9

Introduction to Differentiation

Differentiation concerns the study of the rate of change of a function, or variable, or in other words, if I have a function $f : \mathbb{R} \to \mathbb{R}$ and we increase the variable x just by a little bit, say to $x + h$, how does the value $f(x + h)$ compare to the value $f(x)$.

Example 9.1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 1$. At any point x we can see that it's gradient, or it's rate of change, is $\frac{\Delta y}{\Delta x} = 2$.

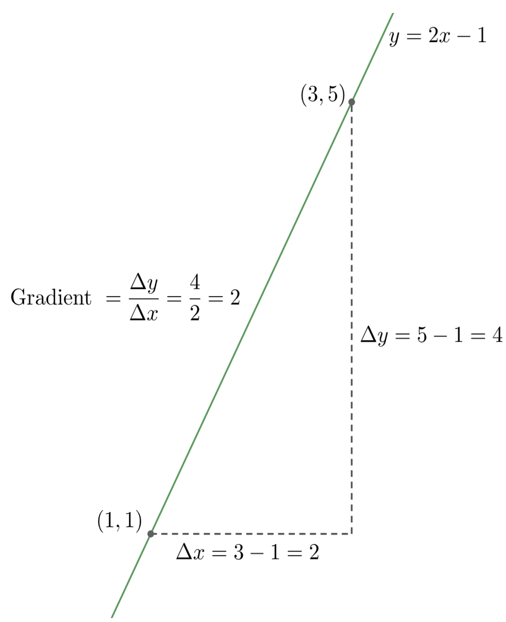

Figure 9.1: Calculating the gradient of a straight line

To calculate the gradient of the straight line, we fix two points. In this case we have chosen the points $(1, 1)$ and $(3, 5)$ which both lie on the straight line $y = 2x - 1$. Now, calculate the difference in their y co-ordinates, which we call Δy, that is

$$\Delta y = 5 - 1 = 4.$$

Next, calculate the difference in their x co-ordinates, which we call Δx, that is

$$\Delta x = 3 - 1 = 2.$$

Now we define the gradient to be the change in y divided by the change in x, or in other words,

$$\text{Gradient } = \frac{\Delta y}{\Delta x} = \frac{4}{2} = 2.$$

In general, a straight line (or linear function) $f : \mathbb{R} \to \mathbb{R}$ will have the form

$$y = mx + c,$$

where m denotes the gradient of the line and c denotes the y-intercept (where the graph intercepts the y-axis, i.e. when $x = 0$). So in the particular case of the line $y = 2x - 1$ above, as we have seen the gradient is 2 and the y-intercept is $c = -1$.

What can be said though if the function is not a linear function, i.e. the graph is not a straight line? In other words, what can be meant by the following question?

Question 9.0.1. Let $f(x) = x^2$. What is the 'gradient' of $f(x)$ at $x = 1$?

Let us start by following a similar process to what we used in the linear case.

Example 9.2. We want to calculate the gradient of lines connecting $(1, 1)$ on the graph of $y = x^2$ to other points on the graph.

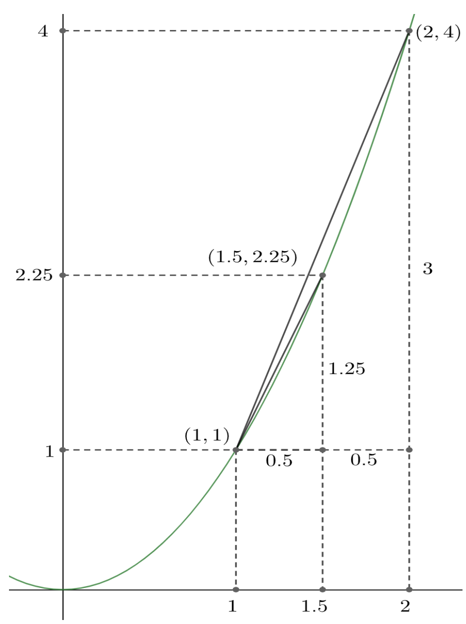

Figure 9.2: Calculating the gradient of line segments of $y = x^2$.

