SCHWARZCHILD BLACK HOLE SOLUTION.

Vijay Chawan, 13th February.

The line element for spherically symmetric object(after some processing)* is written as;

$$ds^2 = A(r, t)\, dt^2 - B\,(r, t)\, dr^2 - r^2\left(d\theta^2 + \text{Sin}^2\,\theta d\phi^2\right).$$

For static case, A and B will be independent of t. Which is the case with Schwarzschild Black Hole.

In[ ]:=

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In[ ]:= coords = {t, r, θ, ϕ}; (*r−θ−ϕ coordinate system*)
(*[+, −, −, −] signature*)

n = 4;            (*dimension of space*)

metric = {{A[r], 0, 0, 0}, {0, -B[r], 0, 0},
   {0, 0, -r^2, 0}, {0, 0, 0, -r^2 * Sin[θ]^2}};

inversemetric = Simplify[Inverse[metric]];

Print["Covarient metric in r−θ−ϕ coordinates"]

metric // MatrixForm

Print["Covarient metric"]

 inversemetric // MatrixForm (*contravarient metric*)
```

Covarient metric in r−θ−ϕ coordinates

Out[ ]//MatrixForm=

$$\begin{pmatrix} A[r] & 0 & 0 & 0 \\ 0 & -B[r] & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \operatorname{Sin}[\theta]^2 \end{pmatrix}$$

Covarient metric

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{A[r]} & 0 & 0 & 0 \\ 0 & -\frac{1}{B[r]} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{\operatorname{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

Calculating Christoffel Connection, Riemann tensor, Ricci tensor using the above metric.

$$\Gamma_{\mu\nu}{}^{\lambda} = 1/2\, g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

$$R_{\sigma\mu\nu}{}^{\lambda} = \partial_{\mu}\Gamma^{\lambda}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\sigma}$$

$$R_{\sigma\nu} = R_{\sigma\lambda\nu}{}^{\lambda}$$

Note: metric[[1,1]] will give the first column and first row element that is tt element of metric.

Christoffel Connection

In[ ]:=

```
crishtoffel[λ_, μ_, ν_] :=
  crishtoffel[λ, μ, ν] = Simplify[1/2 * (Sum_{σ=1}^{4} inversemetric[[λ, σ]] *
       (D[metric[[σ, ν]], coords[[μ]]] + D[metric[[μ, σ]],
          coords[[ν]]] - D[metric[[μ, ν]], coords[[σ]]]))];
```

Riemann tensor

In[ ]:=
```
riemanntensor[λ_, σ_, μ_, ν_] := riemanntensor[λ, σ, μ, ν] =
  Simplify[D[crishtoffel[λ, σ, ν], coords[[μ]]] -
    D[crishtoffel[λ, σ, μ], coords[[ν]]] +
    Sum[(crishtoffel[λ, μ, ρ]*crishtoffel[ρ, σ, ν]), {ρ, 1, 4}] -
    Sum[(crishtoffel[λ, ν, γ]*crishtoffel[γ, σ, μ]), {γ, 1, 4}]];
```

$$\texttt{riemanntensor}[\lambda, \sigma, \mu, \nu] = \texttt{Simplify}\Big[\partial_{\mu}\,\texttt{crishtoffel}[\lambda,\sigma,\nu] - \partial_{\nu}\,\texttt{crishtoffel}[\lambda,\sigma,\mu] + \sum_{\rho=1}^{4}(\texttt{crishtoffel}[\lambda,\mu,\rho]*\texttt{crishtoffel}[\rho,\sigma,\nu]) - \sum_{\gamma=1}^{4}(\texttt{crishtoffel}[\lambda,\nu,\gamma]*\texttt{crishtoffel}[\gamma,\sigma,\mu])\Big];$$

Ricci tensor and Ricci scalar

In[ ]:=
```
riccitensor[μ_, ν_] :=
  riccitensor[μ, ν] = Simplify[Sum[riemanntensor[λ, μ, λ, ν], {λ, 1, 4}]];

ricciscalar =
  Simplify[Sum[inversemetric[[μ, ν]]*riccitensor[μ, ν], {μ, 1, 4}, {ν, 1, 4}]];
```

$$\texttt{riccitensor}[\mu,\nu] = \texttt{Simplify}\Big[\sum_{\lambda=1}^{4}\texttt{riemanntensor}[\lambda,\mu,\lambda,\nu]\Big];$$

$$\texttt{ricciscalar} = \texttt{Simplify}\Big[\sum_{\mu=1}^{4}\sum_{\nu=1}^{4}\texttt{inversemetric}[[\mu,\nu]]*\texttt{riccitensor}[\mu,\nu]\Big];$$

Printing Ricci tensor

In[ ]:=

