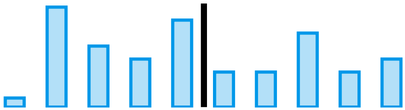
Merge Sort (Divide & Conquer)

Merge Sort

Idea:

1. Split the array into two halves:



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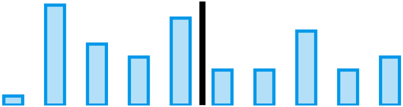
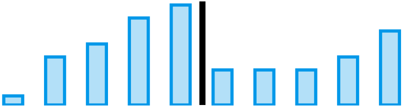


2. Sort each of them recursively:



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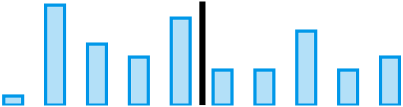

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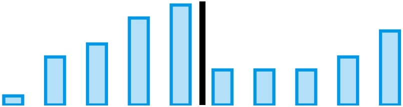
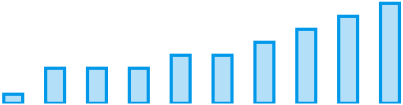


3. Merge the sorted parts:



Example: Merge Sort run

$$\langle 5, 4, 6, 1, 2, 7, 3 \rangle$$

$$\langle 5, 4, 6, 1 \rangle \qquad \langle 2, 7, 3 \rangle$$

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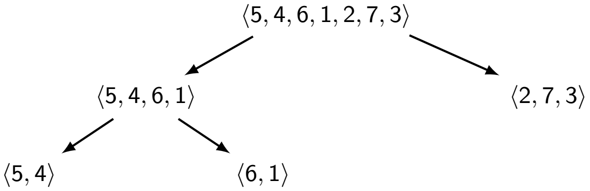

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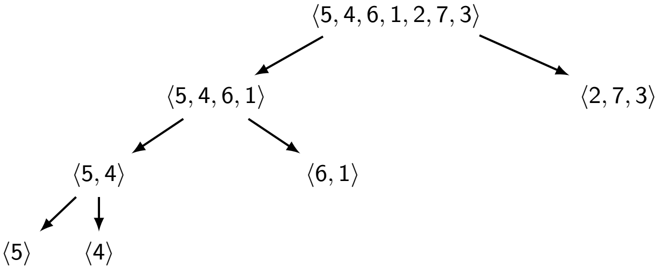

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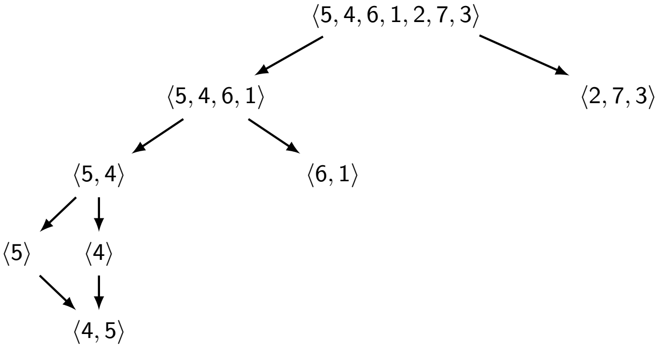

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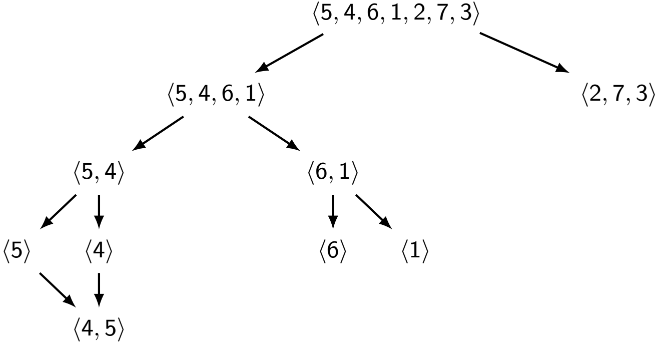

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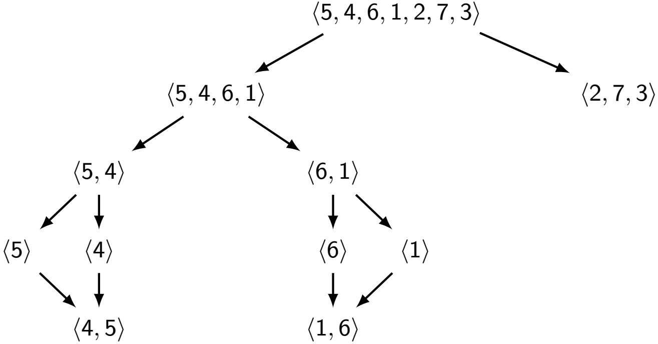

