Mathematical and Logical Foundations of Computer Science

Predicate Logic (Semantics)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**
- Intuitionistic vs. Classical logic
- Type theory

Today

- Semantics of Predicate Logic
- Models
- Variable valuations
- Satisfiability & validity

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- Models
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Further reading:

- Chapter 10 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

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\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
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The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: $\forall$ & $\exists$ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{array}}{Q}\ 1\ [\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: ∀ & ∃ left and right rules

Sequent Calculus rules for quantifiers:

$$\frac{\Gamma \vdash P[x\backslash y]}{\Gamma \vdash \forall x.P} \; [\forall R] \qquad \frac{\Gamma, P[x\backslash t] \vdash Q}{\Gamma, \forall x.P \vdash Q} \; [\forall L]$$

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Conditions:

- for $[\forall R]$: y must not be free in Γ or $\forall x.P$
- for $[\forall L]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists R]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
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Semantics: Assigning meaning/interpretations to formulas

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- a meaningful interpretation for `even` would be
- a meaningful interpretation for `odd` would be
- a meaningful interpretation for `prime` would be

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- a meaningful interpretation for `prime` would be
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1. we have used 0 in our examples as a **constant symbol**, which has no meaning on its own
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If we want to distinguish them, we might use:

1. $\overline{0}$ for the **constant symbol**
2. 0 for the **object of the domain**

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $i < n$
- of predicate symbols p_i of arity j_i, for $i < m$

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a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i
- interpretations $\mathcal{R}_{p_i}$ for function symbols p_i

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $i < n$
- of predicate symbols p_i of arity j_i, for $i < m$

a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i
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Models of predicate logic replace **truth assignments** for propositional logic

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- we might interpret the signature $\langle\langle \texttt{add}\rangle, \langle \texttt{even}\rangle\rangle$
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- we might interpret the signature $\langle\langle \texttt{add}\rangle, \langle \texttt{even}\rangle\rangle$
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For example

- $(x_1 \mapsto d_1), x_2 \mapsto d_2$ maps x_1 to ? and x_2 to ?
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Given a **model** M with domain D and a **variable valuation** v, to assign **meaning** to Predicate Logic formulas, we define two operations:

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Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

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- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x \mapsto d)} P$

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- $\vDash_{M,v} P \rightarrow Q$ iff $\vDash_{M,v} Q$ whenever $\vDash_{M,v} P$
- $\vDash_{M,v} \forall x.P$ iff for every $d \in D$ we have $\vDash_{M,(v,x \mapsto d)} P$
- $\vDash_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\vDash_{M,(v,x \mapsto d)} P$

Semantics of Predicate Logic

For example:

- consider the signature $\langle\langle \mathtt{zero}, \mathtt{succ}, \mathtt{add}\rangle, \langle \mathtt{even}, \mathtt{odd}\rangle\rangle$
- the model M: $\langle \mathbb{N}, \langle 0, +1, +\rangle, \langle\{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}, \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}\rangle\rangle$
- we write $+1$ for the function that given a number increments it by 1
- $+(n, m)$ stands for $n + m$

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What is $\models_{M,\cdot} \mathtt{even}(\mathtt{succ}(\mathtt{zero})) \vee \mathtt{odd}(\mathtt{succ}(\mathtt{zero}))$?

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What is $\models_{M,\cdot} \mathtt{even}(\mathtt{succ}(\mathtt{zero})) \vee \mathtt{odd}(\mathtt{succ}(\mathtt{zero}))$?

- iff $\models_{M,\cdot} \mathtt{even}(\mathtt{succ}(\mathtt{zero}))$ or $\models_{M,\cdot} \mathtt{odd}(\mathtt{succ}(\mathtt{zero}))$

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What is $\models_{M,\cdot} \texttt{even}(\texttt{succ}(\texttt{zero})) \vee \texttt{odd}(\texttt{succ}(\texttt{zero}))$?

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- iff $\langle [\![\mathtt{succ}(\mathtt{zero})]\!]^M_\cdot \rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$ or $\langle [\![\mathtt{succ}(\mathtt{zero})]\!]^M_\cdot \rangle \in \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$
- iff $\langle 1\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$ or $\langle 1\rangle \in \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$
- iff True

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What is $\models_{M,\cdot} \forall x.\mathtt{even}(x)$?

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- iff False, because $1 \notin \{0, 2, 4, \ldots\}$

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Decidability: Validity is not decidable for predicate logic, i.e., there is no algorithm that given a formula P either returns “yes” if P is valid, and otherwise returns “no”, while it is decidable for propositional logic

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Given a deduction system such as Natural deduction, a formula is said to be **provable** if there is a proof of it in that deduction system

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We will not prove them here

Conclusion

What did we cover today?

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- Models
- Variable valuations
- Satisfiability & validity

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Next time?

- Equivalences in Predicate Logic