Glossary

m-interlacing	Superimposition of m Poisson-binomial processes, each with its own (shifted, stretched) *lattice space*, and its own intensity and scaling factor. See pages 11, 24, 34, 35, 36, 37, 38, 61, 63, 69, 75, 76
Anisotropy	A property of a point process: the points are evenly scattered in all directions. The point distribution is stochastically invariant under rotations. See pages 10, 37
Attraction / Repulsion	The larger the scaling factor s, the less repulsion (also called *inhibition*) among the points of the process. Cluster processes discussed here exhibit both attraction (points tend to cluster together) and repulsion among cluster centers (due to the underlying lattice structure). See pages 7, 16, 37
Boundary Effect	Also called edge effect. Bias in estimated point counts or nearest neighbor distances, due to unobserved points located outside but close to the finite window of observations. Also the result of missing points in the window of observations, in simulated point processes especially when the scaling factor s is large. Special techniques are used to handle this problem. See pages 10, 11, 12, 24, 25, 27, 29, 30, 32, 37, 38, 44, 46, 52, 60, 61, 64, 67, 76
Confidence Region	A confidence region of level γ is a 2-D set of minimum area covering a proportion γ of the mass of a bivariate probability distribution. See pages 24, 27, 32, 66, 68
Connected Component	A set of vertices in a graph that are connected to each other by paths. See also nearest neighbor graph. See pages 12, 37, 38, 48, 61, 63, 65, 69, 77, 79
Empirical Distribution	Cumulative frequency histogram attached to a statistic (for instance, nearest neighbor distances), and based on observations. When the number of observations tends to infinity and the bin sizes tends to zero, this step function tends to the theoretical cumulative distribution function of the statistic in question. See pages 8, 17, 24, 31, 33, 38, 46, 48, 50, 54, 64
Ergodicity	A statistic such as the interarrival times is ergodic if it has the same asymptotic distribution, whether it is computed on many observations from a single realization of the process, or averaged across many realizations, each with few observations. See pages 9, 32, 33, 37, 48, 59
Homogeneity	A property of a point process, characterized by an homogeneous intensity function, that is, constant or independent of the location. See pages 9, 10, 11, 61
Identifiability	A models is identifiable if it is uniquely defined by its parameters. Then it is possible to estimate each parameter separately. A trivial example of non-identifiability is when we have two parameters, say α, β, but they only occur in a product $\alpha\beta$. In that case, if $\alpha\beta = 6$, it is impossible to tell whether $\alpha = 2, \beta = 3$ or $\alpha = 1, \beta = 6$. See pages 10, 15, 24, 33, 34, 61
Index Space	Consists of the indices $h, k \in \mathbb{Z}$, attached to the points X_k in one dimension, or (X_h, Y_k) in two dimensions. See pages 6, 10, 11, 31, 44, 46, 52, 76, 82
Intensity	Core parameter of the Poisson-binomial process. Denoted as λ. It represents the granularity of the underlying lattice, that is, the point density. In d dimensions, $\mathrm{E}[N(B)] = 1$ for any hypercube B of length $1/\lambda$. Here N is the point count. When λ is constant (not depending on the location), the process is homogeneous. See pages 6, 15, 23, 25, 32, 37, 38, 40, 41, 61, 69
Interarrival Time	In one dimension, random variable measuring the distance between a point of the process and its closest neighbor to the right, on the real axis. Interarrival times are also called *increments*. See pages 7, 9, 25, 32, 33, 37, 41, 49, 59, 60, 64, 69, 71
Lattice Space	In two dimensions, it consists of the locations $(h/\lambda, k/\lambda)$ with $h, k \in \mathbb{Z}$. The distribution of a point (X_h, Y_k) is centered at $(h/\lambda, k/\lambda)$. The concept can be extended to any dimension. See pages 6, 10, 11, 12, 16, 35, 36, 46, 47, 61, 62, 63, 78
Location-scale	A random variable X has a location-scale distribution with two parameters, the scale s and location μ, if any linear transformation $a + bX$ has a distribution of the same family, with parameters respectively $b^2 s$ and $\mu + a$. Here μ is the expectation and s is proportional to the variance of the distribution. See pages 6, 10