So let us start by choosing some points, which get closer and closer to $(1, 1)$. Say consider the points $(1.1, 1.21)$, $(1.2, 1.44)$, $(1.4, 1.96)$, $(1.6, 2.56)$, $(1.8, 3.24)$ and $(2, 4)$. Let us now compute the gradient of the straight line between $(1, 1)$ with these points.

Coordinates	$(1.1, 1.21)$	$(1.2, 1.44)$	$(1.4, 1.96)$	$(1.6, 2.56)$	$(1.8, 3.24)$	$(2, 4)$
Δx	0.1	0.2	0.4	0.6	0.8	1
Δy	0.21	0.44	0.96	1.56	2.24	3
Grad $= \frac{\Delta y}{\Delta x}$	2.1	2.2	2.4	2.6	2.8	3

Table 9.1: Calculating the gradient of the line segments at $(1, 1)$

From Table 9.1 we can see that as the gap between $(1, 1)$ and the other point gets smaller and smaller, that the gradient of the line segment gets closer and closer to 2. We can try to consider this 'algebraically'. Say we have a 'step size' h (for example $h = 0.1$). Then we can consider the point $(1 + h, f(1 + h)) = (1 + h,(1 + h)^2)$. We now can calculate the gradient of the straight line segment between $(1, 1)$ and $(1 + h,(1 + h)^2)$. Indeed,

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1}{(1+h) - 1} = \frac{1^2 + 2h + h^2 - 1}{h} = \frac{h^2 + 2h}{h} = 2 + h.$$

As before we considered points getting closer and closer to $(1, 1)$, so let us consider smaller and smaller values for our step size h, i.e. taking the limit as $h \to 0$. We then obtain:

$$\text{Gradient} = \lim_{h \to 0} (2 + h) = 2.$$

Therefore, the gradient of the 'tangent line' of $y = x^2$ at $(1, 1)$ is 2. Here, the tangent line of $y = x^2$ at $(1, 1)$ just means the straight line which 'touches' the curve $y = x^2$ at the point $(1, 1)$, see Figure 9.3.

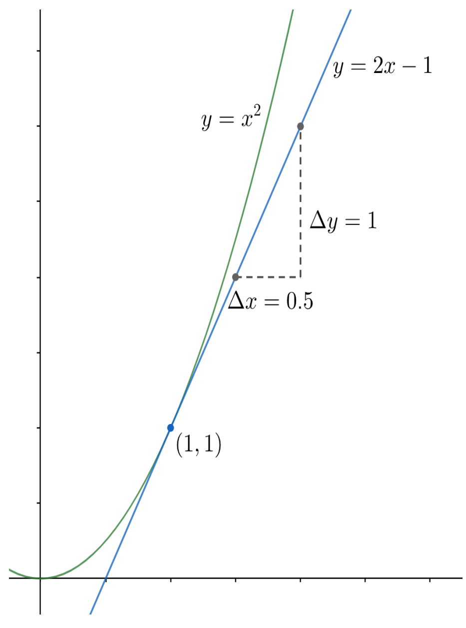

Figure 9.3: Tangent line to $y = x^2$ at $(1, 1)$

Using this example as our motivation, we can now include the definition of the *derivative*.

Definition 9.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$. If the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, then we say that f is *differentiable at the point* x_0; moreover, the value of the limit is called the derivative of f at x_0, and is denoted by $f'(x_0)$.

If f is differentiable at every $x \in \mathbb{R}$, then we just say that f is differentiable and the function $f' : \mathbb{R} \to \mathbb{R}$ is the derivative of f.

Remark 9.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then there are many different ways in which the derivative f' of a function f can be denoted, such as:

$$Df, \ \frac{df}{dx} \text{ and } \frac{d}{dx}f.$$

For the middle notation, we can view *df* as being a "small change in f" and dx as a "small change in x".

In other words, computing the derivative of $f : \mathbb{R} \to \mathbb{R}$ at a given point $(x_0, f(x_0))$ gives you the gradient of the tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$.