Live-session exercises and solutions for Mathematics week 11

Exercise 11A

(a) Do the following vectors form a basis of the 4-tuples?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

(b) If yes, is the basis orthogonal?

(c) Find the coordinates of

$$\begin{pmatrix} 3 \\ 1 \\ 7 \\ -1 \end{pmatrix}$$

with respect to the above vectors. (Hint: orthogonality may be useful.)

[Aside: This basis is very important in my research area, quantum computing. It is called the 'Bell basis' and it plays an important role in quantum teleportation.]

Model answers to the live exercises

Exercise 11A

(a) A basis for an algebra V is a collection of vectors with the folowing two properties:

- Every vector in V can be expressed as a linear combination of vectors in the collection (we also say the collection 'spans' V).
- The collection is linearly independent.

We'll show the second property first. The four vectors are linearly independent if the only solution of

$$a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_3 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is $a_1 = a_2 = a_3 = a_4 = 0$. We can answer this question using Gaussian elimination:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \qquad \text{swap rows 2 and 4}$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \qquad r2 \leftarrow r2 - r1$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \qquad r4 \leftarrow r4 - r3$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \qquad r3 \leftarrow 2 \times r3 + r4$$

$$\mapsto \left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \qquad r1 \leftarrow 2 \times r1 + r2$$

We can read off $a_1 = a_2 = a_3 = a_4 = 0$. Thus the four vectors are linearly independent.

For the first property, there are two options: we can argue that the algebra of 4-tuples has dimension 4, and therefore any linearly independent set containing four vectors must be a basis and span the entire algebra.

Alternatively, we can prove the property directly, very similar to the above. This time, we want to show that there exists a solution of

$$a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_3 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

for every vector $\vec{w}$. Again, using Gaussian elimination:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & w_1 \\ 0 & 0 & 1 & 1 & w_2 \\ 0 & 0 & 1 & -1 & w_3 \\ 1 & -1 & 0 & 0 & w_4 \end{array}\right) \mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & w_1 \\ 1 & -1 & 0 & 0 & w_4 \\ 0 & 0 & 1 & -1 & w_3 \\ 0 & 0 & 1 & 1 & w_2 \end{array}\right) \qquad \text{swap rows 2 and 4}$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & w_1 \\ 0 & -2 & 0 & 0 & w_4 - w_1 \\ 0 & 0 & 1 & -1 & w_3 \\ 0 & 0 & 1 & 1 & w_2 \end{array}\right) \qquad r2 \leftarrow r2 - r1$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & w_1 \\ 0 & -2 & 0 & 0 & w_4 - w_1 \\ 0 & 0 & 1 & -1 & w_3 \\ 0 & 0 & 0 & 2 & w_2 - w_3 \end{array}\right) \qquad r4 \leftarrow r4 - r3$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & w_1 \\ 0 & -2 & 0 & 0 & w_4 - w_1 \\ 0 & 0 & 2 & 0 & w_2 + w_3 \\ 0 & 0 & 0 & 2 & w_2 - w_3 \end{array}\right) \qquad r3 \leftarrow 2 \times r3 + r4$$

$$\mapsto \left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & w_1 + w_4 \\ 0 & -2 & 0 & 0 & w_4 - w_1 \\ 0 & 0 & 2 & 0 & w_2 + w_3 \\ 0 & 0 & 0 & 2 & w_2 - w_3 \end{array}\right) \qquad r1 \leftarrow 2 \times r1 + r2$$

We find

$$a_1 = \frac{w_1 + w_4}{2} \qquad a_2 = \frac{w_1 - w_4}{2} \qquad a_3 = \frac{w_2 + w_3}{2} \quad \text{and} \quad a_4 = \frac{w_2 - w_3}{2}$$

This works for any $\vec{w}$, so the four vectors span the algebra of 4-tuples and they are a basis.

(b) A basis is orthogonal if every pair of distinct vectors in the collection is orthogonal. With four vectors, that means there are six pairs to check.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\rangle = 1 \times 1 + 0 \times 0 + 0 \times 0 + 1 \times (-1) = 1 + 0 + 0 - 1 = 0$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1 \times 0 + 0 \times 1 + 0 \times 1 + 1 \times 0 = 0 + 0 + 0 + 0 = 0$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle = 1 \times 0 + 0 \times 1 + 0 \times (-1) + 1 \times 0 = 0 + 0 + 0 + 0 = 0$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1 \times 0 + 0 \times 1 + 0 \times 1 + (-1) \times 0 = 0 + 0 + 0 + 0 = 0$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle = 1 \times 0 + 0 \times (-1) + 0 \times 1 + (-1) \times 0 = 0 + 0 + 0 + 0 = 0$$

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle = 0 \times 0 + 1 \times 1 + 1 \times (-1) + 0 \times 0 = 0 + 1 - 1 + 0 = 0$$

(c) There are two ways of doing this, depending on how you proved the spanning property of the basis in part (a). Given the full proof (with Gaussian elimination) we wrote out above, we can simply plug in $w_1 = 3$, $w_2 = 1$, $w_3 = 7$,

$w_4 = -1$ to compute

$$a_1 = \frac{3-1}{2} = 1 \qquad a_2 = \frac{3-(-1)}{2} = 2 \qquad a_3 = \frac{1+7}{2} = 4 \quad \text{and} \quad a_4 = \frac{1-7}{2} = -3$$

If you didn't work out the equations but went for the dimension argument instead, then you can now make use of orthogonality to avoid the Gaussian elimination. Note that $\langle \vec{v}, \vec{v} \rangle = 2$ for every $\vec{v}$ in the basis, so:

$$a_1 = \frac{1}{2}\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \\ -1 \end{pmatrix} \right\rangle = \frac{3-1}{2} = 1$$

$$a_2 = \frac{1}{2}\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \\ -1 \end{pmatrix} \right\rangle = \frac{3+1}{2} = 2$$

$$a_3 = \frac{1}{2}\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \\ -1 \end{pmatrix} \right\rangle = \frac{1+7}{2} = 4$$

$$a_3 = \frac{1}{2}\left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \\ -1 \end{pmatrix} \right\rangle = \frac{1-7}{2} = -3$$