Modulo Operator	Sometimes, it is useful to work with point "residues" modulo $\frac{1}{\lambda}$, instead of the original points, due to the nature of the underlying lattice. It magnifies the patterns of the point process. By definition, $X_k \bmod \frac{1}{\lambda} = X_k - \frac{1}{\lambda}\lfloor \lambda X_k \rfloor$ where the brackets represent the integer part function. See pages 36, 38, 40, 63, 76, 78, 83
NN Graph	Nearest neighbor graph. The vertices are the points of the process. Two vertices (the points they represent) are connected if at least one of the two points is nearest neighbor to the other one. This graph is undirected. See pages 22, 37, 65, 69, 79, 89
Point Count	Random variable, denoted as $N(B)$, counting the number of points of the process in a particular set B, typically an interval $[a, b]$ in one dimension, and a square or circle in two dimensions. See pages 5, 7, 24, 30, 32, 33, 37, 38, 44, 49, 59, 60, 62, 64, 67, 69, 71
Point Distribution	Random variable representing how a point of the process is distributed in a domain B; for instance, for a stationary Poisson process, points are uniformly distributed on any compact domain B (say, an interval in one dimension, or a square in two dimensions). See pages 7, 25, 37, 76
Quantile function	Inverse of the cumulative distribution function (CDF) F, denoted as Q. Thus if $P(X < x) = F(x)$, then $P(X < Q(x)) = x$. See pages 6, 13, 14, 38, 54, 57, 65
Scaling Factor	Core parameter of the Poisson-binomial process. Denoted as s, proportional to the variance of the distribution F attached to the points of the process. It measures the level of repulsion among the points (maximum if $s = 0$, minimum if $s = \infty$). In d dimensions, the process is stationary Poisson of intensity λ^d if $s = \infty$, and coincides with the fixed *lattice space* if $s = 0$. See pages 5, 6, 10, 11, 12, 15, 23, 25, 32, 37, 48, 59, 61, 62, 63, 64, 69, 74, 75
Shift vector	The lattice attached to a 2-D Poisson-binomial process consists of the vertices $(\frac{h}{\lambda}, \frac{k}{\lambda})$ with $h, k \in \mathbb{Z}$. A shifted process has its lattice translated by a shift vector (u, v). The new vertices are $(u + \frac{h}{\lambda}, v + \frac{k}{\lambda})$. See page 11, 36, 38, 40, 41, 63, 75, 83
Standardized Process	Poisson-binomial process with intensity $\lambda = 1$, scaling factor $s = 1$, and shifted (if necessary) so that the lattice space coincides with $\mathbb{Z}$ or $\mathbb{Z}^2$. See page 10
State Space	Space where the points of the process are located. Here, $\mathbb{R}$ or $\mathbb{R}^2$. See also *index space* and *lattice space*. See pages 6, 16, 23, 32, 36, 37, 38, 40, 44, 46, 51, 63, 82, 83
Stationarity	Property of a point process: the point distributions in two sets of same shape and area, are identical. The process is stochastically invariant under translations. See pages 6, 8, 11, 24, 29, 63

List of Figures

1 Convergence to stationary Poisson point process of intensity λ . . . 8
2 Four superimposed Poisson-binomial processes: $s = 0$ (left), $s = 5$ (right) . . . 12
3 Radial cluster process ($s = 0.2, \lambda = 1$) with centers in blue; zoom in on the left . . . 15
4 Radial cluster process ($s = 2, \lambda = 1$) with centers in blue; zoom in on the left . . . 16
5 Manufactured marble lacking true lattice randomness (left) . . . 16
6 Locally random permutation σ; $\tau(k)$ is the index of X_k's closest neighbor to the right . . . 17
7 Chaotic function (bottom), and its transform (top) showing the global minimum . . . 18
8 Orbit of η in the complex plane (left), perturbed by a Poisson-binomial process (right) . . . 21
9 Data animations – click on a picture to start a video . . . 22
10 Minimum contrast estimation for (λ, s) . . . 25
11 Confidence region for (p, q) – Hotelling's quantile function on the left . . . 27
12 Period and amplitude of $\phi_\tau(t)$; here $\tau = 1, \lambda = 1.4, s = 0.3$ . . . 29
13 Bias reduction technique to minimize boundary effects . . . 29
14 A new test of independence (R-squared version) . . . 30
15 Radial cluster process ($s = 0.5, \lambda = 1$) with centers in blue; zoom in on the left . . . 35
16 Radial cluster process ($s = 1, \lambda = 1$) with centers in blue; zoom in on the left . . . 35
17 Realization of a 5-interlacing with $s = 0.15$ and $\lambda = 1$: original (left), modulo $2/\lambda$ (right) . . . 36
18 Rayleigh test to assess if a point distribution matches that of a Poisson process . . . 39
19 Unsupervised (left) versus supervised clustering (right) of Figure 17 . . . 39

20 Elbow rule (right) finds $m = 3$ clusters in Brownian motion (left) 43
21 Elbow rule (right) finds $m = 8$ or $m = 11$ "jumps" in left plot 43
22 Each arrow links a point (blue) to its lattice index (red): $s = 0.2$ (left), $s = 1$ (right) 46
23 Distance between a point and its lattice location ($s = 1$) 47
24 Chaotic convergence of partial sums in Formula (19) 66

