

DERIVADAS
PARCIAIS
DE
ORDEM
SUPERIOR

Definição: Sejam
 $z = f(x, y)$ e $(x_0, y_0) \in Df$.

A derivada parcial de
 f em relação a x
no ponto (x_0, y_0) é de-
finida por:

$$\frac{\partial f(x_0, y_0)}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

se este limite existir.

Exemplo: $f(x, y) = x^2 + y$.

$$(x_0, y_0) = (2, -4)$$

$$\frac{\partial f}{\partial x}(2, -4) = \lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x, -4) - f(2, -4)}{\Delta x}$$

- $$\begin{aligned} f(2 + \Delta x, -4) &= (2 + \Delta x)^2 + (-4) \\ &= \cancel{4} + 4\Delta x + \Delta x^2 - \cancel{4} \\ &= 4\Delta x + \Delta x^2 \end{aligned}$$
- $$f(2, -4) = (2)^2 - 4 = \cancel{4} - \cancel{4} = 0$$

$$\frac{\partial f}{\partial x}(2, -4) = \lim_{\Delta x \rightarrow 0} \frac{4\Delta x + \Delta x^2}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0} [4 + \cancel{\Delta x}] = 4$$

Observe que $f(x, y) = x^2 + y^2$

$$\Rightarrow \frac{\partial f}{\partial x}(x, y) = 2x$$

$$\frac{\partial f}{\partial x} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial f}{\partial x}(x, y) = 2x \quad \Bigg| \quad \frac{\partial f}{\partial x}(2, -4) = 4$$

$$f(x,y) = \begin{cases} \frac{x^3 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^4 + 3x^2y^2 + 2xy^2}{(x^2 + y^2)^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$$\hookrightarrow \text{Domínio}\left(\frac{\partial f}{\partial x}\right) = \mathbb{R}^2$$

$$f(x,y) = \begin{cases} \frac{x^3 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} -\frac{2x^2y(1+x)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ \text{?}, & (x,y) = (0,0) \end{cases}$$

↳ Domínio $\mathbb{R}^2 \setminus \{(0,0)\}$

$$f: Df \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial f}{\partial x} := f_x: Df_x \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial (f_x)}{\partial x}(x_0, y_0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) (x_0, y_0)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f_x(x_0 + \Delta x, y_0) - f_x(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\frac{\partial f}{\partial x}(x_0 + \Delta x, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)}{\Delta x}$$

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$$= \lim_{\Delta y \rightarrow 0} \frac{f_x(x_0, y_0 + \Delta y) - f_x(x_0, y_0)}{\Delta y}$$

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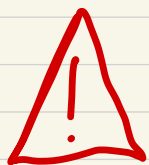


$$\frac{\partial}{\partial x} (f_x)(x_0, y_0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) (x_0, y_0)$$

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$$\frac{\partial}{\partial x} (f_y)(x_0, y_0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) (x_0, y_0)$$

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$\frac{\partial}{\partial x}$

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$$



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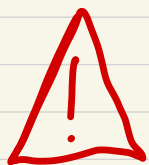
$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

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$\frac{\partial}{\partial y}$

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$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

Exemplo: $f(x, y) = xy - e^x \cos y$

$$\bullet \frac{\partial f}{\partial x}(x, y) = y - e^x \cos y$$

$$\bullet \frac{\partial f}{\partial y}(x, y) = x + e^x \sin y$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = -e^x \cos y$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = 1 + e^x \sin y$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = 1 + e^x \sin y$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = e^x \cos y$$

Exemplo: $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

$$\frac{\partial f}{\partial x}(x, y, z) = \frac{2x}{x^2 + y^2 + z^2}$$

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{2z}{x^2 + y^2 + z^2}$$

$$\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \right) (x, y, z) = \frac{\partial^2 f}{\partial z \partial x} (x, y, z) =$$

$$= \frac{0 \cdot (x^2 + y^2 + z^2) - 2z \cdot (2x)}{(x^2 + y^2 + z^2)^2}$$

$$= (-4xz) / (x^2 + y^2 + z^2)^2$$

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$$= \frac{0 \cdot (x^2 + y^2 + z^2) - 2x \cdot (2z)}{(x^2 + y^2 + z^2)^2}$$

$$= (-4xz) / (x^2 + y^2 + z^2)^2$$

Observemos que nos dois últimos exemplos tivemos as derivadas parciais mistas iguais, isto é,

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$$

$$\frac{\partial^2 f}{\partial z \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial z}(x, y)$$

Definição: Uma função f é de classe C^1 quando suas derivadas parciais são contínuas. Se as derivadas parciais de segunda ordem de f são contínuas, dizemos que f é de classe C^2 .

Teorema: Se $z = f(x, y)$ é de classe C^2 , então suas derivadas parciais mistas são iguais, isto é,

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y).$$

Obs: Este teorema vale para funções de várias variáveis.