Introduction to Probability

Authors
Jordan Chellig

Contents

1

Foundations of Probability

1.1 Experiments and Sample Spaces

The study of probability is motivated by the belief that many different kinds of phenomena are governed by some degree of randomness. The main focus of this course is to use probability as a tool to analyse *random experiments*: Experiments where the outcome can not be determined in advance. An example of a random experiment may be that of rolling a six sided dice, we know the outcomes could be any of numbers from one to six, but we are unable to predict which face is produced, until the experiment (or roll) is actually carried out. For another example, we may take a survey of 100 people and count how many are left-handed, the outcomes of this experiment will be some whole number between 0 and 100. While the outcome of this experiment can not be determined exactly, our intuition says it would be almost impossible to sample 100 people, and for all of them to be left-handed. So while the outcome of this experiment is completely random, we have some idea about the values that are likely to be produced. We are interested in using probability as tool to make precise quantitative predictions about the randomness that occurs around us.

Definition 1.1.1. Suppose we have a random experiment, then the **sample space** of the experiment is the set of all possible outcomes. We usually denote the sample space as Ω.

The main idea from this definition is that the sample space captures every possible outcome that you could observe in the experiment. For example, when rolling a six sided die the sample space would be the numbers from one to six, or equivalently using set notation $\{1, 2, 3, 4, 5, 6\}$. In the survey example, if we count the number of left handed people from a sample of one hundred people, then the sample space is all whole numbers from 0 to 100, or again written symbolically as $\{0, 1, \ldots, 99, 100\}$.

In practice, to find the sample space we need to find a way of representing all possible outcomes that could potentially be observed from the experiment.

Example 1.1.1. Suppose we flip three coins and record the sequence of heads and tails produced. We wish to find the sample space for this experiment.

Once flipped, each coin either produces either a heads (H) or a tails (T). Therefore if we flip three coins, then each coin will either show a H or T. Suppose we flip the three coins, then we could write the outcome as a sequence of $H's$ and $T's$ to represent the heads and tails that appear.

For example if the first coin was heads, the second tails, and the thirds heads, then the outcome would be HTH. So the sample space is made up of all possible sequences of $H's$ and $T's$, of length 3. We can write this symbolically as:

$$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, HHH, TTT\}.$$

Example 1.1.2. Suppose you are sat outside one night counting the number of planes that fly over you. You record the number of planes seen before sunrise, again we wish to find the sample space.

Clearly we have that the number of planes flying overhead is a whole number, and this number has to be non-negative. Therefore, the sample space is all the non-negative whole numbers,

$$\Omega = \{0, 1, 2, 3, 4, \ldots\}.$$

We remark two things about Example 1.1.2, firstly the sample space does not have to be finite. In this example, the sample space was any non-negative whole number, which is an infinite set. The second is that some outcomes in the sample space are ridiculously unlikely: It is almost certain that you would never observe $100,000$ planes flying over in a single night. However, it is still a possible outcome from the experiment, therefore it should be included in the sample space.