16

Introduction to partial differentiation

Previously we have seen functions $f : \mathbb{R} \to \mathbb{R}$ which take an element from the real numbers and map this to another real number. However, we can define functions $f : \mathbb{R}^n \to \mathbb{R}$ in a similar way.

Example 16.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + y, xy)$. This functions takes two inputs, x and y, and gives two outputs, $x + y$ and xy.

Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = \sin^{-1}\left(\sqrt{x^2 + y^2}\right)$. We can view the graph of such a function in $\mathbb{R}^3$.

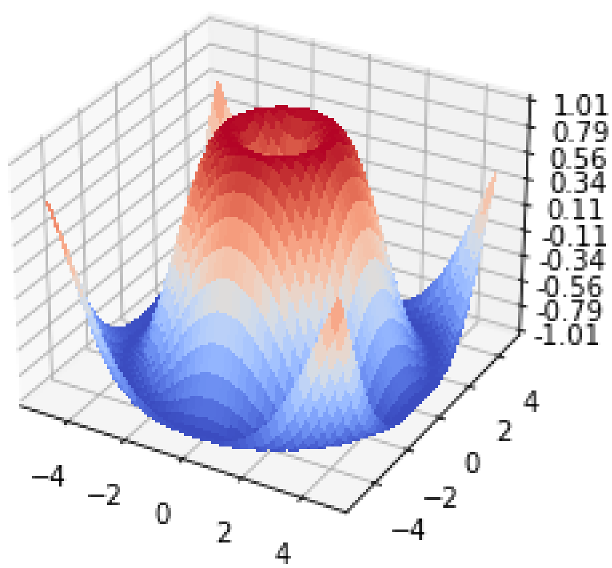

Figure 16.1: Graph of $z = g(x, y)$

Such functions are called multivariable functions. Throughout the rest of this bootcamp and your course, you will be mainly interested in functions $f : \mathbb{R}^n \to \mathbb{R}$, i.e. those functions which take n input values and return 1 output. When we have such a function $f : \mathbb{R}^2 \to \mathbb{R}$, say, can we determine anything about the 'gradient', or derivative, of f? Unfortunately, 'gradient' here doesn't make too much sense, since we cannot choose line segments which we can then calculate the gradient of. However, we could calculate the gradient of 'slices' of f in the x and y direction.

Example 16.2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = x^2 + y^2$. The graph of f can then be seen in Figure 16.2.

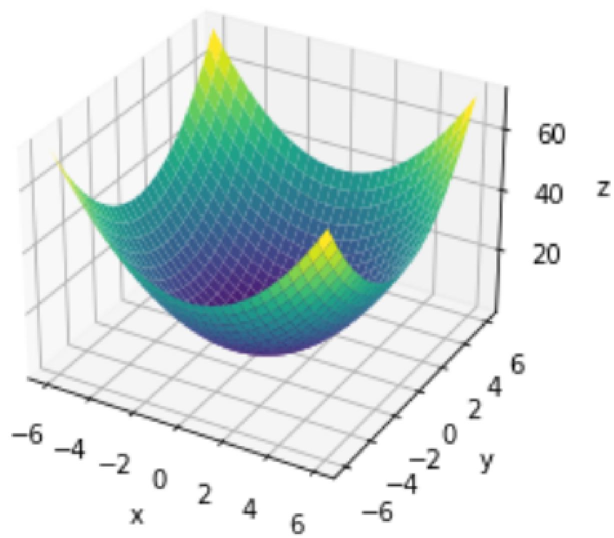

Figure 16.2: Graph of $z = f(x, y)$

Let us first compute a 'slice' of $z = f(x, y)$ in the x-direction at $y = 1$. From Figure 16.2 we can see this will produce the graph of the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2 + 1$. We can compute the derivative of such a function. Indeed,

$$g'(x) = 2x.$$

So this is saying that when $y = 1$, the graph $z = f(x, y)$ grows at a speed of $2x$ in the x-direction.

Similarly, we can compute a 'slice' of $z = f(x, y)$ in the y-direction at $x = 1$. From Figure 16.2 we can see this will produce the graph of the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(y) = y^2 + 1$. Again, we can compute the derivative of such a function. Indeed,

$$h'(y) = 2y.$$

What is then saying is that at $x = 1$, the graph of $z = f(x, y)$ grows at a speed of $2y$ in the y-direction.

We can try this for different values of y and x. Indeed, for each $a \in \mathbb{R}$, let us consider the two functions $g, h : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x^2 + a^2 \quad \text{and} \quad h(y) = a^2 + y^2.$$

Then the graph of g (respectively, h) becomes the 'slice' of $z = f(x, y)$ at $y = a$ (respectively, $x = a$) in the x-direction (respectively, y-direction). Therefore,

$$g'(x) = 2x \quad \text{and} \quad h'(y) = 2y.$$

We can then define the *partial derivatives* of f to be the gradient of $z = f(x, y)$ in the x and y directions, i.e.

$$f_x(x, y) = 2x \quad \text{and } f_y(x, y) = 2y.$$

Definition 16.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $(x_0, y_0) \in \mathbb{R}^2$. Then:

- the *partial derivative of f with respect to x* (at (x_0, y_0)), denoted $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$, by the limit
$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$
- the *partial derivative of f with respect to y* (at (x_0, y_0)), denoted $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$, by the limit
$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$