

Glossary

m -interlacing	Superimposition of m Poisson-binomial processes, each with its own (shifted, stretched) <i>lattice space</i> , and its own intensity and scaling factor. See pages 11, 24, 34, 35, 36, 37, 38, 61, 63, 69, 75, 76
Anisotropy	A property of a point process: the points are evenly scattered in all directions. The point distribution is stochastically invariant under rotations. See pages 10, 37
Attraction / Repulsion	The larger the scaling factor s , the less repulsion (also called <i>inhibition</i>) among the points of the process. Cluster processes discussed here exhibit both attraction (points tend to cluster together) and repulsion among cluster centers (due to the underlying lattice structure). See pages 7, 16, 37
Boundary Effect	Also called edge effect. Bias in estimated point counts or nearest neighbor distances, due to unobserved points located outside but close to the finite window of observations. Also the result of missing points in the window of observations, in simulated point processes especially when the scaling factor s is large. Special techniques are used to handle this problem. See pages 10, 11, 12, 24, 25, 27, 29, 30, 32, 37, 38, 44, 46, 52, 60, 61, 64, 67, 76
Confidence Region	A confidence region of level γ is a 2-D set of minimum area covering a proportion γ of the mass of a bivariate probability distribution. See pages 24, 27, 32, 66, 68
Connected Component	A set of vertices in a graph that are connected to each other by paths. See also <i>nearest neighbor graph</i> . See pages 12, 37, 38, 48, 61, 63, 65, 69, 77, 79
Empirical Distribution	Cumulative frequency histogram attached to a statistic (for instance, nearest neighbor distances), and based on observations. When the number of observations tends to infinity and the bin sizes tends to zero, this step function tends to the theoretical cumulative distribution function of the statistic in question. See pages 8, 17, 24, 31, 33, 38, 46, 48, 50, 54, 64
Ergodicity	A statistic such as the interarrival times is ergodic if it has the same asymptotic distribution, whether it is computed on many observations from a single realization of the process, or averaged across many realizations, each with few observations. See pages 9, 32, 33, 37, 48, 59
Homogeneity	A property of a point process, characterized by an homogeneous intensity function, that is, constant or independent of the location. See pages 9, 10, 11, 61
Identifiability	A models is identifiable if it is uniquely defined by its parameters. Then it is possible to estimate each parameter separately. A trivial example of non-identifiability is when we have two parameters, say α, β , but they only occur in a product $\alpha\beta$. In that case, if $\alpha\beta = 6$, it is impossible to tell whether $\alpha = 2, \beta = 3$ or $\alpha = 1, \beta = 6$. See pages 10, 15, 24, 33, 34, 61
Index Space	Consists of the indices $h, k \in \mathbb{Z}$, attached to the points X_k in one dimension, or (X_h, Y_k) in two dimensions. See pages 6, 10, 11, 31, 44, 46, 52, 76, 82
Intensity	Core parameter of the Poisson-binomial process. Denoted as λ . It represents the granularity of the underlying lattice, that is, the point density. In d dimensions, $E[N(B)] = 1$ for any hypercube B of length $1/\lambda$. Here N is the point count. When λ is constant (not depending on the location), the process is homogeneous. See pages 6, 15, 23, 25, 32, 37, 38, 40, 41, 61, 69
Interarrival Time	In one dimension, random variable measuring the distance between a point of the process and its closest neighbor to the right, on the real axis. Interarrival times are also called <i>increments</i> . See pages 7, 9, 25, 32, 33, 37, 41, 49, 59, 60, 64, 69, 71
Lattice Space	In two dimensions, it consists of the locations $(h/\lambda, k/\lambda)$ with $h, k \in \mathbb{Z}$. The distribution of a point (X_h, Y_k) is centered at $(h/\lambda, k/\lambda)$. The concept can be extended to any dimension. See pages 6, 10, 11, 12, 16, 35, 36, 46, 47, 61, 62, 63, 78
Location-scale	A random variable X has a location-scale distribution with two parameters, the scale s and location μ , if any linear transformation $a + bX$ has a distribution of the same family, with parameters respectively b^2s and $\mu + a$. Here μ is the expectation and s is proportional to the variance of the distribution. See pages 6, 10

