Robotics –
Planning and Motion

Kinematics

COMP52815

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Lecture 4: Learning Objectives

The aim of this lecture is to build a model which will lead to the kinematics.

- Objectives:
  1. Spatial Description
  2. Transformation
     - Rotation
     - Translation

See also:

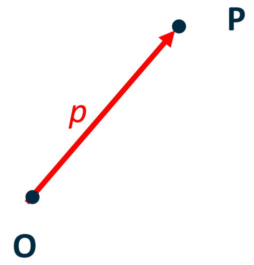
- Robot Modeling and Control, Spong et al, C1
- Robotics: Modelling, Planning and Control, Siciliano et al, C1

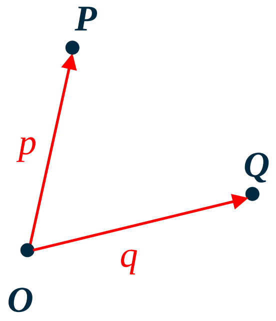
Durham University

Spatial Description

- **Position of a Point:**

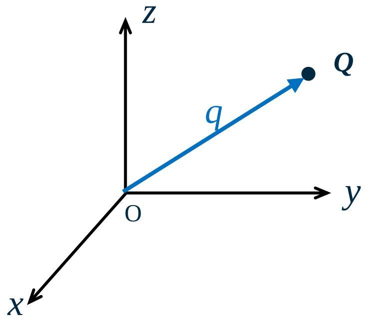
With respect to a fixed origin **O**, the position of a point P is described by the vector **OP** (p).





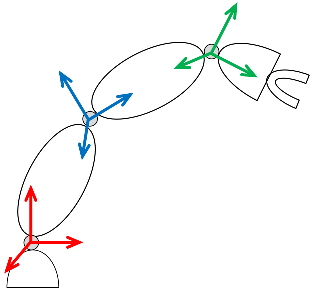
$$p = \overrightarrow{OP}$$
$$q = \overrightarrow{OQ}$$

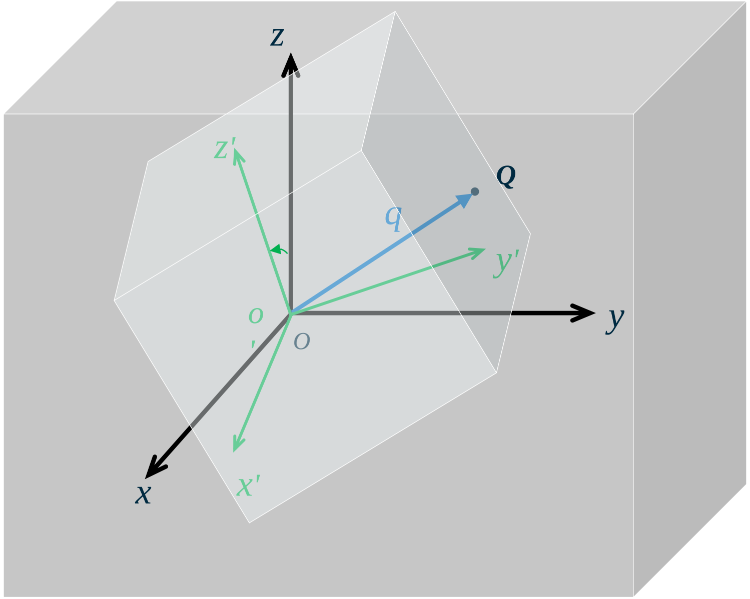
Cartesian Frame



Spatial Description

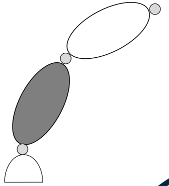
- **Coordinate Frames:**
  - Rotation
  - Translation

