Exercise Sheet 11 Logic

Q1
```
variables P Q R x: Prop
lemma l1 : (R → ¬P) → (Q → R) → (P → ¬Q) :=
begin
  intros hrp hqr hp hq,
  apply hrp, apply hqr,
  assumption, assumption
end
```

Q2

```
variable D : Type*
variable p : D → Prop
variable q : D → D → Prop
lemma l2 :  (∀ x, ∃ y, (p(x) ∨ q x y)) → ∀ x, p(x) ∨∃ y, q x y :=
begin
  intros h x,
  specialize h x,
  destruct h,
  intros y q,
  destruct q,
  intros hp, left,
  assumption,
  intros hq, right,
  existsi y,
  assumption
end
```

Q3

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{(x+z)+1=y+1,\,x\le y,\,x+z=y\vdash (x+z)+1=y+1}\ [Id] \qquad \overline{x\le y,\,x+z=y\vdash x+z=y}\ [Id]}{x+z=y \Rightarrow (x+z)+1=y+1,\,x\le y,\,x+z=y\vdash (x+z)+1=y+1}}{\forall w\,(x+z=y \Rightarrow (x+z)+w=y+w),\,x\le y,\,x+z=y\vdash (x+z)+1=y+1}\ [\forall L]}{\forall y.\forall w.(x+z=y \Rightarrow (x+z)+w=y+w),\,x\le y,\,x+z=y\vdash (x+z)+1=y+1}\ [\forall L]}{}\ [\forall L]}{}$$

The page continues with a large handwritten sequent-calculus proof tree. Its principal inference steps, read from the lower part upward, are:

$$\cfrac{P_1,P_2,P_3,\,x\le y \vdash x<y+1}{\cfrac{P_1,P_2,P_3 \vdash x\le y \Rightarrow x<y+1}{\cfrac{P_1,P_2,P_3 \vdash \forall y.(x\le y \Rightarrow x<y+1)}{P_1,P_2,P_3 \vdash \forall x.\forall y.(x\le y \Rightarrow x<y+1)}\ [\forall R]}\ [\forall R]}\ [{\Rightarrow}R]$$

with the branches using rules labelled $[Id]$, $[\exists R]$, $[\exists L]$, $[W_L]$, $[\wedge_L]$, $[\vee L]$, and $[\forall L]$ over sequents such as

$$x+z=y,\ x<y+1,\ P_3,\ x\le y\vdash x<y+1$$
$$P_3,\ x\le y,\ x+z=y\vdash (x+z)+1=y+1$$
$$P_3,\ x\le y,\ x+z=y\vdash \exists z.(x+z)+1=y+1$$
$$x+z=y,\ \exists z.(x+z)+1=y+1 \Rightarrow x<y+1,\ P_3,\ x\le y\vdash x<y+1$$
$$x+z=y,\ x<y+1 \Leftrightarrow \exists z.(x+z)+1=y+1,\ P_3,\ x\le y\vdash x<y+1$$
$$x+z=y,\ \forall y.(x<y\Leftrightarrow \exists z.(x+z)+1=y),\ P_3,\ x\le y\vdash x<y+1$$
$$\overline{P_2,P_3,\,x\le y\vdash x\le y}\ [Id]$$
$$x+z=y,\ P_2,P_3,\ x\le y\vdash x<y+1$$
$$\exists z.x+z=y,\ P_2,P_3,\ x\le y\vdash x<y+1$$
$$x\le y \Rightarrow \exists z.x+z=y,\ P_2,P_3,\ x\le y\vdash x<y+1$$
$$x\le y \Rightarrow \exists z.x+z=y,\ \exists z.x+z=y\Rightarrow x\le y,\ P_2,P_3,\ x\le y\vdash x<y+1$$
$$x\le y \Leftrightarrow \exists z.x+z=y,\ P_2,P_3,\ x\le y\vdash x<y+1$$
$$\forall y.(x\le y \Leftrightarrow \exists z.x+z=y),\ P_2,P_3,\ x\le y\vdash x<y+1$$

Q4.