Modulo Operator	Sometimes, it is useful to work with point “residues” modulo $\frac{1}{\lambda}$, instead of the original points, due to the nature of the underlying lattice. It magnifies the patterns of the point process. By definition, $X_k \bmod \frac{1}{\lambda} = X_k - \frac{1}{\lambda} \lfloor \lambda X_k \rfloor$ where the brackets represent the integer part function. See pages 36 , 38 , 40 , 63 , 76 , 78 , 83
NN Graph	Nearest neighbor graph. The vertices are the points of the process. Two vertices (the points they represent) are connected if at least one of the two points is nearest neighbor to the other one. This graph is undirected. See pages 22 , 37 , 65 , 69 , 79 , 89
Point Count	Random variable, denoted as $N(B)$, counting the number of points of the process in a particular set B , typically an interval $[a, b]$ in one dimension, and a square or circle in two dimensions. See pages 5 , 7 , 24 , 30 , 32 , 33 , 37 , 38 , 44 , 49 , 59 , 60 , 62 , 64 , 67 , 69 , 71
Point Distribution	Random variable representing how a point of the process is distributed in a domain B ; for instance, for a stationary Poisson process, points are uniformly distributed on any compact domain B (say, an interval in one dimension, or a square in two dimensions). See pages 7 , 25 , 37 , 76
Quantile function	Inverse of the cumulative distribution function (CDF) F , denoted as Q . Thus if $P(X < x) = F(x)$, then $P(X < Q(x)) = x$. See pages 6 , 13 , 14 , 38 , 54 , 57 , 65
Scaling Factor	Core parameter of the Poisson-binomial process. Denoted as s , proportional to the variance of the distribution F attached to the points of the process. It measures the level of repulsion among the points (maximum if $s = 0$, minimum if $s = \infty$). In d dimensions, the process is stationary Poisson of intensity λ^d if $s = \infty$, and coincides with the fixed <i>lattice space</i> if $s = 0$. See pages 5 , 6 , 10 , 11 , 12 , 15 , 23 , 25 , 32 , 37 , 48 , 59 , 61 , 62 , 63 , 64 , 69 , 74 , 75
Shift vector	The lattice attached to a 2-D Poisson-binomial process consists of the vertices $(\frac{h}{\lambda}, \frac{k}{\lambda})$ with $h, k \in \mathbb{Z}$. A shifted process has its lattice translated by a shift vector (u, v) . The new vertices are $(u + \frac{h}{\lambda}, v + \frac{k}{\lambda})$. See page 11 , 36 , 38 , 40 , 41 , 63 , 75 , 83
Standardized Process	Poisson-binomial process with intensity $\lambda = 1$, scaling factor $s = 1$, and shifted (if necessary) so that the lattice space coincides with \mathbb{Z} or \mathbb{Z}^2 . See page 10
State Space	Space where the points of the process are located. Here, \mathbb{R} or \mathbb{R}^2 . See also <i>index space</i> and <i>lattice space</i> . See pages 6 , 16 , 23 , 32 , 36 , 37 , 38 , 40 , 44 , 46 , 51 , 63 , 82 , 83
Stationarity	Property of a point process: the point distributions in two sets of same shape and area, are identical. The process is stochastically invariant under translations. See pages 6 , 8 , 11 , 24 , 29 , 63