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Index

m-interlacing, 11, 24, 34–38, 40, 61, 63, 69, 75, 76
m-mixture, 35–38, 60, 61, 63

anisotropy, 10, 17, 37
attraction (point process), 7, 16
attractor (distribution), 38, 42, 47, 64

Berry-Esseen theorem, 67
Bessel function, 58
Beta function, 15
bias, 44
binomial distribution, 7, 38, 60
boundary effect, 10–12, 17, 24, 25, 27, 29, 30, 32, 34, 37, 38, 44, 46, 52, 60, 61, 64, 67, 76, 82
Brownian motion, 23, 41

Cauchy distribution, 42, 51
censored data, 11, 44, 60
central limit theorem, 26, 42, 52, 56
  multivariate, 67
chaotic convergence, 21, 65
characteristic function, 58, 67
chi-squared distribution, 67
child process, 13, 75
clique (graph theory), 64
cluster process, 11, 13, 36, 37
  on the sphere, 61
clustering, 40
  fractal clustering, 24, 70
  fuzzy, 24
  GPU-based, 24, 36
  supervised, 23, 36
  unsupervised, 23, 36
Cochran's theorem, 67
confidence band, 39
confidence interval, 32, 37
confidence level, 27
confidence region, 27, 32, 66, 68
  dual region, 24, 27, 68
connected components, 12, 37, 38, 48, 61, 63, 65, 69, 77, 79
contour line, 26
convergence acceleration, 65
convolution of distributions, 51, 57, 58
counting measure, 7
covariance matrix, 67
covering (stochastic), 62
cross-validation, 28

data animation, 24
degrees of freedom, 67
density estimation, 14
deviate, 74
Dirichlet eta function, 18, 20, 22, 43, 66, 70
distribution
  binomial, 7, 38, 60
  Cauchy, 42, 51, 52, 74
  chi-squared, 67
  empirical, 54
  exponential-binomial, 5, 49
  Fréchet, 23, 42
  Gaussian, 67
  generalized logistic, 5, 13–15, 33, 56, 65
  half-logistic, 15
  Hotelling, 26
  Laplace, 54, 55, 58, 65
  location-scale, 6, 10
  logistic, 11, 14, 74
  Lévy, 42
  metalog, 15
  modified Bessel, 58
  Poisson, 18
  Poisson-binomial, 5, 7, 13, 18, 47
  Rayleigh, 38, 47, 61
  stable distribution, 58
  triangular, 57
  truncated, 51, 57
  uniform, 53, 74
  Weibull, 23, 38, 42, 47
domain of attraction, 64
dual confidence region, 24, 27, 68
dynamical systems, 9, 23, 43, 48, 64

edge (graph theory), 63
edge effect (statistics), 11, 44
elbow rule, 11, 30, 36, 38, 41
empirical distribution, 8, 17, 24, 31, 33, 38, 46, 48, 50, 54, 64
entropy, 17
ergodicity, 9, 32, 33, 37, 48, 59
extreme values, 42, 46

filtering (image processing), 22–24, 40
fixed point algorithm, 48
Fourier transform, 58
fractal clustering, 22, 24, 70
fractal dimension, 43
Fréchet distribution, 23, 42

Gamma function, 42
Gaussian distribution, 67
  multivariate, 67
GPU-based clustering, 23, 24, 36, 40
graph, 12, 64
  connected components, 12, 37, 63, 69, 77, 79
  edge, 12
  nearest neighbor graph, 22, 37, 65, 69, 79
  node, 12, 64
  path, 12
  random graph, 64
    random nearest neighbor graph, 64
  undirected, 12, 37, 38, 63–65, 69, 79
  vertex, 12
graph theory, 12, 63
grid, 5, 6

hash table, 16, 69, 77
hexagonal lattice, 13
hidden model, 13, 16, 33, 46, 52
high precision computing, 28

histogram equalization, 40
homogeneity, 9, 11, 14, 38, 61
Hotelling distribution, 26, 67

identifiability, 10, 15, 24, 33, 34, 61
independent increments, 9, 37
index, 16, 28, 31, 52
  index discrepancy, 17
  index process, 16
  index space, 6, 11, 31, 44, 46, 52, 76, 82
inhibition (point process), 5
intensity function, 6, 14, 15, 23, 25, 32, 37, 38, 40, 41, 61, 69
interarrival times, 7, 9, 25, 32, 33, 37, 41, 49, 57, 59, 64, 69, 71
  standardized, 58
interlaced processes, 11, 36
inverse model, 13
inverse transform sampling, 14, 54, 65, 74

