Mathematical and Logical Foundations of Computer Science

Lecture 2 - Symbolic Logic

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- **Symbolic logic**
- Propositional logic
- Predicate logic
- Constructive vs. Classical logic
- Type theory

Today

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- Symbolic logic
- Grammars
- (Meta)variables
- Axiom schemata
- Substitution

Symbolic Logics

Symbolic logics are **formal languages** that allow conducting logical reasoning through the **manipulation of symbols**.

> *"Symbolic logic is the development of the most general principles of rational procedure, in ideographic symbols, and in a form which exhibits the connection of these principles one with another." (Irving Lewis in A Survey of Symbolic Logic)*

Pioneered for example by Leibniz, Boole, Frege, etc.

For example:

- **Propositional logic**
- **Predicate logic**
- Higher-order logic

Grammars - BNFs

Two important aspects of a language are:

- its **syntax** describing the well-formed sequences of symbols denoting objects of the language;
- and its **semantics** assigning meaning to those symbols.

This lecture focuses on <u>syntax</u>.

Grammars - BNFs

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The syntax of a language is defined through a **grammar**.

In particular, the language of a symbolic logic is defined by a grammar that allows deriving formulas from collections of symbols (we will see an example in a few slides).

Grammars - BNFs

The grammar of such a language is often defined using a **Backus Naur Form** (BNF). BNFs allow defining **context-free grammars** (i.e., where production rules are context independent). They are collections of **rules** of the form:

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The **arity** of a terminal symbol is the number of arguments it takes.

The **Fixity** of a terminal symbol is the place where it occurs w.r.t. its arguments: **infix** if it occurs in-between its arguments, **prefix** if it occurs before, and **postfix** if it occurs after.

Grammars - BNF example

Example of a BNF for (some) arithmetic expressions:

$$exp ::= num \mid exp + exp \mid exp \times exp$$

where a numeral num is a sequence of digits. Here exp is a non-terminal symbol and $+$, $\times$, $\mathit{0}$, 1, etc., are terminal symbols.

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\begin{array}{lll}
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Fixity: all the above operators are infix.

Grammars - BNF example

Example of a BNF for propositional logic formulas:

$$P ::= a \mid P \rightarrow P \mid P \vee P \mid P \wedge P \mid \neg P$$

where a ranges over a set of atomic propositions (e.g., *"it is raining"*, or *"it is sunny"*). Here P is a non-terminal symbol and $\wedge$, $\vee$, $\rightarrow$, and $\neg$, as well as the atomic propositions, are terminal symbols.

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Derivation:

$$P \mapsto P \vee P \mapsto r \vee P \mapsto r \vee \neg P \mapsto r \vee \neg s$$

Grammars - abstract syntax trees

An expression derived from a BNF grammar can then be seen as a tree, called an **abstract syntax tree**.

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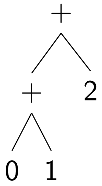
For example, given the grammar:

$$exp ::= num \mid exp + exp \mid exp \times exp$$

an abstract syntax tree corresponding to $0 + 1 + 2$ is:



Grammars - associativity

Note the **ambiguity** in our example: $0 + 1 + 2$.
Does it stand for $(0 + 1) + 2$ or $0 + (1 + 2)$?

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We need to define the **associativity** of the terminal symbols to avoid ambiguities.

- left associativity: $(0 + 1) + 2$
- right associativity: $0 + (1 + 2)$

We will consider the first but we will sometimes use parentheses to avoid ambiguities.

