Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**
- Intuitionistic vs. Classical logic
- Type theory

Today

Equivalences:

- in Natural Deduction
- in the Sequent Calculus
- using semantics

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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

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\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
- p ranges over predicate symbols
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The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

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The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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(\neg P)[x\backslash t] & = & \neg P[x\backslash t] \\
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The additional **conditions** ensure that **free variables do not get captured**.

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$(P_1 \lor P_2)[x\backslash t]$	$=$	$P_1[x\backslash t] \lor P_2[x\backslash t]$
$(P_1 \to P_2)[x\backslash t]$	$=$	$P_1[x\backslash t] \to P_2[x\backslash t]$
$(\forall x.P)[x\backslash t]$	$=$	$\forall x.P$
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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\;[\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\;[\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\;[\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}^{\,1}\\ \vdots \\ Q\end{array}}{Q}\;1\;[\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: ∀ & ∃ left and right rules

Sequent Calculus rules for quantifiers:

$$\frac{\Gamma \vdash P[x\backslash y]}{\Gamma \vdash \forall x.P}\ [\forall R] \qquad \frac{\Gamma, P[x\backslash t] \vdash Q}{\Gamma, \forall x.P \vdash Q}\ [\forall L]$$

$$\frac{\Gamma \vdash P[x\backslash t]}{\Gamma \vdash \exists x.P}\ [\exists R] \qquad \frac{\Gamma, P[x\backslash y] \vdash Q}{\Gamma, \exists x.P \vdash Q}\ [\exists L]$$

Recap: $\forall$ & $\exists$ left and right rules

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Conditions:

- for $[\forall R]$: y must not be free in Γ or $\forall x.P$
- for $[\forall L]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists R]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists L]$: y must not be free in Γ, Q, or $\exists x.P$

Recap: Models

Models: a model provides the interpretation of all symbols

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
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a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i
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Models of predicate logic replace **truth assignments** for propositional logic

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Variable valuations:

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- that map variables to D
- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

- $[\![t]\!]^M_v$ gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

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Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

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Meaning of formulas:

- $\vDash_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\vDash_{M,v} \neg P$ iff $\neg \vDash_{M,v} P$
- $\vDash_{M,v} P \wedge Q$ iff $\vDash_{M,v} P$ and $\vDash_{M,v} Q$
- $\vDash_{M,v} P \vee Q$ iff $\vDash_{M,v} P$ or $\vDash_{M,v} Q$
- $\vDash_{M,v} P \rightarrow Q$ iff $\vDash_{M,v} Q$ whenever $\vDash_{M,v} P$
- $\vDash_{M,v} \forall x.P$ iff for every $d \in D$ we have $\vDash_{M,(v,x \mapsto d)} P$
- $\vDash_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\vDash_{M,(v,x \mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- Implication elimination: $(A \rightarrow B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

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- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$

Logical Equivalences

In addition, the following hold (some hold only classically):

- $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$
- $(\forall x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A \rightarrow B) \leftrightarrow (A \rightarrow \exists x.B)$ if $x \notin \texttt{fv}(A)$

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
- that we can derive A form B

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
- that we can derive A form B

We will prove:

- $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$
- $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$
- $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\frac{}{(\forall x.A) \wedge (\forall x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\frac{\forall x.A \qquad \forall x.B}{(\forall x.A) \wedge (\forall x.B)}\ [\wedge I]$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\qquad\qquad}}{A[x\backslash y]}}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{}{\qquad\qquad}}{\qquad\qquad}}{\forall x.B}}{(\forall x.A) \land (\forall x.B)}\,[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \land B$ or in $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A[x\backslash y] \wedge B[x\backslash y]}}{A[x\backslash y]}[\wedge E_L]}{\forall x.A}[\forall I] \qquad \cfrac{\cfrac{\cfrac{}{}}{}}{\forall x.B}}{(\forall x.A) \wedge (\forall x.B)}[\wedge I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x.A \land B}{A[x\backslash y] \land B[x\backslash y]}\,[\forall E]}{A[x\backslash y]}\,[\land E_L]}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{\qquad\qquad}{\qquad\qquad}}{\qquad}}{\forall x.B}}{(\forall x.A) \land (\forall x.B)}\,[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \land B$ or in $\forall x.A$
- y must not clash with $\texttt{bv}(A \land B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x.A \land B}{A[x\backslash y] \land B[x\backslash y]}\,[\forall E]}{A[x\backslash y]}\,[\land E_L]}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{}{}}{B[x\backslash y]}}{\forall x.B}\,[\forall I]}{(\forall x.A) \land (\forall x.B)}\,[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \land B$ or in $\forall x.A$
- y must not clash with $\mathtt{bv}(A \land B)$
- y must not be free in $\forall x.A \land B$ or in $\forall x.B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
  \dfrac{\dfrac{\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\,[\forall E]}{A[x\backslash y]}\,[\wedge E_L]}{\forall x.A}\,[\forall I]
  \qquad
  \dfrac{\dfrac{\dfrac{}{A[x\backslash y] \wedge B[x\backslash y]}}{B[x\backslash y]}\,[\wedge E_R]}{\forall x.B}\,[\forall I]
}{(\forall x.A) \wedge (\forall x.B)}\,[\wedge I]
$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$
- y must not clash with $\mathtt{bv}(A \wedge B)$
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\;[\forall E]
}{A[x\backslash y]}\;[\wedge E_L]
}{\forall x.A}\;[\forall I]
\qquad
\dfrac{
\dfrac{
\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\;[\forall E]
}{B[x\backslash y]}\;[\wedge E_R]
}{\forall x.B}\;[\forall I]
}{(\forall x.A) \wedge (\forall x.B)}\;[\wedge I]
$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$
- y must not clash with $\mathtt{bv}(A \wedge B)$
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.B$
- y must not clash with $\mathtt{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\forall x.A \wedge B$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A \wedge B}\ [\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\qquad\qquad}}{\qquad}}{A[x\backslash y]} \qquad\qquad \dfrac{\dfrac{\dfrac{}{\qquad\qquad}}{\qquad}}{B[x\backslash y]}}{\dfrac{A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A \wedge B}\;[\forall I]}\;[\wedge I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A}}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{}{}}{B[x\backslash y]}}{A[x\backslash y] \land B[x\backslash y]}\,[\land I]}{\forall x.A \land B}\,[\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \land (\forall x.B)$ or in $\forall x.A \land B$
- y must not clash with $\texttt{bv}(A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall x.A) \land (\forall x.B)}{\forall x.A}\;[\land E_L]}{A[x\backslash y]}\;[\forall E] \qquad \dfrac{\dfrac{}{\qquad\qquad}}{B[x\backslash y]}}{A[x\backslash y] \land B[x\backslash y]}\;[\land I]}{\forall x.A \land B}\;[\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \land (\forall x.B)$ or in $\forall x.A \land B$
- y must not clash with $\texttt{bv}(A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall x.A) \wedge (\forall x.B)}{\forall x.A}\;[\wedge E_L]}{A[x\backslash y]}\;[\forall E] \qquad \dfrac{\dfrac{}{\forall x.B}}{B[x\backslash y]}\;[\forall E]}{A[x\backslash y] \wedge B[x\backslash y]}\;[\wedge I]}{\forall x.A \wedge B}\;[\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$
- y must not clash with $\texttt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{(\forall x.A) \land (\forall x.B)}{\forall x.A}\,[\land E_L]}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{(\forall x.A) \land (\forall x.B)}{\forall x.B}\,[\land E_R]}{B[x\backslash y]}\,[\forall E]}{\dfrac{A[x\backslash y] \land B[x\backslash y]}{\forall x.A \land B}\,[\forall I]}\,[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \land (\forall x.B)$ or in $\forall x.A \land B$
- y must not clash with $\texttt{bv}(A)$
- y must not clash with $\texttt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\forall x.A \land B \vdash (\forall x.A) \land (\forall x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{}{\forall x.A \wedge B \vdash \forall x.A} \qquad \dfrac{}{\forall x.A \wedge B \vdash \forall x.B}}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\ [\wedge R]$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\forall x.A \wedge B \vdash A[x\backslash y]}{\forall x.A \wedge B \vdash \forall x.A}\,[\forall R] \qquad \dfrac{}{\forall x.A \wedge B \vdash \forall x.B}}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\,[\wedge R]$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\qquad\qquad\qquad\qquad}{\qquad\qquad\qquad\qquad}}{A[x\backslash y] \land B[x\backslash y] \vdash A[x\backslash y]}
}{\forall x.A \land B \vdash A[x\backslash y]}\;[\forall L]
}{\forall x.A \land B \vdash \forall x.A}\;[\forall R]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\qquad\qquad\qquad\qquad}{\qquad\qquad\qquad\qquad}}{\qquad\qquad\qquad\qquad}}{\qquad\qquad\qquad\qquad}}{\forall x.A \land B \vdash \forall x.B}
}{\forall x.A \land B \vdash (\forall x.A) \land (\forall x.B)}\;[\land R]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\texttt{bv}(A \land B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{
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\dfrac{
\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}
}{A[x\backslash y] \land B[x\backslash y] \vdash A[x\backslash y]}\;[\land L]
}{\forall x.A \land B \vdash A[x\backslash y]}\;[\forall L]
}{\forall x.A \land B \vdash \forall x.A}\;[\forall R]
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{}
}{}
}{}
}{\forall x.A \land B \vdash \forall x.B}
}{\forall x.A \land B \vdash (\forall x.A) \land (\forall x.B)}\;[\land R]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\mathtt{bv}(A \land B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\,[Id]
}{A[x\backslash y] \wedge B[x\backslash y] \vdash A[x\backslash y]}\,[\wedge L]
}{\forall x.A \wedge B \vdash A[x\backslash y]}\,[\forall L]
}{\forall x.A \wedge B \vdash \forall x.A}\,[\forall R]
\qquad
\dfrac{
\dfrac{
\dfrac{}{}
}{}
}{\forall x.A \wedge B \vdash \forall x.B}
}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\,[\wedge R]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\mathtt{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\,[Id]
}{A[x\backslash y] \wedge B[x\backslash y] \vdash A[x\backslash y]}\,[\wedge L]
}{\forall x.A \wedge B \vdash A[x\backslash y]}\,[\forall L]
}{\forall x.A \wedge B \vdash \forall x.A}\,[\forall R]
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{}
}{}
}{\forall x.A \wedge B \vdash B[x\backslash y]}
}{\forall x.A \wedge B \vdash \forall x.B}\,[\forall R]
}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\,[\wedge R]
}{}
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\texttt{bv}(A \wedge B)$
- y must not be free in the context or $\forall x.B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\;[Id]
}{A[x\backslash y] \wedge B[x\backslash y] \vdash A[x\backslash y]}\;[\wedge L]
}{\forall x.A \wedge B \vdash A[x\backslash y]}\;[\forall L]
}{\forall x.A \wedge B \vdash \forall x.A}\;[\forall R]
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\quad}
}{A[x\backslash y] \wedge B[x\backslash y] \vdash B[x\backslash y]}
}{\forall x.A \wedge B \vdash B[x\backslash y]}\;[\forall L]
}{\forall x.A \wedge B \vdash \forall x.B}\;[\forall R]
}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\;[\wedge R]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\texttt{bv}(A \wedge B)$
- y must not be free in the context or $\forall x.B$
- y must not clash with $\texttt{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y] \wedge B[x\backslash y] \vdash A[x\backslash y]}\,[\wedge L]}{\forall x.A \wedge B \vdash A[x\backslash y]}\,[\forall L]}{\forall x.A \wedge B \vdash \forall x.A}\,[\forall R]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B[x\backslash y]}}{A[x\backslash y] \wedge B[x\backslash y] \vdash B[x\backslash y]}\,[\wedge L]}{\forall x.A \wedge B \vdash B[x\backslash y]}\,[\forall L]}{\forall x.A \wedge B \vdash \forall x.B}\,[\forall R]
}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\,[\wedge R]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\texttt{bv}(A \wedge B)$
- y must not be free in the context or $\forall x.B$
- y must not clash with $\texttt{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y] \wedge B[x\backslash y] \vdash A[x\backslash y]}\,[\wedge L]}{\forall x.A \wedge B \vdash A[x\backslash y]}\,[\forall L]}{\forall x.A \wedge B \vdash \forall x.A}\,[\forall R]
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B[x\backslash y]}\,[Id]}{A[x\backslash y] \wedge B[x\backslash y] \vdash B[x\backslash y]}\,[\wedge L]}{\forall x.A \wedge B \vdash B[x\backslash y]}\,[\forall L]}{\forall x.A \wedge B \vdash \forall x.B}\,[\forall R]
}{\forall x.A \wedge B \vdash (\forall x.A) \wedge (\forall x.B)}\,[\wedge R]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A$
- y must not clash with $\texttt{bv}(A \wedge B)$
- y must not be free in the context or $\forall x.B$
- y must not clash with $\texttt{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\qquad\qquad\qquad \quad \qquad\qquad\qquad}{\qquad\qquad\qquad\qquad}}{\qquad\qquad\qquad\qquad}}{\qquad\qquad\qquad\qquad}}{\qquad\qquad\qquad\qquad}}{\qquad\qquad\qquad\qquad}}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\forall x.A, \forall x.B \vdash \forall x.A \wedge B}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}\ [\wedge L]$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\forall x.A, \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A, \forall x.B \vdash \forall x.A \wedge B}\;[\forall R]}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}\;[\wedge L]$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A \wedge B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}}{\forall x.A, \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}\;[\forall L]}{\forall x.A, \forall x.B \vdash \forall x.A \wedge B}\;[\forall R]}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}\;[\wedge L]$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y] \wedge B[x\backslash y]}{A[x\backslash y], \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}\;[\forall L]}{\forall x.A, \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}\;[\forall L]}{\forall x.A, \forall x.B \vdash \forall x.A \wedge B}\;[\forall R]}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}\;[\wedge L]$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$
- y must not clash with $\texttt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]} \qquad \dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B[x\backslash y]}}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y] \wedge B[x\backslash y]}\,[\wedge R]}{A[x\backslash y], \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}\,[\forall L]}{\forall x.A, \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}\,[\forall L]}{\forall x.A, \forall x.B \vdash \forall x.A \wedge B}\,[\forall R]}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}\,[\wedge L]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$
- y must not clash with $\texttt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y]}[Id] \quad \dfrac{}{A[x\backslash y], B[x\backslash y] \vdash B[x\backslash y]}[Id]}{A[x\backslash y], B[x\backslash y] \vdash A[x\backslash y] \wedge B[x\backslash y]}[\wedge R]}{A[x\backslash y], \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}[\forall L]}{\forall x.A, \forall x.B \vdash A[x\backslash y] \wedge B[x\backslash y]}[\forall L]}{\forall x.A, \forall x.B \vdash \forall x.A \wedge B}[\forall R]}{(\forall x.A) \wedge (\forall x.B) \vdash \forall x.A \wedge B}[\wedge L]
$$

