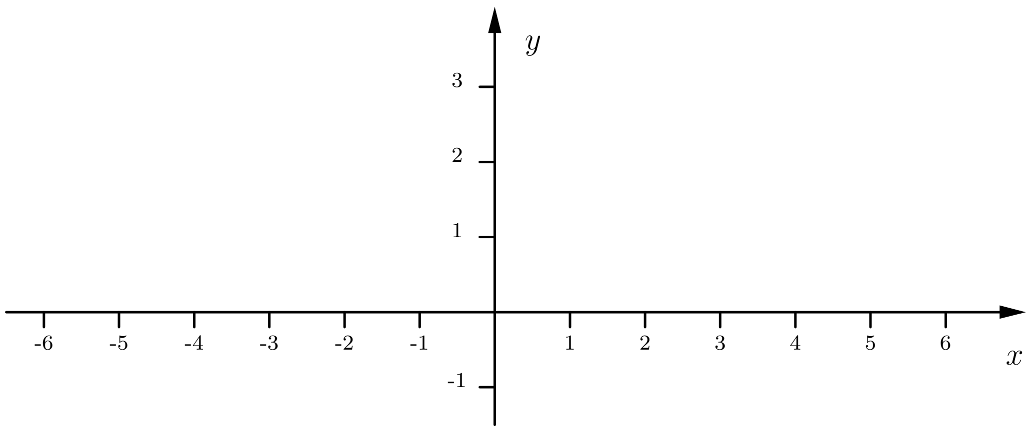
11 Analytic Geometry

11.1 Cartesian coordinate systems and points in the plane

Recall the idea of a **coordinate system**:

110



It consists of two **axes**, usually referred to as the x**-axis** (in horizontal direction) and the y**-axis** (in vertical direction). The axes are at right angles to each other. Their intersection point is called the **origin**. Furthermore, a **unit of length** is chosen on the axes.

A coordinate system allows us to identify a point P in the plane with its **coordinates** $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$: From the origin, go p_1 units along the x-axis (to the right if p_1 is positive, and to the left if it is negative), then go p_2 units parallel to the y-axis (up if p_2 is positive, down if it is negative).[10]

The coordinates of a point are unique once the axes and the unit of length have been chosen.

On a computer screen the origin is usually placed at the top left corner and the y-axis points *downward*.

Distance between points. Given points P and Q with coordinates $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, their **distance** d is computed with the help of the Pythagorean Theorem:

$$d = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

111

This works for negative coordinates just as well as for positive ones.

[10]Most textbooks write the two coordinates of a point in a *row* rather than a *column*, that is, in the form (p_1, p_2). This is purely to save space in printing; I prefer the column form because it is visually more appealing and suggestive.

11.2 Vectors and straight line movements

We can interpret a pair of numbers $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ also as a **movement** of the plane: every point $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is shifted to $P' = \begin{pmatrix} p_1 + v_1 \\ p_2 + v_2 \end{pmatrix}$.

112

Instead of a single movement, we can also see this as moving the points in the plane by v_1 units parallel to the x-axis, and v_2 units parallel to the y-axis.

We use uppercase letters for points: P, Q, R,..., and lowercase letters with arrows for movements: $\vec{v}$, $\vec{w}$, $\vec{u}$,... Instead of "(straight line) movement" we also say **vector**.

We allow vectors to be added to points:

$$P' = P + \vec{v}$$

because the coordinates of P after moving it along $\vec{v}$ are computed as $\begin{pmatrix} p_1 + v_1 \\ p_2 + v_2 \end{pmatrix}$.

A vector has a **length**, again computed by the Pythagorean Theorem:

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$$

which is the distance each point travels under the movement described by $\vec{v}$. A vector of length 1 is called a **unit vector**; $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the **null vector**. For every vector $\vec{v} \neq \vec{0}$ there is a unit vector pointing in the same direction; it is computed as

$$\frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} v_1 / |\vec{v}| \\ v_2 / |\vec{v}| \end{pmatrix}$$

Given two points P and Q we have the movement $\overrightarrow{PQ}$ that moves P into Q; it has the coordinates $\begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$.

Movements as an algebra. Movements can be composed: write $\vec{v} + \vec{w}$ for the movement of first following $\vec{v}$, then $\vec{w}$. The result is another straight line movement with coordinates $\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$.

