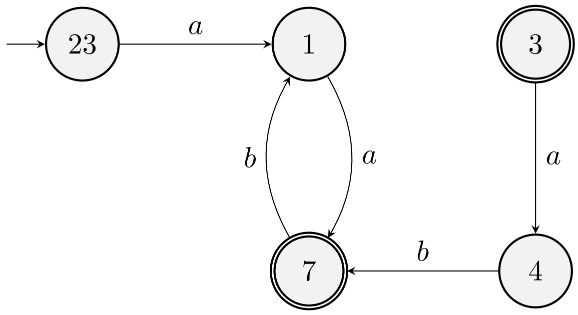
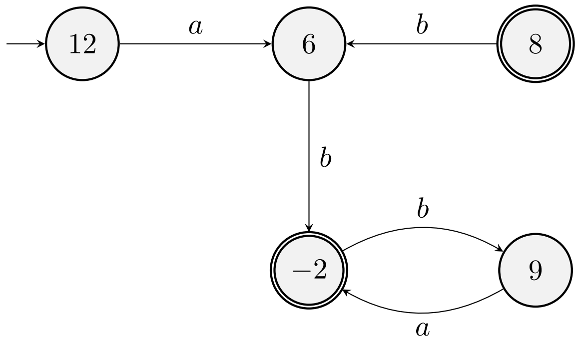
Equivalence, minimal automata, non-regular languages: Problems for Week 2

Exercise 1. *Check which of the following automata over the alphabet $\Sigma = \{a, b\}$ are equivalent. If they are not equivalent, you should give a word that's accepted by one but not by another.*

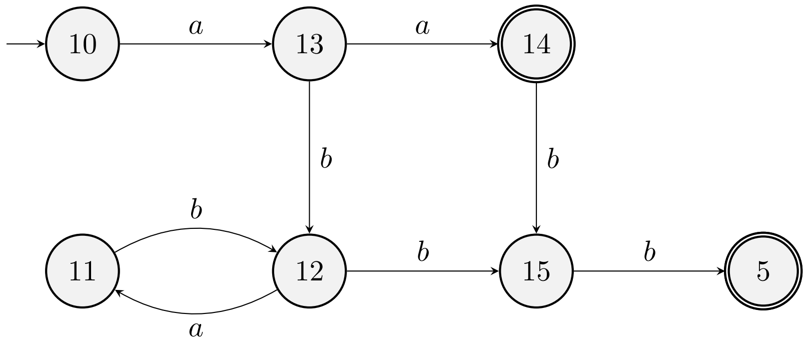
1.



2.



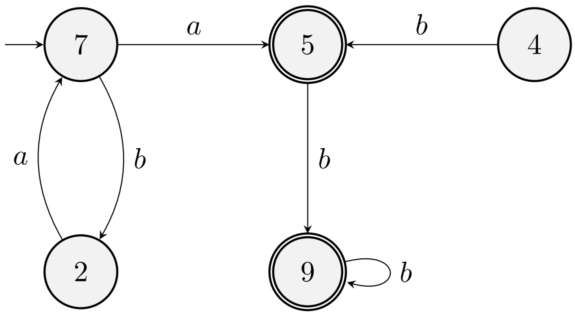
3.



Solution 1. *The second automaton rejects the word aa, which is accepted by the first and third. Hence the second automaton is inequivalent to the others.*

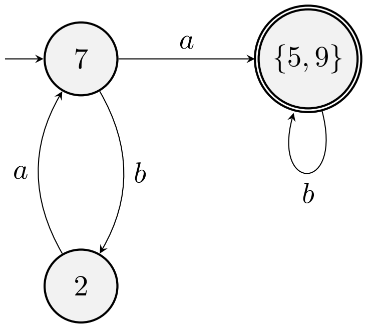
The third automaton accepts $abbb$, which is rejected by the first. Hence these two automata are inequivalent as well.

Exercise 2. *Minimize the following automaton:*



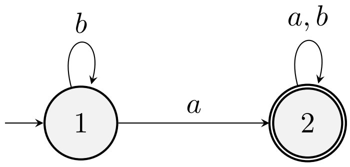
Solution 2.

- *States* 5 *and* 9 *are equivalent: both are accepting (accept* ε*), and accept words* b^n *and reject any word containing an* a*.*
- *State* 4 *is unreachable, and hence can be removed.*
- *States* 2 *and* 7 *are inequivalent to each other, and to* 5 *(and hence* 9*), and thus cannot be unified.*

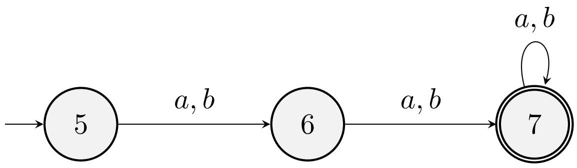


Exercise 3. *The alphabet is* $\{a, b\}$*. Give a DFA for words with at least one* a*, and one for words with at least two characters. By combining these using pairs of states, obtain a DFA for words with at least one* a *and at least two characters.*

