3.7 General Normal Distributions

In previous section we considered the function $\Phi(a)$ which allowed us to computer probabilities of the random variable $Z \sim \mathrm{N}(0,1)$. We shall now extend these ideas in order to consider a general normal random variable $X \sim \mathrm{N}(\mu, \sigma^2)$. The idea is to use the two transformations we considered in Section 3.5.1. Essentially every normal distribution curve is just a translation (changing μ) and a stretch (changing σ^2), of the standard normal curve. Thus any area we want to find under the curve X should be able to be easily transformed into an area under Z. This is captured by the following Lemma:

Lemma 3.7.1. Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Z \sim \mathrm{N}(0,1)$. Then we have for all x,

$$\mathbb{P}(X \leq x) = \mathbb{P}\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Problem 3.7.1. Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Z \sim \mathrm{N}(0,1)$ show that:

$$\mathbb{P}(X \geq x) = 1 - \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Analogously to Lemma 3.6.1 we also have that:

$$\mathbb{P}[(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

One concept to note about the above results is that the denominator term is σ, the standard deviation, rather than the variance. Remember that $X \sim \mathrm{N}(\mu, \sigma^2)$.

With Lemma 3.7.1 we can now calculate probabilities with any normal distribution. The Lemma informally says what equivalent area we would need to find under the standard normal curve to evaluate the original probability. We consider the following example:

Example 3.7.1. Suppose $X \sim \mathrm{N}(3, 25)$, what is:

(i) The $\mathbb{P}(X \leq 2.5)$,

(ii) The $\mathbb{P}(X \geq 4.5)$,

(iii) The $\mathbb{P}(2.5 \leq X \leq 4.5)$.

For each part we apply the lemmas above. Firstly we have that $\mu = 2.5$ and $\sigma^2 = 25$, therefore $\sigma = 5$. Now by applying Lemma 3.7.1 we have that:

$$\mathbb{P}(X \leq 2.5) = \Phi\left(\frac{2.5-3}{5}\right) = \Phi(-0.1) = 0.4602.$$

Similarly for part (ii) we can apply the result from Problem 3.7.1,

$$\mathbb{P}(X \geq 4.5) = 1 - \Phi\left(\frac{4.5-3}{5}\right) = 1 - \Phi(0.3) = 0.3821.$$

Finally for part (iii) we can apply the analogous result of Lemma 3.6.1 to find that:

$$\begin{aligned}\mathbb{P}(2.5 \leq X \leq 4.5) = \Phi\left(\frac{4.5-3}{5}\right) - \Phi\left(\frac{2.5-3}{5}\right) &= \Phi(0.3) - \Phi(-0.1) \\ &= 0.6179 - 0.4602 = 0.1577.\end{aligned}$$