Shortest Paths and Dijkstra's Algorithm

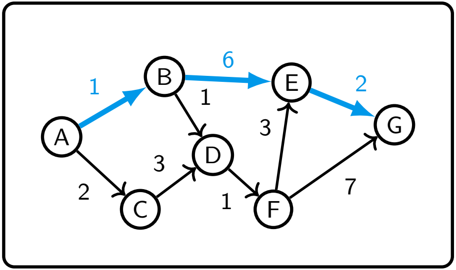
Paths and shortest paths

Recall: A **path** is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that v_i and v_{i+1} are connected by an edge for all $1 \leq i \leq n-1$.

A **shortest path** from A to B is a path for which the sum of the weights along the path is less than or equal to the sum of the weights along any other path from A to B. Note that there may be multiple different shortest paths from A to B. (In unweighted graphs, set weights to 1.)

Example

1. $A \to B \to E \to G$

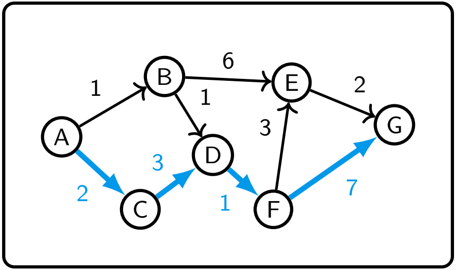


Recall: A **path** is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that v_i and v_{i+1} are connected by an edge for all $1 \leq i \leq n-1$.

A **shortest path** from A to B is a path for which the sum of the weights along the path is less than or equal to the sum of the weights along any other path from A to B. Note that there may be multiple different shortest paths from A to B. (In unweighted graphs, set weights to 1.)

Example

1. $A \to B \to E \to G$
2. $A \to C \to D \to F \to G$
3. …



Paths and shortest paths

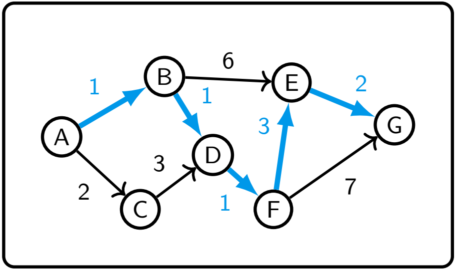
Recall: A **path** is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that v_i and v_{i+1} are connected by an edge for all $1 \leq i \leq n-1$.

A **shortest path** from A to B is a path for which the sum of the weights along the path is less than or equal to the sum of the weights along any other path from A to B. Note that there may be multiple different shortest paths from A to B. (In unweighted graphs, set weights to 1.)

Example

1. $A \to B \to E \to G$
2. $A \to C \to D \to F \to G$
3. ...

The shortest: $A \to B \to D \to F \to E \to G$



Dijkstra's algorithm to find the shortest path from `v` to `z`

For each vertex `w` of the graph other than `v`, we keep track of the following:

i. `d[w] =` the shortest distance from `v` to `w` so far
(Initially: ∞, except `d[v]` $= 0$)

ii. `p[w] =` the predecessor on the path from `v`
(initially: `w` itself, just a convention)

iii. `f[w] =` is computation of `d[w]` *finished*?
(initially: `false`)

The algorithm

The algorithm (idea):

```
set d[v] = 0                                                    (i.e. start on v)
while there are unfinished vertices:
    set w = the yet unfinished vertex with the smallest d[w]
    set f[w] = true                                             (i.e. mark w as finished)
    for every neighbour u of w:
        if d[w] + weight(w,u) < d[u]:
            set d[u] = d[w] + weight(w,u) and p[u] = w
```

(Where `weight(w,u)` is the weight of the edge `w` $\rightarrow$ `u`)

The input of the algorithm is a graph (represented as an adjacency matrix or adjacency lists) and two vertices `v` and `z`. The aim is to find the shortest path from `v` to `z`.

As the algorithm runs it changes the values `d[w]`, `p[w]` and `f[w]`. Initially `d[w] = infinity`, `p[w] = w` and `f[w] = false` for every vertex `w`.

