Mathematical and Logical Foundations of Computer Science

Predicate Logic
(Natural Deduction & Sequent Calculus Proofs)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**
- Intuitionistic vs. Classical logic
- Type theory

Today

- Predicate Logic proofs
- Natural Deduction rules
- Intuitionistic Sequent Calculus rules
- Classical Sequent Calculus rules

Further reading:

- Chapter 8 of `http://leanprover.github.io/logic_and_proof/`
- Chapter 5 of `https://www.paultaylor.eu/stable/prot.pdf`

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$
\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
\end{array}
$$

where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
- p ranges over predicate symbols
- $p(t_1, \ldots, t_n)$ is a well-formed formula only if p has arity n

The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

Recap: Substitution

Substitution is defined recursively on terms and formulas:

$P[x\backslash t]$ substitute all the free occurrences of x in P with t.

$$
\begin{array}{lcl}
x[x\backslash t] & = & t \\
x[y\backslash t] & = & x \\
(f(t_1, \ldots, t_n))[x\backslash t] & = & f(t_1[x\backslash t], \ldots, t_n[x\backslash t]) \\
(p(t_1, \ldots, t_n))[x\backslash t] & = & p(t_1[x\backslash t], \ldots, t_n[x\backslash t]) \\
\hline
(\neg P)[x\backslash t] & = & \neg P[x\backslash t] \\
(P_1 \wedge P_2)[x\backslash t] & = & P_1[x\backslash t] \wedge P_2[x\backslash t] \\
(P_1 \vee P_2)[x\backslash t] & = & P_1[x\backslash t] \vee P_2[x\backslash t] \\
(P_1 \rightarrow P_2)[x\backslash t] & = & P_1[x\backslash t] \rightarrow P_2[x\backslash t] \\
\hline
(\forall x.P)[x\backslash t] & = & \forall x.P \\
(\exists x.P)[x\backslash t] & = & \exists x.P \\
(\forall y.P)[x\backslash t] & = & \forall y.P[x\backslash t], \text{ if } y \notin \mathtt{fv}(t) \\
(\exists y.P)[x\backslash t] & = & \exists y.P[x\backslash t], \text{ if } y \notin \mathtt{fv}(t)
\end{array}
$$

The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I]$$

Condition: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$

$$\frac{\forall x.P}{P[x\backslash t]}\ [\forall E]$$

Condition: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$

$$\frac{P[x\backslash t]}{\exists x.P}\ [\exists I]$$

Condition: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$

$$\dfrac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}\ 1\\ \vdots \\ Q\end{array}}{Q}\ 1\ [\exists E]$$

Condition: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Inference Rule for "for all introduction"

We make checking these conditions more tractable

- **going backward**
- using **contexts** to record hypotheses

Here is a proof of $\forall x. x > 2 \rightarrow x > 2$:

$$\dfrac{\dfrac{\overline{x > 2}^{\ 1}}{x > 2 \rightarrow x > 2}\ 1\ [\rightarrow I]}{\forall x. x > 2 \rightarrow x > 2}\ [\forall I]$$

Context:

- 1: $x > 2$

We can pick any variable we want as the context is empty and our conclusion does not have any free variables

Recap: Sequent Calculus

We have such contexts in the **Sequence Calculus**!

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ [\to L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ [\neg L] \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ [\neg R]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ [\vee L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ [\wedge L] \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{}{A \vdash A}\ [Id] \qquad \frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ [Cut]$$

$$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X] \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W] \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ [C]$$

Recap: Sequent Calculus

In addition we allow using the following **derived rules**:

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \to B, \Gamma_2 \vdash C}\ [\to L] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B}\ [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C}\ [\vee L] \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C}\ [\wedge L]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [W] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [C]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A}\ [Id]$$

All these **derived rules** can be proved/derived using the rules on the previous slide

Sequent Calculus for Predicate Logic

$\forall$ **right**

$$\frac{\Gamma \vdash P[x\backslash y]}{\Gamma \vdash \forall x.P}\ [\forall R]$$

Condition: y must not be free in Γ or in $\forall x.P$

$\forall$ **left**

$$\frac{\Gamma, P[x\backslash t] \vdash Q}{\Gamma, \forall x.P \vdash Q}\ [\forall L]$$

Condition: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$

Sequent Calculus for Predicate Logic

$\exists$ **right**

$$\frac{\Gamma \vdash P[x \backslash t]}{\Gamma \vdash \exists x.P}\ [\exists R]$$

