1.5 Conditional Probability

One of the central ideas of the probability, is how additional knowledge changes the likelihood of different outcomes. For example, at the beginning of a football match an odds maker may given a certain probability of a team winning. If half-way through the match, a player from the team you have bet on receives a red card, the odds of them winning has now changed due to the additional information. Our aim is to give a precise characterisation of how additional information changes the probability of events. Another related idea we explore is independence, not all information should affect the probability of events occurring. For example if I tell you the age of my cat, this has absolutely no bearing on the outcome of the football match, and the way we handle the probabilities should reflect this.

1.5.1 Conditional Events

We start with a motivating example, involving computing probabilities when we are given additional information.

Example 1.5.1. A school is surveying which students take part in sport across two year groups. All students are either in Year 7, or Year 9 and they either take part in sport, or they do not. A total of 500 students were surveyed, 200 of which were in Year 7, and 300 in Year 9. Of those in Year 7, 90 of them played a sport; while of those in Year 9, 120 of them took part in sport. A student is picked at random, with each student chosen being equally likely:

(i) What is the probability they play sport?

(ii) What is the probability they play sport, given that they are in Year 9.

For part (i) we use a similar approach to previous examples. We let S be the event that the student plays sport, in total there are 90 students in Year 7 and 120 in Year 9, which play sport. Therefore there are 210 in total, thus $\mathbb{P}(S) = 210/500$.

For part (ii) we need to think about where we are sampling from. We know the student sampled is in Year 9, and we know that of the 300 students in Year 9, we have that 120 of them play sport. So the probability that this Year 9 student plays sport is 120/300.

The main idea behind conditional events, is that you "restrict" your view of the sample space. In the previous example, we are only interested in Year 9 students, thus we looked at the proportion of those in Year 9 that also played sport, out of all Year 9 students. We now introduce some notation to describe the situation.

Definition 1.5.1. Let Ω be a sample space, and A and B be events. We denote $\mathbb{P}(A|B)$ to be the probability of A, given that B has occurred. Furthermore we have that:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We note that this definition captures the procedure we followed to solve Example 1.5.1 (ii). Essentially we looked at the number of students that were both in Year 9 and played sport, and divided by the number of students that were in Year 9. *We remark that while the definition concerns probabilities, in Example 1.5.1 we used the actual of number of students. You can repeat the example above using probabilities and applying the definition above, you should think about why these methods are equivalent.*

Example 1.5.2. Suppose two fair six sided dice are rolled, what is the probability that one of dice is a six, given that their combined sum is ten?

Again we define events that we are interested in. Let A be the event that one of the dice show a six, and let B be the event that the two dice sum to ten. We are interested in finding $\mathbb{P}(A|B)$, so we apply Definition 1.5.1. Firstly we represent each outcome as an ordered pair, for example $(1, 2)$ represents the outcome that the first die shows a one and the second shows a two. Then we can write all the outcomes where at least one die is a six as,

$$A = \{(6, 1), (6, 2) \dots (6, 6), (1, 6), (2, 6) \dots (5, 6)\}.$$

Furthermore we list all outcomes which sum to ten, hence we have the following:

$$B = \{(6, 4), (5, 5), (4, 6)\}.$$

Thus it follows that,

$$A \cap B = \{(6, 4), (4, 6)\}.$$

Now we note that there are 36 possible outcomes in the sample space, each outcome is equally likely, therefore $\mathbb{P}(A \cap B) = 2/36$, while $\mathbb{P}(B) = 3/36$. Therefore by definition we have that:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/36}{3/36} = \frac{2}{3}.$$

Again you can approach this style of question in a similar vein to Example 1.5.1. Try writing out the entire sample space, and counting the number of outcomes which satisfy both A and B, then divide by those which satisfy B. You should again think why these methods are equivalent.

In the next example we consider how using conditional events can help us find probabilities of events that depend on the outcome of another event.

Example 1.5.3. Suppose we have an urn that contains either red or blue balls. There are 7 red balls in the urn and 5 blue balls. A ball is picked randomly from the urn and its colour is noted. A second ball is then picked from the urn without replacement.

(i) Given that the first ball is red, what is the probability that the second ball picked is red?

(ii) What is the probability that both balls are red?

We start with part (i). Let $R1$ be the event that the first ball picked is red, while let $R2$ be the event that the second ball picked is red. We are looking to compute $\mathbb{P}(R2|R1)$, in may be tempting to apply the formula here, but we should actually think what this event means. If $R1$ has occurred, then it means that a red ball has been removed from the urn. Therefore, the urn now contains 6 red balls and 5 blue balls, therefore the probability a red ball is picked from this urn is,

$$\frac{6}{6+5} = \frac{6}{11}.$$

This is exactly the probability that a red ball is picked second, given that the first ball is red. Therefore $\mathbb{P}(R2|R1) = 6/11$.

In part (ii) we need to find $\mathbb{P}(R2 \cap R1)$, at this point we can apply Definition 1.5.1 as follows:

$$\mathbb{P}(R2|R1) = \frac{\mathbb{P}(R2 \cap R1)}{\mathbb{P}R1} \iff \mathbb{P}(R2 \cap R1) = \mathbb{P}(R2|R1)\mathbb{P}(R1).$$

We computed $\mathbb{P}(R2|R1) = 6/11$ in part (i). For $\mathbb{P}(R1)$ we have that the urn initially contains 7 red balls and 5 blue balls, each equally likely to be chosen. Therefore it follows that $\mathbb{P}(R1) = 7/(7+5) = 7/12$. and hence we have that:

$$\mathbb{P}(R2 \cap R1) = \frac{6}{11} \cdot \frac{7}{12} = \frac{42}{132}.$$