Solutions to Exercise Sheet 2

Model answers to all exercises

Exercise 2.1

0	1	2	3	4
1	1	0	3	0

Exercise 2.2

At the beginning of the `while`-loop we initialise u_x and v_y to 1, and v_x and u_y to 0. Since at this point x contains a and y contains b, the statement is true. Assuming now that the statement holds at the start of the loop body, let us show that it holds at the end as well. This is clear for the first part of the statement, since we put the value that was in y into x and also transfer u_y into u_x, v_y into v_x. For the second part, at the end of the loop body the variable y contains r where $x = ky + r$. We rearrange the last equality to $r = x - ky$ and express x and y according to invariant that holds at the beginning of the loop body:

$$r = x - ky = u_x a + v_x b - k(u_y a + v_y b) = (u_x - ku_y)a + (v_x - kv_y)b$$

These are exactly the coefficients that the program puts into the variables `u_y` and `v_y`.

Exercise 2.3

The answer is of course 3×4^{-1} so all we need to do is to find the multiplicative inverse of 4 in $\mathbb{Z}_{17}$. We could do this by running the extended Euclidean algorithm but 17 is such a small number that we can also solve it by a simple search: $17+1$ is not divisible by four, $2 \times 17 + 1 = 35$ is not divisible by four, $3 \times 17 + 1 = 52$ *is* divisible by four, namely, $52 = 13 \times 4$ and so in this field, $4^{-1} = 13$.

Now we can compute $3 \times 4^{-1} = 3 \times 13 = 39 = 5$ (in $\mathbb{Z}^{17}$, i.e., modulo 17).

Exercise 2.4

We do this in stages. First, we know that $a + 0 = a$ and by commutativity also $0 + a = a$:

$+$	0	1	$\triangle$	$\square$
0	0	1	$\triangle$	$\square$
1	1	0	$\square$	
$\triangle$	$\triangle$		0	
$\square$	$\square$			

From the given value $\square$ on the second row we get: $1 + \triangle = \square$, so we have by associativity, $1 + \square = 1 + (1 + \triangle) = (1+1) + \triangle = 0 + \triangle = \triangle$:

$+$	0	1	$\triangle$	$\square$
0	0	1	$\triangle$	$\square$
1	1	0	$\square$	$\triangle$
$\triangle$	$\triangle$		0	
$\square$	$\square$			

and by commutativity:

$+$	0	1	$\triangle$	$\square$
0	0	1	$\triangle$	$\square$
1	1	0	$\square$	$\triangle$
$\triangle$	$\triangle$	$\square$	0	
$\square$	$\square$	$\triangle$		

For the last entry in the third row we again use what we know already, $\triangle + \square = \triangle + (\triangle + 1) = (\triangle + \triangle) + 1 = 0 + 1 = 1$:

$+$	0	1	$\triangle$	$\square$
0	0	1	$\triangle$	$\square$
1	1	0	$\square$	$\triangle$
$\triangle$	$\triangle$	$\square$	0	1
$\square$	$\square$	$\triangle$	1	

Every row must contain a zero entry because of the additive inverse rule, so the final missing entry is zero:

$+$	0	1	$\triangle$	$\square$
0	0	1	$\triangle$	$\square$
1	1	0	$\square$	$\triangle$
$\triangle$	$\triangle$	$\square$	0	1
$\square$	$\square$	$\triangle$	1	0

For the multiplication table, the first two rows and columns follow from $a * 1 = a$ and $a * 0 = 0$:

$\times$	0	1	$\triangle$	$\square$
0	0	0	0	0
1	0	1	$\triangle$	$\square$
$\triangle$	0	$\triangle$	$\square$	
$\square$	0	$\square$		

Every row (of a non-zero element) must contain a 1 because of the law of the multiplicative inverse, and with commutativity this gives us:

$\times$	0	1	$\triangle$	$\square$
0	0	0	0	0
1	0	1	$\triangle$	$\square$
$\triangle$	0	$\triangle$	$\square$	1
$\square$	0	$\square$	1	

The final missing entry we get from what we have already and associativity, $\square \times \square = (\triangle \times \triangle) \times \square = \triangle \times (\triangle \times \square) = \triangle \times 1 = \triangle$:

$\times$	0	1	$\triangle$	$\square$
0	0	0	0	0
1	0	1	$\triangle$	$\square$
$\triangle$	0	$\triangle$	$\square$	1
$\square$	0	$\square$	1	$\triangle$

Exercise 2.5

We have $123456789_{10} = 111010110111100110100010101_2$, so in binary scientific notation we get $1.11010110111100110100010101 \times 2^{26}$. 32-bit floating point numbers only have space for 23 digits after the (binary) point, so we have to lose some precision and the candidates are either $1.11010110111100110100010 \times 2^{26}$ or $1.11010110111100110100011 \times 2^{26}$. It is clear that the second one is closer to 123456789. In decimal it is 123456792, (while the first one equals 123456784).