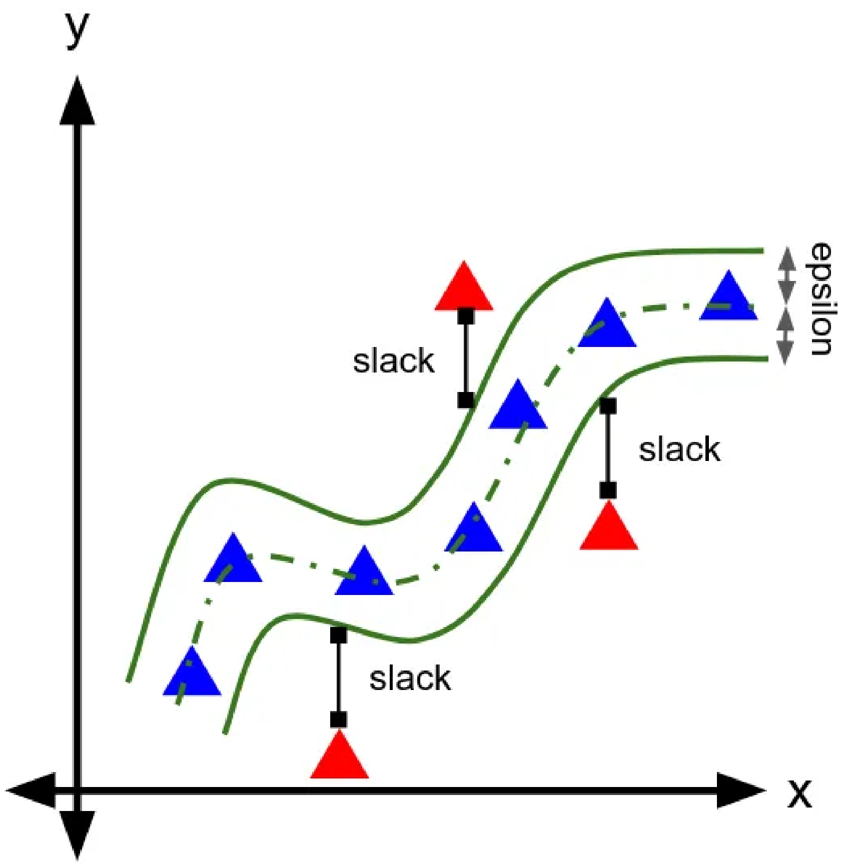
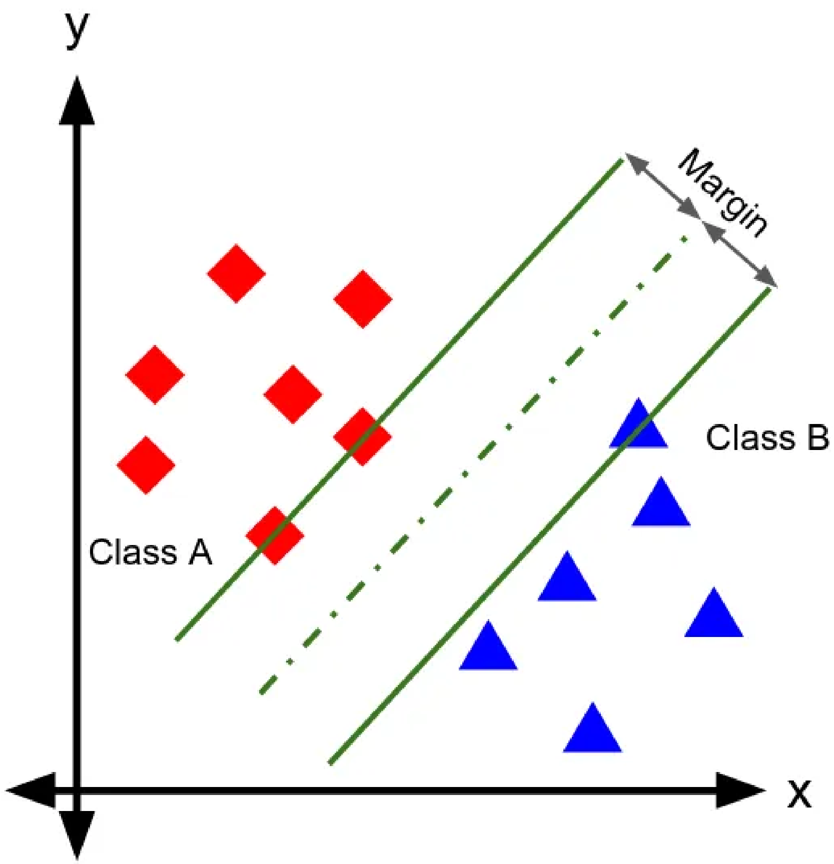
Machine Learning
SVM Regression

Jian Liu



Regression



Classification

Linear Regression

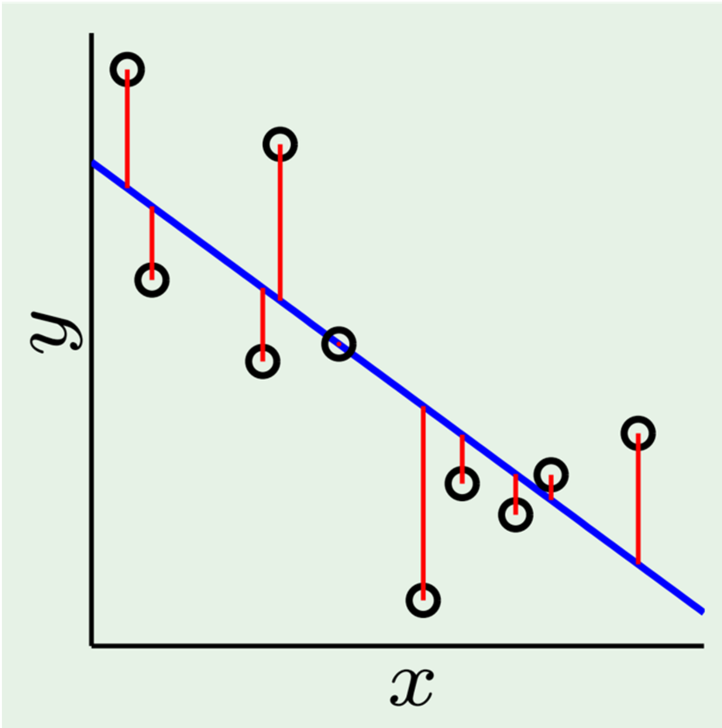
- Given data **X** and target **Y**
- The objective: Find a function that returns the best fit.
- Assume that the relationship between **X** and **Y** is approximately linear. The model can be represented as (**W** represents coefficients and **b** is an intercept)

