Mathematical and Logical Foundations of Computer Science

Lecture 10 - Propositional Logic (Wrap-up)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Syntax of propositional logic
- Natural Deduction
- Sequent Calculus
- Classical reasoning
- Semantics
- Equivalences
- Provability/Validity

Syntax & Informal Semantics

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Lower-case letters are atoms: p, q, r, etc.

Upper-case letters are (meta-)variables: P, Q, R, etc.

Two special atoms:

- $\top$ which stands for True
- $\bot$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \bot$

Syntax

Example of propositions:

- "if x is a number then it is even or odd"
  - atom p: "x is a number"
  - atom q: "x is even"
  - atom r: "x is odd"
  - $p \to q \vee r$
- "if x is even then it is not odd"
  - atom p: "x is even"
  - atom q: "x is odd"
  - $p \to \neg q$
- "if $a = b$ and $b = c$ then $a = c$"
  - atom p: "$a = b$"
  - atom q: "$b = c$"
  - atom r: "$a = c$"
  - $(p \wedge q) \to r$
  - or equivalently: $p \to q \to r$

Precedence & Associativity

Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \rightarrow$.

For example:

- $\neg P \vee Q$ means $(\neg P) \vee Q$
- $P \wedge Q \vee R$ means $(P \wedge Q) \vee R$
- $P \wedge Q \rightarrow Q \wedge P$ means $(P \wedge Q) \rightarrow (Q \wedge P)$

Associativity: all operators are right associative

For example:

- $P \vee Q \vee R$ means $P \vee (Q \vee R)$.
- $P \wedge Q \wedge R$ means $P \wedge (Q \wedge R)$.
- $P \rightarrow Q \rightarrow R$ means $P \rightarrow (Q \rightarrow R)$.

However use parentheses around compound formulas for clarity.

Constructive Natural Deduction

Constructive Natural Deduction rules:

$$\frac{\bot}{A}\;[\bot E] \qquad \frac{}{\top}\;[\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ B\end{array}}{A \to B}\;1\;[\to I] \qquad \frac{A \to B \quad A}{B}\;[\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ \bot\end{array}}{\neg A}\;1\;[\neg I] \qquad \frac{\neg A \quad A}{\bot}\;[\neg E]$$

$$\frac{A}{A \vee B}\;[\vee I_L] \qquad \frac{A}{B \vee A}\;[\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\;[\vee E]$$

$$\frac{A \quad B}{A \wedge B}\;[\wedge I] \qquad \frac{A \wedge B}{B}\;[\wedge E_R] \qquad \frac{A \wedge B}{A}\;[\wedge E_L]$$

Constructive Sequent Calculus

Constructive Sequence Calculus rules:

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ [\to L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ [\neg L] \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ [\neg R]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ [\vee L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ [\wedge L] \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{}{A \vdash A}\ [Id] \qquad \frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ [Cut]$$

$$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X] \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W] \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ [C]$$

Constructive Sequent Calculus

In addition we allow using the following **derived rules**:

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \to B, \Gamma_2 \vdash C}\ [\to L] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B}\ [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C}\ [\vee L] \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C}\ [\wedge L]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [W] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [C]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A}\ [Id]$$

All these **derived rules** can be proved/derived using the rules on the previous slide

Classical Reasoning

Classical Natural Deduction includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

There are two kinds of **classical Sequent Calculus**:

1. we can either add LEM and DNE rules
2. or we can use classical sequents instead

Classical sequents are of the form $\Gamma \vdash \Delta$, where Γ and Δ are both lists of formulas

Classical Sequent Calculus (1st version) includes all the Constructive Sequent Calculus rules, plus:

$$\frac{}{\Gamma \vdash A \vee \neg A}\ [LEM] \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ [DNE]$$

Classical Reasoning

Classical Sequent Calculus (2nd version) rules:

$$\frac{\Gamma \vdash A, \Delta_1 \quad \Gamma, B \vdash \Delta_2}{\Gamma, A \to B \vdash \Delta_1, \Delta_2}\ [\to L] \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ [\to R] \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ [\neg L]$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}\ [\vee L] \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ [\vee R] \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ [\neg R]$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ [\wedge L] \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ [\wedge R] \qquad \frac{}{A \vdash A}\ [Id]$$

$$\frac{\Gamma_1 \vdash B, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [Cut] \qquad \frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ [X_L] \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ [X_R]$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ [W_L] \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ [C_L] \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ [W_R] \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ [C_R]$$

We also allow using the usual derived rules such as for example

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta_1, A, \Delta_2}\ [Id] \qquad \frac{\Gamma, A \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, A \to B, \Delta_2}\ [\to R]$$

