Solutions to Exercise Sheet 7

Exercise 7.1

(a) $2 \in X$ by Rule 1, then $2^2 = 4 \in X$ by Rule 2, then $4^2 = 16 \in X$ by Rule 2, then $16^2 = 256 \in X$ by Rule 2 again.

(b) Checking the base case: 2 has the form $2^1 = 2^{2^0}$ so the statement is true with $n = 0$.

Inductive step: Assume $x \in X$ and by induction hypothesis that x has the form 2^{2^n}. Then $x^2 = (2^{2^n})^2 = 2^{2^n \times 2} = 2^{2^{n+1}}$ and the statement is again true with n replaced by $n+1$.

(c) We use the insight gained in the previous item:

- Base case: Include the pair $(2,0)$ in the relation.
- Inductive step: If the relation already contains the pair (x,n) then also include the pair $(x^2, n+1)$.

Exercise 7.2

(a) We have $9 \in X$ and $15 \in X$, hence $9 - 15 = -6 \in X$. It follows that we also have $15 - (-6) = 21$, $21 - (-6) = 27$ and $27 - (-6) = 33$ in X.

(b) The property is divisibility by 3.

Both 9 and 15 are divisible by 3 and this property holds for the difference of two numbers if it holds for each of them individually. This means it holds for all elements of X. However, 32 is not divisible by 3.

(c) This is not a function because it is not single-valued:

We have $(9,1) \in R$ by the base case, but we also get $(15-9, 0-1) = (6,-1) \in R$, therefore $(9-6, 1-(-1)) = (3,2) \in R$, therefore $(15-3, 0-2) = (12,-2) \in R$ and therefore $(12-3, -2-2) = (9,-4) \in R$.

Exercise 7.3

(a) Obviously, the relation is reflexive as every bracket expression has the same level of nesting as itself. It is also transitive because the order on natural numbers is transitive. It is not anti-symmetric because two different expressions can have the same level of nesting, for example `[[]][]` and `[[]]`.

(b) **base case** $($`[]`$,$`[]`$)$ belongs to R.

inductive case 1 If (s,t) belongs to R then so does $($`[`s`]`$, t)$.

inductive case 2 If (s,t) belongs to R then so does $($`[`s`]`$,$`[`t`]`$)$.

inductive case 3 If (s,t) and (s,t') belong to R then so does (s, tt').

inductive case 4a If (s,t) belongs to R and $s' \in wbb$ then also $(ss', t) \in R$.

inductive case 4b If (s,t) belongs to R and $s' \in wbb$ then also $(s's, t) \in R$.

Exercise 7.4

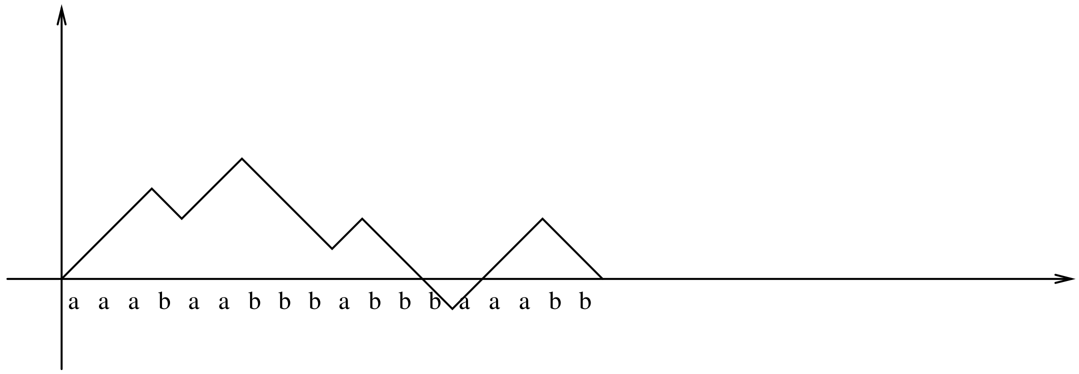
As a grammar, my proposal is

$$S \quad ::= \quad \varepsilon \quad | \quad \mathtt{a}S\mathtt{b}S \quad | \quad \mathtt{b}S\mathtt{a}S$$

Intuitively, the idea is that the string has to start with either `a` or `b`, and for each `a` there must be a "partner" somewhere along the string of the other kind.

It is clear that this grammar produces only strings with an equal number of `a`'s and `b`'s.

Conversely, assume that s is such a string. We need to show that our grammar can generate it. For this we could assign to every initial segment s' the following integer: $c(s') = \#_a(s') - \#_b(s')$ (the number of `a`'s minus the number of `b`'s). Plotting the value of the function c, as we traverse the string from left to right, we would get a graph that looks something like the following:



where each time we read an `a`, we go up one unit and every time we read a `b`, we go down one unit. If we start on the x-axis then we must end up on the x-axis since we go up the same number of times as we go down by assumption. Assuming the string starts with an `a` we find its "partner" by going along the graph and looking for the first occasion when the graph returns to the x-axis. The string in-between must have an equal number of `a`'s and `b`'s, as must the remainder of the string. This means that the string can be thought of as having been generated by the rule $\mathtt{a}S\mathtt{b}S$.