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Monotonicity and the first derivative

This short section highlights some connections between the first derivative and whether a function is increasing or decreasing. First we introduce the natural definition of when we say a function is increasing or decreasing.

Definition 13.1. Let $f : \mathbb{R} \to \mathbb{R}$. We say that a function is:

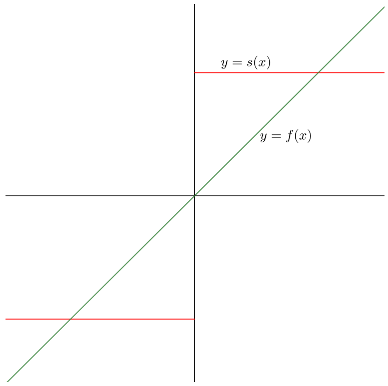
- Increasing if for every $x, y \in \mathbb{R}$ if $x \leq y$ then $f(x) \leq f(y)$;
- Strictly increasing if for every $x, y \in \mathbb{R}$ if $x < y$ then $f(x) < f(y)$;
- Decreasing if for every $x, y \in \mathbb{R}$ if $x \leq y$ then $f(x) \geq f(y)$;
- Strictly decreasing if for every $x, y \in \mathbb{R}$ if $x < y$ then $f(x) > f(y)$.

In other words, a function $f : \mathbb{R} \to \mathbb{R}$ is increasing if the graph $y = f(x)$ "points upwards" and is decreasing if the graph "points downwards". Further, the function is strictly increasing/decreasing if the function is increasing/decreasing and the graph does not have "any flat bits".

Example 13.2. To see that the above definitions coincides with what we intuitively believe an increasing or decreasing function is, let us consider the following functions. Indeed, let $f, h : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$ and $h(x) = -x$. Also, define $s, g : \mathbb{R} \to \mathbb{R}$ be

$$s(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \qquad g(x) = -s(x) = \begin{cases} -1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

We claim that both f and s are increasing whilst both g and h are decreasing. To see this consider the following graphs:



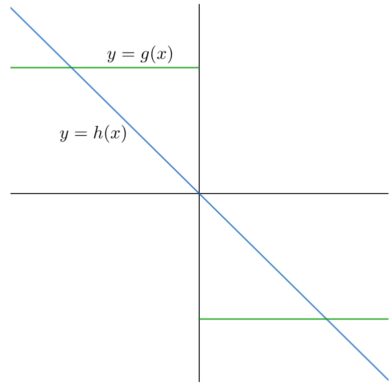


Figure 13.1: Graph of $y = f(x)$ and $y = s(x)$ Figure 13.2: Graph of $y = g(x)$ and $y = h(x)$

From Figure 13.1 we can see that as x increases $f(x)$ or $s(x)$ does not get 'smaller', i.e. the graph is not pointing downwards. Similarly, from Figure 13.2 we can see that as x increases $h(x)$ or $g(x)$ gets 'bigger', i.e. the graph is not pointing upwards.

Let us show that f is strictly increasing and h is strictly decreasing. To see this let $x, y \in \mathbb{R}$ be such that $x < y$. We then have to show that

$$f(x) < f(y) \quad \text{and} \quad h(x) > h(y).$$

The former follows trivially by the definition of f. Indeed, as $x < y$

$$f(x) = x < y = f(y).$$

As $x < y$ was arbitrary we can conclude that f is strictly increasing. Now onto h. Indeed, as $x < y$ note $-x > -y$ and so

$$h(x) = -x > -y = h(y).$$

Therefore, as $x < y$ was arbitrary we can conclude that h is strictly decreasing.

It can be shown, though slightly arduous, that s and g are increasing and decreasing, respectively (try this as an exercise: a cumbersome approach is to follow a similar method to above, but consider all the different cases when $x < 0$, $x = 0$ or $x > 0$ with $y < 0$, $y = 0$, or $y > 0$). However s is not strictly increasing and g is not strictly decreasing. To see that s is not strictly increasing let $x = 1$ and $y = 2$. Then $x < y$, but $s(x) = 1 = s(y)$. Therefore s does not satisfy the property of being strictly increasing. One can check similarly for g (try this as an exercise).

Above we can see that you can check a function is increasing or decreasing simply by following the definition. However this can be slightly tedious. A simpler way is to use the first derivative of a function. To see this let us consider the mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. Then one can easily check that f is strictly increasing.

Now, let us compute its first derivative. Indeed, $f'(x) = 3x^2$. Therefore, for each $x \in \mathbb{R}$ we can see that $f'(x) \geq 0$, so that the gradient of every tangent line to f is non-negative, i.e. every tangent line does not point downwards. This means the graph of $y = f(x)$ never points 'downwards' (Figure 13.3) and so must point upwards, therefore f must be increasing.

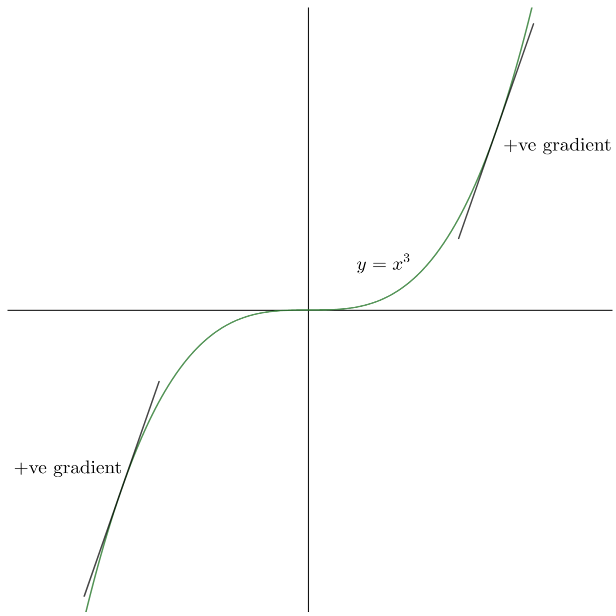


Figure 13.3: Gradient approach to monotonicity

To generalise this, we can say the following.

Lemma 13.0.1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If:

- $f'(x) \geq 0$ for every $x \in \mathbb{R}$, then f is increasing;
- $f'(x) > 0$ for every $x \in \mathbb{R}$, then f is strictly increasing;
- $f'(x) \leq 0$ for every $x \in \mathbb{R}$, then f is decreasing;
- $f'(x) < 0$ for every $x \in \mathbb{R}$, then f is strictly decreasing.

Example 13.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. Then we saw in Example 13.2 that f is an increasing function. On the other hand,

$$f'(x) = 1 \geq 0 \text{ for every } x \in \mathbb{R}.$$

Now, let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$. Then we know that g is a strictly increasing function. Also,

$$g'(x) = 3x^2 \geq 0 \text{ for every } x \in \mathbb{R}.$$

So we can conclude that g is increasing. However, this does not imply that g is strictly increasing function.