22/09 - Aula 16 - Derivando funções trigonométricas

Ex. Calcule a derivada da função $\boxed{y = x^{\frac{1}{x}}}$

Solução:

Método 1: Aplicando o logaritmo natural em ambos membros temos

$$\ln y(x) = \ln x^{\frac{1}{x}}$$

$$\Leftrightarrow \quad \ln y(x) = \frac{1}{x} \cdot \ln x$$

$$\Leftrightarrow \quad \boxed{\ln y(x) = \frac{\ln x}{x}}$$

Derivando em x obtemos

$$\frac{y'(x)}{y(x)} = \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{\cancel{x}} \cdot \cancel{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\Rightarrow \quad y'(x) = y(x) \cdot \left(\frac{1 - \ln x}{x^2}\right) = x^{\frac{1}{x}} \cdot \left(\frac{1 - \ln x}{x^2}\right)$$

Portanto,

$$\boxed{y'(x) = x^{\frac{1}{x}} \cdot \left(\frac{1 - \ln x}{x^2}\right)}$$

Método 2: Seja $u(x) = \frac{1}{x}$ então, $y = x^{u(x)}$

$$y' = \underbrace{\left(x^{u(x)}\right)'}_{\text{Considerando a base constante}} + \underbrace{\left(x^{u(x)}\right)'}_{\text{Considerando o expoente constante}}$$

Fato 1 $(x^a)' = a x^{a-1}$, $a \in \mathbb{R}$ constante

Fato 2 $(a^x)' = a^x \cdot \ln a$, $a > 0$ $\Rightarrow \underbrace{(a^{u(x)})' = a^{u(x)} \cdot \ln a \cdot u'(x)}_{\text{Regra da Cadeia}}$

Logo,

$$y' = (x^{u(x)})' \Rightarrow \quad y' = \overbrace{u(x) \cdot x^{u(x)-1}}^{\text{tombo}} + x^{u(x)} \cdot \ln x \cdot u'(x)$$

$$\Rightarrow \quad y' = \frac{1}{x} \cdot x^{\frac{1}{x}-1} + x^{\frac{1}{x}} \cdot \ln x \cdot \left(\frac{1}{x}\right)'$$

$$\Rightarrow \quad y' = \frac{x^{\frac{1}{x}}}{x^2} + x^{\frac{1}{x}} \ln x \cdot \underbrace{\frac{-1}{x^2}}_{} = x^{\frac{1}{x}} \left(\frac{1 - \ln x}{x^2}\right)$$

□

Teorema 12.1.

$$(\operatorname{sen} x)' = \cos x$$
$$(\cos x)' = -\operatorname{sen} x$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$ Se $f(x) = \operatorname{sen} x \Rightarrow$

$$(\operatorname{sen} x)' = \lim_{h\to 0}\frac{\operatorname{sen}(x+h)-\operatorname{sen} x}{h} =$$

$$= \lim_{h\to 0}\frac{\overbrace{\operatorname{sen} x\cdot\cos h+\operatorname{sen} h\cdot\cos x}^{\operatorname{sen}(x+h)}-\operatorname{sen} x}{h}$$

$$= \lim_{h\to 0}\left\{\operatorname{sen} x\cdot\left[\frac{\cos h-1}{h}\right]+\frac{\operatorname{sen} h}{h}\cdot\cos x\right\}$$

$$= \operatorname{sen} x\cdot\underbrace{\overset{\to 0}{\lim_{h\to 0}\left[\frac{\cos h-1}{h}\right]}}_{(I)}+\cos x\cdot\underbrace{\overset{\to 1}{\lim_{h\to 0}\frac{\operatorname{sen} h}{h}}}_{(II)}$$