Condition: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$

$\exists$ **left**

$$\frac{\Gamma, P[x \backslash y] \vdash Q}{\Gamma, \exists x.P \vdash Q}\ [\exists L]$$

Condition: y must not be free in Γ, Q or in $\exists x.P$

A simple proof

Prove that $(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)$

Here is a proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(x) \vdash p(x)}\ [Id]}{p(x) \vdash p(x) \vee q(x)}\ [\vee R_1]}{\forall z.p(z) \vdash p(x) \vee q(x)}\ [\forall L]}{\forall z.p(z) \vdash \forall x.p(x) \vee q(x)}\ [\forall R]}{\vdash (\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)}\ [\rightarrow R]
$$

A simple proof

More generally, we can prove $(\forall x.P) \to \forall x.P \vee Q$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[x\backslash y] \vdash P[x\backslash y]}\;[Id]}{P[x\backslash y] \vdash P[x\backslash y] \vee Q[x\backslash y]}\;[\vee R_1]}{\forall x.P \vdash P[x\backslash y] \vee Q[x\backslash y]}\;[\forall L]}{\forall x.P \vdash \forall x.P \vee Q}\;[\forall R]}{\vdash (\forall x.P) \to \forall x.P \vee Q}\;[\to R]$$

We assume that y does not occur in P or Q

Another proof involving $\forall$

Prove that $(\forall x.P) \to (\forall x.Q) \to \forall x.P \land Q$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[x\backslash y], Q[x\backslash y] \vdash P[x\backslash y]}\,[Id] \qquad \dfrac{}{P[x\backslash y], Q[x\backslash y] \vdash Q[x\backslash y]}\,[Id]}{P[x\backslash y], Q[x\backslash y] \vdash P[x\backslash y] \land Q[x\backslash y]}\,[\land R]}{P[x\backslash y], \forall x.Q \vdash P[x\backslash y] \land Q[x\backslash y]}\,[\forall L]}{\forall x.P, \forall x.Q \vdash P[x\backslash y] \land Q[x\backslash y]}\,[\forall L]}{\forall x.P, \forall x.Q \vdash \forall x.P \land Q}\,[\forall R]}{\forall x.P \vdash (\forall x.Q) \to \forall x.P \land Q}\,[\to R]}{\vdash (\forall x.P) \to (\forall x.Q) \to \forall x.P \land Q}\,[\to R]
$$

We assume that y does not occur in P or Q

Yet another proof involving $\forall$

Prove that $(\forall x.P \to Q) \to (\forall x.P) \to \forall x.Q$

Here is a proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[x\backslash y] \vdash P[x\backslash y]}\,[Id] \qquad \dfrac{}{Q[x\backslash y], P[x\backslash y] \vdash Q[x\backslash y]}\,[Id]}{P[x\backslash y] \to Q[x\backslash y], P[x\backslash y] \vdash Q[x\backslash y]}\,[\to L]}{P[x\backslash y] \to Q[x\backslash y], \forall x.P \vdash Q[x\backslash y]}\,[\forall L]}{\forall x.P \to Q, \forall x.P \vdash Q[x\backslash y]}\,[\forall L]}{\forall x.P \to Q, \forall x.P \vdash \forall x.Q}\,[\forall R]}{\forall x.P \to Q \vdash (\forall x.P) \to \forall x.Q}\,[\to R]}{\vdash (\forall x.P \to Q) \to (\forall x.P) \to \forall x.Q}\,[\to R]
$$

We assume that y does not occur in P or Q

Classical Sequent Calculus - 1st version

As in Natural Deduction, we can add the following classical (equivalent) rules to the intuitionistic Sequence Calculus for Predicate Logic, to obtain a classical version:

$$\frac{}{\Gamma \vdash P \vee \neg P}\ [LEM]$$

$$\frac{\Gamma \vdash \neg\neg P}{\Gamma \vdash P}\ [DNE]$$

A proof involving $\neg$ and $\forall$

Prove $\forall x.Q$ from the hypotheses $\forall x.\neg Q \to \neg P$ and $\forall x.P$