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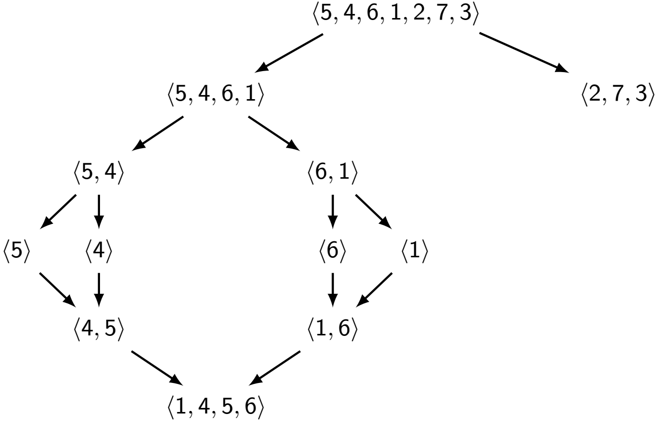

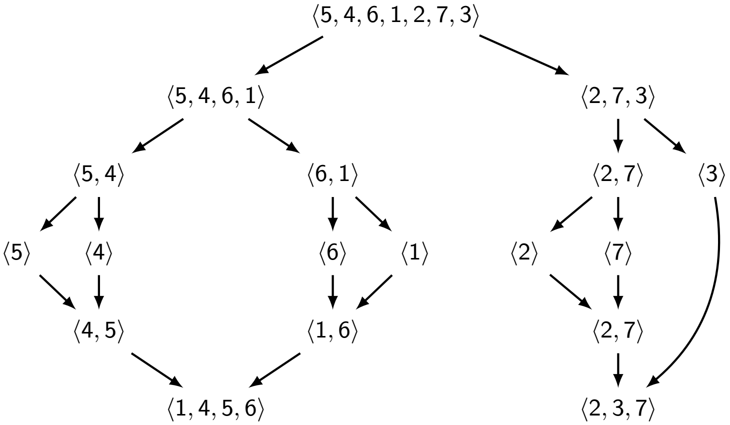
⟨5, 4, 6, 1, 2, 7, 3⟩
⟨5, 4, 6, 1⟩
⟨2, 7, 3⟩
⟨5, 4⟩
⟨6, 1⟩
⟨2, 7⟩
⟨3⟩
⟨5⟩
⟨4⟩
⟨6⟩
⟨1⟩
⟨2⟩
⟨7⟩
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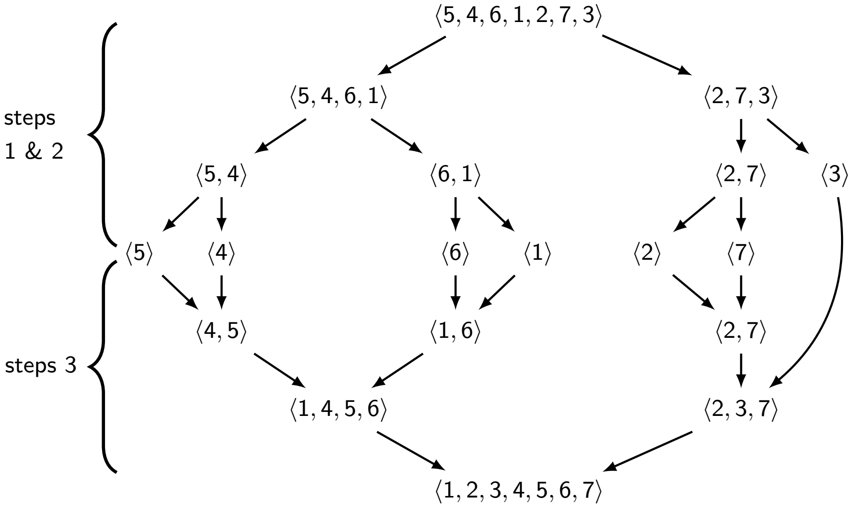
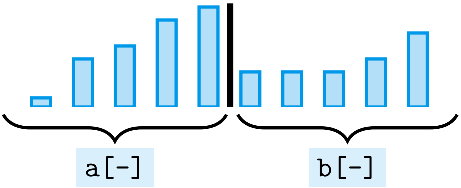


Merging two sorted arrays `a[-]` and `b[-]` efficiently

Idea: In variables `i` and `j` we store the current positions in `a[-]` and `b[-]`, respectively (starting from `i=0` and `j=0`). Then:

1. Allocate a *temporary* array `tmp[-]`, for the result.
2. If `a[i] <= b[j]` then copy `a[i]` to `tmp[i+j]` and `i++`,
3. Otherwise, copy `b[j]` to `tmp[i+j]` and `j++`.

Repeat 2./3. until `i` or `j` reaches the end of `a[-]` or `b[-]`, respectively, and then copy the rest from the other array.



Merging two sorted arrays `a[-]` and `b[-]` efficiently

Merging two sorted arrays is the most important part of merge sort and must be efficient. For example:

Take `a = [1,6,7]` and `b = [3,5]`. Set `i=0` and `j=0`, and allocate `tmp` of length `5`:

1. `a[0]` $\leqslant$ `b[0]`, so set `tmp[0] = a[0]` $(=$ `1` $)$ and `i++`.
2. `a[1]` $>$ `b[0]`, so set `tmp[1] = b[0]` $(=$ `3` $)$ and `j++`.
3. `a[1]` $>$ `b[1]`, so set `tmp[2] = b[1]` $(=$ `5` $)$ and `j++`.