$$f(w_1, \ldots, w_n, b) = y = \mathbf{w} \cdot \mathbf{x} + b + \varepsilon$$

Linear Regression

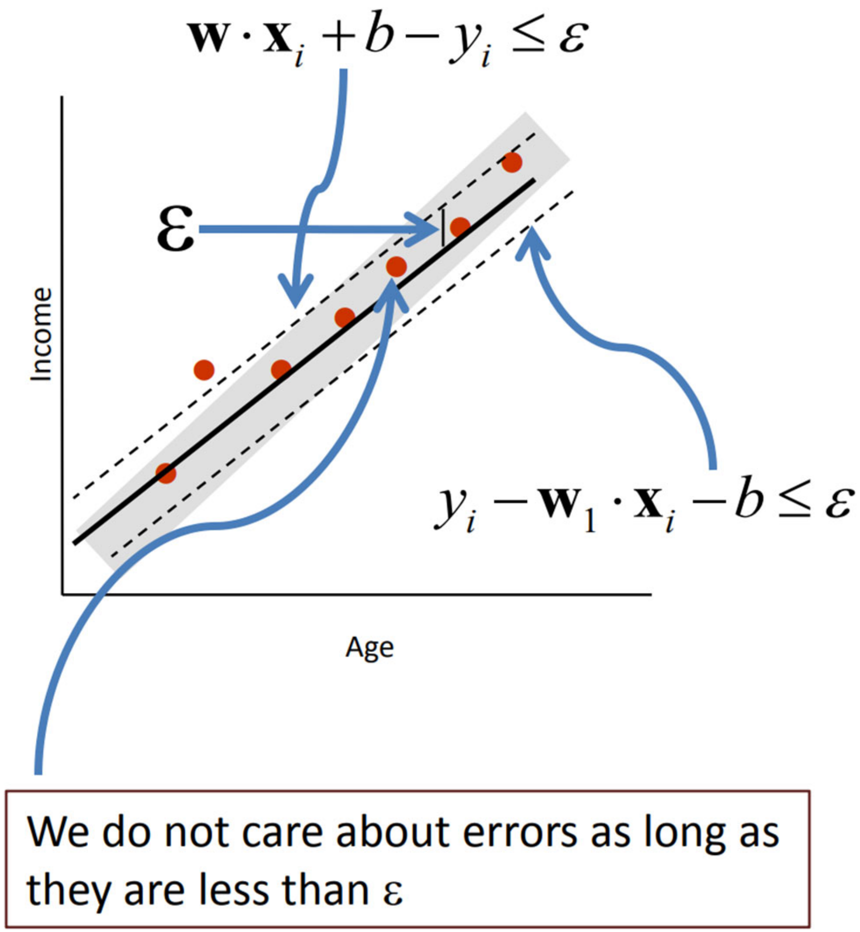
- To find the best fit, we minimize the sum of squared errors -> Least square estimation

$$\min \sum_{i=1}^{m} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{m} (y_i - (\mathbf{w} \cdot \mathbf{x}_i + b))^2$$



Support Vector Regression

- Find a function, f(x),
  with at most ε-deviation from the target y



Support Vector Regression

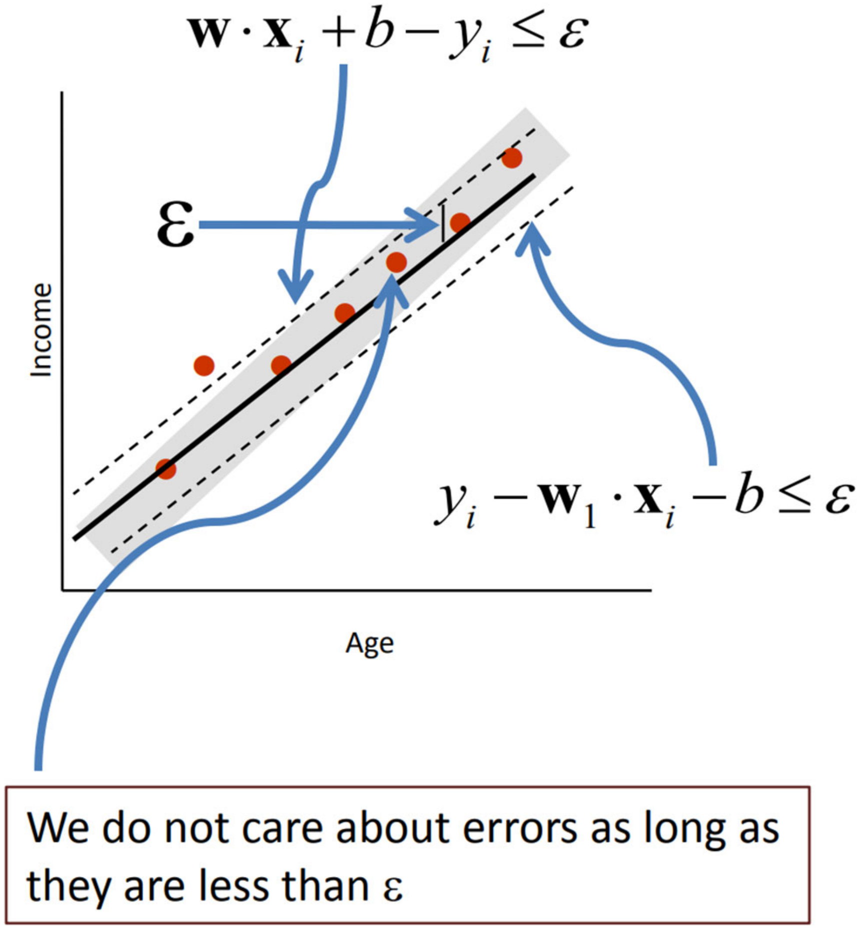
- Find a function, f(x), with at most ε-deviation from the target y

