# Mathematical and Logical Foundations of Computer Science

# Lecture 5b - Propositional Logic
# (Natural Deduction & Sequent Calculus)

Vincent Rahli

(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

# Where are we?

- Symbolic logic
- **Propositional logic**
- Predicate logic
- Constructive vs. Classical logic
- Type theory

# Today

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

**Further reading**

- Section 5 in "Proof and Types"
  https://www.paultaylor.eu/stable/prot.pdf
- Chapter 3 of
  http://leanprover.github.io/logic_and_proof/

# Recap: Propositional logic syntax

**Syntax:**

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

Lower-case letters are atoms: $p$, $q$, $r$, etc.

Upper-case letters stand for any proposition: $P$, $Q$, $R$, etc.

**Two special atoms:**

- $\top$ which stands for True
- $\perp$ which stands for False

**We also introduced four connectives:**

- $P \wedge Q$: we have a proof of both $P$ and $Q$
- $P \vee Q$: we have a proof of at least one of $P$ and $Q$
- $P \rightarrow Q$: if we have a proof of $P$ then we have a proof of $Q$
- $\neg P$: stands for $P \rightarrow \perp$

# Recap: Propositional logic syntax

How would you express these sentences in propositional logic?

- "if $x > 2$ then $x > 1$"
  - atom $p$: "$x > 2$"
  - atom $q$: "$x > 1$"
  - proposition: $p \to q$
- "if $x > 2$ and $x$ is even then $x > 3$"
  - atom $p$: "$x > 2$"
  - atom $q$: "$x$ is even"
  - atom $r$: "$x > 3$
  - proposition: $(p \land q) \to r$
  - we don't need parentheses, and can just write: $p \land q \to r$

# Recap: Natural deduction vs. Sequent Calculus

2 deduction systems for propositional logic (don't mix their rules!)

**Natural Deduction**

- "natural" style of constructing a proof
- start with the given premises
- repeatedly apply the given inference rules
- until you obtain the conclusion
- Can work both forwards and backwards
- "natural" doesn't mean there is a unique proof

**Sequent Calculus**

- hypotheses are made explicit in a **context**
- instead of deriving proposition, we derive **sequents**
  - a sequent is of the form $\Gamma \vdash P$
  - where the environment/context $\Gamma$ is a list of propositions
  - and $P$ is a proposition
  - intuitively: $P$ is true assuming that the formulas in $\Gamma$ are true
- we typically go backward

# Recap: Natural Deduction

**Natural Deduction** rules:

$$\frac{\bot}{A}\;[\bot E] \qquad \frac{}{\top}\;[\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ B\end{array}}{A \to B}\;1\;[\to I] \qquad \frac{A \to B \quad A}{B}\;[\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ \bot\end{array}}{\neg A}\;1\;[\neg I] \qquad \frac{\neg A \quad A}{\bot}\;[\neg E]$$

$$\frac{A}{A \vee B}\;[\vee I_L] \qquad \frac{A}{B \vee A}\;[\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\;[\vee E]$$

$$\frac{A \quad B}{A \wedge B}\;[\wedge I] \qquad \frac{A \wedge B}{B}\;[\wedge E_R] \qquad \frac{A \wedge B}{A}\;[\wedge E_L]$$

# Recap: Sequent Calculus

**Sequence Calculus** rules:

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ [\to L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ [\neg L] \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ [\neg R]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ [\vee L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ [\wedge L] \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{}{A \vdash A}\ [Id] \qquad \frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A}\ [Cut]$$

$$\frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, B, \Delta \vdash C}\ [X] \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ [W] \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ [C]$$

# Recap: Sequent Calculus

In addition we allow using the following **derived rules**:

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \rightarrow B, \Gamma_2 \vdash C}\ [\rightarrow L] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B}\ [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C}\ [\vee L] \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C}\ [\wedge L]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [W] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [C]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A}\ [Id]$$

