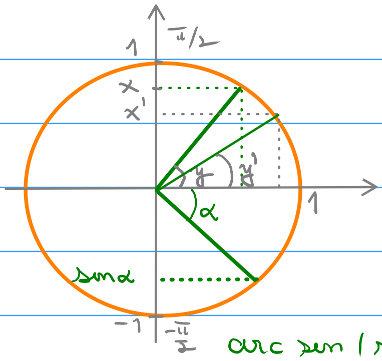
24/09 - Aula 17 - Derivando funções trigonométricas inversas

Proposição 12.2.

$$(\operatorname{arcsen} x)' = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$
$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$
$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}, \quad -\infty < x < +\infty$$



$\operatorname{arc\,sen}(\operatorname{sen}\alpha) = \alpha$

Seja $-1 < x < 1 \Rightarrow \operatorname{arcsen} x = y \Leftrightarrow \boxed{\operatorname{sen} y = x}$

$\operatorname{arc\,sen} x' = y' \Leftrightarrow \operatorname{sen} y' = x'$

$$\boxed{x \in]-1,1[\;\Rightarrow\; \operatorname{arc\,sen} x = y \Leftrightarrow \operatorname{sen} \underbrace{y(x)}_{\operatorname{arcsen} x} = x}$$

Logo, $\begin{cases} \operatorname{sen}(\operatorname{arcsen} x) = \operatorname{sen} y(x) = x & -1 < x < 1 \\ \operatorname{arcsen}(\operatorname{sen} y) = y & -\frac{\pi}{2} < y < \frac{\pi}{2} \end{cases}$

$\frac{d}{dx}(\operatorname{arcsen} x) = \frac{d}{dx} y(x) = y'(x)$ $\qquad \Downarrow$ $\cos y > 0$

$\operatorname{sen} y(x) = x \Rightarrow (\operatorname{sen} y(x))' = (x)'$

$\Leftrightarrow \cos(y(x)) \cdot y'(x) = 1$

$\Leftrightarrow \boxed{y'(x) = \frac{1}{\cos y(x)}}$ $(*)$

Pitágoras $\Rightarrow \cos^2 y + \operatorname{sen}^2 y = 1$

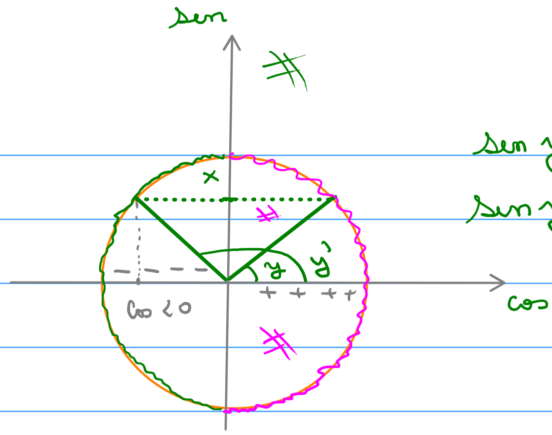
$\cos^2 y = 1 - \operatorname{sen}^2 y$ $\qquad \operatorname{sen}^2 y = (\operatorname{sen} y)^2$

$\cos y = \pm\sqrt{1 - \operatorname{sen}^2 y}$

$\Rightarrow \cos y(x) = +\sqrt{1 - (\operatorname{sen} y(x))^2} = \sqrt{1 - x^2}$

$(*) \Rightarrow y'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow (\operatorname{arcsen} x)' = \frac{1}{\sqrt{\underbrace{1-x^2}_{(+)}}}$

$-1 < x < 1$

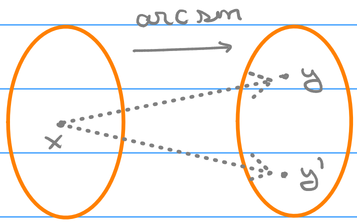


$-1 < x < 1$ $\quad$ $-1 < x < 1$

$\frac{\pi}{2} < y < \frac{3\pi}{2}$ $\quad$ $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\operatorname{sen} y = x \iff y = \operatorname{arc\,sen} x$

$\operatorname{sen} y' = x \iff y' = \operatorname{arc\,sen} x$



Não é função.