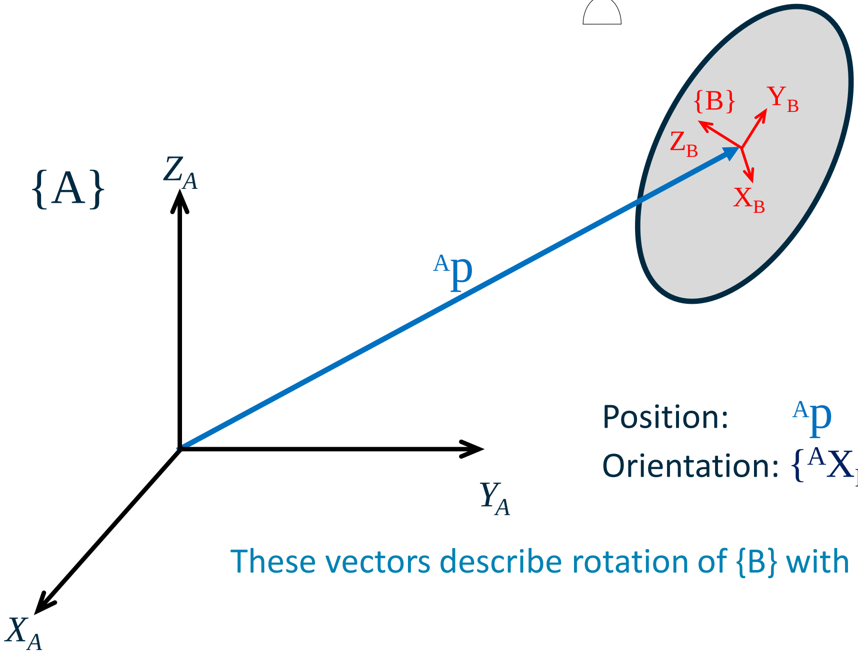




Spatial Description

- **Rigid body configuration:**





Position: ${}^{A}p$

Orientation: $\{{}^{A}X_B, {}^{A}Y_B, {}^{A}Z_B\}$

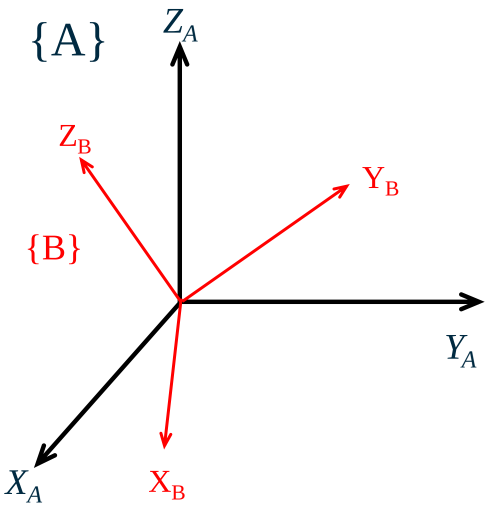
These vectors describe rotation of {B} with respect to {A}.

Transformation

- **Rotation Matrix:**

$$ {}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

- State description: ${}^{A}\hat{X}_{B} = {}^{A}_{B}R \; {}^{B}\hat{X}_{B}$



$$ {}^{A}\hat{X}_{B} = {}^{A}_{B}R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_{B} = {}^{A}_{B}R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_{B} = {}^{A}_{B}R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix}} $$

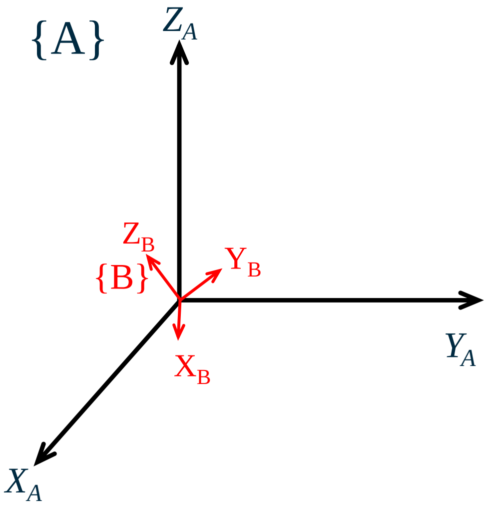
Transformation

- **Rotation Matrix:**

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} $$

- Dot product:

$$ {}^{A}\hat{X}_{B} = \begin{bmatrix} \hat{X}_{B}.\ \hat{X}_{A} \\ \hat{X}_{B}.\ \hat{Y}_{A} \\ \hat{X}_{B}.\ \hat{Z}_{A} \end{bmatrix} \quad {}^{A}\hat{Y}_{B} = \begin{bmatrix} \hat{Y}_{B}.\ \hat{X}_{A} \\ \hat{Y}_{B}.\ \hat{Y}_{A} \\ \hat{Y}_{B}.\ \hat{Z}_{A} \end{bmatrix} \quad {}^{A}\hat{Z}_{B} = \begin{bmatrix} \hat{Z}_{B}.\ \hat{X}_{A} \\ \hat{Z}_{B}.\ \hat{Y}_{A} \\ Z_{B}.\ \hat{Z}_{A} \end{bmatrix} $$



$$ {}^{A}_{B}R = \begin{bmatrix} \hat{X}_{B}.\hat{X}_{A} & \hat{Y}_{B}.\hat{X}_{A} & \hat{Z}_{B}.\hat{X}_{A} \\ \hat{X}_{B}.\hat{Y}_{A} & \hat{Y}_{B}.\hat{Y}_{A} & \hat{Z}_{B}.\hat{Y}_{A} \\ \hat{X}_{B}.\hat{Z}_{A} & \hat{Y}_{B}.\hat{Z}_{A} & \hat{Z}_{B}.\hat{Z}_{A} \end{bmatrix} \Rightarrow {}^{B}X_{A}{}^{T} $$

Transformation

- **Rotation Matrix:**

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_{B} \quad {}^{A}\hat{Y}_{B} \quad {}^{A}\hat{Z}_{B}] = \begin{bmatrix} {}^{B}\hat{X}_{A}{}^{T} \\ {}^{B}\hat{Y}_{A}{}^{T} \\ {}^{B}\hat{Z}_{A}{}^{T} \end{bmatrix} = [{}^{B}\hat{X}_{A} \quad {}^{B}\hat{Y}_{A} \quad {}^{B}\hat{Z}_{A}] = {}^{B}_{A}R^{T} $$

$$ \boxed{{}^{A}_{B}R = {}^{B}_{A}R^{T}} $$

- **Inverse of Rotation Matrix:**

$$ {}^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

- **Orthonormal Matrix**

$$ \boxed{{}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}} $$

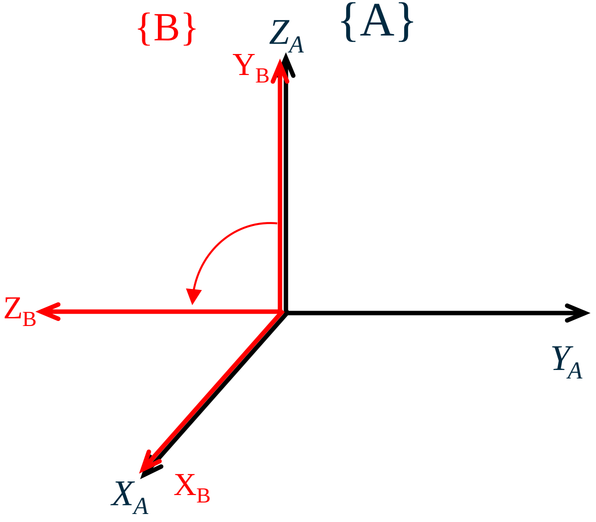
An orthonormal matrix is a square matrix which columns & rows are orthogonal unit vectors