List of Figures

1	Convergence to stationary Poisson point process of intensity λ	8
2	Four superimposed Poisson-binomial processes: $s = 0$ (left), $s = 5$ (right)	12
3	Radial cluster process ($s = 0.2, \lambda = 1$) with centers in blue; zoom in on the left	15
4	Radial cluster process ($s = 2, \lambda = 1$) with centers in blue; zoom in on the left	16
5	Manufactured marble lacking true lattice randomness (left)	16
6	Locally random permutation σ ; $\tau(k)$ is the index of X_k 's closest neighbor to the right	17
7	Chaotic function (bottom), and its transform (top) showing the global minimum	18
8	Orbit of η in the complex plane (left), perturbed by a Poisson-binomial process (right)	21
9	Data animations – click on a picture to start a video	22
10	Minimum contrast estimation for (λ, s)	25
11	Confidence region for (p, q) – Hotelling's quantile function on the left	27
12	Period and amplitude of $\phi_\tau(t)$; here $\tau = 1, \lambda = 1.4, s = 0.3$	29
13	Bias reduction technique to minimize boundary effects	29
14	A new test of independence (R-squared version)	30
15	Radial cluster process ($s = 0.5, \lambda = 1$) with centers in blue; zoom in on the left	35
16	Radial cluster process ($s = 1, \lambda = 1$) with centers in blue; zoom in on the left	35
17	Realization of a 5-interlacing with $s = 0.15$ and $\lambda = 1$: original (left), modulo $2/\lambda$ (right)	36
18	Rayleigh test to assess if a point distribution matches that of a Poisson process	39
19	Unsupervised (left) versus supervised clustering (right) of Figure 17	39

20	Elbow rule (right) finds $m = 3$ clusters in Brownian motion (left)	43
21	Elbow rule (right) finds $m = 8$ or $m = 11$ “jumps” in left plot	43
22	Each arrow links a point (blue) to its lattice index (red): $s = 0.2$ (left), $s = 1$ (right)	46
23	Distance between a point and its lattice location ($s = 1$)	47
24	Chaotic convergence of partial sums in Formula (19)	66

References

- [1] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley, fourth edition, 2016. 64
- [2] José M. Amigó, Roberto Dale, and Piergiulio Tempesta. A generalized permutation entropy for random processes. *Preprint*, pages 1–9, 2012. arXiv:2003.13728. 17
- [3] Luc Anselin. *Point Pattern Analysis: Nearest Neighbor Statistics*. The Center for Spatial Data Science, University of Chicago, 2016. Slide presentation. 13
- [4] Adrian Baddeley. Spatial point processes and their applications. In Weil W., editor, *Stochastic Geometry. Lecture Notes in Mathematics*, pages 1–75. Springer, Berlin, 2007. 13
- [5] Adrian Baddeley and Richard D. Gill. Kaplan-meier estimators of distance distributions for spatial point processes. *Annals of Statistics*, 25(1):263–292, 1997. 44
- [6] David Bailey, Jonathan Borwein, and Neil Calkin. *Experimental Mathematics in Action*. A K Peters, 2007. 17