- pick y such that it does not occur in A or B
- y must not be free in the context or $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$
- y must not clash with $\texttt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\qquad}{} \quad \dfrac{\dfrac{\dfrac{\dfrac{\qquad}{}}{}}{}}{} \quad \dfrac{\dfrac{\dfrac{\dfrac{\qquad}{}}{}}{}}{}}{(\exists x.A) \vee (\exists x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad (\exists x.A) \lor (\exists x.B)}{(\exists x.A) \lor (\exists x.B)}\ 1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{A[x\backslash y] \vee B[x\backslash y] \qquad A[x\backslash y] \to (\exists x.A) \vee (\exists x.B) \qquad B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\;1\;[\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\overline{A[x\backslash y] \vee B[x\backslash y]}^{\,1} \quad \overline{A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)} \quad \overline{B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\;1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\;1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{(\exists x.A) \vee (\exists x.B)}}{A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\;2\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\;[\vee E]}{(\exists x.A) \vee (\exists x.B)}\;1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\;1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{\exists x.A}}{(\exists x.A) \vee (\exists x.B)}\;[\vee I_L]}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\;2\;[\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\;[\vee E]}{(\exists x.A) \vee (\exists x.B)}\;1\;[\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \lor B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \lor B[x\backslash y]$}
\AxiomC{}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$(\exists x.A) \lor (\exists x.B)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{$B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$(\exists x.A) \lor (\exists x.B)$}
\RightLabel{$1\ [\exists E]$}
\BinaryInfC{$(\exists x.A) \lor (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \lor B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \lor B[x\backslash y]$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$(\exists x.A) \lor (\exists x.B)$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{$B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$(\exists x.A) \lor (\exists x.B)$}
\RightLabel{1 $[\exists E]$}
\BinaryInfC{$(\exists x.A) \lor (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \vee B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{2 $[\rightarrow I]$}
\UnaryInfC{$A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{1 $[\exists E]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,2\ [\to I] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{\exists x.B}}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_R]}{B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,3\ [\to I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\,2\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B[x\backslash y]}}{\exists x.B}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_R]}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\,3\ [\rightarrow I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\ [\exists E]
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \vee B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.B$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{$3\ [\to I]$}
\UnaryInfC{$B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{$1\ [\exists E]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\exists x.A \vee B$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{}{\exists x.A \to \exists x.A \lor B} \qquad \dfrac{}{\exists x.B \to \exists x.A \lor B}}{\exists x.A \lor B}\ [\lor E]$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\exists x.A \vee B}{\exists x.A \to \exists x.A \vee B}\ 1\ [\to I] \qquad \exists x.B \to \exists x.A \vee B}{\exists x.A \vee B}\ [\vee E]$$

