22/10 - Aula 26 - Teorema Fundamental do Cálculo

Vimos em aulas anteriores que o método da mudança de variável na integral se aplica para integrais da seguinte forma

$$\rightarrow \int f(\underbrace{\varphi(x)}_{u})\underbrace{\varphi'(x)\,dx}_{du} = \int f(u)\,du = F(u) + C = F(\varphi(x)) + C$$

Seja $u = \varphi(x) \Rightarrow du = \varphi'(x)dx$

Ex.: Calcule a seguinte integral: $\int \sqrt{2x+1}\,dx$

Sol

$$\int \sqrt{\underbrace{2x+1}_{\varphi(x)}}\,dx = \frac{1}{2}\int \sqrt{\underbrace{2x+1}_{\varphi(x)}}\cdot \underbrace{2\,dx}_{\varphi'(x)dx} = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot \frac{u^{\frac{1}{2}+1}}{\underbrace{\frac{1}{2}+1}} + C$$

Note que $\varphi(x) = 2x+1 \Rightarrow \varphi'(x) = 2$

Seja $\boxed{u = \varphi(x)} \Rightarrow \boxed{u = 2x+1} \Rightarrow \underset{\sim}{du} = 2dx$

$$\Rightarrow \int \sqrt{2x+1}\,dx = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot u^{\frac{3}{2}} + C = \frac{1}{3}\cdot(2x+1)^{3/2} + C$$

Logo, $\int \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C \Leftrightarrow \left[\frac{1}{3}(2x+1)^{\frac{3}{2}}\right]'$

$$= \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(2x+1)^{\overbrace{\frac{3}{2}-1}^{1/2}}\cdot\cancel{2} = \sqrt{2x+1}$$

Ex.: $\int \sqrt{2x+1}\cdot dx = \int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int\sqrt{u}\,du = \frac{1}{3}\cdot u^{3/2} + C$

$u = u(x) = 2x+1$ $\qquad = \frac{1}{3}(2x+1)^{3/2} + C$

$\Downarrow$

$du = 2dx \Rightarrow dx = \frac{1}{2}du$

Ex.: $\int \frac{1}{1+(2x+1)^2}\,dx = \frac{1}{2}\int\frac{1}{1+u^2}\cdot du = \frac{1}{2}\operatorname{arctg} u + C$

$u = u(x) = 2x+1$ $\qquad = \frac{1}{2}\operatorname{arctg}(2x+1) + C$

$du = 2dx \Rightarrow dx = \frac{1}{2}du$

Integração por Partes:

$$\int \underbrace{f(x)}_{\text{integrando}} \, \underbrace{dx}_{\text{diferencial de } x}$$

$$\int u(x).v'(x)\,dx = u(x)v(x) - \overbrace{\int u'(x)v(x)\,dx}^{\text{integral}}$$

Exemplo: Encontre a antiderivada da função abaixo $f(x) = e^x \sin x$

Queremos encontrar uma função $F(x)$ tal que

$F'(x) = e^x \operatorname{sen} x$, ou seja, queremos encontrar

$$I = \int \underbrace{e^x}_{u(x)} \underbrace{\operatorname{sen} x}_{v'(x)} \, dx = u(x)v(x) - \int u'(x)v(x)\,dx \overset{*}{=}$$

$\boxed{u = e^x} \Rightarrow \boxed{u'(x) = e^x}$

$\boxed{v'(x) = \operatorname{sen} x} \Rightarrow v(x) = \int v'(x)\,dx = \int \operatorname{sen} x\,dx = -\cos x \Rightarrow \boxed{v(x) = -\cos x}$

$$\overset{*}{=} e^x(-\cos x) - \int e^x.(-\cos x)\,dx = -e^x\cos x + \int e^x \cos x\,dx$$

$$\boxed{I = \int e^x \operatorname{sen} x\,dx = -e^x \cos x + \boxed{\int e^x \cos x\,dx}}$$

$$\int \underbrace{e^x}_{u} \underbrace{\cos x}_{v'} \, dx = uv - \int v\,du = \left\{e^x \operatorname{sen} x - \int \operatorname{sen} x . e^x \, dx\right\}$$