The problem can be written as a convex optimization problem

$$\min \frac{1}{2} \| \mathbf{w} \|^2$$

$$s.t.\ y_i - \mathbf{w}_1 \cdot \mathbf{x}_i - b \le \varepsilon;$$

$$\mathbf{w}_1 \cdot \mathbf{x}_i + b - y_i \le \varepsilon;$$

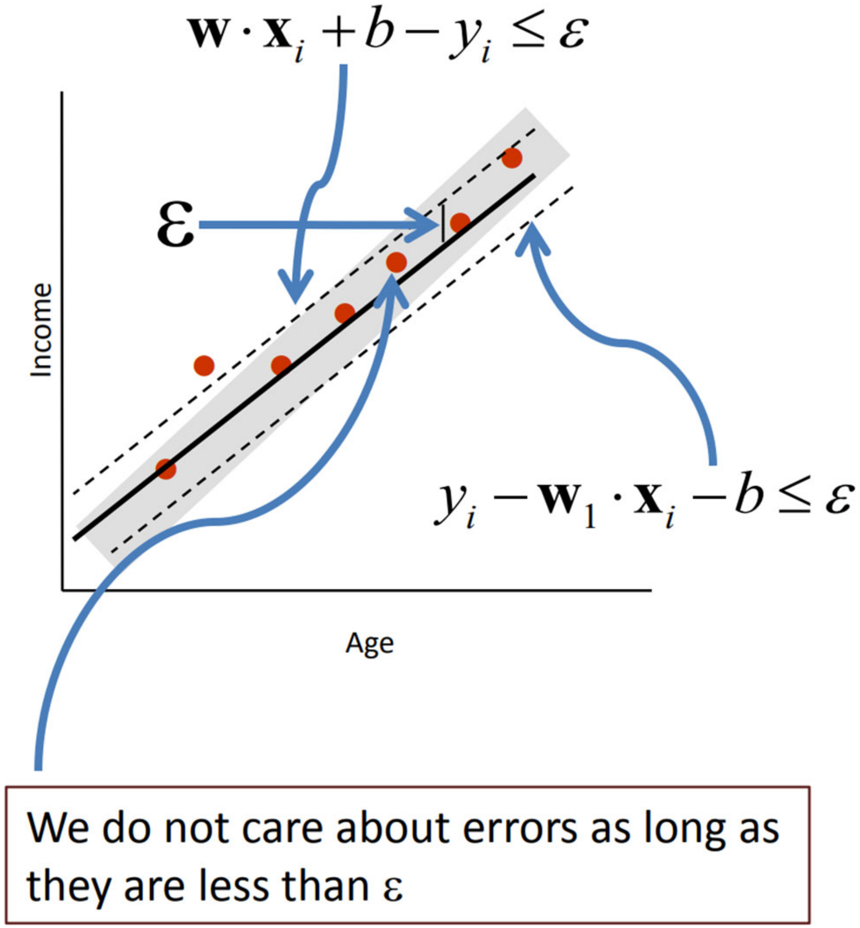


Support Vector Regression

- Find a function, f(x), with at most ε-deviation from the target y

What if the problem is not feasible?

We can introduce slack variables (similar to soft margin loss function).



Support Vector Regression

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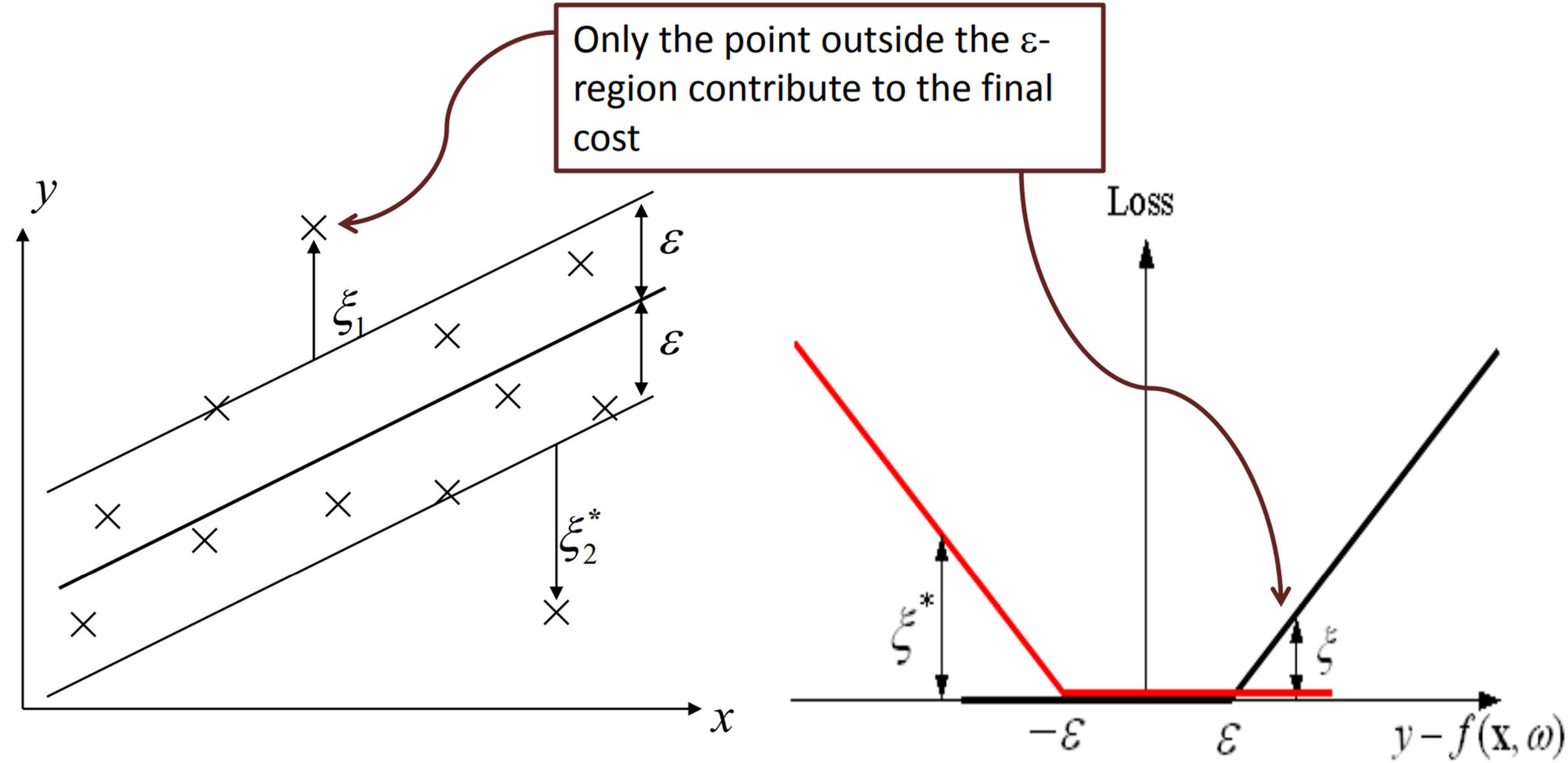
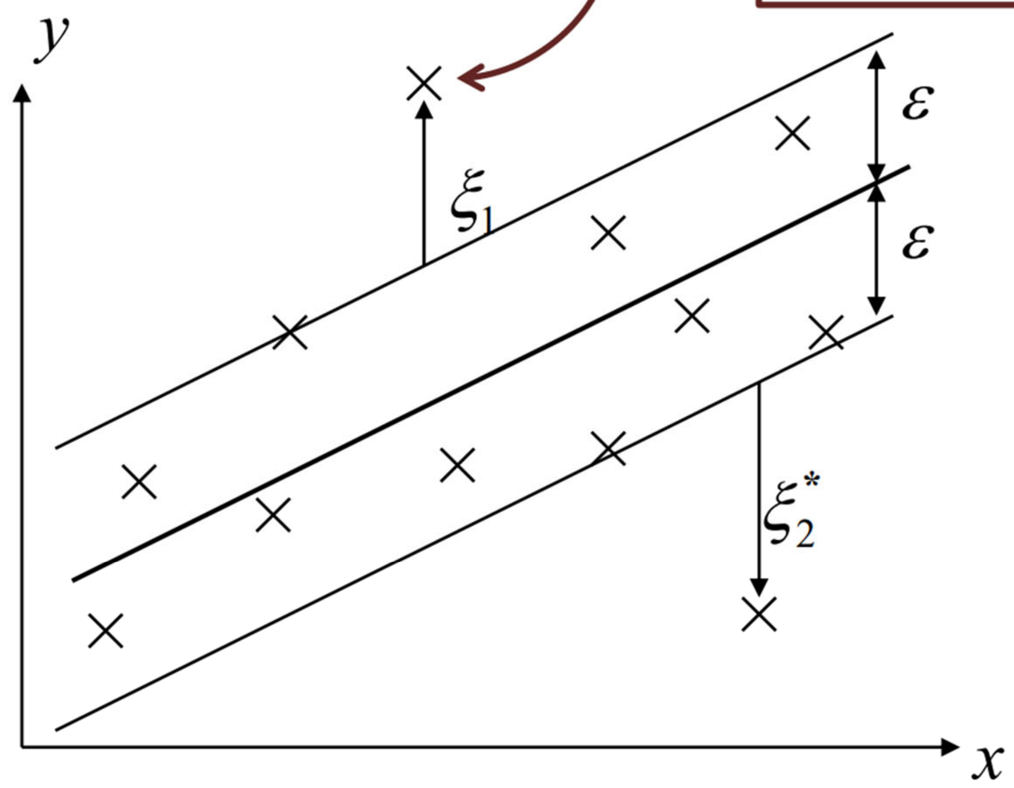


Support Vector Regression

We can introduce slack variables (similar to soft margin loss function).

