1.4 Computing Probabilities from Events

In this section we are interested in finding how to calculate the probability of a given event occurring. Moreover we look at how to find probabilities of events that have been combined using unions, intersections and complements. We start with some notation, for an event A in a sample space, we denote $\mathbb{P}(A)$ to be the probability that the event occurs. For example, in our fair six-sided die roll, we have that $\mathbb{P}(\{3\})$, the probability the die shows a 3, is equal to 1/6. We note in some sense $\mathbb{P}$ acts as a function, which takes in an event and outputs the probability that the event occurs, thus we can talk about $\mathbb{P}$ having some properties:

We note two intuitive properties, first we have that all values are probabilities hence they are non-negative: For any event A, we have that $\mathbb{P}(A) \geq 0$. We also have that for a sample space Ω, one of the outcomes in Ω has to occur, so $\mathbb{P}(\Omega) = 1$.

The next property we have is known as the **inclusion-exclusion principle**.

Lemma 1.4.1. For any two events A and B we have that:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

The inclusion principle is useful for finding the value of either a union or an intersection, when we are given the other. The idea behind the principle is that if we want to the measure the size of $A \cup B$, we can add the size of A and the size of B together. However, when we do this we end up double counting the region $A \cap B$, so we need to remove it once.

Definition 1.4.1. Given two events A and B, we can say that they are **Mutually Exclusive** or equivalently, **Disjoint**, if

$$\mathbb{P}(A \cap B) = 0.$$

Essentially if two events are mutually exclusive, then the probability that they occur at the same time is precisely zero. For example in the six sided dice roll, the events $A = \{2\}$ and $B = \{1, 4\}$ are disjoint. It is impossible for the dice to be both a 2 and either a 1 or a 4 at the same time, thus $\mathbb{P}(A \cap B) = 0$. However the events $C = \{1, 2\}$ and $D = \{1, 3\}$ are not disjoint, as the outcome 3 would cause both C and D to occur at the same time. By combining Lemma 1.4.1 and applying the definition of disjoint events, we arrive at two more central ideas of probability:

Lemma 1.4.2. Let Ω be a sample space, and A and B be two **mutually exclusive** events in Ω. Then we have,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Lemma 1.4.3. Let Ω be a sample space, and A be an event in Ω then:

$$\mathbb{P}(A) + \mathbb{P}(A^c) = 1.$$

On an intuitive level, Lemma 1.4.3 just describes the idea that for any event A, either the events occurs or the event does not occur. Thus their probabilities must sum to one, as their union is the whole sample space. Lemma 1.4.2 is a form of the inclusion-exclusion principle for when the events do not intersect. Essentially if we have two events, and they do not intersect each other, then the sum of the events together is equal to the sum of their parts. You should try to see why both of these facts are true using a Venn-diagram. You should also understand the difference between Lemma 1.4.2 and the inclusion-exclusion principle, the former requires the events to be disjoint, while the latter works for any pair of events.

Examples of Applying The Rules of Probability

In this section we will apply the previous results to see how to compute the probability of more complex events.

Example 1.4.1. The weather is recorded across 100 days in the year. The weather is either recorded as clear, rainy or snowy. It was recorded that 65 days were either rainy or snowy, and 45 days were rainy. A day is picked at random from the sample, with each equally likely, what is the probability of the following?

(i) The weather was clear on that day.

(ii) The chosen day is a snowy day.

We lead with part (i). Let R be the event that the chosen day is rainy, and S be the event that the chosen day is snowy. We note that $R \cup S$ denotes the event that the chosen day is either snowy or rainy. Therefore if the day is neither snowy or rainy, then it must be clear, so we are interested in finding the complement, $(R \cup S)^c$. We observe that as 65 out of the 100 days are either snowy or rainy, it follows that $\mathbb{P}(R \cup S) = 65/100$. We now apply Lemma 1.4.3 to note that,

$$\mathbb{P}(R \cup S) + \mathbb{P}((R \cup S)^c) = 1.$$

Therefore we have that $\mathbb{P}((R \cup S)^c) = 1 - 65/100 = 35/100 = 7/20$.

For part (ii) we proceed in a similar way. We note that $(R \cup S)^c$ are the days which are clear, and R are the days which are rainy. We note that these two sets are disjoint, (the chosen day can not be both rainy and clear), therefore the number of days which are rainy or clear is equal to $35 + 45 = 80$. Therefore the remaining days must be snowy therefore there are $100 - 80 = 20$ snowy days in the sample. Therefore $\mathbb{P}(S) = 20/100 = 1/5$.

