Assignment 4 Week6&7

Wednesday, March 30, 2022 6:02 PM

Exercise 1 *Let $\Sigma = \{a,b\}$ and $u, v \in \Sigma^+$, where Σ^+ is the set of nonempty words over Σ. We say that u is* present *in v if u can be obtained by deleting letters from v. For example, abbba is present in aabbababa. We write $|u|$ to denote the* length *of word u, i.e., the number of characters. For example, $|abbba| = 5$.*

The goal of this exercise is to show that the following decision problem is in NP.

***Input:** $w_0\#w_1\#...\#w_k$ such that $w_i \in \Sigma^+$ for $0 \le i \le k$.*

***Problem:** is there a word $x \in \Sigma^+$ such that $|x| = |w_0|$ and x is present in each w_i for $0 < i \le k$?*

We will decompose the task into several steps.

1. *Let us consider a two-tape deterministic Turing machine $\mathcal{M}_1$ on the input alphabet $I = \{a, b, \#\}$ with initial state 0, tape alphabet $T = \{a, b, \#, \text{␣}\}$, return values $V = \{$`True`$,$`False`$\}$, and whose transition function is represented as the diagram in Figure 1 below.*

   Initially,

   - *the* Main *tape contains a non-empty block of as and bs (representing a word $w \in \Sigma^+$) in between a # on the left, on which the head is positioned, and a # or a ␣ on the right. (Outside of these #s and/or ␣s there could be any symbol in T.)*
   - *the* Aux *tape contains a non-empty block of as and bs (representing a word $x \in \Sigma^+$) and is otherwise blank and the head is located on the ␣ immediately to the left.*

   For example,

Main	•#	a	a	b	b	a	b	a	b	a	#
Aux	•␣	a	b	b	b	a	␣	␣	␣	␣	␣

   In the case where the input block on the Main *tape forms the word $w = a^n$ and the input block on the* Aux *tape forms the word $x = a^m$ for $m, n > 0$, how many steps does the machine $\mathcal{M}_1$ goes through until it returns a value in V? (Returning counts as a step.)* **[3 marks]**

This is in fact the worst case complexity as a function of $n = |w|$ and $m = |x|$ and you can use this fact in the remainder of the exercise.

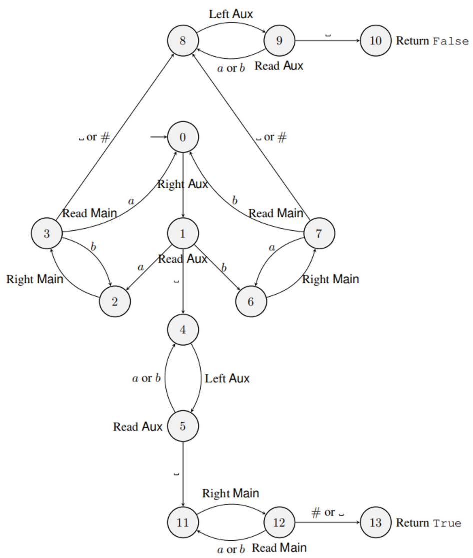

Figure 1: Transition diagram for machine $\mathcal{M}_1$

2. *In this question, we want to design a two-tape nondeterministic Turing machine $\mathcal{M}_2$ that takes as an input a word $w \in \Sigma^+$ and can generate any word $x \in \Sigma^+$ that has the* ***same length*** *as w.*

   Formally, the start configuration is:

   - *the* Main *tape contains a nonempty block of as and bs (representing a word $w \in \Sigma^+$), with a ␣ to the left on which the head is placed and a # to the right, the rest of the tape is blank to the left and can contain any symbol in T beyond # on the right;*
   - *the* Aux *tape is blank.*

   For example,

Main	•␣	a	a	b	b	a	b	a	b	a	#
Aux	•␣	␣	␣	␣	␣	␣	␣	␣	␣	␣	␣

   The machine $\mathcal{M}_2$ should stop when reaching a configuration where:

   - *the* Main *tape is unchanged except for the head which should be placed on the # on the right*
   - *the* Aux *tape contains an arbitrary block of as and bs (representing a word $x \in \Sigma^+$) of the same length as the input block on the* Main *tape and the head is on the first ␣ to the left.*

