2

Random Variables

In the previous chapter we spent a lot of time studying random experiments and their outcomes. However in the real world the actual outcome of an experiment is sometimes not as important as the consequence that the outcome has. For example, if you bet on a roulette wheel you might not be so interested in what exact number produced, but more so whether you are actually winning or losing money. The random experiment is the outcome of the roulette wheel, but what you want to know is how that outcome will affect some other quantity, i.e how much money you have left at the end of the game. This is where the theory of random variables gives us a language for using the outcomes of random experiments. We start with a motivating example:

Example 2.0.1. Suppose we play following game: We roll a six-sided die, if the die shows a one or five then you gain £1, if die shows a two or a three then you gain £2, and if the die shows a six then you gain nothing. We want to analyse the outcomes of this game.

Suppose X represents the amount of money you have after one dice roll. Then the value of X depends entirely on the outcome of the roll. For example is the roll is one or a five then $X = 1$. If the roll is a two; a three; or a four, then $X = 2$. If the roll is a six then $X = 0$. So X is actually a function, which takes in the outcome of the roll as an input, and outputs the amount of money you would make based on what the dice roll was.

In precise terms, X is a function from the sample space of the dice roll (all the six outcomes), into the possible values of money you could have. Written formally,

$$X : \{0, 1, 2, 3, 4, 5, 6\} \to \{-1, 0, 1\}.$$

Furthermore as X is a function, we denote $X(i)$ to be the amount of money you would make if the die showed face i, then $X(1) = X(5) = 1$; $X(2) = X(3) = X(4) = 2$; and $X(6) = 0$.

Introducing function notation is slightly more work than is really necessary for this course. The main intuition to consider is that a random variable uses the outcome of a random experiment, to decide what value to output. Essentially we can reveal the value of a random variable, by actually carrying out the experiment and observing the outcome.

The values that the random variable can take are referred to as the *range* or the *support* of the random variable. In the previous example the support of X was $\{-1, 0, 1\}$. In this course, all random variables will have a range which are either non-negative integers or real numbers. Suppose we have some random variable X, if it can only take a countable number of values, then we refer to X as a *discrete random variable*. If X can take values within some interval of real numbers, then we refer to X as a *continuous random variable*. We will mainly study discrete random variables. Though extra attention is given to continuous random variables in the appendix notes.

2.1 Discrete Random Variables

We being our study of discrete random variables. We recall that a discrete random variable is quantity that depends on the outcome of some random experiment, and can only have countably many outputs. In this course we assume that all discrete random variables can only take values in $\mathbb{N}_0$, the positive counting numbers including zero, $\{0, 1, 2, \ldots\}$. Throughout this course random variables will always be denoted with a capital letter i.e X, Y or Z, while lower case letters will be used to denote the values that the random variable can take. Our first definition focuses on how we describe the probability of random variable taking a certain output.

Definition 2.1.1. Suppose X is a discrete random variable, we denote $f_X(i)$ to be the *probability distribution function* of X. For each $i \in \mathbb{N}_0$ we define $f_X(i)$ as follows:

$$f_X(i) := \mathbb{P}(X = i).$$

For each integer i, the distribution function $f_X(i)$ denotes how likely it is for the random variable X to produce the outcome i. In an analogous way to the first chapter the probability that $X = i$ can be found by considering a sample space of equally likely outcomes:

$$\mathbb{P}(X = i) = \frac{\text{The number of outcomes in the sample space which cause } X \text{ to output } i}{\text{The number of possible outcomes}}.$$

Example 2.1.1. Consider the scenario given in Example 2.0.1, where X denotes the random variable indicating how much money is won at the end of the game. What is $f_X(i)$?

Firstly we recall that X can only take the values $0, 1, 2$. Therefore, for any other i not equal to $0, 1$ or 2, we have that $f_X(i) = 0$. Now suppose $i = 1$, this can only occur if the dice roll is a one or a five. Therefore are two outcomes which cause $X = 1$, out of a possible six outcomes, therefore $\mathbb{P}(X = 1) = 1/3$. Similarly $X = 2$ only if the roll is 2, 3 or 4, again we have that $\mathbb{P}(X = 2) = 1/2$. Finally the die outputs a six if and only if $X = 0$, therefore $\mathbb{P}(X = 0) = 1/6$. Collating this all together have the following:

$$f_X(i) = \begin{cases} 1/6 & \text{if } i = 0; \\ 1/3 & \text{if } i = 1; \\ 1/2 & \text{if } i = 2; \\ 0 & \text{otherwise.} \end{cases}$$