06-35324 *Mathematical and Logical Foundations of Computer Science* University of Birmingham
Autumn Semester 2021 School of Computer Science

Solutions to Exercise Sheet 10

Exercise 10.1

The parametric presentation derived from three points is computed as $X = P + s \cdot \overrightarrow{PQ} + t \cdot \overrightarrow{PR}$, so we get $X = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + t \cdot \begin{pmatrix} 5 \\ -2 \\ -5 \end{pmatrix}$. The coordinates of the normal vector $\vec{n}$ compute as

$$\begin{array}{rclcr} a & = & 10 - 4 & = & 6 \\ b & = & -10 - (-5) & = & -5 \\ c & = & -2 - (-10) & = & 8 \end{array}$$

and the normal form is obtained as $\langle \vec{n}, X \rangle = \langle \vec{n}, P \rangle$, so $6x_1 - 5x_2 + 8x_3 = -6 - 15 + 32 = 11$.

Exercise 10.2

(a) Change the direction vector $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to the normal vector: $\vec{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Setting up the normal form for the line as $\langle \vec{n}, X \rangle = \langle \vec{n}, P \rangle$ with $P = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ gives us $x_1 - 2x_2 = -3$.

(b) See below.

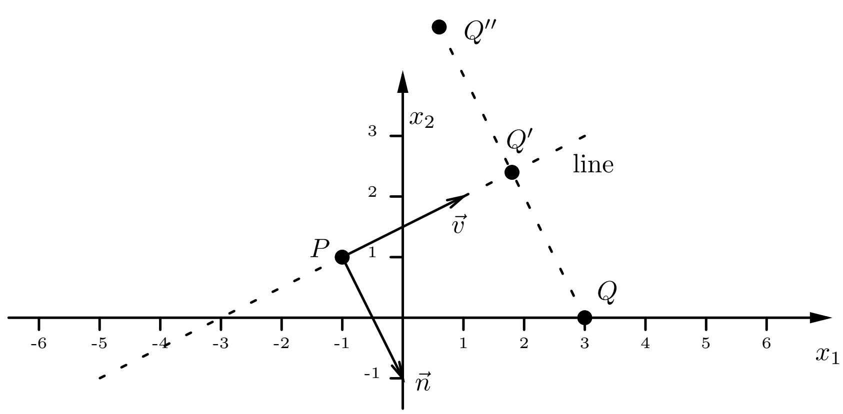
(c) According to the formula in Section 12.5 of the course booklet, the distance is computed as $\frac{d - \langle \vec{n}, Q \rangle}{|\vec{n}|} = \frac{-3-(3+0)}{\sqrt{1^2+2^2}} = \frac{-6}{\sqrt{5}}$.

(d) For this we compute $Q' = Q + \frac{d - \langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \frac{6}{5} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ \frac{12}{5} \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.4 \end{pmatrix}$.

(e) For this we compute $Q'' = Q + 2 \times \frac{d - \langle \vec{n}, Q \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \frac{12}{5} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{24}{5} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 4.8 \end{pmatrix}$.

(f) We use the inner product to check that $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is orthogonal to $\overrightarrow{Q''Q} = \begin{pmatrix} 0.6 - 3 \\ 4.8 - 0 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 4.8 \end{pmatrix}$:
Indeed, $\left\langle \overrightarrow{Q''Q}, \vec{v} \right\rangle = (-2.4) \times 2 + 4.8 \times 1 = 0$

(g)



Exercise 10.3

Two points on the line are

$$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Reflecting these yields

$$A'' = A + 2 \times \frac{d - \langle \vec{n}, A \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 2 \times 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and $$B'' = B + 2 \times \frac{d - \langle \vec{n}, B \rangle}{\langle \vec{n}, \vec{n} \rangle} \cdot \vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \times \frac{-1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

Connecting A'' and B'' gives the parametric representation

$$X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix} \qquad \text{which can be simplified to} \qquad X = s \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Exercise 10.4

Mirroring B on the first diagonal gives us $B' = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Mirroring B' on the x-axis gives us $B'' = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. We need to shoot the ball in the direction towards B'', that is, in direction $\begin{pmatrix} 7 \\ -4 \end{pmatrix}$. This hits the x-axis at $x = -2 + \frac{7}{4} = -\frac{1}{4}$ and then the first diagonal at $\begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$.