   For example,

Main	␣	a	a	b	b	a	b	a	b	a	•#
Aux	•␣	b	b	a	b	a	a	a	a	b	␣

   Give the machine $\mathcal{M}_2$ and briefly explain your solution. (Do not use more than 8 states.) **[3 marks]**

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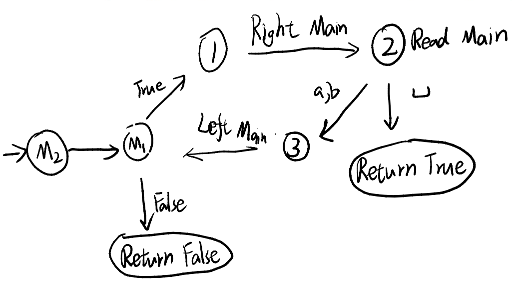
3. *Using machines $\mathcal{M}_1$ and $\mathcal{M}_2$ as macros, design a two-tape nondeterministic Turing machine $\mathcal{M}_3$ for the above decision problem.*

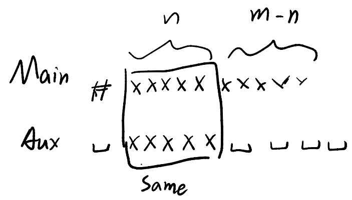
   This means that the machine should start with

   - *a block of as, bs and #s representing the input $w_0\#w_1\#...\#w_k$ on an otherwise blank* Main *tape with the head on the first ␣ to the left of w_0*
   - *and a blank* Aux *tape.*

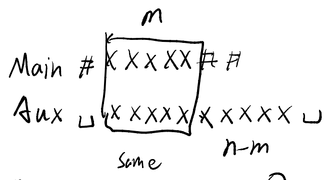
   It should accept *(return `True`) if the question is true for this input and* reject *(return `False`) otherwise. The tape contents and head positions at the end do not matter.*

   Give the machine $\mathcal{M}_3$ and briefly explain your solution. (Do not use more than 5 states.) **[3 marks]**

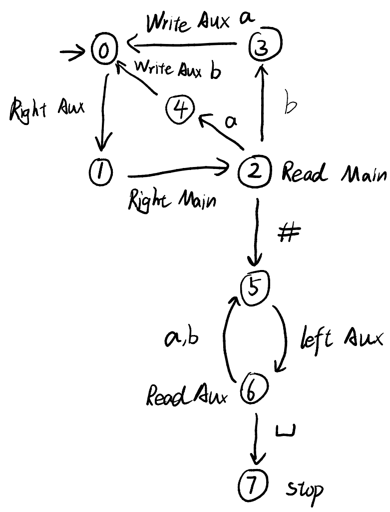




$$\underbrace{(1+n)}_{0\to1}+\overset{1\to2}{n}+\underbrace{n}_{2\to3}+\overset{3\to2}{0}+\underbrace{n}_{3\to0}+\overset{1\to4}{1}+\underbrace{(n+1)}_{4\to5}+\overset{5\to4}{n}+\underbrace{1}_{5\to11}+\overset{11\to12}{(m-n+1)}+\underbrace{(m-n)}_{12\to11}+\overset{12\to13}{1}+\underbrace{1}_{\text{Return}}=7+4n+2m$$



$$\underbrace{(m+1)}_{0\to1}+\overset{1\to2}{(m+1)}+\underbrace{(m+1)}_{2\to3}+\overset{3\to0}{m}+\underbrace{1}_{3\to8}+\overset{8\to9}{(m+1)}+\underbrace{m}_{8\to8}+\overset{9\to10}{1}+\underbrace{1}_{\text{Return}}=6m+7$$



Heads of Aux and Main move at same time when the content of head of Main is a or b then write one value to Aux

When Read Main # which means it reaches end so we move head of Aux to the begining (which is a ␣) then end

␣ a b a b b # b a a a a a b # a b # . . .
␣ ␣ ␣ ␣

␣ a b a b b # b a a a a a b # a b # . . .
␣ a a a a a ␣

4. *Explain briefly why the problem is in* NP. **[3 marks]**

*(**Note:** You may assume that any polytime two-tape nondeterministic Turing machine can be converted into a polytime one-tape nondeterministic machine with the same language. Just as we learnt in lectures for deterministic machines.)*

The value of x should be random but it is too hard to achieve.