Mathematical and Logical Foundations of Computer Science

Lecture 3 - Propositional Logic (Syntax)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

- Propositional logic
- Syntax of the language
- Informal semantics
- Simple proofs

Propositions - informal presentation

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Advantages of formal symbolic language over natural languages are:

- **unambiguous**
- **more concise**

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- negation: $\neg$ (not) — can be defined using $\rightarrow$ and $\bot$

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- The car's brakes failed
  an atomic proposition
- The control software crashed and the car's brakes failed
  a conjunction of 2 atomic propositions
- If the control software crashes, then the car's brakes will fail
  an implication connecting 2 atomic propositions

Propositional logic

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Example of a compound formula: $\neg p \wedge q \wedge q \wedge \neg r$.

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- it can be defined as $P \to \bot$
- if P is true, then $\bot$ (False)
- true iff P is false

Avoiding ambiguities

$P \wedge Q \vee R$

- Is this a well-formed formula?

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$P \land Q \lor R$

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However use parentheses around compound formulas for clarity.

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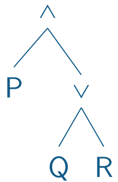

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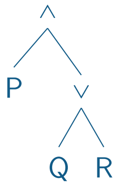

Leaves are atomic propositions and the other nodes are connectives.

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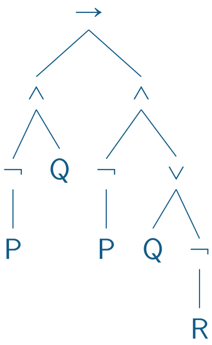

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- That is, the root node of the parse tree
- The main connective of $(P \wedge Q) \vee R$ is $\vee$

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1. If John is at home, then his television is on
2. His television is not on
3. Therefore, John is not at home

Identify atomic propositions:

- $p =$"John is at home"
- $q =$"John's television is on"

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How do we write this argument in propositional logic?

- Premise 1: $p \rightarrow q$
- Premise 2: $\neg q$
- Conclusion: $\neg p$

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- i.e., set of premises separated by commas, then a **turnstile** followed by the conclusion.
- Recall that premises and conclusions are both formulas.
- A sequent is **valid** if the argument has been proven, i.e., if the conclusion is true assuming that the premises are true.

Proofs in Propositional Logic

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1. A **formal** language
   - for representing propositions, arguments
   - here we are using propositional logic
2. A **proof** theory
   - to prove ("infer", "deduce") whether an argument is valid
   - we'll see several different approaches in this module
   - for now (next few lectures): Natural Deduction

Natural Deduction

Natural Deduction

- “natural” style of constructing a proof (like a human would)
- syntactic (rather than semantic) proof method
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Basic idea to prove an argument is valid:

- start with the premises (we can assume these are true)
- repeatedly apply inference rules (which "preserve truth")
- until we have inferred the conclusion

What are inference rules?

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Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

Some simple inference rules

And-introduction:

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True-introduction

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A **proof** is a tree of instances of inference rules.

A simple proof

Negation-elimination, i.e., both A and $\neg A$ cannot be true at same time

Formally, want to prove $A, \neg A \ \vdash \bot$

A **proof** is a tree of instances of inference rules.

Assuming that $\neg A$ is defined as $A \rightarrow \bot$, a proof of the above sequent (or argument) is:

$$\frac{A \quad \neg A}{\bot} \ [\rightarrow E]$$

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Given three hypotheses A, B, C, how can we prove $(A \wedge B) \wedge (A \wedge C)$?

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The rule used at each step is **and-introduction**, i.e., $\wedge I$

Conclusion

What did we cover today?

- Syntax of propositional logic
- Informal semantics of propositional logic formulas
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Next time?

- Natural Deduction