

Exercise Sheet 3 - Solutions

Propositional Logic – Sequent Calculus and Natural Deduction

1. Here is a Natural Deduction proof of $\neg(A \leftrightarrow \neg A)$:

$$\begin{array}{c}
 \frac{\overline{A \leftrightarrow \neg A}^1}{\overline{A \rightarrow \neg A}} [\wedge E_L] \quad \frac{\overline{A}^2}{\neg A} [\rightarrow E] \quad \frac{\overline{A}^2}{\neg A} [\neg E] \quad \frac{\overline{A \leftrightarrow \neg A}^1}{\overline{\neg A \rightarrow A}} [\wedge E_L] \quad \frac{\overline{\neg A}^3}{\neg A} [\rightarrow E] \quad \frac{\overline{A}^3}{\neg A} [\neg E] \\
 \hline
 \frac{\perp^2 [\neg I]}{\neg A} [\neg I] \quad \frac{\perp^3 [\neg I]}{\neg A} [\neg I] \quad \frac{A}{\neg A} [\neg E] \\
 \hline
 \frac{\perp^1 [\neg I]}{\neg(A \leftrightarrow \neg A)}^1 [\neg I]
 \end{array}$$

2. Here is a Sequent Calculus proof of $\neg(A \leftrightarrow \neg A)$:

$$\begin{array}{c}
 \frac{\overline{A \vdash A} [Id]}{A \rightarrow \neg A, A \vdash \perp} [\neg L] \quad \frac{\overline{A \vdash A} [Id]}{A \rightarrow \neg A, A \vdash \perp} [\neg L] \\
 \hline
 \frac{A \rightarrow \neg A, A \vdash \perp}{A \rightarrow \neg A \vdash \neg A} [\neg R] \quad \frac{A \rightarrow \neg A, A \vdash \perp}{A \rightarrow \neg A, A \vdash \perp} [\neg L] \\
 \hline
 \frac{A \rightarrow \neg A, \neg A \rightarrow A \vdash \perp}{A \leftrightarrow \neg A \vdash \perp} [\wedge L] \\
 \hline
 \frac{A \leftrightarrow \neg A \vdash \perp}{\vdash \neg(A \leftrightarrow \neg A)} [\neg R]
 \end{array}$$

3. Here is a Natural Deduction proof of $(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))$:

$$\frac{\frac{\Pi_1 \quad \Pi_2}{(A \vee B) \wedge (A \vee C)} [\wedge I]}{(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))}^1 [\rightarrow I]$$

where Π_1 is

$$\frac{\overline{A \vee (B \wedge C)}^1 \quad \frac{\overline{A}^2}{A \vee B} [\vee I_L] \quad \frac{\frac{\overline{B \wedge C}^3}{B} [\wedge E_L]}{A \vee B} [\vee I_R]}{A \vee B} [\vee E]$$

and where Π_2 is:

$$\frac{\overline{A \vee (B \wedge C)}^1 \quad \frac{\overline{A}^4}{A \vee C} [\vee I_L] \quad \frac{\frac{\overline{B \wedge C}^5}{C} [\wedge E_R]}{A \vee C} [\vee I_R]}{A \vee C} [\vee E]$$

4. Here is a Sequent Calculus proof of $(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))$:

$$\begin{array}{c}
 \frac{\frac{\overline{A \vdash A} \quad [Id]}{A \vdash A \vee B} \quad [\vee R_1] \quad \frac{\frac{\overline{B, C \vdash B} \quad [Id]}{B, C \vdash A \vee B} \quad [\vee R_2]}{B \wedge C \vdash A \vee B} \quad [\wedge L] \quad \frac{\frac{\overline{A \vdash A} \quad [Id]}{A \vdash A \vee C} \quad [\vee R_1] \quad \frac{\frac{\overline{B, C \vdash C} \quad [Id]}{B, C \vdash A \vee C} \quad [\vee R_2]}{B \wedge C \vdash A \vee C} \quad [\wedge L]}{A \vee (B \wedge C) \vdash A \vee B \quad A \vee (B \wedge C) \vdash A \vee C} \quad [\wedge R] \\
 \frac{A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)}{\vdash (A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))} \quad [\rightarrow R]
 \end{array}$$