- $S_1 \to S_2 \to S_3$

$S_1 \triangleq \forall x.\forall y.\, x<y \to x \le y$

$S_2 \triangleq \forall x.\exists y.\, x<y$

$S_3 \triangleq \forall x.\exists y.\, x \le y$

(ND)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.\, x<y \to x\le y}\,1}{\forall y.\, z<y \to z\le y}(\forall E)}{z<\omega \to z\le\omega}(\forall E) \quad \dfrac{}{z<\omega}\,3}{z\le\omega}(\to E) \qquad \dfrac{\dfrac{}{\forall x.\exists y.\, x<y}\,2}{\exists y.\, z<y}(\forall E)\,x\backslash z}{\exists y.\, z\le y}\,3(\exists E)\ \text{fresh } \omega}{\dfrac{\exists y.\, z\le y}{\forall x(\exists y.\, x\le y)}\,2(\to I)}(\forall I)\ \text{fresh } z$$

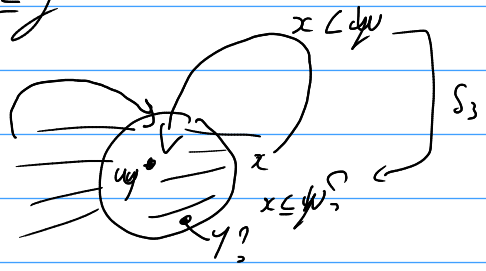
$$\dfrac{S_2 \to S_3}{S_1 \to S_2 \to S_3}\,1(\to I]$$

Context:

1 : S_1

2 : S_2

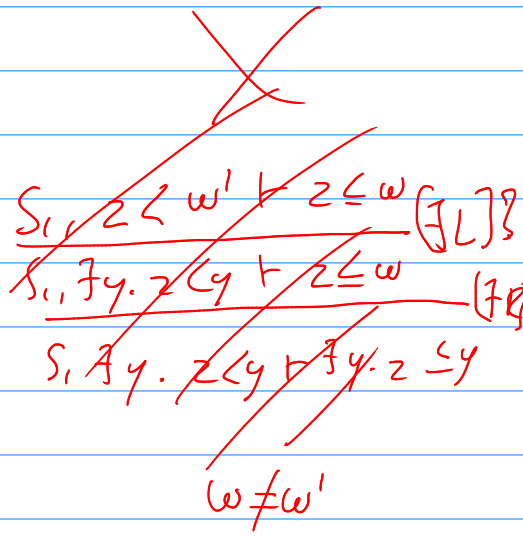
3 : $z<\omega$



Soundness: for all M, $\models_M (S_1 \to (S_2 \to S_3))$

(SC)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{z<\omega \vdash z<\omega}(Id) \quad \dfrac{}{z\le\omega,\, z<\omega \vdash z\le\omega}(Id)}{(z<\omega \to z\le\omega),\, z<\omega \vdash z\le\omega}(\to L)}{(\forall y.\, z<y \to z\le y),\, z<\omega \vdash z\le\omega}(\forall L)}{S_1,\, z<\omega \vdash z\le\omega}(\forall L)}{S_1,\, z<\omega \vdash \exists y.\, z\le y}(\exists R)\bullet}{S_1,\, \exists y.\, z<y \vdash \exists y.\, z\le y}(\exists L)\bullet}{S_1,\, S_2 \vdash \exists y.\, z\le y}(\forall L)}{S_1,\, S_2 \vdash S_3}(\forall R)}{S_1 \vdash S_2 \to S_3}(\to R)}{\vdash S_1 \to (S_2 \to S_3)}(\to R]$$



$\models_{M_1} S_3$ $\quad$ $\neg \models_{M_2} S_3$