Here is a classical proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[x\backslash y], \neg Q[x\backslash y] \vdash \neg Q[x\backslash y]}[Id] \quad \dfrac{\dfrac{}{P[x\backslash y], \neg Q[x\backslash y] \vdash P[x\backslash y]}[Id]}{\neg P[x\backslash y], P[x\backslash y], \neg Q[x\backslash y] \vdash \bot}[\neg L]}{\neg Q[x\backslash y] \to \neg P[x\backslash y], P[x\backslash y], \neg Q[x\backslash y] \vdash \bot}[\to L]}{\neg Q[x\backslash y] \to \neg P[x\backslash y], \forall x.P, \neg Q[x\backslash y] \vdash \bot}[\forall L]}{\forall x.\neg Q \to \neg P, \forall x.P, \neg Q[x\backslash y] \vdash \bot}[\forall L]}{\forall x.\neg Q \to \neg P, \forall x.P \vdash \neg\neg Q[x\backslash y]}[\neg R]}{\forall x.\neg Q \to \neg P, \forall x.P \vdash Q[x\backslash y]}[DNE]}{\forall x.\neg Q \to \neg P, \forall x.P \vdash \forall x.Q}[\forall R]
$$

We assume that y does not occur in P or Q

Classical Sequent Calculus - 2nd version

As for Propositional Logic, we can also obtain a classical version of this Sequent Calculus using classical sequents:

- a classical sequent be of the form $\Gamma \vdash \Delta$
- where Γ and Δ are lists of predicate logic formulas
- rules:

$$\frac{\Gamma \vdash A, \Delta_1 \quad \Gamma, B \vdash \Delta_2}{\Gamma, A \to B \vdash \Delta_1, \Delta_2}\ [\to L] \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ [\to R] \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ [\neg L]$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \lor B \vdash \Delta_1, \Delta_2}\ [\lor L] \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \lor B, \Delta}\ [\lor R] \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ [\neg R]$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\ [\land L] \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \land B, \Delta_1, \Delta_2}\ [\land R] \qquad \frac{}{A \vdash A}\ [Id]$$

$$\frac{\Gamma_1 \vdash B, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [Cut] \qquad \frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ [X_L] \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ [X_R]$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ [W_L] \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ [C_L] \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ [W_R] \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ [C_R]$$

Classical Sequent Calculus - 2nd version

We also allow using the usual derived rules.

In addition:

$$\frac{\Gamma \vdash P[x\backslash y], \Delta}{\Gamma \vdash \forall x.P, \Delta}\ [\forall R] \qquad \frac{\Gamma, P[x\backslash t] \vdash \Delta}{\Gamma, \forall x.P \vdash \Delta}\ [\forall L]$$

$$\frac{\Gamma \vdash P[x\backslash t], \Delta}{\Gamma \vdash \exists x.P, \Delta}\ [\exists R] \qquad \frac{\Gamma, P[x\backslash y] \vdash \Delta}{\Gamma, \exists x.P \vdash \Delta}\ [\exists L]$$

Conditions:

- for $[\forall R]$: y must not be free in Γ, Δ, or $\forall x.P$
- for $[\forall L]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists R]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists L]$: y must not be free in Γ, Δ, or $\exists x.P$

A proof involving $\neg$ and $\forall$ – Revisited

Prove $\forall x.Q$ from the hypotheses $\forall x.\neg Q \rightarrow \neg P$ and $\forall x.P$ using classical sequents

Here is a classical proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[x\backslash y], Q[x\backslash y] \vdash Q[x\backslash y]}\;[Id]}{P[x\backslash y] \vdash \neg Q[x\backslash y], Q[x\backslash y]}\;[\neg R] \qquad \dfrac{\dfrac{}{P[x\backslash y] \vdash P[x\backslash y]}\;[Id]}{\neg P[x\backslash y], P[x\backslash y] \vdash}\;[\neg L]}{\neg Q[x\backslash y] \rightarrow \neg P[x\backslash y], P[x\backslash y] \vdash Q[x\backslash y]}\;[\rightarrow L]}{\neg Q[x\backslash y] \rightarrow \neg P[x\backslash y], \forall x.P \vdash Q[x\backslash y]}\;[\forall L]}{\forall x.\neg Q \rightarrow \neg P, \forall x.P \vdash Q[x\backslash y]}\;[\forall L]}{\forall x.\neg Q \rightarrow \neg P, \forall x.P \vdash \forall x.Q}\;[\forall R]
$$

We assume that y does not occur in P or Q

Conclusion

What did we cover today?

- Predicate Logic proofs
- Natural Deduction proofs
- Intuitionistic Sequent Calculus rules
- Classical Sequent Calculus rules

Classical reasoning in Natural Deduction?

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

Next time?

- Predicate logic – semantics