Glossary

m-interlacing	Superimposition of m Poisson-binomial processes, each with its own (shifted, stretched) *lattice space*, and its own intensity and scaling factor. See pages 11, 24, 34, 35, 36, 37, 38, 61, 63, 69, 75, 76
Anisotropy	A property of a point process: the points are evenly scattered in all directions. The point distribution is stochastically invariant under rotations. See pages 10, 37
Attraction / Repulsion	The larger the scaling factor s, the less repulsion (also called *inhibition*) among the points of the process. Cluster processes discussed here exhibit both attraction (points tend to cluster together) and repulsion among cluster centers (due to the underlying lattice structure). See pages 7, 16, 37
Boundary Effect	Also called edge effect. Bias in estimated point counts or nearest neighbor distances, due to unobserved points located outside but close to the finite window of observations. Also the result of missing points in the window of observations, in simulated point processes especially when the scaling factor s is large. Special techniques are used to handle this problem. See pages 10, 11, 12, 24, 25, 27, 29, 30, 32, 37, 38, 44, 46, 52, 60, 61, 64, 67, 76
Confidence Region	A confidence region of level γ is a 2-D set of minimum area covering a proportion γ of the mass of a bivariate probability distribution. See pages 24, 27, 32, 66, 68
Connected Component	A set of vertices in a graph that are connected to each other by paths. See also nearest neighbor graph. See pages 12, 37, 38, 48, 61, 63, 65, 69, 77, 79
Empirical Distribution	Cumulative frequency histogram attached to a statistic (for instance, nearest neighbor distances), and based on observations. When the number of observations tends to infinity and the bin sizes tends to zero, this step function tends to the theoretical cumulative distribution function of the statistic in question. See pages 8, 17, 24, 31, 33, 38, 46, 48, 50, 54, 64
Ergodicity	A statistic such as the interarrival times is ergodic if it has the same asymptotic distribution, whether it is computed on many observations from a single realization of the process, or averaged across many realizations, each with few observations. See pages 9, 32, 33, 37, 48, 59
Homogeneity	A property of a point process, characterized by an homogeneous intensity function, that is, constant or independent of the location. See pages 9, 10, 11, 61
Identifiability	A models is identifiable if it is uniquely defined by its parameters. Then it is possible to estimate each parameter separately. A trivial example of non-identifiability is when we have two parameters, say α, β, but they only occur in a product $\alpha\beta$. In that case, if $\alpha\beta = 6$, it is impossible to tell whether $\alpha = 2, \beta = 3$ or $\alpha = 1, \beta = 6$. See pages 10, 15, 24, 33, 34, 61
Index Space	Consists of the indices $h, k \in \mathbb{Z}$, attached to the points X_k in one dimension, or (X_h, Y_k) in two dimensions. See pages 6, 10, 11, 31, 44, 46, 52, 76, 82
Intensity	Core parameter of the Poisson-binomial process. Denoted as λ. It represents the granularity of the underlying lattice, that is, the point density. In d dimensions, $\mathrm{E}[N(B)] = 1$ for any hypercube B of length $1/\lambda$. Here N is the point count. When λ is constant (not depending on the location), the process is homogeneous. See pages 6, 15, 23, 25, 32, 37, 38, 40, 41, 61, 69
Interarrival Time	In one dimension, random variable measuring the distance between a point of the process and its closest neighbor to the right, on the real axis. Interarrival times are also called *increments*. See pages 7, 9, 25, 32, 33, 37, 41, 49, 59, 60, 64, 69, 71
Lattice Space	In two dimensions, it consists of the locations $(h/\lambda, k/\lambda)$ with $h, k \in \mathbb{Z}$. The distribution of a point (X_h, Y_k) is centered at $(h/\lambda, k/\lambda)$. The concept can be extended to any dimension. See pages 6, 10, 11, 12, 16, 35, 36, 46, 47, 61, 62, 63, 78
Location-scale	A random variable X has a location-scale distribution with two parameters, the scale s and location μ, if any linear transformation $a + bX$ has a distribution of the same family, with parameters respectively $b^2 s$ and $\mu + a$. Here μ is the expectation and s is proportional to the variance of the distribution. See pages 6, 10

Modulo Operator — Sometimes, it is useful to work with point "residues" modulo $\frac{1}{\lambda}$, instead of the original points, due to the nature of the underlying lattice. It magnifies the patterns of the point process. By definition, $X_k \bmod \frac{1}{\lambda} = X_k - \frac{1}{\lambda}\lfloor \lambda X_k \rfloor$ where the brackets represent the integer part function. See pages 36, 38, 40, 63, 76, 78, 83

NN Graph — Nearest neighbor graph. The vertices are the points of the process. Two vertices (the points they represent) are connected if at least one of the two points is nearest neighbor to the other one. This graph is undirected. See pages 22, 37, 65, 69, 79, 89

Point Count — Random variable, denoted as $N(B)$, counting the number of points of the process in a particular set B, typically an interval $[a, b]$ in one dimension, and a square or circle in two dimensions. See pages 5, 7, 24, 30, 32, 33, 37, 38, 44, 49, 59, 60, 62, 64, 67, 69, 71

Point Distribution — Random variable representing how a point of the process is distributed in a domain B; for instance, for a stationary Poisson process, points are uniformly distributed on any compact domain B (say, an interval in one dimension, or a square in two dimensions). See pages 7, 25, 37, 76

Quantile function — Inverse of the cumulative distribution function (CDF) F, denoted as Q. Thus if $P(X < x) = F(x)$, then $P(X < Q(x)) = x$. See pages 6, 13, 14, 38, 54, 57, 65

Scaling Factor — Core parameter of the Poisson-binomial process. Denoted as s, proportional to the variance of the distribution F attached to the points of the process. It measures the level of repulsion among the points (maximum if $s = 0$, minimum if $s = \infty$). In d dimensions, the process is stationary Poisson of intensity λ^d if $s = \infty$, and coincides with the fixed *lattice space* if $s = 0$. See pages 5, 6, 10, 11, 12, 15, 23, 25, 32, 37, 48, 59, 61, 62, 63, 64, 69, 74, 75

Shift vector — The lattice attached to a 2-D Poisson-binomial process consists of the vertices $(\frac{h}{\lambda}, \frac{k}{\lambda})$ with $h, k \in \mathbb{Z}$. A shifted process has its lattice translated by a shift vector (u, v). The new vertices are $(u + \frac{h}{\lambda}, v + \frac{k}{\lambda})$. See page 11, 36, 38, 40, 41, 63, 75, 83

Standardized Process — Poisson-binomial process with intensity $\lambda = 1$, scaling factor $s = 1$, and shifted (if necessary) so that the lattice space coincides with $\mathbb{Z}$ or $\mathbb{Z}^2$. See page 10

State Space — Space where the points of the process are located. Here, $\mathbb{R}$ or $\mathbb{R}^2$. See also *index space* and *lattice space*. See pages 6, 16, 23, 32, 36, 37, 38, 40, 44, 46, 51, 63, 82, 83

Stationarity — Property of a point process: the point distributions in two sets of same shape and area, are identical. The process is stochastically invariant under translations. See pages 6, 8, 11, 24, 29, 63

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