Exercise Sheet 6 - Solutions
Propositional Logic – Logical Equivalences & Normal Forms & SAT

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1. Let $A = ((p \wedge q) \to r) \to (p \vee \neg r)$. To convert A to a DNF we first compute A's truth table:

p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$\neg r$	$p \vee \neg r$	A
T	**T**	**T**	**T**	**T**	**F**	**T**	**T**
T	**T**	**F**	**T**	**F**	**T**	**T**	**T**
T	**F**	**T**	**F**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**T**	**T**	**T**	**T**
F	**T**	**T**	**F**	**T**	**F**	**F**	**F**
F	**T**	**F**	**F**	**T**	**T**	**T**	**T**
F	**F**	**T**	**F**	**T**	**F**	**F**	**F**
F	**F**	**F**	**F**	**T**	**T**	**T**	**T**

We enumerate all the **T** rows:

- $p \wedge q \wedge r$
- $p \wedge q \wedge \neg r$
- $p \wedge \neg q \wedge r$
- $p \wedge \neg q \wedge \neg r$
- $\neg p \wedge q \wedge \neg r$
- $\neg p \wedge \neg q \wedge \neg r$

Finally, we combine those using $\vee$:

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

2. To convert A to a CNF we first enumerate all the **F** rows:

- $\neg p \wedge q \wedge r$
- $\neg p \wedge \neg q \wedge r$

We combine the negation of those using $\wedge$:

$$\neg(\neg p \wedge q \wedge r) \wedge \neg(\neg p \wedge \neg q \wedge r)$$

We then use de Morgan and double negation elimination to convert this formula to a CNF as follows:

- $\neg(\neg p \wedge q \wedge r) \wedge \neg(\neg p \wedge \neg q \wedge r)$
- $(\neg\neg p \vee \neg q \vee \neg r) \wedge \neg(\neg p \wedge \neg q \wedge r)$ – by de Morgan
- $(\neg\neg p \vee \neg q \vee \neg r) \wedge (\neg\neg p \vee \neg\neg q \vee \neg r)$ – by de Morgan

- $(p \vee \neg q \vee \neg r) \wedge (\neg\neg p \vee \neg\neg q \vee \neg r)$ – by DNE
- $(p \vee \neg q \vee \neg r) \wedge (p \vee \neg\neg q \vee \neg r)$ – by DNE
- $(p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r)$ – by DNE

3. We will now convert A to a CNF using equivalences
   - $((p \wedge q) \rightarrow r) \rightarrow (p \vee \neg r)$
   - $(\neg(p \wedge q) \vee r) \rightarrow (p \vee \neg r)$ – implication elimination
   - $\neg(\neg(p \wedge q) \vee r) \vee (p \vee \neg r)$ – implication elimination
   - $(\neg\neg(p \wedge q) \wedge \neg r) \vee (p \vee \neg r)$ – de Morgan
   - $((p \wedge q) \wedge \neg r) \vee (p \vee \neg r)$ – DNE
   - $(p \vee \neg r) \vee ((p \wedge q) \wedge \neg r)$ – commutativity of $\vee$
   - $((p \vee \neg r) \vee (p \wedge q)) \wedge ((p \vee \neg r) \vee \neg r)$ – distributivity of $\vee$ over $\wedge$
   - $(((p \vee \neg r) \vee p) \wedge ((p \vee \neg r) \vee q)) \wedge ((p \vee \neg r) \vee \neg r)$ – distributivity of $\vee$ over $\wedge$
   - $(((p \vee \neg r) \vee p) \wedge ((p \vee \neg r) \vee q)) \wedge (p \vee \neg r \vee \neg r)$ – associativity of $\vee$
   - $((p \vee \neg r \vee p) \wedge ((p \vee \neg r) \vee q)) \wedge (p \vee \neg r \vee \neg r)$ – associativity of $\vee$
   - $((p \vee \neg r \vee p) \wedge (p \vee \neg r \vee q)) \wedge (p \vee \neg r \vee \neg r)$ – associativity of $\vee$
   - $(p \vee \neg r \vee p) \wedge (p \vee \neg r \vee q) \wedge (p \vee \neg r \vee \neg r)$ – associativity of $\wedge$

   Through idempotence, commutativity and associativity, this could even be simplified further to:

   $$(p \vee \neg r) \wedge (p \vee q \vee \neg r)$$

4. Here is a possible run of the algorithm:
   - $(p \vee t \vee s) \wedge (q \vee r \vee \neg s \vee \neg t) \wedge (p \vee \neg q \vee s) \wedge (p \vee q \vee r \vee \neg t)$ $\wedge (q \vee r \vee \neg s) \wedge (\neg p \vee \neg s \vee \neg t) \wedge (\neg p \vee \neg q \vee s \vee \neg r) \wedge (\neg r \vee t)$
     - **select** $p = \mathbf{T}$
     - $(q \vee r \vee \neg s \vee \neg t) \wedge (q \vee r \vee \neg s) \wedge (\neg s \vee \neg t) \wedge (q \vee s \vee \neg r) \wedge (\neg q \vee s \vee \neg r) \wedge (\neg r \vee t)$
       - **select** $q = \mathbf{T}$
       - $(\neg s \vee \neg t) \wedge (s \vee \neg r) \wedge (\neg r \vee t)$
         - **select** $r = \mathbf{T}$
         - $(\neg s \vee \neg t) \wedge (s) \wedge (t)$
           - **select** $s = \mathbf{T}$
           - $(\neg t) \wedge (t)$
             - **select** $t = \mathbf{F}$
             - $()$ – empty clause – **backtrack**
             - **select** $t = \mathbf{T}$
             - $()$ – empty clause – **backtrack**
           - **select** $s = \mathbf{F}$
           - $() \wedge (t)$ – empty clause – **backtrack**
         - **select** $r = \mathbf{F}$
         - $(\neg s \vee \neg t)$
           - **select** $s = \mathbf{F}$
           - no more clauses – **SAT**

   Therefore this formula is satisfiable. It is for example satisfied by the valuation:

   $$p = \mathbf{T}, q = \mathbf{T}, r = \mathbf{F}, s = \mathbf{F}, t = \mathbf{T}$$