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Stationary points: Maxima, minima and points of inflexion

In the last section we were able to utilise the first derivative to show when a function is increasing or decreasing, i.e. the graph does not change direction. In this section we want to now consider when the graph *does* change direction. Consider the following function in Figure 15.1.

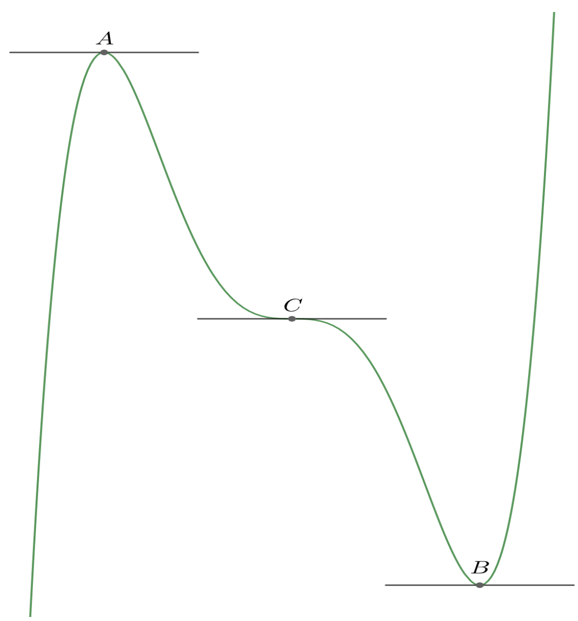

Figure 15.1: Different stationary points

Consider the point A. We can see that to the left of A, the function is increasing, that is the gradient of every tangent line to the left of A is positive. Now, to the right of A for points between A and C we can see that the function is decreasing, that is the gradient of every tangent line to the right of A is decreasing. So, the gradient to left of A is negative, but to the right of A it is positive. Therefore, we can see that the gradient of the tangent line at A is zero, that is the tangent is parallel to the x-axis.

Now, as one can see from Figure 15.1 that this also happens at B and C. Such points are called *stationary points*.

Definition 15.1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $x_0 \in \mathbb{R}$. We say that $(x_0, f(x_0))$ is a *stationary point (of f)* if the tangent line to the graph at $(x_0, f(x_0))$ is horizontal, i.e.

$$f'(x_0) = 0.$$

Example 15.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^5 - 5x^3$. The the graph of f is given in Figure 15.1. Let us determine the values of A, B and C. To compute this we need to first calculate $f'(x)$. Indeed,

$$f'(x) = 15x^4 - 15x^2.$$

We now need to solve $f'(x) = 0$. So determine which x satisfy $15x^4 - 15x^2 = 0$. So,

$$\begin{aligned} 15x^4 - 15x^2 &= 0 \\ 15x^2(x^2 - 1) &= 0 \quad \text{(factorising out } 15x^2\text{)} \\ 15x^2(x-1)(x+1) &= 0 \quad \text{(difference of two squares).} \end{aligned}$$

Therefore, the solutions are given by $x = 0$, $x = 1$ and $x = -1$. Now we need to calculate $f(0)$, $f(1)$, $f(-1)$. Plugging in the values to the above expression of $f(x)$, we see $f(0) = 0$, $f(1) = -2$ and $f(-1) = 2$. So,

$$A = (-1, 2), \quad B = (1, -2) \quad \text{and} \quad C = (0, 0).$$

As we can see from Figure 15.1 the graph does not change direction at C. In fact, it is decreasing to the left of C and it is decreasing to the right of C. Such stationary points are called *inflexion points*. If a stationary point is not an inflexion point, it is called a *turning points*. So in our case, A and B are turning points and C is an inflexion point.

Every turning point is a stationary point, but not every stationary point is a turning point.

But how can we determine when a stationary point is a turning point and when it is a point of inflexion? To answer this, we must define the different types of turning points. Again, let us consider Figure 15.1. We say that A is a *(local) maximum*; this is because if we look immediately to the left of A, A is higher, and if we look immediately to the right of A, we still have that A is higher. In a similar nature, we say that B is a *(local) minimum*.

Definition 15.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $x_0 \in \mathbb{R}$ be such that $f'(x_0) = 0$. We say that:

- $(x_0, f(x_0))$ is a *(local) maximum* if close to $(x_0, f(x_0))$ we have $f(x_0)$ is the largest value of f;
- $(x_0, f(x_0))$ is a *(local) minimum* if close to $(x_0, f(x_0))$ we have $f(x_0)$ is the smallest value of f.

In either case, if it is clear from context, we generally drop the local and simply say minimum and maximum.

Example 15.4. How can we characterise when a function attains a minimum or maximum value? Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, see Figure 15.2.

First, let us determine the stationary points of this function. To do this, we need to solve $f'(x) = 0$, i.e. $2x = 0$. Therefore, the only stationary point occurs when $x = 0$. As $f(0) = 0$, we see the only stationary point is $(0, 0)$. From Figure 15.2 we can clearly see that this point is a minimum point.

However, if we look to points slightly to the left of $(0, 0)$ we can see that the gradient of any tangent line will be negative. Similarly, we can see that to the right of $(0, 0)$ the gradient of any tangent line will be positive. Hence, $f'(x)$ changes from negative, to zero and then to positive. Therefore $f'(x)$ must be increasing.

