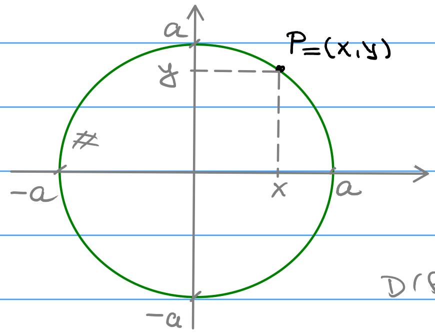
29/10 - Aula 29 - Integral Definida- Métodos da Substituição e Integração por Partes

Exemplo: Calcule a área delimitada pela circunferência de equação $x^2+y^2=a^2$, $a>0$ real

$A=\pi.a^2$



$x^2+y^2=a^2$

$y=\pm\sqrt{a^2-x^2}$, $y\geq 0$

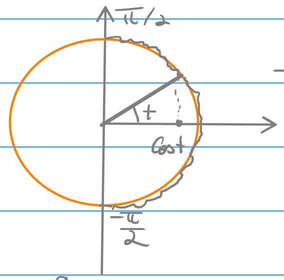
Seja $\boxed{y=f(x)=+\sqrt{a^2-x^2}}$

$D(f)=\{x\in\mathbb{R} \mid -a\leq x\leq a\}$

$$A=2\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$$

Aplicaremos o método da substituição ou mudança de variável.

Seja $x=a\,\text{sen}\,t$, $-\frac{\pi}{2}\leq t\leq\frac{\pi}{2} \Rightarrow \underbrace{-a\leq x\leq a}$



$-1\leq \text{sen}\,t\leq 1 \Rightarrow -a\leq \underbrace{a\,\text{sen}\,t}_{x}\leq a$

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos t\,.a\cos t\,dt = a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt$$

(1) Seja $x=a\,\text{sen}\,t \Rightarrow t=\text{arcsen}\left(\frac{x}{a}\right)$

Se $x=-a \Rightarrow t=\text{arcsen}\left(\frac{-a}{a}\right)=\text{arcsen}\,(-1)=-\frac{\pi}{2}$

Se $x=a \Rightarrow t=\frac{\pi}{2}$ (exercício), $0\leq\cos t\leq 1$

(2) $\sqrt{a^2-x^2}=\sqrt{a^2-(a\,\text{sen}\,t)^2}=\sqrt{a^2-a^2\text{sen}^2 t}=a\sqrt{1-\text{sen}^2 t}=a\sqrt{\cos^2 t}$

$=a.|\cos t| = a\cos t$

(3) Se $x = a \, \text{sen}\, t \Rightarrow \boxed{dx = a \cos t \, dt}$

Então, $A = 2\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt \overset{\text{Fato}}{=}$

(*) $\boxed{\text{Fato: } \cos^2 t = \frac{1}{2}\cdot(1 + \cos 2t)}$

$\text{sen}^2 t + \cos^2 t = 1$

$|\cos t| = \sqrt{\cos^2 t} = \sqrt{1 - \text{sen}^2 t}$

$$= \cancel{2}a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cancel{2}}(1 + \cos 2t)\, dt = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2t)\, dt$$

$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot dt + a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t \, dt$$

$$\overset{\text{TFC}}{=} a^2 \cdot [t]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + a^2 \left[\frac{\text{sen}\, 2t}{2}\right]_{-\frac{\pi}{2}}^{\pi/2}$$

$$= a^2 \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + \frac{a^2}{2}\cdot\left(\text{sen}(\cancel{2}\cdot\pi/\cancel{2}) - \text{sen}(\cancel{2}(-\pi/\cancel{2}))\right), \quad \text{sen}\, \pi = 0$$

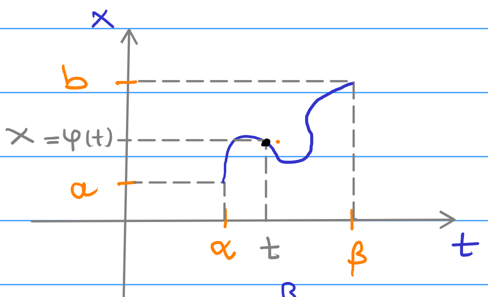
$$= a^2\pi + 0 = \pi a^2 \Rightarrow \boxed{A = \pi a^2}$$