Only the point outside the ε-region contribute to the final cost



$$J(w) = \frac{1}{2} w' w + C \sum_{1}^{N} (\xi + \xi^*);$$

$$y_i - (x_i w + b) \leq \epsilon + \xi_i$$
$$(x_i w + b) - y_i \leq \epsilon + \xi_i^*$$
$$\xi^* \geq 0$$
$$\xi_i \geq 0$$

Support Vector Regression

We can introduce slack variables (similar to soft margin loss function).

M~ujwufwfr jyjw C:

- Fx C nshwjfxjx1tzwytqjwfshj ktw utnsyx tzyxnij tk ϵ fqxt nshwjfxjx3
- Fx C fuuwtfhmjx 51ymj ytqjwfshj fuuwtfhmjx 5 fsi ymj jvzfynts htqqfuxjx nsyt ymj xnr uqnknji -fqymtzlm xtr jynr jx nskjfxngqj.tsj3

$$J(w) = \frac{1}{2}w'w + C\sum_{1}^{N}(\xi + \xi^*);$$

$$y_i - (x_i w + b) \leq \epsilon + \xi_i$$

$$(x_i w + b) - y_i \leq \epsilon + \xi_i^*$$

$$\xi^* \geq 0$$

$$\xi_i \geq 0$$

Optimizing the Lagrangian

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) - \sum_{i=1}^{\ell}(\eta_i\xi_i + \eta_i^*\xi_i^*)$$

$$-\sum_{i=1}^{\ell}\alpha_i(\varepsilon + \xi_i - y_i + \langle w, x_i\rangle + b)$$

$$-\sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \xi_i^* + y_i - \langle w, x_i\rangle - b)$$

Lagrange multipliers $\alpha_i^{(*)}, \eta_i^{(*)} \geq 0.$

Optimizing the Lagrangian

The partial derivatives of L with respect to the variables

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

$$\partial_{\xi_i^{(*)}} L = C - \alpha_i^{(*)} - \eta_i^{(*)} = 0$$

Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell}(\alpha_i^* - \alpha_i) = 0$$

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$$\begin{aligned} L := {} & \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) - \sum_{i=1}^{\ell}(\eta_i\xi_i + \eta_i^*\xi_i^*) \\ & - \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \xi_i - y_i + \langle w, x_i\rangle + b) \\ & - \sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \xi_i^* + y_i - \langle w, x_i\rangle - b) \end{aligned}$$

Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

$$\boxed{\partial_{\xi_i^{(*)}} L = C - \alpha_i^{(*)} - \eta_i^{(*)} = 0}$$

$$L := \frac{1}{2}\|w\|^2 + C \sum_{i=1}^{\ell} (\xi_i + \xi_i^*) - \boxed{\sum_{i=1}^{\ell} (\eta_i \xi_i + \eta_i^* \xi_i^*)}$$

$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \xi_i - y_i + \langle w, x_i \rangle + b)$$

$$- \sum_{i=1}^{\ell} \alpha_i^* (\varepsilon + \xi_i^* + y_i - \langle w, x_i \rangle - b)$$

Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell}(\alpha_i^* - \alpha_i) = 0$$

$$\partial_w L = w - \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i = 0$$

$$\partial_{\xi_i^{(*)}} L = \boxed{C - \alpha_i^{(*)}} - \eta_i^{(*)} = 0$$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) - \sum_{i=1}^{\ell}(\boxed{\eta_i}\xi_i + \eta_i^*\xi_i^*)$$

$$- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \xi_i - y_i + \langle w, x_i\rangle + b)$$

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Optimizing the Lagrangian

$$C - \alpha_i^{(*)}.$$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) - \sum_{i=1}^{\ell}(\boxed{\eta_i}\xi_i + \eta_i^*\xi_i^*)$$

$$- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \xi_i - y_i + \langle w, x_i\rangle + b)$$

$$- \sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \xi_i^* + y_i - \langle w, x_i\rangle - b)$$

Optimizing the Lagrangian

$$\boxed{C - \alpha_i^{(*)}}$$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\boxed{\eta_i}\cancel{\xi_i} + \eta_i^*\cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \cancel{\xi_i} - y_i + \langle w, x_i\rangle + b)$$

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Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

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$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \cancel{\xi_i} - y_i + \langle w, x_i \rangle + b)$$

$$- \sum_{i=1}^{\ell} \alpha_i^* (\varepsilon + \cancel{\xi_i^*} + y_i - \langle w, x_i \rangle - b)$$

Optimizing the Lagrangian

$$\partial_b L = \boxed{\sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0}$$

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$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \cancel{\xi_i} - y_i + \langle w, x_i \rangle + b)$$

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Optimizing the Lagrangian

$$\partial_b L = \boxed{\sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0}$$

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

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$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \cancel{\xi_i} - y_i + \cancel{\langle w, x_i \rangle + b})$$

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Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell} \alpha_i(\varepsilon + \cancel{\xi_i} - y_i + \cancel{\langle w, x_i \rangle + b})$$

$$- \sum_{i=1}^{\ell} \alpha_i^*(\varepsilon + \cancel{\xi_i^*} + y_i - \cancel{\langle w, x_i \rangle - b})$$

Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0 \quad \rightarrow \quad w = \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i$$

$$\begin{aligned} L := & \boxed{\frac{1}{2}\|w\|^2} + C \sum_{i=1}^{\ell} (\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell} (\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*}) \\ & - \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \cancel{\xi_i} - y_i + \cancel{\langle w, x_i \rangle + b}) \\ & - \sum_{i=1}^{\ell} \alpha_i^* (\varepsilon + \cancel{\xi_i^*} + y_i - \cancel{\langle w, x_i \rangle - b}) \end{aligned}$$

Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i = 0 \longrightarrow w = \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i$$

$$L := \boxed{\frac{1}{2}\|w\|^2} + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i\cancel{\xi_i} + \eta_i^*\cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell}\boxed{\alpha_i(\varepsilon} + \cancel{\xi_i} \boxed{- y_i} + \cancel{\langle w, x_i\rangle + b})$$

$$- \sum_{i=1}^{\ell}\boxed{\alpha_i^*(\varepsilon} + \cancel{\xi_i^*} \boxed{+ y_i} - \cancel{\langle w, x_i\rangle + b})$$

Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i = 0 \;\rightarrow\; w = \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i$$

$$\text{maximize} \quad \begin{cases} -\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle \\ -\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \cancel{\xi_i} - y_i + \cancel{\langle w, x_i\rangle + b})$$

$$- \sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \cancel{\xi_i^*} + y_i - \cancel{\langle w, x_i\rangle - b})$$

