2.4.1 Linearity of Expectation

In this section we briefly discuss the linearity of expectation. Which gives us a way to simplify the calculation of expectation for sums of random variables.

Lemma 2.4.2. Suppose X and Y are random variables. Then we have that:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

In general, suppose that $X_1, X_2, \ldots X_k$ is a collection of random variables. Then we have that,

$$\mathbb{E}\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} \mathbb{E}[X_i].$$

The linearity of expectation is useful for when we need to calculate the expectation of a sum of multiple outcomes. It is important to note that we require no further conditions on the random variables. The linearity of expectation always applies, even if we know nothing regarding to dependency of the variables. We show an application in the following example:

Example 2.4.3. Suppose we play the following game: We roll a fair six-sided die 100 times, where each roll is independent of any other roll. For every roll that shows a six we gain £1. Suppose X is a random variable representing the total amount of money we have after 100 dice rolls, what is the expectation of X, i.e the expected amount of money you would earn after playing this game.

The main challenge with this style of problem is that it is difficult to compute the distribution function for X. One technique is to write the random variable X as a sum of simpler random variables. Suppose X_i is the random variable which is 1, if the i^{th} dice roll is a 6, and 0 otherwise. Then we can rewrite X as sum of individual dice rolls,

$$X = \sum_{i=0}^{100} X_i.$$

We have now represented the total winnings as a sum of 100 individual dice rolls. We apply the linearity of expectation to find that:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{100} X_i\right] = \sum_{i=1}^{100} \mathbb{E}[X_i].$$

Now we note that X_i is equal to one whenever the dice shows a six, and zero otherwise. Therefore we have that,

$$\mathbb{E}[X_i] = \sum_{k=0}^{1} k \cdot \mathbb{P}(X_i = k) = 0 \cdot \mathbb{P}[X_i = 0] + 1 \cdot \mathbb{P}[X_i = 1] = \frac{1}{6}.$$

By substituting the value of $\mathbb{E}[X_i]$ into the formula for linearity of expectation, we have that:

$$\mathbb{E}[X] = \sum_{i=1}^{100} \mathbb{E}[X_i] = \sum_{i=1}^{100} \frac{1}{6} = 100 \times \frac{1}{6} = \frac{50}{3} \approx 16.67.$$

Where the third equallity follows from the fact that:

$$\sum_{i=1}^{100} \frac{1}{6} = \underbrace{\frac{1}{6} + \frac{1}{6} + \ldots + \frac{1}{6}}_{\text{100-times}} = 100 \times \frac{1}{6}.$$