Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences continued)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**
- Intuitionistic vs. Classical logic
- Type theory

Today

Equivalences:

- in Natural Deduction
- in the Sequent Calculus
- rewriting using “known” equivalences
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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

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\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
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The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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The additional **conditions** ensure that **free variables do not get captured**.

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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{matrix} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{matrix}}{Q}\ 1\ [\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: ∀ & ∃ left and right rules

Sequent Calculus rules for quantifiers:

$$\frac{\Gamma \vdash P[x \backslash y]}{\Gamma \vdash \forall x.P}\ [\forall R] \qquad \frac{\Gamma, P[x \backslash t] \vdash Q}{\Gamma, \forall x.P \vdash Q}\ [\forall L]$$

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Conditions:

- for $[\forall R]$: y must not be free in Γ or $\forall x.P$
- for $[\forall L]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists R]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

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- of a non-empty domain D
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Models of predicate logic replace **truth assignments** for propositional logic

Variable valuations:

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- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

- $[\![t]\!]_v^M$ gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

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Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

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Meaning of formulas:

- $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]^M_v, \ldots, [\![t_n]\!]^M_v \rangle \in \mathcal{R}_p$
- $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
- $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x \mapsto d)} P$
- $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x \mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- Implication elimination: $(A \rightarrow B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

Recap: Logical Equivalences

In addition, the following hold (some hold only classically):

- $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$
- $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$
- $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$
- $(\forall x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
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Recap: Logical Equivalences

As before: if ($P \leftrightarrow Q$ or $Q \leftrightarrow P$) and P occurs in A, then replacing P by Q in A leads to a formula B, such that $A \leftrightarrow B$

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Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v, $\models_{M,v} P$ iff $\models_{M,v} Q$

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- that we can derive B form A
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We will use the following result:

Lemma (L1): if $x \notin \texttt{fv}(A)$ then $A[x\backslash t] = A$

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Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{A}{\forall x.A}\ [\forall I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\forall x.A}{A[x\backslash y]}\ [\forall E]$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
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$$\frac{}{\exists x.A}$$

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$$\dfrac{A[x\backslash y]}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A

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Here is a proof of the right-to-left implication (constructive):

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$$\dfrac{}{A}$$

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Here is a proof of the right-to-left implication (constructive):

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- by L1, because $x \notin \texttt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \quad A}{A}$$

- by L1, because $x \notin \texttt{fv}(A)$ then $A[x\backslash y] = A$
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Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

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- pick y such that it does not occur in A
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Here is a proof of the left-to-right implication (constructive):

$$\frac{\exists x.A \qquad \dfrac{}{A[x\backslash y]}\ 1}{A}\ 1\ [\exists E]$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
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Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$ in the Sequent Calculus

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Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\quad}}{\quad}}{\quad}}{\quad}}{\forall x.A \vee B \vdash (\forall x.A) \vee B}$$

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{}{\forall x.A \vee B \vdash (\forall x.A), B}}{\forall x.A \vee B \vdash (\forall x.A) \vee B}\ [\vee R]$$

Logical Equivalences

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Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\forall x.A \vee B \vdash A[x\backslash y], B}{\forall x.A \vee B \vdash (\forall x.A), B}\ [\forall R]}{\forall x.A \vee B \vdash (\forall x.A) \vee B}\ [\vee R]$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove that $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \lor B[x\backslash y] \vdash A[x\backslash y], B}}{\forall x.A \lor B \vdash A[x\backslash y], B}\ [\forall L]}{\forall x.A \lor B \vdash (\forall x.A), B}\ [\forall R]}{\forall x.A \lor B \vdash (\forall x.A) \lor B}\ [\lor R]$$

- pick y such that it does not occur in A or B

Logical Equivalences

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]} \qquad \dfrac{}{B[x\backslash y] \vdash B}}{A[x\backslash y] \lor B[x\backslash y] \vdash A[x\backslash y], B}\ [\lor L]}{\forall x.A \lor B \vdash A[x\backslash y], B}\ [\forall L]}{\forall x.A \lor B \vdash (\forall x.A), B}\ [\forall R]}{\forall x.A \lor B \vdash (\forall x.A) \lor B}\ [\lor R]$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}[Id] \quad \dfrac{}{B[x\backslash y] \vdash B}}{A[x\backslash y] \vee B[x\backslash y] \vdash A[x\backslash y], B}[\vee L]}{\forall x.A \vee B \vdash A[x\backslash y], B}[\forall L]}{\forall x.A \vee B \vdash (\forall x.A), B}[\forall R]}{\forall x.A \vee B \vdash (\forall x.A) \vee B}[\vee R]$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}[Id] \quad \dfrac{}{B \vdash B}}{A[x\backslash y] \lor B[x\backslash y] \vdash A[x\backslash y], B}[\lor L]}{\forall x.A \lor B \vdash A[x\backslash y], B}[\forall L]}{\forall x.A \lor B \vdash (\forall x.A), B}[\forall R]}{\forall x.A \lor B \vdash (\forall x.A) \lor B}[\lor R]$$

