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Higher order derivatives

Given function $f : \mathbb{R} \to \mathbb{R}$, we can now construct another function $f' : \mathbb{R} \to \mathbb{R}$ (the derivative of f). We can now iterate this construction, and consider the derivative f'' of f', the derivative f''' of f'', and so on. These functions are known as *higher-order derivatives* of the function f; specifically, f'' is called the *second derivative* of f, f''' is call the *third derivative* of f, and so on. In this context, the derivative f' of f is also called the *first derivative* of f, and the function f itself can be thought of as the *zeroth derivative* of f.

Clearly the notation f', f'', f''', etc., for the subsequent derivative of a function f becomes a bit cumbersome if one wants to consider derivatives of very high order; in this case the alternative notation $f^{(n)}$ for the nth derivative can be used instead (so $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, and so on). Using this notation, a precise definition of the nth derivative of a function can be given recursively as follows.

Definition 14.1. Let $f : \mathbb{R} \to \mathbb{R}$. For each $n \in \mathbb{N} \cup \{0\}$, the *nth derivative* $f^{(n)}$ of f is defined by

$$f^{(n)} = \begin{cases} f, & \text{if } n = 0, \\ \left(f^{(n-1)}\right)', & \text{if } n > 0, \end{cases}$$

where $\left(f^{(n-1)}\right)'$ denotes the derivative of $f^{(n-1)}$.

Remark 14.2. The alternative notation for higher-order derivatives that corresponds to the notation $\frac{df}{dx}$ for the first derivative is as follows: for all $n \in \mathbb{N}$, the *nth* derivative $f^{(n)}$ of f can also be denoted by

$$\frac{d^n f}{dx^n} \quad \text{or} \quad \frac{d^n}{dx^n} f.$$

Example 14.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 2x + 3$. Then,

$$f'(x) = \frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left(x^2 + 2x + 3\right) = 2x + 2.$$

Next,

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}(2x + 2) = 2.$$

Then,

$$f^{(3)}(x) = \frac{d^3 f}{dx^3} = \frac{d}{dx}\left(f''(x)\right) = \frac{d}{dx}(2) = 0.$$

Example 14.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{3x}$. Then, applying the chain rule we can see that

$$f'(x) = \frac{d}{dx}\left(e^{3x}\right) = 3e^{3x}.$$

Applying the chain rule again, we obtain

$$f''(x) = \frac{d^2 f}{dx^2}\frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left(3e^{3x}\right) = 3 \cdot 3e^{3x} = 9e^{3x}.$$