Robotics –
Planning and Motion

Kinematics

COMP52815

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机器人一
规划与运动

运动学

COMP52815

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Lecture 4: Learning Objectives

The aim of this lecture is to build a model which will lead to the kinematics.

- Objectives:
  1. Spatial Description
  2. Transformation
     - Rotation
     - Translation

See also:

- Robot Modeling and Control, Spong et al, C1
- Robotics: Modelling, Planning and Control, Siciliano et al, C1

Durham University

第 4 讲：学习目标

本次讲座的目的是构建一个模型，该模型将导致运动学。

- 目标：

  1. 空间描述 2.转型

     - 旋转
     - 译本

另请参阅：

- 机器人建模与控制，Spong 等人，C1
- 机器人技术：建模、规划和控制，Siciliano 等人，C1

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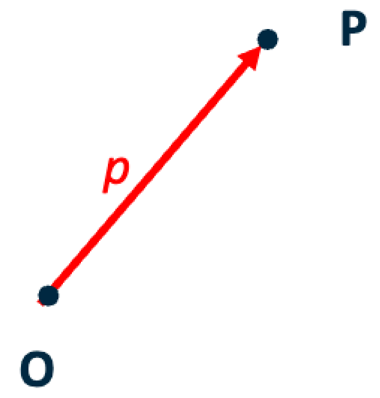
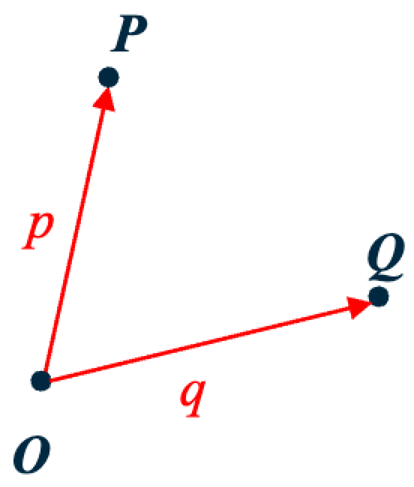
Spatial Description

- **Position of a Point:**

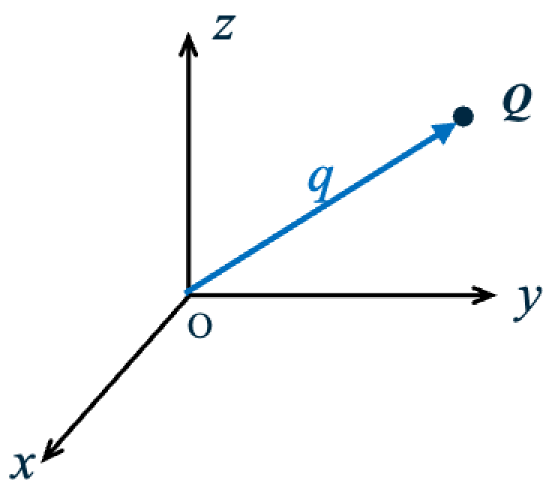
With respect to a fixed origin **O**, the position of a point P is described by the vector **OP** (*p*).





$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

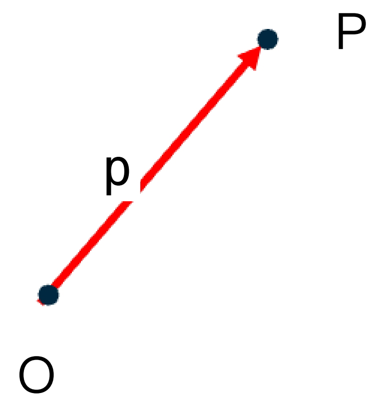
Cartesian Frame

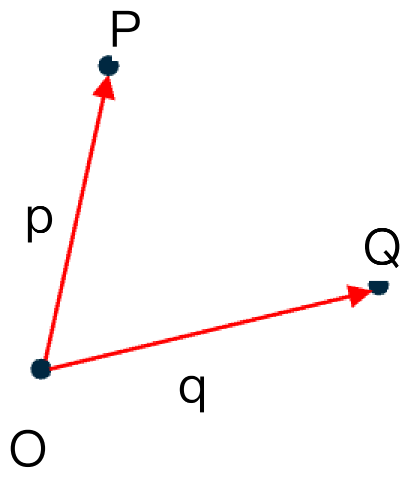


空间描述

- **点的位置：**

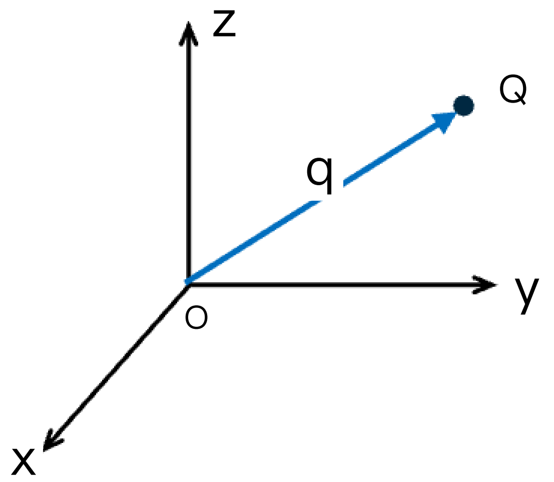
相对于固定原点 O，点 P 的位置由矢量 OP （p） 描述。





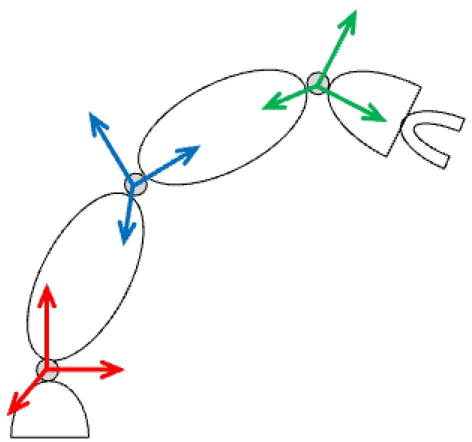
$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

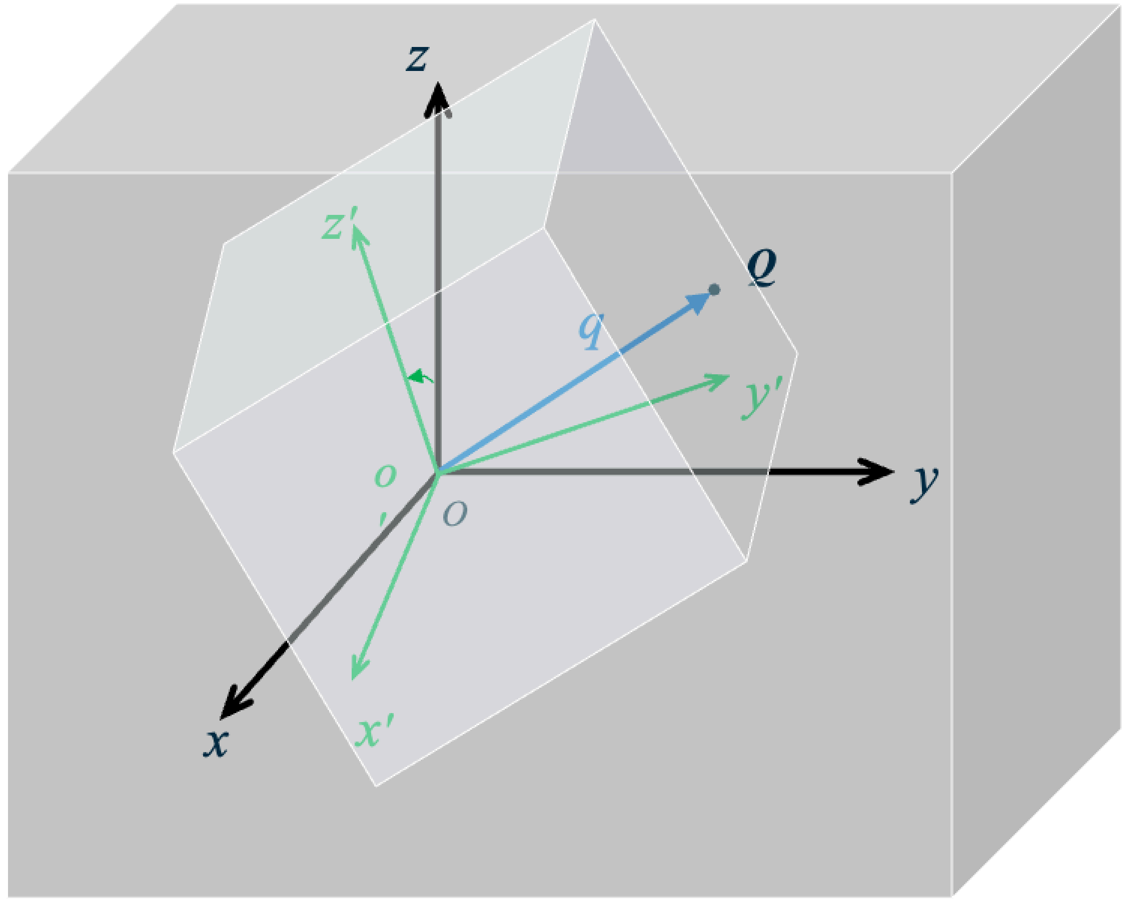
笛卡尔框架



Spatial Description

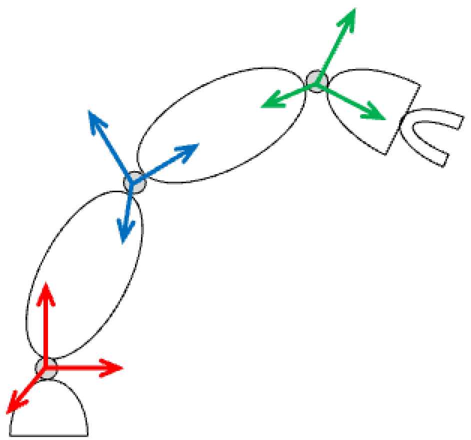
- **Coordinate Frames:**
  - Rotation
  - Translation