All these **derived rules** can be proved/derived using the rules on the previous slide

# Recap: Proofs

**Natural Deduction**

introduction/elimination rules

natural proofs

$$\dfrac{\dfrac{\overline{A}^{\ 1}}{\vdots}\quad}{} $$

$$\dfrac{\begin{matrix}\overline{A}^{\,1} \\ \vdots \\ B\end{matrix}}{A \to B}\ 1\ [\to I]$$

**Sequent Calculus**

right/left rules

amenable to automation

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

- in the Sequent Calculus the discharged hypothesis $A$ is kept in the context!
- all the available hypotheses are always kept in the context part of sequents
- a proposition provable in one system is provable in the other

# Example 1

Provide a Natural Deduction proof of $(A \wedge B) \rightarrow (B \wedge A)$

Here is an example of a backward proof:

$$\dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{\,1}}{B}\,[\wedge E_R] \qquad \dfrac{\overline{A \wedge B}^{\,1}}{A}\,[\wedge E_L]}{B \wedge A}\,[\wedge I]}{(A \wedge B) \rightarrow (B \wedge A)}\,1\ [\rightarrow I]$$

How do we know where the introduced hypotheses ($A \wedge B$ above) will be used in the proof?

We typically don't so we can keep track of them on the side while doing the proof as follows:

# Example 1

Let us prove $(A \wedge B) \rightarrow (B \wedge A)$ again:

$$\dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{\,1}}{B}\,[\wedge E_R] \quad \dfrac{\overline{A \wedge B}^{\,1}}{A}\,[\wedge E_L]}{B \wedge A}\,[\wedge I]}{(A \wedge B) \rightarrow (B \wedge A)}\,1\ [\rightarrow I]$$

**Hypotheses**:

- hypothesis $1$: $A \wedge B$

**This can be achieved using the Sequent Calculus!**

# Example 1

Provide a Sequent Calculus proof of $(A \wedge B) \rightarrow (B \wedge A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,[Id] \qquad \dfrac{}{A, B \vdash A}\,[Id]}{A, B \vdash B \wedge A}\,[\wedge R]}{A \wedge B \vdash B \wedge A}\,[\wedge L]}{\vdash (A \wedge B) \rightarrow (B \wedge A)}\,[\rightarrow R]$$

# Example 2

Provide a Natural Deduction proof of

$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

We will keep track of our hypotheses on the side

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vee C}^{\,3} \quad \dfrac{\dfrac{\dfrac{\overline{A\to B}^{\,1}\quad \overline{A}^{\,4}}{B}\,[\to E]}{B\vee D}\,[\vee I_L]}{A\to(B\vee D)}\,4\,[\to I] \quad \dfrac{\dfrac{\dfrac{\overline{C\to D}^{\,2}\quad \overline{C}^{\,5}}{D}\,[\to E]}{B\vee D}\,[\vee I_R]}{C\to(B\vee D)}\,5\,[\to I]}{B\vee D}\,[\vee E]}{(A\vee C)\to(B\vee D)}\,3\,[\to I]}{(C\to D)\to(A\vee C)\to(B\vee D)}\,2\,[\to I]}{(A\to B)\to(C\to D)\to(A\vee C)\to(B\vee D)}\,1\,[\to I]
$$

**Hypotheses**:

- hyp. $1$: $A \to B$
- hyp. $2$: $C \to D$
- hyp. $3$: $A \vee C$
- hyp. $4$: $A$
- hyp. $5$: $C$

- If an hypothesis is introduced in a branch, make sure you don't use it in another branch (e.g. $4$ cannot be used in the far right branch)
- This is enforced by sequents in the Sequent Calculus

