3.2 Continuous Random Variables

A continuous random variable is one which can take any value within some interval of the real numbers. For example suppose that we record the height of a random person, then this recorded height is a random variable, say H. Then H is a continuous random variable which can be any real value between 0 and positive infinity. Or the exact temperature T recorded on a given day, can be any real value.

One main challenge with continuous random variables is that asking exact probabilities do not really make sense. For example if X is a continuous random variable, then for any value $x \in \mathbb{R}$, we have that $\mathbb{P}(X = x) = 0$. Intuitively think about throwing a dart at a dart board. The dart will definitely hit the board; however, if you specify a point on the board, the probability that the dart hits that specific point exactly to any arbitrary precision is zero. Therefore when talking about probabilities for continuous random variables, we should only ever consider cumulative probabilities, i.e $\mathbb{P}(X \leq x), \mathbb{P}(X \geq x)$ or $\mathbb{P}(x \leq X \leq y)$. Any probabilities of the form $\mathbb{P}(X = x)$ are always equal to zero. See section 3.3 for a more in-depth discussion on why this is the case.

3.3 Intuitive Differences with Discrete Random Variables

This section makes the discussion about continuous random variables having a zero probability at given points rigorous. *This section is conceptually more challenging, do feel free to skip over it. Just make sure you are happy that continuous random variables always have zero probability at single points.*

We motivate this idea with the following experiment: Suppose we pick a random real number from the interval $[0, 1]$ with each number being equally likely to be chosen. Let X be the outcome of this experiment, then what is $\mathbb{P}[X = 0.5]$?

We have that X is a continuous random variable, with support in $[0, 1]$. Now the question is, can the random variable X ever equal 0.5 exactly? Let us define the notation $[x]_i$ to be the number in the i^{th} decimal place of x. For example $[0.4354]_2 = 3$ and $[0.50145]_1 = 5$. One way to think about this, is to remember that $0.5 = 0.5000000\ldots$, with zeros going on infinitely. Thus $[0.5]_1 = 5$ and $[0.5]_2 = [0.5]_3 = 0$ and so on. Thus if $X = 0.5$, then $[X]_1 = 5, [X]_2 = 0, [X]_3 = 0$ off to infinity. Picking a random number uniformly in $[0, 1]$ would be equivalent to picking each decimal place of X one at a time, with each value from 0 to 9 being equally likely, and each decimal place being independent. Therefore we have,

$$\begin{aligned}
\mathbb{P}(X = 0.5) &= \mathbb{P}(\{[X]_1 = 5\} \cap \{[X]_2 = 0\} \cap \{[X]_3 = 0\} \cap \{[X]_4 = 0\} \cap \ldots) \\
&= \mathbb{P}([X]_1 = 5) \times \mathbb{P}([X]_2 = 0) \times \mathbb{P}([X]_3 = 0) \times \mathbb{P}([X]_4 = 0) \times \ldots \\
&= \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \ldots \\
&= \lim_{n \to \infty} \left(\frac{1}{10}\right)^n \\
&= 0.
\end{aligned}$$

The final equality follows, as it is an infinite product of numbers strictly less than one. Do not worry too much about this calculation the main idea to take away is that if we have a continuous random variable $\mathbb{P}(X = i)$ for any number i is always zero. As a result this seems quite counter-intuitive. For example, if we have a dart board, the probability the dart hits any specific point is zero, i.e the perfect centre of bulls-eye. However the dart will hit the board with probability one. The contradiction we have here is that we have not asked the right question.

Consider the previous example above, where we pick a random number from $[0, 1]$. Suppose we instead ask, what is $\mathbb{P}[X \leq 0.5]$? Well in this case we know that the interval $[0, 0.5]$ is precisely half the length of the interval $[0, 1]$. So there exactly a 50% chance that the point chosen is in the lower half of the interval. Therefore as we would expect $\mathbb{P}[X \leq 0.5] = 0.5$.

Similarly suppose we want to find $\mathbb{P}[0.4 \leq X \leq 0.6]$. Well we are asking what is the probability $X \in [0.4, 0.6]$ this interval has length $0.6 - 0.4 = 0.2$. Therefore it takes us 20% of the whole interval, hence we have that $\mathbb{P}[0.4 \leq X \leq 0.6] = 0.2$.

We remark that we can also approach this problem using the decimal notation above. If $0.4 \leq X \leq 0.6$, then we must have that $[X]_1$ is either 4 or 5. For the remaining decimal places it does not matter what they are, the outcome will always be between 0.4 and 0.6. These two events are disjoint, therefore:

$$\mathbb{P}(0.4 \leq X \leq 0.6) = \mathbb{P}(\{[X]_1 = 0.4\} \cup \{[X]_1 = 0.5\}) = \mathbb{P}(X_1 = 0.4) + \mathbb{P}(X_1 = 0.5).$$

Now for $[X]_1$ we remark that any of values from 0 to 9 can be chosen, and all are equally likely. Hence

$$\mathbb{P}([X]_1 = 4) = \mathbb{P}([X]_1 = 5) = 0.1.$$

Therefore it follows that:

$$\mathbb{P}(0.4 \leq X \leq 0.6) = 0.1 + 0.1 = 0.2.$$

We remark these calculations only follow because the distribution of X is uniform, i.e every value is equally likely to be chosen.