Seja $F(x)$ uma primitiva de $f(x)$ no intervalo $[a, b]$

$\boxed{F'(x) = f(x)}$, $\forall x \in [a, b]$

ou seja, $\int f(x)\, dx = F(x) + C$

Seja $x = \varphi(t)$, $\alpha \leq t \leq \beta$, com $a = \varphi(\alpha)$ e $b = \varphi(\beta)$



$$\boxed{\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt}$$

$F'(x) = f(x) \Rightarrow F'(\varphi(t)) = f(\varphi(t))$

pois $x = \varphi(t)$

$$\int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt = \int_\alpha^\beta F'(\varphi(t)) \cdot \varphi'(t)\, dt = \int_\alpha^\beta (F(\varphi(t)))'\, dt$$

$$\overset{\text{TFC}}{=} F(\underbrace{\varphi(\beta)}_{b}) - F(\underbrace{\varphi(\alpha)}_{a}) = F(b) - F(a) \underset{\text{TFC}}{=} \int_a^b f(x)\, dx$$

□

Integração definida, por partes :

Recorde que

$$\int_a^b f'(x)\,dx \underset{TFC}{=} F(b) - F(a) = f(b) - f(a) = [f(x)]_a^b$$

onde $F(x)$ é primitiva de $f'(x)$ $\Rightarrow$ $F(x) = f(x)$ pois $F'(x) = f'(x)$

$$\int_a^b u(x).v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x).v(x)\,dx$$

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x) \Rightarrow$$

$$\int_a^b (u(x)v(x))'\,dx = \int_a^b (u'(x)v(x) + u(x)v'(x))\,dx \Rightarrow$$

$$u(b)v(b) - u(a)v(a) = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx \Rightarrow$$

$$\boxed{\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx}$$

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Ex. Calcule $\int_0^{\pi/2} \operatorname{sen}x.\cos^2 x\,dx$ por mudança de variável.

Seja $\boxed{u = \cos^2 x} \Rightarrow du = 2\cos x.(-\operatorname{sen}x)\,dx \Rightarrow dx = -\dfrac{1}{2\cos x\operatorname{sen}x}\,du$

$$\int_0^{\pi/2} \operatorname{sen}x.\cos^2 x\,dx = \int_1^0 \cancel{\operatorname{sen}x}.u.\frac{-1}{2\cos x\,\cancel{\operatorname{sen}x}}\,du$$

$x = 0 \Rightarrow u = \cos^2 0 = 1$

$x = \frac{\pi}{2} \Rightarrow u = \cos^2\frac{\pi}{2} = 0$

$$= -\frac{1}{2}\int_1^0 \frac{u}{\cos x}\,du = -\frac{1}{2}\int_1^0 \frac{u}{\sqrt{u}}\,du$$

$u = \cos^2 x \Rightarrow \cos x = \sqrt{u}$

$$= +\frac{1}{2}\int_0^1 u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1$$

$$= \frac{1}{3}[1 - 0] = \frac{1}{3}$$

$\operatorname{sen}x\,dx = -du$

$$\int_0^{\pi/2} \operatorname{sen}x \cdot \cos^2 x \, dx = -\int_1^0 u^2 \, du = +\int_0^1 u^2 du = \frac{u^3}{3}\Big|_0^1$$

Seja $u = \cos x \Rightarrow du = -\operatorname{sen}x\,dx$

$x = 0 \Rightarrow u = \cos 0 = 1$ $\qquad = \frac{1}{3} - \frac{0}{3} = 1/3$

$x = \pi/2 \Rightarrow u = \cos \pi/2 = 0$

Ex. Calcule $\displaystyle\int_0^{\pi/2} \frac{dx}{3 + 2\cos x}$

Dica: Use que $\cos x = \dfrac{1 - tg^2(\frac{x}{2})}{1 + tg^2(\frac{x}{2})}$, faça $u = tg(\frac{x}{2})$

Sol:

$$\frac{1}{3 + 2\cos x} = \frac{1}{3 + 2 \cdot \left[\dfrac{1 - tg^2(\frac{x}{2})}{1 + tg^2(\frac{x}{2})}\right]} \overset{mmc}{=} \frac{1}{\dfrac{3 + 3tg^2(x/2) + 2 - 2tg^2(x/2)}{1 + tg^2(x/2)}}$$

$$= \frac{1 + tg^2(x/2)}{5 + tg^2(x/2)} \Rightarrow$$

$$\int_0^{\pi/2} \frac{dx}{3 + 2\cos x} = \int_0^{\pi/2} \frac{1 + tg^2(x/2)}{5 + tg^2(x/2)} \, dx \; ; \qquad \text{Seja } u = tg(\tfrac{x}{2})$$