113

Movements can also be extended or shrunk by a **factor** or **scalar**. For example,

$$\frac{1}{2} \cdot \vec{v} = \begin{pmatrix} v_1/2 \\ v_2/2 \end{pmatrix}$$

is the movement that shifts points in the same direction as $\vec{v}$ but only half as far. If the scalar is negative then the new vector points in the *opposite direction*. This is called **scalar multiplication**.

The laws of vector algebra.

114: Laws of vector addition

$$\begin{aligned} \vec{u} + \vec{v} &= \vec{v} + \vec{u} \\ \vec{u} + (\vec{v} + \vec{w}) &= (\vec{u} + \vec{v}) + \vec{w} \end{aligned}$$

115: Law for the null vector

$$\vec{v} + \vec{0} = \vec{v}$$

116: Laws of scalar multiplication

$$\begin{aligned} 1 \cdot \vec{v} &= \vec{v} \\ 0 \cdot \vec{v} &= \vec{0} \\ s \cdot \vec{0} &= \vec{0} \\ (s+t) \cdot \vec{v} &= s \cdot \vec{v} + t \cdot \vec{v} \\ s \cdot (\vec{v} + \vec{w}) &= s \cdot \vec{v} + s \cdot \vec{w} \\ (st) \cdot \vec{v} &= s \cdot (t \cdot \vec{v}) \end{aligned}$$

Note how the laws of scalar multiplication are similar to those in a ring; the only law that is missing, is $a \times b = b \times a$; this is because it makes no sense to exchange a scalar (a number) for a vector and vice versa. We also have additive inverses:

117

On the other hand, the idea of a multiplicative inverse makes no sense.

11.3 Analytic geometry in the plane

Representing straight lines. Given a point P and a vector $\vec{v} \neq \vec{0}$, we can consider all points X that one can reach from P by following some distance along the direction of $\vec{v}$:

118

All such points lie on a straight line. We write

$$X = P + s \cdot \vec{v}$$

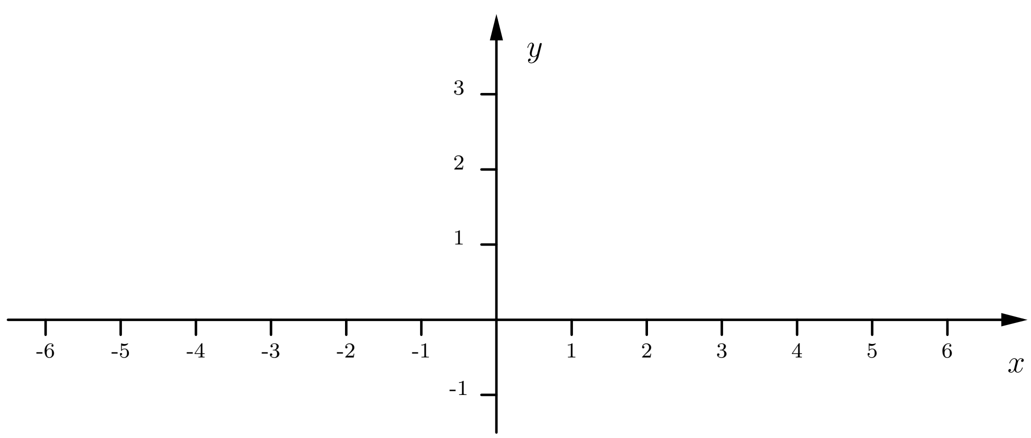
for this. Note that s is allowed to be any positive or negative number.

The expression $X = P + s \cdot \vec{v}$ is called a **parametric representation** of a line where the **parameter** is s. It can also be said to be a **generating expression** because in any application we would have P and $\vec{v}$ given as pairs of numbers, and the presentation would allow us to *generate* points on the line by simply choosing some value for the scalar s and computing the resulting coordinates for X. Example:

119

$$X = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

120



If we are given two different points P and Q in the plane then this defines a straight line whose generating expression is $X = P + s \cdot \overrightarrow{PQ}$.

