12 The inner product

12.1 Equations in analytic geometry

A line from a linear equation. We start with an example. Consider the linear equation

$$x_1 - 2x_2 = 3$$

According to Chapter 10, the solutions are given by

$$\begin{array}{rcl} x_2 & : & \text{choose freely} \\ x_1 & = & 3 + 2x_2 \end{array}$$

We write the possible solutions in vector form:

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and we see that the solutions all lie on a line. This is true in general and worth highlighting:

Theorem

The solution space of a linear equation in two unknowns is a line in 2D.

A plane from a linear equation. Again, we start with an example. Consider the linear equation

$$x_1 - 2x_2 + 3x_3 = 5$$

The solutions are given by

$$\begin{array}{rcl} x_3 & : & \text{choose freely} \\ x_2 & : & \text{choose freely} \\ x_1 & = & 5 + 2x_2 - 3x_3 \end{array}$$

and in vector form we get:

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Theorem

The solution space of a linear equation in three unknowns is a plane in 3D.

Intersection tasks revisited. Intersection tasks become very easy when one of the geometric entities is given in the parametric representation and the other as an equation. We illustrate with an example, intersecting a line with a plane in 3D:

$$\begin{array}{rl} \text{line:} & X = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \\ \text{plane:} & x_1 - 2x_2 + 3x_3 = 5 \end{array}$$

We substitute the line into the equation, using the fact that the coordinates of points on the line are

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2+s \\ 1 \\ 1-3s \end{pmatrix}$$

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The intersection point A has coordinates

We could ask why this calculation is so much simpler than what we did in Chapter 11, where we had to use Gaussian elimination. One way to understand this is to remember that a parametric representation *generates points*. As a picture:

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In contrast, an equation *tests* a given point:

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If we intersect a geometric object given in a parametric representation with one given by an equation, then these two fit together perfectly to give us a condition on the parameter s:

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12.2 Translating between the parametric and equational representations.

In the last item we have seen that the equational representation is computationally very helpful, but how does one get it when the geometric object is given in parametric form? This can be answered in complete generality.

Lines. If we are given the parametric representation of a line in 2D

$$X = P + s \cdot \vec{v} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and we are looking for an equation

$$ax_1 + bx_2 = d$$

which describes the same line, then all we need to do is set

$$\begin{aligned} a &= -v_2 \\ b &= v_1 \\ d &= ap_1 + bp_2 \end{aligned}$$

Let's test that this formula works:

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Planes. If we are given the parametric representation of a plane in 3D

$$X = P + s \cdot \vec{v} + t \cdot \vec{w} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + t \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

and we are looking for an equation

$$ax_1 + bx_2 + cx_3 = d$$

which describes the same plane, then all we need to do is set

$$\begin{aligned} a &= v_2w_3 - v_3w_2 \\ b &= v_3w_1 - v_1w_3 \\ c &= v_1w_2 - v_2w_1 \\ d &= ap_1 + bp_2 + cp_3 \end{aligned}$$

For the translation in the other direction I gave examples in items 12.1 and 12.1 already; it is the same as solving the equation.

Practical advice

In the exam I expect you to be able to

- translate between the parametric and the equational representation of a line or a plane;
- solve intersection tasks that involve an equational representation of a line or a plane.

Notes

The formulas in Section 12.2 may seem "magic" and hard to remember. The following may help:

- For the coefficients of the equation of a line, just exchange the two coordinates of the direction vector and change the sign of one of them.
- For the coefficients of the equation of a plane, remember the **determinant**:

  $$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

  which can be read out as "multiply the two entries on the main diagonal and subtract from this the product of the two entries on the other diagonal."

  The three coefficients for the equation of the plane can be written as determinants:

  $$a = \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} \qquad b = -\begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} \qquad c = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

In both cases we get the right-hand side d of the equation by plugging the coordinates of the point P into the equation.

12.3 The geometric interpretation of equational presentations.

Given a linear equation

$$ax_1 + bx_2 + cx_3 = d$$

we assemble the coefficients in a vector:

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Geometrically, this vector is **orthogonal**[12] ("at right angles") to the plane — unlike $\vec{v}$ and $\vec{w}$ from the parametric presentation which lie *inside* the plane. A 2D picture:

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This is called the **normal** or a **normal vector** to the plane.[13] Note that a normal must not be the null vector but otherwise it can be arbitrarily long, and also, it doesn't matter whether it points up or down.

The right-hand side d of the equation also has a geometric interpretation. It is negative if the origin is on the same side of the plane as the normal and positive otherwise. Its value tells us something about the **distance** of the origin from the plane, because

Distance of origin from plane or line

$$\text{distance} = \frac{d}{|\vec{n}|}$$

(Remember that $|\vec{n}|$ is the length of $\vec{n}$ computed as $\sqrt{a^2+b^2+c^2}$.)

From this we see that if we make $\vec{n}$ have length 1 then d gives us exactly the distance; that's why some books insist that a normal should always have length 1.

Note that the expression $d/|\vec{n}|$ could be positive or negative contradicting the everyday use of the word "distance." That is because it is giving us additional information about which side of the plane the origin is located.

