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Language of Sets

What is a set?

Informally speaking, a *set* is a (finite or infinite) collection of objects, called elements of the set. We use brackets { and } to denote the beginning and the end of the set.

Example 1.1. The set of members of the Beatles is given by

$$\{\text{John Lennon}, \text{Paul McCartney}, \text{Ringo Starr}, \text{George Harrison}\}.$$

In mathematics we work with objects of various kinds and correspondingly we may consider sets of such objects.

Example 1.2. The set of 'positive integers' less than or equal to 5 is given by $\{1, 2, 3, 4, 5\}$.

We mainly use latin capital letters, such as $A, B, C, \ldots$, to denote sets. We write $x \in A$ to say that "x is an element of A"; correspondingly, we write $x \notin A$ to say that "x is not an element of A".

Example 1.3. The set of 'positive integers' less than or equal to 5 is given by $A = \{1, 2, 3, 4, 5\}$. Here $4 \in A$, but $6 \notin A$.

Typical examples of sets are given by:

- $\emptyset$ which denotes the *empty set*, that is, the set that has no elements;
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ which denotes the set of *natural numbers* (or counting numbers); i.e. the positive integers $1, 2, 3, \ldots$;
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ which denotes the set of *integers*, including 0, the natural numbers $\mathbb{N}$, as well as their negatives $-\mathbb{N}$;
- $\mathbb{Q}$ which denotes the set of *rational numbers*, i.e. those numbers which can be written as a fraction p/q where $p, q \in \mathbb{Z}$ and $q \neq 0$;

- $\mathbb{R}$ which denotes the set of *real numbers*, which include the rational numbers as well as the *irrational numbers*, i.e. those that are not rational such as $\sqrt{2}$ or π.

Another way of defining sets is through the *set-builder notation*, i.e. by specifying a property that the elements of the set must satisfy; for example,

$$P = \{x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z}\}$$

says that P is the set of all integers x that are a multiple of 2, that is, the set of *even* integers.

Definition 1.4. Let A and B be two sets. We say that:

- A is a *subset* of B, written $A \subseteq B$, if every element of A is also an element of B;
- A is *equal* to B, written $A = B$, if A and B have the same elements, that is, every element of A is also an element of B, and every element of B is also an element of A.

Note that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Note also that, according to the above definition of equality,

$$\{1, 2, 3\} = \{3, 1, 2\};$$

in other words, the order in which we list the elements of a set does not matter, and different re-orderings of the same elements define the same set. Similarly, we can see that

$$\{1, 2, 3, 3\} = \{1, 2, 3\}.$$

That is, we do not count the same element twice in a set.

Definition 1.5. Let A and B be two sets. Then:

- The *union* of A and B, denoted by $A \cup B$, is the set defined by

  $$x \in A \cup B \text{ if and only if } x \in A \text{ or } x \in B;$$

  in other words, if x is an element of the union $A \cup B$, then either x is an element of A and/or an element of B.

- The *intersection* of A and B, denoted by $A \cap B$, is the set of common elements of A and B, that is,

  $$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B.$$

- The *difference* of A and B, written $A \setminus B$, is the set of the elements of A that are not elements of B, that is,

  $$x \in A \setminus B \text{ if and only if } x \in A \text{ and } x \notin B.$$

Example 1.6. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Then:

1. $A \cup B = \{1, 2, 3, 4, 5\}$;
2. $A \cap B = \{3\}$;
3. $A \setminus B = \{1, 2\}$.

For (i) recall that $A \cup B$ is the collection of all elements that are in A or B. Hence we can write $A \cup B = \{1, 2, 3, 3, 4, 5\}$. But remember we do not write repeats of an element in a set. Therefore, as 3 is repeated, we may remove one of these and write $A \cup B = \{1, 2, 3, 4, 5\}$.

For (ii) recall that $A \cap B$ is the collection of elements that are in *both* A and B. The only such element in both A and B is 3. Therefore $A \cap B = \{3\}$.

For (ii) recall that $A \setminus B$ is the collection of element of A which are not elements of B. The only element of A which is also an element of B, by (ii), is 3. Therefore $A \setminus B = \{1, 2\}$.