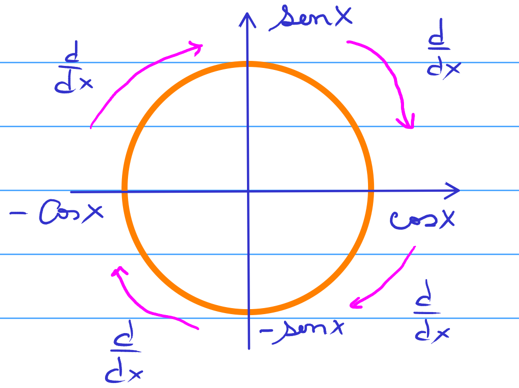
(I) $$\frac{\cos h-1}{h} = \frac{\cos h-1}{h}\cdot\frac{\cos h+1}{\cos h+1} = \frac{\cos^2 h-1}{h(\cos h+1)} = \frac{-\operatorname{sen}^2 h}{h\cdot(\cos h+1)} \overset{(*)}{=}$$

$\operatorname{sen}^2 h+\cos^2 h = 1 \Rightarrow \cos^2 h - 1 = -\operatorname{sen}^2 h$

$$\overset{(*)}{=} -\frac{\operatorname{sen} h}{h}\cdot\frac{\operatorname{sen} h}{\cos h+1}$$

Logo, $$\lim_{h\to 0}\frac{\cos h-1}{h} = -\underbrace{\lim_{h\to 0}\frac{\operatorname{sen} h}{h}}_{=1}\cdot\underbrace{\lim_{h\to 0}\frac{\operatorname{sen} h}{\cos h+1}}_{=\frac{\operatorname{sen} 0}{0+1}=\frac{0}{1}=0} = 0$$

(II) $$\lim_{h\to 0}\frac{\operatorname{sen} h}{h} = 1$$ (Limite Fundamental)



Logo,

$$(\operatorname{sen} x)' = \cos x$$

Analogamente, $(\cos x)' = -\operatorname{sen} x$

$$\frac{d}{dx}(-\operatorname{sen} x) = -\frac{d}{dx}\operatorname{sen} x = -\cos x$$

Proposição 12.1.

$$(\operatorname{tg} x)' = \sec^2 x$$
$$(\operatorname{cotg} x)' = -\operatorname{cosec}^2 x \quad \checkmark$$
$$(\sec x)' = \sec x \operatorname{tg} x$$
$$(\operatorname{cosec} x)' = -\operatorname{cosec} x \operatorname{cotg} x$$

• $(\operatorname{tg} x)' = \left(\dfrac{\operatorname{sen} x}{\cos x}\right)' = \dfrac{(\operatorname{sen} x)' \cos x - \operatorname{sen} x \cdot (\cos x)'}{\cos^2 x}$

$= \dfrac{\cos x \cdot \cos x - \operatorname{sen} x \cdot (-\operatorname{sen} x)}{\cos^2 x}$

$= \dfrac{\operatorname{sen}^2 x + \cos^2 x}{\cos^2 x} = \left(\dfrac{1}{\cos x}\right)^2 = (\sec x)^2 = \sec^2 x$

•• $(\operatorname{cotg} x)' = \left(\dfrac{1}{\operatorname{tg} x}\right)' = \dfrac{0 \cdot \operatorname{tg} x - 1 \cdot (\operatorname{tg} x)'}{\operatorname{tg}^2 x} = \dfrac{-\sec^2 x}{\operatorname{tg}^2 x}$

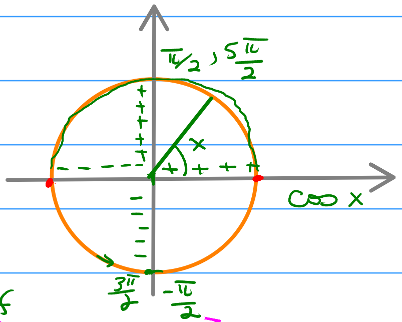
$= \dfrac{-\dfrac{1}{\cos^2 x}}{\dfrac{\operatorname{sen}^2 x}{\cos^2 x}} = \dfrac{-\cancel{\cos^2 x}}{\cancel{\cos^2 x} \cdot \operatorname{sen}^2 x} = \dfrac{-1}{\operatorname{sen}^2 x} = -\left(\dfrac{1}{\operatorname{sen} x}\right)^2 = -\operatorname{cossec}^2 x$

