Exercise Sheet 11 - Solutions
Predicate Logic – Verification

1.
```
lemma l1 : (R → ¬P) → (Q → R) → (P → ¬Q) :=
begin
  intros h1 h2 hp hq,
  apply h1,
  -- 1st branch
  apply h2, assumption,
  -- 2nd branch
  assumption,
end
```

2.
```
lemma l2 : (∀ x, ∃ y, (p(x) ∨ q x y)) → ∀ x, p(x) ∨ ∃ y, q x y :=
begin
  intros h x,
  specialize h x,
  destruct h, intros y q,
  destruct q,
  -- 1st branch
  intros hp,
  left, assumption,
  -- 2nd branch
  intros hq,
  right, existsi y, assumption,
end
```

3.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P_2, P_3, x \leq y \vdash x \leq y}\,[Id] \quad \Pi_1}{x \leq y \rightarrow \exists z.x+z=y, P_2, P_3, x \leq y \vdash x < y+1}\,[\rightarrow L]}{x \leq y \rightarrow \exists z.x+z=y, (\exists z.x+z=y) \rightarrow x \leq y, P_2, P_3, x \leq y \vdash x < y+1}\,[W_L]}{x \leq y \leftrightarrow \exists z.x+z=y, P_2, P_3, x \leq y \vdash x < y+1}\,[\wedge L]}{\forall y.(x \leq y \leftrightarrow \exists z.x+z=y), P_2, P_3, x \leq y \vdash x < y+1}\,[\forall L]}{P_1, P_2, P_3, x \leq y \vdash x < y+1}\,[\forall L]}{P_1, P_2, P_3 \vdash x \leq y \rightarrow x < y+1}\,[\rightarrow R]}{P_1, P_2, P_3 \vdash \forall y.(x \leq y \rightarrow x < y+1)}\,[\forall R]}{P_1, P_2, P_3 \vdash \forall x.\forall y.(x \leq y \rightarrow x < y+1)}\,[\forall R]
$$

where Π_1 is:

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_2 \quad \dfrac{}{x+z=y, x<y+1, P_3, x \leq y \vdash x < y+1}\,[Id]}{x+z=y, (\exists z.(x+z)+1=y+1) \rightarrow x < y+1, P_3, x \leq y \vdash x < y+1}\,[\rightarrow L]}{x+z=y, x<y+1 \rightarrow \exists z.(x+z)+1=y+1, (\exists z.(x+z)+1=y+1) \rightarrow x<y+1, P_3, x \leq y \vdash x < y+1}\,[W_L]}{x+z=y, x<y+1 \leftrightarrow \exists z.(x+z)+1=y+1, P_3, x \leq y \vdash x < y+1}\,[\wedge L]}{x+z=y, \forall y.(x<y \leftrightarrow \exists z.(x+z)+1=y), P_3, x \leq y \vdash x < y+1}\,[\forall L]}{x+z=y, P_2, P_3, x \leq y \vdash x < y+1}\,[\forall L]}{\exists z.x+z=y, P_2, P_3, x \leq y \vdash x < y+1}\,[\exists L]
$$

and Π_2 is:

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\dfrac{}{x+z=y, x \leq y \vdash x+z=y}\;[Id]
\quad
\dfrac{}{x+z=y, (x+z)+1=y+1, x \leq y \vdash (x+z)+1=y+1}\;[Id]
}{x+z=y, x+z=y \to (x+z)+1=y+1, x \leq y \vdash (x+z)+1=y+1}\;[\to L]
}{x+z=y, \forall w.(x+z=y \to (x+z)+w=y+w), x \leq y \vdash (x+z)+1=y+1}\;[\forall L]
}{x+z=y, \forall y.\forall w.(x+z=y \to (x+z)+w=y+w), x \leq y \vdash (x+z)+1=y+1}\;[\forall L]
}{x+z=y, P_3, x \leq y \vdash (x+z)+1=y+1}\;[\forall L]
}{x+z=y, P_3, x \leq y \vdash \exists z.(x+z)+1=y+1}\;[\exists R]
$$

4.

```
def P1 : Prop := ∀ (x y : ℕ), x ≤ y ↔ ∃ z, x+z=y
def P2 : Prop := ∀ (x y : ℕ), x < y ↔ ∃ z, (x+z)+1=y
def P3 : Prop := ∀ (x y z : ℕ), x = y → x+z = y+z

lemma l3 : P1 → P2 → P3 → ∀ (x y : ℕ), x ≤ y → x < y+1 :=
begin
  intros p1 p2 p3 x y h,
  specialize p1 x y,
  destruct p1, intros h1 h2, clear p1 h2,
  specialize h1 h,
  destruct h1, intros z h2, clear h1,
  specialize p2 x (y+1),
  destruct p2, intros q1 q2, clear p2 q1,
  apply q2, clear q2,
  existsi z,
  apply p3, assumption
end
```