$\Downarrow$

$x = 0 \Rightarrow u = tg(0) = 0$ $\qquad du = \sec^2(\frac{x}{2}) \cdot \frac{1}{2} \, dx$

$x = \pi/2 \Rightarrow u = tg\left(\frac{\pi/2}{2}\right) = tg\left(\frac{\pi}{4}\right) = 1$

$du = \frac{1}{2}(1 + tg^2(x/2))\,dx$

$du = \frac{1}{2}(1 + u^2)\,dx$

$\Downarrow$

$$= \int_0^1 \frac{\cancel{1 + u^2}}{5 + u^2} \frac{2}{\cancel{1 + u^2}} \, du \qquad \boxed{dx = \frac{2}{1 + u^2} du}$$

$$= 2\int_0^1 \frac{1}{5 + u^2} \, du =$$

Seja $u = \sqrt{5}\,v \Rightarrow u = 0 \Rightarrow v = 0$ e $u = 1 \Rightarrow v = 1/\sqrt{5}$

$$= 2\int_0^{1/\sqrt{5}} \frac{1}{5 + (\sqrt{5}v)^2} \cdot \sqrt{5}\,dv = \frac{\sqrt{5}\,2}{5} \int_0^{1/\sqrt{5}} \frac{1}{1 + v^2} \cdot dv$$

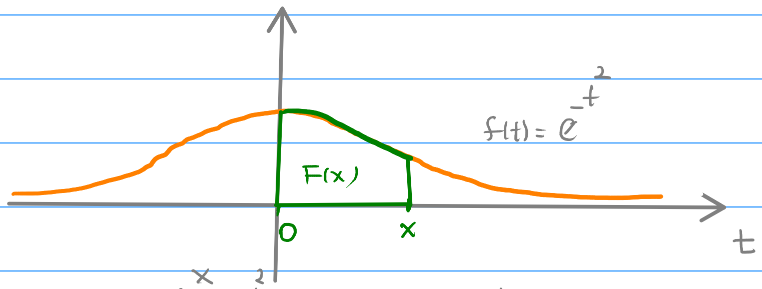
$$= \frac{2\sqrt{5}}{5} \cdot \left[\arctan v\right]_0^{\frac{1}{\sqrt{5}}} = \frac{2\sqrt{5}}{5} \cdot \left[\arctan \frac{1}{\sqrt{5}} - \overbrace{\arctan 0}^{=0}\right]$$

Como $\frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} \Rightarrow \; = \frac{2}{\sqrt{5}} \arctan\left(\frac{1}{\sqrt{5}}\right)$ □

Aplicando o teorema fundamental do cálculo para definir funções.

Seja $F(x) = \int_a^x f(t)\,dt \overset{TFC}{\Longrightarrow} F'(x) = f(x)$

Ex 1: $F(x) = \int_0^x e^{-t^2}\,dt = \int_0^x f(t)\,dt$, onde $f(t) = e^{-t^2}$



$f(t)$ é par $\Leftrightarrow f(-t) = f(t)$

$$F(x) = \int_0^x e^{-t^2}\,dt \overset{TFC}{\Longrightarrow} F'(x) = f(x) = e^{-x^2}$$

Ex Seja $g(x) = \int_0^{x^2} \sqrt{1+t^3}\,dt$, calcule $g'(x)$

$f(t) = \sqrt{1+t^3}$, $[0, x^2]$

Seja $F(u) = \int_0^u \sqrt{1+t^3}\,dt \Rightarrow \boxed{g(x) = F(x^2)}$

De fato, $F(x^2) = \int_0^{x^2} \sqrt{1+t^3}\,dt = g(x)$

$$\Rightarrow \boxed{\frac{d}{dx} g(x) = \frac{d}{dx}\left[F(x^2)\right] \underset{R.C}{=} F'(x^2)\cdot(x^2)' = 2x\,F'(x^2)}$$

Mas, $F'(u) = \frac{d}{du}\int_0^u \sqrt{1+t^3}\,dt \overset{TFC}{=} \sqrt{1+u^3} \Rightarrow F'(x^2) = \sqrt{1+(x^2)^3}$

$$= \sqrt{1+x^6}$$

$$g'(x) = 2x\,F'(x^2) = 2x\sqrt{1+x^6}$$

Logo,
$$\boxed{\frac{d}{dx}\int_0^{x^2} \sqrt{1+t^3}\,dt = 2x\sqrt{1+x^6}}$$

<u>Ex.</u> Seja $g(x) = \displaystyle\int_{x^2}^{x^3} \frac{1}{\ln t}\,dt$, $(x>0)$ calcule $g'(x)$