1.7.1 Independence

In the final part of this chapter we consider the case where the knowledge of one event does **not** impact the likelihood of another. As an intuitive example suppose you are betting on a football match, there is some associated probability that team you bet on wins. If I was then to tell you my which was my favourite outfit in this years Met Gala, you would be correct in thinking that this has no bearing on the likelihood of you winning your bet. In a more gamified example, if we roll a fair dice twice in a row, knowledge of the first roll tells us absolutely nothing about what the second roll will produce. This notion is called **independence**, and we define it as follows:

Definition 1.7.1. Suppose A and B are two events in a sample space, then we say that A and B are independent if and only if

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

Essentially this definition captures the above discussion. Suppose we have some event A, and we have some knowledge about B. Then the probability of A given that B has occurred is unchanged from as if we did not have any additional information at all. We remark that this is precisely equivalent if we swap the roles of A and B, i.e $\mathbb{P}(B|A) = \mathbb{P}(B)$, you can prove this directly from the definition of conditional probability. Furthermore if A and B are independent, then we also have the following:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Again this follows from the definition of conditional probability, and you should attempt to show why this is true. It is also equivalent to the definition of the independence, and is usually the way you should check if two events are independent. We note this rule also extends to multiple events, where every pair of events are independent.

Lemma 1.7.2. Suppose $A_1, A_2, \ldots A_k$ are a set of k pairwise independent events (i.e every pair of events are independent) then we have the following:

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_k) = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \ldots \times \mathbb{P}(A_k).$$

While this may look like a convenient way to compute probabilities, it must be stressed that this rule only holds when the events are independent. A common error is to assume events are independent and use the above. If you are unsure whether events are independent then you will need to use some of the earlier techniques that we have considered.

Example 1.7.3. Suppose one hundred fair six sided-dice are rolled. What is the probability that all of them show an even number.

Firstly we note that all of the dice rolls are independent. If we roll a die, then the outcomes of the other dices are completely unaffected. Let A_i be the event that i^{th} die shows an even number, for $1 \leq i \leq 100$. We note that for every i we have that $\mathbb{P}(A_i) = 1/2$. Therefore by Lemma 1.7.2 we have:

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_{100}) = \mathbb{P}(A_1)\mathbb{P}(A_2) \times \ldots \times \mathbb{P}(A_{100}) = (1/2)^{100} \approx 7.9 \times 10^{-31}.$$

Example 1.7.4. Suppose we take the following survey. We ask a group of people whether they are a fan of Ru Paul's Drag Race$^{\mathrm{TM}}$. When this survey was conducted in Manchester it was found that 45% of respondents were a fan of the show. While in London 65% of respondents were drag race fans. We take a random sample of five Londoners and ten Mancunians, what is the probability that they are all fans of the show?

As usual we need to define a set of events, we will denote L_i for $i \in \{1 \ldots 5\}$ to be the probability that the i^{th} Londoner is a fan of the show. I.e L_1 denotes whether the first person is a fan of the show, L_2 denotes whether the second person is a fan of the show, all the way up to L_5 denoting whether the fifth person is a fan of the show. Now as 65% of Londoners were a fan of the show; then for each of these 5 people, we know that $\mathbb{P}(L_i) = 0.65$, for each $i \in \{1, 2, \ldots, 5\}$.

Similarly we will denote M_i for $i \in \{1, 2, \ldots, 10\}$ to be the events that the i^{th} Mancunian is a fan of the show. Again M_1 represents the first person, up until M_{10}. Again as 45% of respondents said they were a fan of the show, we have that $\mathbb{P}(M_i) = 0.45$ for all $i \in \{1, 2, \ldots, 10\}$.

Now we are looking for the event that all of the 15 respondents are a fan of the show. Hence we are looking for:

$$\mathbb{P}(L_1 \cap \ldots \cap L_5 \cap M_1 \cap M_2 \cap \ldots \cap M_{10}).$$

Now by using the definition of independent events, and noting that all London events having probability 0.65, along with all Manchester events having probability 0.45, we have that:

$$\mathbb{P}(L_1 \cap \ldots \cap L_5 \cap M_1 \cap M_2 \cap \ldots \cap M_{10}) = \mathbb{P}(L_1)^5 \times \mathbb{P}(M_1)^{10} = 0.65^5 \times 0.45^{10} = 0.0000395.$$