Robotics –
Planning and Motion

Manipulator Kinematics

COMP52815

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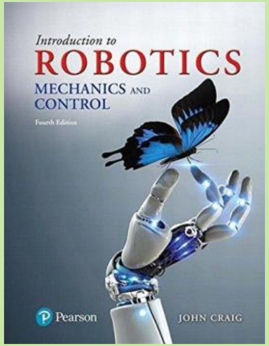
Lecture 5: Learning Objectives

The aim of this lecture is to build a model which will lead to the kinematics.

- Objectives
  - Link Description
  - Denavit-Hartenberg (D-H parameters)
  - Manipulator Kinematics

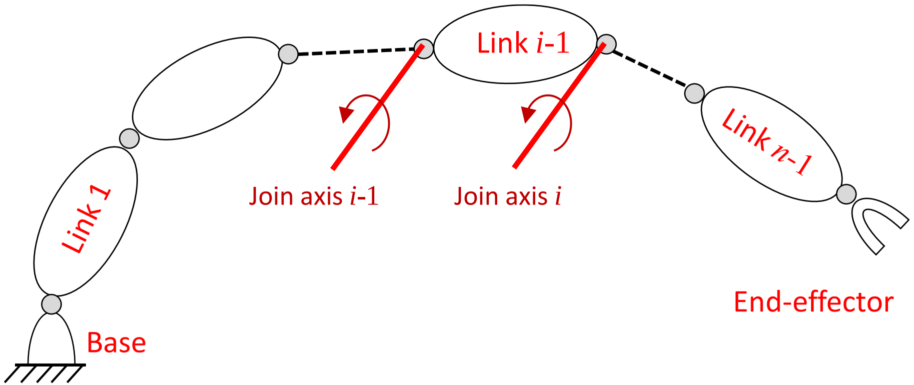
- John J. Craig, “Introduction to Robotics- Mechanics & Control”, 3rd Edition, *Pearson Education International*, 2005, C3, p.62



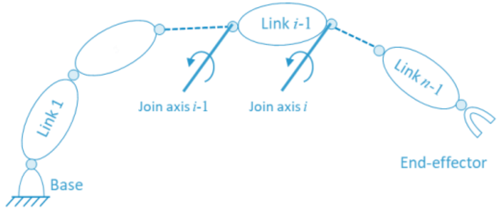
Durham University

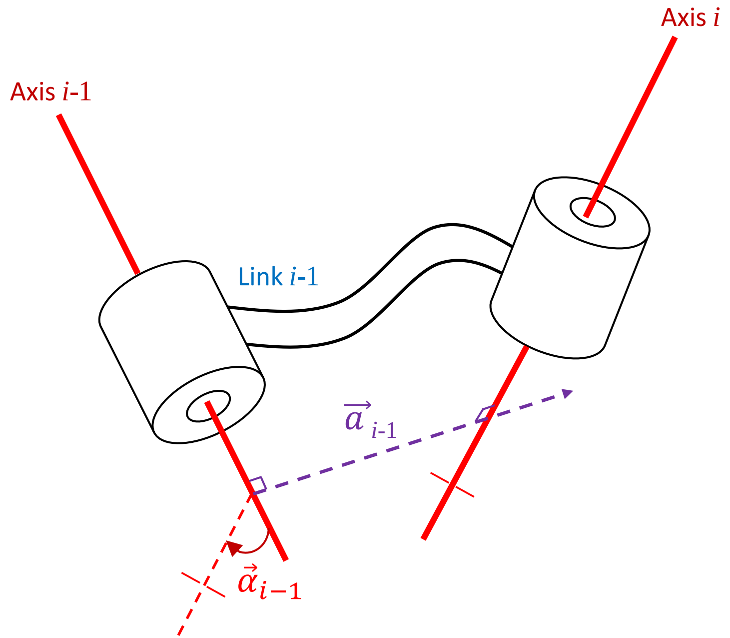
Link Description

- Manipulator:



Link Description



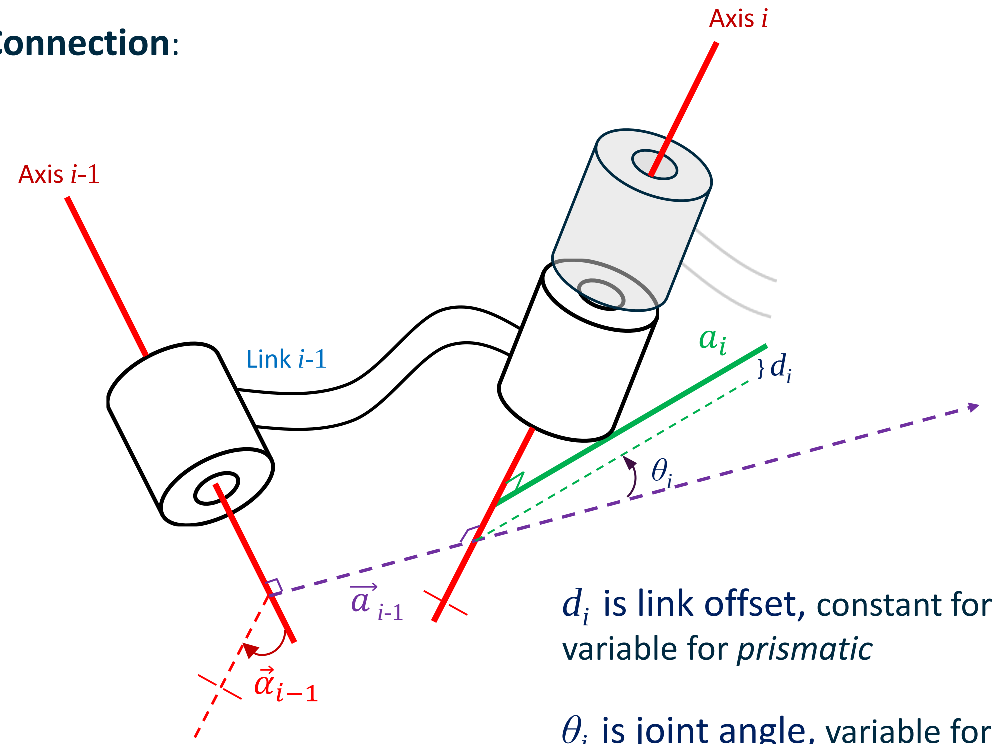


$\vec{a}_{i-1}$: Link Length – mutual perpendicular

$\vec{\alpha}_{i-1}$: Link Twist – angle between axes i and i-1

Link Description

Link Connection:

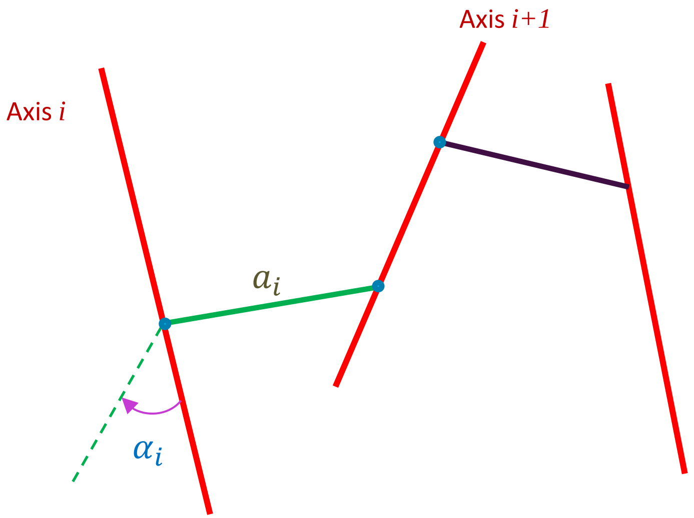


d_i is link offset, constant for *revolting* joint and variable for *prismatic*

θ_i is joint angle, variable for *revolting* joint and constant for *prismatic*

Link Description

First and last links:



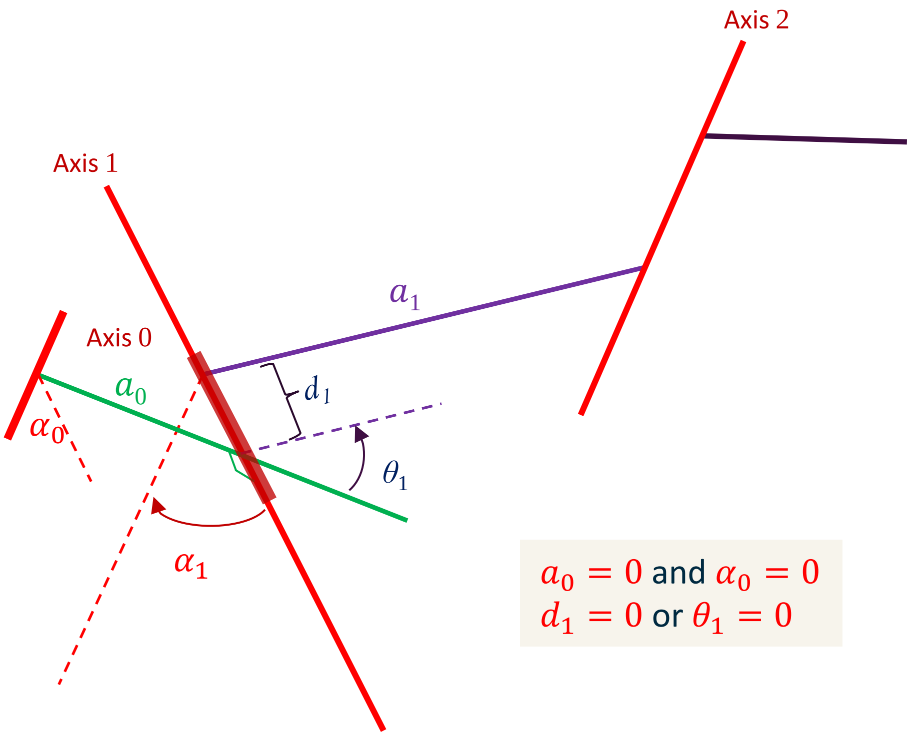
a_i and α_i depend on joint axes *i* and *i*+1

Axes 1 to n:
$a_1, a_2, \cdots, a_{n-1}$
and,
$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$