Kolmogorov-Smirnov test, 30, 64

Laplace distribution, 58
lattice, 5, 6, 10, 13
  Bravais lattice, 62
  congruent lattices, 63
  hexagonal, 13, 62
  lattice group (group theory), 63
  lattice index, 76
  lattice space, 10–12, 16, 35, 36, 46, 47, 61, 63, 78
  perturbed lattice, 5
  semi-regular, 63
  shifted, 12, 13
  stretched, 12
  vertex, 13
law of large numbers, 26
Le Cam's theorem, 13, 19, 52, 58
location-scale distribution, 6, 10
logistic map, 48
Lévy distribution, 42
Lévy flight, 42

Mahalanobis transformation, 11
marked point process, 10
metalog distribution, 15
minimum contrast estimation, 24, 25, 33
mixture model, 11, 36
model-free inference, 28
modulo operator (point processes), 36, 38, 40, 63, 76, 78, 83
moment generating function, 8, 15

nearest neighbors, 5, 32, 37, 44, 48, 50, 61, 64, 77
  nearest neighbor distances, 24, 35, 37, 39, 44, 47, 61, 63, 69, 75, 77
  nearest neighbor graph, 22, 37, 65, 69, 79
    random nearest neighbor graph, 64
neural network, 22–24, 40
normal distribution, 67
  multivariate, 67
numerical stability, 28, 48, 53, 57, 66

order statistics, 46, 53
outliers, 46
overfitting, 33

palette optimization, 24
parent process, 13, 37, 74
partition, 60
permutation
  entropy, 17
  random permutation, 13, 17
perturbed lattice process, 5
point count distribution, 5, 7, 24, 30, 32, 33, 37, 38, 44, 49, 59, 60, 62, 64, 67, 69, 71
point distribution, 7, 25, 37, 76
point process, 6
  anisotropic, 10
  attractive, 7
  binomial, 5, 7
  cluster process, 13
    child process, 13
    Matérn, 13
    Neyman-Scott, 13
    parent process, 13
  ergodic, 9
  intensity, 69
  interarrival times, 69
  interlaced, 11
  marked process, 10
  mixture, 11, 36, 60, 63
  non-homogeneous, 14
  perturbed lattice process, 5, 12, 13, 16
  point count distribution, 5, 60
  Poisson, 5, 18, 46, 51
    non homogeneous, 9
  Poisson-binomial, 5, 6, 18
  radial, 13
  renewal process, 9, 13
  repulsive, 7, 15
  shifted, 10, 36
  stationary, 6, 8
  stretched, 10, 12
  superimposed, 11, 36, 63
  thinned, 10
point process operations, 10, 13
Poisson distribution, 18
Poisson point process, 18
Poisson-binomial distribution, 18, 47
Poisson-binomial point process, 18
  standardized, 10
proxy space, 26
pseudo-random number generator, 28, 48

quantile, 14
  fundamental theorem, 15, 54
  quantile function, 6, 13, 14, 38, 54, 57, 65

radial distribution, 13, 36
random function, 13, 18, 20, 23
random graph, 64
random lines, 61
random numbers, 48

random permutation, 17
random walk, 42
Rayleigh distribution, 38, 47, 61
Rayleigh test, 38
records, 46
renewal process, 9, 13
repulsion (point process), 5, 7, 15, 39
resampling, 28, 39
Riemann hypothesis, 21, 22
Riemann zeta function, 13, 18, 20, 43

sample size, 27
scaling factor, 5, 6, 11, 12, 15, 23, 25, 32, 37, 44, 46, 48, 59, 61–64, 69, 74, 75
shift vector, 11, 36, 38, 40, 41, 63, 75, 83
shifted process, 10, 40, 41, 63
simulation, 27, 48
spatial process, 36
spatial statistics, 13
stable distribution, 42, 52, 58
standardized arrival times, 58
standardized point process, 10, 38
state space, 6, 16, 23, 32, 36–38, 40, 44, 46, 51, 82, 83
stationarity, 6, 8, 11, 24, 29, 59, 63
stochastic convergence, 24
stochastic geometry, 61, 62
stochastic residues, 36
stretching (point process), 10, 12, 38, 75
superimposition (point processes), 11, 36
symbolic math, 48

tessellation, 62
thinning (point process), 10
tiling (spatial processes), 63
training set, 23
transcendental number, 48
truncated distribution, 51, 57

vertex (graph theory), 12, 61, 63, 64
visualization, 12
Voronoi tessellation, 62

Weibull distribution, 23, 38, 42, 47
Wiener process, 41