Mathematical and Logical Foundations of Computer Science

Lecture 4 - Propositional Logic (Natural Deduction)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Natural Deduction proofs

Recap: Connectives & Special Atomic Propositions

Syntax

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Two special atoms:

- $\top$ which stands for True
- $\bot$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \bot$

Recap: Proofs in Propositional Logic

For **formal proofs**, we need two things

1. A **formal** language
   - for representing propositions, arguments
   - here we are using propositional logic
2. A **proof** theory
   - to prove ("infer", "deduce") whether an argument is valid
   - inference rules, which are the building blocks of proofs

Recap: What are inference rules?

Inference rules are the tools we are allowed to use

Careful with the rules you assume otherwise you might be able to prove false statements!

Example of an inference rule (**and-introduction** rule):

$$\frac{A \qquad B}{A \wedge B}\ [\wedge I]$$

These are **rule schemata**, where here A and B are **metavariables** ranging over all possible propositions.

Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

Recap: Some simple inference rules

And-introduction

$$\frac{A \qquad B}{A \wedge B}\ [\wedge I]$$

implication-elimination

$$\frac{A \to B \qquad A}{B}\ [\to E]$$

False-elimination

$$\frac{\bot}{A}\ [\bot\, E]$$

True-introduction

$$\frac{}{\top}\ [\top I]$$

Recap: A simple proof

Negation-elimination, i.e., both A and $\neg A$ cannot be true at same time

Formally, want to prove $A, \neg A \ \vdash \perp$

A **proof** is a tree of instances of inference rules.

Assuming that $\neg A$ is defined as $A \rightarrow \perp$, a proof of the above sequent (or argument) is:

$$\frac{A \quad \neg A}{\perp} \ [\rightarrow E]$$

Recap: Another simple proof

Given three hypotheses A, B, C, how can we prove $(A \wedge B) \wedge (A \wedge C)$?

Here is a proof:

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\,[\wedge I] \qquad \dfrac{A \quad C}{A \wedge C}\,[\wedge I]}{(A \wedge B) \wedge (A \wedge C)}\,[\wedge I]$$

The rule used at each step is **and-introduction**, i.e., $[\wedge I]$

Natural Deduction

Framework

- "natural" style of constructing a proof
- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion

Two key points:

- Can work both forwards and backwards
- Natural doesn't mean there is unique proof

Introduced by **Gentzen** in 1934
and further studied by **Prawitz** in 1965.

Slightly confusing aspect of natural Deduction

Discharging/cancellation of hypothesis

$$\dfrac{\dfrac{A\overline{A}^{\ 1}}{\vdots}\quad B}{A \to B}\ 1\ [\to I]$$

This is the "implication-introduction" rule.

We don't have to make use of A in which case we can just omit it:

$$\frac{B}{A \to B}$$

Cancelling hypothesis continued

Given the hypothesis A, C how can we prove $B \to ((A \land B) \land (A \land C))$?

Here is a proof:

$$\dfrac{\dfrac{\dfrac{A \quad B\overline{B}^{\,1}}{A \land B}[\land I] \quad \dfrac{A \quad C}{A \land C}[\land I]}{(A \land B) \land (A \land C)}[\land I]}{B \to ((A \land B) \land (A \land C))}\,1\;[\to I]$$

At this point, we can also cancel another hypothesis, say A

This gives a proof of

$$A \to (B \to ((A \land B) \land (A \land C)))$$

using the hypothesis C only

Cancelling hypothesis continued

We proved it forward, but we can also prove it backward:

$$\dfrac{\dfrac{\dfrac{A \quad \overline{B}^{\,1}}{A \wedge B}\,[\wedge I] \qquad \dfrac{A \quad C}{A \wedge C}\,[\wedge I]}{(A \wedge B) \wedge (A \wedge C)}\,[\wedge I]}{B \rightarrow ((A \wedge B) \wedge (A \wedge C))}\ 1\ [\rightarrow I]$$