Optimizing the Lagrangian

$$\text{maximize} \quad \begin{cases} \boxed{-\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle} \\ \boxed{-\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*)} + \boxed{\sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*)} \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

$$L := \boxed{\frac{1}{2}\|w\|^2} + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i\cancel{\xi_i} + \eta_i^*\cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell}\boxed{\alpha_i(\varepsilon} + \cancel{\xi_i} \boxed{- y_i} + \cancel{\langle w, x_i\rangle + b})$$

$$- \sum_{i=1}^{\ell}\boxed{\alpha_i^*(\varepsilon} + \cancel{\xi_i^*} \boxed{+ y_i} - \cancel{\langle w, x_i\rangle - b})$$

Optimizing the Lagrangian

$$\text{maximize} \quad \begin{cases} -\frac{1}{2} \sum_{i,j=1}^{\ell} (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*) \langle x_i, x_j \rangle \\ -\varepsilon \sum_{i=1}^{\ell} (\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i (\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Optimizing the Lagrangian

↓

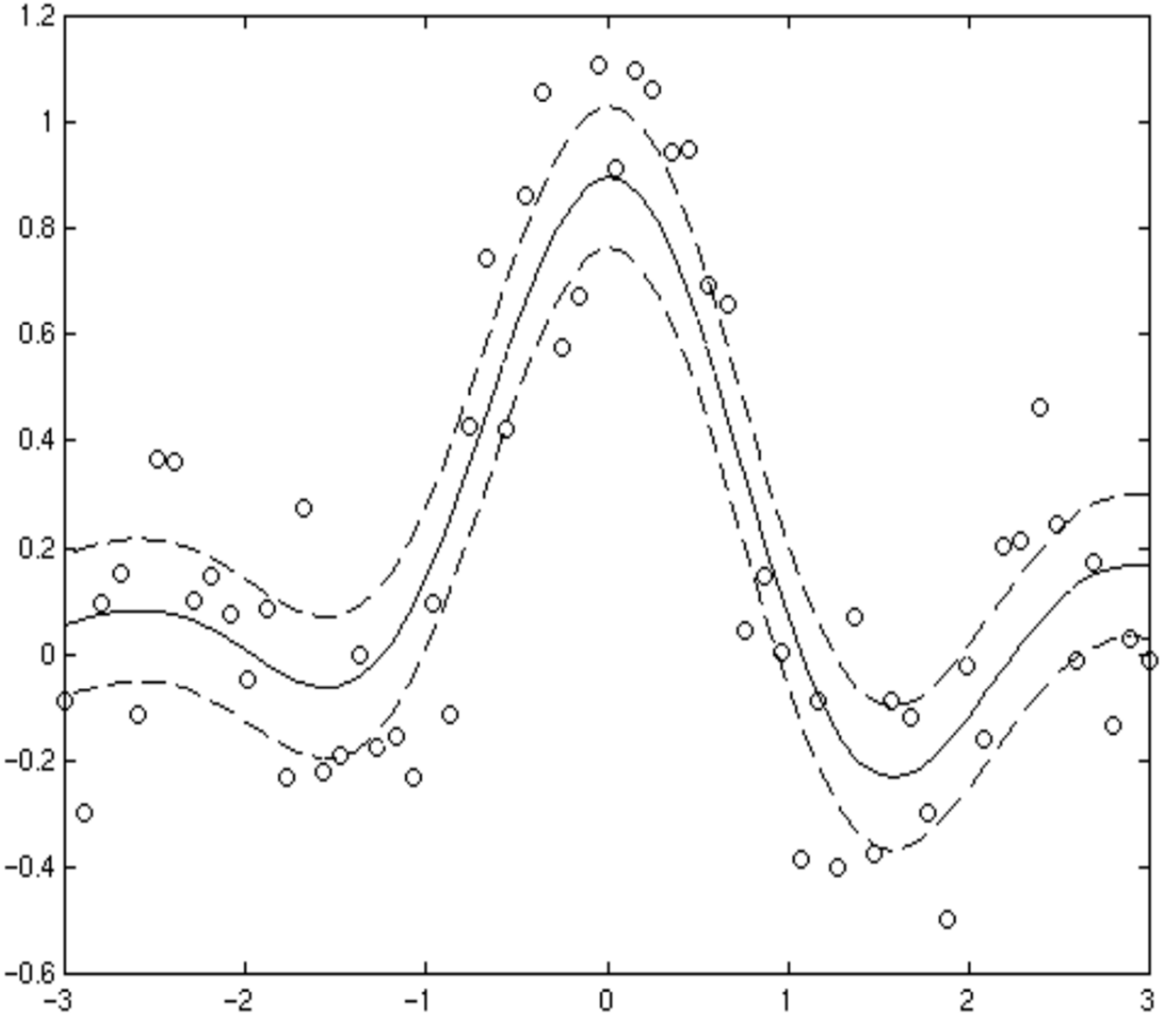
Dual optimization

$$\text{maximize} \quad \begin{cases} -\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\boxed{\langle x_i, x_j\rangle} \\ -\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Now we can use the similar tricks as in SVM

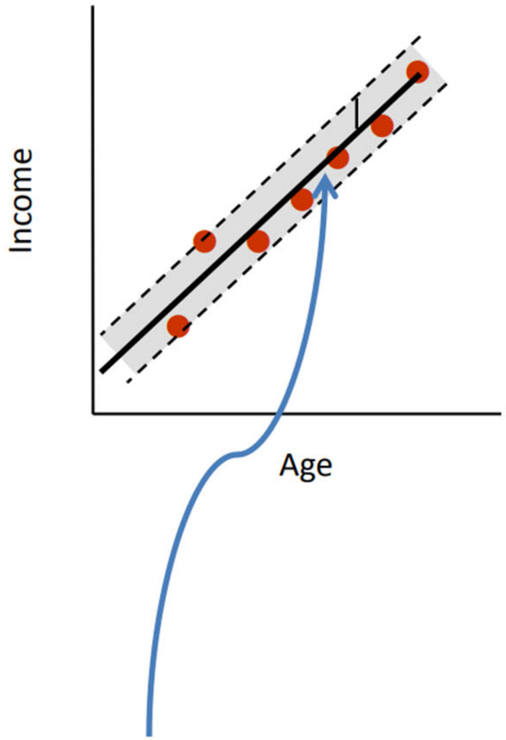
How about a non-linear case?



How about a non-linear case?

