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Ordered pairs and Cartesian product

We have seen above that, for any two given objects x and y, the sets $\{x, y\}$ and $\{y, x\}$ are the same. If we want to "keep track" of the order, instead of the set $\{x, y\}$ we must use the *ordered pair* (x, y) – note the different kind of parentheses! Given an ordered pair (x, y), we say that x is the *first element* (or *first component*) of the pair and y is the *second element* (or *second component*) of the pair. For two ordered pairs (x, y) and (z, w) we have

$$(x, y) = (z, w) \text{ if and only if } x = z \text{ and } y = w;$$

in other words, two ordered pairs are the same if and only if they have the same first element and the same second element. Here the order of the elements matters! In particular, $(x, y) \neq (y, x)$, unless $x = y$.

Example 2.1. Let $A = \{1, 2\}$, $B = \{2, 1\}$, $C = (1, 2)$ and $D = (2, 1)$. Then as the order of elements in a *set* do not matter it follows

$$A = B.$$

However,

$$C \neq D.$$

To see this we just need to compare C and D component-wise. The first component of C is 1 and the first component of D is 2. Since $1 \neq 2$ we can see that $C \neq D$.

A typical example where order matters is when considering points in the plane: the point with coordinates $(1, 2)$ is not the same as the point with coordinates $(2, 1)$, see Figure 2.1. Using Cartesian coordinates, it is actually natural to think of the plane as the set of all ordered pairs of real numbers: here we are identifying each point of the plane with its Cartesian coordinates.

This example leads us to the definition of the *Cartesian product* of two sets.

Definition 2.2. Let A and B be two sets. We define the *Cartesian product of A and B*, denoted $A \times B$, as the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. When $A = B$, we also write A^2 instead of $A \times A$.

According to our discussion about coordinates, we can identify the sets of points of the plane with the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ of ordered pairs of real numbers.

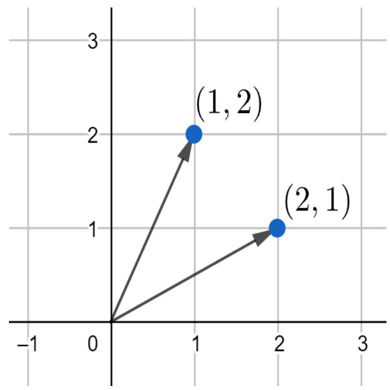

Figure 2.1: Showing two different points in the (Cartesian) plane.

Correspondingly,

$$C = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$$

is the set of points (x, y) of the plane that satisfy the equation $x^2 + y^2 = 1$; in other words, C is the set of the points of the circle of radius 1 centered at the origin $(0, 0)$, see Figure 2.2.

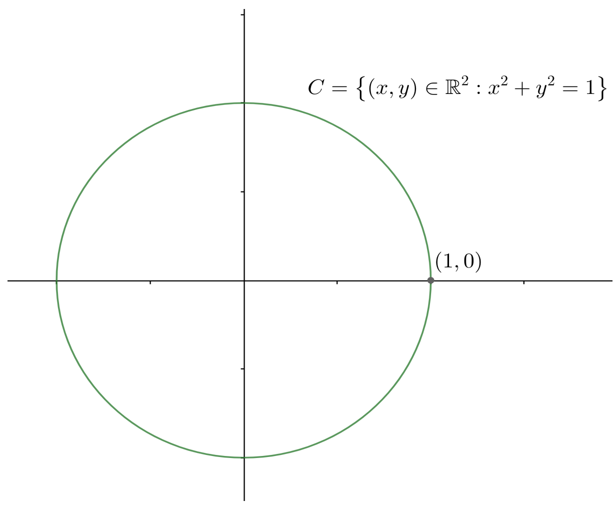

Figure 2.2: The set C, which is the unit circle in the plane

Note that, in addition to ordered pairs (x, y), we can speak of ordered *triples* (x, y, z), ordered *quadruples* (x, y, z, w), etc. For instance, $\mathbb{R}^3$ denotes the set of ordered triples of real numbers, which can be identified (via Cartesian coordinates) with the set of points of three-dimensional space.