Mathematical and Logical Foundations of Computer Science

Lecture 3 - Propositional Logic (Syntax)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Propositional logic
- Syntax of the language
- Informal semantics
- Simple proofs

Propositions - informal presentation

Propositional logic is a **symbolic logic** to reason about logical statements called **propositions** that can (in principle) be true or false.

Propositions are built by combining atomic propositions using the and, or, not, and implies logical connectives.

Are these examples of propositions?

- Birmingham is north of London Yes
- Is Birmingham north of London? No
- $8 \times 7 = 42$ Yes
- Every even natural number > 2 is the sum of two primes Yes
- Please mind the gap No

Arguments - informal presentation

Let an **argument** be a list of propositions, the last of which is called the conclusion and the others are called premises.

An argument is **valid** if and only if (iff) whenever the premises are true, then so is the conclusion

In propositional logic true propositions can be derived from other true propositions through the use of derivation rules.

For example:

1. If John is at home, then his television is on.
2. His television is not on.
3. Therefore, John is not at home.

Valid? Yes

Arguments - informal presentation

More examples:

1. You can eat a burger or pasta.
2. You ate a burger.
3. Therefore, you did not eat pasta.

Valid? No Because you could eat both. In propositional logic, or is not exclusive as it is often the case in English.

1. If the control software crashes, then the car's brakes will fail.
2. The car's brakes failed.
3. Therefore, the control software crashed.

valid? No The car's brakes could have failed for another reason.

1. If the control software crashes, then the car's brakes will fail.
2. The control software did not crash.
3. Therefore, the car's brakes did not fail.

valid? No The car's brakes could have failed for another reason.

Formalizing logical statements and arguments

We want to formalise such statements and arguments.

We will take a **symbolic approach**.

It will allow us proving the (in)validity of statements generally.

Advantages of formal symbolic language over natural languages are:

- **unambiguous**
- **more concise**

Propositional Logic

Symbols:

- atomic propositions (true/false atomic statements)
- combined using logical connectives

Atomic propositions (atoms)

- propositions that cannot be broken into smaller parts
- Let $p, q, r, \ldots$ be atomic propositions
- two special atoms: $\top$ stands for True, $\bot$ stands for False

Logical Connectives

- conjunction: $\wedge$ (and)
- disjunction: $\vee$ (or)
- implication: $\rightarrow$ (if .... then / implies)
- negation: $\neg$ (not) — can be defined using $\rightarrow$ and $\bot$

Propositions - informal examples

What are the atomic propositions and connectives?

- The car's brakes failed
  an atomic proposition
- The control software crashed and the car's brakes failed
  a conjunction of 2 atomic propositions
- If the control software crashes, then the car's brakes will fail
  an implication connecting 2 atomic propositions

Propositional logic

The syntax of propositional logic formulas (called propositions) is defined by the following grammar:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \to P \mid \neg P$$

where a ranges over **atomic propositions**.

Atomic propositions are formulas.

If P and Q are formulas, then

- $P \wedge Q$ is a formula
- $P \vee Q$ is a formula
- $P \to Q$ is a formula
- $\neg P$ is a formula

Those are called **compound formulas**.

Example of a compound formula: $\neg p \wedge q \wedge q \wedge \neg r$.

Connectives - informal semantics

Conjunction: $P \wedge Q$, i.e., P and Q

- true if both individual propositions P and Q are true

Disjunction: $P \vee Q$, i.e., P or Q

- true if one or both individual propositions P and Q are true
- also sometimes called "inclusive or"
- Note: Or in English is often an "exclusive or" (i.e. where one or the other is true, but not both)
- e.g., "Your mark will be pass or fail"
- but logical disjunction is always defined as above

Connectives - informal semantics

Implication: $P \rightarrow Q$, i.e., P implies Q

- means: if P is true then Q must be true too
- if P is false, we can conclude nothing about Q
- P is the antecedent, Q is the consequent

Negation: $\neg P$, i.e., not P

- it can be defined as $P \rightarrow \perp$
- if P is true, then $\perp$ (False)
- true iff P is false

Avoiding ambiguities

$P \wedge Q \vee R$

- Is this a well-formed formula? Yes
- what does it mean?
- $(P \wedge Q) \vee R$?
- $P \wedge (Q \vee R)$?
- We don't know.

In general use parentheses to avoid ambiguities.
Use either $(P \wedge Q) \vee R$ or $P \wedge (Q \vee R)$.

Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \rightarrow$.
For example, $\neg P \vee Q$ means $(\neg P) \vee Q$.

Associativity: all operators are right associative
For example, $P \vee Q \vee R$ means $P \vee (Q \vee R)$.

However use parentheses around compound formulas for clarity.

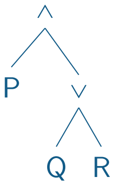
Parse Trees

Parentheses help clarify how formulas are derived given the propositional logic's grammar:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

The parse tree for $(P \wedge Q) \vee R$ is:

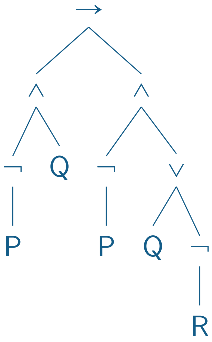
while the parse tree for $P \wedge (Q \vee R)$ is:



Leaves are atomic propositions and the other nodes are connectives.

Parse Trees

What it the parse tree for: $(\neg P \wedge Q) \rightarrow (\neg P \wedge (Q \vee \neg R))$?



Scope and Main connective

Scope of a connective

- The connective itself, plus what it connects
- That is, the sub-tree of the parse tree rooted at the connective
- The scope of $\wedge$ in $(P \wedge Q) \vee R$ is $P \wedge Q$

Main connective of a formula

- The connective whose scope is the whole formula
- That is, the root node of the parse tree
- The main connective of $(P \wedge Q) \vee R$ is $\vee$

Arguments in Propositional Logic

Example argument

1. If John is at home, then his television is on
2. His television is not on
3. Therefore, John is not at home

Identify atomic propositions:

- $p =$"John is at home"
- $q =$"John's television is on"

How do we write this argument in propositional logic?

- Premise 1: $p \to q$
- Premise 2: $\neg q$
- Conclusion: $\neg p$

Arguments in Propositional Logic

Example argument

- Premise 1: $p \to q$
- Premise 2: $\neg q$
- Conclusion: $\neg p$

Notation: written as a **sequent**

- $p \to q, \neg q \vdash \neg p$
- i.e., set of premises separated by commas, then a **turnstile** followed by the conclusion.
- Recall that premises and conclusions are both formulas.
- A sequent is **valid** if the argument has been proven, i.e., if the conclusion is true assuming that the premises are true.

Proofs in Propositional Logic

For **formal proofs** we need two things

1. A **formal** language
   - for representing propositions, arguments
   - here we are using propositional logic
2. A **proof** theory
   - to prove ("infer", "deduce") whether an argument is valid
   - we'll see several different approaches in this module
   - for now (next few lectures): Natural Deduction

Natural Deduction

Natural Deduction

- "natural" style of constructing a proof (like a human would)
- syntactic (rather than semantic) proof method
- proofs are constructed by applying inference rules

Basic idea to prove an argument is valid:

- start with the premises (we can assume these are true)
- repeatedly apply inference rules (which "preserve truth")
- until we have inferred the conclusion

What are inference rules?

Inference rules are the tools we have/are allowed to use

Example of an inference rule:

$$\frac{A \quad B}{A \wedge B}\ [\wedge I]$$

Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

Some simple inference rules

And-introduction:

$$\frac{A \quad B}{A \wedge B} \; [\wedge I]$$

Implication-elimination

$$\frac{A \quad A \rightarrow B}{B} \; [\rightarrow E]$$

False-elimination

$$\frac{\bot}{A} \; [\bot E]$$

True-introduction

$$\frac{}{\top} \; [\top I]$$

A simple proof

Negation-elimination, i.e., both A and $\neg A$ cannot be true at same time

Formally, want to prove $A, \neg A \ \vdash \perp$

A **proof** is a tree of instances of inference rules.

Assuming that $\neg A$ is defined as $A \rightarrow \perp$, a proof of the above sequent (or argument) is:

$$\frac{A \quad \neg A}{\perp} \ [\rightarrow E]$$

Another simple proof

Given three hypotheses A, B, C, how can we prove $(A \wedge B) \wedge (A \wedge C)$?

Here is a proof:

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\,[\wedge I] \qquad \dfrac{A \quad C}{A \wedge C}\,[\wedge I]}{(A \wedge B) \wedge (A \wedge C)}\,[\wedge I]$$

The rule used at each step is **and-introduction**, i.e., $\wedge I$

Conclusion

What did we cover today?

- Syntax of propositional logic
- Informal semantics of propositional logic formulas
- Simple Natural Deduction proofs

Next time?

- Natural Deduction