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Partial differentiation: In practice

Recall the definition of the partial derivative for functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Definition 17.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $(x_0, y_0) \in \mathbb{R}^2$. Then:

- the *partial derivative of f with respect to x* (at (x_0, y_0)), denoted $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$, by the limit
$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$
- the *partial derivative of f with respect to y* (at (x_0, y_0)), denoted $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$, by the limit
$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Essentially, the partial derivative gives the gradient of the graph of $f(x, y)$ at (x_0, y_0) in the x and y directions, respectively. As we saw with derivatives, there are generally nice ways in which to compute these, rather than just relying on the definition.

To compute a partial derivative, consider the other variables as constants,

and then differentiate as you normally would.

Example 17.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^3$. Let us first compute $f_x(x, y)$. Indeed, we treat the variable y as a constant, i.e. so it differentiates to 0. Therefore,

$$f_x(x, y) = \frac{\partial}{\partial x}(f(x, y)) = \frac{\partial}{\partial x}\left(x^2 + y^3\right) = \frac{d}{dx}\left(x^2\right) + 0 = 2x.$$

In the above, since y is a constant, it follows y^3 differentiates to 0.

Similarly, we can compute $f_y(x, y)$. Indeed, we treat the variable x as a constant. Therefore,

$$f_y(x, y) = \frac{\partial}{\partial y}(f(x, y)) = \frac{\partial}{\partial y}\left(x^2 + y^3\right) = 0 + \frac{d}{dy}\left(y^3\right) = 3y^2.$$

Again, in the above, since we viewed x as a constant, then x^2 differentiates to 0.

Now, let us define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = x^2 y + x$. Let us compute both g_x first. Again, we treat y as a constant. Therefore,

$$g_x(x,y) = \frac{\partial}{\partial x}\left(g(x,y)\right) = \frac{\partial}{\partial x}\left(x^2y + x\right) = y\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(1) = 2xy + 1.$$

In the above, since y is a constant, we can pull it out and applying the differential to the 'important' bit, that is x^2.

Similarly, let us compute $g_y(x,y)$. Indeed, we treat the variable y as a constant. Therefore,

$$g_y(x,y) = \frac{\partial}{\partial y}\left(g(x,y)\right) = \frac{\partial}{\partial y}\left(x^2y + x\right) = x^2\frac{d}{dx}(y) + 0 = x^2.$$

For the first part, as x^2 is a constant we can take it out and differentiate the 'important' bit, that is y. For the second we treat x as a constant, so with respect to y, it differentiates to 0.

So far we have seen partial differentiation with respect to functions $f : \mathbb{R}^2 \to \mathbb{R}$. However, there is nothing stopping us for considering partial derivatives for functions $f : \mathbb{R}^n \to \mathbb{R}$. The idea is still the same. When we want to compute the partial derivatives with respect to the variables $(x, y, z, \ldots)$ we proceed as before: we view the other variables as constants and differentiate how we normally would.

Example 17.3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = xy + yz + 2x^2y^3z$. Then:

$$f_x(x,y,z) = \frac{\partial}{\partial x}\left(f(x,y,z)\right) = \frac{\partial}{\partial x}\left(xy + yz + 2x^2y^3z\right) = y + 4xy^3z.$$

$$f_y(x,y,z) = \frac{\partial}{\partial y}\left(f(x,y,z)\right) = \frac{\partial}{\partial y}\left(xy + yz + 2x^2y^3z\right) = x + z + 6x^2y^2z.$$

$$f_z(x,y,z) = \frac{\partial}{\partial z}\left(f(x,y,z)\right) = \frac{\partial}{\partial z}\left(xy + yz + 2x^2y^3z\right) = y + 2x^2y^3.$$

17.1 Higher order derivatives

If you recall when we first introduced differentiation in Chapter 3 after we had defined the first derivative of a function, we defined the notion of higher order derivatives; namely $\frac{d^2}{dx^2}$ and $\frac{d^2}{dy^2}$. We can do similar things in the context of partial derivatives.

Definition 17.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Then:

- $f_{xx}(x,y) = \dfrac{\partial^2}{\partial x^2}\left(f(x,y)\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(f(x,y)\right)\right);$
- $f_{yy}(x,y) = \dfrac{\partial^2}{\partial y^2}\left(f(x,y)\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\left(f(x,y)\right)\right);$
- $f_{yx}(x,y) = \dfrac{\partial^2}{\partial x \partial y}\left(f(x,y)\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y}\left(f(x,y)\right)\right);$

- $f_{xy}(x,y) = \dfrac{\partial}{\partial y \partial x}\left(f(x,y)\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x}\left(f(x,y)\right)\right)$.

You may observe that we have two further notions of derivatives in the case for a function $f : \mathbb{R}^2 \to \mathbb{R}$; we call these the mixed derivatives of f. The notation of these derivatives may vary from text to text, but mainly they adopt the following convention: if we write the notation f_{xy} what we mean is to compute the x derivative first, and then the y derivative, i.e.

$$\frac{\partial^2}{\partial y \partial x}\left(f(x,y)\right) = f_{xy}(x,y)$$

Observe that the order in which we write down the derivative as a 'fraction' is the reverse order to how we write the f_{xy} notation. The latter notation tells you the order in which to differentiate if you read the variables from the left to the right, i.e. differentiate with respect to x first and then y, whereas the quotient notation is the reverse.