- 1: $\exists x.A$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{\exists x.A \lor B}}{\exists x.A \lor B}\ 2\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\ 1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\exists x.B \to \exists x.A \lor B}}{\exists x.A \lor B}\ [\lor E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x \backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}{}^{1} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{\exists x.A \vee B}}{\exists x.A \vee B}{\scriptstyle 2\ [\exists E]}}{\exists x.A \to \exists x.A \vee B}{\scriptstyle 1\ [\to I]} \qquad \dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}{\scriptstyle [\vee E]}$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \exists x.B \rightarrow \exists x.A \vee B}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\ 1 \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}}{A[x\backslash y] \vee B[x\backslash y]}\ [\vee I_L]}{\exists x.A \vee B}\ [\exists I]}{\exists x.A \vee B}\ 2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\ 1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{} \quad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\ [\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\begin{prooftree}
\AxiomC{$(\exists x.A) \lor (\exists x.B)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$A[x\backslash y] \lor B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A \lor B$}
\RightLabel{2 $[\exists E]$}
\BinaryInfC{$\exists x.A \lor B$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$\exists x.A \to \exists x.A \lor B$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\BinaryInfC{}
\UnaryInfC{$\exists x.B \to \exists x.A \lor B$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$\exists x.A \lor B$}
\end{prooftree}
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_L]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{} \quad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{\exists x.A \lor B}}{\exists x.B \to \exists x.A \lor B}\,3\ [\to I]}{\exists x.A \lor B}\,[\lor E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad
\dfrac{\dfrac{\overline{\exists x.A}^{\,1} \qquad \dfrac{\dfrac{\overline{A[x\backslash y]}^{\,2}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I]
\qquad
\dfrac{\dfrac{\overline{\exists x.B} \qquad \dfrac{\dfrac{\overline{}}{}}{\exists x.A \vee B}}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \vee B}\,3\ [\to I]}
{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\overline{\exists x.A}^{\,1} \quad \dfrac{\dfrac{\overline{A[x\backslash y]}^{\,2}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\overline{\exists x.B}^{\,3} \quad \dfrac{\dfrac{\overline{}}{}}{\exists x.A \vee B}}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \vee B}\,3\ [\to I]}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,{\scriptstyle 1} \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,{\scriptstyle 2}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,{\scriptstyle 2}\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,{\scriptstyle 1}\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,{\scriptstyle 3} \quad \dfrac{\dfrac{\dfrac{}{\quad}}{A[x\backslash y] \vee B[x\backslash y]}}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,{\scriptstyle 4}\ [\exists E]}{\exists x.B \to \exists x.A \vee B}\,{\scriptstyle 3}\ [\to I]}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,{\scriptstyle 1} \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,{\scriptstyle 2}}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_L]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,{\scriptstyle 2}\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\,{\scriptstyle 1}\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,{\scriptstyle 3} \qquad \dfrac{\dfrac{\dfrac{}{B[x\backslash y]}}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_R]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,{\scriptstyle 4}\ [\exists E]}{\exists x.B \to \exists x.A \lor B}\,{\scriptstyle 3}\ [\to I]}{\exists x.A \lor B}\,[\lor E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,3 \qquad \dfrac{\dfrac{\dfrac{}{B[x\backslash y]}\,4}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_R]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \rightarrow \exists x.A \vee B}\,3\ [\rightarrow I]}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
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- 3: $\exists x.B$
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Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in the Sequent Calculus

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{}{\exists x.A \vee B \vdash (\exists x.A) \vee (\exists x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{A[x\backslash y] \vee B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}{\exists x.A \vee B \vdash (\exists x.A) \vee (\exists x.B)}\ [\exists L]$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{A[x\backslash y] \vdash (\exists x.A) \lor (\exists x.B) \qquad\qquad B[x\backslash y] \vdash (\exists x.A) \lor (\exists x.B)}{A[x\backslash y] \lor B[x\backslash y] \vdash (\exists x.A) \lor (\exists x.B)}\,[\lor L]}{\exists x.A \lor B \vdash (\exists x.A) \lor (\exists x.B)}\,[\exists L]$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{A[x\backslash y] \vdash \exists x.A}}{A[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee R_1] \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}}{A[x\backslash y] \vee B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee L]}{\exists x.A \vee B \vdash (\exists x.A) \vee (\exists x.B)}\,[\exists L]
$$

- pick y such that it does not occur in A or B

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Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}}{A[x\backslash y] \vdash \exists x.A}\;[\exists R]}{A[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\;[\vee R_1] \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}}{A[x\backslash y] \vee B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\;[\vee L]}{\exists x.A \vee B \vdash (\exists x.A) \vee (\exists x.B)}\;[\exists L]
$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A[x\backslash y] \vdash A[x\backslash y]$}
\RightLabel{$[\exists R]$}
\UnaryInfC{$A[x\backslash y] \vdash \exists x.A$}
\RightLabel{$[\lor R_1]$}
\UnaryInfC{$A[x\backslash y] \vdash (\exists x.A) \lor (\exists x.B)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$B[x\backslash y] \vdash (\exists x.A) \lor (\exists x.B)$}
\RightLabel{$[\lor L]$}
\BinaryInfC{$A[x\backslash y] \lor B[x\backslash y] \vdash (\exists x.A) \lor (\exists x.B)$}
\RightLabel{$[\exists L]$}
\UnaryInfC{$\exists x.A \lor B \vdash (\exists x.A) \lor (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y] \vdash \exists x.A}\,[\exists R]}{A[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee R_1] \qquad \dfrac{\dfrac{\dfrac{}{}}{B[x\backslash y] \vdash \exists x.B}}{B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee R_2]}{A[x\backslash y] \vee B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee L]}{\exists x.A \vee B \vdash (\exists x.A) \vee (\exists x.B)}\,[\exists L]
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$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y] \vdash \exists x.A}\,[\exists R]}{A[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee R_1] \qquad \dfrac{\dfrac{\dfrac{}{B[x\backslash y] \vdash B[x\backslash y]}}{B[x\backslash y] \vdash \exists x.B}\,[\exists R]}{B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}\,[\vee R_2]}{\dfrac{A[x\backslash y] \vee B[x\backslash y] \vdash (\exists x.A) \vee (\exists x.B)}{\exists x.A \vee B \vdash (\exists x.A) \vee (\exists x.B)}\,[\exists L]}\,[\vee L]$$

- pick y such that it does not occur in A or B

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$$\dfrac{\qquad\qquad}{(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B}$$

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Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in the Sequent Calculus

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{}{\exists x.A \vdash \exists x.A \vee B} \qquad \dfrac{}{\exists x.B \vdash \exists x.A \vee B}}{(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B}\ [\vee L]$$

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$$\dfrac{\dfrac{A[x\backslash y] \vdash \exists x.A \vee B}{\exists x.A \vdash \exists x.A \vee B}\,[\exists L] \qquad \dfrac{}{\exists x.B \vdash \exists x.A \vee B}}{(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B}\,[\vee L]$$

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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{}}{A[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]}
    }{A[x\backslash y] \vdash \exists x.A \vee B}\;[\exists R]
  }{\exists x.A \vdash \exists x.A \vee B}\;[\exists L]
  \qquad
  \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{\exists x.B \vdash \exists x.A \vee B}
}{(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B}\;[\vee L]
$$

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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A[x\backslash y] \vdash A[x\backslash y]$}
\RightLabel{$[\vee R_1]$}
\UnaryInfC{$A[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]$}
\RightLabel{$[\exists R]$}
\UnaryInfC{$A[x\backslash y] \vdash \exists x.A \vee B$}
\RightLabel{$[\exists L]$}
\UnaryInfC{$\exists x.A \vdash \exists x.A \vee B$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{$\exists x.B \vdash \exists x.A \vee B$}
\RightLabel{$[\vee L]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B$}
\end{prooftree}
$$

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\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A[x\backslash y] \vdash A[x\backslash y]$}
\RightLabel{$[\vee R_1]$}
\UnaryInfC{$A[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]$}
\RightLabel{$[\exists R]$}
\UnaryInfC{$A[x\backslash y] \vdash \exists x.A \vee B$}
\RightLabel{$[\exists L]$}
\UnaryInfC{$\exists x.A \vdash \exists x.A \vee B$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$B[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]$}
\RightLabel{$[\exists R]$}
\UnaryInfC{$B[x\backslash y] \vdash \exists x.A \vee B$}
\RightLabel{$[\exists L]$}
\UnaryInfC{$\exists x.B \vdash \exists x.A \vee B$}
\RightLabel{$[\vee L]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B

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$$
\dfrac{
\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]}\,[\vee R_1]}{\dfrac{A[x\backslash y] \vdash \exists x.A \vee B}{\exists x.A \vdash \exists x.A \vee B}\,[\exists L]}\,[\exists R]
\qquad
\dfrac{\dfrac{\dfrac{}{B[x\backslash y] \vdash B[x\backslash y]}}{B[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]}\,[\vee R_2]}{\dfrac{B[x\backslash y] \vdash \exists x.A \vee B}{\exists x.B \vdash \exists x.A \vee B}\,[\exists L]}\,[\exists R]
}{(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B}\,[\vee L]
$$

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\dfrac{
\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{A[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]}\,[\vee R_1]}{\dfrac{A[x\backslash y] \vdash \exists x.A \vee B}{\exists x.A \vdash \exists x.A \vee B}\,[\exists L]}\,[\exists R]
\quad
\dfrac{\dfrac{\dfrac{}{B[x\backslash y] \vdash B[x\backslash y]}\,[Id]}{B[x\backslash y] \vdash A[x\backslash y] \vee B[x\backslash y]}\,[\vee R_2]}{\dfrac{B[x\backslash y] \vdash \exists x.A \vee B}{\exists x.B \vdash \exists x.A \vee B}\,[\exists L]}\,[\exists R]
}{(\exists x.A) \vee (\exists x.B) \vdash \exists x.A \vee B}\,[\vee L]
$$

- pick y such that it does not occur in A or B

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\exists x.\neg A$$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\overline{\neg\neg(\exists x.\neg A)}}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\bot}{\neg\neg(\exists x.\neg A)}\ 1\ [\neg I]}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

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Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{}{\forall x.A}}{\bot}\ [\neg E]}{\neg\neg(\exists x.\neg A)}\ 1\ [\neg I]}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

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$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{A[x\backslash y]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A

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Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\begin{prooftree}
\AxiomC{$\neg\forall x.A$}
\AxiomC{}
\UnaryInfC{$\neg\neg A[x\backslash y]$}
\RightLabel{$[DNE]$}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\forall I]$}
\UnaryInfC{$\forall x.A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$1\ [\neg I]$}
\UnaryInfC{$\neg\neg(\exists x.\neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\exists x.\neg A$}
\end{prooftree}
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A

Logical Equivalences

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Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\perp}{\neg\neg A[x\backslash y]}\;2\ [\neg I]}{A[x\backslash y]}\;[DNE]}{\forall x.A}\;[\forall I]}{\perp}\;[\neg E]}{\neg\neg(\exists x.\neg A)}\;1\ [\neg I]}{\exists x.\neg A}\;[DNE]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

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$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)} \qquad \dfrac{\dfrac{}{\ }}{\exists x.\neg A}}{\bot}\,[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]
$$

- 1: $\neg(\exists x.\neg A)$
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$$
\begin{prooftree}
\AxiomC{$\neg\forall x.A$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\exists x.\neg A)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$\exists x.\neg A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\neg I]$}
\UnaryInfC{$\neg\neg A[x\backslash y]$}
\RightLabel{$[DNE]$}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\forall I]$}
\UnaryInfC{$\forall x.A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg\neg(\exists x.\neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\exists x.\neg A$}
\end{prooftree}
$$