- [7] N. Balakrishnan and C.R. Rao (Editors). *Order Statistics: Theory and Methods*. North-Holland, 1998. 47, 53, 64
- [8] B. Bollobas and P. Erdős. Cliques in random graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 80(3):419–427, 1976. 64
- [9] Miklos Bona. *Combinatorics of Permutations*. Routledge, second edition, 2012. 17
- [10] Jonathan Borwein and David Bailey. *Mathematics by Experiment*. A K Peters, 2008. 17
- [11] Bartłomiej Błaszczyszyn and Dhandapani Yogeshwaran. Clustering and percolation of point processes. *Preprint*, pages 1–20, 2013. Project Euclid. 13
- [12] Bartłomiej Błaszczyszyn and Dhandapani Yogeshwaran. On comparison of clustering properties of point processes. *Preprint*, pages 1–26, 2013. arXiv:1111.6017. 13
- [13] Bartłomiej Błaszczyszyn and Dhandapani Yogeshwaran. Clustering comparison of point processes with applications to random geometric models. *Preprint*, pages 1–44, 2014. arXiv:1212.5285. 13
- [14] Oliver Chikumbo and Vincent Granville. Optimal clustering and cluster identity in understanding high-dimensional data spaces with tightly distributed points. *Machine Learning and Knowledge Extraction*, 1(2):715–744, 2019. 43
- [15] Yves Coudène. *Ergodic Theory and Dynamical Systems*. Springer, 2016. 9
- [16] Noel Cressie. *Statistic for Spatial Data*. Wiley, revised edition, 2015. 13
- [17] H.A. David and H.N. Nagaraja. *Order Statistics*. Wiley, third edition, 2003. 53
- [18] Tilman M. Davies and Martin L. Hazelton. Assessing minimum contrast parameter estimation for spatial and spatiotemporal log-Gaussian Cox processes. *Statistica Neerlandica*, 67(4):355–389, 2013. 25
- [19] Robert Devaney. *An Introduction to Chaotic Dynamical Systems*. Chapman and Hall/CRC, third edition, 2021. 9
- [20] D.J.Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes – Volume I: Elementary Theory and Methods*. Springer, second edition, 2013. 13
- [21] D.J.Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes – Volume II: General Theory and Structure*. Springer, second edition, 2014. 13
- [22] David Coupier (Editor). *Stochastic Geometry: Modern Research Frontiers*. Wiley, 2019. 62
- [23] Ding-Geng Chen (Editor), Jianguo Sun (Editor), and Karl E. Peace (Editor). *Interval-Censored Time-to-Event Data: Methods and Applications*. Chapman and Hall/CRC, 2012. 11
- [24] Bradley Efron. Bootstrap methods: Another look at the jackknife. *Annals of Statistics*, 7(1):1–26, 1979. 24
- [25] Paul Erdős and Alfréd Rényi. On the evolution of random graphs. In *Publication of the Mathematical Institute of the Hungarian Academy of Sciences*, volume 5, pages 17–61, 1960. 64
- [26] W. Feller. On the Kolmogorov-Smirnov limit theorems for empirical distributions. *Annals of Mathematical Statistics*, 19(2):177–189, 1948. 39, 64