Semantics

A **valuation** ϕ assigns **T** or **F** with each atom

A valuation is **extended** to all formulas as follows:

- $\phi(\top) = \mathbf{T}$
- $\phi(\bot) = \mathbf{F}$
- $\phi(A \vee B) = \mathbf{T}$ iff either $\phi(A) = \mathbf{T}$ or $\phi(B) = \mathbf{T}$
- $\phi(A \wedge B) = \mathbf{T}$ iff both $\phi(A) = \mathbf{T}$ and $\phi(B) = \mathbf{T}$
- $\phi(A \rightarrow B) = \mathbf{T}$ iff $\phi(B) = \mathbf{T}$ whenever $\phi(A) = \mathbf{T}$
- $\phi(\neg A) = \mathbf{T}$ iff $\phi(A) = \mathbf{F}$

Satisfaction & validity:

- Given a valuation ϕ, we say that ϕ **satisfies** A if $\phi(A) = \mathbf{T}$
- A is **satisfiable** if there exists a valuation ϕ on atomic propositions such that $\phi(A) = \mathbf{T}$
- A is **valid** if $\phi(A) = \mathbf{T}$ for all possible valuations ϕ

Truth Tables

We can use **truth tables** to check whether propositions are valid:

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

P	Q	$P \to Q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

A proposition is (semantically) valid if the last column in its truth table only contains **T**

Validity

All three techniques can be used to prove the validity of propositions:

- a **Natural Deduction** proof (syntactic validity)
- a **Sequent Calculus** proof (syntactic validity)
- a **truth table** with only **T** in the last column (semantical validity)

We saw that:

- a formula A is provable in **Natural Deduction**
- **iff** A is provable in the **Sequent Calculus**
- **iff** A is **semantically valid**

This is true about the classical versions of these deduction systems

Logical equivalences

Let $A \leftrightarrow B$ be defined as $(A \to B) \wedge (B \to A)$

- it means that A and B are logically equivalent
- this is called a "bi-implication"
- read as "A if and only if B"

We will now prove:

- Distributivity of $\wedge$ over $\vee$:
  $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Double negation elimination as an equivalence: $\neg\neg A \leftrightarrow A$

You can also try proving the distributivity of $\vee$ over $\wedge$:
$(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$

Provability/Validity

Provide a constructive Natural Deduction proof of the following equivalence: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

Left-to-right implication:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \wedge (B \vee C)$}
\RightLabel{$[\wedge E_R]$}
\UnaryInfC{$B \vee C$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \wedge (B \vee C)$}
\RightLabel{$[\wedge E_L]$}
\UnaryInfC{A}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{B}
\RightLabel{$[\wedge I]$}
\BinaryInfC{$A \wedge B$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$(A \wedge B) \vee (A \wedge C)$}
\RightLabel{2 $[\rightarrow I]$}
\UnaryInfC{$B \rightarrow (A \wedge B) \vee (A \wedge C)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \wedge (B \vee C)$}
\RightLabel{$[\wedge E_L]$}
\UnaryInfC{A}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{C}
\RightLabel{$[\wedge I]$}
\BinaryInfC{$A \wedge C$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$(A \wedge B) \vee (A \wedge C)$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$C \rightarrow (A \wedge B) \vee (A \wedge C)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$(A \wedge B) \vee (A \wedge C)$}
\RightLabel{1 $[\rightarrow I]$}
\UnaryInfC{$(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$}
\end{prooftree}
$$

Provability/Validity

Right-to-left implication:

$$\dfrac{\dfrac{\dfrac{\overline{(A \land B) \lor (A \land C)}^{\;1} \quad \Pi_1 \quad \Pi_2}{A}\;[\lor E] \qquad \dfrac{\overline{(A \land B) \lor (A \land C)}^{\;1} \quad \Pi_3 \quad \Pi_4}{B \lor C}\;[\lor E]}{A \land (B \lor C)}\;[\land I]}{((A \land B) \lor (A \land C)) \to (A \land (B \lor C))}\;1\;[\to I]$$

where Π_1 is:

$$\dfrac{\dfrac{\overline{A \land B}^{\;2}}{A}\;[\land E_L]}{(A \land B) \to A}\;2\;[\to I]$$

where Π_2 is:

$$\dfrac{\dfrac{\overline{A \land C}^{\;3}}{A}\;[\land E_L]}{(A \land C) \to A}\;3\;[\to I]$$

where Π_3 is:

$$\dfrac{\dfrac{\dfrac{\overline{A \land B}^{\;4}}{B}\;[\land E_R]}{B \lor C}\;[\lor I_L]}{(A \land B) \to (B \lor C)}\;4\;[\to I]$$

where Π_4 is:

$$\dfrac{\dfrac{\dfrac{\overline{A \land C}^{\;5}}{C}\;[\land E_R]}{B \lor C}\;[\lor I_R]}{(A \land C) \to (B \lor C)}\;5\;[\to I]$$