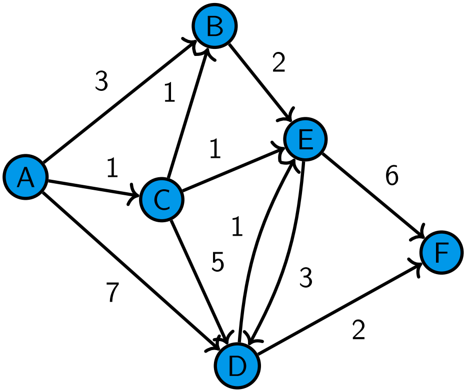
The arrays `d` and `f` obeys the following *invariant*:

- `d[w]` is the length of the shortest path from `v` to `w` when using only the finished vertices (i.e. those `w` such that `f[w] == true`).
- If `w` is finished then `d[w]` is the actual length of the shortest path from `v` to `w`.

After the algorithm finishes, we compute the found shortest path by using the array `p`. Lastly, `weight(w,u)` is the weight of the edge `w` $\to$ `u` obtained from the adjacency matrix/lists of the graph.

Example: Execution of Dijkstra's algorithm

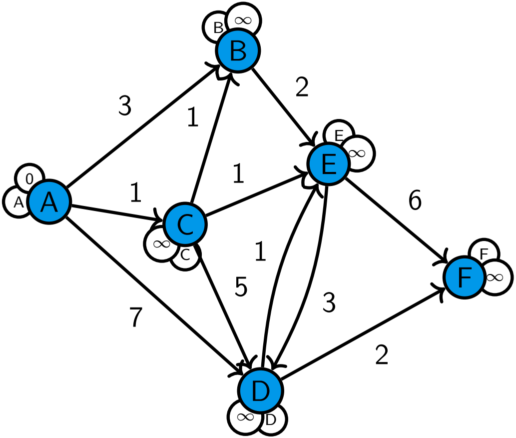
Shortest Path $A \to F$



Example: Execution of Dijkstra's algorithm

Shortest Path $A \to F$

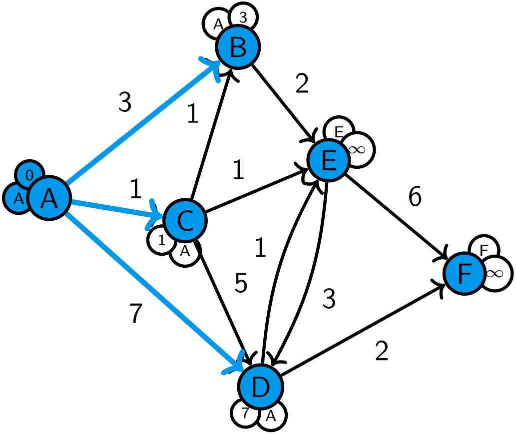
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	



Example: Execution of Dijkstra's algorithm

Shortest Path $A \to F$

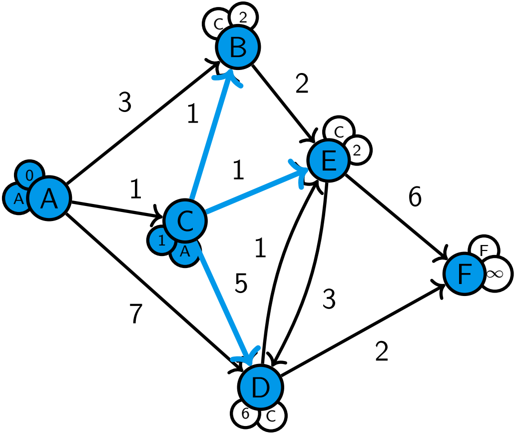
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A



Example: Execution of Dijkstra's algorithm

Shortest Path $A \rightarrow F$

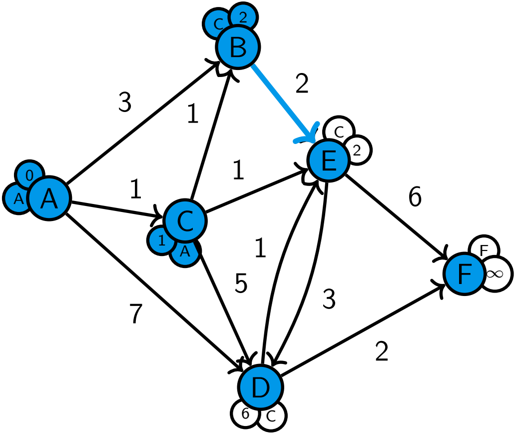
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A
0,A,✓	2,C	1,A,✓	6,C	2,C	∞,F	C



