# 5 Exercises, with Solutions

While the purpose of these exercises is to strengthen the learning experience and to generate out-of-the-box thinking, perhaps even more importantly, they provide additional methodological and technical material, complementing and extending the main text.

Starred exercises are more difficult. Several of the problems require only simulations, statistical analysis, and testing hypotheses on a computer. They are marked as [S] and should help you hone your machine learning and computing skills; they may not be easier or less challenging than the mathematical problems. Exercises involving mathematics or probability theory are marked as [M], while those combining both simulations and mathematics are marked as [MS]. Solutions or hints are provided for each problem.

## 5.1 Full List

Table 7 provides a listing all the exercises. To access any exercise, click on its red number, to the left. The NN abbreviation stands for "nearest neighbors".

| | | | |
|---|---|---|---|
| 1 | Point count, Laplace distribution | 15 | Distribution of NN distances |
| 2 * | Convergence to Poisson process | 16 | Cell networks: coverage problem |
| 3 * | Limit of generalized logistic distribution | 17 | Optimum circle covering of the plane |
| 4 | Small paradox | 18 | Interlaced lattices, lattice mixtures, NN |
| 5 | Exact distribution of interarrival times | 19 * | Lattice topology and algebra |
| 6 * | Retrieving $F$ from interarrival times | 20 ** | NN graph: size of connected components |
| 7 * | Poisson limit of Poisson-binomial distribution | 21 | NN graph: maximum clique problem |
| 8 | A few simple theorems | 22 | Computing moments using the CDF |
| 9 | Testing stationarity, independent increments | 23 | Simulations: generalized logistic distribution |
| 10 | Interdependencies in point counts | 24 | Riemann Hypothesis |
| 11 | Boundary effect | 25 * | Convergence acceleration of math series |
| 12 | A curious, Poisson-like point process | 26 | Fast image filtering algorithm |
| 13 * | Poisson-binomial process on the sphere | 27 ** | Confidence regions: theory, computations |
| 14 | Taxonomy of point processes | 28 * | Minimum set covering 90% of a distribution |

Table 7: List of exercises

## 5.2 Probability Distributions, Limits and Convergence

The focus here is on the distribution $F$ including some of its limiting cases, the distribution of arrival times, and convergence to the Poisson process. The Laplace, generalized logistic, Borel, and Poisson-binomial distributions are investigated.

**Exercise 1** **[M] Point count, Laplace distribution**. If $F$ is a Laplace distribution and $\lambda = 1$, find $E[N(B)]$, where $B = [a, b]$ is an interval with $\lfloor a \rfloor \leq \lfloor b \rfloor < \lfloor a \rfloor + 1$. Here the brackets represent the integer part function, and $F_s(x) = F(x/s)$. See Theorem 4.8, solving the same problem with a uniform rather than Laplace distribution.

**Solution**

Let $p_k = F_s(b-k) - F_s(a-k)$ with $s > 0$, and let sgn stands for the sign function, with $\text{sgn}(0) = 0$. Here

$$F_s(x-k) = \frac{1}{2} + \frac{1}{2}\text{sgn}(x-k)\Big[1 - \exp\Big(-\frac{1}{s} \cdot |x-k|\Big)\Big]$$

We have three cases:

- If $k \leq a < b$ then $p_k = \frac{1}{2}\Big[\exp(-(a-k)/s) - \exp(-(b-k)/s)\Big]$
- If $a \leq k \leq b$ then $p_k = 1 - \frac{1}{2}\Big[\exp(-(b-k)/s) + \exp((a-k)/s)\Big]$
- If $a < b \leq k$ then $p_k = \frac{1}{2}\Big[\exp((b-k)/s) - \exp((a-k)/s)\Big]$