At this point `i` $= 1$, `j` $= 2$ and the first three values stored in `tmp` are `[1,3,5]`.

Since `j` is at the end of `b`, we are done with `b` and we copy the remaining values from `a` into `tmp`. Then, `tmp` stores `[1,3,5,6,7]`.

Merge Sort (pseudocode)

```
mergesort(a, n) {
    mergesort_run(a, 0, n-1)
}

void mergesort_run(a, left, right) {
    if (left < right){
      mid = (left + right) div 2

      mergesort_run(a, left, mid)
      mergesort_run(a, mid+1, right)

      merge(a, left, mid, right)
    }
}
```

The pseudocode we present here tries to avoid some of the unnecessary allocations of new arrays. Namely, when running recursive calls of merge sort, we do not allocate two new arrays for the two halves, we only compute the `left`-most and `right`-most positions of those halves, with respect to the original array `arr`.

Initially we call `mergesort_run` to sort all elements of the array, that is, we want to sort elements on positions

`0`, `1`, `2`, `3`, ..., `n-1`

In order to sort this, we run merge sort twice, first time for the positions

`0`, `1`, `2`, `3`, ..., `mid`

and the second time for positions

`(mid+1)+0`, `(mid+1)+1`, `(mid+1)+2`, ..., `n-1`.

(In further recursive calls are those `left` and `right` bounds recomputed accordingly.)

Merging (pseudocode)

```
merge(array a, int left, int mid, int right) {
    create new array b of size right-left+1
    bcount = 0
    lcount = left
    rcount = mid+1
    while ( (lcount <= mid) and (rcount <= right) ) {
        if ( a[lcount] <= a[rcount] )
            b[bcount++] = a[lcount++]
        else
            b[bcount++] = a[rcount++]
    }
    if ( lcount > mid )
        while ( rcount <= right )
            b[bcount++] = a[rcount++]
    else
        while ( lcount <= mid )
            b[bcount++] = a[lcount++]
    for ( bcount = 0 ; bcount < right-left+1 ; bcount++ )
        a[left+bcount] = b[bcount]
}
```

Time Complexity of Mergesort

Merging two arrays of lengths n_1 and n_2 is in $O(n_1 + n_2)$

Sizes of recursive calls:

Merging:

$$
\left.
\begin{array}{cccccccccl}
 & & & n & & & & & & O(\frac{n}{2} + \frac{n}{2}) = O(n) \\
 & & \frac{n}{2} & & & \frac{n}{2} & & & & O(2 \cdot (\frac{n}{4} + \frac{n}{4})) = O(n) \\
 & \frac{n}{4} & & \frac{n}{4} & & \frac{n}{4} & & \frac{n}{4} & & O(4 \cdot (\frac{n}{8} + \frac{n}{8})) = O(n) \\
 & / \backslash & & / \backslash & & / \backslash & & / \backslash & & \\
\vdots\ \vdots & & \vdots\ \vdots & & \vdots\ \vdots & & \vdots\ \vdots & & & \\
|\ \ | & & |\ \ | & & |\ \ | & & |\ \ | & & & \\
1\ \ 1 & & 1\ \ 1 & & 1\ \ 1 & & 1\ \ 1 & & &
\end{array}
\right\}
$$

Each level requires $O(n)$ for merging.

If $n = 2^k$, then we have $k = \log_2 n$ levels $\implies$ $O(n \log n)$ is the time complexity of merge sort.

(This is the Worst/Best/Average Case complexity.)

Let us analyse the running time of merge sort for an array of size n and for simplicity we assume that $n = 2^k$. First, we run the algorithm recursively for two halves. Putting the running time of those two recursive calls aside, after both recursive calls finish, we merge the result in time $O(\frac{n}{2} + \frac{n}{2})$.

Okay, so what about the recursive calls? To sort $\frac{n}{2}$-many entries, we split them in half and sort both $\frac{n}{4}$-big parts independently. Again, after we finish, we merge in time $O(\frac{n}{4} + \frac{n}{4})$. However, this time, merging of $\frac{n}{2}$-many entries happens twice and, therefore, in total it runs in $O(2 \times (\frac{n}{4} + \frac{n}{4})) = O(2 \times \frac{n}{2}) = O(n)$.

Similarly, we have 4 subproblems of size $\frac{n}{4}$, each of them is merging their subproblems in time $O(\frac{n}{8} + \frac{n}{8})$. In total, all calls of `merge` for subproblems of size $\frac{n}{4}$ take $O(4 \times (\frac{n}{8} + \frac{n}{8})) = O(n)$. ... We see that it always takes $O(n)$ to merge all subproblems of the same size (= those on the same level of the recursion).

Since the height of the tree is $O(\log n)$ and each level requires $O(n)$ time for all merging, the time complexity is $O(n \log n)$. Notice that this analysis does not depend on the particular data, so it is the Worst, Best and Average Case.

Stability of Mergesort

The splitting phase of mergesort does not change the order of any items.

So long as merging phase merges the left with the right in that order and takes values from the leftmost sub-array before the rightmost one when values are equal (as the pseudocode above does) then different elements with the same values do not change their relative order.

Therefore mergesort is stable.