Mathematical and Logical Foundations of Computer Science

Lecture 10 - Propositional Logic (Wrap-up)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Syntax of propositional logic
- Natural Deduction
- Sequent Calculus
- Classical reasoning
- Semantics
- Equivalences
- Provability/Validity

Syntax & Informal Semantics

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We also introduced four connectives:
- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \to Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \to \bot$

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- "if x is a number then it is even or odd"
  - atom p: "x is a number"
  - atom q: "x is even"
  - atom r: "x is odd"
  - $p \rightarrow q \vee r$

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- "if x is a number then it is even or odd"
    - atom p: "x is a number"
    - atom q: "x is even"
    - atom r: "x is odd"
    - $p \to q \lor r$
- "if x is even then it is not odd"
    - atom p: "x is even"
    - atom q: "x is odd"
    - $p \to \neg q$

Syntax

Example of propositions:

- "if x is a number then it is even or odd"
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    - $p \rightarrow q \vee r$
- "if x is even then it is not odd"
    - atom p: "x is even"
    - atom q: "x is odd"
    - $p \rightarrow \neg q$
- "if $a = b$ and $b = c$ then $a = c$"
    - atom p: "$a = b$"
    - atom q: "$b = c$"
    - atom r: "$a = c$"
    - $(p \wedge q) \rightarrow r$
    - or equivalently: $p \rightarrow q \rightarrow r$

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For example:

- $\neg P \vee Q$ means $(\neg P) \vee Q$
- $P \wedge Q \vee R$ means $(P \wedge Q) \vee R$
- $P \wedge Q \rightarrow Q \wedge P$ means $(P \wedge Q) \rightarrow (Q \wedge P)$

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For example:

- $P \vee Q \vee R$ means $P \vee (Q \vee R)$.
- $P \wedge Q \wedge R$ means $P \wedge (Q \wedge R)$.
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- $P \wedge Q \wedge R$ means $P \wedge (Q \wedge R)$.
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However use parentheses around compound formulas for clarity.

Constructive Natural Deduction

Constructive Natural Deduction rules:

$$\frac{\bot}{A} \ [\bot E] \qquad \frac{}{\top} \ [\top I] \qquad \frac{\overset{\overline{A}^{\ 1}}{\vdots}}{\frac{B}{A \to B}} \ 1 \ [\to I] \qquad \frac{A \to B \quad A}{B} \ [\to E]$$

$$\frac{\overset{\overline{A}^{\ 1}}{\underset{\bot}{\vdots}}}{\neg A} \ 1 \ [\neg I] \qquad \frac{\neg A \quad A}{\bot} \ [\neg E]$$

$$\frac{A}{A \lor B} \ [\lor I_L] \qquad \frac{A}{B \lor A} \ [\lor I_R] \qquad \frac{A \lor B \quad A \to C \quad B \to C}{C} \ [\lor E]$$

$$\frac{A \quad B}{A \land B} \ [\land I] \qquad \frac{A \land B}{B} \ [\land E_R] \qquad \frac{A \land B}{A} \ [\land E_L]$$

Constructive Sequent Calculus

Constructive Sequence Calculus rules:

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} \ [\to L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \ [\to R]$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B} \ [\neg L] \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A} \ [\neg R]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \ [\vee L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \ [\vee R_1] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A} \ [\vee R_2]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \ [\wedge L] \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \ [\wedge R]$$

$$\frac{}{A \vdash A} \ [Id] \qquad \frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A} \ [Cut]$$

$$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C} \ [X] \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \ [W] \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \ [C]$$

Constructive Sequent Calculus

In addition we allow using the following **derived rules**:

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \to B, \Gamma_2 \vdash C} \ [\to L] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B} \ [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C} \ [\vee L] \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C} \ [\wedge L]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B} \ [W] \qquad\qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B} \ [C]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A} \ [Id]$$

All these **derived rules** can be proved/derived using the rules on the previous slide

Classical Reasoning

Classical Natural Deduction includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A} \;\; [LEM] \qquad \frac{\neg\neg A}{A} \;\; [DNE]$$

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Classical sequents are of the form $\Gamma \vdash \Delta$, where Γ and Δ are both lists of formulas

Classical Sequent Calculus (1st version) includes all the Constructive Sequent Calculus rules, plus:

$$\frac{}{\Gamma \vdash A \vee \neg A} \ [LEM] \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} \ [DNE]$$

Classical Reasoning

Classical Sequent Calculus (2nd version) rules:

$$\frac{\Gamma \vdash A, \Delta_1 \quad \Gamma, B \vdash \Delta_2}{\Gamma, A \to B \vdash \Delta_1, \Delta_2} \ [\to L] \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \ [\to R] \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \ [\neg L]$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2} \ [\vee L] \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \ [\vee R] \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \ [\neg R]$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \ [\wedge L] \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2} \ [\wedge R] \qquad \frac{}{A \vdash A} \ [Id]$$

$$\frac{\Gamma_1 \vdash B, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \ [Cut] \qquad \frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta} \ [X_L] \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2} \ [X_R]$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \ [W_L] \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \ [C_L] \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \ [W_R] \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \ [C_R]$$