In any case, because of this geometric interpretation, the equational presentation is also called the **normal form** of the line (in 2D) or plane (in 3D).

A new notation. The expression $ax_1 + bx_2 + cx_3$ on the left-hand side of an equation (describing a plane) can be viewed as obtaining a *number* from the two *tuples* $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. It is called the **inner product**[14] of $\vec{n}$ and X, and is denoted by $\langle \vec{n}, X \rangle$. Let's highlight the definition:

[12] Another commonly used word for this is **perpendicular**.

[13] The name "normal" is historical and of no geometric significance in itself.

[14] Also called **scalar product** (because numbers are also called scalars) or **dot product** (because it is sometimes written as $\vec{n} \cdot X$.

The inner product of two tuples

$$\text{If } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \text{ then } \langle \vec{v}, \vec{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Note that this is a purely formal definition for us and we don't need to know whether the tuples in question are to be interpreted as vectors or as the coordinates of some point. Let's compute an example:

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$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$$\langle \vec{v}, \vec{w} \rangle =$$

Given this simple definition it is easy to check the following laws (which are analogous to the laws of multiplication of numbers, but note that there can not be an associativity law):

The laws of the inner product

$$\begin{aligned} \langle \vec{0}, \vec{v} \rangle &= 0 \\ \langle \vec{v}, \vec{w} \rangle &= \langle \vec{w}, \vec{v} \rangle \\ \langle \vec{v} + \vec{w}, \vec{x} \rangle &= \langle \vec{v}, \vec{x} \rangle + \langle \vec{w}, \vec{x} \rangle \\ \langle s \cdot \vec{v}, \vec{w} \rangle &= s \times \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

If you know the normal to a plane and a point P on it, then the normal form can be written as

The normal form of a line/plane

$$\langle \vec{n}, X \rangle = \langle \vec{n}, P \rangle$$

where we have used the fact that the three-tuple $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ can be read either as a vector $\vec{x}$ or as the coordinates of a point X.

12.4 Geometric properties of the inner product

Orthogonality test. Two vectors $\vec{v}$ and $\vec{w}$ are orthogonal ("at right angles") to each other exactly if $\langle \vec{v}, \vec{w} \rangle = 0$.

Angles. More generally, we have $\langle \vec{v}, \vec{w} \rangle = |\vec{v}| \times |\vec{w}| \times \cos(\alpha)$ where α is the angle between the vectors $\vec{v}$ and $\vec{w}$. From the formula one can see that $\langle \vec{v}, \vec{w} \rangle$ is positive exactly if the two vectors form an *acute* angle, and negative if that angle is *obtuse*.

Inner product and length. For any vector $\vec{v}$ we have

$$|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

which we can also write as

$$|\vec{v}|^2 = \langle \vec{v}, \vec{v} \rangle$$

Projection property. Given $\vec{n} \neq \vec{0}$ and $\vec{v}$ we can compute the number

$$\frac{\langle \vec{n}, \vec{v} \rangle}{\langle \vec{n}, \vec{n} \rangle}$$

If we stretch $\vec{n}$ by this value, that is, if we compute

$$\frac{\langle \vec{n}, \vec{v} \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n}$$

then we get the **orthogonal projection** of $\vec{v}$ onto the line defined by $\vec{n}$. A picture:

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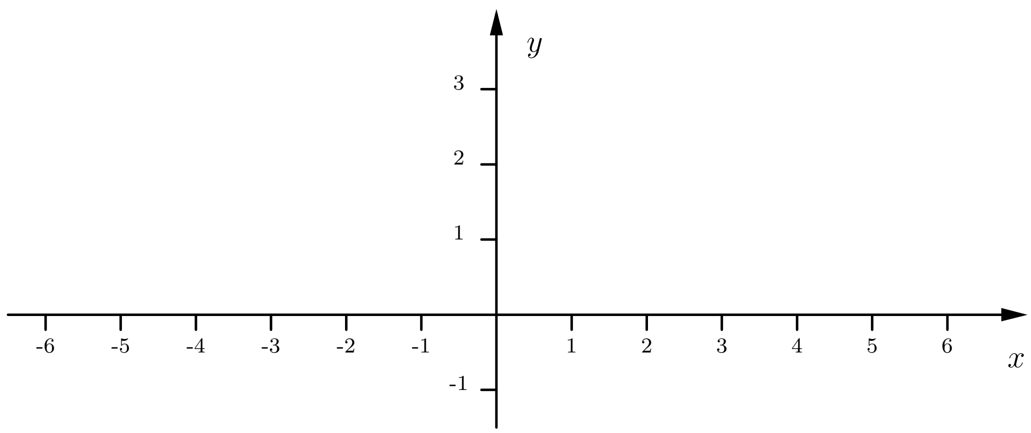

12.5 Graphics-related tasks

We will now use these geometric properties to solve some common graphics-related problems:

Task 1: Computing the distance from a plane. If $\langle \vec{n}, X \rangle = \langle \vec{n}, P \rangle = d$ is the normal form of a plane E and Q a point, then the distance of Q to E (measured in the direction of the normal, that is, orthogonal to the plane) is given by

Distance from plane given by $\langle \vec{n}, X \rangle = d$

$$\frac{d - \langle \vec{n}, Q \rangle}{|\vec{n}|} = \frac{\langle \vec{n}, P \rangle - \langle \vec{n}, Q \rangle}{|\vec{n}|} = \frac{\left\langle \vec{n}, \overrightarrow{QP} \right\rangle}{|\vec{n}|}$$

If this expression returns 0 then Q is a point *in the plane*.

