Mathematical and Logical Foundations of Computer Science

Lecture 5 - Propositional Logic (Sequent Calculus)

Vincent Rahli

(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus rules
- Sequent Calculus proofs

See Section 5 in "Proof and Types"

https://www.paultaylor.eu/stable/prot.pdf

Recap: Propositional logic syntax

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Two special atoms:

- $\top$ which stands for True
- $\bot$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \bot$

Recap: Natural deduction

Framework

- "natural" style of constructing a proof
- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion

Two key points:

- Can work both forwards and backwards
- Natural doesn't mean there is unique proof

Introduced by **Gentzen** in 1934
and further studied by **Prawitz** in 1965.

Recap: Introduction & Elimination rules

Rules for $\rightarrow$ (implication)

- implication-introduction

$$\dfrac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \rightarrow B}\ 1\ [\rightarrow I]$$

- implication-elimination

$$\dfrac{A \rightarrow B \qquad A}{B}\ [\rightarrow E]$$

Forward & backward reasoning

Prove the following:

$$(P \wedge Q) \rightarrow R \quad \vdash \quad P \rightarrow (Q \rightarrow R)$$

Here is a proof (starting backward):

$$\dfrac{\dfrac{\dfrac{(P \wedge Q) \rightarrow R \qquad \dfrac{\overline{P}^{\,1} \qquad \overline{Q}^{\,2}}{P \wedge Q}\,[\wedge I]}{R}\,[\rightarrow E]}{Q \rightarrow R}\,2\ [\rightarrow I]}{P \rightarrow Q \rightarrow R}\,1\ [\rightarrow I]$$

We went backward up to R.

Going forward, it would also have been unclear which rule to apply to R.

Forward & backward reasoning

Derive B from $A \wedge B \wedge C$

Here is a proof (starting backward):

$$\cfrac{\cfrac{A \wedge B \wedge C}{B \wedge C}\ [\wedge E]}{B}\ [\wedge E]$$

It was not clear which rule to use to prove B, which is why we went forward.

Sequent Calculus - History

Gentzen introduced **natural deduction** in 1934 in his attempts to prove the consistency of first order number theory (predicate logic + induction on numbers)

Unfortunately, Gentzen's attempt to prove **the cut elimination theorem** (the Hauptsatz) for natural deduction failed—a key theorem in his consistency proof.

Gentzen then introduced a **Sequent Calculus** for which he proved the Hauptsatz.

Prawitz later (in 1965) succeeded in proving the Hauptsatz directly.

Sequent calculi are often (not always) amenable to proof automation, as they provide proofs enough structure to specify **proof search procedures**.

Here we will see that it allows us proving propositions backward only.

Sequents

The Sequent Calculus has **left/right** rules instead of **elimination/introduction** rules.

We saw that **sequents** can be used to state arguments with premises on the left and the conclusion on the right, e.g.:

$$P \to Q, P \vdash Q$$

We use Γ and Δ for lists of formulas separated by commas.

We will **eliminate** connectives from the **premises** (the **left**) and **introduce** connectives from the **conclusion** (the **right**).

Sequent Calculus vs. Natural Deduction (implication)

Natural Deduction

$$\frac{A \to B \qquad A}{B}\ [\to E]$$

$$\dfrac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I]$$

Sequent Calculus

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ [\to L]$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

Sequent Calculus vs. Natural Deduction (negation)

Natural Deduction

$$\frac{A \qquad \neg A}{\perp}\ [\neg E]$$

$$\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots} \atop \perp}{\neg A}\ 1\ [\neg I]$$

Sequent Calculus

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ [\neg L]$$

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A}\ [\neg R]$$

Sequent Calculus vs. Natural Deduction (disjunction)

Natural Deduction

$$\frac{A}{A \vee B}\ [\vee I_L]$$

$$\frac{A}{B \vee A}\ [\vee I_R]$$

$$\frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

Sequent Calculus

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1]$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ [\vee L]$$

Sequent Calculus vs. Natural Deduction (conjunction)

Natural Deduction

$$\frac{A \quad B}{A \wedge B}\ [\wedge I]$$

$$\frac{A \wedge B}{B}\ [\wedge E_R]$$

$$\frac{A \wedge B}{A}\ [\wedge E_L]$$

Sequent Calculus

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ [\wedge L]$$

Attempt at a proof

How can we prove $A, A \rightarrow B \vdash B$?

$$\frac{A \vdash A \quad A, B \vdash B}{A, A \rightarrow B \vdash B} \; [\rightarrow L]$$

What do we do now?

We need further rules

Identity and structural rules

Identity $$\frac{}{A \vdash A}\ [Id]$$

Exchange $$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X]$$

Weakening $$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W]$$

Contraction $$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ [C]$$

We also add this useful but not necessary rule

Cut: $$\frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ [Cut]$$

2nd attempt at a proof

How can we prove $A, A \to B \vdash B$?

$$
\dfrac{\dfrac{}{A \vdash A}\,[Id] \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,[Id]}{B, A \vdash B}\,[W]}{A, B \vdash B}\,[X]}{A, A \to B \vdash B}\,[\to L]
$$

As the sort of reasoning done in the right branch comes up often, we instead make use of the following **derivable** rule:

$$
\dfrac{}{\Gamma, A, \Delta \vdash A}\,[Id]
$$

Derivable rules

A **derivable** rule such as:

$$\frac{}{\Gamma, A, \Delta \vdash A}\ [Id]$$

is a rule such that the premises are the unproved hypotheses of a proof, and the conclusion is the conclusion of that proof.

The above alternative $[Id]$ rule is derivable by:

- using $[X]$ a number of times to move A to the left of Γ
- using $[W]$ a number of time to remove Γ, Δ
- and finally using the original $[Id]$ rule once

Similarly, such **alternative left rules** are also derivable:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \land B, \Delta \vdash C}\ [\land L]$$

Example of a Sequent Calculus proof

Provide a Sequent Calculus proof of the following:

$$(P \wedge Q) \to R \vdash P \to (Q \to R)$$

Here is a proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q \vdash P}\,[Id] \quad \dfrac{}{P, Q \vdash Q}\,[Id]}{P, Q \vdash P \wedge Q}\,[\wedge R] \quad \dfrac{}{R, P, Q \vdash R}\,[Id]}{(P \wedge Q) \to R, P, Q \vdash R}\,[\to L]}{\dfrac{(P \wedge Q) \to R, P \vdash Q \to R}{(P \wedge Q) \to R \vdash P \to (Q \to R)}\,[\to R]}\,[\to R]
$$

Note the use of derived rules!

Another example of a Sequent Calculus proof

Provide a Sequent Calculus proof of the following:

$$\neg A \vee B, A \vdash B$$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\ [Id]}{\neg A, A \vdash B}\ [\neg L] \qquad \dfrac{}{B, A \vdash B}\ [Id]}{\neg A \vee B, A \vdash B}\ [\vee L]$$

Sequent Calculus & Natural Deduction

Theorem: The following **correspondence** holds:

- Given a **Sequent Calculus proof** of $\Gamma \vdash A$, one can derive a **natural deduction proof** of A under the hypotheses in Γ.
- Given a **natural deduction proof** of A under the hypotheses in Γ one can derive a **Sequent Calculus proof** of $\Gamma \vdash A$.

Conclusion

What did we cover today?

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus rules
- Sequent Calculus proofs

Next time?

- Sequent Calculus & Natural Deduction