- [27] Peter J. Forrester and Anthony Mays. Finite size corrections in random matrix theory and Odlyzko’s data set for the Riemann zeros. *Proceedings of the Royal Society A*, 471:1–21, 2015. arXiv:1506.06531. 22
- [28] Guilherme França and André LeClair. Statistical and other properties of Riemann zeros based on an explicit equation for the n -th zero on the critical line. *Preprint*, pages 1–26, 2014. arXiv:1307.8395. 22
- [29] Vincent Garcia, Eric Debreuve, and Michel Barlaud. Fast k nearest neighbor search using GPU. In *IEEE Computer Society Conference on Computer Vision and Pattern Recognition Workshops*, Anchorage, AK, 2008. 40, 84
- [30] Minas Gjoka, Emily Smith, and Carter Butts. Estimating clique composition and size distributions from sampled network data. *Preprint*, pages 1–9, 2013. arXiv:1308.3297. 64
- [31] B.V. Gnedenko and A. N. Kolmogorov. *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, 1954. 42
- [32] Michel Goemans and Jan Vondrák. Stochastic covering and adaptivity. In *Proceedings of the 7th Latin American Theoretical Informatics Symposium*, pages 532–543, Valdivia, Chile, 2006. 62
- [33] M. Golzy, M. Markatou, and Arti Shivram. Algorithms for clustering on the sphere: Advances & applications. In *Proceedings of the World Congress on Engineering and Computer Science*, volume 1, pages 1–6, San Francisco, USA, 2016. 61
- [34] R. Goodman. *Introduction to Stochastic Models*. Dover, second edition, 2006. 8
- [35] Vincent Granville. Estimation of the intensity of a Poisson point process by means of nearest neighbor distances. *Statistica Neerlandica*, 52(2):112–124, 1998. 14
- [36] Vincent Granville. *Applied Stochastic Processes, Chaos Modeling, and Probabilistic Properties of Numeration Systems*. Data Science Central, 2018. 9, 17, 42
- [37] Vincent Granville. *Statistics: New Foundations, Toolbox, and Machine Learning Recipes*. Data Science Central, 2019. 25, 28, 39
- [38] Vincent Granville, Mirko Krivanek, and Jean-Paul Rasson. Simulated annealing: A proof of convergence. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 16:652–656, 1996. 40
- [39] Peter Hall. *Introduction to the theory of coverage processes*. Wiley, 1988. 62
- [40] K. Hartmann, J. Krois, and B. Waske. *Statistics and Geospatial Data Analysis*. Freie Universität Berlin, 2018. E-Learning Project SOGA. 31
- [41] Jane Hawkins. *Ergodic Dynamics: From Basic Theory to Applications*. Springer, 2021. 9
- [42] Nicholas J. Higham. *Accuracy and Stability of Numerical Algorithms*. Society for Industrial and Applied Mathematics, 2002. 57
- [43] Zhiqiu Hu and Rong-Cai Yang. A new distribution-free approach to constructing the confidence region for multiple parameters. *PLOS One*, 8(12), 2013. 28
- [44] Aleksandar Ivić. *The Riemann’s Zeta Function: Theory and Applications*. Dover, reprint edition, 2003. 22
- [45] Timothy D. Johnson. Introduction to spatial point processes. *Preprint*, page 2008. NeuroImaging Statistics Oxford (NISOx) group. 13
- [46] Richard Kershner. The number of circles covering a set. *American Journal of Mathematics*, 61(2):665–671, 1939. 62
- [47] Michael A. Klatt, Jaeuk Kim, and Salvatore Torquato. Cloaking the underlying long-range order of randomly perturbed lattices. *Physical Review Series E*, 101(3):1–10, 2020. 53
- [48] Denis Kojevnikov, Vadim Marmer, and Kyungchul Song. Limit theorems for network dependent random variables. *Journal of Econometrics*, 222(2):419–427, 2021. 13
- [49] Samuel Kotz, Tomasz Kozubowski, and Krzysztof Podgorski. *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance*. Springer, 2001. 58
- [50] K. Krishnamoorthy. *Handbook of Statistical Distributions with Applications*. Routledge, second edition, 2015. 74
- [51] Faraj Lagum. *Stochastic Geometry-Based Tools for Spatial Modeling and Planning of Future Cellular Networks*. PhD thesis, Carleton University, 2018. 13
- [52] Günther Last and Mathew Penrose. *Lectures on the Poisson Process*. Cambridge University Press, 2017. 13
- [53] André LeClair. Riemann hypothesis and random walks: The zeta case. *Symmetry*, 13:1–13, 2021. 22
- [54] G. Last M.A. Klatt and D. Yogeshwaran. Hyperuniform and rigid stable matchings. *Random Structures and Algorithms*, 2:439–473, 2020. 13