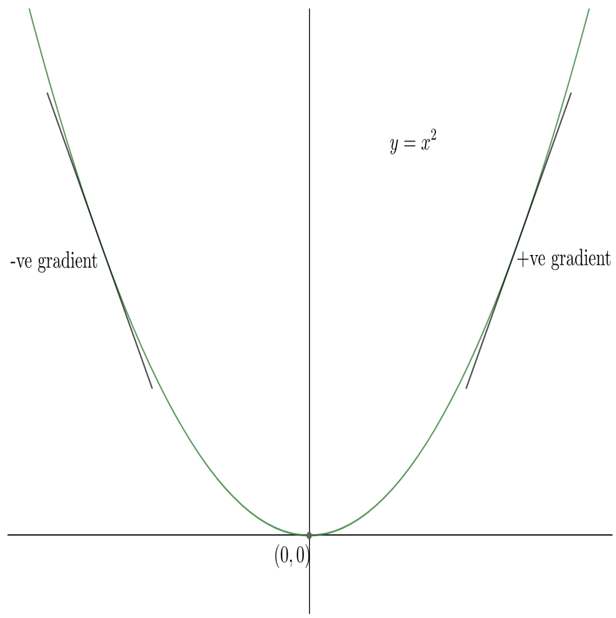

Figure 15.2: Characterising a minimum point

For a function to be increasing we must have its derivative is non-negative, i.e. $f''(x) \geq 0$. However we cannot simply determine whether a function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, or just increasing. This is due to the points where the derivative is equal to zero may cause some issues.

Nonetheless, if we see that $f''(x) > 0$ then this issue won't occur. Therefore, we can conclude that if a stationary point occurs for some x *and* $f''(x) > 0$, then this stationary point must be a minimum point.

Lemma 15.0.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and $x_0 \in \mathbb{R}$ be such that $f'(x_0) = 0$, i.e. $(x_0, f(x_0))$ is a stationary point. If:

- $f''(x_0) > 0$, then $(x_0, f(x_0))$ is a minimum point;
- $f''(x_0) < 0$, then $(x_0, f(x_0))$ is a maximum point.

Example 15.5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 3x + 2$. Determine the stationary points of f and characterise them.

Step 1: Calculate $f'(x)$.

So by our rules of differentiation, we obtain $f'(x) = 3x^2 - 3$.

Step 2: Solve $f'(x) = 0$.

That is, we need to solve $3x^2 - 3 = 0$. Dividing both sides by 3, we need $x^2 - 1 = 0$. Therefore, the difference of two squares yields $(x-1)(x+1) = 0$. Hence $x = 1$ or $x = -1$.

Step 3: Calculate y values.

That is, calculate $f(1)$ and $f(-1)$. Well,

$$f(1) = 1^3 - 3 \cdot 1 + 2 = 0 \quad \text{and} \quad f(-1) = (-1)^3 - 3(-1) + 2 = 4.$$

So our stationary points occur at $(1, 0)$ and $(-1, 4)$.

Step 4: Determine $f''(x)$.

Recall from Step 1 that $f'(x) = 3x^2 - 3$. So, differentiating again yields $f''(x) = 6x$.

Step 5: Evaluate $f''(x)$ at the stationary points.

Evaluating at $(1, 0)$: $f''(1) = 6 \cdot 1 = 6 > 0$. Therefore $(1, 0)$ is a minimum point.
Evaluating at $(-1, 4)$: $f''(-1) = 6 \cdot (-1) = -6 < 0$. Therefore $(-1, 4)$ is a maximum point.

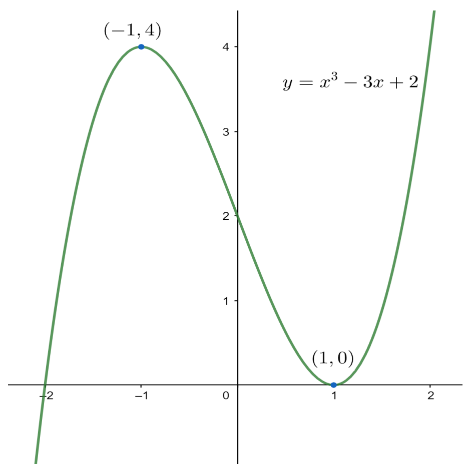

Figure 15.3: Graph of $y = f(x) = x^3 - 3x + 2$.

As mentioned previously, if $f''(x_0) = 0$ then we cannot say much about the type of stationary point we have without further investigation.

Example 15.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then we can show the unique stationary point occurs at $(0, 0)$. Further, $f''(x) = 6x$ and so $f''(0) = 0$. But, from a simple sketch, we can see that $(0, 0)$ is an inflexion point.

Now, let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^4$. Then, the unique stationary point, again, occurs at $(0, 0)$. Further, $g''(x) = 12x^2$ and so $g''(0) = 0$. But, from a sketch, we can see that

$(0, 0)$ is a minimum point.

Finally, let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(x) = -x^4$. Then, the unique stationary point, again, occurs at $(0, 0)$. Further, $h''(x) = -12x^2$ and so $h''(0) = 0$. But, from a sketch, we can see that $(0, 0)$ is a maximum point.