12

Rules of Differentiation

So far we have seen how to differentiate simple functions, such as x^4. We have seen how to combine simple functions together to create more complicated functions. This section gives you some useful rules which allows you to determine the derivative for functions which are the combination of some simple functions.

Lemma 12.0.1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Then

(i) $\dfrac{d}{dx}\left(f(x) + g(x)\right) = f'(x) = g'(x)$.

(ii) (Product rule) Then $\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$.

(iii) (Quotient rule) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

(iv) (Chain rule) $\dfrac{d}{dx}\left(f \circ g\right)(x) = f'\left(g(x)\right) \cdot g'(x)$.

Example 12.1.

- Let us differentiate $h(x) = 3x^2 + 2x + 1$. We can use rule (i) in Lemma 12.0.1.

$$h'(x) = \frac{d}{dx}\left(3x^2 + 2x + 1\right) = \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}\left(2x\right) + \frac{d}{dx}(1) = 6x^{2-1} + 2 + 0 = 6x + 2.$$

- Now let $u(x) = x^2\cos(x)$. Then we can use the product rule to determine $u'(x)$. So let $f(x) = x^2$ and $g(x) = \cos(x)$. Then

$$\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}(x^2) = 2x \quad \text{and} \quad \frac{d}{dx}\left(g(x)\right) = \frac{d}{dx}\left(\cos(x)\right) = -\sin(x).$$

Then using (ii) we get:

$$\begin{aligned}\frac{du}{dx} = \frac{d}{dx}\left(\left(fg\right)(x)\right) = f(x)\frac{d}{dx}\left(g(x)\right) + g(x)\frac{d}{dx}\left(f(x)\right) &= x^2 \cdot (-\sin(x)) + \cos(x) \cdot (2x) \\ &= -2x^2\sin(x) + 2x\cos(x).\end{aligned}$$

- Next, let $v(x) = \dfrac{2x+1}{x^2+2x+1}$. We can use the quotient rule to determine $v'(x)$. So let $f(x) = 2x+1$ and $g(x) = x^2+2x+1$. Then

$$\frac{d}{dx}\left(f(x)\right) = f'(x) = \frac{d}{dx}\left(2x+1\right) = 2 \quad \text{and} \quad \begin{aligned}\frac{d}{dx}\left(g(x)\right) = g'(x) &= \frac{d}{dx}\left(x^2+2x+1\right) \\ &= 2x+2.\end{aligned}$$

Then, by (iii) we get:

$$\begin{aligned}\frac{d}{dx}\left(\left(\frac{f}{g}\right)(x)\right) = \frac{g(x)\frac{d}{dx}\left(f(x)\right) - f(x)\dfrac{d}{dx}\left(g(x)\right)}{\left(g(x)\right)^2} &= \frac{(x^2+2x+1)\cdot(2) - (2x+1)\cdot(2x+2)}{\left(x^2+2x+1\right)^2} \\ &= \frac{(2x^2+4x+2)-(4x^2+6x+2)}{\left((x+1)^2\right)^2} \\ &= \frac{-2x\left(x+1\right)}{(x+1)^4} \\ &= \frac{-2x}{(x+1)^3}.\end{aligned}$$

- Finally, let us compute the derivative of e^{3x}. Let $f(x) = e^x$ and $g(x) = 3x$. Then we have $\left(f \circ g\right)(x) = f\left(g(x)\right) = e^{3x}$. Therefore we can apply the chain rule. First let us calculate $f'(x)$ and $g'(x)$.

$$\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left(e^x\right) = e^x \quad \text{and} \quad \frac{d}{dx}(g(x)) = \frac{d}{dx}(3x) = 3.$$

Therefore, applying (iv) we get

$$\frac{d}{dx}\left(e^{3x}\right) = \frac{d}{dx}\left(\left(f \circ g\right)(x)\right) = f'\left(g(x)\right) \cdot g'(x) = f'\left(3x\right)\cdot 3 = 3e^{3x}.$$

A quick way to remember the chain rule is the following. Suppose we want to calculate the derivative of $y = f(g(x))$. Let $u = g(x)$ and then $y = f(u)$. Also,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

Looking at the derivatives as 'fractions', then it appears that the du will cancel out.

We can easy compute the derivative of the inverse of an invertible function.

Lemma 12.0.2 (Derivative of the inverse)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be an invertible function. Then,

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Example 12.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x+6$. Then, $f'(x) = \dfrac{d}{dx}\left(3x+6\right) = 3$

and so $f^{-1}(x) = \frac{1}{3}x - 2$. Therefore,

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{d}{dx}\left(\frac{1}{3}x - 2\right) = \frac{1}{3}.$$

On the other hand, note $f'(x) = 3$. Therefore, $f'(f^{-1}(x)) = 3$ and so

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{3} = \frac{1}{f'(f^{-1}(x))}.$$

Example 12.3. Let $f : \mathbb{R} \to (0, +\infty)$ be defined by $f(x) = e^x$. Then $f^{-1} : (0, +\infty) \to \mathbb{R}$ is given by $f^{-1}(x) = \ln(x)$.

Also, $f'(x) = e^x$. Therefore,

$$f'\left(f^{-1}(x)\right) = e^{\ln(x)} = x.$$

From this we can see,

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{x}.$$