- [55] J. Mateu, C. Comas, and M.A. Calduch. Testing for spatial stationarity in point patterns. In *International Workshop on Spatio-Temporal Modeling*, 2010. 9
- [56] Jorge Mateu, Frederic P Schoenberg, and David M Diez. On distances between point patterns and their applications. *Preprint*, pages 1–29, 2010. 13
- [57] Natarajan Meghanathan. Distribution of maximal clique size of the vertices for theoretical small-world networks and real-world networks. *Preprint*, pages 1–20, 2015. arXiv:1508.01668. 64
- [58] Jesper Møller. Introduction to spatial point processes and simulation-based inference. In *International Center for Pure and Applied Mathematics (Lecture Notes)*, Lomé, Togo, 2018. 13, 25, 33
- [59] Jesper Møller and Frederic Paik Schoenberg. Thinning spatial point processes into Poisson processes. *Random Structures and Algorithms*, 42:347–358, 2010. 10
- [60] Jesper Møller and Rasmus P. Waagepetersen. *An Introduction to Simulation-Based Inference for Spatial Point Processes*. Springer, 2003. 13
- [61] Jesper Møller and Rasmus P. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. CRC Press, 2007. 13
- [62] S. Ghosh N., Miyoshi, and T. Shirai. Disordered complex networks: energy optimal lattices and persistent homology. *Preprint*, pages 1–44, 2020. arXiv:2009.08811. 5
- [63] Saralees Nadarajah. A modified Bessel distribution of the second kind. *Statistica*, 67(4):405–413, 2007. 58
- [64] Melvyn B. Nathanson. *Additive Number Theory: The Classical Bases*. Springer, reprint edition, 2010. 63
- [65] D Noviyanti and H P Lestari. The study of circumsphere and insphere of a regular polyhedron. *Journal of Physics: Conference Series*, 1581:1–10, 2020. 61
- [66] Yosihiko Ogata. Cluster analysis of spatial point patterns: posterior distribution of parents inferred from offspring. *Japanese Journal of Statistics and Data Science*, 3:367–390, 2020. 13
- [67] Vamsi Paruchuri, Arjan Duresi, and Raj Jain. Optimized flooding protocol for ad hoc networks. *Preprint*, pages 1–10, 2003. arXiv:cs/0311013v1. 62
- [68] Yuval Peres and Allan Sly. Rigidity and tolerance for perturbed lattices. *Preprint*, pages 1–20, 2020. arXiv:1409.4490. 5, 13
- [69] Brian Ripley. *Stochastic Simulation*. Wiley, 1987. 74
- [70] Peter Shirley and Chris Wyman. Generating stratified random lines in a square. *Journal of Computer Graphics Techniques*, 6(2):48–54, 2017. 61
- [71] Karl Sigman. Notes on the Poisson process. New York NY, 2009. IEOR 6711: Columbia University course. 9, 13
- [72] Luuk Spreuwers. *Image Filtering with Neural Networks: Applications and Performance Evaluation*. PhD thesis, University of Twente, 1992. 40
- [73] J. Michael Steele. Le Cam’s inequality and Poisson approximations. *The American Mathematical Monthly*, 101(1):48–54, 1994. 19, 52
- [74] Dietrich Stoyan, Wilfrid S. Kendall, Sung Nok Chiu, and Joseph Mecke. *Stochastic Geometry and Its Applications*. Wiley, 2013. 62
- [75] Anna Talgat, Mustafa A. Kishk, and Mohamed-Slim Alouini. Nearest neighbor and contact distance distribution for binomial point process on spherical surfaces. *IEEE Communications Letters*, 24(12):2659–2663, 2020. 61
- [76] Gerald Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. American Mathematical Society, third edition, 2015. 17
- [77] Remco van der Hofstad. *Random Graphs and Complex Networks*. Cambridge University Press, 2016. 64
- [78] Robert Williams. *The Geometrical Foundation of Natural Structure: A Source Book of Design*. Dover, 1979. 62
- [79] Oren Yakir. Recovering the lattice from its random perturbations. *Preprint*, pages 1–18, 2020. arXiv:2002.01508. 13, 53
- [80] Ruqiang Yan, Yongbin Liub, and Robert Gao. Permutation entropy: A nonlinear statistical measure for status characterization of rotary machines. *Mechanical Systems and Signal Processing*, 29:474–484, 2012. 17
- [81] D. Yogeshwaran. Geometry and topology of the boolean model on a stationary point processes : A brief survey. *Preprint*, pages 1–13, 2018. Researchgate. 13
- [82] Tonglin Zhang. A Kolmogorov-Smirnov type test for independence between marks and points of marked point processes. *Electronic Journal of Statistics*, 8(2):2557–2584, 2014. 30