- **Linear case**

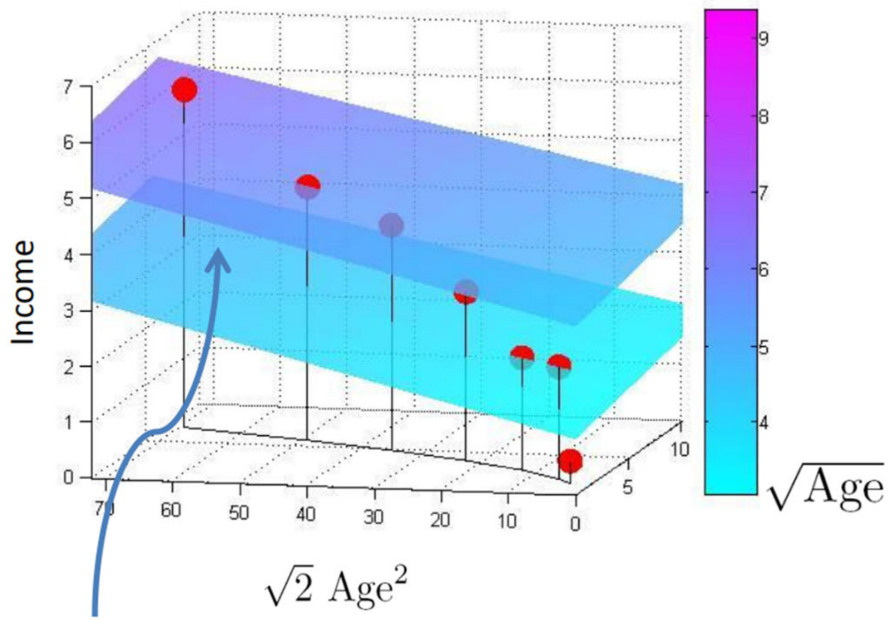
$$f: age \rightarrow income$$



$$y_i = \mathbf{w}_1 \cdot \mathbf{x}_i + b$$

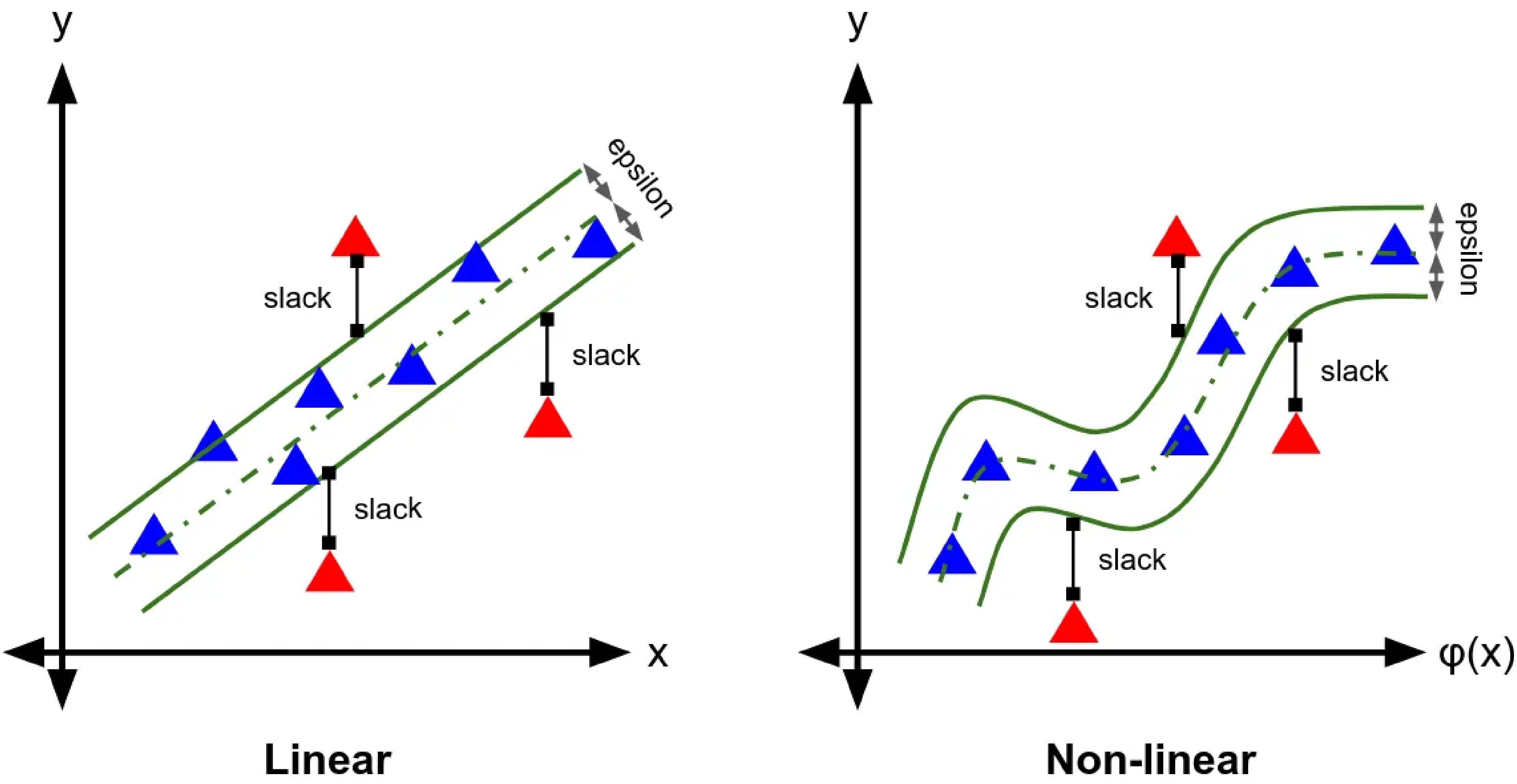
- **Non-linear case**
  - Map data into a higher dimensional space, e.g.,

$$f: (\sqrt{age}, \sqrt{2}age^2) \rightarrow income$$



$$y_i = \mathbf{w}_1 \sqrt{\mathbf{x}_i} + \mathbf{w}_2 \sqrt{2}\mathbf{x}_i^2 + b$$

Linear vs Non-linear



Dual problem

$$\min \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{m}(\xi_i + \xi_i^*)$$

$$s.t.\begin{cases} y_i - (\mathbf{w}\cdot\mathbf{x}_i) - b \le \varepsilon + \xi_i \\ (\mathbf{w}\cdot\mathbf{x}_i) + b - y_i \le \varepsilon + \xi_i^* \\ \xi_i \;, \xi_i^* \ge 0, i = 1,\ldots,m \end{cases}$$

