Solutions to Exercise Sheet 11

Exercise 11.1

We need to show that we can add two solutions and get another solution, so assume that both $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ satisfy the equation, and consider $C = A + B = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 \end{pmatrix}$:

$$2(a_1+b_1) - (a_2+b_2) - 3(a_3+b_3) + 2(a_4+b_4) = 2a_1 + 2b_1 - a_2 - b_2 - 3a_3 - 3b_3 + 2a_4 + 2b_4 =$$

$$= 2a_1 - a_2 - 3a_3 + 2a_4 + 2b_1 - b_2 - 3b_3 + 2b_4 = 0 + 0 = 0$$

For scalar multiplication we do the same, we consider $s \cdot A = \begin{pmatrix} sa_1 \\ sa_2 \\ sa_3 \\ sa_4 \end{pmatrix}$:

$$2sa_1 - sa_2 - 3sa_3 + 2sa_4 = s(2a_1 - a_2 - 3a_3 + 2a_4) = s \times 0 = 0$$

From Gaussian elimination we know that the solutions all take the following form:

$$x_4 : \text{f.c.} \quad x_3 : \text{f.c.} \quad x_2 : \text{f.c.} \quad x_1 = \frac{1}{2}(x_2 + 3x_3 - 2x_4)$$

We make some particular choices to get the vectors:

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

These three vectors are linearly independent because if

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + a_3 \cdot \vec{v}_3 = \begin{pmatrix} a_3 + 3a_2 - a_1 \\ 2a_3 \\ 2a_2 \\ a_1 \end{pmatrix} = \vec{0}$$

then by looking at the last three coordinates we see that $a_1 = a_2 = a_3 = 0$ must be the case. So the criterion of Theorem 8 is satisfied.

We also see that the three vectors generate every solution, since if the choice for x_4 is a_4, the choice for x_3 is a_3, and the choice for x_2 is a_2, then the solution can be written as a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$:

$$X = \frac{a_2}{2} \cdot \vec{v}_3 + \frac{a_3}{2} \cdot \vec{v}_2 + a_4 \cdot \vec{v}_1 = \begin{pmatrix} \frac{1}{2}(a_2 + 3a_3 - 2a_4) \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Altogether, then, we have a basis for the space of solutions that contains three vectors, so it is three-dimensional.

Exercise 11.2

This is very similar to the argument in Section 13.6 in the course booklet. We use the independence criterion of Theorem 8, so assume $\sum_{i=1}^{n} a_i \cdot \vec{v}_i = \vec{0}$. We use both sides of this identity in an inner product with a vector $\vec{v}_k$, as we did in Box 150:

$$\left\langle \sum_{i=1}^{n} a_i \cdot \vec{v}_i, \vec{v}_k \right\rangle = \left\langle \vec{0}, \vec{v}_k \right\rangle$$

The laws of the inner product allow us to rewrite both sides of the equation:

$$\sum_{i=1}^{n} a_i \times \langle \vec{v}_i, \vec{v}_k \rangle = 0$$

On the left side, by orthogonality, only the term $\langle \vec{v}_k, \vec{v}_k \rangle$ is different from zero, so the equation reduces to

$$a_k \times \langle \vec{v}_k, \vec{v}_k \rangle = 0$$

We divide by $\langle \vec{v}_k, \vec{v}_k \rangle$ (which we can because of positive definiteness) and obtain $a_k = 0$.

We can do this calculation for every $k = 1, \ldots, n$, so the independence criterion of Theorem 8 is satisfied.

Exercise 11.3

$$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \cdot \vec{w}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{7}{14} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ -1/2 \end{pmatrix}$$

To avoid fractions, we stretch $\vec{w}_2$ by a factor of 2 to obtain $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

$$\vec{w}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \cdot \vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} \cdot \vec{w}_2 = \begin{pmatrix} 5 \\ 12 \\ -5 \end{pmatrix} - \frac{14}{14} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{20}{10} \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ -6 \end{pmatrix}$$

Again, we can simplify the last vector to $\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$.

We can now calculate the inner product between any two of $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ to see that we obtain zero in each case.