- pick y such that it does not occur in A or B
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}[Id] \qquad \dfrac{}{B \vdash B}[Id]}{A[x\backslash y] \vee B[x\backslash y] \vdash A[x\backslash y], B}[\vee L]}{\forall x.A \vee B \vdash A[x\backslash y], B}[\forall L]}{\forall x.A \vee B \vdash (\forall x.A), B}[\forall R]}{\forall x.A \vee B \vdash (\forall x.A) \vee B}[\vee R]$$

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Prove that $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{}{(\forall x.A) \lor B \vdash \forall x.A \lor B}$$

Logical Equivalences

Prove that $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\forall x.A) \lor B \vdash A[x\backslash y] \lor B[x\backslash y]}{(\forall x.A) \lor B \vdash \forall x.A \lor B}\ [\forall R]$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove that $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{}{\forall x.A \vdash A[x\backslash y] \lor B[x\backslash y]} \qquad \dfrac{}{B \vdash A[x\backslash y] \lor B[x\backslash y]}}{(\forall x.A) \lor B \vdash A[x\backslash y] \lor B[x\backslash y]}\ [\lor L]}{(\forall x.A) \lor B \vdash \forall x.A \lor B}\ [\forall R]$$

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Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{\forall x.A \vdash A[x\backslash y]}}{\forall x.A \vdash A[x\backslash y] \lor B[x\backslash y]}\;[\lor R_1] \qquad \dfrac{}{B \vdash A[x\backslash y] \lor B[x\backslash y]}}{(\forall x.A) \lor B \vdash A[x\backslash y] \lor B[x\backslash y]}\;[\lor L]}{(\forall x.A) \lor B \vdash \forall x.A \lor B}\;[\forall R]
$$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}}{\forall x.A \vdash A[x\backslash y]}\ [\forall L]}{\forall x.A \vdash A[x\backslash y] \vee B[x\backslash y]}\ [\vee R_1] \quad \dfrac{}{B \vdash A[x\backslash y] \vee B[x\backslash y]}}{\dfrac{(\forall x.A) \vee B \vdash A[x\backslash y] \vee B[x\backslash y]}{(\forall x.A) \vee B \vdash \forall x.A \vee B}\ [\forall R]}\ [\vee L]$$

- pick y such that it does not occur in A or B

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\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\ [Id]}{\forall x.A \vdash A[x\backslash y]}\ [\forall L]}{\forall x.A \vdash A[x\backslash y] \vee B[x\backslash y]}\ [\vee R_1] \qquad \dfrac{}{B \vdash A[x\backslash y] \vee B[x\backslash y]}}{\dfrac{(\forall x.A) \vee B \vdash A[x\backslash y] \vee B[x\backslash y]}{(\forall x.A) \vee B \vdash \forall x.A \vee B}\ [\forall R]}\ [\vee L]
$$

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$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\;[Id]
}{\forall x.A \vdash A[x\backslash y]}\;[\forall L]
}{\forall x.A \vdash A[x\backslash y] \vee B[x\backslash y]}\;[\vee R_1]
\qquad
\dfrac{
\dfrac{}{B \vdash B[x\backslash y]}
}{B \vdash A[x\backslash y] \vee B[x\backslash y]}\;[\vee R_2]
}{
\dfrac{(\forall x.A) \vee B \vdash A[x\backslash y] \vee B[x\backslash y]}{(\forall x.A) \vee B \vdash \forall x.A \vee B}\;[\forall R]
}\;[\vee L]
$$

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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\,[Id]
    }{\forall x.A \vdash A[x\backslash y]}\,[\forall L]
  }{\forall x.A \vdash A[x\backslash y] \vee B[x\backslash y]}\,[\vee R_1]
  \qquad
  \dfrac{
    \dfrac{}{B \vdash B}\,[Id]
  }{B \vdash A[x\backslash y] \vee B[x\backslash y]}\,[\vee R_2]
}{
  \dfrac{(\forall x.A) \vee B \vdash A[x\backslash y] \vee B[x\backslash y]}{(\forall x.A) \vee B \vdash \forall x.A \vee B}\,[\forall R]
}\,[\vee L]
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Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Logical Equivalences

Prove that $(\exists x.A \land B) \leftrightarrow ((\exists x.A) \land B)$ if $x \notin \texttt{fv}(B)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\exists x.A \land B \vdash (\exists x.A) \land B}$$