Aplicação: Esboce o gráfico da função : $f(x) = \operatorname{sen} x$

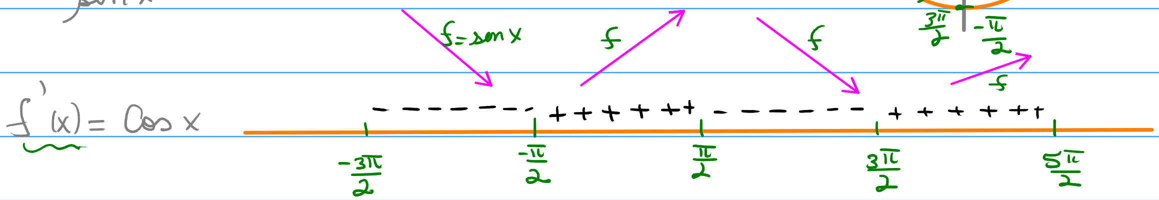
$f(x) = \operatorname{sen} x$ $\qquad\qquad$ $\cos x = 0 \iff x = \dfrac{\pi}{2} + k\pi$

$f'(x) = \cos x$

$f''(x) = -\operatorname{sen} x$



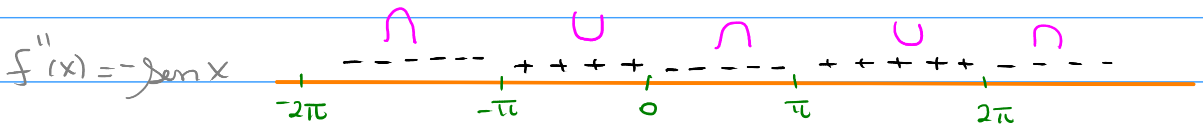
$\dfrac{3\pi}{2} + \pi = \dfrac{5\pi}{2}$



Pontos críticos de $f(x) = \operatorname{sen} x$ é $C = \{x \in \mathbb{R} \mid (\operatorname{sen} x)' = \cos x = 0\}$

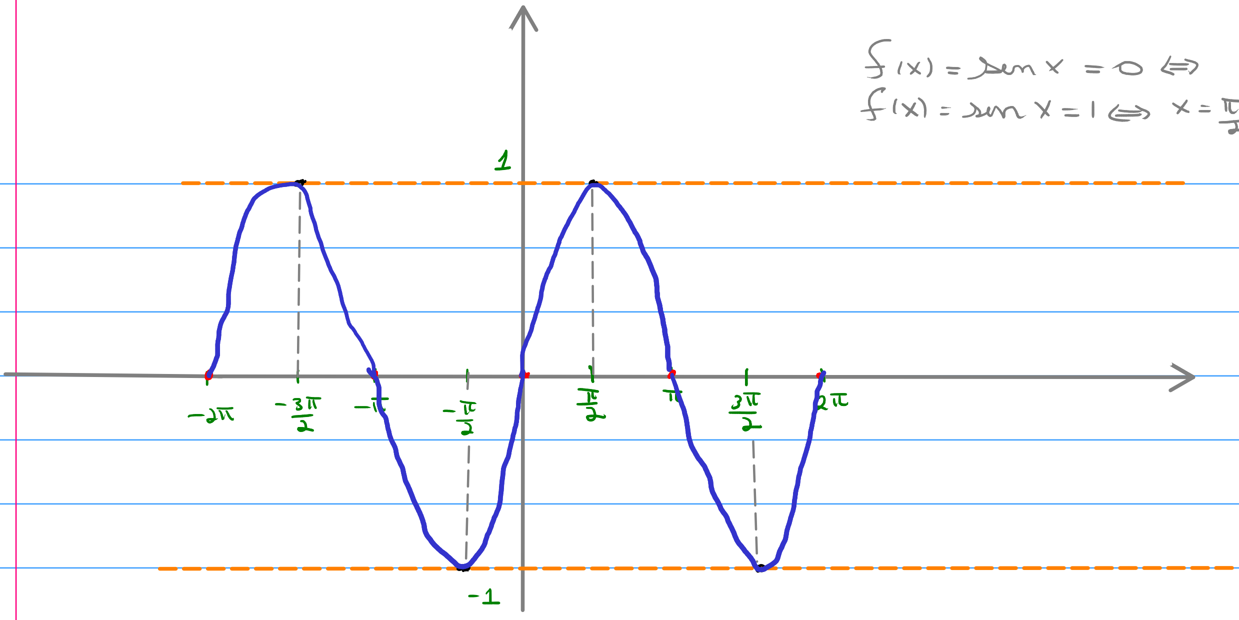
$= \left\{x \in \mathbb{R} \mid x = \dfrac{\pi}{2} + k\pi = \dfrac{(2k+1)\pi}{2}\right\}$

