10

Differentiation from first principles

Recall the definition of the derivative of a function.

Definition 10.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$. If the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, then we say that f is *differentiable at the point* x_0; moreover, the value of the limit is called the derivative of f at x_0, and is denoted by $f'(x_0)$.

If f is differentiable at every $x \in \mathbb{R}$, then we just say that f is differentiable and the function $f' : \mathbb{R} \to \mathbb{R}$ is the derivative of f.

Remark 10.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then there are many different ways in which the derivative f' of a function f can be denoted, such as:

$$Df, \ \frac{df}{dx} \text{ and } \frac{d}{dx}f.$$

For the middle notation, we can view df as being a "small change in f" and dx as a "small change in x".

In other words, computing the derivative of $f : \mathbb{R} \to \mathbb{R}$ at a given point $(x_0, f(x_0))$ gives you the gradient of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$.

Example 10.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and fix $x_0 \in \mathbb{R}$ (for example $x_0 = 1$). Then,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} = \lim_{h \to 0} \frac{2x_0 h + h^2}{h} = \lim_{h \to 0}(2x_0 + h) = 2x_0.$$

Let us now consider the tangent line to $y = x^2$ at x_0. The derivative $f'(x_0)$ gives the value of the gradient of the tangent line at x_0. Therefore, the gradient of the tangent line to $y = x^2$ at x_0 is $2x_0$. In the particular case when $x_0 = 1$, we see that the gradient is 2.

Example 10.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^2 + 2x + 1$ and $x_0 \in \mathbb{R}$ (again, for ease you may consider $x_0 = 1$). Then,

$$\begin{aligned}
f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \to 0} \frac{\left(3\left(x_0 + h\right)^2 + 2\left(x_0 + h\right) + 1\right) - \left(3x_0^2 + 2x_0 + 1\right)}{h} \\
&= \lim_{h \to 0} \frac{\left(\cancel{3x_0^2} + 6hx_0 + 3h^2 + \cancel{2x_0} + 2h + \cancel{1}\right) - \cancel{3x_0^2} - \cancel{2x_0} - \cancel{1}}{h} \\
&= \lim_{h \to 0} \frac{6hx_0 + 2h + 3h^2}{h} \\
&= \lim_{h \to 0} \left(6x_0 + 2 + 3h\right) \\
&= 6x_0 + 2.
\end{aligned}$$

Therefore, the gradient of the tangent line to $y = 3x^2 + 2x + 1$ at $(x_0, f(x_0))$ is given by $6x_0 + 2$.