Logical Equivalences

Prove that $(\exists x.A \land B) \leftrightarrow ((\exists x.A) \land B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{A[x\backslash y] \land B[x\backslash y] \vdash (\exists x.A) \land B}{\exists x.A \land B \vdash (\exists x.A) \land B}\ [\exists L]$$

- pick y such that it does not occur in A or B

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Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{A[x\backslash y], B[x\backslash y] \vdash (\exists x.A) \wedge B}}{A[x\backslash y] \wedge B[x\backslash y] \vdash (\exists x.A) \wedge B}[\wedge L]}{\exists x.A \wedge B \vdash (\exists x.A) \wedge B}[\exists L]$$

- pick y such that it does not occur in A or B

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Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{A[x\backslash y], B[x\backslash y] \vdash \exists x.A} \qquad \dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B}}{A[x\backslash y], B[x\backslash y] \vdash (\exists x.A) \wedge B}\,[\wedge R]}{A[x\backslash y] \wedge B[x\backslash y] \vdash (\exists x.A) \wedge B}\,[\wedge L]}{\exists x.A \wedge B \vdash (\exists x.A) \wedge B}\,[\exists L]$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}}{A[x\backslash y], B[x\backslash y] \vdash \exists x.A}[\exists R] \quad \dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B}}{A[x\backslash y], B[x\backslash y] \vdash (\exists x.A) \wedge B}[\wedge R]}{A[x\backslash y] \wedge B[x\backslash y] \vdash (\exists x.A) \wedge B}[\wedge L]}{\exists x.A \wedge B \vdash (\exists x.A) \wedge B}[\exists L]
$$

- pick y such that it does not occur in A or B

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Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\;[Id]
      }{A[x\backslash y], B[x\backslash y] \vdash \exists x.A}\;[\exists R]
      \qquad
      \dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B}
    }{A[x\backslash y], B[x\backslash y] \vdash (\exists x.A) \wedge B}\;[\wedge R]
  }{A[x\backslash y] \wedge B[x\backslash y] \vdash (\exists x.A) \wedge B}\;[\wedge L]
}{\exists x.A \wedge B \vdash (\exists x.A) \wedge B}\;[\exists L]
$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y], B[x\backslash y] \vdash \exists x.A}\,[\exists R] \qquad \dfrac{}{A[x\backslash y], B \vdash B}\,[Id]}{\dfrac{A[x\backslash y], B[x\backslash y] \vdash (\exists x.A) \wedge B}{A[x\backslash y] \wedge B[x\backslash y] \vdash (\exists x.A) \wedge B}\,[\wedge L]}\,[\wedge R]}{\exists x.A \wedge B \vdash (\exists x.A) \wedge B}\,[\exists L]
$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \texttt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Logical Equivalences

Prove that $(\exists x.A \land B) \leftrightarrow ((\exists x.A) \land B)$ if $x \notin \texttt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{(\exists x.A) \land B \vdash \exists x.A \land B}$$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{}{(\exists x.A), B \vdash \exists x.A \wedge B}}{(\exists x.A) \wedge B \vdash \exists x.A \wedge B}\ [\wedge L]$$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{A[x\backslash y], B \vdash \exists x.A \wedge B}{(\exists x.A), B \vdash \exists x.A \wedge B}\ [\exists L]}{(\exists x.A) \wedge B \vdash \exists x.A \wedge B}\ [\wedge L]$$

- pick y such that it does not occur in A or B

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Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B \vdash A[x\backslash y] \wedge B[x\backslash y]}}{A[x\backslash y], B \vdash \exists x.A \wedge B}\,[\exists R]}{(\exists x.A), B \vdash \exists x.A \wedge B}\,[\exists L]}{(\exists x.A) \wedge B \vdash \exists x.A \wedge B}\,[\wedge L]
$$

- pick y such that it does not occur in A or B

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Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B \vdash A[x\backslash y]} \qquad \dfrac{}{A[x\backslash y], B \vdash B[x\backslash y]}}{A[x\backslash y], B \vdash A[x\backslash y] \land B[x\backslash y]}\ [\land R]}{A[x\backslash y], B \vdash \exists x.A \land B}\ [\exists R]}{(\exists x.A), B \vdash \exists x.A \land B}\ [\exists L]}{(\exists x.A) \land B \vdash \exists x.A \land B}\ [\land L]
$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B \vdash A[x\backslash y]}[Id] \quad \dfrac{}{A[x\backslash y], B \vdash B[x\backslash y]}}{A[x\backslash y], B \vdash A[x\backslash y] \wedge B[x\backslash y]}[\wedge R]}{A[x\backslash y], B \vdash \exists x.A \wedge B}[\exists R]}{(\exists x.A), B \vdash \exists x.A \wedge B}[\exists L]}{(\exists x.A) \wedge B \vdash \exists x.A \wedge B}[\wedge L]$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B \vdash A[x\backslash y]}[Id] \quad \dfrac{}{A[x\backslash y], B \vdash B}[Id]}{A[x\backslash y], B \vdash A[x\backslash y] \wedge B[x\backslash y]}[\wedge R]}{A[x\backslash y], B \vdash \exists x.A \wedge B}[\exists R]}{(\exists x.A), B \vdash \exists x.A \wedge B}[\exists L]}{(\exists x.A) \wedge B \vdash \exists x.A \wedge B}[\wedge L]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \mathtt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$

