06-35324 *Mathematical and Logical Foundations of Computer Science* University of Birmingham
Autumn Semester 2021 School of Computer Science
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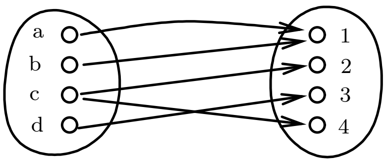
Solutions to Exercise Sheet 6

Exercise 6.1

(a) 

(b) It is not, because it is not single valued: There are two outputs assigned to input c: 2 and 4.

(c) It is not, again because it is not single valued: There are two outputs assigned to input 1: a and b.

(d) $R \cap (A' \times B') = \{(b,1),(c,2),(d,3)\}$. This is defined for all three inputs and produces exactly one output in each case, so it is a function. It is injective, because the three outputs are all different, and it is surjective, because every element of B' is reached (appears as an output). Being injective and surjective, it is also bijective.

(e) Necessarily, B' must have more elements than A'. Looking at the picture from part (a), one can see that $A' = \{b,c\}$ and $B' = \{1,2,3\}$ do the trick: 3 is not in the range of the function.

(f) Now we are looking for a situation where A' has more elements than B'. Again, by staring at the diagram one may come up with the example $A' = \{a,b,c\}$ and $B' = \{1,2\}$. This is not injective as 1 appears twice in the output.

Exercise 6.2

(a) If we assume that a variable of type `String` can contain strings of arbitrary length, then this is a well-defined function. It is injective since the trailing `aa` could be chopped off again, but it is not surjective, as the output has at least two characters. So it is not bijective.

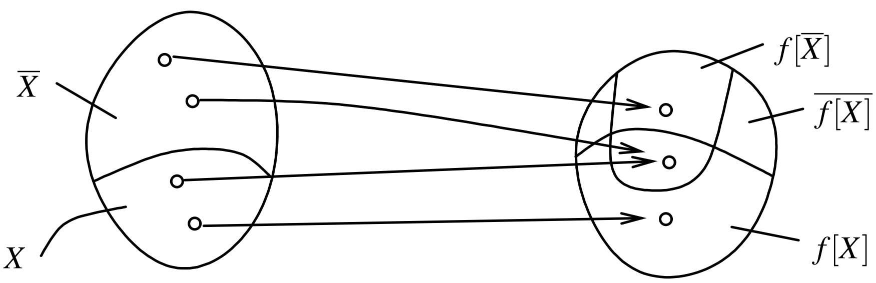
In practice, Java virtual machine implementations do not allow strings of arbitrary length to be created. This means that the method does not implement a function as an exception will be raised for maximally long input strings.

(b) This is a function if we consider the limit on the length of strings allowed. It is not injective as there are different strings of the same length. It is also not surjective because the length can never be negative (but every number greater equal 0 is reached). In any case, it is certainly not bijective.

(c) This is not a function because it will return a different string each time it is called.

Exercise 6.3

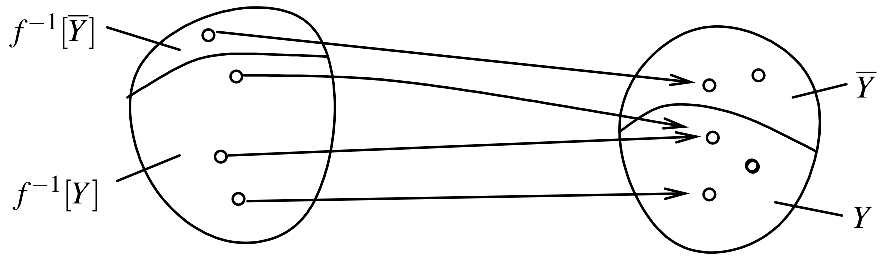
(a)



(b) The equation is not correct because the value of the function at a point outside X need not be outside $f[X]$. It can take any value it likes.

Example: $A = B = \{1,2\}$, $X = \{1\}$, $f(1) = f(2) = 1$. Then $f[\overline{X}] = \{1\}$ but $\overline{f[X]} = \overline{\{1\}} = \{2\}$.

(c)



(d) We argue the two subset relationships by showing that an arbitrary element from the left-hand set also belongs to the right-hand set:

$$\begin{array}{rl}
x \in f^{-1}[\overline{Y}] \text{ means} & f(x) \in \overline{Y} \\
\text{which means} & f(x) \notin Y \\
\text{which means} & x \notin f^{-1}[Y] \\
\text{which means} & x \in \overline{f^{-1}[Y]} \\
\text{so} & f^{-1}[\overline{Y}] \subseteq \overline{f^{-1}[Y]}
\end{array}$$

And for the other containment:

$$\begin{array}{rl}
x \in \overline{f^{-1}[Y]} \text{ means} & x \notin f^{-1}[Y] \\
\text{which means} & f(x) \notin Y \\
\text{which means} & f(x) \in \overline{Y} \\
\text{which means} & x \in f^{-1}[\overline{Y}] \\
\text{so} & \overline{f^{-1}[Y]} \subseteq f^{-1}[\overline{Y}]
\end{array}$$

Exercise 6.4

The circle has to be moved in direction of the x-axis by one radius so that it touches the y-axis at the origin. Then a line through the origin will intersect the circle at the origin and one further point. We associate the "further point" with the line. This doesn't work for a vertical line, i.e., the y-axis, so in that case we associate the origin to the line.

The mapping is bijective since any line through the origin either meets this circle in two points, or is a tangent to it and touches it in the origin. This means that if we take a point P on the circle then we can reconstruct the line l it came from: either it is different from the origin and then the line is the unique one going through origin and P, or it equals the origin and in that case the corresponding line is the y-axis.