Transformation

- **Example:**

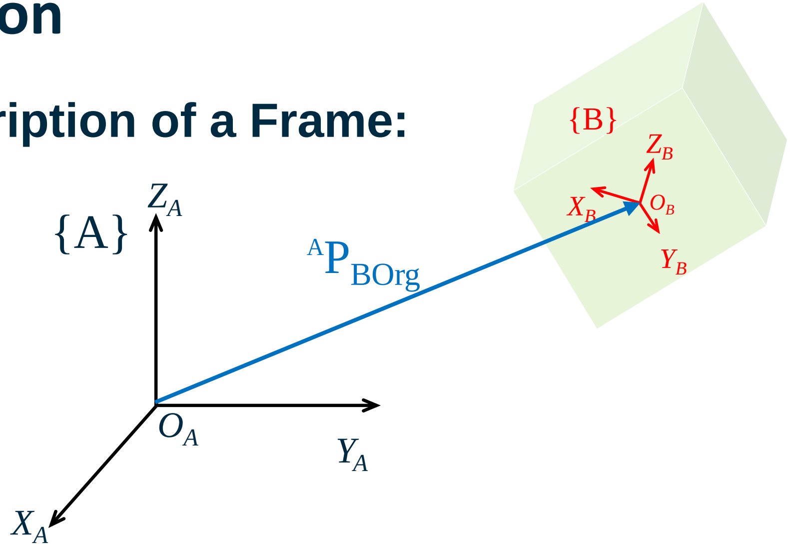
$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} $$

$$ \begin{array}{cc} & \begin{array}{ccc} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \\ \downarrow & \downarrow & \downarrow \end{array} \\ {}^{A}_{B}R = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow {}^{B}\hat{X}_{A}{}^{T} \\ \leftarrow {}^{B}\hat{Y}_{A}{}^{T} \\ \leftarrow {}^{B}\hat{Z}_{A}{}^{T} \end{array} \end{array} $$



Transformation

- **Description of a Frame:**

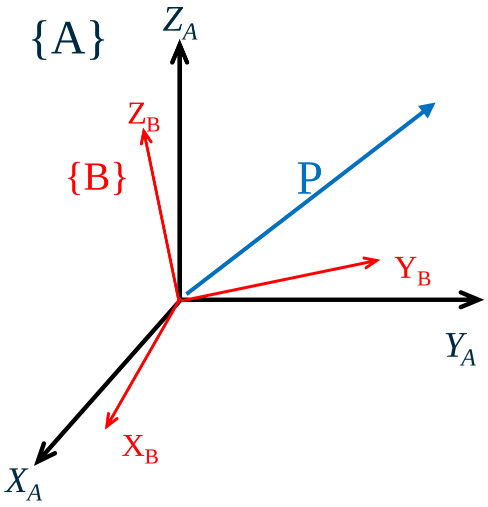


Frame {B}: ${}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B\,, \quad {}^{A}P_{BOrg}$

$$\{\mathrm{B}\} = \{{}^{A}_{B}R \quad {}^{A}P_{BOrg}\}$$

Transformation

- **Mapping:**
  - *Changing descriptions from frame to frame*
- **Rotations**



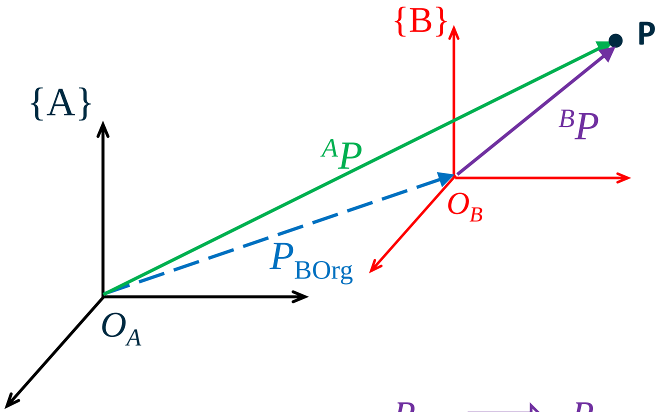
If P is in {B}: ^BP

$$^AP = \begin{bmatrix} {}^B\hat{X}_A \cdot {}^BP \\ {}^B\hat{Y}_A \cdot {}^BP \\ {}^B\hat{Z}_A \cdot {}^BP \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} \cdot {}^BP$$