Provability/Validity

Provide a constructive Sequent Calculus proof of the following equivalence: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

Left-to-right implication:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash A}[Id] \quad \dfrac{}{A, B \vdash B}[Id]}{A, B \vdash A \wedge B}[\wedge R]}{A, B \vdash (A \wedge B) \vee (A \wedge C)}[\vee R_1] \quad \dfrac{\dfrac{\dfrac{}{A, C \vdash A}[Id] \quad \dfrac{}{A, C \vdash C}[Id]}{A, C \vdash A \wedge C}[\wedge R]}{A, C \vdash (A \wedge B) \vee (A \wedge C)}[\vee R_2]}{\dfrac{\dfrac{A, B \vee C \vdash (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}[\wedge L]}{\vdash (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))}[\rightarrow R]}[\vee L]
$$

Provability/Validity

Right-to-left implication:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{A, B \vdash A}\,[Id]
      \qquad
      \dfrac{\dfrac{}{A, B \vdash B}\,[Id]}{A, B \vdash B \lor C}\,[\lor R_1]
    }{A, B \vdash A \land (B \lor C)}\,[\land R]
  }{A \land B \vdash A \land (B \lor C)}\,[\land L]
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{A, C \vdash A}\,[Id]
      \qquad
      \dfrac{\dfrac{}{A, C \vdash C}\,[Id]}{A, C \vdash B \lor C}\,[\lor R_2]
    }{A, C \vdash A \land (B \lor C)}\,[\land R]
  }{A \land C \vdash A \land (B \lor C)}\,[\land L]
}{
  \dfrac{(A \land B) \lor (A \land C) \vdash A \land (B \lor C)}{\vdash ((A \land B) \lor (A \land C)) \to (A \land (B \lor C))}\,[\to R]
}\,[\lor L]
$$

Provability/Validity

Prove that $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$ is valid using a truth table

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge C)$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

The 5th and last columns are identical, so the two formulas are equivalent

Provability/Validity

Provide a classical Natural Deduction proof of the following equivalence: $\neg\neg A \leftrightarrow A$

$$
\dfrac{\dfrac{\dfrac{\overline{\neg\neg A}^{\,1}}{A}\,[DNE]}{\neg\neg A \to A}\,1\,[\to I] \qquad \dfrac{\dfrac{\dfrac{\overline{\neg A}^{\,3} \quad \overline{A}^{\,2}}{\bot}\,[\neg E]}{\neg\neg A}\,3\,[\neg I]}{A \to \neg\neg A}\,2\,[\to I]}{\neg\neg A \leftrightarrow A}\,[\wedge I]
$$

Provability/Validity

Provide a classical Sequent Calculus (1st version) proof of the following equivalence: $\neg\neg A \leftrightarrow A$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\neg\neg A \vdash \neg\neg A}\,[Id]
}{\neg\neg A \vdash A}\,[DNE]
}{\vdash \neg\neg A \rightarrow A}\,[\rightarrow R]
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A \vdash A}\,[Id]
}{A, \neg A \vdash \bot}\,[\neg L]
}{A \vdash \neg\neg A}\,[\neg R]
}{\vdash A \rightarrow \neg\neg A}\,[\rightarrow R]
}{\vdash \neg\neg A \leftrightarrow A}\,[\wedge R]
$$

Provability/Validity

Provide a classical Sequent Calculus (2nd version) proof of the following equivalence: $\neg\neg A \leftrightarrow A$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A \vdash A$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$\vdash \neg A, A$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$\neg\neg A \vdash A$}
\RightLabel{$[\rightarrow R]$}
\UnaryInfC{$\vdash \neg\neg A \rightarrow A$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A \vdash A$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$A, \neg A \vdash$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$A \vdash \neg\neg A$}
\RightLabel{$[\rightarrow R]$}
\UnaryInfC{$\vdash A \rightarrow \neg\neg A$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$\vdash \neg\neg A \leftrightarrow A$}
\end{prooftree}
$$

Provability/Validity

Prove that $\neg\neg A \leftrightarrow A$ is valid using a truth table

A	$\neg A$	$\neg\neg A$
T	**F**	**T**
F	**T**	**F**

The 1st and last columns are identical, so the two formulas are equivalent

Conclusion

What did we cover today?

- Syntax of propositional logic
- Natural Deduction
- Sequent Calculus
- Classical reasoning
- Semantics
- Equivalences
- Provability/Validity

Next time?

- Predicate logic (syntax)