Index

- m*-interlacing, 11, 24, 34–38, 40, 61, 63, 69, 75, 76
- m*-mixture, 35–38, 60, 61, 63
- anisotropy, 10, 17, 37
- attraction (point process), 7, 16
- attractor (distribution), 38, 42, 47, 64
- Berry-Esseen theorem, 67
- Bessel function, 58
- Beta function, 15
- bias, 44
- binomial distribution, 7, 38, 60
- boundary effect, 10–12, 17, 24, 25, 27, 29, 30, 32, 34, 37, 38, 44, 46, 52, 60, 61, 64, 67, 76, 82
- Brownian motion, 23, 41
- Cauchy distribution, 42, 51
- censored data, 11, 44, 60
- central limit theorem, 26, 42, 52, 56
 - multivariate, 67
- chaotic convergence, 21, 65
- characteristic function, 58, 67
- chi-squared distribution, 67
- child process, 13, 75
- clique (graph theory), 64
- cluster process, 11, 13, 36, 37
 - on the sphere, 61
- clustering, 40
 - fractal clustering, 24, 70
 - fuzzy, 24
 - GPU-based, 24, 36
 - supervised, 23, 36
 - unsupervised, 23, 36
- Cochran’s theorem, 67
- confidence band, 39
- confidence interval, 32, 37
- confidence level, 27
- confidence region, 27, 32, 66, 68
 - dual region, 24, 27, 68
- connected components, 12, 37, 38, 48, 61, 63, 65, 69, 77, 79
- contour line, 26
- convergence acceleration, 65
- convolution of distributions, 51, 57, 58
- counting measure, 7
- covariance matrix, 67
- covering (stochastic), 62
- cross-validation, 28
- data animation, 24
- degrees of freedom, 67
- density estimation, 14
- deviate, 74
- Dirichlet eta function, 18, 20, 22, 43, 66, 70
- distribution
 - binomial, 7, 38, 60
 - Cauchy, 42, 51, 52, 74
 - chi-squared, 67
 - empirical, 54
 - exponential-binomial, 5, 49
 - Fréchet, 23, 42
 - Gaussian, 67
 - generalized logistic, 5, 13–15, 33, 56, 65
 - half-logistic, 15
 - Hotelling, 26
 - Laplace, 54, 55, 58, 65
 - location-scale, 6, 10
 - logistic, 11, 14, 74
 - Lévy, 42
 - metalog, 15
 - modified Bessel, 58
 - Poisson, 18
 - Poisson-binomial, 5, 7, 13, 18, 47
 - Rayleigh, 38, 47, 61
 - stable distribution, 58
 - triangular, 57
 - truncated, 51, 57
 - uniform, 53, 74
 - Weibull, 23, 38, 42, 47
- domain of attraction, 64
- dual confidence region, 24, 27, 68
- dynamical systems, 9, 23, 43, 48, 64
- edge (graph theory), 63
- edge effect (statistics), 11, 44
- elbow rule, 11, 30, 36, 38, 41
- empirical distribution, 8, 17, 24, 31, 33, 38, 46, 48, 50, 54, 64
- entropy, 17
- ergodicity, 9, 32, 33, 37, 48, 59
- extreme values, 42, 46
- filtering (image processing), 22–24, 40
- fixed point algorithm, 48
- Fourier transform, 58
- fractal clustering, 22, 24, 70
- fractal dimension, 43
- Fréchet distribution, 23, 42
- Gamma function, 42
- Gaussian distribution, 67
 - multivariate, 67
- GPU-based clustering, 23, 24, 36, 40
- graph, 12, 64
 - connected components, 12, 37, 63, 69, 77, 79
 - edge, 12
 - nearest neighbor graph, 22, 37, 65, 69, 79
 - node, 12, 64
 - path, 12
 - random graph, 64
 - random nearest neighbor graph, 64
 - undirected, 12, 37, 38, 63–65, 69, 79
 - vertex, 12
- graph theory, 12, 63
- grid, 5, 6
- hash table, 16, 69, 77
- hexagonal lattice, 13
- hidden model, 13, 16, 33, 46, 52
- high precision computing, 28