$$^AP = {}^A_BR \;\; {}^BP$$

Transformation

- **Translation:**



$\overrightarrow{O_B P} \Longrightarrow \overrightarrow{O_A P}$

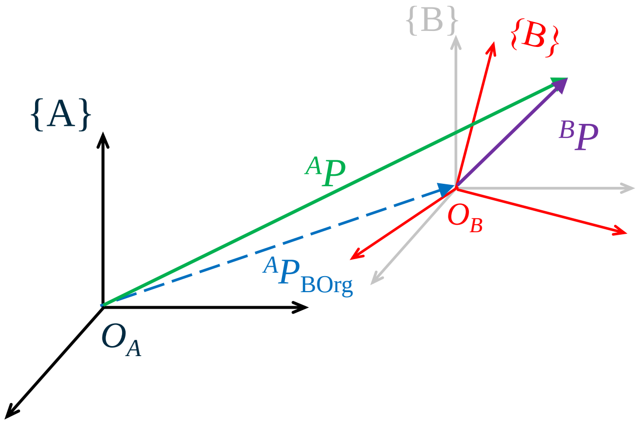
These are two different vectors

P_{BOrg}: $P_{O_B} \Longrightarrow P_{O_A}$

$$\boxed{^{A}P_{OA} = {}^{A}P_{OB} + {}^{A}P_{BOrg}}$$

Transformation

- **General Transformation:**



$$^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BOrg}$$

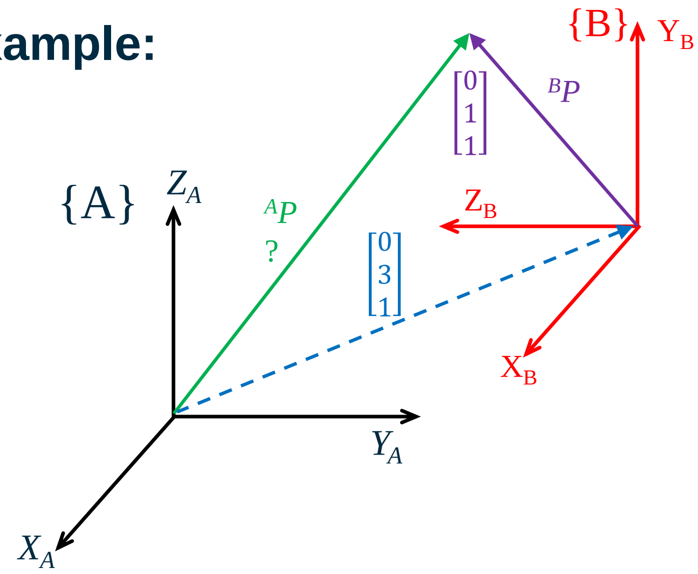
$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BOrg} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$

- Homogeneous Transformation:

$$^{A}P_{(4x1)} = {}^{A}_{B}T_{(4x4)} \; {}^{B}P_{(4x1)}$$

Transformation

- **Example:**



$$^{A}_{B}T = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -1 & 3\\0 & 1 & 0 & 1\\0 & 0 & 0 & 1\end{bmatrix}$$

$$^{B}P = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$$

$$^{A}P = \, ^{A}_{B}T \cdot {}^{B}P \quad \Rightarrow \quad ^{A}P = \begin{bmatrix}0\\2\\2\\1\end{bmatrix}$$

Transformation

- **General Operators:**

$$P_2 = \left[\begin{array}{ccc:c} & R_k(\theta) & & Q \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] P_1$$