We also allow using the usual derived rules such as for example

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta_1, A, \Delta_2} \ [Id] \qquad \frac{\Gamma, A \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, A \to B, \Delta_2} \ [\to R]$$

Semantics

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- A is **satisfiable** if there exists a valuation ϕ on atomic propositions such that $\phi(A) = $ **T**
- A is **valid** if $\phi(A) = $ **T** for all possible valuations ϕ

Truth Tables

We can use **truth tables** to check whether propositions are valid:

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

P	Q	$P \rightarrow Q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

A proposition is (semantically) valid if the last column in its truth table only contains **T**

Validity

All three techniques can be used to prove the validity of propositions:

- ‣ a **Natural Deduction** proof (syntactic validity)
- ‣ a **Sequent Calculus** proof (syntactic validity)
- ‣ a **truth table** with only **T** in the last column (semantical validity)

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We saw that:

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- ▸ **iff** A is provable in the **Sequent Calculus**
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This is true about the classical versions of these deduction systems

Logical equivalences

Let $A \leftrightarrow B$ be defined as $(A \rightarrow B) \land (B \rightarrow A)$

- it means that A and B are logically equivalent
- this is called a "bi-implication"
- read as "A if and only if B"

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We will now prove:

- Distributivity of $\wedge$ over $\vee$:
  $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Double negation elimination as an equivalence: $\neg \neg A \leftrightarrow A$

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You can also try proving the distributivity of $\vee$ over $\wedge$:
$(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$

Provability/Validity

Provide a constructive Natural Deduction proof of the following equivalence: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

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Left-to-right implication:

$$
\cfrac{
  \cfrac{
    \cfrac{\overline{A \wedge (B \vee C)}\ 1}{B \vee C}\ [\wedge E_R]
    \quad
    \cfrac{
      \cfrac{\cfrac{\overline{A \wedge (B \vee C)}\ 1}{A}\ [\wedge E_L] \quad \overline{B}\ 2}{A \wedge B}\ [\wedge I]
    }{
      \cfrac{(A \wedge B) \vee (A \wedge C)}{B \to (A \wedge B) \vee (A \wedge C)}\ 2\ [\to I]
    }\ [\vee I_L]
    \quad
    \cfrac{
      \cfrac{\cfrac{\overline{A \wedge (B \vee C)}\ 1}{A}\ [\wedge E_L] \quad \overline{C}\ 3}{A \wedge C}\ [\wedge I]
    }{
      \cfrac{(A \wedge B) \vee (A \wedge C)}{C \to (A \wedge B) \vee (A \wedge C)}\ 3\ [\to I]
    }\ [\vee I_R]
  }{(A \wedge B) \vee (A \wedge C)}\ [\vee E]
}{(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))}\ 1\ [\to I]
$$

Provability/Validity

Right-to-left implication:

$$
\cfrac{
  \cfrac{
    \cfrac{\overline{(A \wedge B) \vee (A \wedge C)}\ 1 \quad \Pi_1 \quad \Pi_2}{A}\ [\vee E]
    \quad
    \cfrac{\overline{(A \wedge B) \vee (A \wedge C)}\ 1 \quad \Pi_3 \quad \Pi_4}{B \vee C}\ [\vee E]
  }{A \wedge (B \vee C)}\ [\wedge I]
}{((A \wedge B) \vee (A \wedge C)) \to (A \wedge (B \vee C))}\ 1\ [\to I]
$$

where Π_1 is: where Π_2 is:

$$
\cfrac{\cfrac{\overline{A \wedge B}\ 2}{A}\ [\wedge E_L]}{(A \wedge B) \to A}\ 2\ [\to I]
\qquad
\cfrac{\cfrac{\overline{A \wedge C}\ 3}{A}\ [\wedge E_L]}{(A \wedge C) \to A}\ 3\ [\to I]
$$

where Π_3 is: where Π_4 is:

$$
\cfrac{\cfrac{\cfrac{\overline{A \wedge B}\ 4}{B}\ [\wedge E_R]}{B \vee C}\ [\vee I_L]}{(A \wedge B) \to (B \vee C)}\ 4\ [\to I]
\qquad
\cfrac{\cfrac{\cfrac{\overline{A \wedge C}\ 5}{C}\ [\wedge E_R]}{B \vee C}\ [\vee I_R]}{(A \wedge C) \to (B \vee C)}\ 5\ [\to I]
$$

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Provide a constructive Sequent Calculus proof of the following
equivalence: $(A \land (B \lor C)) \leftrightarrow ((A \land B) \lor (A \land C))$

