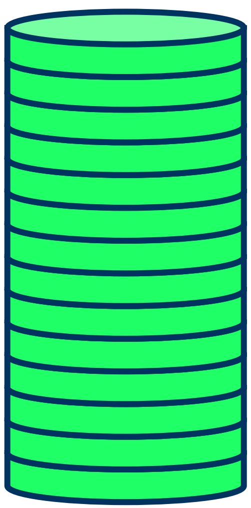
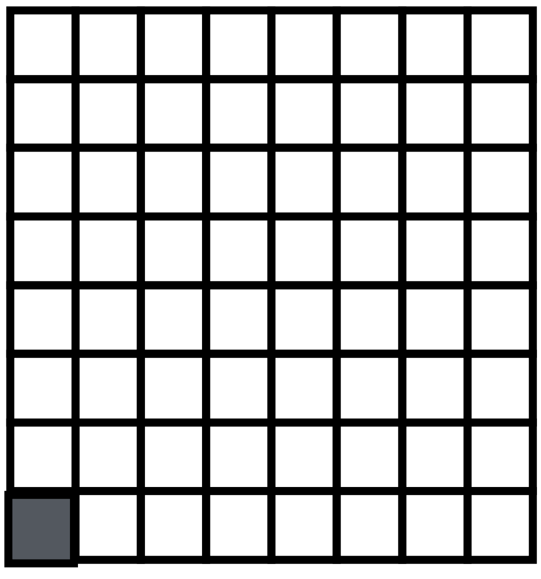
Automated Game Playing by (Intelligent) Machines: Part III





Combining Games Together

Nim is a Sum of Games

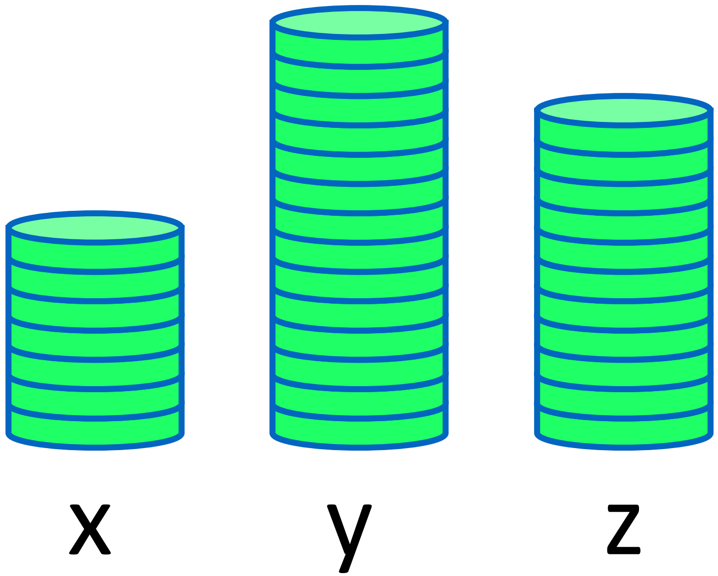
3-pile nim is like the “sum” (or combination) of three 1-pile nim games.

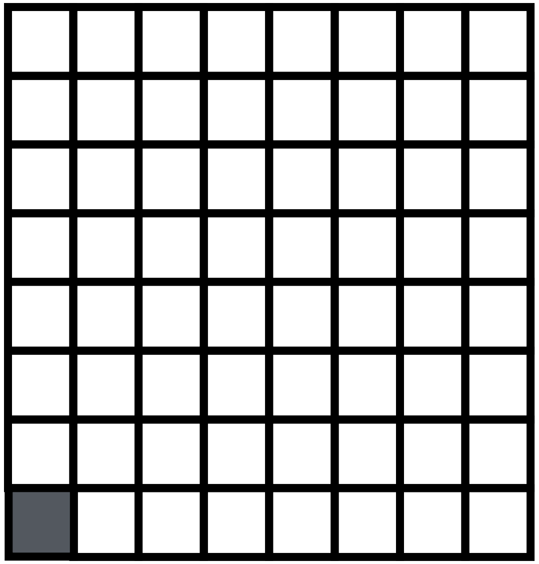
At each step, pick one of the three games (that is non-terminal) and make a move in that game.

The position is terminal iff all 3 games are in terminal positions.

Other Combinations

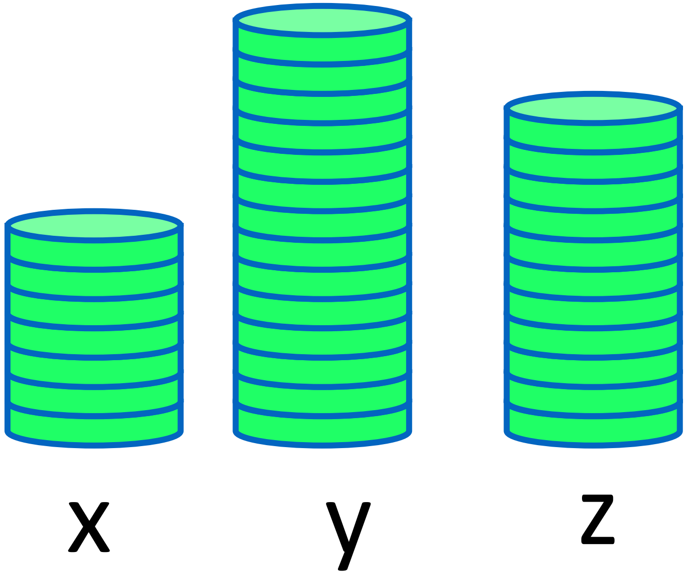
I can create a new game by combining Nim and Chomp

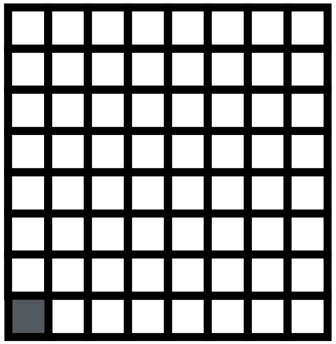


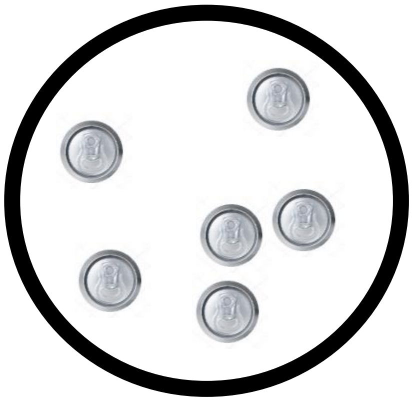


A player can make a move in any of these. If no move exists (both games terminal), next player loses

Combine Even More?







Combine Nim, Chomp and Soda can game? What is the winning strategy?

Why combine? Sometimes even a single game can be seen as a combination of multiple games (like 3 pile Nim is just a combination of 1 pile Nim)

Sum (or Combination) of games

A and B are games. Game A+B is a new game where the allowed moves are to pick one of the two games A or B (that is in a non-terminal position) and make a legal move in that game.

Terminal positions of A+B ≡ Positions that terminal in *both* A and B.

Note: The sum operator is

- commutative [A+B = B+A]
- associative [(A+B) + C = A + (B+C)]

Analyzing Games

We assign a number to positions of any (normal, impartial) game.

This number is called the Nimber of the position.

Computed using the MEX function.

The MEX

The “MEX” of a finite set of natural numbers is the Minimum EXcluded element.

MEX {0, 1, 2, 4, 5, 6} = 3

MEX {1, 3, 5, 7, 9} = 0

MEX {} = 0

Definition of Nimber

The Nimber of a game (position) G (denoted N(G)) is defined inductively as follows:

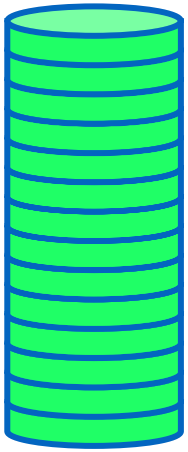
$N(G) = 0$ if G is terminal

$N(G) = \text{MEX}\{N(G_1), N(G_2), \dots N(G_n)\}$

where $G_1, G_2, \dots G_n$ are the successor positions of G (i.e. the positions resulting from all the allowed moves)

Example: Nimber for 1-pile Nim Game

Let P_k denote the position that is a pile of k chips in the game of (one-pile) Nim.



k chips

What is the value of $N(P_k)$?

Theorem: Nimber of a position with a single pile of k chips, $N(P_k) = k$

Proof: Use induction.

Base case is when k=0. $N(P_0) = 0$. (Terminal position).

$N(P_1) = MEX(N(P_0)) = 1$ (just for intuition)

When $k>0$ the set of possible positions after a legal move is P_{k-1}, P_{k-2}, … P_0.

By induction these positions have Nimbers k-1, k-2, … 0.

The MEX of these is k.

Key Theorem (No Proof)

Theorem: Any game/position G is a P-position *if and only if* N(G)=0. (for any game)

Also Keep in Mind: Given G, next player can move to a position G' with any desired value of N(G') < N(G)

How to Win Any Game

Make sure you move the game in a state G s.t. N(G)=0.

1) For atomic games: analyze, try to compute Nimber for any position

2) For sum of games: use the Nimber theorem!

The Nimber Theorem

Theorem: Let A and B be two impartial normal games. Then:

$$N(A+B) = N(A) \oplus N(B)$$

Beauty of Nimbers

They *completely capture* what you need to know about a game in order to add it to another game.

This often allows you to compute winning strategies, and can speed up game search.

Application to Nim

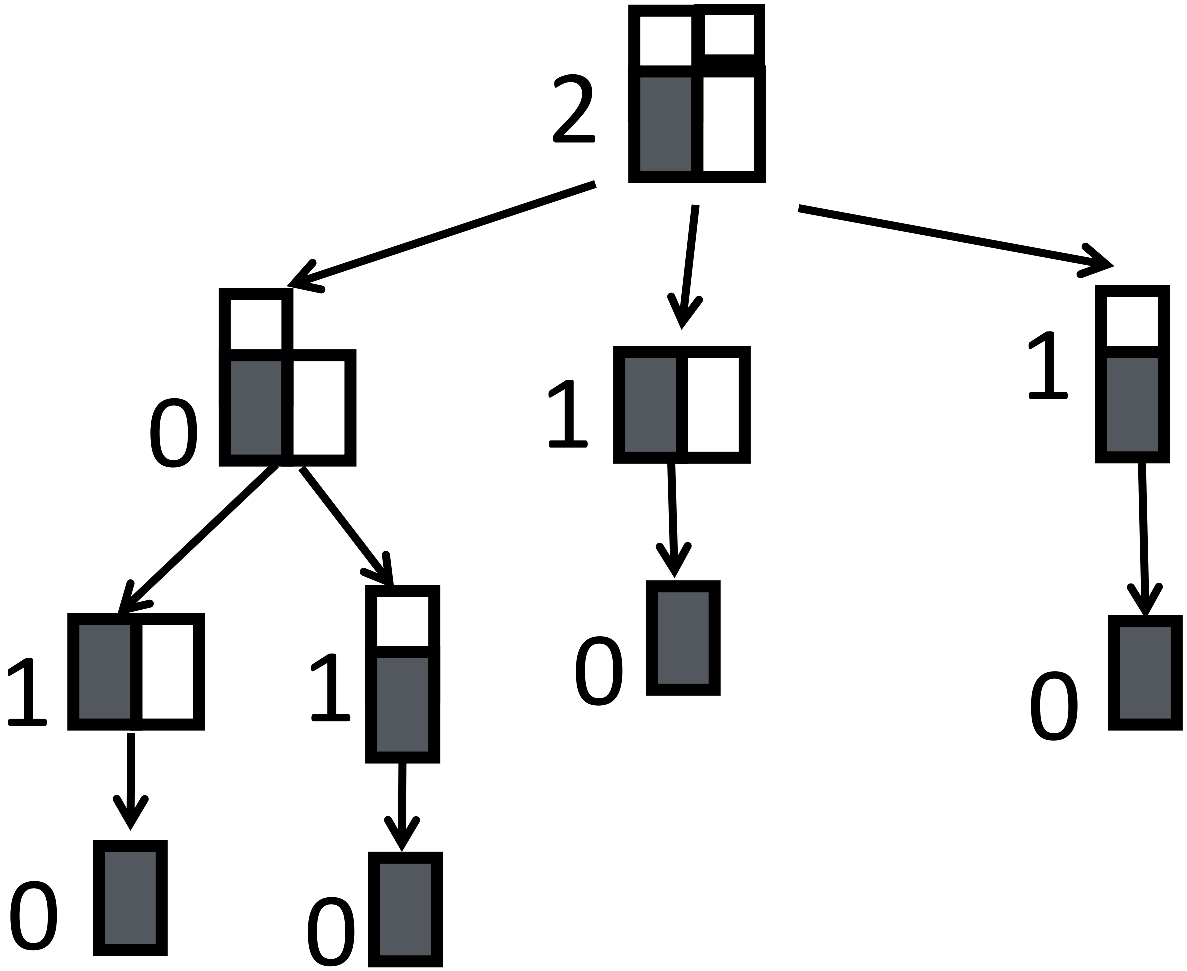
Game of Nim is just the sum of several games. We have shown that for pile of size a, $N(P_a) = a$.

