8

Images and Pre-Images of Sets

For a function $f : A \to B$, we have already defined the image $f(A)$. More generally, we can give a meaning to the expression $f(S)$ for all subsets S of the domain A.

Definition 8.1. Let $f : A \to B$ be a function.

(i) For all $S \subseteq A$, the *image of S via f* is the subset $f(S)$ of B whose elements are the images via f of the elements of S. In other words,

$$f(S) = \{f(x) : x \in S\}.$$

(ii) For all $T \subseteq B$, the *preimage of T via f* is the subset $f^{-1}(T)$ of A whose elements are the preimages via f of the elements of T. In other words,

$$f^{-1}(T) = \bigcup_{y \in T} f^{-1}(y) = \{x \in A : f(x) \in T\}.$$

The notation $f^{-1}(T)$ for the preimage of a subset T of the codomain B of $f : A \to B$ may be somewhat misleading, since we previously used the symbol f^{-1} to denote the inverse of f. Please note that the preimage $f^{-1}(T)$ of a set is defined irrespective of whether the function f is invertible.

Essentially what these two definitions state are the following:

- The *image* of a set, S, is the collection of outputs where the inputs are the elements in S.
- The *preimage* of a set, T, is the collection of inputs where the outputs are the elements in T.

Example 8.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Define $A = \{4\}$ and $B = \{1, -1, 2\}$. Then:

- $f(A) = \{f(4)\} = \{4^2\} = \{16\}$.
- $f(B) = f(\{1, -1, 2\}) = \{f(1), f(-1), f(2)\} = \{1^2, (-1)^2, 2^2\} = \{1, 1, 4\} = \{1, 4\}$.

Let us now compute $f^{-1}(A)$ and $f^{-1}(B)$. We will first compute $f^{-1}(A)$. Following the definition,

$$f^{-1}(A) = \left\{f^{-1}(4)\right\} = \{x \in \mathbb{R} : f(x) = 4\} = \{-2, 2\}.$$

Arguing similarly,

$$f^{-1}(B) = f^{-1}(1) \cup f^{-1}(-1) \cup f^{-1}(2).$$

So to calculate $f^{-1}(B)$ we need to compute $f^{-1}(1)$, $f^{-1}(-1)$ and $f^{-1}(2)$. That is, as $f : \mathbb{R} \to \mathbb{R}$,

- $f^{-1}(1) = \{x \in \mathbb{R} : f(x) = 1\} = \{1, -1\}$.
- $f^{-1}(-1) = \{x \in \mathbb{R} : f(x) = -1\} = \emptyset$.
- $f^{-1}(2) = \{x \in \mathbb{R} : f(x) = 2\} = \left\{-\sqrt{2}, \sqrt{2}\right\}$.

Therefore,

$$f^{-1}(B) = f^{-1}(1) \cup f^{-1}(-1) \cup f^{-1}(2) = \{1, -1\} \cup \emptyset \cup \left\{-\sqrt{2}, \sqrt{2}\right\} = \left\{-\sqrt{2}, -1, 1, \sqrt{2}\right\}.$$