Vem Euler

Situação 1

$\boxed{\operatorname{arc\,sen} x = y \iff \operatorname{sen} y = x}$

$-1 < x < 1$

$\frac{\pi}{2} < y < \frac{3\pi}{2}$

$\Downarrow$

$(\operatorname{arc\,sen} x)' = -\frac{1}{\sqrt{1-x^2}}$

Situação 2

$\boxed{\operatorname{arc\,sen} x = y \iff \operatorname{sen} y = x}$

$-1 < x < 1$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Downarrow$

$(\operatorname{arc\,sen} x)' = +\frac{1}{\sqrt{1-x^2}}$

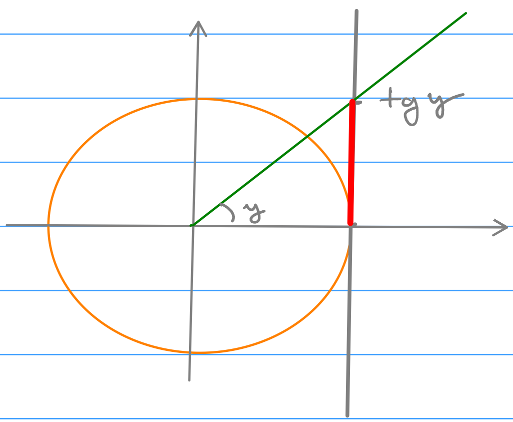
Obs. 1) $f(x) = \operatorname{sen} u(x) \Rightarrow f'(x) = \frac{d}{dx} \operatorname{sen} u(x) = \frac{d}{du} \operatorname{sen} u \cdot \frac{du}{dx} =$

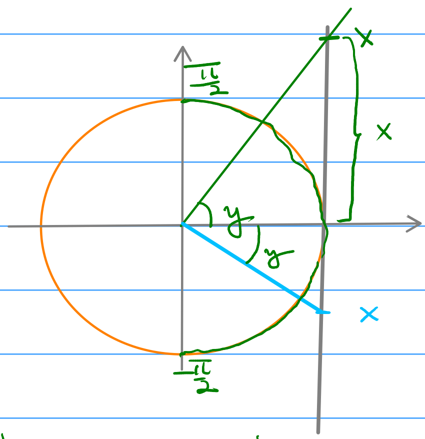
$= \cos u \cdot u'(x) \Rightarrow (\operatorname{sen} u(x))' = \cos u(x) \cdot u'(x)$

2) $f(x) = \cos u(x) \Rightarrow f'(x) = (\cos u(x))' = -\operatorname{sen} u(x) \cdot u'(x)$

3) $f(x) = \operatorname{tg} u(x) \Rightarrow f'(x) = \operatorname{tg}' u(x) \cdot u'(x) = \sec^2 u(x) \cdot u'(x)$

$\boxed{\text{Estudem a definição de } \operatorname{arc\,cos} x = y \iff \cos y(x) = x}$





$-\infty < x < \infty$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\operatorname{arctg} x = y \iff \operatorname{tg} y = x$

$$y(x) = \text{arctg}\, x \Rightarrow \boxed{\text{tg}\, y(x) = x} \Rightarrow (\text{tg}\, y(x))' = 1$$

$$\text{tg}'\, y(x) \cdot y'(x) = 1 \Rightarrow \sec^2 y(x) \cdot y'(x) = 1 \Rightarrow \boxed{y'(x) = \frac{1}{\sec^2 y(x)}}$$

Vamos escrever $\sec^2 y(x)$ em função de $\underline{x}$

Identidade trigonométrica: $\sec^2 y - \text{tg}^2 y = 1$

Logo, $\sec^2 y = 1 + \text{tg}^2 y = 1 + (\text{tg}\, y)^2 = 1 + x^2$

Portanto,

$$\boxed{(\text{arctg}\, x)' = \frac{1}{1+x^2}}$$

$-\infty < x < \infty$

$\frac{-\pi}{2} < y < \frac{\pi}{2}$

Derivada em cadeia envolvendo funções trigonométricas : $u = u(x)$

1. $(\text{sen}\, u)' = \cos u \cdot u$
2. $(\cos u)' = -\text{sen}\, u \cdot u'$
3. $(\text{tg}\, u)' = \sec^2 u \cdot u'$
4. $(\text{cotg}\, u)' = -(\text{cosec}^2 u) \cdot u'$
5. $(\sec u)' = (\sec u \cdot \text{tg}\, u) \cdot u'$
6. $(\text{cosec}\, u)' = -(\text{cosec}\, u \cdot \text{cotg}\, u) \cdot u'$
7. $(\text{arc sen}\, u) = \frac{u'}{\sqrt{1-u^2}}$
8. $(\text{arc cos}\, u) = \frac{-u'}{\sqrt{1-u^2}}$
9. $(\text{arc tg}\, u)' = \frac{u'}{1+u^2}$