Example: Execution of Dijkstra's algorithm

Shortest Path $A \to F$

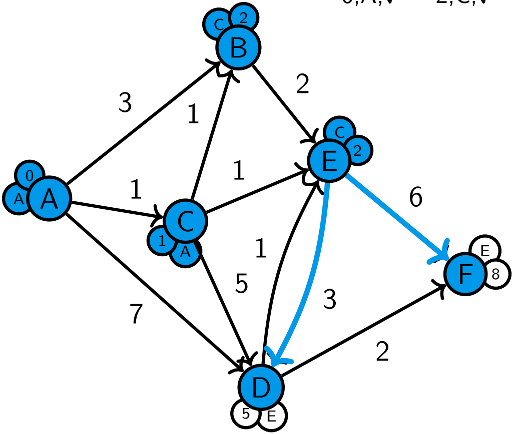
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A
0,A,✓	2,C	1,A,✓	6,C	2,C	∞,F	C
0,A,✓	2,C,✓	1,A,✓	6,C	2,C	∞,F	B



Example: Execution of Dijkstra's algorithm

Shortest Path $A \rightarrow F$

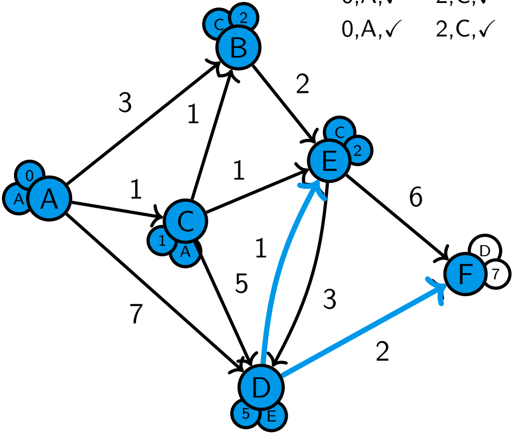
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A
0,A,✓	2,C	1,A,✓	6,C	2,C	∞,F	C
0,A,✓	2,C,✓	1,A,✓	6,C	2,C	∞,F	B
0,A,✓	2,C,✓	1,A,✓	5,E	2,C,✓	8,E	E



Example: Execution of Dijkstra's algorithm

Shortest Path $A \to F$

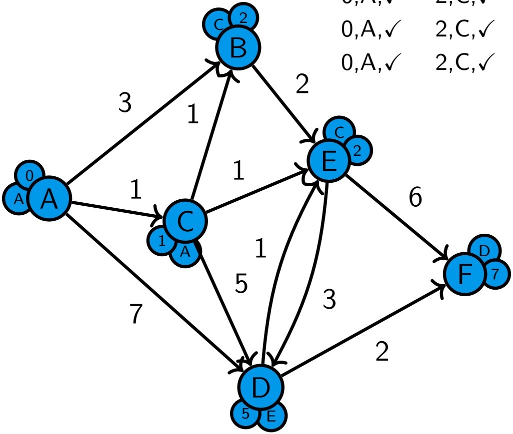
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A
0,A,✓	2,C	1,A,✓	6,C	2,C	∞,F	C
0,A,✓	2,C,✓	1,A,✓	6,C	2,C	∞,F	B
0,A,✓	2,C,✓	1,A,✓	5,E	2,C,✓	8,E	E
0,A,✓	2,C,✓	1,A,✓	5,E,✓	2,C,✓	7,D	D