- histogram equalization, 40
- homogeneity, 9, 11, 14, 38, 61
- Hotelling distribution, 26, 67
- identifiability, 10, 15, 24, 33, 34, 61
- independent increments, 9, 37
- index, 16, 28, 31, 52
 - index discrepancy, 17
 - index process, 16
 - index space, 6, 11, 31, 44, 46, 52, 76, 82
- inhibition (point process), 5
- intensity function, 6, 14, 15, 23, 25, 32, 37, 38, 40, 41, 61, 69
- interarrival times, 7, 9, 25, 32, 33, 37, 41, 49, 57, 59, 64, 69, 71
 - standardized, 58
- interlaced processes, 11, 36
- inverse model, 13
- inverse transform sampling, 14, 54, 65, 74
- Kolmogorov-Smirnov test, 30, 64
- Laplace distribution, 58
- lattice, 5, 6, 10, 13
 - Bravais lattice, 62
 - congruent lattices, 63
 - hexagonal, 13, 62
 - lattice group (group theory), 63
 - lattice index, 76
 - lattice space, 10–12, 16, 35, 36, 46, 47, 61, 63, 78
 - perturbed lattice, 5
 - semi-regular, 63
 - shifted, 12, 13
 - stretched, 12
 - vertex, 13
- law of large numbers, 26
- Le Cam's theorem, 13, 19, 52, 58
- location-scale distribution, 6, 10
- logistic map, 48
- Lévy distribution, 42
- Lévy flight, 42
- Mahalanobis transformation, 11
- marked point process, 10
- metalog distribution, 15
- minimum contrast estimation, 24, 25, 33
- mixture model, 11, 36
- model-free inference, 28
- modulo operator (point processes), 36, 38, 40, 63, 76, 78, 83
- moment generating function, 8, 15
- nearest neighbors, 5, 32, 37, 44, 48, 50, 61, 64, 77
 - nearest neighbor distances, 24, 35, 37, 39, 44, 47, 61, 63, 69, 75, 77
 - nearest neighbor graph, 22, 37, 65, 69, 79
 - random nearest neighbor graph, 64
- neural network, 22–24, 40
- normal distribution, 67
 - multivariate, 67
- numerical stability, 28, 48, 53, 57, 66
- order statistics, 46, 53
- outliers, 46
- overfitting, 33
- palette optimization, 24
- parent process, 13, 37, 74
- partition, 60
- permutation
 - entropy, 17
 - random permutation, 13, 17
- perturbed lattice process, 5
- point count distribution, 5, 7, 24, 30, 32, 33, 37, 38, 44, 49, 59, 60, 62, 64, 67, 69, 71
- point distribution, 7, 25, 37, 76
- point process, 6
 - anisotropic, 10
 - attractive, 7
 - binomial, 5, 7
 - cluster process, 13
 - child process, 13
 - Matérn, 13
 - Neyman-Scott, 13
 - parent process, 13
 - ergodic, 9
 - intensity, 69
 - interarrival times, 69
 - interlaced, 11
 - marked process, 10
 - mixture, 11, 36, 60, 63
 - non-homogeneous, 14
 - perturbed lattice process, 5, 12, 13, 16
 - point count distribution, 5, 60
 - Poisson, 5, 18, 46, 51
 - non homogeneous, 9
 - Poisson-binomial, 5, 6, 18
 - radial, 13
 - renewal process, 9, 13
 - repulsive, 7, 15
 - shifted, 10, 36
 - stationary, 6, 8
 - stretched, 10, 12
 - superimposed, 11, 36, 63
 - thinned, 10
- point process operations, 10, 13
- Poisson distribution, 18
- Poisson point process, 18
- Poisson-binomial distribution, 18, 47
- Poisson-binomial point process, 18
 - standardized, 10
- proxy space, 26
- pseudo-random number generator, 28, 48
- quantile, 14
 - fundamental theorem, 15, 54
 - quantile function, 6, 13, 14, 38, 54, 57, 65
- radial distribution, 13, 36
- random function, 13, 18, 20, 23
- random graph, 64
- random lines, 61
- random numbers, 48

- random permutation, 17
- random walk, 42
- Rayleigh distribution, 38, 47, 61
- Rayleigh test, 38
- records, 46
- renewal process, 9, 13
- repulsion (point process), 5, 7, 15, 39
- resampling, 28, 39
- Riemann hypothesis, 21, 22
- Riemann zeta function, 13, 18, 20, 43

- sample size, 27
- scaling factor, 5, 6, 11, 12, 15, 23, 25, 32, 37, 44, 46, 48, 59, 61–64, 69, 74, 75
- shift vector, 11, 36, 38, 40, 41, 63, 75, 83
- shifted process, 10, 40, 41, 63
- simulation, 27, 48
- spatial process, 36
- spatial statistics, 13
- stable distribution, 42, 52, 58
- standardized arrival times, 58
- standardized point process, 10, 38
- state space, 6, 16, 23, 32, 36–38, 40, 44, 46, 51, 82, 83
- stationarity, 6, 8, 11, 24, 29, 59, 63
- stochastic convergence, 24
- stochastic geometry, 61, 62
- stochastic residues, 36
- stretching (point process), 10, 12, 38, 75
- superimposition (point processes), 11, 36
- symbolic math, 48

- tessellation, 62
- thinning (point process), 10
- tiling (spatial processes), 63
- training set, 23
- transcendental number, 48
- truncated distribution, 51, 57

- vertex (graph theory), 12, 61, 63, 64
- visualization, 12
- Voronoi tessellation, 62

- Weibull distribution, 23, 38, 42, 47
- Wiener process, 41