3

Intervals in the real line

It is convenient to introduce notation for particular subsets of $\mathbb{R}$, called intervals. First we consider bounded intervals, whose elements are those real numbers between two given numbers $a, b \in \mathbb{R}$, called *endpoints* of the interval. There are several types of such intervals, depending on whether each endpoint is included or not included:

Definition 3.1. Let $a, b \in \mathbb{R}$ with $a \leq b$. Then:

(i) the open interval, denoted (a, b), is the set

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}\,;$$

(ii) the closed interval, denoted $[a, b]$, is the set

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}\,;$$

(iii) the half-open intervals

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}\,;$$
$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}\,.$$

Example 3.2. Let $a = 1$ and $b = 2$. Then:

1. $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$. So it is the interval which contains every number that is larger than 1 but also smaller than 2, but not including the endpoints.

2. $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. So it is the interval which contains every number that is larger than 1 but also smaller than 2. However this also include the 'endpoints' 1 and 2. So,

$$(1, 2) \subseteq [1, 2],$$

and

$$[1, 2] = (1, 2) \cup \{1, 2\}.$$

3. $(1, 2] = \{x \in \mathbb{R} : 1 < x \leq 2\}$. So it is the interval which contains every number that is

strictly larger than 1, but also smaller than 2. However this interval also includes just the 'endpoint' 2.

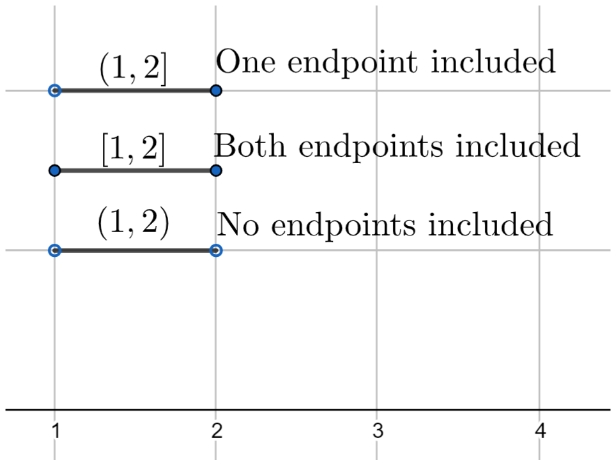

Figure 3.1: Highlighting the different intervals in Example 3.2

Comment. There is an unfortunate notation clash between open intervals and ordered pairs. For example, the expressions $(1, 2)$ may denote either:

- the ordered pairs with first component 1 and second component 2, or;
- the open interval with end points 1 and 2.

Usually the context make it clear which of the two is meant. Other texts may avoid this clash by using different notation; e.g. sometimes the notation $]a, b[$ is used for the open interval with end points a and b.

In addition, we can consider *unbounded intervals*.

Definition 3.3. Let $a \in \mathbb{R}$. Then:

(i) the open half-lines

$$(a, +\infty) = \{x \in \mathbb{R} : x > a\}\,,$$
$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}\,;$$

(ii) the closed half-lines

$$[a, +\infty) = \{x \in \mathbb{R} : x \geq a\}\,,$$
$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}\,;$$

(iii) the real line

$$(-\infty, \infty) = \mathbb{R}.$$

The symbol ∞ reads "infinity". You may like to think of $-\infty$ and ∞ as "endpoints" of the real line $\mathbb{R}$.