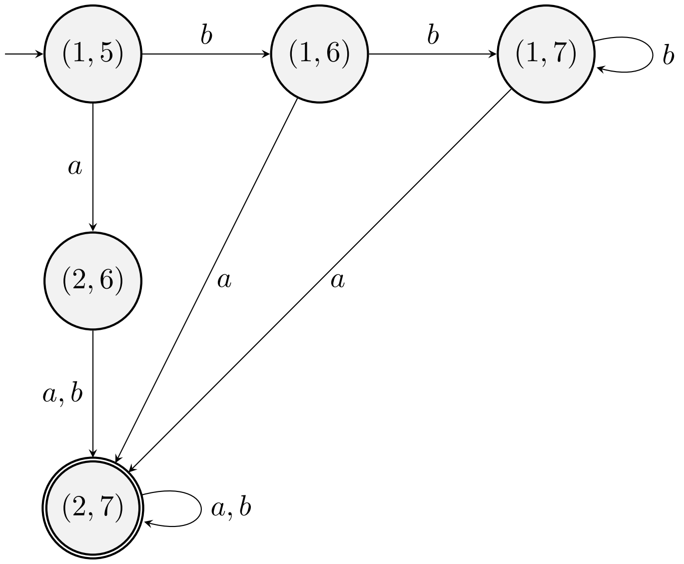
Solution 3. *An automaton for words with at least one* a*:*



An automaton for words with at least two characters:



This yields the following automaton for the intersection:



State $(2,7)$ is accepting because it consists of only accepting states 2 and 7.

Exercise 4. *Consider the following language over the alphabet $\Sigma = \{a, b\}$:*

$$L = \{w | w \text{ contains the same number of } a\text{'s and } b\text{'s}\}$$

Show that L is non-regular.

Solution 4.

- *Suppose that we are given a DFA D that recognizes L.*
- *Consider x_n the state of D reached after reading a^n. State x_n accepts the word b^n, but not the word b^m for $m < n$.*
- *Hence all x_n are inequivalent to x_m for $m < n$.*
- *Hence the DFA D has infinitely many different states, a contradiction to its assumed finiteness.*

Exercise 5. *Are the following languages over $\Sigma = \{a, b\}$ regular ? Why (not)?*

1. $L = \{a^m b^n | m > n\}$
2. $L = \{a^m b^n | m < n\}$
3. $L = \{w | \textit{length}(w) \text{ is a square number}\}$

Solution 5.

1. *L is non-regular. Proof:*
    - *Suppose that we are given a DFA D that recognizes L.*
    - *Consider the state x_n of D reached after reading a^{n+1}. State x_n accepts the word b^n, but not the word b^m for $m > n$.*
    - *Hence all x_n are inequivalent to x_m for $m > n$.*
    - *Hence the DFA D has infinitely many different states, a contradiction to its assumed finiteness.*
2. *L is non-regular. Proof:*
    - *Suppose that we are given a DFA D that recognizes L.*
    - *For $n > 1$, consider the state x_n of D reached after reading a^{n-1}. State x_n accepts the word b^n, but not the word b^m for $m < n$.*
    - *Hence all x_n are inequivalent to x_m for $1 < m < n$.*
    - *Hence the DFA D has infinitely many different states, a contradiction to its assumed finiteness.*

3. *L is non-regular. Proof:*
    - *Suppose that we are given a DFA D that recognizes L.*
    - *Consider the state x_n of D reached after reading $a^{(n^2)}$. State x_n accepts the word $a^{(2n+1)}$, but not the word $a^{(2m+1)}$ for $m < n$. This is because the next square number after n^2 is $(n+1)^2 = n^2 + 2n + 1$.*
    - *Hence all x_n are inequivalent to x_m for $m < n$.*
    - *Hence the DFA D has infinitely many different states, a contradiction to its assumed finiteness.*

Exercise 6. *For any string $w = w_1 w_2 \ldots w_n$, the **reverse of** w, written w^R, is the string w in reverse order, $w_n \ldots w_2 w_1$. For any language L, let $L^R = \{w^R | w \in L\}$. Show that if L is regular, so is L^R.*

Solution 6. *Suppose L is recognized by a regexp E. We construct a new regexp E^R that recognizes L^R, by induction:*

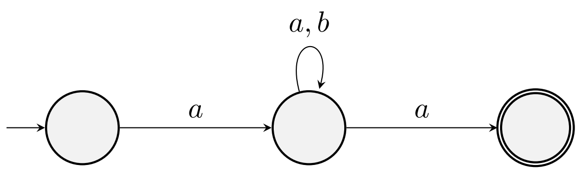
1. *If E is a character x of the alphabet, we set E^R to be x.*
2. *If E is $E_0|E_1$, we set E^R to be $E_0^R|E_1^R$.*
3. *If E is $E_0 E_1$, we set E^R to be $E_1^R E_0^R$. (This is the only thing that changes between E and E^R.)*
4. *If E is $(E_0)^*$, we set E^R to be $(E_0^R)^*$.*

Exercise 7. *Let $\Sigma = \{a, b\}$.*

1. *Let $L_1 = \{a^k u a^k | k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that L_1 is regular.*
2. *Let $L_2 = \{a^k b u a^k | k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that L_2 is not regular.*

Solution 7.

1. *Note that L_1 is equivalently written as $L_1 = \{ava | v \in \Sigma^*\}$. (We need at least one a at the beginning and one at the end, the others are absorbed into v.) An automaton for this is easy to build:*



2. *Suppose that we have a DFA accepting this language. For any $n \in \mathbb{N}$, let x_n be the state reached from the initial state after reading $a^n b$. For $m < n$, if we start at x_m and read in a^m we reach an accepting state, but if we start at x_n and read in a^m we reach a rejecting state, so x_m is not equivalent to x_n. Hence there are infinitely many states, contradicting the assumed finiteness of the DFA.*