Exercise Sheet 10 - Solutions
Predicate Logic – Equivalences

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1. (a) We first prove the left-to-right implication:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x.(p(x)\lor\exists y.q(x,y))}^{\,1}}{p(z)\lor\exists y.q(z,y)}\,[\forall E] \qquad \dfrac{\dfrac{\dfrac{\overline{p(z)}^{\,2}}{p(z)\lor q(z,z)}\,[\lor I_L]}{\exists y.(p(z)\lor q(z,y))}\,[\exists I]}{p(z)\to\exists y.(p(z)\lor q(z,y))}\;2\;[\to I] \qquad \dfrac{\dfrac{\overline{\exists y.q(z,y)}^{\,3} \quad \dfrac{\dfrac{\overline{q(z,y)}^{\,4}}{p(z)\lor q(z,y)}\,[\lor I_R]}{\exists y.(p(z)\lor q(z,y))}\,[\exists I]}{\exists y.(p(z)\lor q(z,y))}\;4\;[\exists E]}{(\exists y.q(z,y))\to\exists y.(p(z)\lor q(z,y))}\;3\;[\to I]}{\exists y.(p(z)\lor q(z,y))}\,[\lor E]}{\forall x.\exists y.(p(x)\lor q(x,y))}\,[\forall I]}{(\forall x.(p(x)\lor\exists y.q(x,y)))\to(\forall x.\exists y.(p(x)\lor q(x,y)))}\;1\;[\to I]$$

   (b) We now prove the right-to-left implication:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x.\exists y.(p(x)\lor q(x,y))}^{\,1}}{\exists y.(p(x)\lor q(x,y))}\,[\forall E] \qquad \dfrac{\overline{p(x)\lor q(x,y)}^{\,2} \quad \dfrac{\dfrac{\overline{p(x)}^{\,3}}{p(x)\lor\exists y.q(x,y)}\,[\lor I_L]}{p(x)\to(p(x)\lor\exists y.q(x,y))}\;3\;[\to I] \quad \dfrac{\dfrac{\dfrac{\overline{q(x,y)}^{\,4}}{\exists y.q(x,y)}\,[\exists I]}{p(x)\lor\exists y.q(x,y)}\,[\lor I_R]}{q(x,y)\to(p(x)\lor\exists y.q(x,y))}\;4\;[\to I]}{p(x)\lor\exists y.q(x,y)}\,[\lor E]}{p(x)\lor\exists y.q(x,y)}\;2\;[\exists E]}{\forall x.(p(x)\lor\exists y.q(x,y))}\,[\forall I]}{(\forall x.\exists y.(p(x)\lor q(x,y)))\to(\forall x.(p(x)\lor\exists y.q(x,y)))}\;1\;[\to I]$$

2. (a) We first prove the left-to-right implication:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(x)\vdash p(x)}\,[Id]}{p(x)\vdash p(x)\lor q(x,x)}\,[\lor R_1]}{p(x)\vdash\exists y.(p(x)\lor q(x,y))}\,[\exists R] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{q(x,y)\vdash q(x,y)}\,[Id]}{q(x,y)\vdash p(x)\lor q(x,y)}\,[\lor R_2]}{q(x,y)\vdash\exists y.(p(x)\lor q(x,y))}\,[\exists R]}{\exists y.q(x,y)\vdash\exists y.(p(x)\lor q(x,y))}\,[\exists L]}{p(x)\lor\exists y.q(x,y)\vdash\exists y.(p(x)\lor q(x,y))}\,[\lor L]}{\forall x.(p(x)\lor\exists y.q(x,y))\vdash\exists y.(p(x)\lor q(x,y))}\,[\forall L]}{\forall x.(p(x)\lor\exists y.q(x,y))\vdash\forall x.\exists y.(p(x)\lor q(x,y))}\,[\forall R]}{\vdash(\forall x.(p(x)\lor\exists y.q(x,y)))\to\forall x.\exists y.(p(x)\lor q(x,y))}\,[\to R]$$