The result of the computation could be negative; if so, then the normal $\vec{n}$ is pointing *towards the side on which Q is located*. This property is used in graphics engines to determine if a point is before a plane (and therefore visible) or behind it (and therefore hidden).

Task 2: Nearest neighbour. If $\langle \vec{n}, X \rangle = \langle \vec{n}, P \rangle = d$ is the normal form of a plane E and Q a point, then we ask which point on E is closest to Q. If you have sufficient spatial awareness then you know that we find this point if we move from Q to E *in the direction of the normal $\vec{n}$.* The formula for it is

Nearest neighbour to Q on plane given by $\langle \vec{n}, X \rangle = d$

$$Q' \quad = \quad Q + \frac{d - \langle \vec{n}, Q \rangle}{|\vec{n}|} \cdot \frac{\vec{n}}{|\vec{n}|} \quad = \quad Q + \frac{d - \langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n}$$

If the plane goes through the origin, the right-hand side d of the normal form is 0 and the formula simplifies to $Q' = Q - \frac{\langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n}$, and you will see the connection to the projection formula in the previous item: To reach Q' we travel back in direction of $\vec{n}$ towards the plane. A 2D picture:

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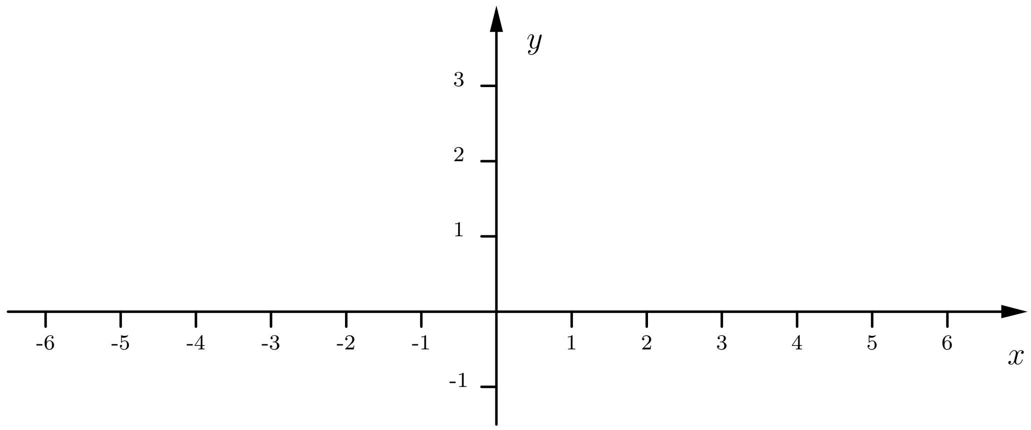

Task 3: Reflecting a point at a plane. With the same notation as before, the formula is very similar to that for the nearest neighbour, except that we go *twice* the distance in direction of the normal:

Point Q reflected on plane given by $\langle \vec{n}, X \rangle = d$

$$Q'' \quad = \quad Q + 2 \times \frac{d - \langle \vec{n}, Q \rangle}{|\vec{n}|} \cdot \frac{\vec{n}}{|\vec{n}|} \quad = \quad Q + 2 \times \frac{d - \langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n}$$

Task 4: Reflecting a line at a plane. This is now very easy: Just pick two (different) points on the line and reflect those according to the previous item. Then connect the two mirror images, and you are done!

Example. Let's do the example in 2D so that you can draw a picture and check my computations; so instead of using a plane for reflection we use a line E, given by

$$x_1 - 2x_2 = 1$$

and the line L that we want to reflect as

$$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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Exercises

1. Compute the normal form of the line $X = P + s \cdot \vec{v} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + s \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.
   Draw a picture in a coordinate system that shows P, $\vec{v}$, and the normal $\vec{n}$.

2. Given the following three points

$$P = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad Q = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad R = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

   find the parametric representation of the plane determined by them, and convert this into the normal form.

Practical advice

In the exam I expect you to be able to use the technology of this chapter to

- sketch the position of a line given in normal form in a coordinate system;
- determine whether two vectors are at right angles to each other;

- determine whether two points are on the same side of a plane or on different sides;
- compute the distance of a point to a plane;
- compute the nearest neighbour of a point on a plane;
- reflect a point at a plane;
- reflect a line at a plane;
- solve a *new* geometric task similar to the ones above.

Notes

- For the exam you need to remember the formula for the inner product of two vectors (given in Section 12.3 above).
- You also need to remember the orthogonality test that the inner product provides.
- You *don't* need to memorise the formulas for distance, nearest neighbour, or reflection; they will be given on the exam paper.