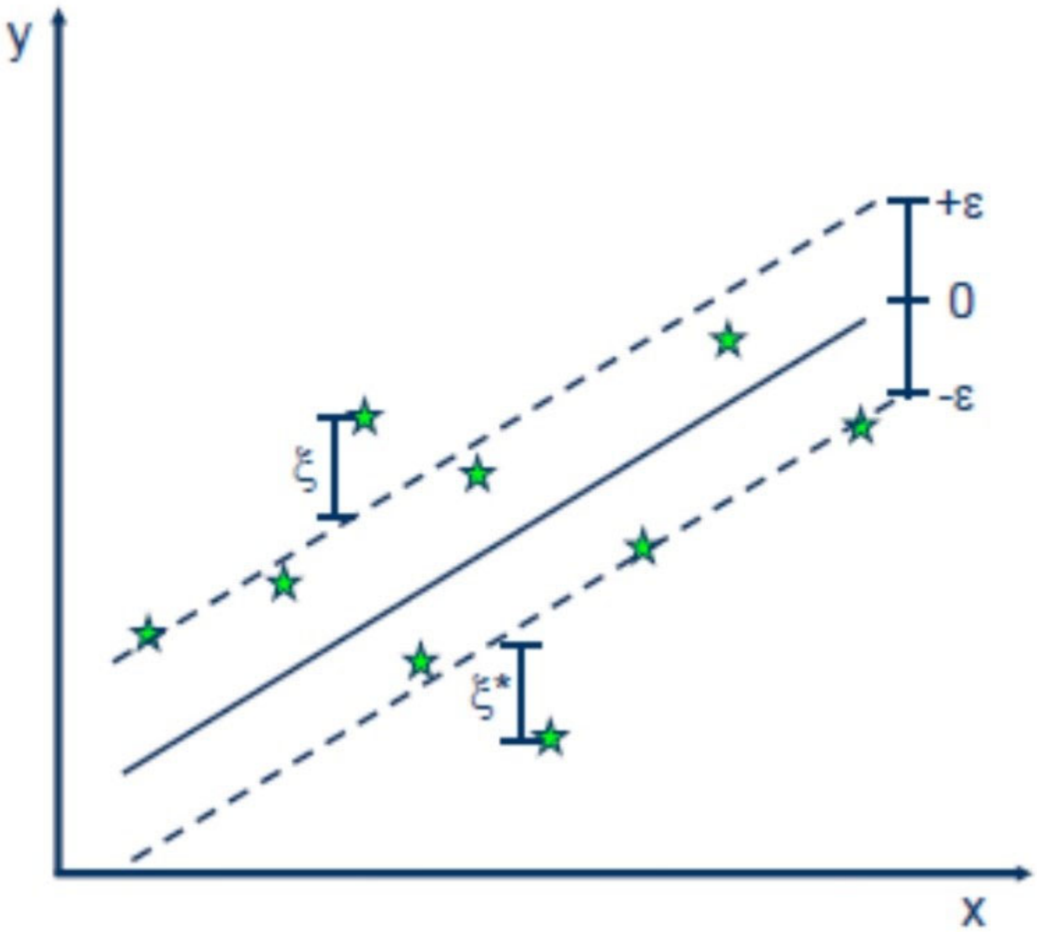
$$\max\begin{cases} \frac{1}{2}\sum_{i,j=1}^{m}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle \\ -\varepsilon\sum_{i=1}^{m}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{m} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$s.t.\sum_{i=1}^{m}(\alpha_i - \alpha_i^*) = 0;\;\; 0 \le \alpha_i, \alpha_i^* \le C$$

Primal variables: w for each feature dim	Dual variables: α, α* for each data point
Complexity: the dim of the input space	Complexity: Number of support vectors

Primal **Dual**

Dual problem



$$\max \begin{cases} \frac{1}{2}\sum_{i,j=1}^{m}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle \\ -\varepsilon\sum_{i=1}^{m}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{m} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

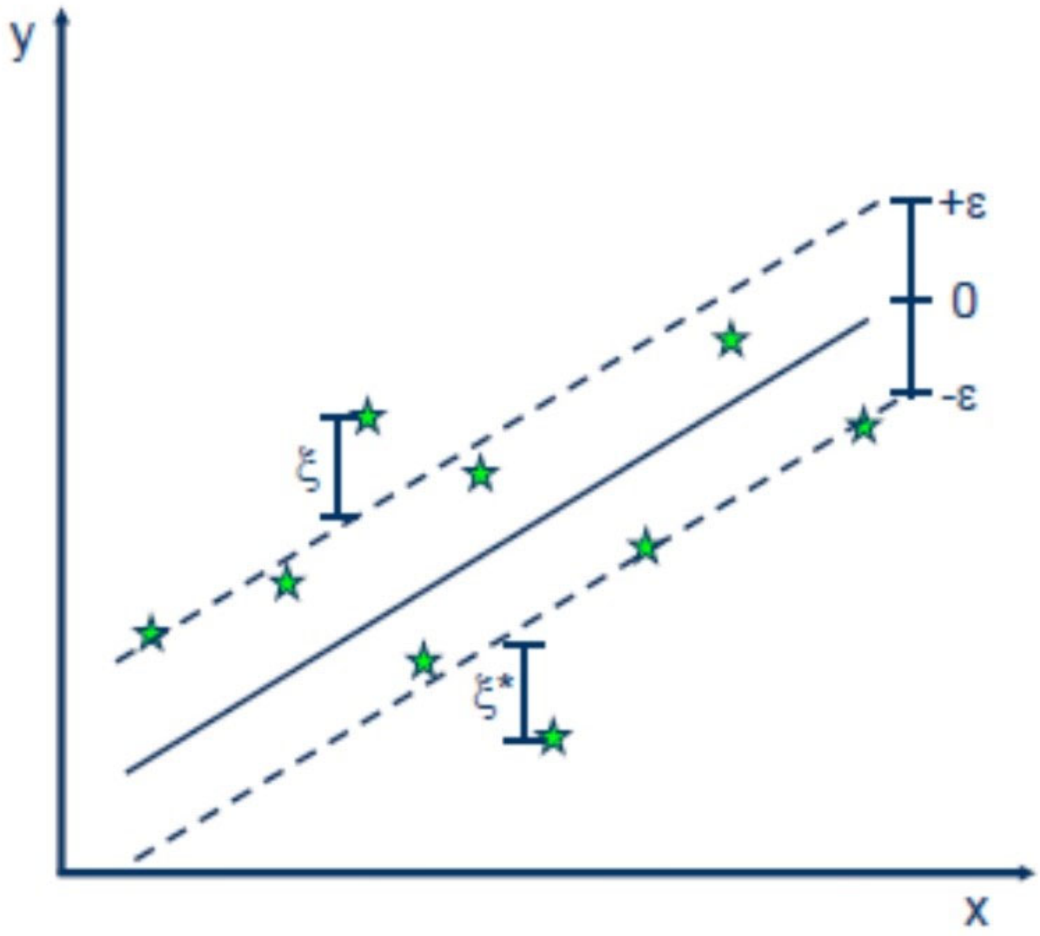
$$s.t.\sum_{i=1}^{m}(\alpha_i - \alpha_i^*) = 0;\ \ 0 \le \alpha_i, \alpha_i^* \le C$$

Dual variables: α, α^* for each data point

Complexity: Number of support vectors

Dual

Dual problem



Kernel trick

K(xi, xj)