If the piles are of size a, b, and c, the nimber of this position, by the Nimber Theorem, is just $a \oplus b \oplus c$.

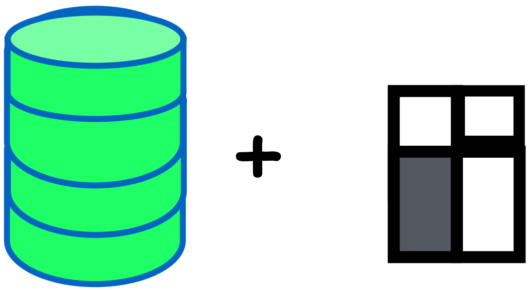
So it's a P-position if and only if $a \oplus b \oplus c = 0$, which is what Bouton proved.

Application to Chomp

What is the nimber of this chomp game?



What if we add this to a nim pile of size 4?



$$4 \oplus 2 = 6$$

Is this an N-position or a P-position?

Nimber $\neq 0$. So it is an N-position.
How do you win?

If we remove two chips from the nim pile, then the nimber is 0, giving a P-position. This is the unique winning move in this position.

The Game of Dawson’s Kayles

Start with a row of n bowling pins:

A move consists of knocking down 2 neighboring pins.

The last player to move wins.

Note: an isolated pin is stuck and can never be removed.

How do we analyze this game?

Given a row of n pins: there are n-1 possible moves [(x,y) denotes x pins to left and y pins to right of the two pins that are knocked down]

$(0,n-2), (1,n-3), \ldots, (n-3,1), (n-2,0)$

So the nimber of a row of n pins, denoted N(n) is:

0 if $n=0$ or if $n=1$

Else $\mathrm{MEX}\{N(0)\oplus N(n-2), N(1)\oplus N(n-3), \ldots\ N(n-2)\oplus N(0)\}$

Let's work out some small values.....

n	0	1	2	3	4	5	6	7	8	9	10	11	12
N(n)	0	0	1	1	2	0	3	1	1	0	3	3	2

MEX{ N(0)⊕N(n-2), N(1)⊕N(n-3), … N(n-2)⊕N(0) }

The table has period 34

If going first, start with 2,3,4,6…. Pins!

Read on Your Own: Treblecross Game

Tic-Tac-Toe on a line with only X’s allowed.

First player to form 3-in-a-row wins.

							X								

Read: http://lbv-pc.blogspot.com/2012/07/treblecross.html

What We have Learnt

- Winning strategy for any combination of games
- Even single games can be seen as combination of smaller games
- Nimber of a position represents everything that I need to know about that position
- Makes it much easier to write computer programs to find winning strategies

Questions?