[\to I]}{(B \to C \to \neg A) \to A \to \neg(B \land C)}\;1\;[\to I]
$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \land C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$B \to C \to \neg A$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{B}
\RightLabel{$[\to E]$}
\BinaryInfC{$C \to \neg A$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{C}
\RightLabel{$[\to E]$}
\BinaryInfC{$\neg A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{A}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$3\ [\neg I]$}
\UnaryInfC{$\neg(B \wedge C)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$A \to \neg(B \wedge C)$}
\RightLabel{$1\ [\to I]$}
\UnaryInfC{$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$B \to C \to \neg A$}
\AxiomC{}
\UnaryInfC{$B \wedge C$}
\RightLabel{$[\wedge E_L]$}
\UnaryInfC{B}
\RightLabel{$[\to E]$}
\BinaryInfC{$C \to \neg A$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{C}
\RightLabel{$[\to E]$}
\BinaryInfC{$\neg A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{A}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{3 $[\neg I]$}
\UnaryInfC{$\neg(B \wedge C)$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$A \to \neg(B \wedge C)$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$}
\end{prooftree}
$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C \to \neg A}\,1 \qquad \dfrac{\dfrac{}{B \wedge C}\,3}{B}\,[\wedge E_L]}{C \to \neg A}\,[\to E] \qquad \dfrac{\dfrac{}{}}{C}}{\neg A}\,[\to E] \qquad \dfrac{}{A}\,2}{\bot}\,[\neg E]}{\neg(B \wedge C)}\,3\ [\neg I]}{A \to \neg(B \wedge C)}\,2\ [\to I]}{(B \to C \to \neg A) \to A \to \neg(B \wedge C)}\,1\ [\to I]
$$

Hypotheses:
- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \wedge C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

We will keep track of our hypotheses on the side

$$
\dfrac{
  \dfrac{
    \dfrac{
      \overline{B \to C \to \neg A}^{\,1}
      \qquad
      \dfrac{\overline{B \land C}^{\,3}}{B}\;[\land E_L]
    }{C \to \neg A}\;[\to E]
    \qquad
    \dfrac{\overline{B \land C}}{C}\;[\land E_R]
  }{\neg A}\;[\to E]
  \qquad
  \overline{A}^{\,2}
}{
  \dfrac{
    \dfrac{
      \dfrac{\bot}{\neg(B \land C)}\;3\,[\neg I]
    }{A \to \neg(B \land C)}\;2\,[\to I]
  }{(B \to C \to \neg A) \to A \to \neg(B \land C)}\;1\,[\to I]
}\;[\neg E]
$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \land C$

Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

We will keep track of our hypotheses on the side

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C \to \neg A}\,1 \qquad \dfrac{\dfrac{}{B \land C}\,3}{B}\,[\land E_L]}{C \to \neg A}\,[\to E] \qquad \dfrac{\dfrac{}{B \land C}\,3}{C}\,[\land E_R]}{\neg A}\,[\to E] \qquad \dfrac{}{A}\,2}{\bot}\,[\neg E]
$$

$$
\dfrac{\dfrac{\dfrac{\bot}{\neg(B \land C)}\,3\ [\neg I]}{A \to \neg(B \land C)}\,2\ [\to I]}{(B \to C \to \neg A) \to A \to \neg(B \land C)}\,1\ [\to I]
$$

Hypotheses:

- hyp. 1: $B \to C \to \neg A$
- hyp. 2: A
- hyp. 3: $B \land C$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

Here is a proof:

$$\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$
Here is a proof:

$$\dfrac{\dfrac{}{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}\ [\to R]$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

Here is a proof:

$$\dfrac{\dfrac{\overline{B \to C \to \neg A, A \vdash \neg(B \wedge C)}}{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}\;[\to R]}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}\;[\to R]$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$
Here is a proof:

$$
\dfrac{\dfrac{\dfrac{\overline{B \to C \to \neg A, A, B \wedge C \vdash \bot}}{B \to C \to \neg A, A \vdash \neg(B \wedge C)}\;[\neg R]}{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}\;[\to R]}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}\;[\to R]
$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

Here is a proof:

$$
\cfrac{\cfrac{\cfrac{\cfrac{B \to C \to \neg A, A, B, C \vdash \bot}{B \to C \to \neg A, A, B \wedge C \vdash \bot}\;[\wedge L]}{B \to C \to \neg A, A \vdash \neg(B \wedge C)}\;[\neg R]}{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}\;[\to R]}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}\;[\to R]
$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

Here is a proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A, B, C \vdash B$}
\AxiomC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\BinaryInfC{$C \to \neg A, A, B, C \vdash \bot$}
\RightLabel{$[\to L]$}
\BinaryInfC{$B \to C \to \neg A, A, B, C \vdash \bot$}
\RightLabel{$[\land L]$}
\UnaryInfC{$B \to C \to \neg A, A, B \land C \vdash \bot$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$B \to C \to \neg A, A \vdash \neg(B \land C)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$B \to C \to \neg A \vdash A \to \neg(B \land C)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$\vdash (B \to C \to \neg A) \to A \to \neg(B \land C)$}
\end{prooftree}
$$

Example 3

Provide a Sequent Calculus proof of

$(B \to C \to \neg A) \to A \to \neg(B \land C)$

Here is a proof:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A, B, C \vdash B}\,[Id] \qquad \cfrac{}{C \to \neg A, A, B, C \vdash \bot}}{B \to C \to \neg A, A, B, C \vdash \bot}\,[\to L]}{B \to C \to \neg A, A, B \land C \vdash \bot}\,[\land L]}{B \to C \to \neg A, A \vdash \neg(B \land C)}\,[\neg R]}{B \to C \to \neg A \vdash A \to \neg(B \land C)}\,[\to R]}{\vdash (B \to C \to \neg A) \to A \to \neg(B \land C)}\,[\to R]
$$