# Example 2

Provide a Sequent Calculus proof of
$(A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{C \to D, A \vdash A}\;[Id]
\quad
\dfrac{\dfrac{}{B, C \to D, A \vdash B}\;[Id]}{B, C \to D, A \vdash B \vee D}\;[\vee R_1]
}{A \to B, C \to D, A \vdash B \vee D}\;[\to L]
\quad
\dfrac{
\dfrac{}{A \to B, C \vdash C}\;[Id]
\quad
\dfrac{\dfrac{}{A \to B, D, C \vdash D}\;[Id]}{A \to B, D, C \vdash B \vee D}\;[\vee R_2]
}{A \to B, C \to D, C \vdash B \vee D}\;[\to L]
}{A \to B, C \to D, A \vee C \vdash B \vee D}\;[\vee L]
}{A \to B, C \to D \vdash (A \vee C) \to (B \vee D)}\;[\to R]
}{A \to B \vdash (C \to D) \to (A \vee C) \to (B \vee D)}\;[\to R]
}{\vdash (A \to B) \to (C \to D) \to (A \vee C) \to (B \vee D)}\;[\to R]
$$

# Example 3

Provide a Natural Deduction proof of
$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$

We will keep track of our hypotheses on the side

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$1$}
\UnaryInfC{$B \to C \to \neg A$}
\AxiomC{}
\RightLabel{$3$}
\UnaryInfC{$B \wedge C$}
\RightLabel{$[\wedge E_L]$}
\UnaryInfC{$B$}
\RightLabel{$[\to E]$}
\BinaryInfC{$C \to \neg A$}
\AxiomC{}
\RightLabel{$3$}
\UnaryInfC{$B \wedge C$}
\RightLabel{$[\wedge E_R]$}
\UnaryInfC{$C$}
\RightLabel{$[\to E]$}
\BinaryInfC{$\neg A$}
\AxiomC{}
\RightLabel{$2$}
\UnaryInfC{$A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$3\ [\neg I]$}
\UnaryInfC{$\neg(B \wedge C)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$A \to \neg(B \wedge C)$}
\RightLabel{$1\ [\to I]$}
\UnaryInfC{$(B \to C \to \neg A) \to A \to \neg(B \wedge C)$}
\end{prooftree}
$$

**Hypotheses**:

- hyp. $1$: $B \to C \to \neg A$
- hyp. $2$: $A$
- hyp. $3$: $B \wedge C$

# Example 3

Provide a Sequent Calculus proof of
$(B \to C \to \neg A) \to A \to \neg(B \land C)$

Here is a proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B, C \vdash B$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B, C \vdash C$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A, B, C \vdash A$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$\neg A, A, B, C \vdash \perp$}
\RightLabel{$[\to L]$}
\BinaryInfC{$C \to \neg A, A, B, C \vdash \perp$}
\RightLabel{$[\to L]$}
\BinaryInfC{$B \to C \to \neg A, A, B, C \vdash \perp$}
\RightLabel{$[\land L]$}
\UnaryInfC{$B \to C \to \neg A, A, B \land C \vdash \perp$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$B \to C \to \neg A, A \vdash \neg(B \land C)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$B \to C \to \neg A \vdash A \to \neg(B \land C)$}
\RightLabel{$[\to R]$}
\UnaryInfC{$\vdash (B \to C \to \neg A) \to A \to \neg(B \land C)$}
\end{prooftree}
$$

Note that compared to the Natural Deduction proof, we only have to eliminate $B \land C$ once here

# Natural Deduction and Sequent Calculus

- sequents are useful to keep track of available hypotheses
- however, we have to keep hypotheses around all the time
- the Sequent Calculus provides more structure to proofs
- however, it is less "natural"

# Conclusion

### What did we cover today?

- Sequent Calculus vs. Natural Deduction
- Sequent Calculus proofs
- Natural Deduction proofs

### Further reading

- Section 5 in “Proof and Types”
  `https://www.paultaylor.eu/stable/prot.pdf`
- Chapter 3 of
  `http://leanprover.github.io/logic_and_proof/`

### Next time?

- Classical reasoning