Please be careful with this, as it is very easy to make mistakes (just as I did in the recorded lecture!)

Example 17.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 2xy^2 + 3xy$. Then:

$$f_x(x,y) = \frac{\partial}{\partial x}\left(f(x,y)\right) = \frac{\partial}{\partial x}\left(2xy^2 + 3xy\right) = 2y^2 + 3y.$$

Similarly,

$$f_y(x,y) = \frac{\partial}{\partial y}\left(f(x,y)\right) = \frac{\partial}{\partial y}\left(2xy^2 + 3xy\right) = 4xy + 3x.$$

Let us now calculate the second order derivatives. Indeed,

$$f_{xx}(x,y) = \frac{\partial^2}{\partial x^2}\left(f(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(f(x,y)\right)\right) = \frac{\partial}{\partial x}\left(f_x(x,y)\right) = \frac{\partial}{\partial x}\left(2y^2 + 3y\right) = 0.$$

and

$$f_{yy}(x,y) = \frac{\partial^2}{\partial y^2}\left(f(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(f(x,y)\right)\right) = \frac{\partial}{\partial y}\left(f_y(x,y)\right) = \frac{\partial}{\partial y}\left(4xy + 3x\right) = 4x.$$

Let us now compute the mixed derivatives for f. Remember the difference in the order given in the notation!

$$f_{xy}(x,y) = \frac{\partial^2}{\partial y \partial x}\left(f(x,y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(f(x,y)\right)\right) = \frac{\partial}{\partial y}\left(f_x(x,y)\right) = \frac{\partial}{\partial y}\left(2y^2 + 3y\right) = 4y + 3.$$

Similarly,

$$f_{yx}(x,y) = \frac{\partial^2}{\partial x \partial y}\left(f(x,y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(f(x,y)\right)\right) = \frac{\partial}{\partial x}\left(f_y(x,y)\right) = \frac{\partial}{\partial x}\left(4xy + 3x\right) = 4y + 3.$$

You may notice that in this case $f_{xy}(x,y) = f_{yx}(x,y)$. This (unfortunately) does not happen always. However, when f behaves sufficiently nicely, so with most cases you should see throughout the rest of this course, this will happen. So in such cases, it does not matter in which order you take the mixed derivatives!

As before with differentiation of functions $f : \mathbb{R} \to \mathbb{R}$ we introduced the notion of third and fourth order derivatives. We can do the same here with partial differentiation, but the ideas remain the same but the definitions become cumbersome and do not provide any additional information. So we will just calculate such derivatives in the following example.

Example 17.6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = xy + yz + 2x^2y^3z$. Then from Example 17.3 we know:

$$f_x(x,y,z) = y + 4xy^3z, \quad f_y(x,y,z) = x + z + 6x^2y^2z \quad \text{and} \quad f_z(x,y,z) = y + 2x^2y^3.$$

Let us first compute now compute f_{xx}, f_{xy}, f_{xz}. Indeed,

$$f_{xx}(x,y,z) = \frac{\partial}{\partial x}\left(f_x(x,y,z)\right) = \frac{\partial}{\partial x}\left(y + 4xy^3z\right) = 4y^3z.$$

$$f_{xy}(x,y,z) = \frac{\partial}{\partial y}\left(f_x(x,y,z)\right) = \frac{\partial}{\partial y}\left(y + 4xy^3z\right) = 1 + 12xy^2z.$$

$$f_{xz}(x,y,z) = \frac{\partial}{\partial z}\left(f_x(x,y,z)\right) = \frac{\partial}{\partial z}\left(y + 4xy^3z\right) = 4xy^3.$$

One may argue similarly to obtain all the second order derivatives. Indeed,

$$f_{yx}(x,y,z) = 1 + 12xy^2z, \quad f_{yy}(x,y,z) = 12x^2yz \quad \text{and} \quad f_{yz}(x,y,z) = 1 + 6x^2y^2$$

and

$$f_{zx}(x,y,z) = 4xy^3, \quad f_{zy}(x,y,z) = 1 + 6x^2y^2 \quad \text{and} \quad f_{zz}(x,y,z) = 0.$$

It is a good exercise (which I would recommend doing) to double check the above mixed derivatives. Let us now compute a few third order derivatives (the rest are left as a (cumbersome) exercise). Indeed,

$$f_{xxx}(x,y,z) = \frac{\partial}{\partial x}\left(f_{xx}(x,y,z)\right) = \frac{\partial}{\partial x}\left(4y^3z\right) = 0,$$

$$f_{xxy}(x,y,z) = \frac{\partial}{\partial y}\left(f_{xx}(x,y,z)\right) = \frac{\partial}{\partial y}\left(4y^3z\right) = 12y^2z,$$

$$f_{xxz}(x,y,z) = \frac{\partial}{\partial z}\left(f_{xx}(x,y,z)\right) = \frac{\partial}{\partial z}\left(4y^3z\right) = 4y^3,$$

and so on.