Convention: $a_0 = a_n = 0$ and $\alpha_0 = \alpha_n = 0$

Link Description

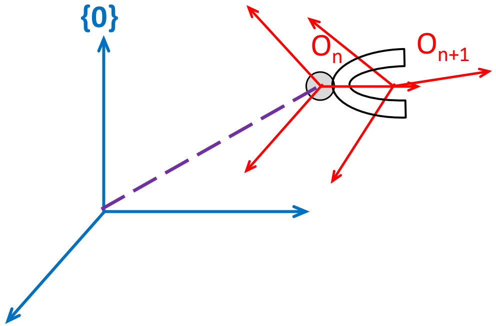
First link:



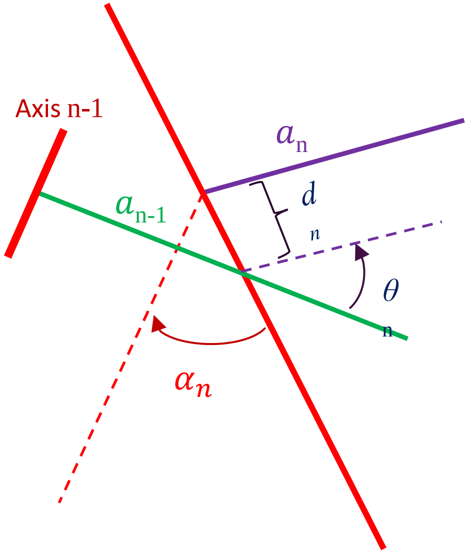
$a_0 = 0$ and $\alpha_0 = 0$
$d_1 = 0$ or $\theta_1 = 0$

Link Description

End-effector frame:



<u>Last</u> link:



$a_n = 0$, $\alpha_n = 0$
$d_n = 0$, $\theta_n = 0$

Link Description

Denavit-Hartenberg (D-H) Parameters:

Four D-H parameters are $(\alpha_i,\ a_i,\ d_i,\ \theta_i)$

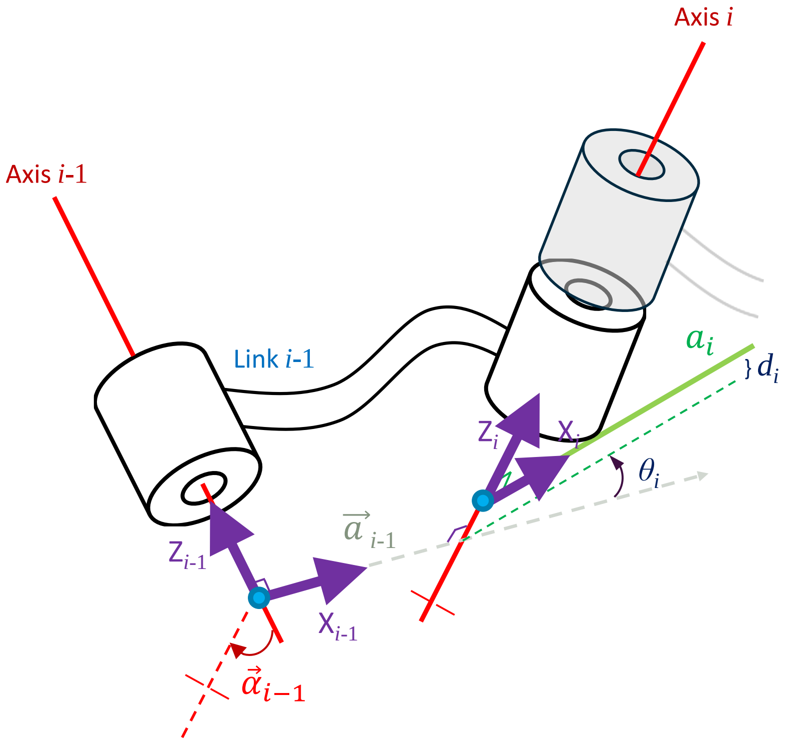
- Three fixed link parameters and

- One joint variable: $\begin{cases} \theta_i & \text{Revolute joint} \\ d_i & \text{Prismatic joint} \end{cases}$

- α_i and a_i describe the link i
- d_i and θ_i describe connection between the links

Link Description

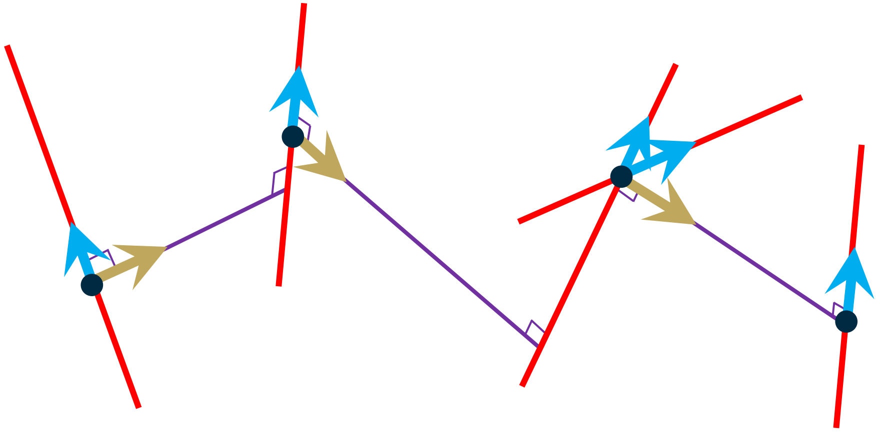
Frame attachment :



Link Description

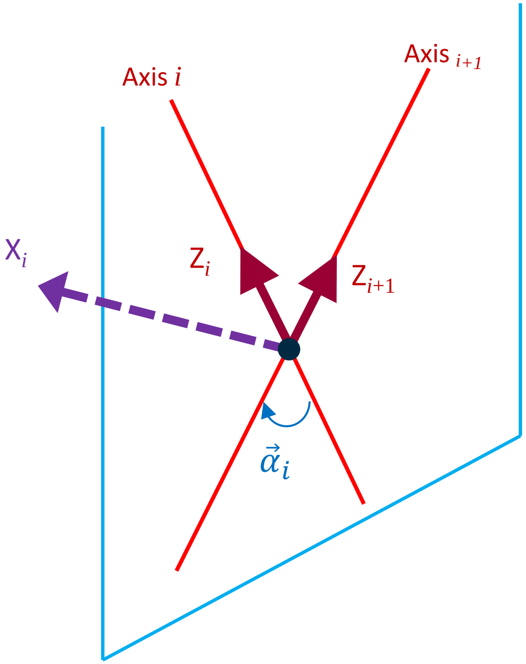
Frame attachment:

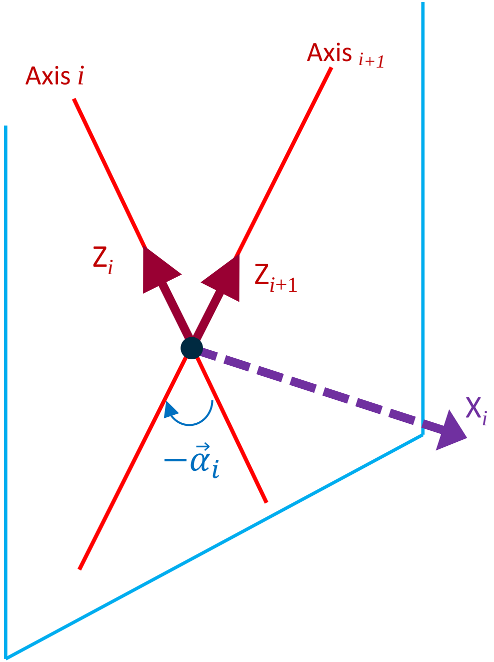
1. Common Normals
2. Origins
3. Z-axis
4. X-axis



Link Description

Intersecting Joint Axes:





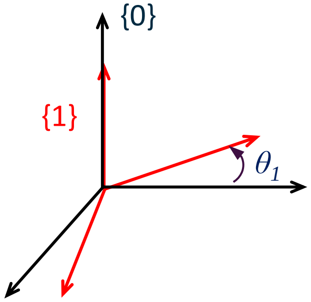
Link Description

Example, First Link:

Revolute:

$z_0 = z_1$

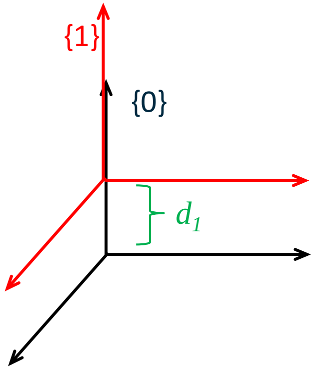
- Set: $a_0 = 0$, $\alpha_0 = 0$, $d_1 = 0$
- θ_1 is variable



Prismatic:

$z_0 = z_1$

- Set: $a_0 = 0$, $\alpha_0 = 0$, $\theta_1 = 0$
- d_1 is variable

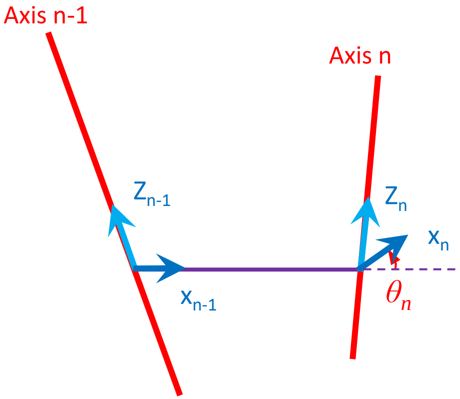


Link Description

Example, <u>Last</u> Link:

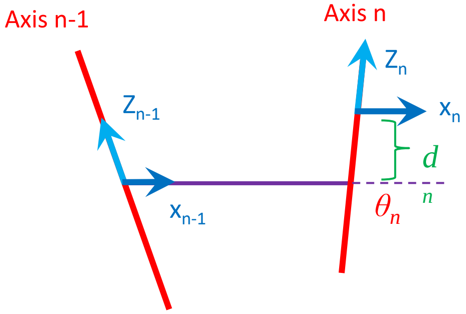
Revolute:

- Set: $d_n = 0$
- θ_n is <u>variable</u>



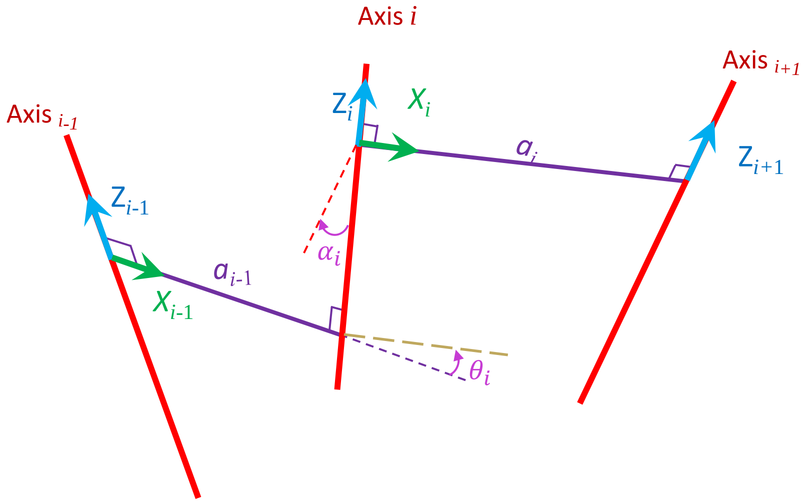
Prismatic:

- Set: $\theta_n = 0$
- d_n is <u>variable</u>



Link Description

Summary:



α_i : angle between z_i and z_{i+1} about x_i
a_i : distance between z_i and z_{i+1} along x_i

d_i : distance between x_{i-1} and x_i along z_i
θ_i : angle between x_{i-1} and x_i about z_i

Link Description

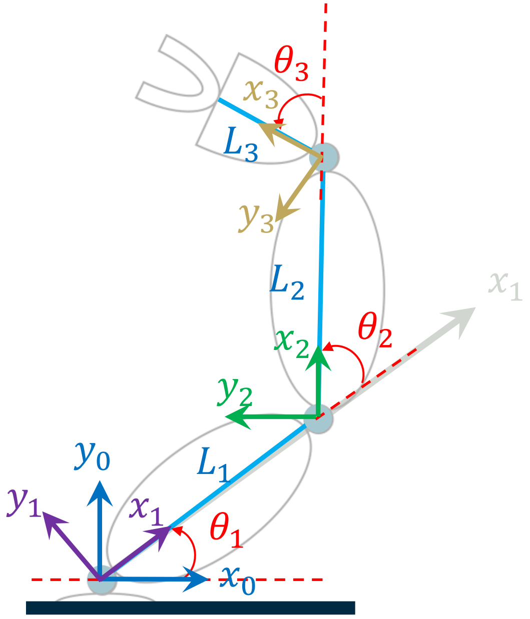
Example, RRR manipulator:

- Z axes
- X and Y axes

- D-H parameters:

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

D-H parameter:
α_i : angle between z_i and z_{i+1} about x_i
a_i : distance between z_i and z_{i+1} along x_i
d_i : distance between x_{i-1} and x_i along z_i
θ_i : angle between x_{i-1} and x_i about z_i



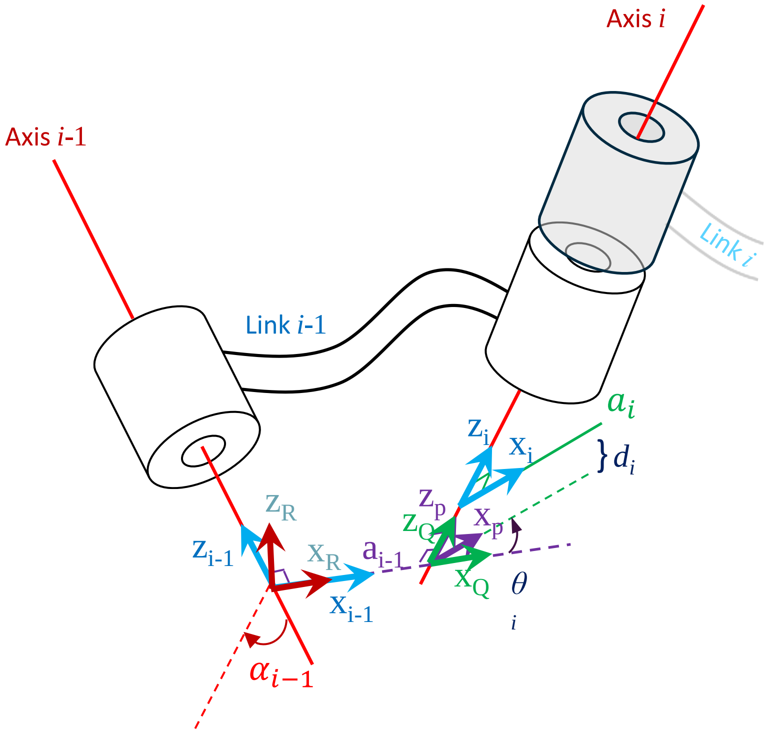
Link Description

Forward Kinematics:

$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T $$

$$ {}^{i-1}_{i}T(\alpha_{i-1}, a_{i-1}, \theta_i, d_i) = R_x(\alpha_{i-1}) D_x(a_{i-1}) R_z(\theta_i) D_z(d_i) $$

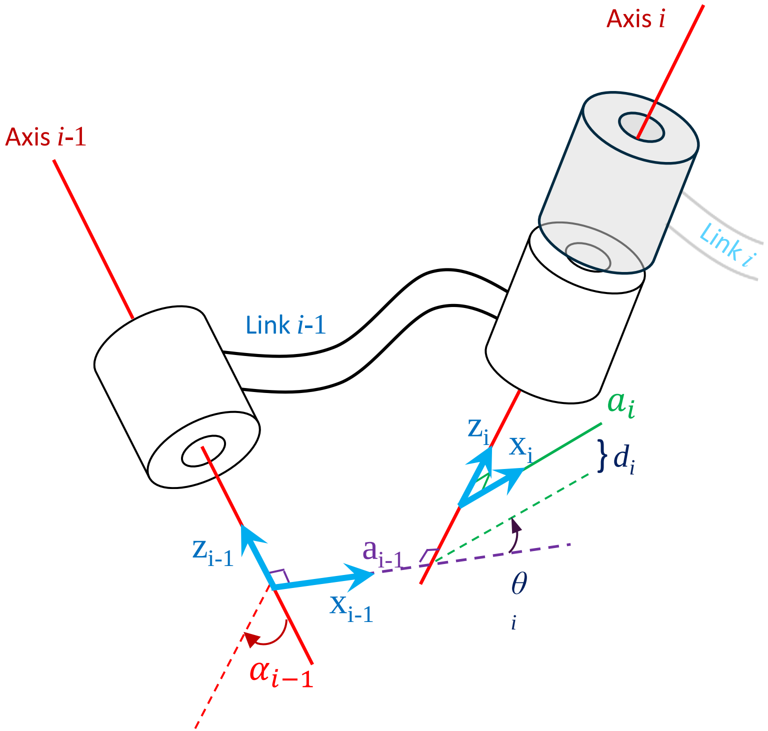
$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Link Description

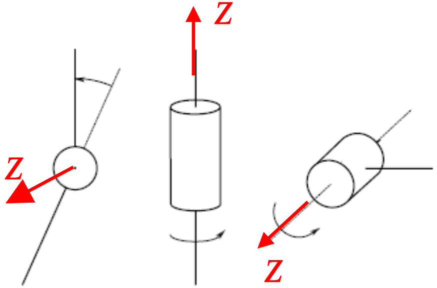
Forward Kinematics:

$$ {}^{0}_{N}T = {}^{0}_{1}T \ {}^{1}_{2}T \ \cdots \ {}^{N-1}_{N}T $$

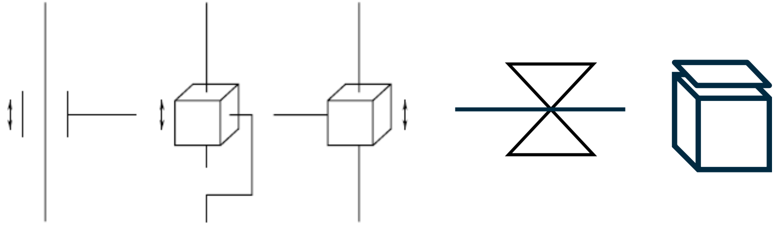


Symbols:

Revolute Joints:

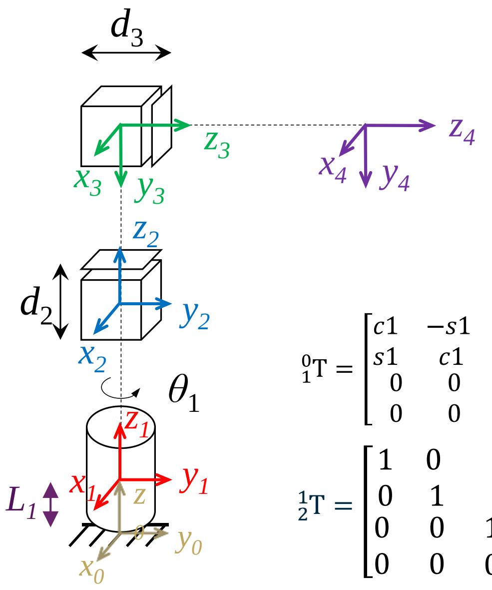


Prismatic joints:



Forward Kinematics

- **Example, RPP:**



$$ {}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & L1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

D-H parameter:

α_i : angle between z_i and z_{i+1} about x_i

a_i : distance between z_i and z_{i+1} along x_i

d_i : distance between x_{i-1} and x_i along z_i

θ_i : angle between x_{i-1} and x_i about z_i

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	L1	θ_1
2	0	0	d_2	0
3	-90°	0	d_3	0

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$${}^{0}_{N}\mathrm{T} = {}^{0}_{1}\mathrm{T}\ {}^{1}_{2}\mathrm{T} \cdots {}^{N-1}_{N}\mathrm{T}$$

$${}^{0}_{1}T \qquad {}^{0}_{1}\mathrm{T} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & L1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{2}T = {}^{0}_{1}T\,{}^{1}_{2}T \qquad {}^{0}_{2}\mathrm{T} = {}^{0}_{1}\mathrm{T}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & L1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

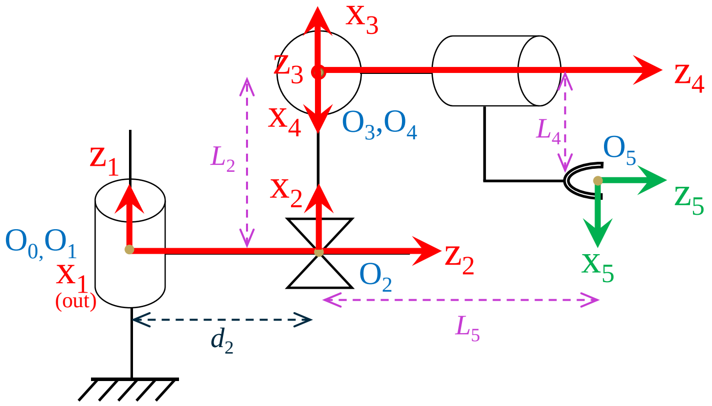
$${}^{0}_{3}T = {}^{0}_{2}T\,{}^{2}_{3}T \qquad {}^{0}_{3}\mathrm{T} = {}^{0}_{2}\mathrm{T}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1 & 0 & -s1 & -s1d_3 \\ s1 & 0 & c1 & c1d_3 \\ 0 & -1 & 0 & L1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example- RPRR:

(a) Find DH parameters

(b) Find each ${}^{i-1}_{i}\mathrm{T}$

(c) Find ${}^{0}_{5}\mathrm{T}$



D-H parameter:

α_i : angle between z_i and z_{i+1} about x_i

a_i : distance between z_i and z_{i+1} along x_i

d_i : distance between x_{i-1} and x_i along z_i

θ_i : angle between x_{i-1} and x_i about z_i

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90°	0	d_2	-90°
3	-90°	L_2	0	θ_3
4	90°	0	0	θ_4
5	0	L_4	L_5	0

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90°	0	d_2	-90°
3	-90°	L_2	0	θ_3
4	90°	0	0	θ_4
5	0	L_4	L_5	0

$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^{1}_{2}T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^{2}_{3}T = \begin{bmatrix} c3 & -s3 & 0 & L_2 \\ 0 & 0 & 1 & 0 \\ -s3 & -c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} c4 & -s4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^{4}_{5}T = \begin{bmatrix} 1 & 0 & 0 & L_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$${}^{0}_{N}\mathrm{T} = {}^{0}_{1}\mathrm{T}\ {}^{1}_{2}\mathrm{T}\ \cdots\ {}^{N-1}_{N}\mathrm{T}$$

${}^{0}_{2}T$

$${}^{0}_{1}\mathrm{T} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}\mathrm{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{0}_{2}\mathrm{T} = \begin{bmatrix} 0 & c1 & -s1 & -s1d_2 \\ 0 & s1 & c1 & c1d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

${}^{0}_{3}T$

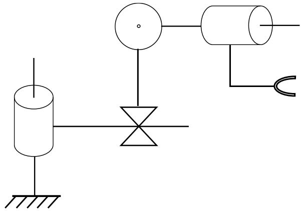
$${}^{0}_{2}\mathrm{T} = \begin{bmatrix} 0 & c1 & -s1 & -s1d_2 \\ 0 & s1 & c1 & c1d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}\mathrm{T} = \begin{bmatrix} c3 & -s3 & 0 & L_2 \\ 0 & 0 & 1 & 0 \\ -s3 & -c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{0}_{3}\mathrm{T} = \begin{bmatrix} s1s3 & s1c3 & c1 & -s1d_2 \\ -c1s3 & -c1c3 & s1 & c1d_2 \\ c3 & -s3 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

${}^{0}_{4}T$

$${}^{0}_{3}\mathrm{T} = \begin{bmatrix} s1s3 & s1c3 & c1 & -s1d_2 \\ -c1s3 & -c1c3 & s1 & c1d_2 \\ c3 & -s3 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{3}_{4}\mathrm{T} = \begin{bmatrix} c4 & -s4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{0}_{4}\mathrm{T} = \begin{bmatrix} s1s3c4 + c1s4 & c1c4 - s1s3s4 & -c3s1 & -s1d_2 \\ -c1s3c4 + s4s1 & s1c4 + c1s3s4 & c1c3 & c1d_2 \\ c3c4 & -c3s4 & s3 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

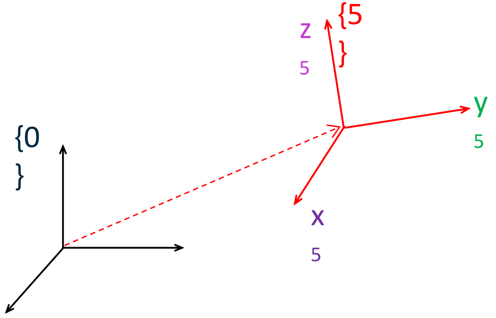
$${}^{0}_{5}T = {}^{0}_{4}T\,{}^{4}_{5}T$$

$${}^{0}_{5}\mathrm{T} = \begin{bmatrix} s1s3c4 + c1s4 & c1c4 - s1s3s4 & -c3s1 & L_4s1s3c4 + ((-L_5c3 - d_2)s1 + L_4c1s4) \\ -c1s3c4 + s4s1 & s1c4 + c1s3s4 & c1c3 & -L_4c1s3c4 + (L_4s1s4) + (L_5c3 + d_2)c1 \\ c3c4 & -c3s4 & s3 & L_4c3c4 + (L_5s3 + L_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$
{}^{0}_{5}\mathrm{T} = \begin{bmatrix} s1s3c4 + c1s4 & c1c4 - s1s3s4 & -c3s1 & L_4s1s3c4 + ((-L_5c3 - d_2)s1 + L_4c1s4) \\ -c1s3c4 + s4s1 & s1c4 + c1s3s4 & c1c3 & -L_4c1s3c4 + (L_4s1s4 + (L_5c3 + d_2)c1 \\ c3c4 & -c3s4 & s3 & L_4c3c4 + (L_5s3 + L_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
x = \begin{bmatrix} x_p \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} L_4s1s3c4 + ((-L_5c3 - d_2)s1 + L_4c1s4) \\ -L_4c1s3c4 + (L_4s1s4 + (L_5c3 + d_2)c1 \\ L_4c3c4 + (L_5s3 + L_2) \\ s1s3c4 + c1s4 \\ -c1s3c4 + s4s1 \\ c3c4 \\ c1c4 - s1s3s4 \\ s1c4 + c1s3s4 \\ -c3s4 \\ -c3s1 \\ c1c3 \\ s3 \end{bmatrix}
$$