Exemplos

(A) $(\text{arctg}(x^2+1))' = \text{arctg}'(x^2+1) \cdot (x^2+1)'$

$$= \frac{1}{1+(x^2+1)^2} \cdot 2x = \frac{2x}{1+x^4+2x^2+1}$$

$$(\text{arctg}\underbrace{(x^2+1)}_{u(x)})' = (\text{arc tg}\, u)' = \frac{u'}{1+u^2} = \frac{2x}{1+(x^2+1)^2}$$

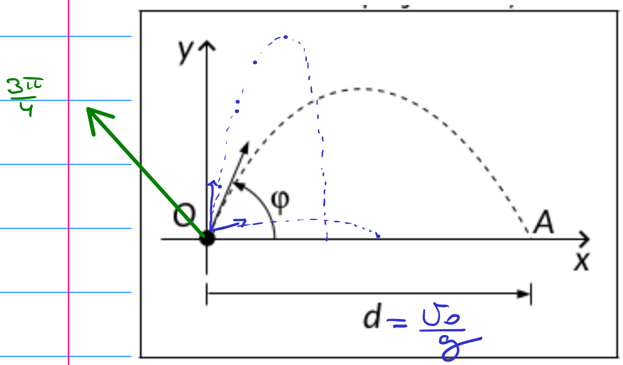
$u = x^2+1 \Rightarrow u' = 2x$

2. A distância $d = OA$ (veja figura 12.4) que um projétil alcança, quando disparado de um canhão com velocidade inicial v_0, por um cano inclinado com um ângulo de elevação φ em relação ao chão (horizontal), é dada pela fórmula

$$d = \frac{v_0}{g} \operatorname{sen} 2\varphi$$

sendo g a aceleração da gravidade local.

Qual é o ângulo φ que proporciona alcance máximo? *Resposta.* 45°.



$$d(\varphi) = \frac{v_0}{g} . \operatorname{sen}(2\varphi) \qquad 0 \leq \varphi \leq \frac{\pi}{2}$$

φ: variável independente

$$d'(\varphi) = \frac{v_0}{g} (\operatorname{sen} 2\varphi)' = \frac{v_0}{g} \cos(2\varphi) . (2\varphi)'$$

$$\Rightarrow d'(\varphi) = \frac{v_0}{g} \cos(2\varphi) . 2$$

$$\Rightarrow d'(\varphi) = 0 \Leftrightarrow \frac{2v_0}{\underbrace{g}_{\neq 0}} \cos(2\varphi) = 0 \Rightarrow \cos(2\varphi) = 0 \Rightarrow 2\varphi = \frac{\pi}{2} \Rightarrow \boxed{\varphi = \frac{\pi}{4}}$$

$$d''(\varphi) = \left(\frac{2v_0}{g} \cos(2\varphi)\right)' = \frac{2v_0}{g} (-\operatorname{sen} 2\varphi) . 2 = -\frac{4v_0}{g} \operatorname{sen}(2\varphi)$$

$$d''\left(\frac{\pi}{4}\right) = -\frac{4v_0}{g} . \operatorname{sen}\left(2 . \frac{\pi}{4}\right) = -\frac{4v_0}{g} . \underbrace{\operatorname{sen} \frac{\pi}{2}}_{=1} = -\frac{4v_0}{g} < 0$$

$\Rightarrow \varphi = \frac{\pi}{4}$ é um ponto de máximo para a função $d(\varphi)$

$$\Rightarrow \text{A maior distância} = d\left(\frac{\pi}{4}\right) = \frac{v_0}{g} \operatorname{sen}\left(2 . \frac{\pi}{4}\right) = \frac{v_0}{g}$$

$$\cos(2\varphi) = 0 \Rightarrow 2\varphi = \frac{3\pi}{2} \Rightarrow \varphi = \frac{3\pi}{4} \Rightarrow d''\left(\frac{3\pi}{4}\right) = -\frac{4v_0}{g} \underbrace{\operatorname{sen}\left(\frac{3\pi}{2}\right)}_{-1} = \frac{4v_0}{g}$$

$\Rightarrow \frac{3\pi}{4}$ ponto de mínimo.

