Solutions to Exercise Sheet 8

Exercise 8.1

(a)

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 4 & 4 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

...and we may read off the solution $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$.

(b)

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 1 & 1 & 3 & 2 \\ -1 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{solution:} \quad \begin{array}{rcl} x_3 & = & 1 \\ x_2 & = & \text{chosen freely} \\ x_1 & = & -1 - x_2 \end{array}$$

(c)

$$\left(\begin{array}{ccc|c} 2 & -2 & -1 & 5 \\ -4 & -1 & -1 & -3 \\ 2 & 3 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -2 & -1 & 5 \\ 0 & -5 & -3 & 7 \\ 0 & 5 & 3 & -6 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -2 & -1 & 5 \\ 0 & -5 & -3 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

There is no solution, because the last line indicates a contradiction: $0 = 1$.

Exercise 8.2

$$\left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 2 & -4 & 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 3 & -6 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 0 & -4 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 0 & -4 & 3 & 0 & 1 \\ 0 & 0 & 4 & -8 & 4 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 0 & -4 & 3 & 0 & 1 \\ 0 & 0 & 0 & -5 & 4 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 0 & -4 & 3 & 0 & 1 \\ 0 & 0 & 0 & -5 & 4 & 1 \\ 0 & 0 & 0 & 5 & -10 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 0 & 1 \\ 0 & 0 & -4 & 3 & 0 & 1 \\ 0 & 0 & 0 & -5 & 4 & 1 \\ 0 & 0 & 0 & 0 & -6 & 1 \end{array}\right)$$

$$\text{solution:} \quad \begin{array}{rcl} x_5 & = & 1/(-6) = -1/6 \\ x_4 & = & (1 - 4 \times (-1/6))/(-5) = -2/6 \\ x_3 & = & (1 - 3 \times (-2/6))/(-4) = -3/6 \\ x_2 & = & (1 - 2 \times (-3/6))/(-3) = -4/6 \\ x_1 & = & (1 - (-4/6))/(-2) = -5/6 \end{array}$$

Exercise 8.3

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

We get $x_1 = x_2 = x_3 = x_4 = 1$.

Exercise 8.4

If a is different from zero then the system has exactly one solution. If $a = 0$ and $b \neq 0$ then there is no solution. If both are zero then there are infinitely many solutions (assuming this is a system over $\mathbb{Q}$, not some finite field).