   (b) We now prove the right-to-left implication:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(x)\vdash p(x)}\,[Id]}{p(x)\vdash p(x)\lor\exists y.q(x,y)}\,[\lor R_1] \qquad \dfrac{\dfrac{\dfrac{}{q(x,y)\vdash q(x,y)}\,[Id]}{q(x,y)\vdash\exists y.q(x,y)}\,[\exists R]}{q(x,y)\vdash p(x)\lor\exists y.q(x,y)}\,[\lor R_2]}{p(x)\lor q(x,y)\vdash p(x)\lor\exists y.q(x,y)}\,[\lor L]}{\exists y.(p(x)\lor q(x,y))\vdash p(x)\lor\exists y.q(x,y)}\,[\exists L]}{\forall x.\exists y.(p(x)\lor q(x,y))\vdash p(x)\lor\exists y.q(x,y)}\,[\forall L]}{\forall x.\exists y.(p(x)\lor q(x,y))\vdash\forall x.(p(x)\lor\exists y.q(x,y))}\,[\forall R]}{\vdash\forall x.\exists y.(p(x)\lor q(x,y))\to(\forall x.(p(x)\lor\exists y.q(x,y)))}\,[\to R]$$

3. Let P be $(\forall x.p(x) \wedge q) \rightarrow ((\forall x.p(x)) \wedge q)$. To prove that P is valid, we have to prove that for all models M, $\vDash_{M,\cdot} P$ is true. Assume a model M, and let us prove that $\vDash_{M,\cdot} P$ is true. M is of the form $\langle D, \langle\rangle, \langle \mathcal{R}_p, \mathcal{R}_q \rangle\rangle$.

   - $\vDash_{M,\cdot} P$
   - iff if $\vDash_{M,\cdot} \forall x.p(x) \wedge q(x)$ then $\vDash_{M,\cdot} \forall x.p(x)$
   - iff if $\vDash_{M,\cdot} \forall x.p(x) \wedge q(x)$ then for all $d \in D$, $\vDash_{M,x\mapsto d} p(x)$
   - iff if $\vDash_{M,\cdot} \forall x.p(x) \wedge q(x)$ then for all $d \in D$, $\langle [\![x]\!]^M_{x\mapsto d} \rangle \in \mathcal{R}_p$
   - iff if $\vDash_{M,\cdot} \forall x.p(x) \wedge q(x)$ then for all $d \in D$, $\langle d \rangle \in \mathcal{R}_p$
   - iff if (for all e, $\vDash_{M,x\mapsto e} p(x) \wedge q(x)$) then for all $d \in D$, $\langle d \rangle \in \mathcal{R}_p$
   - iff if (for all e, $\vDash_{M,x\mapsto e} p(x)$ and $\vDash_{M,x\mapsto e} q(x)$) then for all $d \in D$, $\langle d \rangle \in \mathcal{R}_p$
   - iff if (for all e, $\langle [\![x]\!]^M_{x\mapsto e} \rangle \in \mathcal{R}_p$ and $\langle [\![x]\!]^M_{x\mapsto e} \rangle \in \mathcal{R}_q$) then for all $d \in D$, $\langle d \rangle \in \mathcal{R}_p$
   - iff if (for all e, $\langle e \rangle \in \mathcal{R}_p$ and $\langle e \rangle \in \mathcal{R}_q$) then for all $d \in D$, $\langle d \rangle \in \mathcal{R}_p$
   - iff True
     - Indeed assume (for all e, $\langle e \rangle \in \mathcal{R}_p$ and $\langle e \rangle \in \mathcal{R}_q$) and $d \in D$ and let us prove $\langle d \rangle \in \mathcal{R}_p$
     - Let us instantiate the "for all" assumption with d. We obtain that $\langle d \rangle \in \mathcal{R}_p$ and $\langle d \rangle \in \mathcal{R}_q$. Therefore, our conclusion $\langle d \rangle \in \mathcal{R}_p$ is true.