We remark that in the preceding example, all of the events we defined were disjoint. This was due to the fact that each day could only have one type of weather. This may necessarily not always be the case as we will see in the following example:

Example 1.4.2. We are comparing followers on Instagram. I have 120 followers, while you have 240. In total between us we have 310 followers. We pick one of our followers at random, with each being equally likely, what is the probability that the person follows both of us? Represent the number of followers we have using a Venn-diagram.

Again we start defining events that we are interested in. Let A denote the event that the person picked follows me, while let B denote the event that the person picked follows you. As we are interested in the event that the person follows both of us, we are looking for $\mathbb{P}(A \cap B)$. Therefore we need to apply the inclusion-exclusion principle:

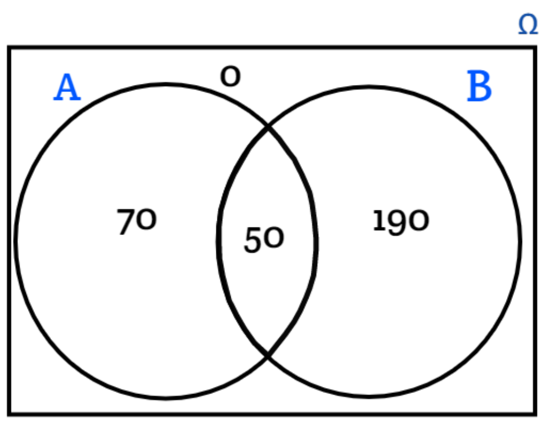
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

As we are picking from the set of our total followers, the person we picked will certainly follow either me or you. Therefore $\mathbb{P}(A \cup B) = 1$. Now 120 people follow me out of a possible 310, so the probability they follow me, $\mathbb{P}(A) = 120/310$, by a similar argument $\mathbb{P}(B) = 240/310$. Plugging these numbers into the inclusion-exclusion principle we have:

$$1 = 120/310 + 240/310 - \mathbb{P}(A \cap B).$$

Therefore we have that $\mathbb{P}(A \cap B) = 50/310$.

To draw the Venn diagram, we have two events which indicate whether a person follows me or follows you. These events also have a non-empty intersection. There are 50 followers that we have in common, therefore there are 120 - 50 = 70 people that only follow me. While there are 240 - 50 = 190 people that will only follow you. As the sample space is only from our collective followers, there are zero people that do not follow either of us. Therefore we can represent the information as follows:



In a final example, we again consider cases where the sets are no longer disjoint, but this time there may also be outcomes that lie outside of the defined events.

Example 1.4.3. Suppose a school of 500 students is sampled. They are asked whether they play piano and whether they play violin. Of the responses; 85 mentioned playing piano; 65 mentioned playing violin; while 370 of the responses played neither. A student is picked at random, all equally likely, what is the probability of the following?

1. The student plays at least one instrument?

2. The student plays both piano and violin.

Again we start by defining events. Let V be the event that the chosen student plays violin, while let P be the event that the chosen student plays piano. We need to find $\mathbb{P}(P \cup V)$, the event that the student plays either piano or violin. We note that $(P \cup V)^c$ is the event that the student plays neither instrument. So by Lemma 1.4.3 we have:

$$\mathbb{P}\left((P \cup V)^c\right) + \mathbb{P}(P \cup V) = 1.$$

We know that 370 students do not play an instrument, so $\mathbb{P}((P \cup V)^c) = 370/500$; therefore by the above,

$$\mathbb{P}(P \cup V) = 1 - 370/500 = 130/500.$$

For part (ii) we are interested in finding $\mathbb{P}(V \cap P)$ so we apply inclusion-exclusion. We have $\mathbb{P}(V) = 65/500$, $\mathbb{P}(P) = 85/500$. Therefore by using part (i) and inclusion exclusion we have:

$$130/500 = 65/500 + 85/500 - \mathbb{P}(V \cap P).$$

By re-arranging, we have that,

$$\mathbb{P}(V \cap P) = 20/500 = 1/25.$$

Furthermore we can represent the responses using a Venn-diagram. As $\mathbb{P}(V \cap P) = 20/500$ we know that 20 students must play both instruments. If 85 responses mention playing piano, then 85 - 20 = 65 students must play piano only. Similarly 65 responses mention playing violin, so $65 - 20 = 45$ students must play violin only. We can now create a Venn-diagram to represent the 500 students: