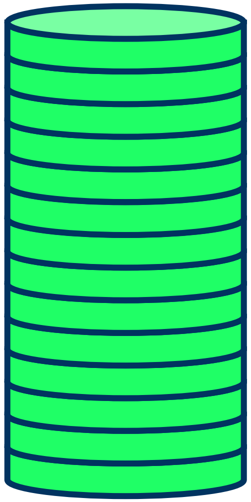
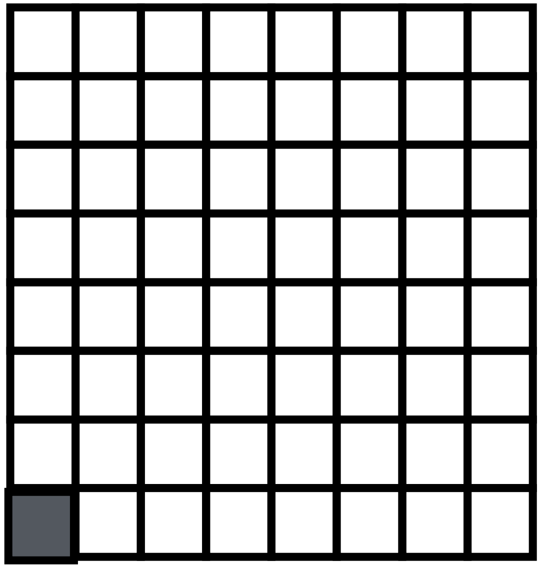
Automated Game Playing by (Intelligent) Machines: Part II





Combinatorial Games

- A set of positions (position = state of the game)
- Two players (know the state)
- Rules specify for each position which moves to other positions are legal moves
    - we restrict to "impartial" games (same moves available for both players)
- The players alternate moving
- A terminal position is one in which there are no moves
- The game must reach terminal position and end in a finite number of moves.
- (No draws!)

Normal Versus Misère

Games ends by reaching a terminal position from which there are no moves.

Normal Play Rule: The last player to move wins

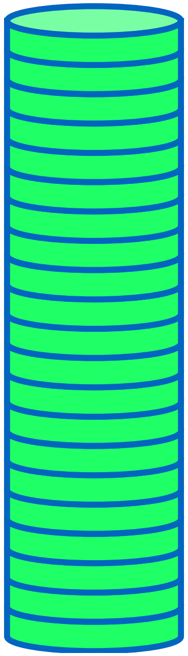
Misère Play Rule: The last player to move loses

P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved the game into that position)

N-Position: Positions that are winning for the Next player (the player who is about to move from the current position)

21 Chips Game

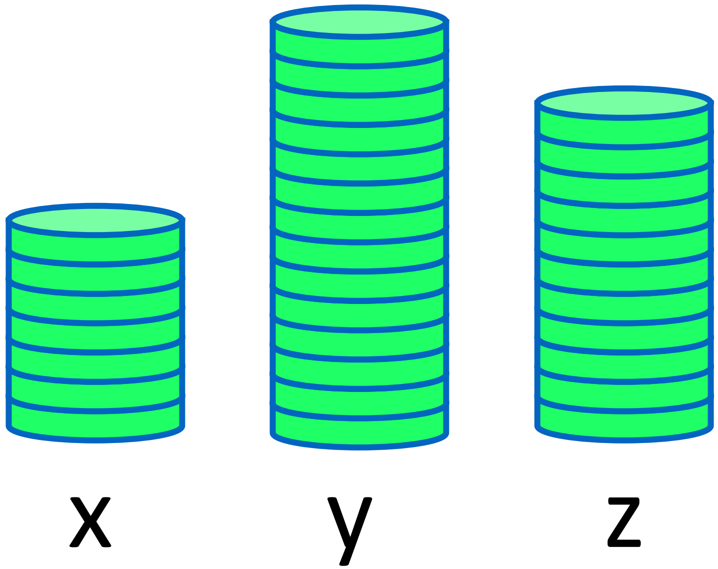


21 chips

0, 4, 8, 12, 16, … are P-positions; if a player moves resulting in that position, that player can win the game

21 chips is an N-position
("First (next) player wins")

The Game of Nim



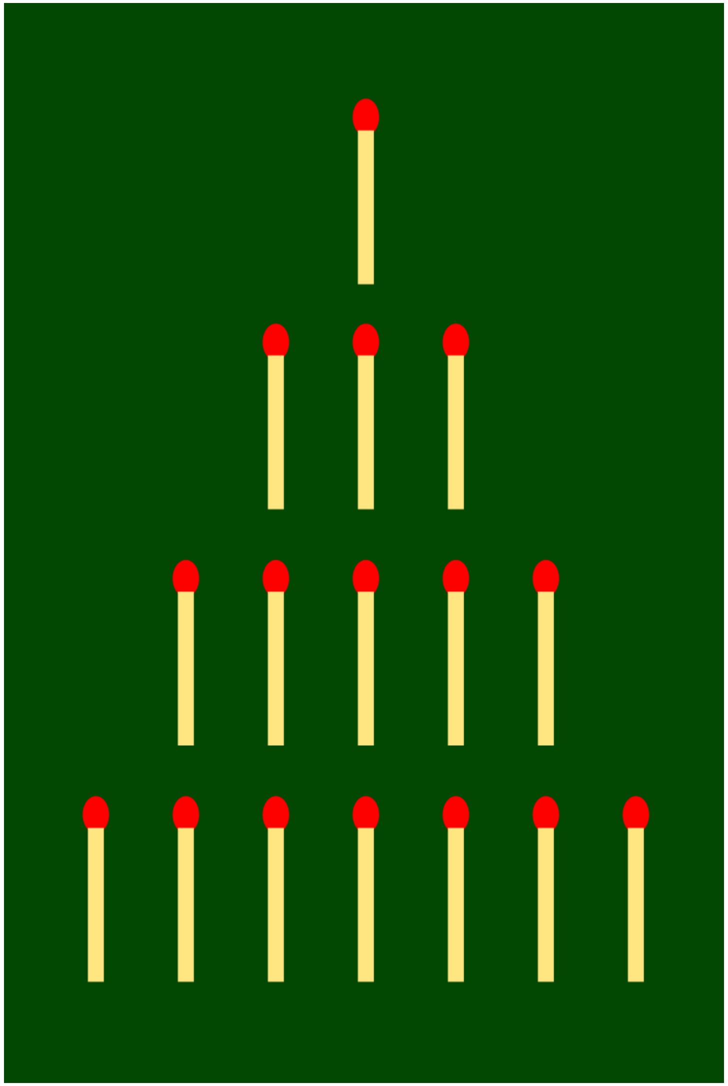
Two players take turns moving

Winner is the last player to remove chips

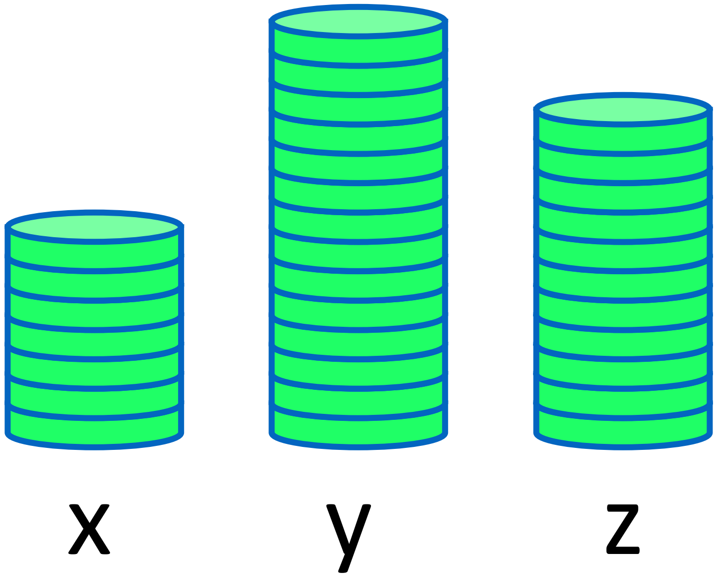
A move consists of selecting a pile and removing one or more chips from it. No limit.

(In one move, you cannot remove chips from more than one pile.)

Example: Matchsticks Game



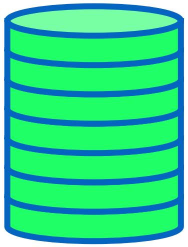
Analyzing Simple Positions



We use (x, y, z) to denote this position

(0,0,0): P-position

One-Pile Nim



What happens in positions of the form (x,0,0)?
(with x>0)

The next player can just take the entire pile, so (x,0,0) is an N-position

Two-Pile Nim

P-positions: two piles have an equal number of chips. Just run Mirroring!

Opponent's turn: must change s.t. two piles have different number of chips

Given this, can easily go to equal number of chips (perhaps the terminal position)

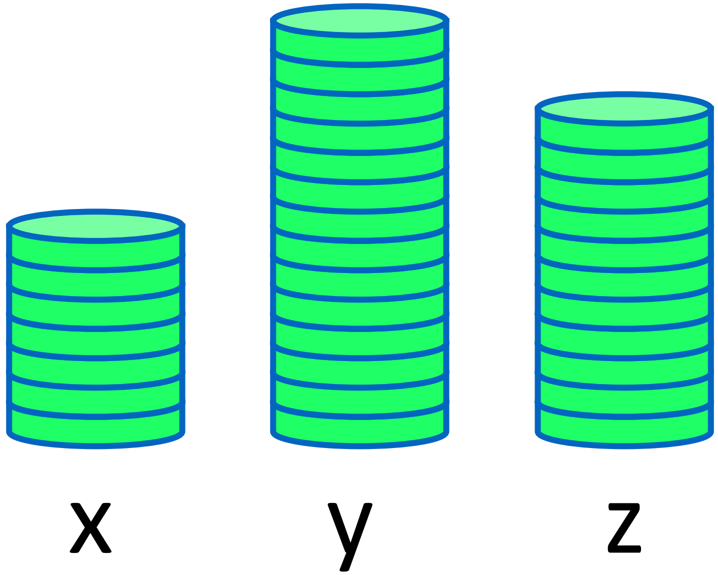
Two-Pile Nim

Theorem: P-positions are those for which the two piles have an equal number of chips.
(and N-positions $\equiv$ unequal piles)

Proof Idea (optional):

1. If I (previous player) left the piles with equal number of chips, I can mirror my opponent! I win and hence P-position.
2. If I left the piles with unequal number of chips, my opponent can make them equal. Then mirror me and win. I lose and hence N-position.

3-Piles? Even More?



Large number of piles? Large number of chips? Tricky?

Detour: XOR gate and Nim-Sum

XOR function

XOR Gate:

$0 \oplus 0 = 0$

$0 \oplus 1 = 1$

$1 \oplus 0 = 1$

$1 \oplus 1 = 0$

$$\begin{array}{r} 010110 \\ \oplus\ 110011 \\ \hline 100101 \end{array}$$

Key property: $x \oplus x = 0$

Key property: $x \oplus 0 = x$

Also keep in mind: you can convert any decimal number (e.g. 6) to binary (e.g. 110) and vice versa

Define Nim-Sum

The nim-sum of two non-negative integers is their bitwise XOR + back to decimal

We will use $\oplus$ to denote the nim-sum

$2 \oplus 3 = 10 \oplus 11 = 01 = 1$

$5 \oplus 3 = 101 \oplus 011 = 110 = 6$

$7 \oplus 4 = 111 \oplus 100 = 011 = 3$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

Some (More) Basic Properties

$x \oplus 0 = x$ $\qquad$ $x \oplus x = 0$ (cancellation)

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$ $\qquad$ (cancellation)

3-Pile Nim

Bouton's Theorem (1902): A position (x,y,z) in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Generalization to n-Piles: A position $(x_1,x_2,\ldots,x_n)$ in Nim is a P-position if and only if $x_1 \oplus x_2 \oplus \ldots \oplus x_n = 0$

Examples

- Two piles, $x = y$. Nim-Sum = 0. Hence P-position as expected.

- What if $x = 7$, $y = 4$, $z = 2$?

$$7 \oplus 4 = 111 \oplus 100 = 011 = 3$$

$$3 \oplus 2 = 11 \oplus 10 = 01 = 1$$

So this is an N-position

- What if we have (7, 4, 2, 1)? Nim-sim of these 4 is 0. Hence P-position!

But How to Play?

- Very simple: I will leave piles in a position where Nim-sum is 0

- Next player changes Nim-sum to non-zero

- I will change it back to 0. And eventually all piles will be 0 with me winning!

- Questions: Say I leave piles where Nim-sum is 0.
  1) Why will my opponent not be able to keep Nim-sum 0?
  2) How can I change Nim-sum non-zero to 0?

Opponent Can't go to Zero

1) Every move from a position with Nim-sum 0 is to a position with Nim-sum non-zero

If (x,y,z) has 0, and x is changed to $x' \neq x$, then we cannot have

$$x \oplus y \oplus z = 0 = x' \oplus y \oplus z$$

Because then $x = x'$ Cancellation

I can go to Zero (Optional)

(2) From each position with nim-sum non zero, there is a move to a position with 0. More tricky!

$$
\begin{array}{r}
001010001 \\
100010111 \\
\oplus\ \underline{111010000} \\
010010110
\end{array}
\quad \Rightarrow \quad
\begin{array}{r}
001010001 \\
100010111 \\
\oplus\ \underline{101000110} \\
000000000
\end{array}
$$

Look at leftmost column where nim-sum bit is 1

Change one of the numbers with a 1 in that column so that everything adds up to zero. You are reducing that number!

What We have Learnt

Any number of piles, any number of chips: we have now learnt about the winning strategy

- If I go first, I want to start with Nim-sum non-zero
- If I go second, I want to start with Nim-sum zero

Fun Exercise: Write a program to play Nim game for 3 or more piles

Questions?