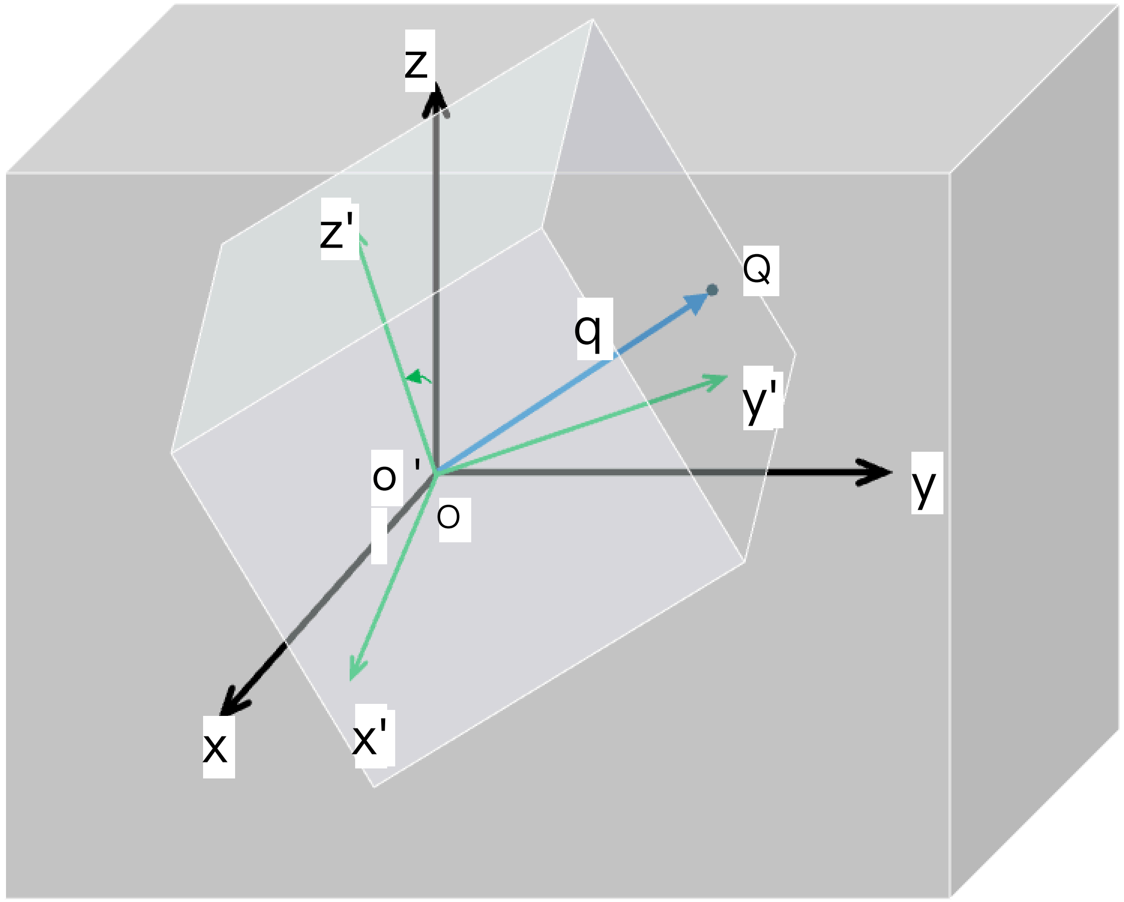




空间描述

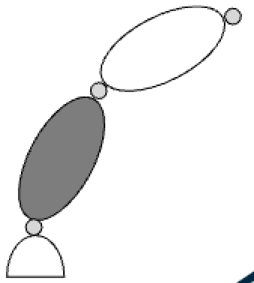
- 坐标框架：
  一旋转
  一译本

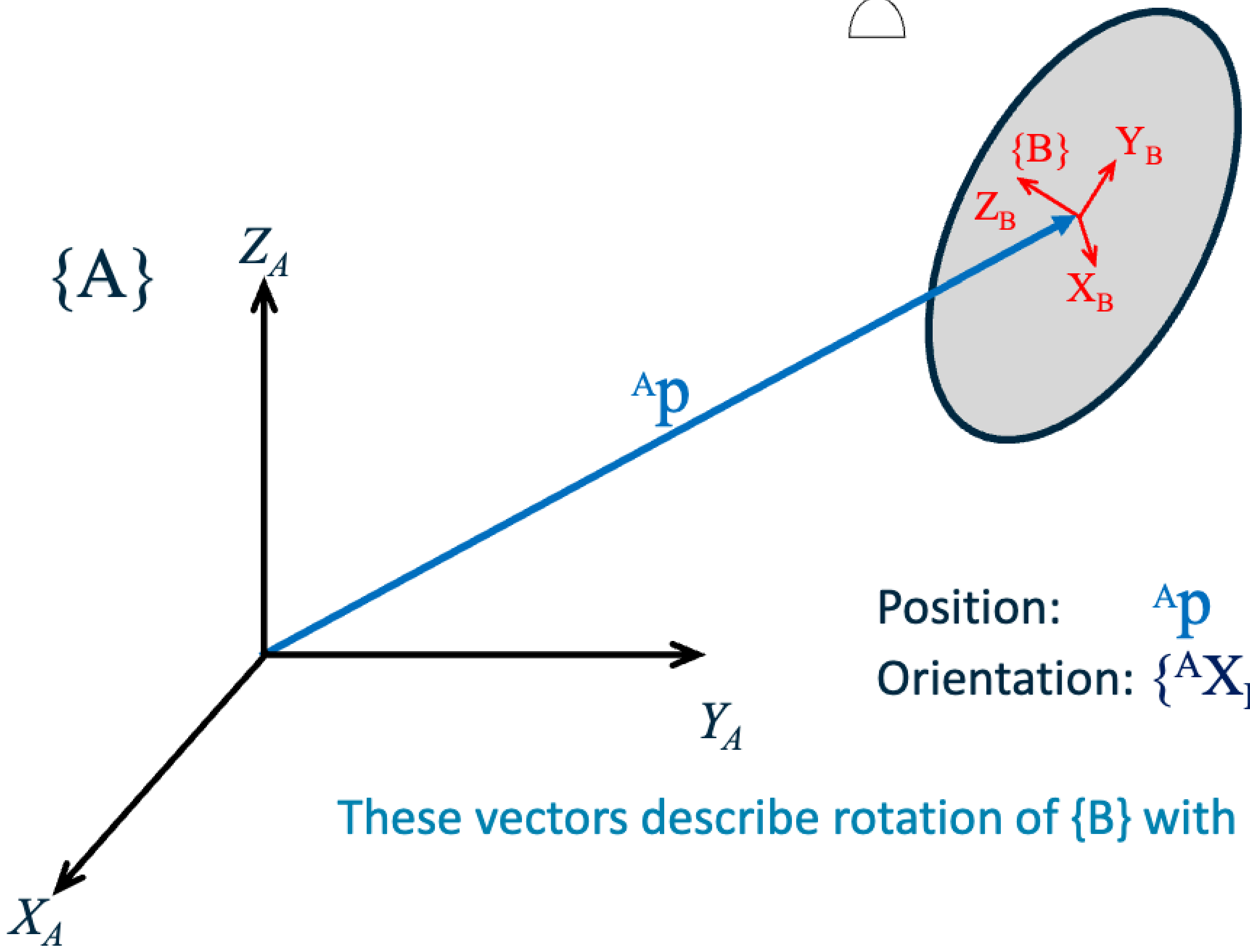




Spatial Description

- **Rigid body configuration:**





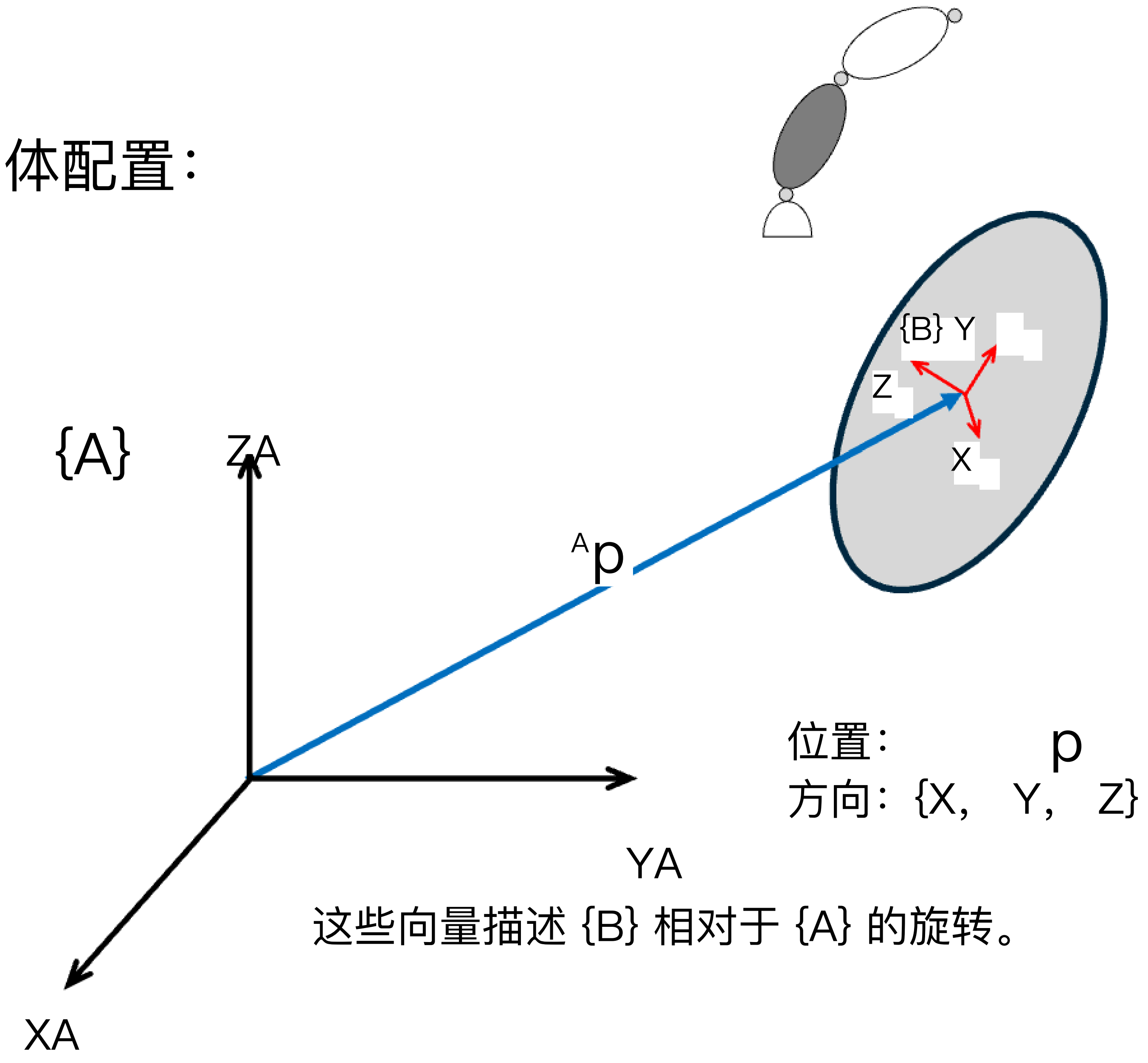
Position: $^A\mathbf{p}$

Orientation: $\{^A X_B\, ,\, ^A Y_B\, ,\, ^A Z_B\}$

These vectors describe rotation of {B} with respect to {A}.

空间描述

- 刚体配置：



位置： p

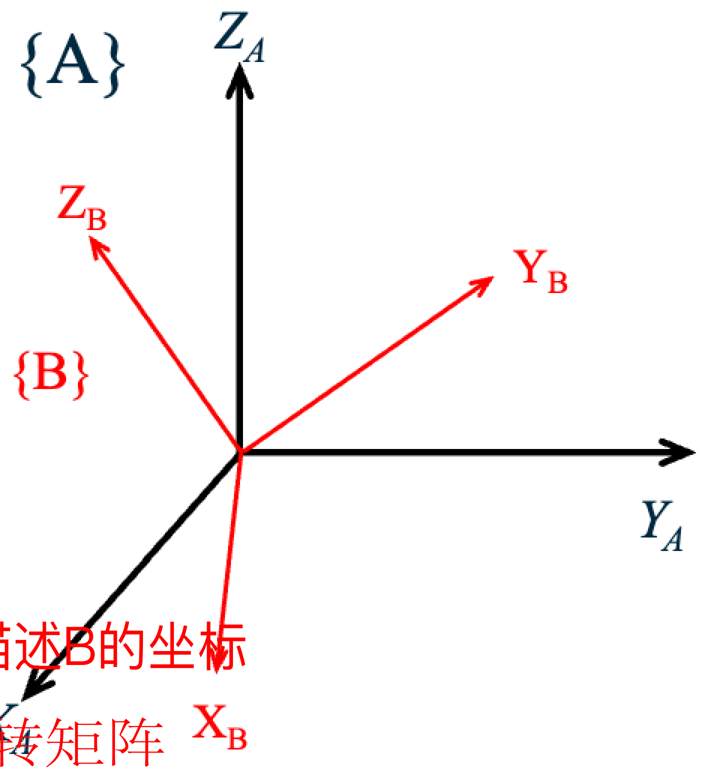
方向：{X, Y, Z}

这些向量描述 {B} 相对于 {A} 的旋转。

Transformation

- **Rotation Matrix:**

$$ {}^A_B R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$



- State description: ${}^A\hat{X}_B = {}^A_B R \; {}^B\hat{X}_B$ 在B中描述B的坐标

在A中描述B的坐标　　用于描述B->A的旋转矩阵

$$ {}^A\hat{X}_B = {}^A_B R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^A\hat{Y}_B = {}^A_B R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad {}^A\hat{Z}_B = {}^A_B R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ \boxed{ {}^A_B R = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} } $$

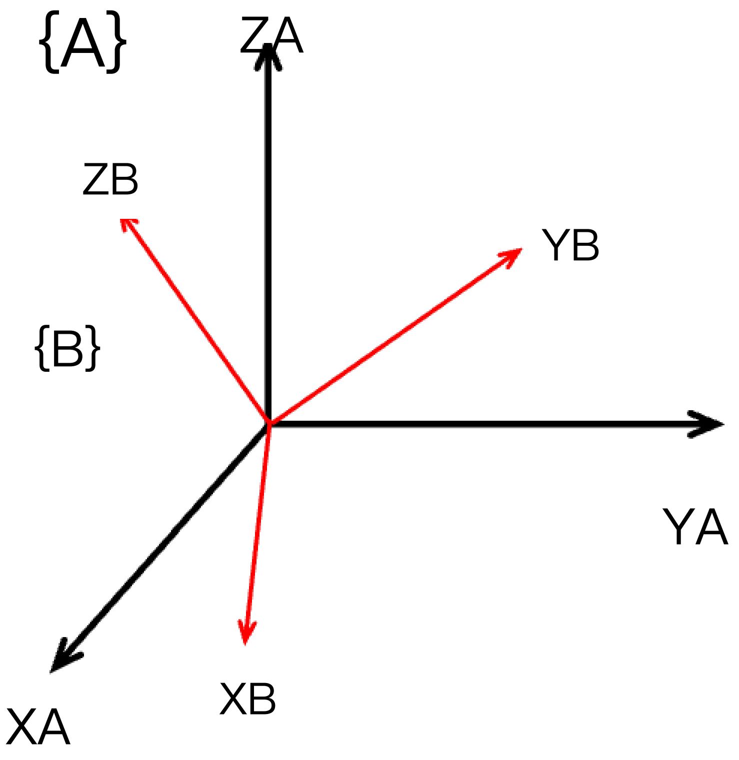
坐标系 B 的 x–轴、y–轴、z–轴，在坐标系 A 中的表

转型

- 旋转矩阵：

$$ {}^{A}_{B}R = \begin{matrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{matrix} $$

- 状态描述： ${}^{A}XB = BR \quad XB$



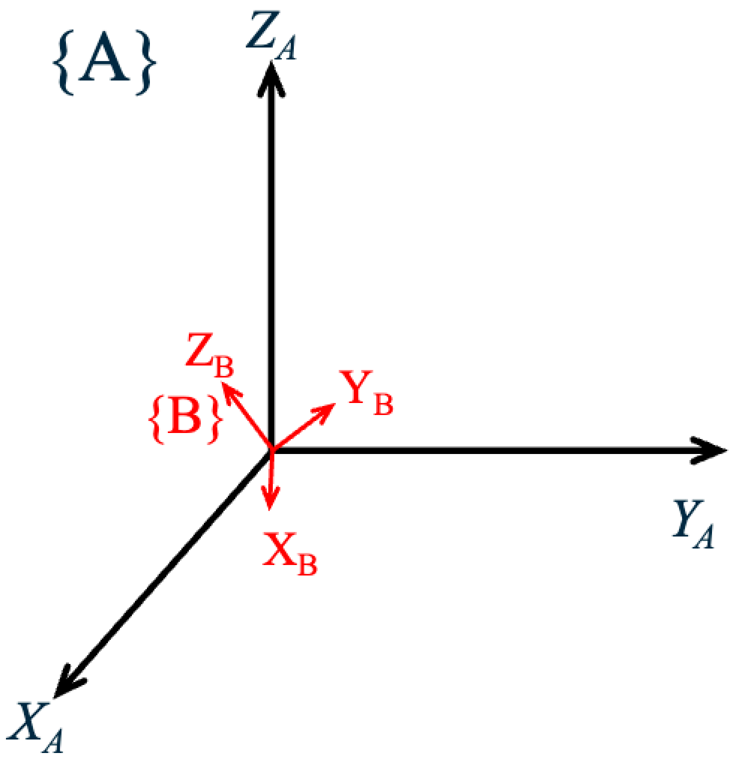
$$ {}^{A}\hat{X}_{B} = R \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \quad {}^{A}\hat{Y}_{B} = {}_{B}R \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \quad {}^{A}\hat{Z}_{B} = R \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} $$

$$ \boxed{{}^{A}_{B}R = \hat{X} \quad \hat{Y} \quad \hat{Z}} $$

Transformation

- **Rotation Matrix:**

$$ {}^{A}_{B}R = \left[{}^{A}\hat{X}_{B} \quad {}^{A}\hat{Y}_{B} \quad {}^{A}\hat{Z}_{B} \right] $$



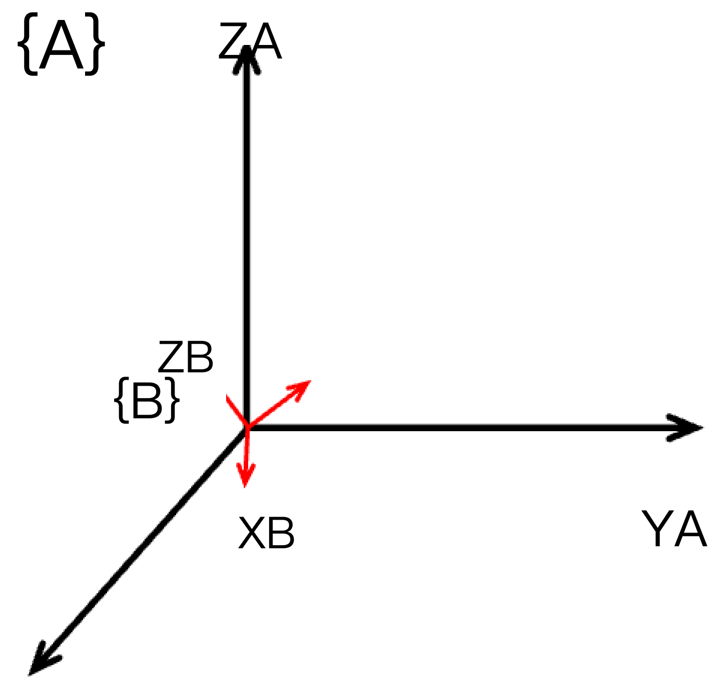
- Dot product:

$$ {}^{A}\hat{X}_{B} = \begin{bmatrix} \hat{X}_B.\ \hat{X}_A \\ \hat{X}_B.\ \hat{Y}_A \\ \hat{X}_B.\ \hat{Z}_A \end{bmatrix} \quad {}^{A}\hat{Y}_{B} = \begin{bmatrix} \hat{Y}_B.\ \hat{X}_A \\ \hat{Y}_B.\ \hat{Y}_A \\ \hat{Y}_B.\ \hat{Z}_A \end{bmatrix} \quad {}^{A}\hat{Z}_{B} = \begin{bmatrix} \hat{Z}_B.\ \hat{X}_A \\ \hat{Z}_B.\ \hat{Y}_A \\ Z_B.\ \hat{Z}_A \end{bmatrix} $$

$$ {}^{A}_{B}R = \begin{bmatrix} \boxed{\hat{X}_B.\hat{X}_A \quad \hat{Y}_B.\hat{X}_A \quad \hat{Z}_B.\hat{X}_A} \\ \hat{X}_B.\hat{Y}_A \quad \hat{Y}_B.\hat{Y}_A \quad \hat{Z}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A \quad \hat{Y}_B.\hat{Z}_A \quad \hat{Z}_B.\hat{Z}_A \end{bmatrix} \Rightarrow {}^{B}X_{A}{}^{T} $$

转置矩阵，想当于在B中描述A的坐标

转型



- 旋转矩阵：

$$ {}^{A}_{B}R = \quad \hat{X} \quad \hat{Y} \quad \hat{Z} $$

- 点积：X

^{A}XB = [XB. XA / XB. YA / XB.ϖZA] YB = [YB 的。/ YB 的。/ YB 的。] XA ^{A}ZB YA= ϖZA [ϖB. / ZB ϖ / 的。]

YA

ZB.ϖZ

A

$$ {}^{A}_{B}R = $$

XB.XA YB XA的。ZB ϖ XA的。

XB.YA YB YA的。ZB ϖ YA的。

XB.ϖZA YB.ϖZA ϖZB.ϖZA

${}^{B}X^{T}$

Transformation

- **Rotation Matrix:**

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_A{}^{T} \\ {}^{B}\hat{Y}_A{}^{T} \\ {}^{B}\hat{Z}_A{}^{T} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_A & {}^{B}\hat{Y}_A & {}^{B}\hat{Z}_A \end{bmatrix} = {}^{B}_{A}R^{T} $$

$$ \boxed{{}^{A}_{B}R = {}^{B}_{A}R^{T}} $$

- **Inverse of Rotation Matrix:**

$$ {}^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

- **Orthonormal Matrix**

$$ \boxed{{}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}} $$

An orthonormal matrix is a square matrix which columns & rows are orthogonal unit vectors

转型

- 旋转矩阵：

$${}^A_B R = [\ XB \quad YB \quad ZB\ ] = \begin{bmatrix} {}^B XA^T \\ {}^B YA^T \\ {}^B ZA^T \end{bmatrix} = {}^B[\ XA \quad YA \quad ZA\ ] = AR$$

$$\boxed{{}^A_B R = R}$$

- Inverse of Rotation Matrix （旋转矩阵的倒数）：

$${}^A_B R = R = R$$

- 正交矩阵

$$\boxed{{}^A_B R = R}$$

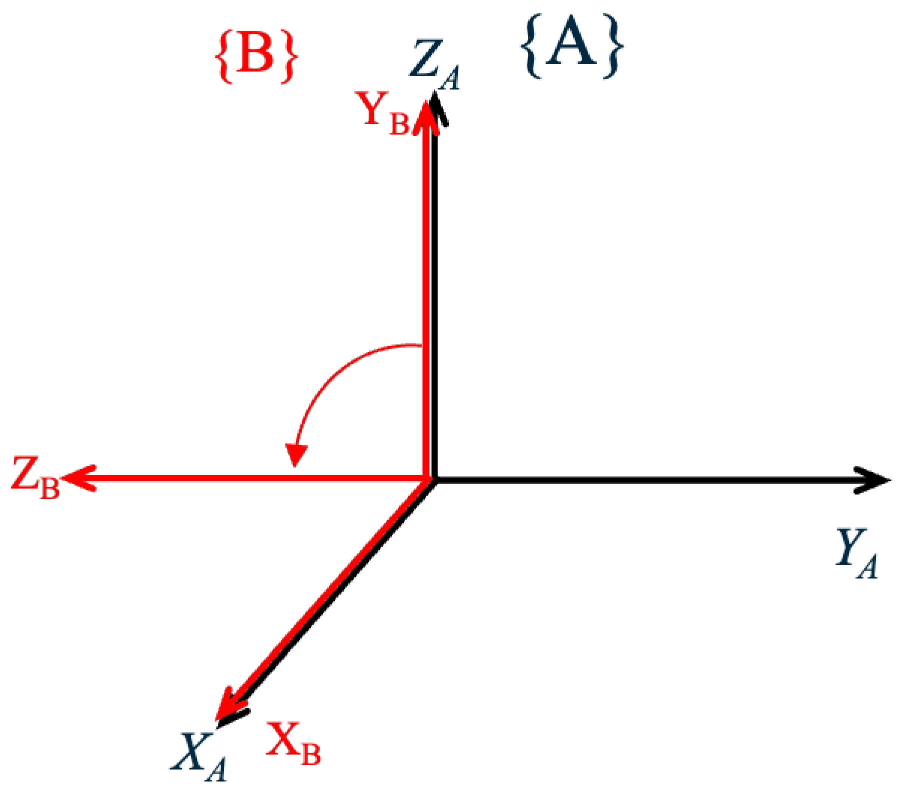
正交矩阵是一个方阵，其中列和行是正交单位向量

Transformation

- **Example:**

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} $$

$$ \begin{array}{cc} & \begin{array}{ccc} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \\ \downarrow & \downarrow & \downarrow \end{array} & \\ {}^{A}_{B}R = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{array}{l} \leftarrow {}^{B}\hat{X}_{A}{}^{T} \\ \leftarrow {}^{B}\hat{Y}_{A}{}^{T} \\ \leftarrow {}^{B}\hat{Z}_{A}{}^{T} \end{array} \end{array} $$