(g) $y = tg^2x \cdot sec^3x \Rightarrow y' = ?$

$$y' = (tg^2x \cdot sec^3x)' = (tg^2x)' \cdot sec^3x + (tg^2x) \cdot \underline{(sec^3x)'}$$

Regra do Produto

- $(\underbrace{tg}_{u}{}^2x)' = (u^2)' = 2u^1 \cdot u' = 2(tg\,x) \cdot (tg\,x)' = \boxed{2\,tg\,x \cdot sec^2x}$

- $(sec^3x)' = 3\,sec^2x \cdot (sec\,x)' = 3\,sec^2x \cdot (sec\,x \cdot tg\,x) = \boxed{3\,sec^3x \cdot tg\,x}$

Logo, $y' = 2\,tg\,x \cdot sec^2x \cdot sec^3x + tg^2x \cdot 3\,sec^3x \cdot tg\,x$

$$\Rightarrow \boxed{y' = 2\,tg\,x \cdot sec^5x + 3\,tg^3x \cdot sec^3x}$$

(h) $f(x) = tg^3(2x+1) \Rightarrow f'(x) = (tg^3(2x+1))'$

$$\Rightarrow \begin{cases} f'(x) = (u^3)' = \dfrac{d}{dx}(u^3) = \dfrac{d}{du}(u^3) \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx} \\ u = tg\,v \\ v = 2x+1 \end{cases}$$

$$= 3u^2 \cdot (tg\,v)' \cdot (2x+1)'$$
$$= 3u^2 \cdot sec^2v \cdot 2$$
$$= 6\,u^2 sec^2 v$$
$$= 6\,tg^2(2x+1) \cdot sec^2(2x+1)$$

$$(tg^3(2x+1))' = 3 \cdot tg^2(2x+1) \cdot (tg(2x+1))' = 3 \cdot tg^2(2x+1) \cdot sec^2(2x+1) \cdot 2$$

Exercício: Calcule o seguinte limite.

$$\lim_{x \to 0^+} sen\,x \cdot sen\,\frac{1}{x} = sen\,0 \cdot sen\,\frac{1}{\underset{\pm\infty}{0}}$$

Recorde que $\displaystyle\lim_{x \to 0} \frac{sen\,x}{x} = 1$

$$sen\,x \cdot sen\,\frac{1}{x} = \frac{sen\,x}{x} \cdot x\,sen\left(\frac{1}{x}\right)$$

$$\lim_{x\to 0}\left[\frac{\operatorname{sen} x}{x}\cdot x\operatorname{sen}\left(\frac{1}{x}\right)\right] = \underbrace{\lim_{x\to 0}\frac{\operatorname{sen} x}{x}}_{=1}\cdot\underbrace{\lim_{x\to 0} x\operatorname{sen}\frac{1}{x}}_{=0} = 0$$

Limite Fundamental

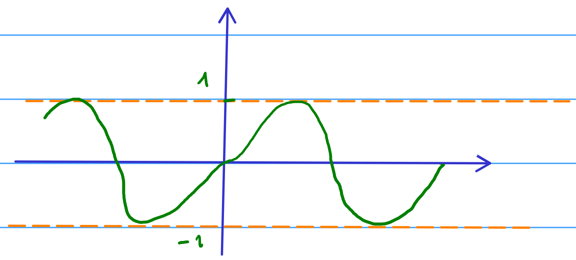
Resta calcular $\displaystyle\lim_{x\to 0} x\operatorname{sen}\left(\frac{1}{x}\right) = \lim_{x\to 0}\frac{\operatorname{sen}(1/x)}{\frac{1}{x}} = \lim_{u\to ?}\frac{\operatorname{sen} u}{u}$

Seja $u = \frac{1}{x}$

$$\left.\begin{array}{l} (\mathrm{I})\ \displaystyle\lim_{x\to 0^+}\frac{\operatorname{sen}\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{u\to+\infty}\frac{\operatorname{sen} u}{u} = 0 \\ \\ (\mathrm{II})\ \displaystyle\lim_{x\to 0^-}\frac{\operatorname{sen}\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{u\to-\infty}\frac{\operatorname{sen} u}{u} = 0 \end{array}\right\} \Rightarrow \lim_{x\to 0}\frac{\operatorname{sen} 1/x}{1/x} = 0$$

Fato: $\displaystyle\lim_{u\to+\infty}\frac{\operatorname{sen} u}{u} = 0$

Note que $-1 \leq \operatorname{sen} u \leq 1$ para todo $u \in \mathbb{R}$



$\Rightarrow$ Se $u > 0$, dividindo por u obtemos $\displaystyle -\frac{1}{u} \leq \frac{\operatorname{sen} u}{u} \leq \frac{1}{u}$

$$\lim_{u\to+\infty}\left(-\frac{1}{u}\right) = 0 = \lim_{u\to+\infty}\left(\frac{1}{u}\right) \qquad \begin{array}{ccc} -\frac{1}{u} & \frac{\operatorname{sen} u}{u} & \frac{1}{u} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

Pelo teorema do confronto $\displaystyle\lim_{u\to+\infty}\frac{\operatorname{sen} u}{u} = 0$

Analogamente, $\displaystyle\lim_{u\to-\infty}\frac{\operatorname{sen} u}{u} = 0$