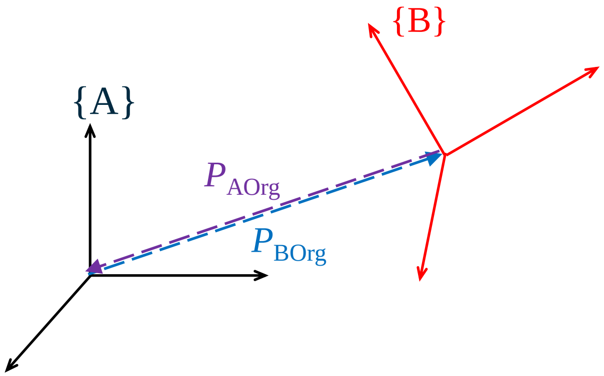
$$\boxed{P_2 = T\ P_1}$$

Transformation

- **Inverse Transform:**

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BOrg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

$R^{-1} = R^{T}$



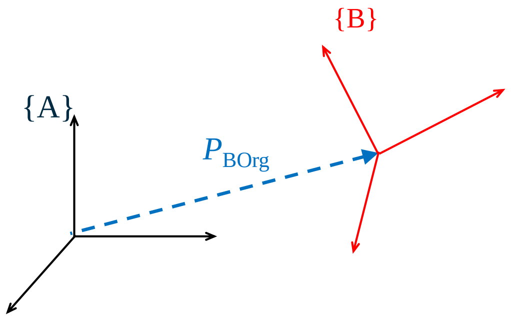
$^{B}P_{AOrg}$

$$ {}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{BOrg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Transformation

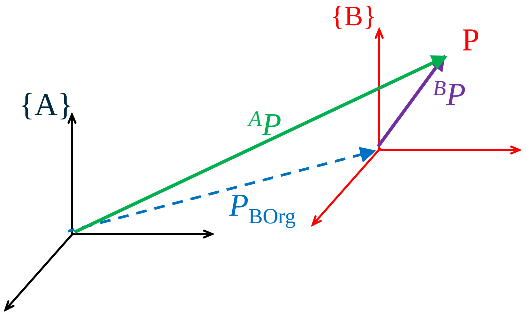
- **Homogeneous Transform Interpretations:**
  - Description of a frame

$$ {}^A_BT: \{B\} = \{{}^A_BR \quad {}^AP_{BOrg}\} $$

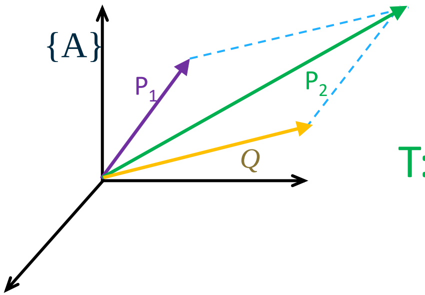


  - Transform mapping

$$ {}^A_BT: {}^BP \rightarrow {}^AP $$



  - Transform operator



$$ T: P_1 \rightarrow P_2 $$

Transformation

- **Compound Transformation:**

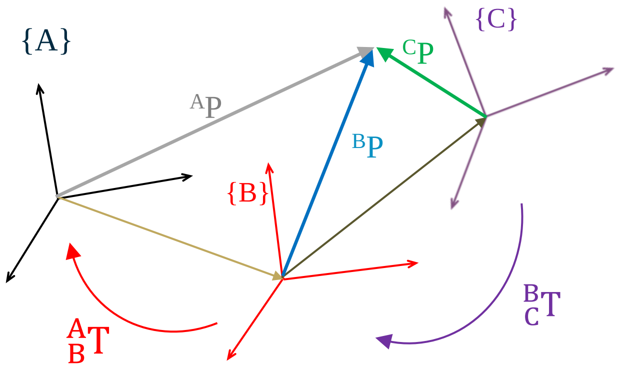
$$ {}^{B}P = {}^{B}_{C}T\ {}^{C}P $$

$$ {}^{A}P = {}^{A}_{B}T\ {}^{B}P $$

$$ {}^{A}P = {}^{A}_{B}T\ {}^{B}_{C}T\ {}^{C}P $$

$$ \boxed{{}^{A}_{C}T = {}^{A}_{B}T\ {}^{B}_{C}T} $$

$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}_{B}R\ {}^{B}P_{COrg} + {}^{A}P_{BOrg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