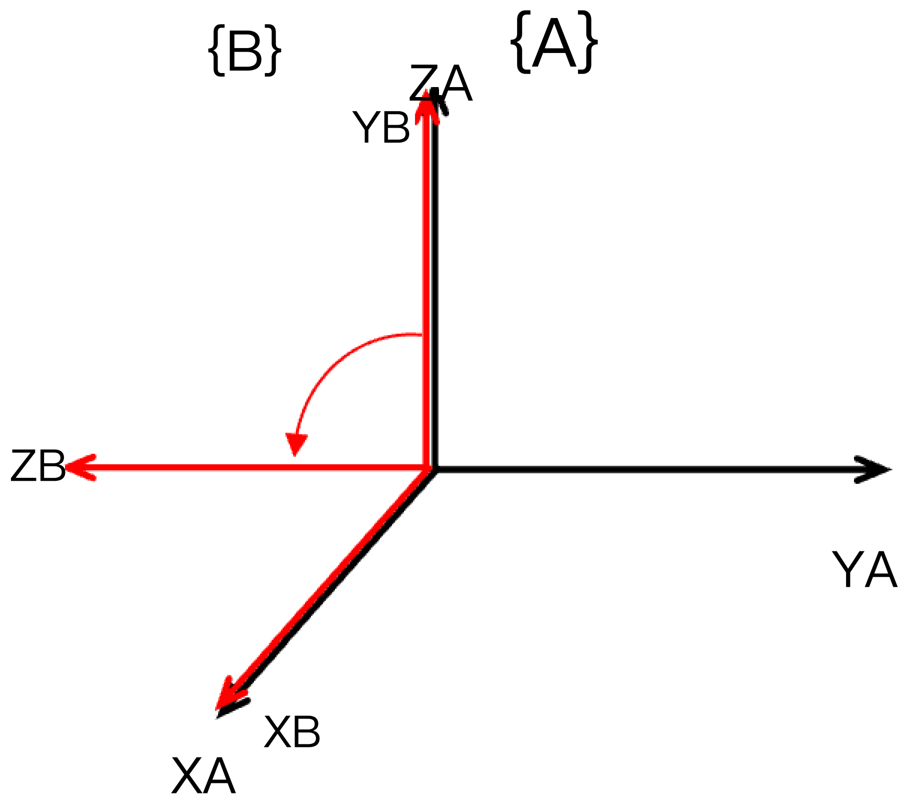


转型

- 例：

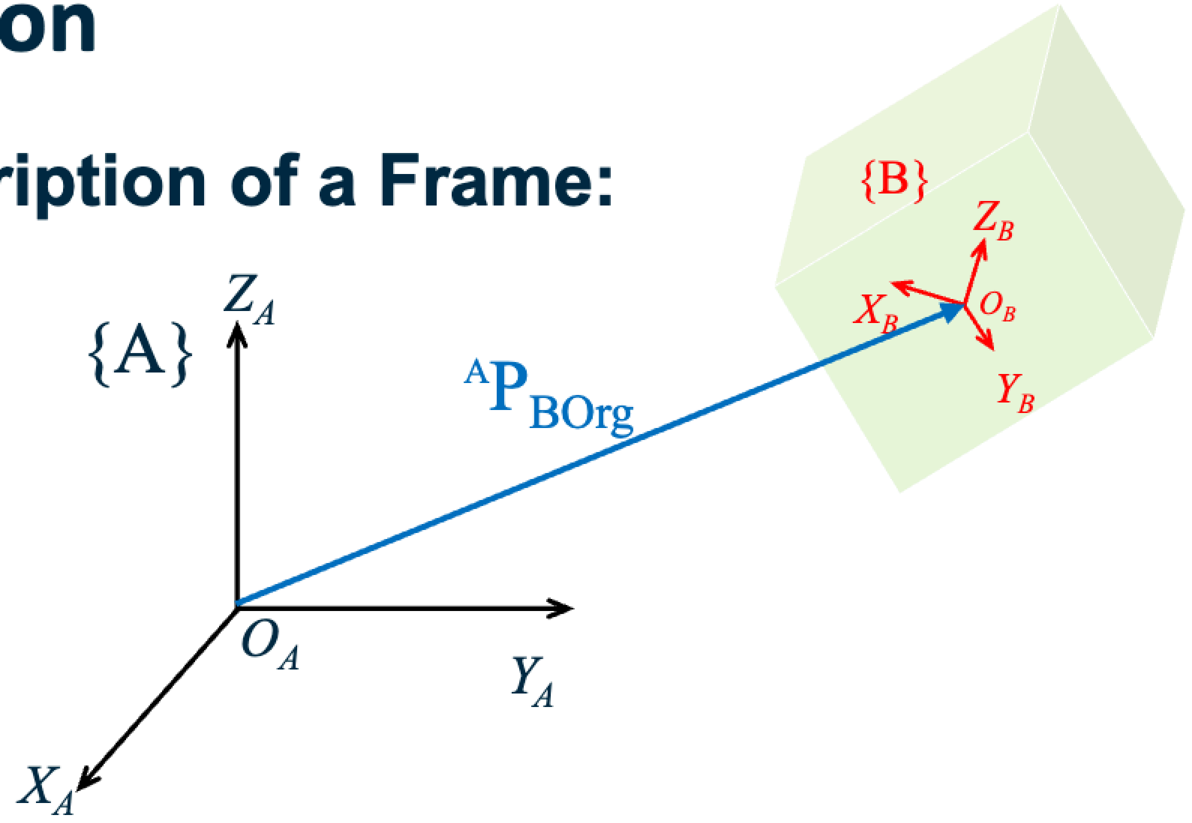
$$ {}^{A}_{B}R = \quad \hat{X} \qquad \hat{Y} \qquad \hat{Z} $$

$$ {}^{A}_{B}R = \begin{array}{cl} \begin{matrix} {}^{A}XB & YB & \varpi ZB \end{matrix} & \\ \left[\begin{matrix} 1\ 0\ 0 \\ 0\ 0\ -1\ 0\ 1 \\ 0\ 1\ 0 \end{matrix}\right] & \begin{matrix} XA^{T} \\ {}^{B}YA^{T} \\ {}^{B}\varpi ZA^{T} \end{matrix} \end{array} $$



Transformation

- **Description of a Frame:**



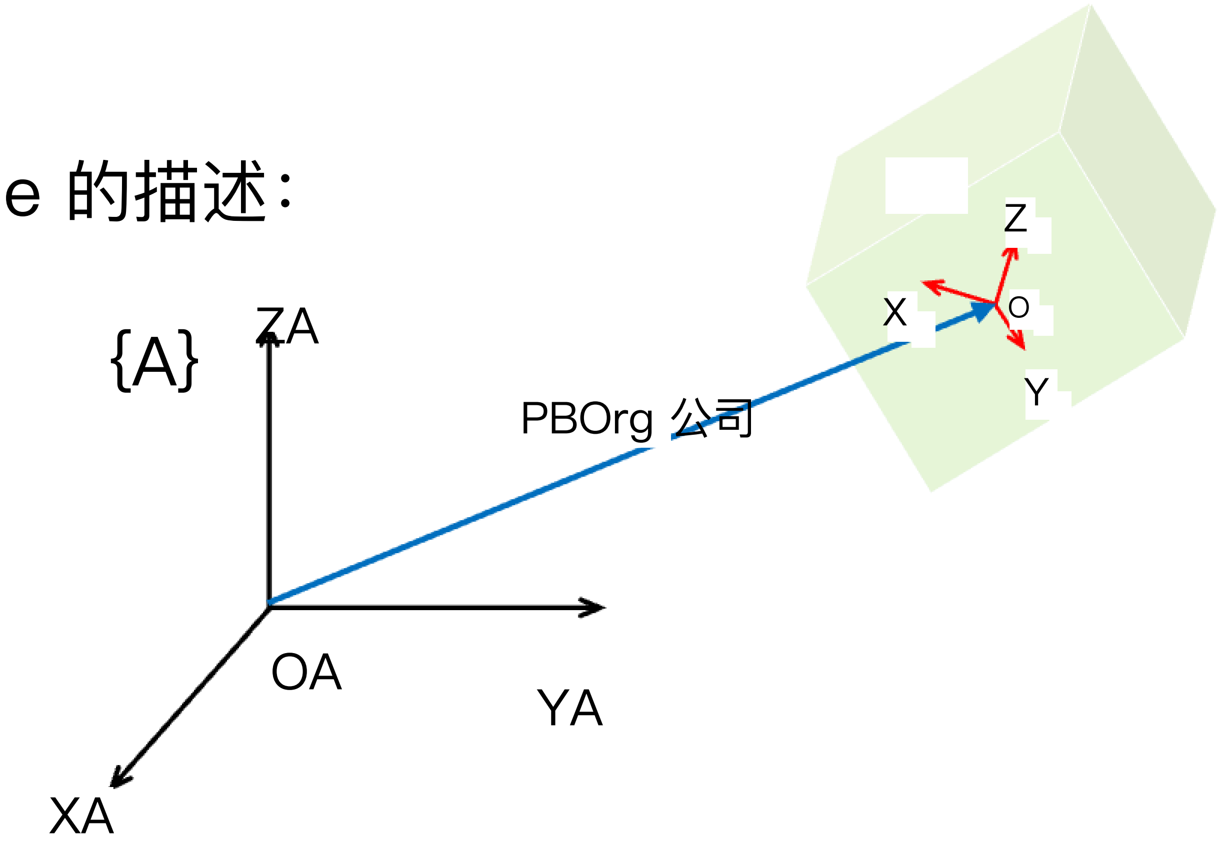
P是在A中描述B的坐标原点

Frame {B}: ${}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B,\quad {}^{A}P_{BOrg}$

$$\{B\} = \{{}^{A}_{B}R \quad {}^{A}P_{BOrg}\}$$

转型

- Frame 的描述：



帧 {B}: ^{A}XB， YB， ZB， PBOrg

{B = {R P}

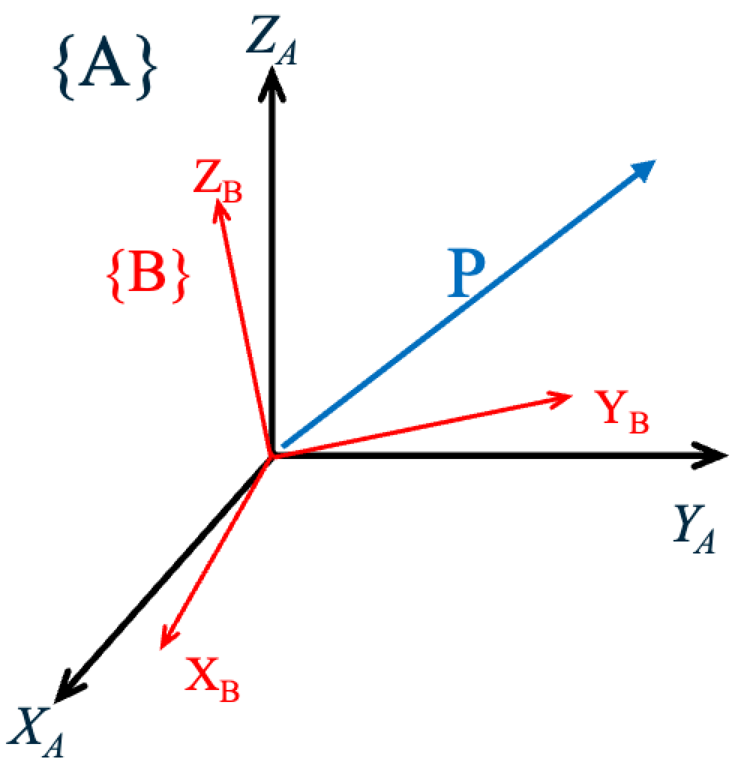
Transformation

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

点乘的结果为：

$$A \odot B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$

- **Mapping:**
  - *Changing descriptions from frame to frame*
- **Rotations**



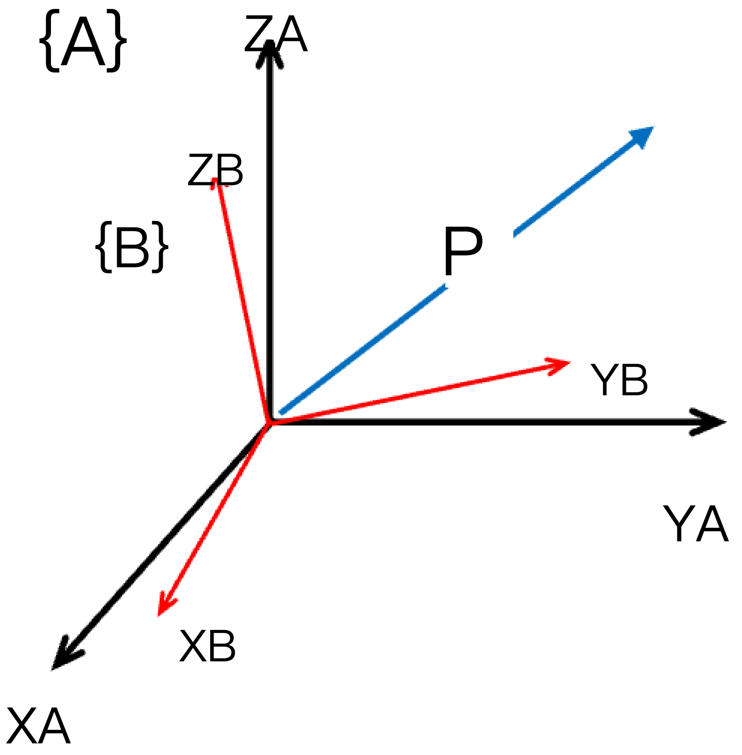
If P is in {B}: ${}^BP$

$${}^AP = \begin{bmatrix} {}^B\hat{X}_A \cdot {}^BP \\ {}^B\hat{Y}_A \cdot {}^BP \\ {}^B\hat{Z}_A \cdot {}^BP \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} \cdot {}^BP$$

