Exercise Sheet 9
Predicate Logic – Sequent Calculus & Semantics

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1. Provide a Sequent Calculus proof of $(\exists x.p(x)) \to (\forall x.\forall y.p(x) \to q(x,y)) \to (\exists x.\forall y.q(x,y))$

2. Consider the following domain and signature:

   - Domain: D
   - Functions: $\texttt{pi1}, \texttt{pi2}, \texttt{swap}$ (arity 1); $\texttt{pair}$ (arity 2)
   - Predicates: $=$ (arity 2)

   Consider the following formulas that capture the "definition" of `swap`, and part of the specifications of the other symbols:

   - let A_1 be $\forall x.\forall y.\texttt{swap}(\texttt{pair}(x,y)) = \texttt{pair}(y,x)$
   - let A_2 be $\forall x.\forall y.x = y \to \texttt{pi1}(x) = \texttt{pi1}(y)$
   - let A_3 be $\forall x.\forall y.\texttt{pi1}(\texttt{pair}(x,y)) = x$
   - let A_4 be $\forall x.\forall y.\forall z.x = y \to y = z \to x = z$

   The goal is to prove the following property of `swap`, which we call C:

   $$\forall x.\forall y.\texttt{pi1}(\texttt{swap}(\texttt{pair}(x,y))) = y$$

   i.e., provide a Sequent Calculus proof of $A_1, A_2, A_3, A_4 \vdash C$

3. Consider the following:

   - the signature $\langle\langle \texttt{zero}, \texttt{succ}\rangle, \langle \texttt{even}, \texttt{odd}, >\rangle\rangle$
   - the model M: $\langle \mathbb{N}, \langle 0, +1\rangle, \langle \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}, \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}, \{\langle 1,0\rangle, \langle 2,0\rangle, \langle 2,1\rangle, \dots\}\rangle\rangle$
   - we write $+1$ for the function that given a number increments it by 1

   Prove that $\vDash_{M,\cdot} \forall x.\texttt{even}(x) \to \exists y.\texttt{odd}(y) \wedge y > x$. Detail your answer as we did in the lectures.

4. Consider the above signature and model M Prove that $\neg \vDash_{M,\cdot} \forall x.\texttt{even}(x) \to \texttt{even}(\texttt{succ}(x))$. Detail your answer as we did in the lectures.