Example: Execution of Dijkstra's algorithm

Shortest Path $A \to F$

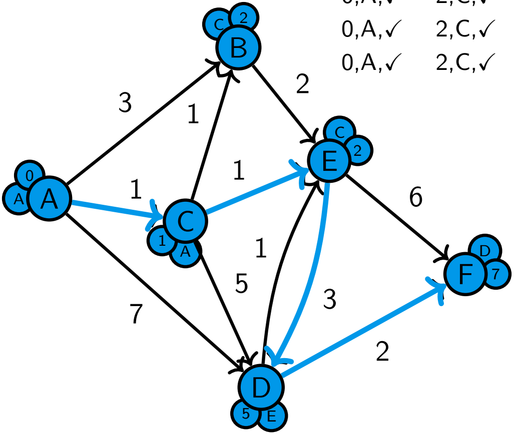
A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A
0,A,✓	2,C	1,A,✓	6,C	2,C	∞,F	C
0,A,✓	2,C,✓	1,A,✓	6,C	2,C	∞,F	B
0,A,✓	2,C,✓	1,A,✓	5,E	2,C,✓	8,E	E
0,A,✓	2,C,✓	1,A,✓	5,E,✓	2,C,✓	7,D	D
0,A,✓	2,C,✓	1,A,✓	5,E,✓	2,C,✓	7,D,✓	F



Example: Execution of Dijkstra's algorithm

Shortest Path $A \to F$

A	B	C	D	E	F	finished
0,A	∞,B	∞, C	∞,D	∞,E	∞,F	
0,A,✓	3,A	1,A	7,A	∞,E	∞,F	A
0,A,✓	2,C	1,A,✓	6,C	2,C	∞,F	C
0,A,✓	2,C,✓	1,A,✓	6,C	2,C	∞,F	B
0,A,✓	2,C,✓	1,A,✓	5,E	2,C,✓	8,E	E
0,A,✓	2,C,✓	1,A,✓	5,E,✓	2,C,✓	7,D	D
0,A,✓	2,C,✓	1,A,✓	5,E,✓	2,C,✓	7,D,✓	F



The shortest path from A to F is obtained (in the reversed order) by reading out `p[w]` 's, starting from F:

$A \to C \to E \to D \to F$.

Every iteration of the algorithm corresponds to one row in the table and each such row shows the content of the three arrays `d[-]`, `p[-]` and `f[-]`. (Check marks denote finished vertices.)

In the graph, the two circles adjacent to a vertex mark the current state of `d[w]` and `p[w]`. They turn blue whenever the vertex is marked as finished.

Dijkstra's time complexity (adjacency matrix)

$n =$ the number of vertices, $m =$ the total number of edges.

We do the following *up to n* times:

a. Mark `w` as finished.

b. Update every neighbour of `w`.

c. Find `w` which is unfinished and with the smallest `d[w]`.

Representing the graph by an *adjacency matrix*, means that, over all n outer loops, it takes:

- $O(n)$ to do step a
- $O(n^2)$ to do step b
- $O(n^2)$ to do step c by going through all vertices.

$\Longrightarrow$ The time complexity is $O(n^2)$.

Dijkstra's time complexity (adjacency lists)

We do the following *up to n*-times:

a. Mark `w` as finished.

b. Update every neighbour of `w`.

c. Find `w` which is unfinished and with the smallest `d[w]`.

With *adjacency lists*, executions of step b. will (in total) update

neighbours of the 1st selected `w`,
neighbours of the 2nd selected `w`,
neighbours of the 3nd selected `w`,
...

Over all iterations combined we update m-many times $\Rightarrow O(m)$

Representing the graph by an *adjacency list*, means that, over all n outer loops, it takes:

- $O(n)$ to do step a
- $O(m)$ to do step b
- $O(n^2)$ to do step c by going through all vertices.

$\Longrightarrow$ The time complexity is $O(n^2)$ (Note: $m \leq n^2$ in a simple graph)

Dijkstra's time complexity (adjacency lists)

Speeding up step c

Use min-priority queue: The priority of `u` is `d[u]`.

- Initialise the queue by inserting all nodes into it
- Call `deleteMin` to find the unfinished node with smallest `d[w]`
  - once per iteration, i.e. up to n times in total
- Whenever `d[u]` changes, we `update` the priority of `u`.