$${}^AP = {}^A_BR \ {}^BP$$

转型

- 映射：
  - 从帧到帧更改描述
- 旋转



如果 P 在 {B} 中： P

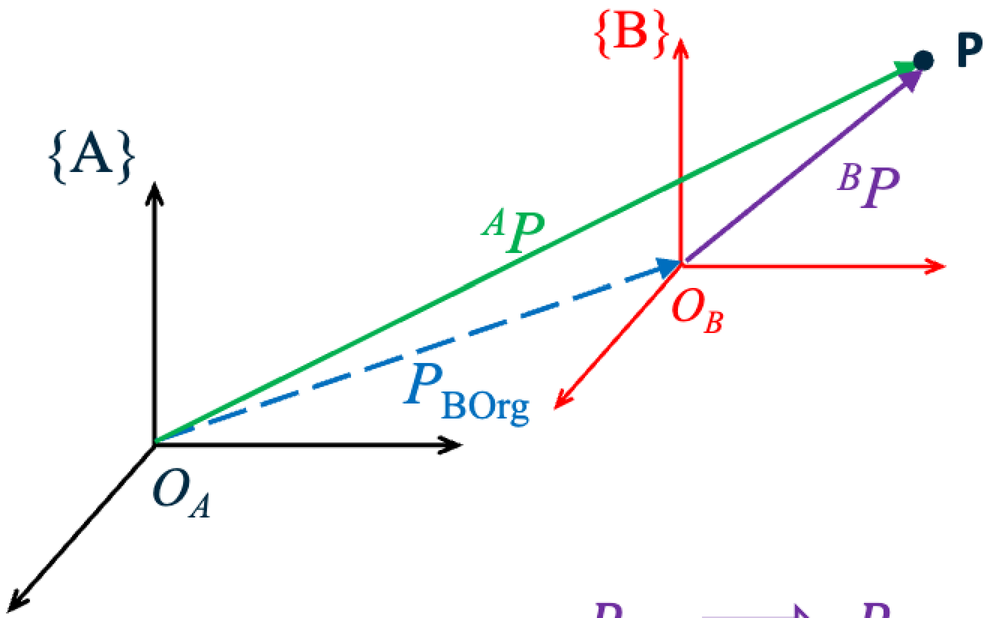
$$^{A}P = \begin{bmatrix} {}^{B}XA \cdot {}^{B}P \\ {}^{B}YA \cdot {}^{B}P \\ {}^{B}A\,P \end{bmatrix} = \begin{bmatrix} Y \\ \sigma\text{o}A^{T} \end{bmatrix} \cdot P$$

$$^{A}P = R\,P$$

Transformation

- **Translation:**

对于B坐标系中的点P，在A坐标系中描述OP = 在B坐标系中描述P+ 在A坐标系中描述B的坐标原点



$\overrightarrow{O_B P} \Longrightarrow \overrightarrow{O_A P}$

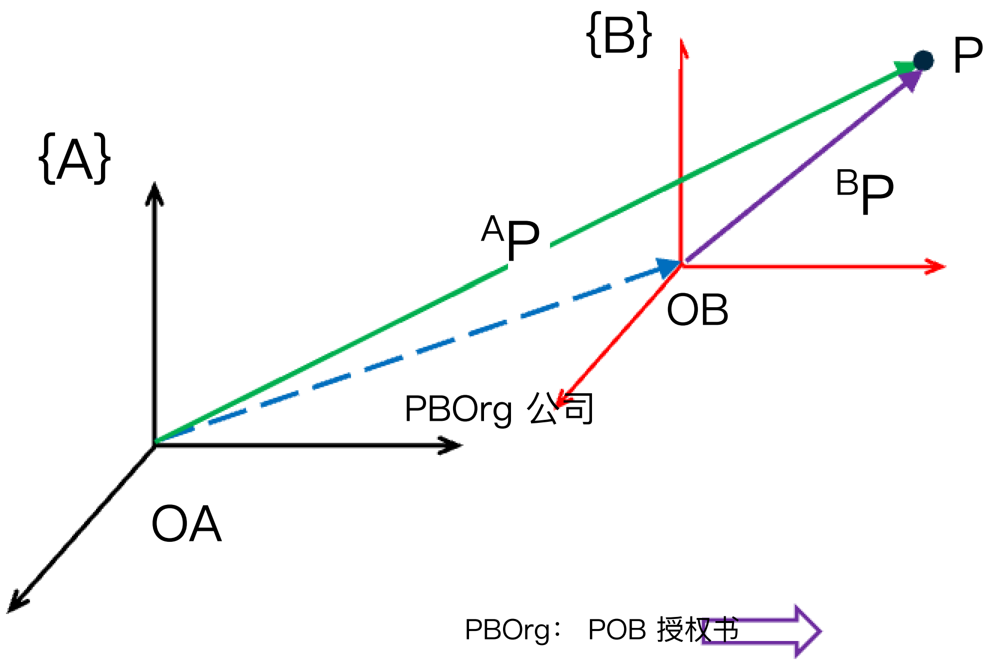
These are two different vectors

P_{BOrg}: $P_{O_B} \Longrightarrow P_{O_A}$

$$\boxed{{}^{A}P_{OA} = {}^{A}P_{OB} + {}^{A}P_{BOrg}}$$

转型

- 译本：



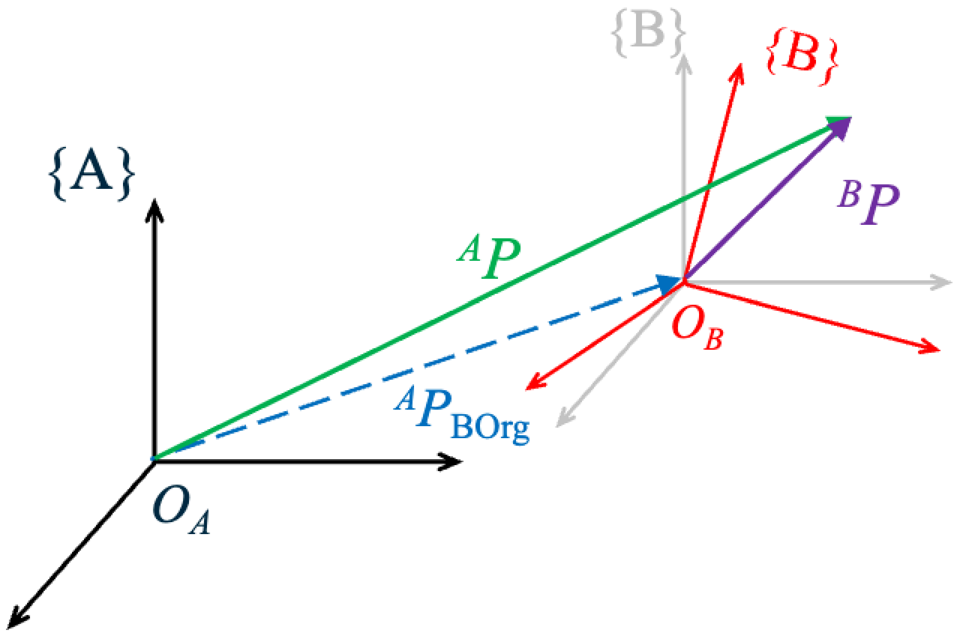
OBP OAP

这是两种不同的向量

APOA = POB + PBOrg

Transformation

- **General Transformation:**



$$^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BOrg}$$

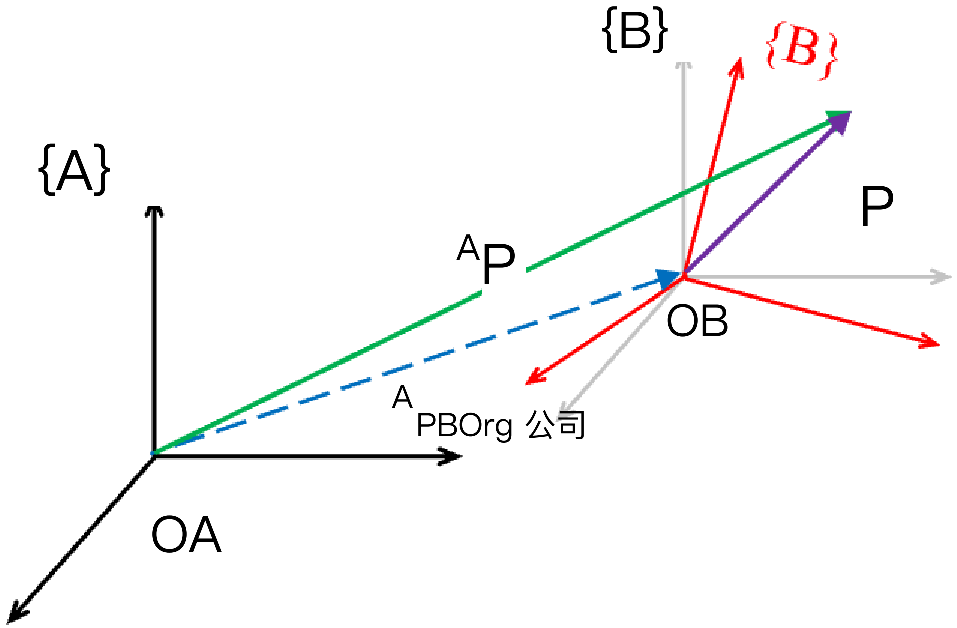
$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BOrg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

