Mathematical and Logical Foundations of Computer Science

Week 2 - Additional Examples

Vincent Rahli

University of Birmingham 2020/21

Propositions

Upper case letters stand for variables, while lower case letters stand for atomic propositions. We will often use variables rather than specific atomic propositions.

Here are examples of propositions:

- $P \to Q \to R$
  - it stands for $P \to (Q \to R)$
  - because $\to$ is right-associative
  - this is different from $(P \to Q) \to R$
  - it uses 2 variables P and Q
- $p \to q \to r$
  - it stands for $p \to (q \to r)$
  - because $\to$ is right-associative
  - this is different from $(p \to q) \to r$
  - it uses 2 atoms p and q

Propositions

Here are further examples of propositions:

- $P \wedge Q \to P \vee Q$
  - it stands for $(P \wedge Q) \to (P \vee Q)$
  - because $\wedge$ and $\vee$ have precedence over $\to$
  - this is different from $P \wedge (Q \to P \vee Q)$
  - it uses 2 variables P and Q
- $\neg\neg P \to P$
  - it stands for $(\neg(\neg P)) \to P$
  - because $\neg$ has precedence over $\to$
  - this is different from $\neg\neg(P \to P)$
  - it uses 1 variable P

Propositions

Those are not propositions:

- $P \wedge \wedge Q$
  - this is not derivable from the syntax of propositional logic
  - it is not possible to have two $\wedge$ next to each other
- $P \rightarrow \wedge Q$
  - this is not derivable from the syntax of propositional logic
  - it is not possible to have a $\wedge$ next to a $\rightarrow$

Natural Deduction Proofs

Here is a Natural Deduction proof of $A \vee B \vdash C \rightarrow (A \vee B) \wedge C$

$$\cfrac{\cfrac{A \vee B \qquad \overline{C}^{\,1}}{(A \vee B) \wedge C}\,[\wedge I]}{C \rightarrow (A \vee B) \wedge C}\,1\ [\rightarrow I]$$

Note that $C \rightarrow (A \vee B) \wedge C$ is read as $C \rightarrow ((A \vee B) \wedge C)$

Natural Deduction Proofs

Here is a Natural Deduction proof of $Q \vdash (Q \to R) \to R$

$$\dfrac{\dfrac{\dfrac{}{Q \to R}{\scriptstyle 1} \qquad Q}{R}\,[\to E]}{(Q \to R) \to R}\,{\scriptstyle 1}\,[\to I]$$

Natural Deduction Proofs

Here is a Natural Deduction proof of $\vdash \neg(A \wedge B) \rightarrow A \rightarrow \neg B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg(A \wedge B)}^{\,1} \qquad \dfrac{\overline{A}^{\,2} \qquad \overline{B}^{\,3}}{A \wedge B}\,[\wedge I]}{\bot}\,[\neg E]}{\neg B}\,3\ [\neg I]}{A \rightarrow \neg B}\,2\ [\rightarrow I]}{\neg(A \wedge B) \rightarrow A \rightarrow \neg B}\,1\ [\rightarrow I]
$$

Note that $\neg(A \wedge B) \rightarrow A \rightarrow \neg B$ is read as $\neg(A \wedge B) \rightarrow (A \rightarrow \neg B)$