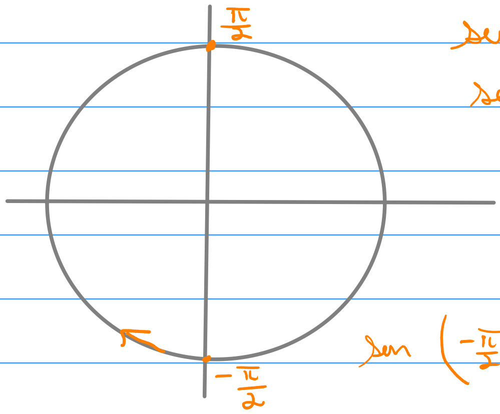
$= \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dots\right\}$



$f(x) = \operatorname{sen} x = 0 \iff x = k\pi$

$f(x) = \operatorname{sen} x = 1 \iff x = \frac{\pi}{2} + 2k\pi$





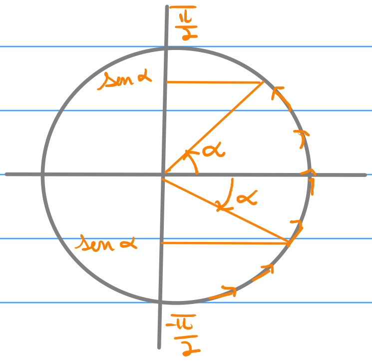
$\operatorname{sen} \frac{\pi}{2} = 1$

$\operatorname{sen}\left(\frac{\pi}{2} + 2\pi\right) = \operatorname{sen} \frac{5\pi}{2} = 1$

$\operatorname{sen}\left(-\frac{\pi}{2}\right) = -\operatorname{sen} \frac{\pi}{2} = -1$

$\operatorname{sen}\left(-\frac{\pi}{2} - 2\pi\right) = \operatorname{sen}\left(-\frac{5\pi}{2}\right) = -\operatorname{sen} \frac{5\pi}{2} = -1$

Funções trigonométricas inversas e suas derivadas

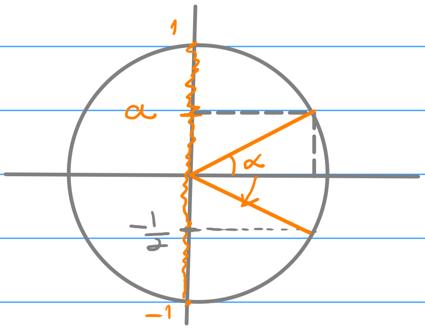


Se $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \Rightarrow \operatorname{sen} \alpha$ é injetora.

$\operatorname{sen} \alpha = a \iff \alpha = \operatorname{arcsen} a$

Seja $a \in [-1, 1] \Rightarrow \operatorname{arcsen} a = \alpha$

$\operatorname{sen} \alpha = a$



$$\boxed{\operatorname{arcsen} a = \operatorname{sen}^{-1} a}$$

$\operatorname{arcsen} 1 = \alpha \iff \operatorname{sen} \alpha = 1 \iff \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \alpha = \frac{\pi}{2}$

$\operatorname{arcsen} 1 = \frac{\pi}{2}$, $\operatorname{arcsen}\left(-\frac{1}{2}\right) = \alpha \iff \operatorname{sen} \alpha = -\frac{1}{2} \Rightarrow \alpha = -\frac{\pi}{6}$

$\operatorname{sen}\left(-\frac{\pi}{6}\right) = -\operatorname{sen}\left(\frac{\pi}{6}\right) = -\operatorname{sen}\left(30^\circ\right) = -\frac{1}{2}$