- 1: $\neg(\exists x.\neg A)$
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$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)}\;1 \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]}}{\exists x.\neg A}\;[\exists I]}{\bot}\;[\neg E]}{\neg\neg A[x\backslash y]}\;2\;[\neg I]}{A[x\backslash y]}\;[DNE]}{\forall x.A}\;[\forall I]}{\bot}\;[\neg E]}{\neg\neg(\exists x.\neg A)}\;1\;[\neg I]}{\exists x.\neg A}\;[DNE]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
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$$
\begin{prooftree}
\AxiomC{$\neg\forall x.A$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\exists x.\neg A)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.\neg A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$2\ [\neg I]$}
\UnaryInfC{$\neg\neg A[x\backslash y]$}
\RightLabel{$[DNE]$}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\forall I]$}
\UnaryInfC{$\forall x.A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$1\ [\neg I]$}
\UnaryInfC{$\neg\neg(\exists x.\neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\exists x.\neg A$}
\end{prooftree}
$$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{}{}}{}}{}}{}}{\dfrac{}{\neg\forall x.A}}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{}\quad\dfrac{\dfrac{}{}}{}}{}}{\bot}}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{\quad}{\quad} \qquad \dfrac{\dfrac{\quad}{\quad}}{\quad}}{\bot}}{\bot}\ 2\ [\exists E]}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

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$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]} \qquad \dfrac{\dfrac{}{}}{A[x\backslash y]}}{\bot}\ [\neg E]}{\bot}\ 2\ [\exists E]}{\neg\forall x.A}\ 1\ [\neg I]$$

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$$
\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]}\,2 \qquad \dfrac{\dfrac{}{\quad}}{A[x\backslash y]}}{\bot}\,[\neg E]}{\dfrac{\bot}{\neg\forall x.A}\,1\ [\neg I]}\,2\ [\exists E]
$$

- 1: $\forall x.A$
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- 1: $\forall x.A$
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Here is a proof of the left-to-right implication (2nd classical version):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\neg\forall x.A \vdash \exists x.\neg A}$$

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Prove the logical equivalence $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$ in the Sequent Calculus

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$$\dfrac{\vdash \forall x.A, \exists x.\neg A}{\neg \forall x.A \vdash \exists x.\neg A}\ [\neg L]$$

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$$\dfrac{\dfrac{\vdash A[x\backslash y], \exists x.\neg A}{\vdash \forall x.A, \exists x.\neg A}\;[\forall R]}{\neg\forall x.A \vdash \exists x.\neg A}\;[\neg L]$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{}}{\vdash A[x\backslash y], \neg A[x\backslash y]}}{\vdash A[x\backslash y], \exists x.\neg A}\,[\exists R]}{\vdash \forall x.A, \exists x.\neg A}\,[\forall R]}{\neg\forall x.A \vdash \exists x.\neg A}\,[\neg L]
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$$\dfrac{\exists x.\neg A, \forall x.A \vdash \bot}{\exists x.\neg A \vdash \neg\forall x.A}\ [\neg R]$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{\neg A[x\backslash y], \forall x.A \vdash \bot}}{\exists x.\neg A, \forall x.A \vdash \bot}\ [\exists L]}{\exists x.\neg A \vdash \neg\forall x.A}\ [\neg R]$$

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Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\neg A[x\backslash y]}{\forall x.\neg A}\ [\forall I]$$

- pick y such that it does not occur in A

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Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\bot}{\neg A[x\backslash y]}\ 1\ [\neg I]}{\forall x.\neg A}\ [\forall I]$$

- pick y such that it does not occur in A
- 1: $A[x\backslash y]$

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Here is a proof of the left-to-right implication (constructive):

$$\cfrac{\cfrac{\cfrac{\neg\exists x.A \qquad \cfrac{\overline{}}{\exists x.A}}{\bot}\;[\neg E]}{\neg A[x\backslash y]}\;1\ [\neg I]}{\forall x.\neg A}\;[\forall I]$$

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$$\overline{\neg\exists x.A}$$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\perp}{\neg\exists x.A}\ 1\ [\neg I]$$

- 1: $\exists x.A$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\overline{\exists x.A} \qquad \dfrac{\overline{} \qquad \overline{}}{\bot}}{\bot}\ 2\ [\exists E]}{\neg\exists x.A}\ 1\ [\neg I]$$

- 1: $\exists x.A$
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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\overline{\exists x.A}^{\;1} \qquad \dfrac{\overline{\neg A[x\backslash y]} \qquad \overline{A[x\backslash y]}}{\bot}\;[\neg E]}{\bot}\;2\ [\exists E]}{\neg\exists x.A}\;1\ [\neg I]$$

- 1: $\exists x.A$
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$$\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\forall x.\neg A}{\neg A[x\backslash y]}\,[\forall E] \qquad \dfrac{}{A[x\backslash y]}}{\bot}\,[\neg E]}{\dfrac{\bot}{\neg\exists x.A}\,1\ [\neg I]}\,2\ [\exists E]$$

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\dfrac{\dfrac{}{\exists x.A}\ 1 \qquad \dfrac{\dfrac{\forall x.\neg A}{\neg A[x\backslash y]}\ [\forall E] \qquad \dfrac{}{A[x\backslash y]}\ 2}{\bot}\ [\neg E]}{\dfrac{\bot}{\neg\exists x.A}\ 1\ [\neg I]}\ 2\ [\exists E]
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Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\neg\exists x.A \vdash \neg A[x\backslash y]}{\neg\exists x.A \vdash \forall x.\neg A}\ [\forall R]$$

