Exercise Sheet 8 - Solutions
Predicate Logic – Natural Deduction

---

1. $\forall x.\forall y.\texttt{max}(x,y) \geq \texttt{min}(x,y)$

2. $\forall x.\neg\exists y.\neg(x=y) \wedge \texttt{max}(x,y) = \texttt{min}(x,y)$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x.\forall y.x > y \vee \neg(x > y)}{\forall y.x+z > y \vee \neg(x+z > y)}\,[\forall E]}{x+z > y+z \vee \neg(x+z > y+z)}\,[\forall E] \quad \Pi_1 \quad \Pi_3}{\texttt{min}(x+z,y+z) \geq z}\,[\vee E]}{\forall z.\texttt{min}(x+z,y+z) \geq z}\,[\forall I]}{\forall y.\forall z.\texttt{min}(x+z,y+z) \geq z}\,[\forall I]}{\forall x.\forall y.\forall z.\texttt{min}(x+z,y+z) \geq z}\,[\forall I]$$

where Π_1 is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x.\forall y.\forall z.x = y \to y \geq z \to x \geq z}{\forall v.\forall w.\texttt{min}(x+z,y+z) = v \to v \geq w \to \texttt{min}(x+z,y+z) \geq w}\,[\forall E]}{\forall w.\texttt{min}(x+z,y+z) = y+z \to y+z \geq w \to \texttt{min}(x+z,y+z) \geq w}\,[\forall E]}{\texttt{min}(x+z,y+z) = y+z \to y+z \geq z \to \texttt{min}(x+z,y+z) \geq z}\,[\forall E] \quad \Pi_2}{y+z \geq z \to \texttt{min}(x+z,y+z) \geq z}\,[\to E] \quad \dfrac{\dfrac{\forall x.\forall y.x+y \geq y}{\forall w.y+w \geq w}\,[\forall E]}{y+z \geq z}\,[\forall E]}{\texttt{min}(x+z,y+z) \geq z}\,[\to E]}{x+z > y+z \to \texttt{min}(x+z,y+z) \geq z}\,1\,[\to I]$$

where Π_2 is

$$\dfrac{\dfrac{\dfrac{\forall x.\forall y.x > y \to \texttt{min}(x,y) = y}{\forall y.x+z > y \to \texttt{min}(x+z,y) = y}\,[\forall E]}{x+z > y+z \to \texttt{min}(x+z,y+z) = y+z}\,[\forall E] \quad \dfrac{}{x+z > y+z}\,1}{\texttt{min}(x+z,y+z) = y+z}\,[\to E]$$

where Π_3 is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x.\forall y.\forall z.x = y \to y \geq z \to x \geq z}{\forall v.\forall w.\texttt{min}(x+z,y+z) = v \to v \geq w \to \texttt{min}(x+z,y+z) \geq w}\,[\forall E]}{\forall w.\texttt{min}(x+z,y+z) = x+z \to x+z \geq w \to \texttt{min}(x+z,y+z) \geq w}\,[\forall E]}{\texttt{min}(x+z,y+z) = x+z \to x+z \geq z \to \texttt{min}(x+z,y+z) \geq z}\,[\forall E] \quad \Pi_4}{x+z \geq z \to \texttt{min}(x+z,y+z) \geq z}\,[\to E] \quad \dfrac{\dfrac{\forall x.\forall y.x+y \geq y}{\forall y.x+y \geq y}\,[\forall E]}{x+z \geq z}\,[\forall E]}{\texttt{min}(x+z,y+z) \geq z}\,[\to E]}{\neg(x+z > y+z) \to \texttt{min}(x+z,y+z) \geq z}\,2\,[\to I]$$

where Π_4 is

$$\dfrac{\dfrac{\dfrac{\forall x.\forall y.\neg(x > y) \to \texttt{min}(x,y) = x}{\forall y.\neg(x+z > y) \to \texttt{min}(x+z,y) = x+z}\,[\forall E]}{\neg(x+z > y+z) \to \texttt{min}(x+z,y+z) = x+z}\,[\forall E] \quad \dfrac{}{\neg(x+z > y+z)}\,2}{\texttt{min}(x+z,y+z) = x+z}\,[\to E]$$

4.

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{\exists x.p(x)}\,{}^{2}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\forall x.p(x) \to q(x)}\,{}^{1}}{p(x) \to q(x)}\,[\forall E]
        \qquad
        \dfrac{}{p(x)}\,{}^{3}
      }{q(x)}\,[\to E]
    }{\exists x.q(x)}\,[\exists I]
    }{\exists x.q(x)}\;3\;[\exists E]
  }{(\exists x.p(x)) \to \exists x.q(x)}\;2\;[\to I]
}{(\forall x.p(x) \to q(x)) \to (\exists x.p(x)) \to \exists x.q(x)}\;1\;[\to I]
$$