Exercise Sheet 9 - Solutions
Predicate Logic – Sequent Calculus & Semantics

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1. Here is a Sequent Calculus proof of $(\exists x.p(x)) \to (\forall x.\forall y.p(x) \to q(x,y)) \to (\exists x.\forall y.q(x,y))$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$p(x) \vdash p(x)$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$p(x), q(x,y) \vdash q(x,y)$}
\RightLabel{$[\to L]$}
\BinaryInfC{$p(x), p(x) \to q(x,y) \vdash q(x,y)$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$p(x), \forall y.p(x) \to q(x,y) \vdash q(x,y)$}
\RightLabel{$[\forall R]$}
\UnaryInfC{$p(x), \forall y.p(x) \to q(x,y) \vdash \forall y.q(x,y)$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$p(x), \forall x.\forall y.p(x) \to q(x,y) \vdash \forall y.q(x,y)$}
\RightLabel{$[\exists R]$}
\UnaryInfC{$p(x), \forall x.\forall y.p(x) \to q(x,y) \vdash \exists x.\forall y.q(x,y)$}
\RightLabel{$[\exists L]$}
\UnaryInfC{$\exists x.p(x), \forall x.\forall y.p(x) \to q(x,y) \vdash \exists x.\forall y.q(x,y)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$\exists x.p(x) \vdash (\forall x.\forall y.p(x) \to q(x,y)) \to (\exists x.\forall y.q(x,y))$}
\RightLabel{$[\to R]$}
\UnaryInfC{$\vdash (\exists x.p(x)) \to (\forall x.\forall y.p(x) \to q(x,y)) \to (\exists x.\forall y.q(x,y))$}
\end{prooftree}
$$

2. Here is a proof of $A_1, A_2, A_3, A_4 \vdash C$:

$$
\begin{prooftree}
\AxiomC{Π_1}
\AxiomC{Π_2}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A_1, A_2, A_3, \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\to L]$}
\BinaryInfC{$A_1, A_2, A_3, \texttt{pi1}(\texttt{pair}(y,x)) = y \to \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\to L]$}
\BinaryInfC{$A_1, A_2, A_3, \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x)) \to \texttt{pi1}(\texttt{pair}(y,x)) = y \to \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_2, A_3, \forall z.\texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x)) \to \texttt{pi1}(\texttt{pair}(y,x)) = z \to \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = z \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_2, A_3, \forall w.\forall z.\texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = w \to w = z \to \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = z \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_2, A_3, A_4 \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\forall R]$}
\UnaryInfC{$A_1, A_2, A_3, A_4 \vdash \forall y.\texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$}
\RightLabel{$[\forall R]$}
\UnaryInfC{$A_1, A_2, A_3, A_4 \vdash C$}
\end{prooftree}
$$

where Π_1 is

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$\texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x), A_3 \vdash \texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x)$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$\forall y.\texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x), A_3 \vdash \texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x)$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_3 \vdash \texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x)$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A_1, \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x)), A_3 \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x))$}
\RightLabel{$[\to L]$}
\BinaryInfC{$A_1, \texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x) \to \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x)), A_3 \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x))$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, \forall w.\texttt{swap}(\texttt{pair}(x,y)) = w \to \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(w), A_3 \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x))$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_2, A_3 \vdash \texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = \texttt{pi1}(\texttt{pair}(y,x))$}
\end{prooftree}
$$

where Π_2 is

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A_1, A_2, \texttt{pi1}(\texttt{pair}(y,x)) = y \vdash \texttt{pi1}(\texttt{pair}(y,x)) = y$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_2, \forall w.\texttt{pi1}(\texttt{pair}(y,w)) = y \vdash \texttt{pi1}(\texttt{pair}(y,x)) = y$}
\RightLabel{$[\forall L]$}
\UnaryInfC{$A_1, A_2, A_3 \vdash \texttt{pi1}(\texttt{pair}(y,x)) = y$}
\end{prooftree}
$$

3.
- $\vDash_{M,\cdot} \forall x.\mathtt{even}(x) \to \exists y.\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, $\vDash_{M,x\mapsto n} \mathtt{even}(x) \to \exists y.\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, if $\vDash_{M,x\mapsto n} \mathtt{even}(x)$ then $\vDash_{M,x\mapsto n} \exists y.\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, if $\langle [\![x]\!]^M_{x\mapsto n}\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$ then $\vDash_{M,x\mapsto n} \exists y.\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, if $\langle n\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$ then $\vDash_{M,x\mapsto n} \exists y.\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $\vDash_{M,x\mapsto n} \exists y.\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then there exists a $m \in \mathbb{N}$ such that $\vDash_{M,x\mapsto n,y\mapsto m}$ $\mathtt{odd}(y) \land y > x$
- iff for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then there exists a $m \in \mathbb{N}$ such that $\vDash_{M,x\mapsto n,y\mapsto m} \mathtt{odd}(y)$ and $\vDash_{M,x\mapsto n,y\mapsto m} y > x$
- iff for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then there exists a $m \in \mathbb{N}$ such that $\langle [\![y]\!]^M_{x\mapsto n,y\mapsto m}\rangle \in \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}$ and $\langle [\![y]\!]^M_{x\mapsto n,y\mapsto m}, [\![x]\!]^M_{x\mapsto n,y\mapsto m}\rangle \in \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \dots\}$
- iff for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then there exists a $m \in \mathbb{N}$ such that $\langle m\rangle \in \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}$ and $\langle m, n\rangle \in \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \dots\}$
- iff for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then there exists a $m \in \mathbb{N}$ such that $m \in \{1, 3, 5, \dots\}$ and $\langle m, n\rangle \in \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \dots\}$
- True, because given $n \in \{0, 2, 4, \dots\}$, we can pick m to be $n + 1$, which is in $\{1, 3, 5, \dots\}$, and which also satisfy $\langle n + 1, n\rangle \in \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \dots\}$

4.
- $\neg \vDash_{M,\cdot} \forall x.\mathtt{even}(x) \to \mathtt{even}(\mathtt{succ}(x))$
- iff it is not true that for all $n \in \mathbb{N}$, $\vDash_{M,x\mapsto n} \mathtt{even}(x) \to \mathtt{even}(\mathtt{succ}(x))$
- iff it is not true that for all $n \in \mathbb{N}$, if $\vDash_{M,x\mapsto n} \mathtt{even}(x)$ then $\vDash_{M,x\mapsto n} \mathtt{even}(\mathtt{succ}(x))$
- iff it is not true that for all $n \in \mathbb{N}$, if $\langle [\![x]\!]^M_{x\mapsto n}\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$ then $\vDash_{M,x\mapsto n} \mathtt{even}(\mathtt{succ}(x))$
- iff it is not true that for all $n \in \mathbb{N}$, if $\langle n\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$ then $\vDash_{M,x\mapsto n} \mathtt{even}(\mathtt{succ}(x))$
- iff it is not true that for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $\vDash_{M,x\mapsto n} \mathtt{even}(\mathtt{succ}(x))$
- iff it is not true that for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $\langle [\![\mathtt{succ}(x)]\!]^M_{x\mapsto n}\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff it is not true that for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $\langle +1([\![x]\!]^M_{x\mapsto n})\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff it is not true that for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $\langle +1(n)\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff it is not true that for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $\langle n + 1\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff it is not true that for all $n \in \mathbb{N}$, if $n \in \{0, 2, 4, \dots\}$ then $n + 1 \in \{0, 2, 4, \dots\}$
- True, because for example $0 + 1$ is not in $\{0, 2, 4, \dots\}$