Exercise Sheet 4 - Solutions
Propositional Logic – Classical Reasoning & Semantics

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1. Here is a proof of $(A \vee \neg A) \to (\neg\neg A \to A)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, \neg\neg A \vdash A}\,[Id] \qquad \dfrac{\dfrac{}{\neg A \vdash \neg A}\,[Id]}{\neg A, \neg\neg A \vdash A}\,[\neg L]}{A \vee \neg A, \neg\neg A \vdash A}\,[\vee L]}{A \vee \neg A \vdash \neg\neg A \to A}\,[\to R]}{\vdash (A \vee \neg A) \to (\neg\neg A \to A)}\,[\to R]$$

2. Here is a proof that $((P \to\bot) \to P) \to P$ implies $\neg\neg P \to P$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}\,[Id] \qquad \dfrac{}{\bot, P \vdash\bot}\,[Id]}{P \to\bot, P \vdash\bot}\,[\to L]}{P \to\bot\vdash \neg P}\,[\neg R]}{\neg\neg P, P \to\bot\vdash P}\,[\neg L]}{\neg\neg P \vdash (P \to\bot) \to P}\,[\to R] \qquad \dfrac{}{P, \neg\neg P \vdash P}\,[Id]}{((P \to\bot) \to P) \to P, \neg\neg P \vdash P}\,[\to L]}{((P \to\bot) \to P) \to P \vdash \neg\neg P \to P}\,[\to R]}{\vdash (((P \to\bot) \to P) \to P) \to \neg\neg P \to P}\,[\to R]$$

Here is a proof that $\neg\neg P \to P$ implies $((P \to\bot) \to P) \to P$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}\,[Id]}{\neg P, P \vdash\bot}\,[\neg L]}{\neg P \vdash P \to\bot}\,[\to R] \qquad \dfrac{\dfrac{}{P \vdash P}\,[Id]}{P, \neg P \vdash\bot}\,[\neg L]}{(P \to\bot) \to P, \neg P \vdash\bot}\,[\to L]}{(P \to\bot) \to P \vdash \neg\neg P}\,[\neg R] \qquad \dfrac{}{P, (P \to\bot) \to P \vdash P}\,[Id]}{\neg\neg P \to P, (P \to\bot) \to P \vdash P}\,[\to L]}{\neg\neg P \to P \vdash ((P \to\bot) \to P) \to P}\,[\to R]}{\vdash (\neg\neg P \to P) \to ((P \to\bot) \to P) \to P}\,[\to R]$$

3. Here is a proof of $\neg(A \wedge B) \to (\neg A \vee \neg B)$ in the classical version of the Natural Deduction:

$$\dfrac{\dfrac{\dfrac{}{A \vee \neg A}\,[LEM] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \wedge B)}\,1 \quad \dfrac{\dfrac{}{A}\,2 \quad \dfrac{}{B}\,4}{A \wedge B}\,[\wedge I]}{\bot}\,[\neg E]}{\neg B}\,4\ [\neg I]}{\neg A \vee \neg B}\,[\vee I_R]}{A \to \neg A \vee \neg B}\,2\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\neg A}\,3}{\neg A \vee \neg B}\,[\vee I_L]}{\neg A \to \neg A \vee \neg B}\,3\ [\to I]}{\neg A \vee \neg B}\,[\vee E]}{\neg(A \wedge B) \to (\neg A \vee \neg B)}\,1\ [\to I]$$

Here is a proof of $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$ in the 1st classical version of the Sequent Calculus:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[LEM]$}
\UnaryInfC{$\neg(A \wedge B) \vdash A \vee \neg A$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B \vdash A$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B \vdash B$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$A, B \vdash A \wedge B$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$\neg(A \wedge B), A, B \vdash \bot$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$\neg(A \wedge B), A \vdash \neg B$}
\RightLabel{$[\vee R_2]$}
\UnaryInfC{$\neg(A \wedge B), A \vdash \neg A \vee \neg B$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$\neg(A \wedge B), \neg A \vdash \neg A$}
\RightLabel{$[\vee R_1]$}
\UnaryInfC{$\neg(A \wedge B), \neg A \vdash \neg A \vee \neg B$}
\RightLabel{$[\vee L]$}
\BinaryInfC{$\neg(A \wedge B), A \vee \neg A \vdash \neg A \vee \neg B$}
\RightLabel{$[Cut]$}
\BinaryInfC{$\neg(A \wedge B) \vdash \neg A \vee \neg B$}
\RightLabel{$[\rightarrow R]$}
\UnaryInfC{$\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$}
\end{prooftree}
$$

Here is a proof of $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$ in the 2nd classical version of the Sequent Calculus:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B \vdash A$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B \vdash B$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$A, B \vdash A \wedge B$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$A \vdash \neg B, A \wedge B$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$\vdash \neg A, \neg B, A \wedge B$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$\neg(A \wedge B) \vdash \neg A, \neg B$}
\RightLabel{$[\vee R]$}
\UnaryInfC{$\neg(A \wedge B) \vdash \neg A \vee \neg B$}
\RightLabel{$[\rightarrow R]$}
\UnaryInfC{$\vdash \neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$}
\end{prooftree}
$$

4.