Prove that $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \texttt{fv}(B)$

Prove that $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\forall x.A \to B$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$

Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$

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- $\forall x.A \to B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$

Logical Equivalences

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- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
- $\leftrightarrow (\neg\exists x.A) \vee B$ – using $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
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- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$
- $\leftrightarrow (\neg \exists x.A) \vee B$ – using $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$
- $\leftrightarrow (\exists x.A) \rightarrow B$ – using implication elimination

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $\leftrightarrow (\neg\forall x.A) \vee B$ – using $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
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- $\leftrightarrow (\neg\forall x.A) \vee B$ – using $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$
- $\leftrightarrow (\forall x.A) \rightarrow B$ – using implication elimination

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \rightarrow B) \leftrightarrow (A \rightarrow \exists x.B)$ if $x \notin \mathtt{fv}(A)$

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \to B) \leftrightarrow (A \to \exists x.B)$ if $x \notin \mathtt{fv}(A)$

We will use following result:

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A \to B) \leftrightarrow (A \to \exists x.B)$ if $x \notin \texttt{fv}(A)$

We will use following result:

Lemma (L2): if $x \notin \texttt{fv}(A)$, then $\models_{M,v,x \mapsto d} A$ iff $\models_{M,v} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \rightarrow B$ then $\models_{M,v} A \rightarrow \forall x.B$
  - to prove: $\models_{M,v} A \rightarrow \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

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  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

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  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

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  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x \mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x \mapsto e} B$ whenever $\models_{M,v,x \mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x \mapsto d} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

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  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$
  - instantiating this assumption with d gives us: $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$
  - instantiating this assumption with d gives us: $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$
  - therefore, because $\models_{M,v,x\mapsto d} A$ is true, $\models_{M,v,x\mapsto d} B$ is also true

Logical Equivalences

Right-to-left implication:

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$

Logical Equivalences

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- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
  - to prove: $\models_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
  - to prove: $\models_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\models_{M,v,x \mapsto d} B$ whenever $\models_{M,v,x \mapsto d} A$
  - assume $d \in D$ and $\models_{M,v,x \mapsto d} A$, and prove $\models_{M,v,x \mapsto d} B$

Logical Equivalences

Right-to-left implication:

- if $\vDash_{M,v} A \rightarrow \forall x.B$ then $\vDash_{M,v} \forall x.A \rightarrow B$
  - to prove: $\vDash_{M,v} \forall x.A \rightarrow B$, i.e., for all $d \in D$, $\vDash_{M,v,x\mapsto d} B$ whenever $\vDash_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove $\vDash_{M,v,x\mapsto d} B$
  - by L2, we can assume $\vDash_{M,v} A$

Logical Equivalences

Right-to-left implication:

- if $\vDash_{M,v} A \to \forall x.B$ then $\vDash_{M,v} \forall x.A \to B$
  - to prove: $\vDash_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\vDash_{M,v,x\mapsto d} B$ whenever $\vDash_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove $\vDash_{M,v,x\mapsto d} B$
  - by L2, we can assume $\vDash_{M,v} A$
  - assumption: $\vDash_{M,v} A \to \forall x.B$, i.e., $\vDash_{M,v} \forall x.B$ whenever $\vDash_{M,v} A$

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
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  - assume $d \in D$ and $\models_{M,v,x\mapsto d} A$, and prove $\models_{M,v,x\mapsto d} B$
  - by L2, we can assume $\models_{M,v} A$
  - assumption: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - because $\models_{M,v} A$, we can assume $\models_{M,v} \forall x.B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$

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Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
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  - assume $d \in D$ and $\models_{M,v,x\mapsto d} A$, and prove $\models_{M,v,x\mapsto d} B$
  - by L2, we can assume $\models_{M,v} A$
  - assumption: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - because $\models_{M,v} A$, we can assume $\models_{M,v} \forall x.B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$
  - instantiating this assumption using d, we get to assume $\models_{M,v,x\mapsto d} B$, which is what we wanted to prove

Conclusion

What did we cover today?

- Equivalence using Natural Deduction
- Equivalence using the Sequent Calculus
- Rewriting using “known” equivalences
- Equivalences using semantics

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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

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Next time?

- Theorem Proving