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$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{\cfrac{}{A, B \vdash A}\,[Id] \quad \cfrac{}{A, B \vdash B}\,[Id]}{A, B \vdash A \wedge B}\,[\wedge R]
    }{A, B \vdash (A \wedge B) \vee (A \wedge C)}\,[\vee R_1]
    \quad
    \cfrac{
      \cfrac{\cfrac{}{A, C \vdash A}\,[Id] \quad \cfrac{}{A, C \vdash C}\,[Id]}{A, C \vdash A \wedge C}\,[\wedge R]
    }{A, C \vdash (A \wedge B) \vee (A \wedge C)}\,[\vee R_2]
  }{
    \cfrac{
      \cfrac{A, B \vee C \vdash (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}\,[\wedge L]
    }{\vdash (A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))}\,[\to R]
  }{}\,[\vee L]
}{}
$$

Provability/Validity

Right-to-left implication:

$$
\cfrac{
  \cfrac{
    \cfrac{}{A, B \vdash A}\ [Id] \qquad
    \cfrac{\cfrac{}{A, B \vdash B}\ [Id]}{A, B \vdash B \vee C}\ [\vee R_1]
  }{
    \cfrac{A, B \vdash A \wedge (B \vee C)}{A \wedge B \vdash A \wedge (B \vee C)}\ [\wedge L]
  }\ [\wedge R]
  \qquad
  \cfrac{
    \cfrac{}{A, C \vdash A}\ [Id] \qquad
    \cfrac{\cfrac{}{A, C \vdash C}\ [Id]}{A, C \vdash B \vee C}\ [\vee R_2]
  }{
    \cfrac{A, C \vdash A \wedge (B \vee C)}{A \wedge C \vdash A \wedge (B \vee C)}\ [\wedge L]
  }\ [\wedge R]
}{
  \cfrac{(A \wedge B) \vee (A \wedge C) \vdash A \wedge (B \vee C)}{\vdash ((A \wedge B) \vee (A \wedge C)) \to (A \wedge (B \vee C))}\ [\to R]
}\ [\vee L]
$$

Provability/Validity

Prove that $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$ is valid using a truth table

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A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge C)$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

The 5th and last columns are identical, so the two formulas are equivalent

Provability/Validity

Provide a classical Natural Deduction proof of the following equivalence: $\neg\neg A \leftrightarrow A$

Provability/Validity

Provide a classical Natural Deduction proof of the following equivalence: $\neg\neg A \leftrightarrow A$

$$
\cfrac{
\cfrac{
\cfrac{\overline{}\ 1}{\cfrac{\neg\neg A}{A}\ [DNE]}
}{\neg\neg A \to A}\ 1\ [\to I]
\quad
\cfrac{
\cfrac{
\cfrac{\overline{}\ 3 \quad \overline{}\ 2}{\cfrac{\neg A \quad A}{\bot}\ [\neg E]}
}{\cfrac{\neg\neg A}{}\ 3\ [\neg I]}
}{A \to \neg\neg A}\ 2\ [\to I]
}{\neg\neg A \leftrightarrow A}\ [\wedge I]
$$

Provability/Validity

Provide a classical Sequent Calculus (1st version) proof of the following equivalence: $\neg\neg A \leftrightarrow A$

Provability/Validity

Provide a classical Sequent Calculus (1st version) proof of the
following equivalence: $\neg\neg A \leftrightarrow A$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{}{\neg\neg A \vdash \neg\neg A} \; [Id]
    }{\neg\neg A \vdash A} \; [DNE]
  }{\vdash \neg\neg A \to A} \; [\to R]
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{}{A \vdash A} \; [Id]
      }{A, \neg A \vdash \bot} \; [\neg L]
    }{A \vdash \neg\neg A} \; [\neg R]
  }{\vdash A \to \neg\neg A} \; [\to R]
}{\vdash \neg\neg A \leftrightarrow A} \; [\wedge R]
$$

Provability/Validity

Provide a classical Sequent Calculus (2nd version) proof of the following equivalence: $\neg\neg A \leftrightarrow A$

Provability/Validity

Provide a classical Sequent Calculus (2nd version) proof of the following equivalence: $\neg\neg A \leftrightarrow A$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{\overline{A \vdash A}\;[Id]}{\vdash \neg A, A}\;[\neg R]
    }{\neg\neg A \vdash A}\;[\neg L]
  }{\vdash \neg\neg A \to A}\;[\to R]
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{\overline{A \vdash A}\;[Id]}{A, \neg A \vdash}\;[\neg L]
    }{A \vdash \neg\neg A}\;[\neg R]
  }{\vdash A \to \neg\neg A}\;[\to R]
}{\vdash \neg\neg A \leftrightarrow A}\;[\wedge R]
$$

Provability/Validity

Prove that $\neg\neg A \leftrightarrow A$ is valid using a truth table

Provability/Validity

Prove that $\neg\neg A \leftrightarrow A$ is valid using a truth table

A	$\neg A$	$\neg\neg A$
T	**F**	**T**
F	**T**	**F**

The 1st and last columns are identical, so the two formulas are equivalent

Conclusion

What did we cover today?

- Syntax of propositional logic
- Natural Deduction
- Sequent Calculus
- Classical reasoning
- Semantics
- Equivalences
- Provability/Validity

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What did we cover today?

- ▸ Syntax of propositional logic
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Next time?

- ▸ Predicate logic (syntax)