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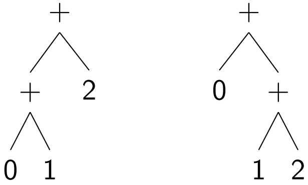

Grammars - precedence

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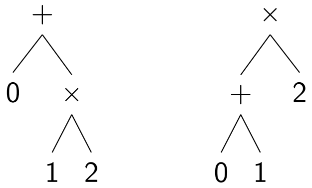

Grammars - example

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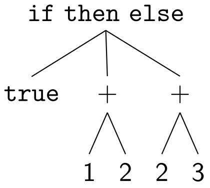
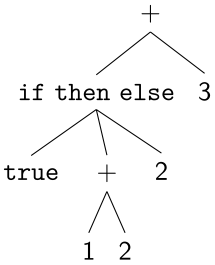
Again this is ambiguous. Without knowing which operator has precedence over the other, it could be either of the two:

$\texttt{if true then } (1 + 2) \texttt{ else } (2 + 3)$



$(\texttt{if true then } 1 + 2 \texttt{ else } 2) + 3$



Grammars - associativity, precedence, parentheses

To avoid ambiguities:

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Parentheses are sometimes necessary:

- using left associativity $0 + 1 + 2$ stands for $(0 + 1) + 2$
- we need parentheses to express $0 + (1 + 2)$

Grammars - example

Given the grammar:

$$P ::= a \mid P \rightarrow P \mid P \vee P \mid P \wedge P \mid \neg P$$

what is the abstract syntax tree for $(\neg P) \wedge (Q \vee R)$?

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For example, we might write $P \rightarrow P$ to mean that P implies P whatever the proposition P is: "it is rainy" → "it is rainy" is true; "it is sunny" → "it is sunny" is true; etc.

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Notation. Given the grammar:

$$exp ::= num \mid exp + exp \mid exp \times exp$$

one typically allows exp, exp_0, exp_1, ..., exp', exp'', ..., as variables ranging over all possible arithmetic expressions derivable using the above rule.

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Technical details:

- The expressions of the language captured by the above grammar, has all the ones that cannot be derived further, i.e., that do not contain **non-terminal** symbols.

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- $exp + exp$ is not part of this language but is useful to capture a **collection** of expressions.
- Why is it called a "metavariable"? A metavariable is a variable within the language, called the **metatheory**, used to describe and study a theory at hand.

(Meta)variables

For example, let us consider the following grammar:

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where equalities are used to state that two expressions are equal.

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Some equalities are assumed to hold in our simple logic through **axioms**, such as $0 + 0 = 0$, $1 + 0 = 1$, $2 + 0 = 2$, etc.

Axiom schemata

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$$exp + 0 = exp$$

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- $2 + 0 = 2$
- $(1 + 2) + 0 = 1 + 2$
- etc.

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For example, we can state:

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using the variables P and Q.

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By replacing P by "2 is prime" and Q by "2 is even", we can obtain the following instance of this formula:

$$(2 \text{ is prime} \wedge 2 \text{ is even}) \to 2 \text{ is prime}$$

Substitution

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A **substitution** is a mapping (e.g., a key/value map), that maps metavariables to arithmetic expressions.
The **substitution operation** is the operation that replaces all occurrences of the keys by the corresponding values (the 1st key/value pair is considered if a key occurs more than once).

Substitution

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For example:

- The substitution $exp \backslash 1$ maps exp to 1.
- $exp_1 \backslash 0, exp_2 \backslash 1$ maps exp_1 to 0 and exp_2 to 1.
- $exp_1 \backslash 0, exp_2 \backslash 1, exp_1 \backslash 1$ also maps exp_1 to 0 and exp_2 to 1.

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The substitution operation, written $eq[s]$, takes an equality eq and a substitution s, and replaces all occurrences of the keys of s by the corresponding values in eq.

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For example: $(exp + 0 = exp)[exp \backslash 1]$ returns $1 + 0 = 1$.

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and as we allow variables in expressions:

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\begin{array}{lcl}
v[s] & = & v, \text{ if } v \text{ is not a key of } s \\
v[s] & = & e, \text{ if } s \text{ maps } v \text{ to } e
\end{array}
$$

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Conclusion

What did we cover today?

- A formal language such as a symbolic logic has a syntax captured by a grammar (e.g., a BNF).
- (Meta)variables are used to capture collections of axioms (as axiom schemata) of symbolic logics.
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Next time?

- Propositional logic - Syntax