6

Common Functions

This short section provides examples of common functions most that have studied Maths at some point will have seen. To begin we define what is meant by a ‘constant function’.

Definition 6.1. A function $f : A \to B$ is said to be *constant* if there exists $b \in B$ such that $f(x) = b$ for each $x \in A$, i.e. f is constant if its image $f(A)$ consists of exactly one element.

Example 6.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1$. Then f is the constant function which takes on only the value of 1.

Some other useful functions that you may be familiar with are the sine and cosine functions.

Example 6.3. The functions $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ are the functions with the following graphs.

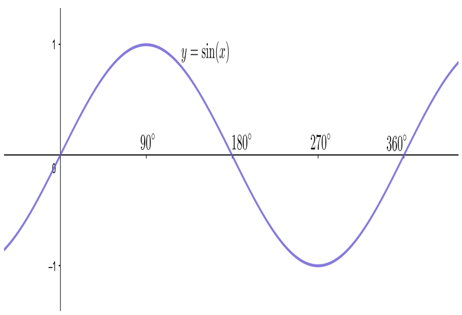

Figure 6.1: The graph of $y = \sin(x)$

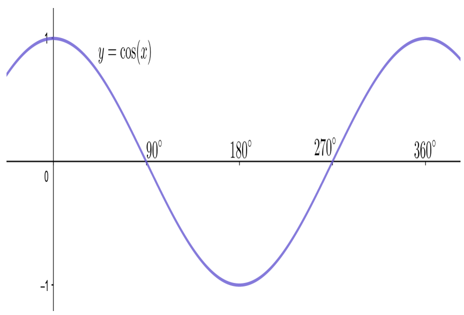

Figure 6.2: The graph of $y = \cos(x)$

Both of these functions take values in $[-1, 1]$ and ‘repeat’ every 360°. Also, cos is a ‘translation’ of sin (to see this, observe if we pulled the graph of $y = \sin(x)$ to the left by 90°, then we would get the graph of $y = \cos(x)$.

Another useful function that you may have seen is the exponential functions.

Example 6.4. Let us consider for a second we are placing rice on a chess board. The rule we choose is the following: the next space on the chess board needs to have twice as many pieces of rice on it than the previous. How many pieces of rice will be on the final square?

Well we start off with 2 pieces of the first square. Then $4 = 2 \times 2 = 2^2$ on the second square. On the third square there will be $8 = 4 \times 2 = 2^3$, and so on. We can then say that on the nth square, there will be 2^n pieces of rice on there. We can extend this to a function $f : \mathbb{R} \to \mathbb{R}$ by defining $f(x) = 2^x$.

What was so special about starting off with 2 pieces of rice? We could have chosen 3 or 100 or 2.6. So we can define a family of functions, called exponential functions. These are the functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a^x$ for some $a \in \mathbb{R}$.

Of particular interest to us will be a special exponential function, where we take $a \approx 2.71828 \cdots =: e$. We denote this number by e, after the mathematician Euler. Then we define $\exp : \mathbb{R} \to \mathbb{R}$ by $\exp(x) = e^x$.

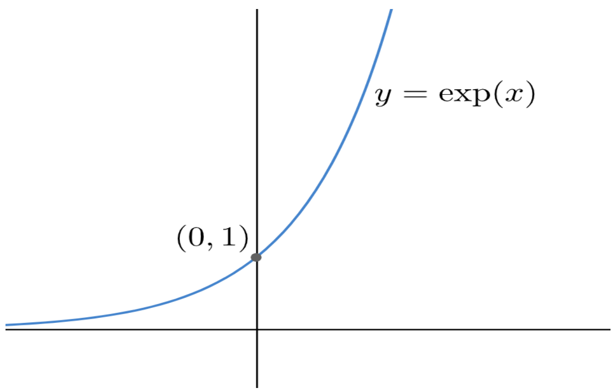

Figure 6.3: Graph of the function $y = \exp(x)$

The exponential function e^x has many interesting properties. First we can see that $e^x > 0$ for every $x \in \mathbb{R}$, i.e. the function only takes on positive values. Also, you can see that the graph 'grows' very quickly when $x > 0$ and the graph 'decays' to zero very quickly for $x < 0$. We will return to some more properties of e^x later. The number e is then defined to be

$$e = \exp(1) = e^1.$$

To continue this section we shall introduce some simple, yet very important, ways of constructing new functions from previous ones.

Definition 6.5. Let $f : A \to B$ and $S \subseteq A$. We define the *restriction of f to S*, denoted by $f\big|_S$, to be the function from S to B by

$$f\big|_S(x) = f(x)$$

for each $x \in S$.

Example 6.6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x \quad \text{and} \quad g(x) = |x|.$$

Then $f \neq g$ (that is the two functions are not equal) since $f(-1) = -1 \neq 1 = g(-1)$. However, for each $x \in [0, +\infty)$ it follows

$$f(x) = x = |x| = g(x),$$

therefore $f\big|_{[0,+\infty)} = g\big|_{[0,+\infty)}$.

Definition 6.7. Let $f : A \to B$ and $g : C \to D$ be two functions such that $f(A) \subseteq C$. We define the *composition of* f *and* g as the function $g \circ f : A \to D$ such that

$$(g \circ f)(x) = g(f(x)),$$

for each $x \in A$.

In other words, the image via the composition $g \circ f$ if an element $x \in A$ is obtained by first applying f to x, thus obtaining an element $y = f(x) \in C$, and then applying g to y.

Example 6.8. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x + 1, \quad \text{and} \quad g(x) = 3x + 2$$

for each $x \in \mathbb{R}$. Then for each $x \in \mathbb{R}$ observe

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = (3x + 2) + 1 = 3x + 3,$$

while

$$(g \circ f)(x) = g(f(x)) = g(x + 1) = 3(x + 1) + 2 = 3x + 5.$$

From this one can see that $f \circ g \neq g \circ f$ since $(g \circ f)(x) \neq (f \circ g)(x)$ for every $x \in \mathbb{R}$. (Note: only one such $x \in \mathbb{R}$ needs to exist for the functions to not be equal!)

From the above example one can see that composition of functions is *not commutative*; i.e. the order in which the composition is applied matters!

Definition 6.9. Given a set A, the function from A to A which maps each element $x \in A$ to itself is called the *identity function* of A and is denoted by id_A. In other words, $\mathrm{id}_A : A \to A$ is the function

$$\mathrm{id}_A(x) = x,$$

for each $x \in A$.

Example 6.10. Let $f : A \to B$ be a function and $S \subseteq A$. Then $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$, since

$$(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x) = \mathrm{id}_B(f(x)) = (\mathrm{id}_B \circ f)(x),$$

for each $x \in A$. Further,

$$f\big|_S = f \circ \mathrm{id}_S.$$