7

Invertibility of Functions

So far we have seen that functions map elements from a set A to elements of a set B. One may question whether you can 'reverse' this process, that is does an 'inverse function' exist in the sense that if $f : A \to B$ maps elements from A to B does there exist a function $g : B \to A$ which maps elements to B so that f and g 'cancel' each other out? We now introduce the definition of an invertible function.

Definition 7.1. A function $f : A \to B$ is said to be *invertible* if there exists a function $g : B \to A$ such that

$$g \circ f = id\big|_A \text{ and } f \circ g = id\big|_B.$$

Such a function g is called the *inverse* of f. We use the notation f^{-1} for the inverse of f.

Example 7.2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is invertible and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{1}{2}x$ is its inverse. Why is this the case? Let us check the definition. We will first show that $(g \circ f)(x) = \mathrm{id}_{\mathbb{R}}(x) = x$. Indeed,

$$(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2} \cdot (2x) = x.$$

We now need to check the other direction, that is $(f \circ g)(x) = \mathrm{id}_{\mathbb{R}}(x) = x$. Indeed,

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x.$$

Therefore f and g satisfy the required properties and so $g = f^{-1}$.

Example 7.3. Please note that <u>both conditions</u> in Definition 7.1 must be satisfied for g to be the inverse of f. For example, let $f : [0, +\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to [0, +\infty)$ be defined by

$$f(x) = \sqrt{x}, \quad \text{and} \quad g(x) = x^2.$$

Then for each $x \in [0, +\infty)$,

$$g(f(x)) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x,$$

that is, $g \circ f = \mathrm{id}_{[0,+\infty)}$. However g is not the inverse of f: indeed, for each $y \in \mathbb{R}$,

$$f(g(y)) = f\left(x^2\right) = \sqrt{y^2} = |y|;$$

in particular, $(f \circ g)(-1) = 1 \neq -1$, so $f \circ g \neq \mathrm{id}_{\mathbb{R}}$.

Exercise 7.4. Let $f : [0, +\infty) \to [0, +\infty)$ and $g : \mathbb{R} \to [0, +\infty)$ be defined by

$$f(x) = \sqrt{x}, \quad \text{and} \quad g(x) = x^2.$$

Show that $f^{-1} = g\big|_{0,+\infty)}$.

In general, for most functions $f : \mathbb{R} \to \mathbb{R}$ which are invertible, there is an easy way to compute the inverse function. We motivate this by the following example.

Example 7.5. Let us consider the equation of the straight line $y = 3x + 6$, so we have a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = y = 3x + 6$. If we want to compute f^{-1} we can interchange the 'roles' played by both x and y. That is, consider the line given by

$$x = 3y + 6.$$

Let us now rearrange for y.

$$\begin{aligned} & x = 3y + 6 \\ \implies & 3y = x - 6 \quad \text{(subtract 6 from both sides)} \\ \implies & y = \frac{1}{3}(x - 6) = \frac{1}{3}x - 2 \quad \text{(divide both sides by 3)}. \end{aligned}$$

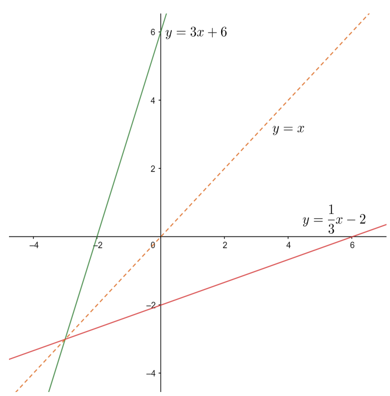

Figure 7.1: Reflecting the graph of $y = f(x)$ about the line $y = x$

If we define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \dfrac{1}{3}x - 2$, we claim that $g = f^{-1}$. As per the definition, to check we need to see that $(f \circ g)(x) = (g \circ f)(x) = x$ for each $x \in \mathbb{R}$. Let us first check $f \circ g$. Indeed,

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3}x - 2\right) = 3\left(\frac{1}{3}x - 2\right) + 6 = (x - 6) + 6 = x.$$

Let us now check $g \circ f$. Indeed,

$$(g \circ f)(x) = g(f(x)) = g(3x+6) = \frac{1}{3}(3x+6) - 2 = (x+2) - 2 = x.$$

Therefore, as claimed, $g = f^{-1}$.

One thing to also notice, in such a case, what we have done is 'reflect' the graph about the line $y = x$ to determine the inverse of the original function, see Figure 7.1.

Example 7.6. Recall the exponential function $\exp : \mathbb{R} \to (0, +\infty)$ defined in Example **??**. Then it can be shown that exp is invertible and its inverse is given by the *natural logarithm* $\ln : (0, +\infty) \to \mathbb{R}$. Recall that we can reflect the graph of a function to find the graph of its inverse. Therefore, the graph of $y = \ln(x)$ is given in the following diagram.

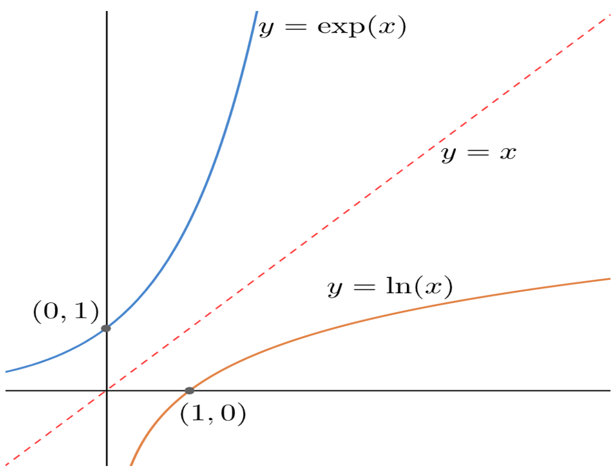

Figure 7.2: Graph showing $y = e^x$ and $y = \ln(x)$

Following the above mentioned properties of exp we can say the following about ln. We can see that $\ln(x) < 0$ for $x \in (0, 1)$ and $\ln(x) \geq 0$ for $x \geq 1$. Also, we can see that $\ln(x)$ grows quite slowly for $x \geq 1$. As ln is the inverse of exp, we also have

$$e^{\ln(x)} = \ln(e^x) = x. \tag{7.1}$$

In particular, taking $x = 1$, we get that

$$\ln(e^1) = \ln(e) = 1.$$