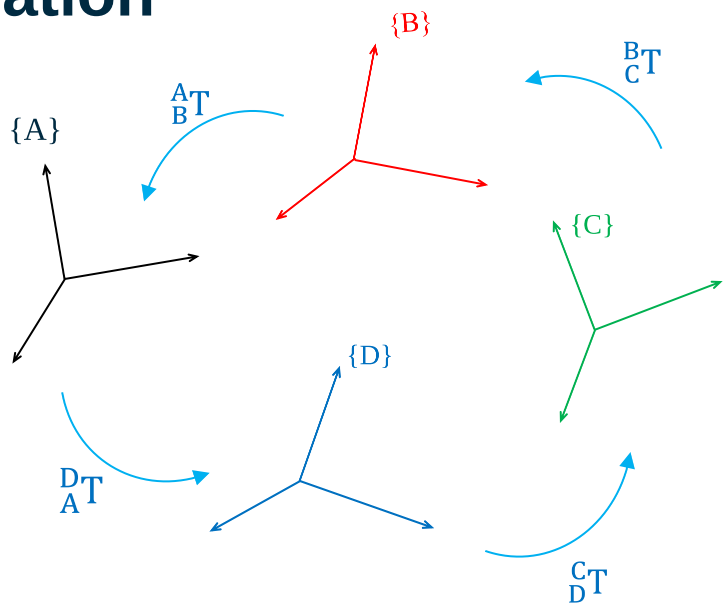


Transformation

- Transform Equation:

$$\boxed{{}^{A}_{B}T\ {}^{B}_{C}T\ {}^{C}_{D}T\ {}^{D}_{A}T = I}$$

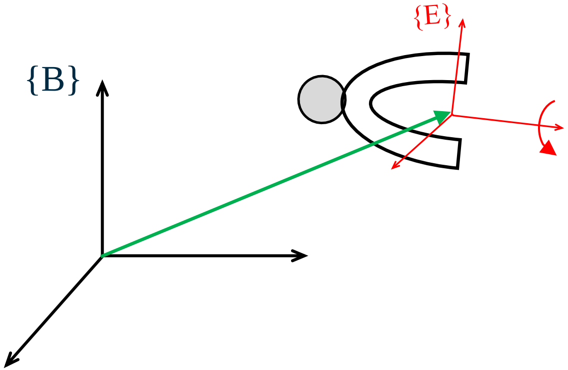
$$\Rightarrow\ {}^{B}_{A}T = {}^{B}_{C}T\ {}^{C}_{D}T\ {}^{D}_{A}T$$



Representations

End-effector Configuration:

$^{B}_{E}T$: Position + Orientation



End-effectors configuration parameters:

$$X = \begin{bmatrix} X_P \\ X_R \end{bmatrix} \begin{matrix} \leftarrow \text{Position} \\ \leftarrow \text{Orientation} \end{matrix}$$

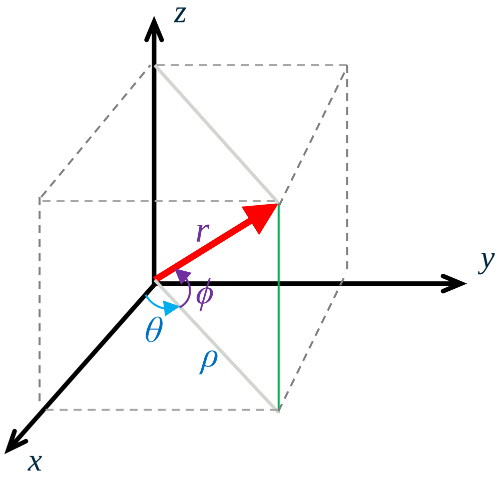
Representations

- **Position representation:**

  - ❑ Cartesian: (x, y, z)

  - ❑ Cylindrical: (ρ, θ, z)

  - ❑ Spherical: (r, θ, ϕ)



Lecture 4 Summary

- Spatial description
- Coordinate Frames
- Rotation matrix
- Transformation