Intersection of two lines. Given two straight lines described by $X = P + s \cdot \vec{v}$ and $Y = Q + t \cdot \vec{w}$ we can find their **point of intersection** as follows:

The intersection point satisfies

$$X = Y$$

so

$$\begin{pmatrix} p_1 + sv_1 \\ p_2 + sv_2 \end{pmatrix} = P + s \cdot \vec{v} = Q + t \cdot \vec{w} = \begin{pmatrix} q_1 + tw_1 \\ q_2 + tw_2 \end{pmatrix}$$

This gives us two ordinary equations, one for each coordinate:

$$\begin{aligned} p_1 + sv_1 &= q_1 + tw_1 \\ p_2 + sv_2 &= q_2 + tw_2 \end{aligned}$$

The unknowns are s and t; s will tell us how far in direction $\vec{v}$ we have to move from P in order to reach the point of intersection. Likewise, t will tell us how far in direction $\vec{w}$ we have to move from Q in order to reach the intersection point. Obviously, it will be enough to compute **either** s or t to find the coordinates of the intersection point. We do so through the system of linear equations which we obtain by considering each coordinate separately:

$$\begin{aligned} v_1 s - w_1 t &= q_1 - p_1 \\ v_2 s - w_2 t &= q_2 - p_2 \end{aligned}$$

We can attack this with Gaussian elimination. Example:

121

The two lines intersect at point $A = Q + 1 \cdot \vec{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

11.4 Analytic geometry in three dimensions

Cartesian coordinate systems and points in space. For a coordinate system in three dimensions we need a third axis, usually called the z**-axis**. To make it "cartesian" the z-axis must be orthogonal (at right angles) to the other two axes, and the unit of length must be chosen the same on all three.

An example would be to let the x-axis be horizontal pointing to the East, the y-axis horizontal pointing North, and the z-axis pointing vertically upwards.

Once a coordinate system has been chosen, points in space are determined by *three coordinates*: $P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$. The distance between two points $P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ is computed as

$$\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

Vectors and lines. A movement along a straight line for a certain distance is again called a **vector**; it is determined by its three coordinates. The length of the vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is

$$\sqrt{v_1^2 + v_2^2 + v_3^2}$$

Any two points P and Q determine a vector $\overrightarrow{PQ}$ whose coordinates are $\begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$.

The generating expression for a line remains the same

$$X = P + s \cdot \vec{v}$$

but we must remember that P and $\vec{v}$ now have three, rather than two coordinates.

Planes. The parametric representation of a plane has the form

$$X = P + s \cdot \vec{v} + t \cdot \vec{w}$$

where P is a point in space, and $\vec{v}$ and $\vec{w}$ are vectors (neither of which is the null vector). Furthermore, $\vec{w}$ must not point in the same (or in the opposite) direction as $\vec{v}$, or otherwise only a line is generated.

Three points P, Q, and R which are not all on the same line determine a plane:

$$X = P + s \cdot \overrightarrow{PQ} + t \cdot \overrightarrow{PR}$$

122

Intersection tasks. Consider the following tasks:

1. test whether a given point lies on a given line or plane;
2. find the intersection point between two lines;
3. find the intersection point between a line and a plane;
4. find the intersection line between two planes.

All of these can be solved in the same way as described in Item 11.3 on page 68, that is, by expressing the task as a system of linear equations, one for each coordinate.

We do an example for task 4. Consider the following two planes

$$\begin{aligned} X &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \\ Y &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + r \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + q \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

We equate them and transform them into *three* linear equations for the *four* unknowns s, t, r, and q:

123

We solve the system with Gaussian elimination:

124

We find that q can be chosen freely and r computes to $-2+q/2$. Substituting this into the equation of the second plane, we obtain

125

which simplifies to

$$Y = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + q \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

(where we have doubled the length of the vector component to avoid fractions). We could likewise have computed the relationship between s and t and would have found a representation of the line of intersection from the parametric representation of the first plane.

11.5 Lines and planes given by points

Two-point description of a line. Two (separate) points P and Q determine a line. We can easily translate this into the parametric representation: The vector that moves P into Q is denoted by $\overrightarrow{PQ}$ and has the coordinates $\begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$ in 2D and $\begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$ in 3D. The parametric representation for the line through P and Q can therefore be written as

$$X = P + s \cdot \overrightarrow{PQ}$$

Using the laws of vector algebra we can write this as follows:

126

Two-point description of a line

$$X = (1-s) \cdot P + s \cdot Q$$

(Sometimes this is written as $X = s \cdot P + t \cdot Q$ with the side condition $s + t = 1$.)

In this calculation we have applied scalar multiplication to points, something that we said before we would never do. Our excuse is that the operation we are defining here involves *two points and a scalar*; indeed, this operation is called a **convex combination of points**[11], especially when $0 \leq s \leq 1$. A picture is also helpful:

127

We see that the **line segment** between P and Q is characterised by the property that s is between 0 and 1. Convex combination is the basis for defining **Bezier curves**.