$\Longrightarrow$ total time complexity of step c

$= O(n \times \text{“cost of deleteMin”} + m \times \text{“cost of update”})$

- Using Binary Heap: $O(n \log n + m \log n)$
- Using Fibonacci Heap: $O(n \log n + m)$

What is omitted in the analysis is the time complexity of initialising the heap. This is usually done by `heapify` and its time complexity was always $O(n)$ for all heaps we had. Alternatively, we can do `insert` n-times which will result in the time complexity $O(n \log n)$ or $O(n)$ depending on the heap that we are using. Either way, the initialisation will not play any role in the total time complexity.

Dijkstra's time complexity – comparison

Adjacency matrices	Adjacency lists	
	Binary Heaps	Fibonacci Heaps
$O(n^2)$	$O((n+m)\log n)$	$O((n\log n)+m)$

Min-priority queues:

- Binary heaps: both `update` and `deleteMin` are in $O(\log n)$.
- Fibonacci heaps: `update` is in $O(1)$ and `deleteMin` is in $O(\log n)$ (both amortized).

Remark: Dijkstra's algorithm works only if all weights are ≥ 0.

Remark: If the graph is *dense*, that is if the number of edges, m, is approximately n^2, then using adjacency lists together with binary heaps has the time complexity $O((n + n^2)\log n) = O(n^2 \log n)$ which is slower than just using adjacency matrices. This problem disappears when using Fibonacci heaps where, for dense graphs, the time complexity becomes $O(n \log n + n^2) = O(n^2)$.

On the other hand, if the graph is not dense, using adjacency lists with Binary Heaps or Fibonacci Heaps is faster than using adjacency matrices.

Dijkstra's algorithm (pseudocode with adjacency matrix)

```
dijkstra_with_matrix(int[][] G, int v, int z) {
    n = G.length;
    d = new int[n]; p = new int[n]; f = new bool[n];

    for (int w = 0; w < n; w++) {
        d[w] = infty;    p[w] = w;    f[w] = false;
    }
    d[v] = 0;

    while (true) {
        w = min_unfinished(d, f);
        if (w == -1)
            break;

        for (int u = 0; u < n; u++)
            update(w, u, d, p);

        f[w] = true;
    }
    // compute results in desired form
    return compute_result(v, z, G, d, p);
}
```

```
int min_unfinished(int[] d, bool[] f) {
    int min = infty;
    int idx = -1;

    for (int i=0; i < d.length; i++) {
        if ( (not f[i]) && d[i] < min) {
            idx = i;
            min = d[i]
        }
    }

    return idx;
}
```

```
void update(w, u, G, d, p) {
    if (d[w] + G[w][u] < d[u]) {
        d[u] = d[w] + G[w][u];
        p[u] = w;
    }
}
```

Dijkstra's algorithm (pseudocode with adjacency lists)

```
dijkstra_with_lists(List<Edge>[] N, int v, int z) {
    n = G.length;
    d = new int[n];    p = new int[n];
    Q = new MinPriorityQueue();

    for (int w = 0; w < n; w++) {
        d[w] = infty;    p[w] = w;
        Q.add(w, d[w]);
    }
    d[v] = 0;
    Q.update(v, 0);

    while (Q.notEmpty()) {
        w = Q.deleteMin()

        for (Edge e : N[w]) { // iterate over edges to neighbours
            u = e.target;
            if (d[w] + e.weight < d[u]) { // should we update?
                d[u] = d[w] + e.weight;
                p[u] = w;
                Q.update(u, d[u]);
            }
        }
    }
    return compute_result(v, z, G, d, p);
}
```

```
class Edge {
  // target node
  int target;

  int weight;
}
```

The initialisation happens on lines 6–9.

Lines 10–11 make sure that the first selected `w` will be `v`.

We use the class `Edge` to store neighbours together with the weight of the edge that connects them. For example, if the vertex `A` has neighbours B, C and D with the edge $A \rightarrow B$ of weight 3, $A \rightarrow C$ of weight 1, and $A \rightarrow D$ of weight 8, then we will have that the linked list `N[v]` stores `Edge(B, 3)`, `Edge(C, 1)` and `Edge(D, 8)`.