- Homogeneous Transformation:

$$^{A}P_{(4x1)} = {}^{A}_{B}T_{(4x4)} \; {}^{B}P_{(4x1)}$$

转型

- 一般转换：



$$^{A}P = R\ P + P$$

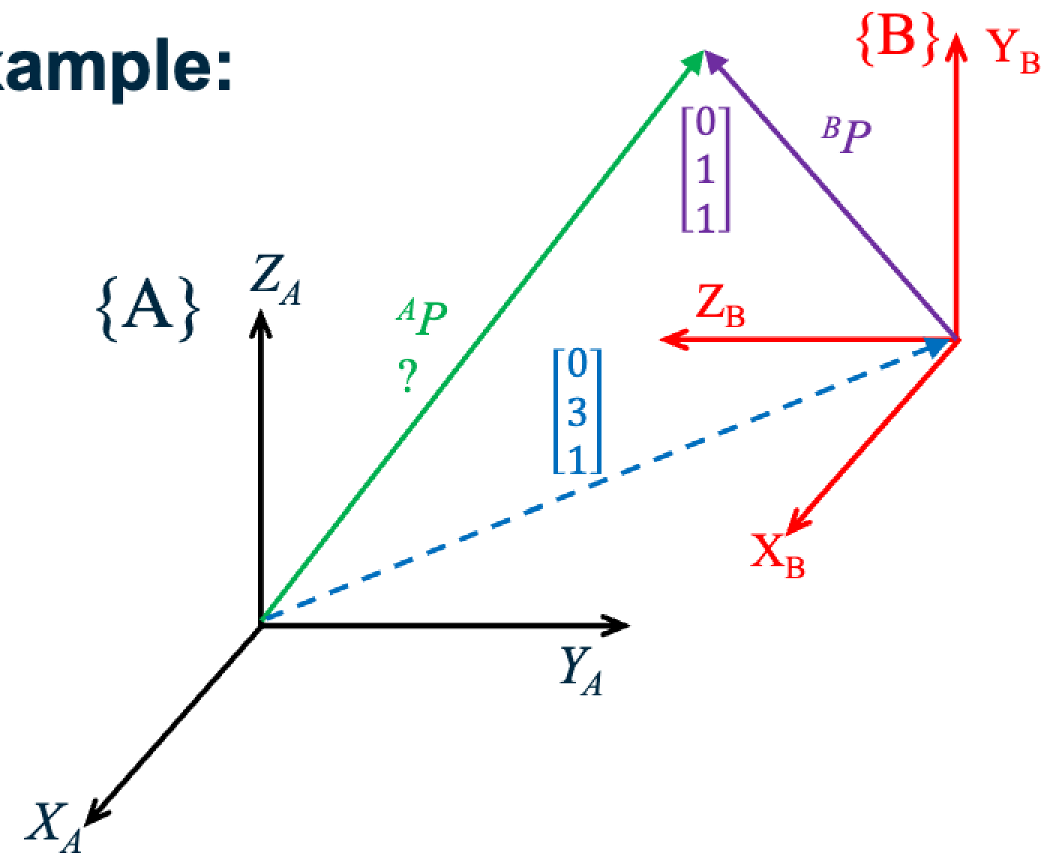
$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = B \begin{bmatrix} A\ R\ PBOrg \\ 0\ 0\ 0\ 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

- 同构变换：

$$^{A}P = TP$$

Transformation

- **Example:**



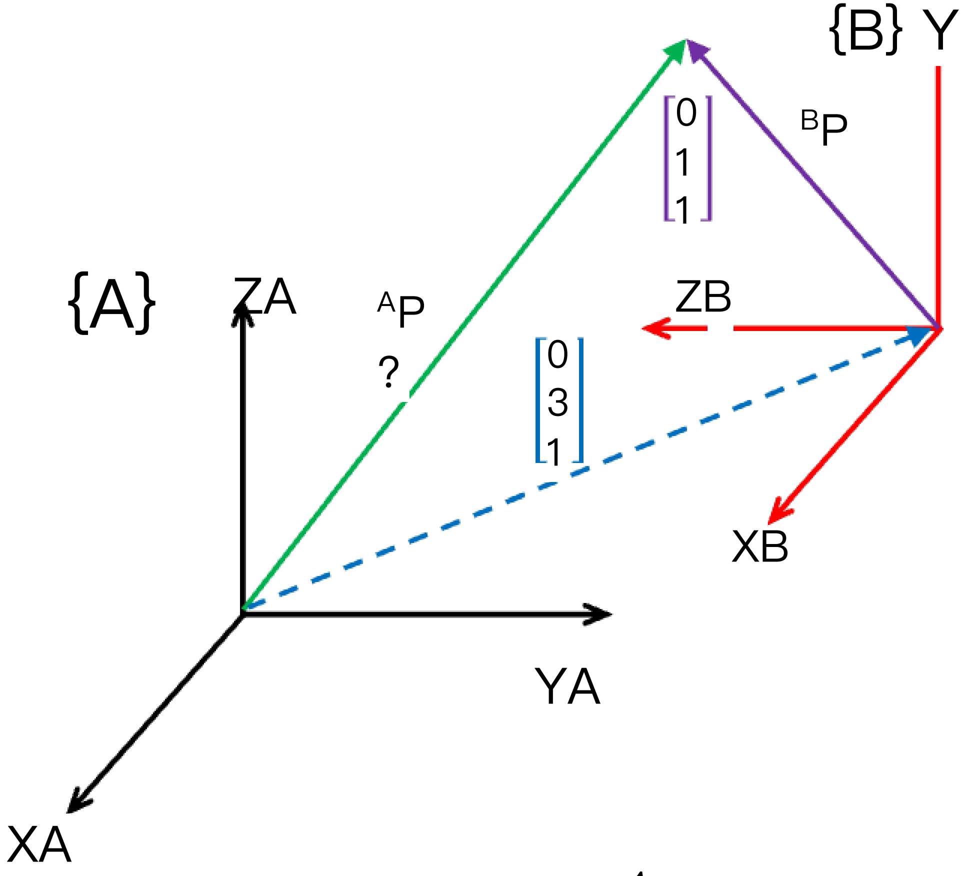
$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}P = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} $$

$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}P \Rightarrow {}^{A}P = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} $$

转型

- 例：



$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}P = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} $$

$$ {}^{A}\mathrm{P} = \mathrm{BT} \cdot \mathrm{P}\ \mathrm{P} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} $$

Transformation

- **General Operators:**

$$P_2 = \left[\begin{array}{ccc:c} & R_k(\theta) & & Q \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] P_1$$

$$\boxed{P_2 = T \; P_1}$$

转型

- 一般操作员：

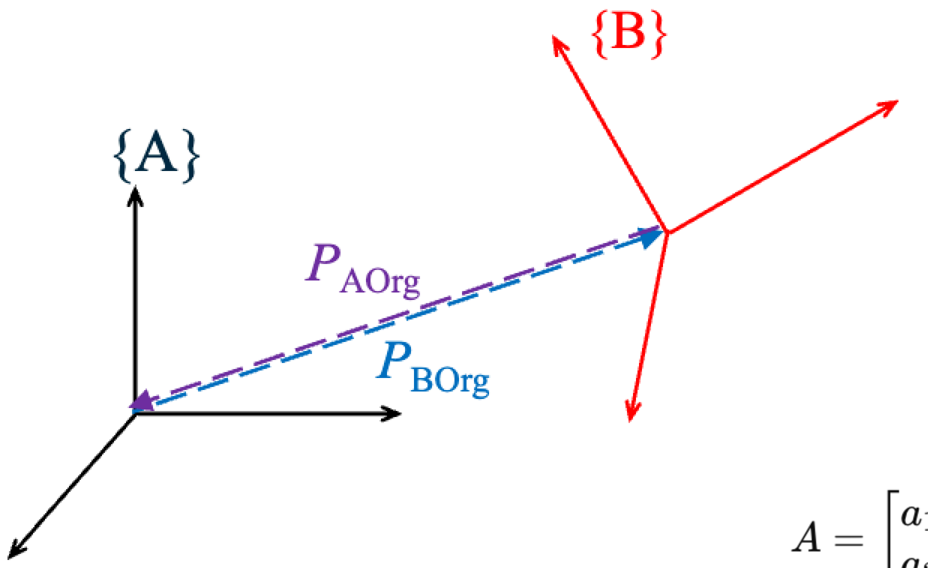
$$P2=\left[\begin{array}{c:c} R_k(\theta) & Q \\ \hdashline 0\ 0\ 0 & 1 \end{array}\right] P1$$

$$\boxed{P2=T\ P}$$

Transformation

逆变换，已知B到A的rotation，如何写出A到B的transformation

- **Inverse Transform:**



$$ {}^A_B T = \begin{bmatrix} & {}^A_B R & & {}^A P_{BOrg} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$R^{-1} = R^T$

$$ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} $$

乘积）是：

$$ A \odot B = \begin{bmatrix} a_{11} \cdot b_{11} & a_{12} \cdot b_{12} \\ a_{21} \cdot b_{21} & a_{22} \cdot b_{22} \end{bmatrix} $$

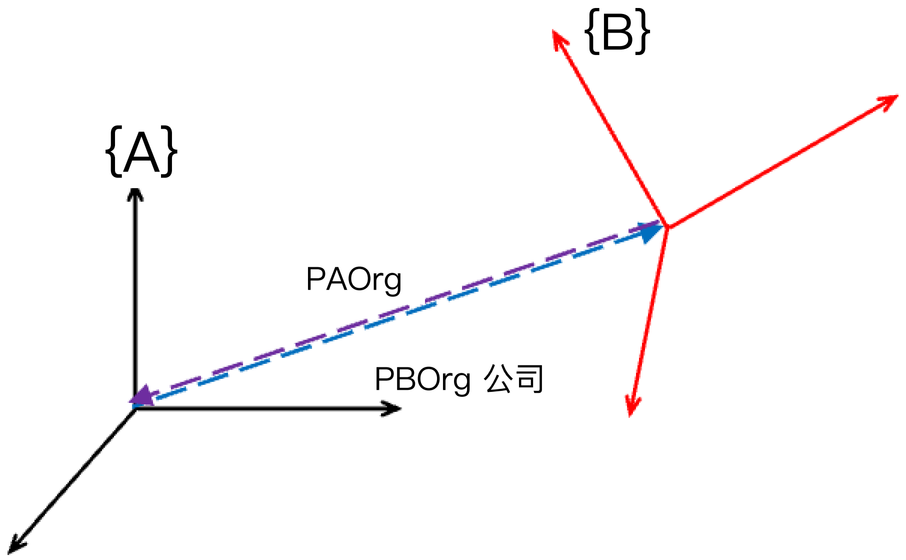
$$ {}^A_B T^{-1} = {}^B_A T = \begin{bmatrix} & {}^A_B R^T & & \overbrace{-{}^A_B R^T \cdot {}^A P_{BOrg}}^{{}^B P_{AOrg}} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

转型

- 逆变换 ：

$${}^{A}_{B}T = \begin{bmatrix} & \text{A R PBOrg} & \\ 0 & 0\ 0\ 1 & \end{bmatrix}$$

R= R



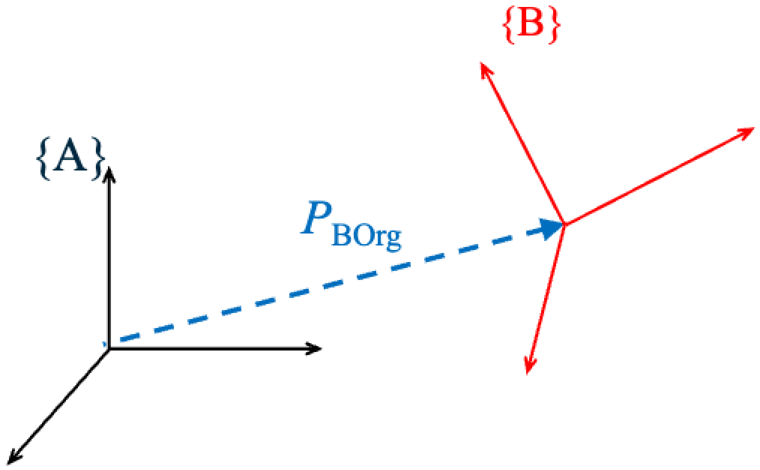
^{B}PAOrg

$${}^{A}_{B}\text{T= 在 =} \begin{bmatrix} & \text{A R—BR• PBOrg} & \\ 0 & 0\ 0\ 1 & \end{bmatrix}$$