- pick y such that it does not occur in A

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Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in the Sequent Calculus

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\overline{\overline{\overline{\qquad}}}}{\neg\exists x.A, A[x\backslash y] \vdash \bot}}{\neg\exists x.A \vdash \neg A[x\backslash y]}\ [\neg R]}{\neg\exists x.A \vdash \forall x.\neg A}\ [\forall R]$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\quad}}{A[x\backslash y] \vdash \exists x.A}}{\neg\exists x.A, A[x\backslash y] \vdash \bot}\,[\neg L]}{\neg\exists x.A \vdash \neg A[x\backslash y]}\,[\neg R]}{\neg\exists x.A \vdash \forall x.\neg A}\,[\forall R]$$

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$$\dfrac{\forall x.\neg A, \exists x.A \vdash \bot}{\forall x.\neg A \vdash \neg\exists x.A}\ [\neg R]$$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\forall x.\neg A, A[x\backslash y] \vdash \bot}{\forall x.\neg A, \exists x.A \vdash \bot}\,[\exists L]}{\forall x.\neg A \vdash \neg\exists x.A}\,[\neg R]$$

- pick y such that it does not occur in A
- we have to use $[\exists L]$ before $[\forall L]$ because y must not be free in the context

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$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A[x\backslash y], A[x\backslash y] \vdash \bot}}{\forall x.\neg A, A[x\backslash y] \vdash \bot}\;[\forall L]}{\forall x.\neg A, \exists x.A \vdash \bot}\;[\exists L]}{\forall x.\neg A \vdash \neg\exists x.A}\;[\neg R]
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}}{\neg A[x\backslash y], A[x\backslash y] \vdash \bot}[\neg L]}{\forall x.\neg A, A[x\backslash y] \vdash \bot}[\forall L]}{\forall x.\neg A, \exists x.A \vdash \bot}[\exists L]}{\forall x.\neg A \vdash \neg\exists x.A}[\neg R]$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \vdash A[x\backslash y]}\,[Id]}{\neg A[x\backslash y], A[x\backslash y] \vdash \bot}\,[\neg L]}{\forall x.\neg A, A[x\backslash y] \vdash \bot}\,[\forall L]}{\forall x.\neg A, \exists x.A \vdash \bot}\,[\exists L]}{\forall x.\neg A \vdash \neg\exists x.A}\,[\neg R]
$$

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Logical Equivalences

As before: if ($P \leftrightarrow Q$ or $Q \leftrightarrow P$) and P occurs in A, then replacing P by Q in A leads to a formula B, such that $A \leftrightarrow B$

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Logical Equivalences

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Also,

Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v, $\models_{M,v} P$ iff $\models_{M,v} Q$

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Example: prove $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

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  - assume $d \in D$ and $\models_{M,v,x\mapsto d} A$, and prove a contradiction

Logical Equivalences

Example: prove $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

- if $\vDash_{M,v} \neg\exists x.A$ then $\vDash_{M,v} \forall x.\neg A$
  - to prove: $\vDash_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\vDash_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove a contradiction
  - assumption: $\vDash_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\vDash_{M,v,x\mapsto e} A$

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Example: prove $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

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  - assumption: $\models_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\models_{M,v,x\mapsto e} A$
  - contradiction! there is one: take $e = d$

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  - assume $d \in D$ and $\models_{M,v,x\mapsto d} A$, and prove a contradiction
  - assumption: $\models_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\models_{M,v,x\mapsto e} A$
  - contradiction! there is one: take $e = d$
- if $\models_{M,v} \forall x.\neg A$ then $\models_{M,v} \neg\exists x.A$
  - to prove: $\models_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\models_{M,v,x\mapsto e} A$

Logical Equivalences

Example: prove $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$

- if $\vDash_{M,v} \neg \exists x.A$ then $\vDash_{M,v} \forall x.\neg A$
  - to prove: $\vDash_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\vDash_{M,v,x \mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x \mapsto d} A$, and prove a contradiction
  - assumption: $\vDash_{M,v} \neg \exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\vDash_{M,v,x \mapsto e} A$
  - contradiction! there is one: take $e = d$
- if $\vDash_{M,v} \forall x.\neg A$ then $\vDash_{M,v} \neg \exists x.A$
  - to prove: $\vDash_{M,v} \neg \exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\vDash_{M,v,x \mapsto e} A$
  - assume that there exists a $e \in D$ such that $\vDash_{M,v,x \mapsto e} A$, and prove a contradiction

Logical Equivalences

Example: prove $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

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  - assumption: $\models_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\models_{M,v,x\mapsto d} A$
  - therefore, instantiating this assumption with e: it is not the case that $\models_{M,v,x\mapsto e} A$

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  - contradiction!

Conclusion

What did we cover today?

- Equivalence using Natural Deduction
- Equivalence using the Sequent Calculus
- Equivalences using semantics

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Further reading:

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  http://leanprover.github.io/logic_and_proof/

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Next time?

- Predicate Logic – Equivalences