Solutions to Exercise Sheet 1

Model answers to all exercises

Exercise 1.1

We start the induction at 4 since the statement is not true for 0–3. At 4 we have $2^4 = 16 < 24 = 4!$. Good. Now we assume that the statement is true for some a (which we also need to assume to be greater than 1). Then we can argue for the successor of a as follows:

$$\begin{array}{rcll} 2^{s(a)} & = & 2 \times 2^a & \text{by the definition of exponentiation} \\ & < & 2 \times a! & \text{by the assumption } P(a) \\ & \leq & s(a) \times a! & \text{since we assumed } a > 1 \\ & = & (s(a))! & \text{by the definition of factorial} \end{array}$$

Exercise 1.2

We assume $a \times b = 0$. Now, either $a = 0$ or $a \neq 0$. In the first case, the conclusion is already true. In the second case we have $a \times b = 0$ by assumption and $a \times 0 = 0$ by annihilation, hence $a \times b = a \times 0$. Using the cancellation law for multiplication (which we are allowed to apply because $a \neq 0$ by assumption), $b = 0$ follows, and so the conclusion has been shown in this case as well.

Exercise 1.3

Here is my list:

- $a \leq a$ is always true.
- If $a \leq b$ and $b \leq c$ then $a \leq c$.
- For any a and b, either $a \leq b$ or $b \leq a$ is true.
- For any c, if $a \leq b$ then $a + c \leq b + c$.
- If $a \leq b$ then $-b \leq -a$.
- For any $c \geq 0$, if $a \leq b$ then $a \times c \leq b \times c$.

Exercise 1.4

(a) (i) If $c > d$.
  (ii) If $c < d$.
  (iii) If $c = d$.

(b) We would like to write this as

$$(c,d) \equiv (c',d') \overset{\text{def}}{\iff} c - d = c' - d'$$

but we are not allowed to use subtraction. So we rearrange to get

$$(c,d) \equiv (c',d') \overset{\text{def}}{\iff} c + d' = c' + d$$

Similarly, we define comparison:

$$(c,d) \leq (c',d') \overset{\text{def}}{\iff} c + d' \leq c' + d$$

(c) (i) $(c,d)+(c',d') \stackrel{\text{def}}{=\!=} (c+c',d+d')$

(ii) $-(c,d) \stackrel{\text{def}}{=\!=} (d,c)$

(iii) $(c,d)\times(c',d') \stackrel{\text{def}}{=\!=} (cc'+dd',cd'+dc')$
(You get this by "cheating" and using ordinary algebra:
$(c-d)\times(c'-d') = cc'-cd'-dc'+dd' = (cc'+dd')-(cd'+dc')$.)