Transformation

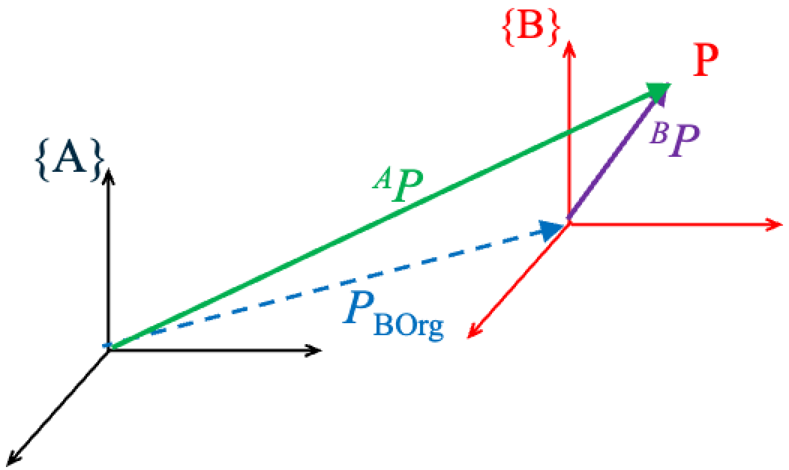
- **Homogeneous Transform Interpretations:**
  - Description of a frame

$$ {}^A_BT: \{B\} = \{ {}^A_BR \quad {}^AP_{BOrg} \} $$

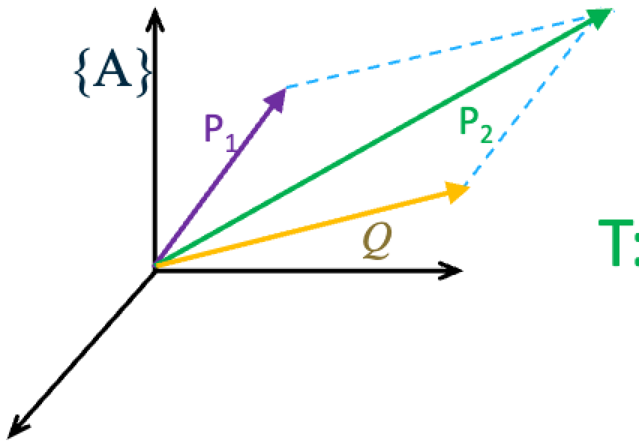


  - Transform mapping

$$ {}^A_B\mathrm{T}: {}^B\mathrm{P} \rightarrow {}^A\mathrm{P} $$



  - Transform operator

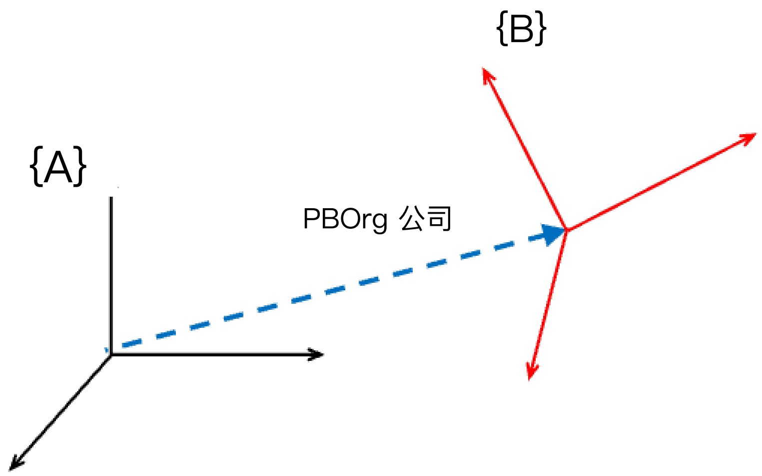


$\mathrm{T}: \mathrm{P}_1 \rightarrow \mathrm{P}_2$

转型

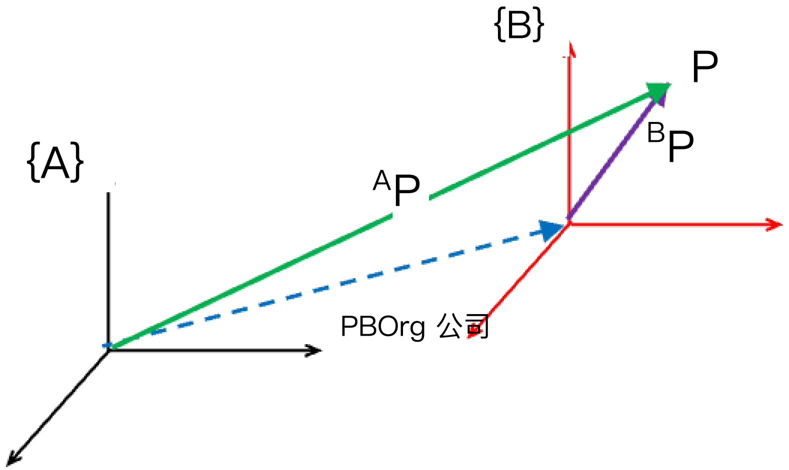
- 齐次变换解释 :
  - o 帧描述

$${}^{A}_{B}T\text{:}\quad B = \{BR\ PBOrg\}$$

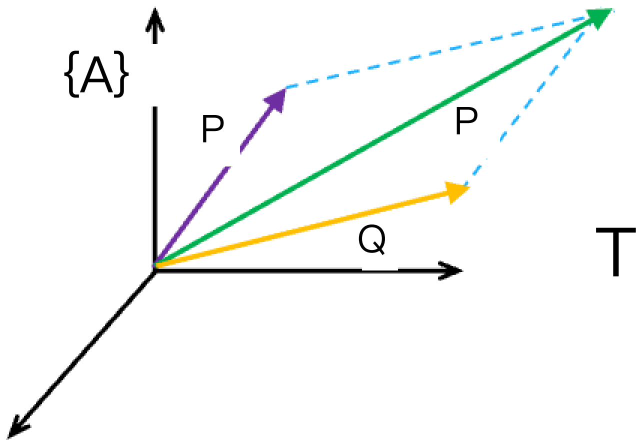


  - o 变换映射

$${}^{A}_{B}T\text{:}$$



  - o Transform 运算符



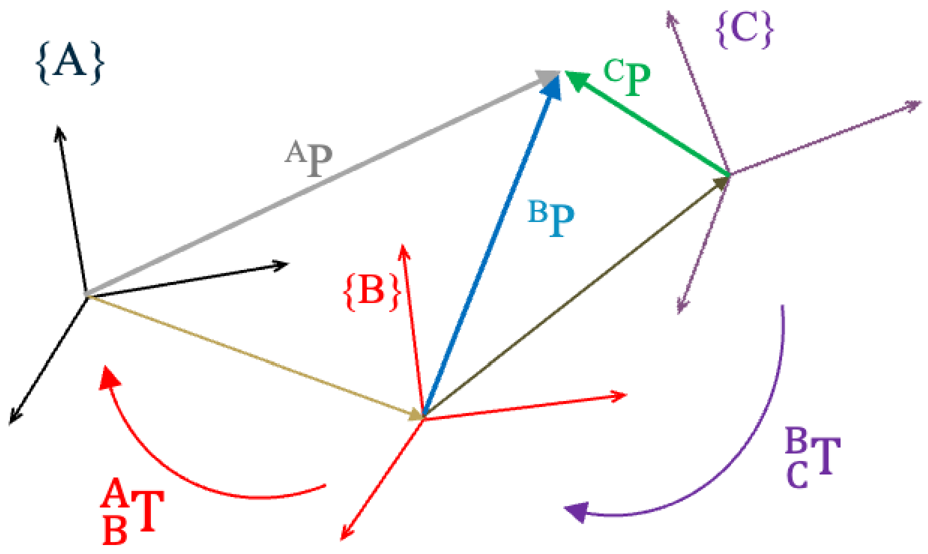
T： P⟶ P

Transformation

- **Compound Transformation:**

$$^{B}P = {}^{B}_{C}T\, {}^{C}P$$

$$^{A}P = {}^{A}_{B}T\, {}^{B}P$$



$$^{A}P = {}^{A}_{B}T\; {}^{B}_{C}T\; {}^{C}P$$

$$\boxed{{}^{A}_{C}T = {}^{A}_{B}T\; {}^{B}_{C}T}$$

$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\, {}^{B}_{C}R & {}^{A}_{B}R\; {}^{B}P_{COrg} + {}^{A}P_{BOrg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

转型

- 化合物转化：

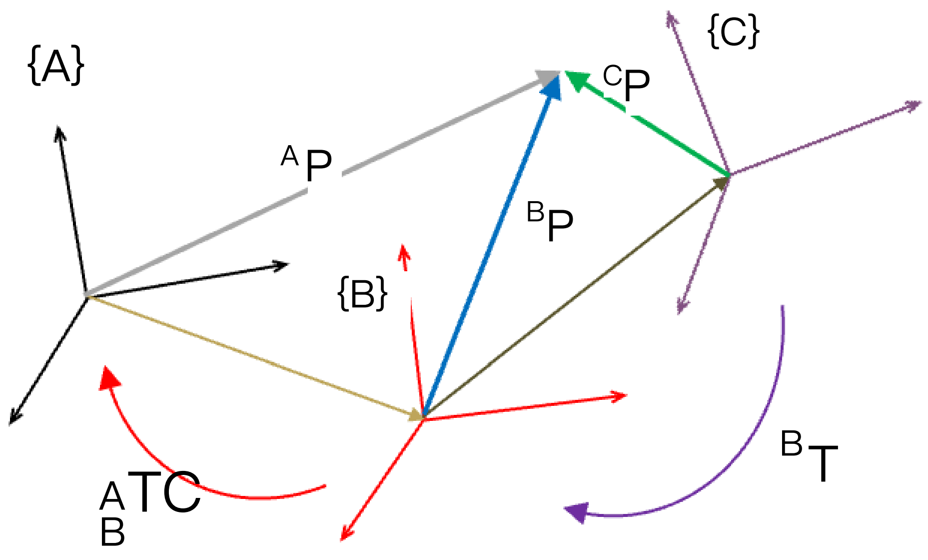
$$^{B}P = T\ P$$

$$^{A}P = T\ P$$

$$^{A}P = T\ T\ P$$

$$^{A}_{C}T = T\ T$$

$$^{A}_{C}T = \begin{bmatrix} {}^{A}R\ R\ R\ P & COrg & +^{A} & P博格 \\ 0\ 0\ 0\ 1 & & & \end{bmatrix}$$



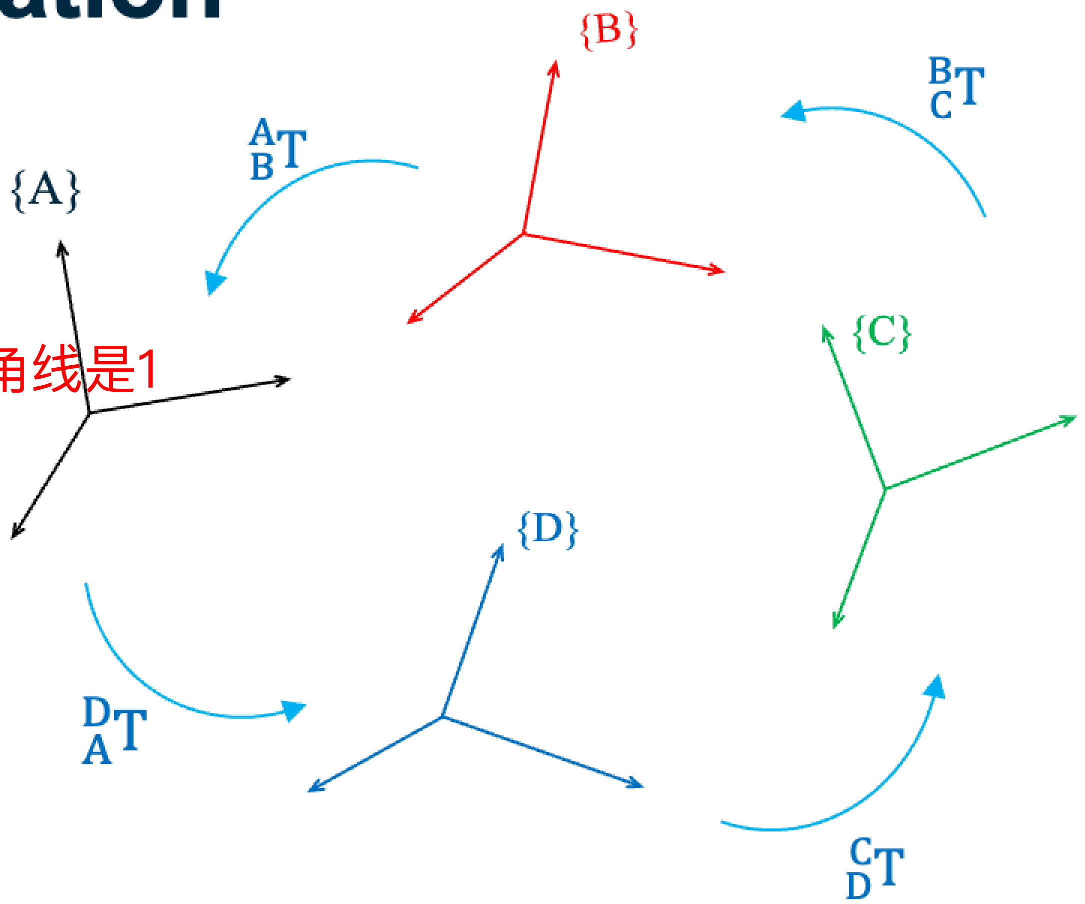
Transformation

- Transform Equation:

I是单位矩阵，对角线是1

$$\boxed{{}^{A}_{B}T\ {}^{B}_{C}T\ {}^{C}_{D}T\ {}^{D}_{A}T = I}$$

$$\Rightarrow\ {}^{B}_{A}T = {}^{B}_{C}T\ {}^{C}_{D}T\ {}^{D}_{A}T$$

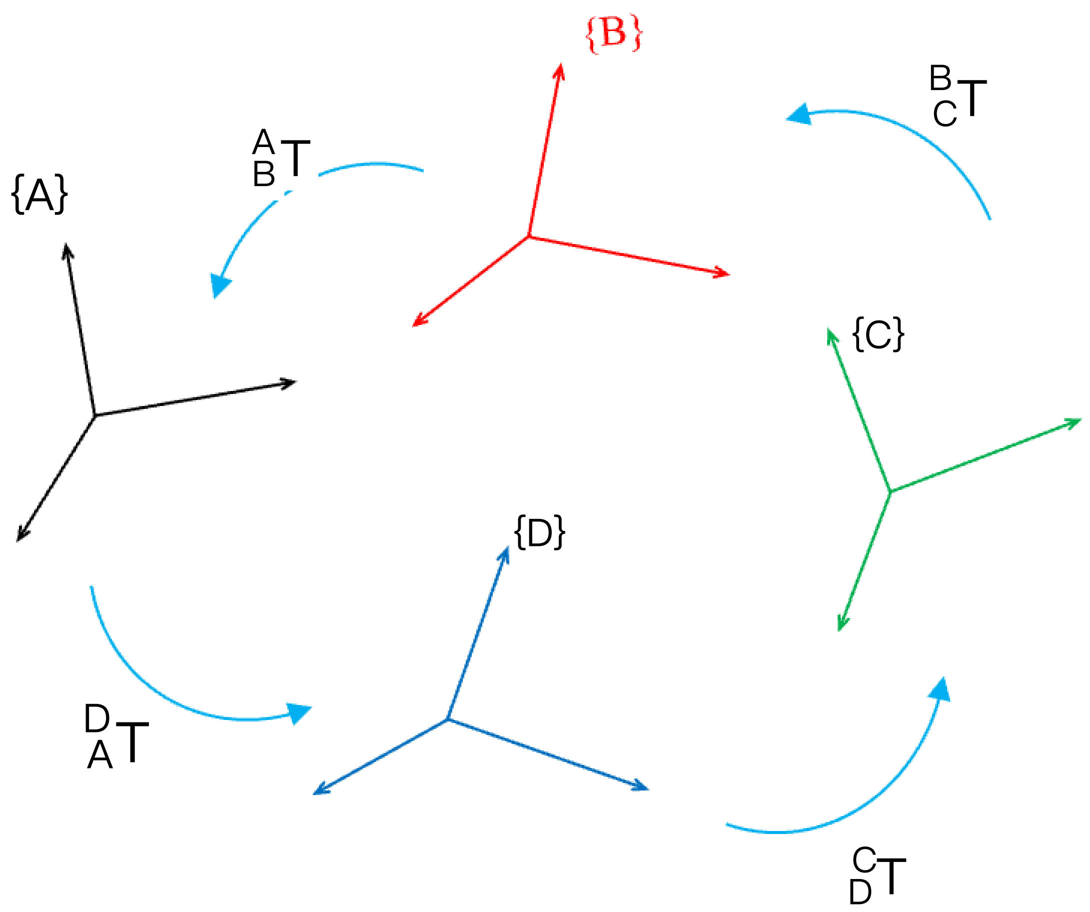


转型

- 变换方程:

$$ {}^A_B T\, {}^B_C T\, {}^C_D T\, {}^D_A T = I $$

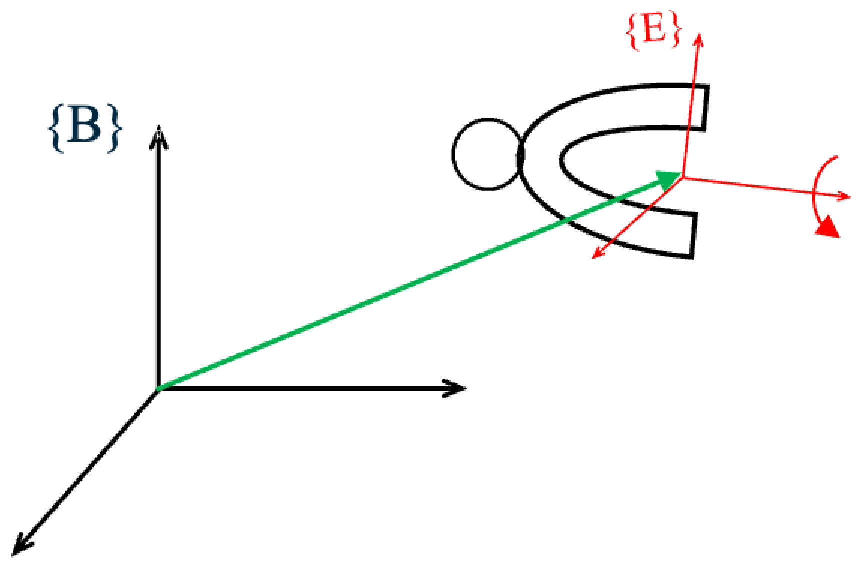
⇒ ${}^B_A T = CT\ DT\ AT$



Representations

End-effector Configuration:

${}^{B}_{E}T$: Position + Orientation



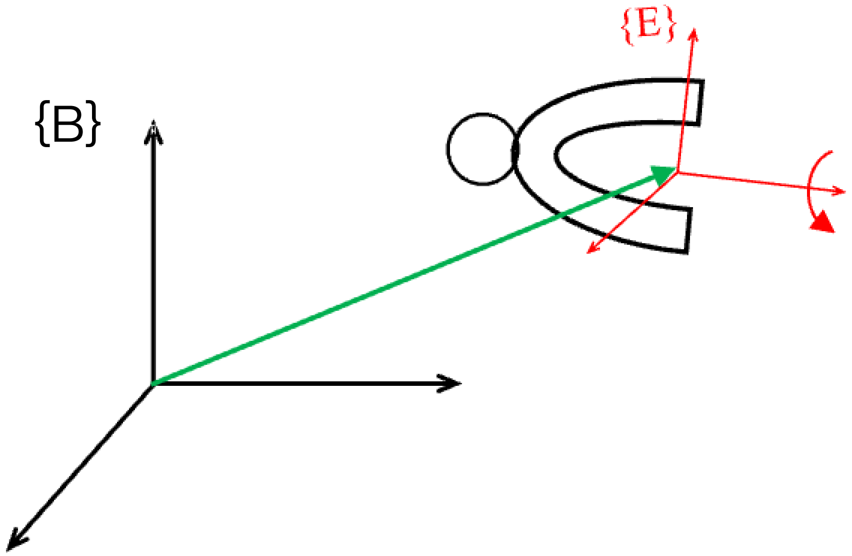
End-effectors configuration parameters:

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix} \begin{matrix} \leftarrow \text{Position} \\ \leftarrow \text{Orientation} \end{matrix}$$

交涉

末端执行器配置：

$^{B}_{E}T$： 位置 + 方向



末端执行器配置参数：

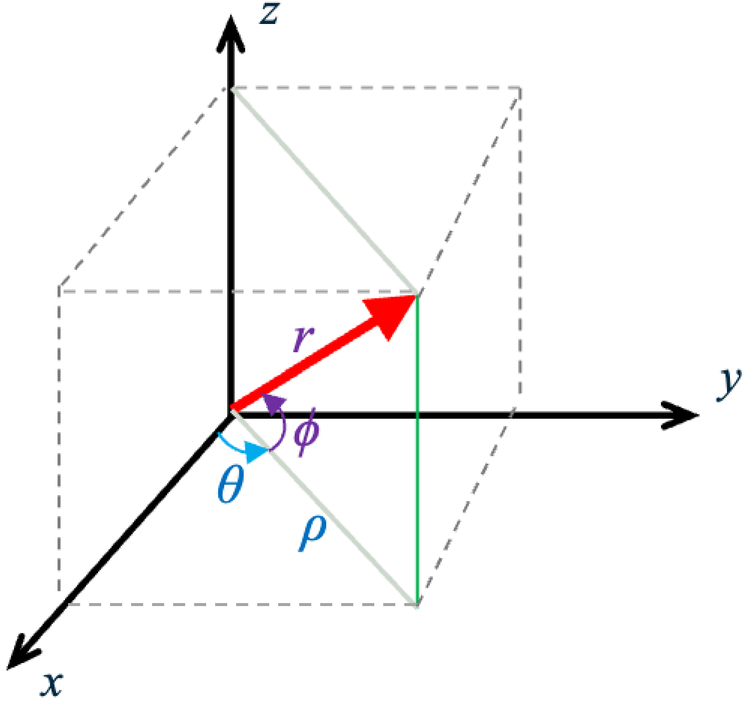
$$X = \begin{bmatrix} X \\ X \end{bmatrix}$$

位置方向

Representations

- **Position representation:**

- ❑ Cartesian: (x, y, z)
- ❑ Cylindrical: (ρ, θ, z)
- ❑ Spherical: (r, θ, ϕ)



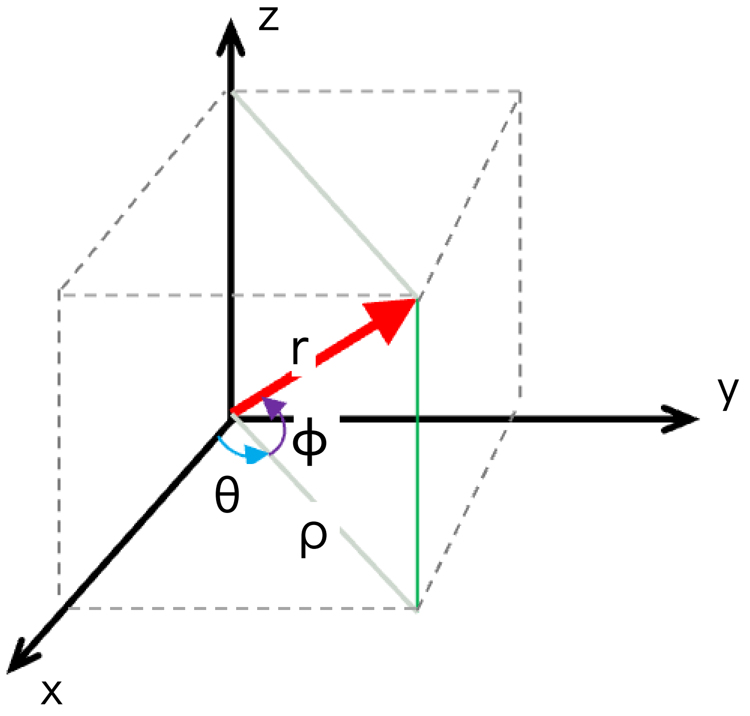
交涉

- 位置表示：

❑ 笛卡尔：　　　(x、y、z)

❑ 圆柱形：（ρ，　θ，　z）

❑ 球形：　　　(r、θ、ϕ)



Lecture 4 Summary

- Spatial description
- Coordinate Frames
- Rotation matrix
- Transformation

第 4 讲 总结

- 空间描述
- 坐标框架
- 旋转矩阵
- 转型