Example 3

Provide a Sequent Calculus proof of

$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

Here is a proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, C \vdash B}\,[Id] \qquad \dfrac{\dfrac{}{A, B, C \vdash C} \qquad \dfrac{\dfrac{}{}}{\neg A, A, B, C \vdash \bot}}{C \to \neg A, A, B, C \vdash \bot}\,[\to L]}{B \to C \to \neg A, A, B, C \vdash \bot}\,[\to L]}{B \to C \to \neg A, A, B \wedge C \vdash \bot}\,[\wedge L]}{B \to C \to \neg A, A \vdash \neg(B \wedge C)}\,[\neg R]}{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}\,[\to R]}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}\,[\to R]
$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$
Here is a proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B, C \vdash B$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B, C \vdash C$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$\neg A, A, B, C \vdash \perp$}
\RightLabel{$[\to L]$}
\BinaryInfC{$C \to \neg A, A, B, C \vdash \perp$}
\RightLabel{$[\to L]$}
\BinaryInfC{$B \to C \to \neg A, A, B, C \vdash \perp$}
\RightLabel{$[\wedge L]$}
\UnaryInfC{$B \to C \to \neg A, A, B \wedge C \vdash \perp$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$B \to C \to \neg A, A \vdash \neg(B \wedge C)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$B \to C \to \neg A \vdash A \to \neg(B \wedge C)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)$}
\end{prooftree}
$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

Here is a proof:

$$
\dfrac{\dfrac{}{A, B, C \vdash B}[Id] \quad \dfrac{\dfrac{}{A, B, C \vdash C}[Id] \quad \dfrac{\dfrac{}{A, B, C \vdash A}}{\neg A, A, B, C \vdash \bot}[\neg L]}{C \to \neg A, A, B, C \vdash \bot}[\to L]}{\dfrac{B \to C \to \neg A, A, B, C \vdash \bot}{\dfrac{B \to C \to \neg A, A, B \land C \vdash \bot}{\dfrac{B \to C \to \neg A, A \vdash \neg(B \land C)}{\dfrac{B \to C \to \neg A \vdash A \to \neg(B \land C)}{\vdash (B \to C \to \neg A) \to A \to \neg(B \land C)}[\to R]}[\to R]}[\neg R]}[\land L]}[\to L]
$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$
Here is a proof:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{A, B, C \vdash B}[Id]
          \quad
          \dfrac{
            \dfrac{}{A, B, C \vdash C}[Id]
            \quad
            \dfrac{\dfrac{}{A, B, C \vdash A}[Id]}{\neg A, A, B, C \vdash \bot}[\neg L]
          }{C \to \neg A, A, B, C \vdash \bot}[\to L]
        }{B \to C \to \neg A, A, B, C \vdash \bot}[\to L]
      }{B \to C \to \neg A, A, B \wedge C \vdash \bot}[\wedge L]
    }{B \to C \to \neg A, A \vdash \neg(B \wedge C)}[\neg R]
  }{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}[\to R]
}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}[\to R]
$$

Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$
Here is a proof:

$$
\dfrac{
  \dfrac{}{A, B, C \vdash B}\,[Id]
  \qquad
  \dfrac{
    \dfrac{}{A, B, C \vdash C}\,[Id]
    \qquad
    \dfrac{\dfrac{}{A, B, C \vdash A}\,[Id]}{\neg A, A, B, C \vdash \bot}\,[\neg L]
  }{C \to \neg A, A, B, C \vdash \bot}\,[\to L]
}{
  \dfrac{B \to C \to \neg A, A, B, C \vdash \bot}{
    \dfrac{B \to C \to \neg A, A, B \wedge C \vdash \bot}{
      \dfrac{B \to C \to \neg A, A \vdash \neg(B \wedge C)}{
        \dfrac{B \to C \to \neg A \vdash A \to \neg(B \wedge C)}{\vdash (B \to C \to \neg A) \to A \to \neg(B \wedge C)}\,[\to R]
      }\,[\to R]
    }\,[\neg R]
  }\,[\wedge L]
}\,[\to L]
$$

Note that compared to the Natural Deduction proof, we only have to eliminate $B \wedge C$ once here

Natural Deduction and Sequent Calculus

- sequents are useful to keep track of available hypotheses

Natural Deduction and Sequent Calculus

- sequents are useful to keep track of available hypotheses
- however, we have to keep hypotheses around all the time

Natural Deduction and Sequent Calculus

- sequents are useful to keep track of available hypotheses
- however, we have to keep hypotheses around all the time
- the Sequent Calculus provides more structure to proofs

Natural Deduction and Sequent Calculus

- sequents are useful to keep track of available hypotheses
- however, we have to keep hypotheses around all the time
- the Sequent Calculus provides more structure to proofs
- however, it is less "natural"

Conclusion

What did we cover today?

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

Conclusion

What did we cover today?

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

Further reading

- Section 5 in “Proof and Types”
  https://www.paultaylor.eu/stable/prot.pdf
- Chapter 3 of
  http://leanprover.github.io/logic_and_proof/

Conclusion

What did we cover today?

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

Further reading

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- Chapter 3 of
  http://leanprover.github.io/logic_and_proof/

Next time?

- Classical reasoning