Three-point description of a plane. We do exactly the same as for lines but start with three points P, Q and R. A parametric representation of the plane through these three points is given by

$$X = P + s \cdot \overrightarrow{PQ} + t \cdot \overrightarrow{PR}$$

Applying vector algebra just as in Box 126 we get

Three-point description of a plane

$$X = (1-s-t) \cdot P + s \cdot Q + t \cdot R$$

The part of the plane where all three parameters s, t and $1-s-t$ are between 0 and 1 is exactly the triangle with corners P, Q and R.

[11] Another term for this is **weighted average**.

11.6 Algebras of vectors (aka, vector spaces)

We have discussed the operations and laws of vectors in subsection 11.2 above. To repeat: We have a null vector, can add two vectors, and we can multiply vectors with a scalar. We can also solve the equation $\vec{v}+\vec{x}=\vec{0}$ for $\vec{x}$: Simply set $\vec{x}=(-1)\cdot\vec{v}$. (So here scalar multiplication immediately provides us with negation and, more generally, subtraction.) Any structure that carries these two operations and satisfies these laws is called an **algebra of vectors** or a **vector space**.

Although we humans find it difficult to imagine spaces of more than 3 dimensions, we can readily believe that the movements of n-dimensional space form an algebra of vectors: This would work exactly as in the 2- and 3-dimensional case. However, the idea of an algebra of vectors is more general and more abstract than "straight-line movements of a space". Indeed, we can go one step further and realise that the scalars could come from any field $\mathbb{K}$, for example GF(2). One then speaks of an "algebra of vectors over $\mathbb{K}$."

Subalgebras. If $\vec{v}$ is an element of some vector algebra then we can generate a **subalgebra** (or more precisely, a **subalgebra of vectors**) by considering all vectors of the form $s\cdot\vec{v}$. This contains the null vector (because $\vec{0}=0\cdot\vec{v}$), is closed under addition, because

128

and is closed under scalar multiplication because

129

where we are using two of the laws of scalar multiplication. The same would be true if we took two vectors $\vec{v}$ and $\vec{w}$ and formed all expressions of the form $s\cdot\vec{v}+t\cdot\vec{w}$.

Another way of looking at our parametric representations, then, is to say that we pick a point and allow all movements from a *subalgebra* to act on this point. This construction leads to what in general is called an **affine subspace**. In other words, lines and planes are affine subspaces of 2D/3D.

Exercises

1. Compute the distance between $P=\begin{pmatrix}-1\\2\end{pmatrix}$ and $Q=\begin{pmatrix}2\\6\end{pmatrix}$.
2. Compute the point in which the line
$$X=\begin{pmatrix}-1\\1\\-2\end{pmatrix}+s\cdot\begin{pmatrix}2\\-2\\3\end{pmatrix}$$
and the plane
$$Y=\begin{pmatrix}4\\0\\-1\end{pmatrix}+t\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}+u\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}$$
meet.
3. Compute the line of intersection between the two planes
$$X=\begin{pmatrix}3\\4\\0\end{pmatrix}+s\cdot\begin{pmatrix}2\\2\\3\end{pmatrix}+t\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}\qquad Y=\begin{pmatrix}-1\\3\\-2\end{pmatrix}+u\cdot\begin{pmatrix}1\\-1\\-1\end{pmatrix}+r\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Practical advice

In the exam I expect you to be able to

- set up the generating expression for a line (or a plane) when given two (respectively, three) points;

- test whether a given point lies on a given line (or plane);
- find the intersection point between two lines or a line and a plane;
- find the intersection line between two planes;
- know the laws of vector algebra;
- do all of these things also over GF(2).

Notes

- Always write the coordinates of points and vectors in the column form. This will help you to avoid mistakes such as confusing coordinates (particularly in 3D).
- A line has infinitely many generating expressions. Any point P on the line and any vector $\vec{v}$ pointing in the direction of the line will do. This has two consequences:
  - Your solution and my model solution could look different although they both describe the same line.
  - Sometimes your computation will produce a direction vector with fractions as coordinates. You can avoid these by simply stretching the direction vector. For example, the two generating expressions

$$P + s \cdot \begin{pmatrix} 2/3 \\ 1/2 \end{pmatrix} \qquad \text{and} \qquad P + s \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

    describe the same line.
- The same is true for planes in 3D, only more so!
- When computing the intersection point between two lines (or a line and a plane, or two planes) with Gaussian elimination, make sure you are not using the same parameter name twice! (Note how I used r and q in Section 11.4 for the second plane.)