Comprehensive set of inference rules

Rules for $\rightarrow$ (implication)

- implication-introduction

$$\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots}\\ B}{A \rightarrow B}\ 1\ [\rightarrow I]$$

- implication-elimination

$$\dfrac{A \rightarrow B \qquad A}{B}\ [\rightarrow E]$$

Comprehensive set of inference rules

Rules for $\neg$ (not)

- Negation-introduction

$$\dfrac{\begin{array}{c}\overline{A}^{\;1}\\ \vdots\\ \bot\end{array}}{\neg A}\;1\;[\neg I]$$

- Negation-elimination

$$\dfrac{A \qquad \neg A}{\bot}\;[\neg E]$$

Comprehensive set of inference rules

Rules for $\vee$ (or)

- or-introduction (for any formula B)

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad\qquad \frac{A}{B \vee A}\ [\vee I_R]$$

- or-elimination

$$\frac{A \vee B \qquad A \to C \qquad B \to C}{C}\ [\vee E]$$

More comprehensive set of inference rules

Rules for $\wedge$ (and)

- and-introduction

$$\frac{A \qquad B}{A \wedge B}\ [\wedge I]$$

- and-elimination

$$\frac{A \wedge B}{B}\ [\wedge E_R] \qquad\qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

A simple natural Deduction proof

Given $A \to B$ and $B \to C$, give a proof of $A \to C$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{A\overline{A}^{\,1} \qquad A \to B}{B}\,[\to E] \qquad B \to C}{C}\,[\to E]}{A \to C}\,1\,[\to I]$$

And backward?

$$\dfrac{\dfrac{\dfrac{\overline{A}^{\,1} \qquad A \to B}{B}\,[\to E] \qquad B \to C}{C}\,[\to E]}{A \to C}\,1\,[\to I]$$

We also need to go forward to prove C

Another simple natural Deduction proof

Given $\neg A \vee B$ and A, how do we derive B?

Here is a proof:

$$\dfrac{\neg A \vee B \qquad \dfrac{\dfrac{\dfrac{A \quad \neg A\overline{\neg A}^{\,1}}{\bot}\,[\neg E]}{B}\,[\bot\, E]}{\neg A \to B}\,1\,[\to I] \qquad \dfrac{B\overline{B}^{\,2}}{B \to B}\,2\,[\to I]}{B}\,[\vee E]$$

Backward? We go forward because we are left with just B

$$\dfrac{\neg A \vee B \qquad \dfrac{\dfrac{\dfrac{A \quad \overline{\neg A}^{\,1}}{\bot}\,[\neg E]}{B}\,[\bot\, E]}{\neg A \to B}\,1\,[\to I] \qquad \dfrac{\overline{B}^{\,2}}{B \to B}\,2\,[\to I]}{B}\,[\vee E]$$

Forward & backward reasoning in Natural Deduction

We typically go both **forward and backward** in proofs

Show $(B \wedge A)$ given the hypothesis $(A \wedge B)$

Here is a proof:

$$\dfrac{\dfrac{A \wedge B}{B}\;[\wedge I] \qquad \dfrac{A \wedge B}{A}\;[\wedge I]}{B \wedge A}\;[\wedge I]$$

Complicated looking question

Prove the following:

$R\ , (P \to Q) \land (Q \to P)\ , Q \to Z\ , R \to P \vdash Z$

Here is a proof:

$$\dfrac{Q \to Z \qquad \dfrac{\dfrac{R \quad R \to P}{P}\ [\to E] \qquad \dfrac{P \to Q \land Q \to P}{P \to Q}\ [\land E]}{Q}\ [\to E]}{Z}\ [\to E]$$

Conclusion

What did we cover today?

- Natural Deduction rules for propositional logic
- Natural Deduction proofs
- Forward & backward reasoning

Next time?

- Sequent calculus