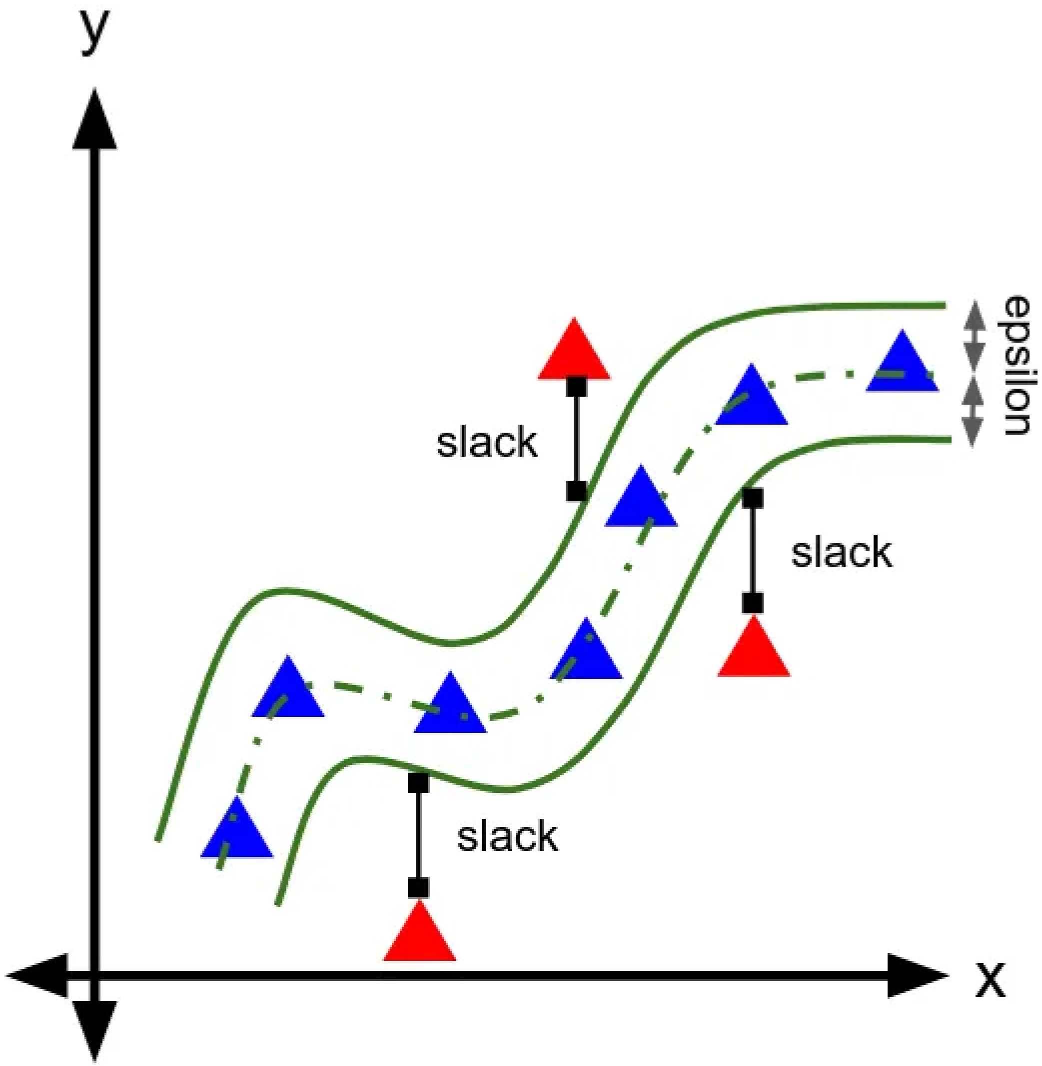
$$\max \begin{cases} \frac{1}{2}\sum_{i,j=1}^{m}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle\varphi(\mathbf{x}_i),\varphi(\mathbf{x}_j)\rangle \\ -\varepsilon\sum_{i=1}^{m}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{m} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$s.t.\sum_{i=1}^{m}(\alpha_i - \alpha_i^*) = 0;\ \ 0 \le \alpha_i, \alpha_i^* \le C$$

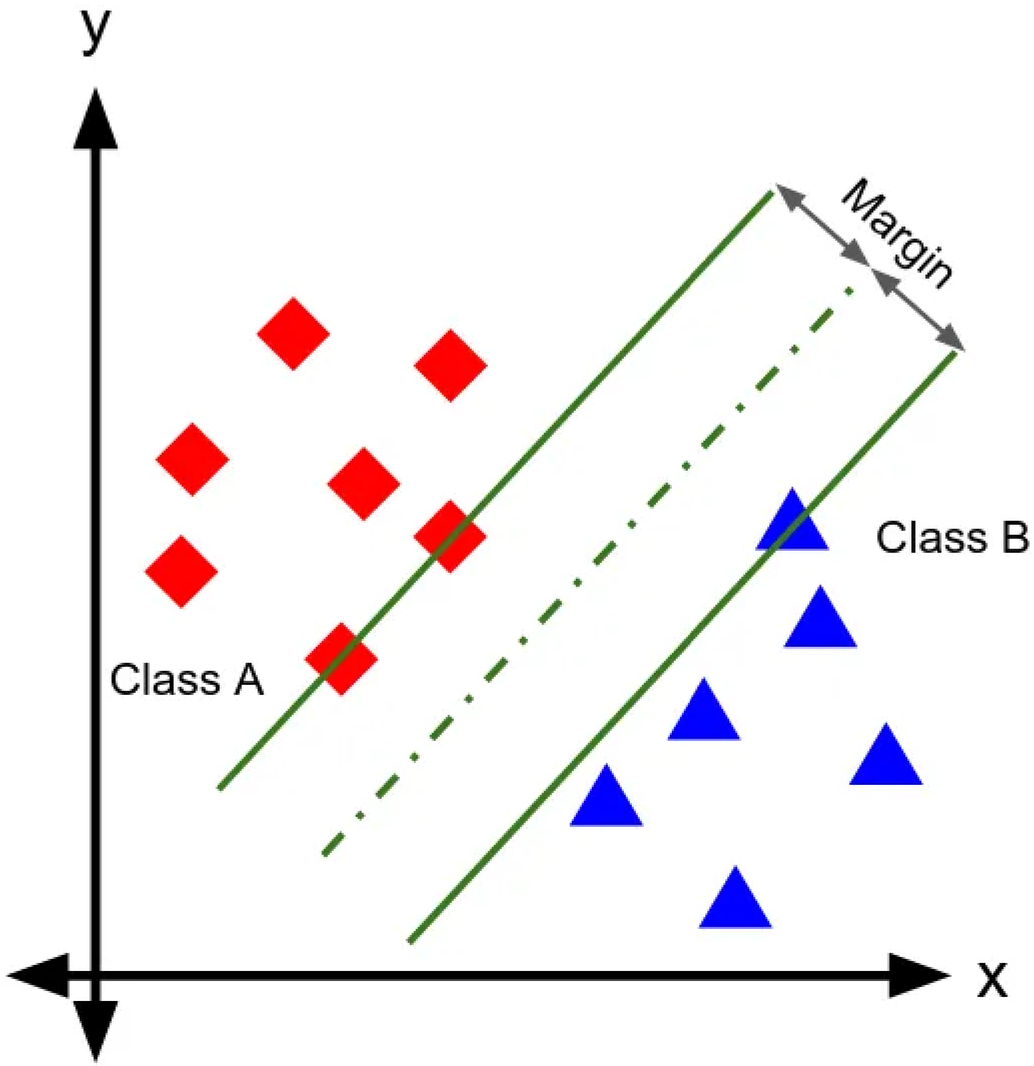
	Dual variables: α, α* for each data point
	Complexity: Number of support vectors

Dual

SVM: Regression vs Classification



Regression



Classification

Summary

- Linear regression tries to minimize the error between the real and predicted value.
- SVR tries to fit the best line within a threshold value (a tube).
- The threshold value is the distance between the hyperplane and boundary line.
- Observations within the threshold of epsilon produce no error, only the observation outside of the epsilon range produce error – sparse kernel machines