Example *A*, RRR:

(a) Find DH parameters

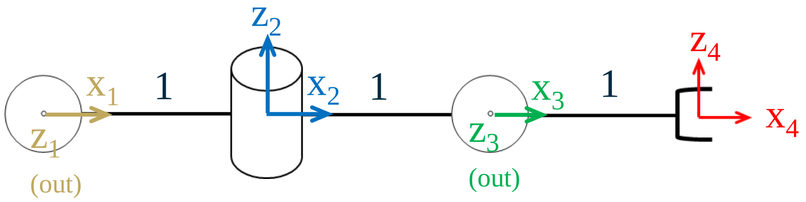
(b) Find each ${}^{i-1}_{i}\mathrm{T}$ (${}^{0}_{1}\mathrm{T}$, ${}^{1}_{2}\mathrm{T}$, ${}^{2}_{3}\mathrm{T}$, ${}^{3}_{4}\mathrm{T}$)

(c) Find ${}^{0}_{5}\mathrm{T}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1				
2				
3				
4				

$$
{}^{0}_{1}\mathrm{T} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^{1}_{2}\mathrm{T} = \begin{bmatrix} c2 & -s2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -s2 & -c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^{2}_{3}\mathrm{T} = \begin{bmatrix} c3 & -s3 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^{3}_{4}\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{4}T = \begin{bmatrix} c1c2c3 - s1s3 & -c1s2 & -c1c2s3 - s1c3 & c1c2c3 - s1s3 + c1c2 + c1 \\ s1c2c3 + c1s3 & -s1s2 & -s1c2s3 + c1c3 & s1c2c3 + c1s3 + s1c2 + s1 \\ -s2c3 & -c2 & s2s3 & s2c3 - s2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Assignment:

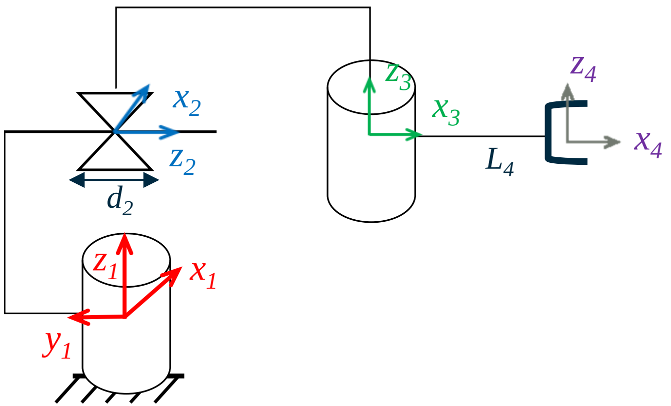
- ➢ Find ${}^{0}_{2}\mathrm{T}$, ${}^{0}_{3}\mathrm{T}$, ${}^{0}_{4}\mathrm{T}$

Example *B*, RPR:

(a) Find DH parameters

(b) Find each ${}^{i-1}_{i}\mathrm{T}$

(c) Find ${}^{0}_{5}\mathrm{T}$



$${}^{0}_{1}\mathrm{T} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1				
2				
3				
4				

$${}^{2}_{3}\mathrm{T} = \begin{bmatrix} c3 & -s3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s3 & -c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

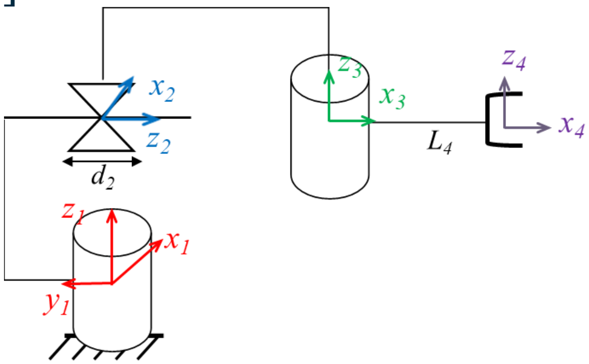
$${}^{3}_{4}\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & L_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{i-1}_{i}\mathrm{T} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} c3 & -s3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s3 & -c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & L_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{2}T = \begin{bmatrix} c1 & 0 & s1 & s1d2 \\ s1 & 0 & -c1 & -c1d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{0}_{3}T = \begin{bmatrix} c1c3 - s1s3 & -c1s3 & 0 & s1d2 \\ c1s3 + s1c3 & c1c3 - s1s3 & 0 & -c1d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{4}T = \begin{bmatrix} c1c3 - s1s3 & -c1s3 & 0 & L4c1c3 - L4s1s3 + s1d2 \\ c1s3 + s1c3 & c1c3 - s1s3 & 0 & L4s1c3 + L4c1s3 - c1d2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Lecture 5 Summary

- Link Description
- D-H Parameters
- Forward Kinematics