Exercise Sheet 4 - Mathematics

Unassessed exercises

Write out your answers to all exercises and submit via Canvas by next week, Tuesday, 11am. (We will review a sample of answers but not be able to give feedback to everyone.)

Exercise 4.1

Let $A = \{-1, 0, 1\}$ and $B = \{0, 2, 4\}$. List the elements of $A \cup B$, $A \cap B$, $A \setminus B$, and $A \times B$.

Exercise 4.2

List all elements of the set $\{x \in \mathbb{N} \mid x \leq 100\} \cap \{x \in \mathbb{N} \mid x = y^3 \text{ for some } y \in \mathbb{N}\}$.

Exercise 4.3

Find a one-to-one correspondence between $\mathscr{P}\mathbb{N}$ and the set of infinite lists whose entries are either 0 or 1.

Exercise 4.4

The purpose of this exercise is to show that Java (and many other programming languages) provides an implementation of a Boolean algebra. The elements of this Boolean algebra are all possible bit patterns in 32-bit registers.

(a) So how many elements are there?

For the operations $\wedge$, $\vee$, and $\overline{}$, each bit is interpreted as a truth value, exactly as explained in the section on *Boolean circuits* at the end of Chapter 6.4, that is, "0" stands for `false` and "1" stands for `true`. The Boolean algebra operations are then acting "bit-wise", that is, if one arguments is $\mathbf{x} = x_{31}x_{30}\ldots x_1x_0$ and the other is $\mathbf{y} = y_{31}y_{30}\ldots y_1y_0$, then $\mathbf{x} \wedge \mathbf{y}$ is the bit vector

$$(x_{31} \wedge y_{31})\ (x_{30} \wedge y_{30})\ (\ldots\ x_1 \wedge y_1)\ (x_0 \wedge y_0)$$

and analogously for $\vee$ and $\overline{}$.

(b) For

$$\mathbf{x} = 1111\ 1110\ 1101\ 1100\ 1011\ 1010\ 1001\ 1000 \qquad \text{and} \qquad \mathbf{y} = 0000\ 0001\ 0010\ 0011\ 0100\ 0101\ 0110\ 0111$$

compute $\mathbf{x} \wedge \mathbf{y}$, $\mathbf{x} \vee \mathbf{y}$, and $\overline{\mathbf{x}}$

(c) Argue that the Boolean algebra laws are satisfied.

(d) Use the online documentation to find the operator symbols Java uses for $\wedge$, $\vee$, and $\overline{}$.

(e) Can you think of a use for these operators?

Exercise 4.5

Draw the Venn diagram for the term $(A \setminus B) \cup (B \setminus A)$.

We abbreviate $(A \setminus B) \cup (B \setminus A)$ as $A \bigtriangleup B$. Draw the Venn diagram for $(A \bigtriangleup B) \bigtriangleup C$.

What logical operation does it correspond to?

Exercise 4.6

Consider the operation $A \bigtriangleup B$ from the previous exercise.

(a) Define it for any Boolean algebra.

(b) Show that it satisfies $(A \bigtriangleup B) \wedge C = (A \wedge C) \bigtriangleup (B \wedge C)$.
(Advice: Start transforming the right hand side.)

(c) Give an example which shows that $(A \wedge B) \bigtriangleup C = (A \bigtriangleup C) \wedge (B \bigtriangleup C)$ may fail.

(d) Bonus question: Show that a Boolean algebra satisfies the ring laws (Box 16 in Section 2.2) if we use $\bigtriangleup$ for $+$, and $\wedge$ for $\times$. What do you use for 0 and 1?