```
riccitensormetric =
  {{riccitensor[1, 1], riccitensor[1, 2], riccitensor[1, 3], riccitensor[1, 4]},
   {riccitensor[2, 1], riccitensor[2, 2], riccitensor[2, 3], riccitensor[2, 4]},
   {riccitensor[3, 1], riccitensor[3, 2], riccitensor[3, 3], riccitensor[3, 4]},
   {riccitensor[4, 1], riccitensor[4, 2], riccitensor[4, 3], riccitensor[4, 4]}};

riccitensormetric // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{A'[r]\,B'[r]}{4\,B[r]^2}+\frac{\frac{A'[r]}{r}-\frac{A'[r]^2}{4\,A[r]}+\frac{A''[r]}{2}}{B[r]} & 0 & 0 & \\ 0 & \frac{A[r]\,(4\,A[r]+r\,A'[r])\,B'[r]+r\,B[r]\left(A'[r]^2-2\,A[r]\,A''[r]\right)}{4\,r\,A[r]^2\,B[r]} & 0 & \\ 0 & 0 & \frac{1}{2}\left(2-\frac{2+\frac{r\,A'[r]}{A[r]}}{B[r]}+\frac{r\,B'[r]}{B[r]^2}\right) & \\ 0 & 0 & 0 & \frac{\mathrm{Sin}[\theta]^2\,(-r\,B[r]\,A'[}{} \end{pmatrix}$$

The Ricci tensor components,

$$R_{11} = -\frac{A'[r]\,B'[r]}{4\,B[r]^2}+\frac{\frac{A'[r]}{r}-\frac{A'[r]^2}{4\,A[r]}+\frac{A''[r]}{2}}{B[r]}$$

$$R_{22} = \frac{A[r]\,(4\,A[r]+r\,A'[r])\,B'[r]+r\,B[r]\left(A'[r]^2-2\,A[r]\,A''[r]\right)}{4\,r\,A[r]^2\,B[r]}$$

$$R_{33} = \frac{1}{2}\left(2-\frac{2+\frac{r\,A'[r]}{A[r]}}{B[r]}+\frac{r\,B'[r]}{B[r]^2}\right)$$

$$R_{44} = \frac{\mathrm{Sin}[\theta]^2 \left(-r\, B[r]\, A'[r] + A[r] \left(-2\, B[r] + 2\, B[r]^2 + r\, B'[r]\right)\right)}{2\, A[r]\, B[r]^2}$$

R_{11} can be written as $\frac{A''}{2B} + \frac{A'}{\mathrm{rB}} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right)$

R_{22} can be written as $-\frac{A''}{2A} + \frac{B'}{\mathrm{rB}} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right)$

R_{33} can be written as $1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)$

1). Multiply $R_{11}$with B/A and add to R_{22}:

The result should be $\frac{B'}{A} = -\frac{A'}{B}$, thus, A*B = constant. The space is asymptotically flat, thus, for r tends to infinity B and A tends to 1, this gives the constant to be 1.

2). Use $\frac{B'}{A} = -\frac{A'}{B}$ is R_{33}:

The result is A + r*A’ = 1 which implies d(rA)/dr = d(r + constant), that is rA = r + constant.

Thus, A = 1 + constant/r and B = 1/(1 + constant/r).

To find the constant now.

In the weak field limit, g_{00}= (1 - 2GM/r), where r >> 2GM.

In our case, A = 1 + constant/r. have the same form as that of g_{00} weak field limit case, thus constant = 2GM.

The line element is then,

$$ds^2 = (1 - 2\,GM/r)\,dt^2 - (1 - 2\,GM/r)^{-1}\,dr^2 - r^2\left(d\theta^2 + \text{Sin}[\theta]^2\,d\phi^2\right)$$

Hence, Schwarzschild Solution is achieved.