Mathematical and Logical Foundations of Computer Science

Lecture 5 - Propositional Logic (Sequent Calculus)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus rules
- Sequent Calculus proofs

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- Sequent Calculus rules
- Sequent Calculus proofs

See Section 5 in “Proof and Types”

https://www.paultaylor.eu/stable/prot.pdf

Recap: Propositional logic syntax

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

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- $\top$ which stands for True
- $\bot$ which stands for False

Recap: Propositional logic syntax

Syntax:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Two special atoms:

- $\top$ which stands for True
- $\perp$ which stands for False

We also introduced four connectives:

- $P \wedge Q$: we have a proof of both P and Q
- $P \vee Q$: we have a proof of at least one of P and Q
- $P \rightarrow Q$: if we have a proof of P then we have a proof of Q
- $\neg P$: stands for $P \rightarrow \perp$

Recap: Natural deduction

Framework

- "natural" style of constructing a proof
- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion

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Two key points:

- Can work both forwards and backwards
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- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion

Two key points:

- Can work both forwards and backwards
- Natural doesn't mean there is unique proof

Introduced by **Gentzen** in 1934
and further studied by **Prawitz** in 1965.

Recap: Introduction & Elimination rules

Rules for $\rightarrow$ (implication)

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Rules for → (implication)

- implication-introduction

$$\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots} \atop B}{A \to B}\ 1\ [\to I]$$

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Rules for $\rightarrow$ (implication)

- implication-introduction

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\cfrac{\begin{array}{c} \overline{A}^{\,1} \\ \vdots \\ B \end{array}}{A \rightarrow B}\; 1\ [\rightarrow I]
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- implication-elimination

$$
\frac{A \rightarrow B \qquad A}{B}\; [\rightarrow E]
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Forward & backward reasoning

Prove the following:

$$(P \wedge Q) \to R \quad \vdash \quad P \to (Q \to R)$$

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We went backward up to R.

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We went backward up to R.

Going forward, it would also have been unclear which rule to apply to R.

Forward & backward reasoning

Derive B from $A \wedge B \wedge C$

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Here is a proof (starting backward):

$$\dfrac{\dfrac{}{}}{B}$$

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Here is a proof (starting backward):

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Forward & backward reasoning

Derive B from $A \wedge B \wedge C$

Here is a proof (starting backward):

$$\dfrac{\dfrac{A \wedge B \wedge C}{B \wedge C}\ [\wedge E]}{B}\ [\wedge E]$$

It was not clear which rule to use to prove B, which is why we went forward.

Sequent Calculus - History

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Sequent calculi are often (not always) amenable to proof automation, as they provide proofs enough structure to specify **proof search procedures**.

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Here we will see that it allows us proving propositions backward only.

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We will **eliminate** connectives from the **premises** (the **left**) and **introduce** connectives from the **conclusion** (the **right**).

Sequent Calculus vs. Natural Deduction (implication)

Natural Deduction

$$\frac{A \to B \qquad A}{B}\ [\to E]$$

$$\dfrac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I]$$

Sequent Calculus

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ [\to L]$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

Sequent Calculus vs. Natural Deduction (negation)

Natural Deduction

$$\frac{A \qquad \neg A}{\bot}\ [\neg E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ \bot\end{array}}{\neg A}\ 1\ [\neg I]$$

Sequent Calculus

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ [\neg L]$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ [\neg R]$$

Sequent Calculus vs. Natural Deduction (disjunction)

Natural Deduction

$$\frac{A}{A \vee B}\ [\vee I_L]$$

$$\frac{A}{B \vee A}\ [\vee I_R]$$

$$\frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

Sequent Calculus

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1]$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ [\vee L]$$

Sequent Calculus vs. Natural Deduction (conjunction)

Natural Deduction

$$\frac{A \quad B}{A \wedge B}\ [\wedge I]$$

$$\frac{A \wedge B}{B}\ [\wedge E_R]$$

$$\frac{A \wedge B}{A}\ [\wedge E_L]$$

Sequent Calculus

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ [\wedge L]$$

Attempt at a proof

How can we prove $A, A \to B \vdash B$?

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$$\frac{}{A, A \to B \vdash B}$$

Attempt at a proof

How can we prove $A, A \rightarrow B \vdash B$?

$$\cfrac{}{A, A \rightarrow B \vdash B}\ [\rightarrow L]$$

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How can we prove $A, A \to B \vdash B$?

$$\frac{A \vdash A \quad A, B \vdash B}{A, A \to B \vdash B}\ [\to L]$$

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What do we do now?

We need further rules

Identity and structural rules

Identity $$\frac{}{A \vdash A}\ [Id]$$

Identity and structural rules

Identity

$$\frac{}{A \vdash A}\ [Id]$$

Exchange

$$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X]$$

Identity and structural rules

Identity $$\frac{}{A \vdash A}\ [Id]$$

Exchange $$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X]$$

Weakening $$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W]$$

Identity and structural rules

Identity $$\frac{}{A \vdash A}\ [Id]$$

Exchange $$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X]$$

Weakening $$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W]$$

Contraction $$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ [C]$$

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We also add this useful but not necessary rule

Identity and structural rules

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Cut: $$\frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ [Cut]$$

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$$\dfrac{}{\Gamma, A, \Delta \vdash A}\,[Id]$$

Derivable rules

A **derivable** rule such as:

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is a rule such that the premises are the unproved hypotheses of a proof, and the conclusion is the conclusion of that proof.

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The above alternative $[Id]$ rule is derivable by:

- using $[X]$ a number of times to move A to the left of Γ
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Similarly, such **alternative left rules** are also derivable:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}\ [\wedge L]$$

Example of a Sequent Calculus proof

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Here is a proof:

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Note the use of derived rules!

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Sequent Calculus & Natural Deduction

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Theorem: The following **correspondence** holds:

- Given a **Sequent Calculus proof** of $\Gamma \vdash A$, one can derive a **natural deduction proof** of A under the hypotheses in Γ.
- Given a **natural deduction proof** of A under the hypotheses in Γ one can derive a **Sequent Calculus proof** of $\Gamma \vdash A$.

Conclusion

What did we cover today?

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus rules
- Sequent Calculus proofs

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Next time?

- Sequent Calculus & Natural Deduction