Mathematical and Logical Foundations of Computer Science
— Summary of Lecture 1 —

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The pitfalls of computer arithmetic

- We started by observing that computer programs can give surprising answers to simple calculational questions: in mathematics, $100,000^2 = 10^{10}$, but our C program said $1,410,065,408$ instead.

- We learned that the reason for this is, that `unsigned int` variables in C programs are represented as 32-bit memory cells, and that any digits that don't fit into 32 bits are ignored. In other words, the result of a computer calculation using `unsigned int` is only correct up to multiples of 2^{32}.

- This prompted us to ask what are the precise properties of mathematical numbers, and what are the properties of the numbers that can be stored in an `unsigned int`.

The arithmetic properties of natural numbers

$$
\begin{array}{rclrcll}
a + 0 & = & a & a \times 1 & = & a & \text{(neutral elements)} \\
a + b & = & b + a & a \times b & = & b \times a & \text{(commutativity)} \\
(a + b) + c & = & a + (b + c) & (a \times b) \times c & = & a \times (b \times c) & \text{(associativity)}
\end{array}
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a + c = b + c & \Longrightarrow & a = b & \text{(additive cancellation)} \\
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But: Multiplicative cancellation does not hold for computer integers!

Peano's Axioms

Idea: write down axioms for "successor" rather than addition and multiplication

1. 0 is a natural number.
2. If a is a natural number then so is $s(a)$.
3. A number of the form $s(a)$ is always different from 0.
4. If $s(a)$ and $s(b)$ are equal, then a and b are equal.
5. If $P(x)$ is a property of natural numbers that

   (ground case) holds of 0, and
   (inductive step) holds of $s(x)$ whenever it holds of x,

   then P holds of all the natural numbers.

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1,2,4,5 hold for computer integers.

3 does not hold for computer integers