$u = e^x$; $u' = e^x$

$v' = \cos x$; $v = \operatorname{sen} x$

Logo,

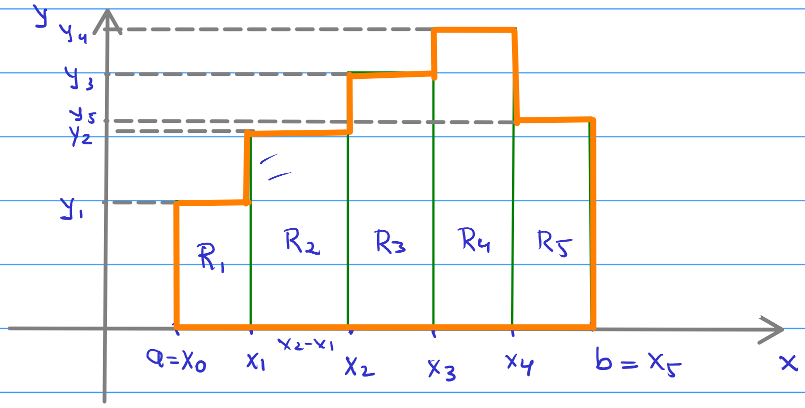
$$I = -e^x \cos x + \left\{e^x \operatorname{sen} x - \int e^x \operatorname{sen} x\,dx\right\} \Leftrightarrow$$

$$I = e^x(\operatorname{sen} x - \cos x) - I \Leftrightarrow$$

$$2I = e^x(\operatorname{sen} x - \cos x)$$

$$I = \frac{1}{2}e^x(\operatorname{sen} x - \cos x) \Rightarrow \boxed{\underbrace{\int e^x \operatorname{sen} x\,dx}_{I} = \frac{1}{2}e^x(\operatorname{sen} x - \cos x) + C}$$

Teorema Fundamental do Cálculo:



$a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b$ $\qquad \Delta x_i = x_i - x_{i-1}$

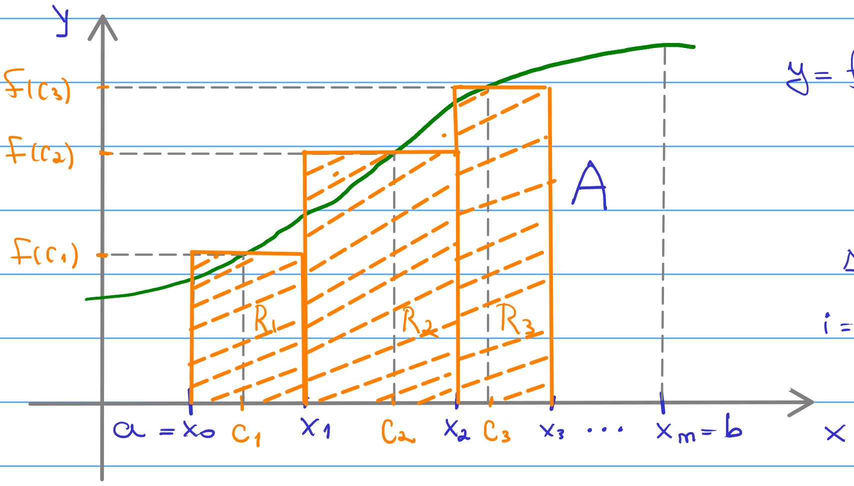
$$[a,b] = \bigcup_{n=1}^{5} [x_m, x_{n-1}]$$

$R_1 = y_1 (x_1 - x_0)$; $R_3 = y_3 (x_3 - x_2)$; $R_5 = y_5 (x_5 - x_4)$

$R_2 = y_2 (x_2 - x_1)$; $R_4 = y_4 (x_4 - x_3)$

$$\sum_{i=1}^{5} \underbrace{y_i \cdot \Delta x_i}_{R_i} = y_1 \Delta x_1 + y_2 \Delta x_2 + y_3 \Delta x_3 + y_4 \Delta x_4 + y_5 \Delta x_5$$

$$= y_1 (x_1 - x_0) + y_2 \cdot (x_2 - x_1) + y_3 \cdot (x_3 - x_2) + \cdots + y_5 (x_5 - x_4)$$



$y = f(x)$

$\Delta x_i = x_i - x_{i-1}$

$i = 2 \Rightarrow \Delta x_2 = x_2 - x_{2-1}$

$= x_2 - x_1$

$R_1 = (x_1 - x_0) \cdot f(c_1) = f(c_1) \Delta x_1$

$R_2 = (x_2 - x_1) f(c_2) = f(c_2) \Delta x_2$

$R_3 = (x_3 - x_1) f(c_3) = f(c_3) \Delta x_3$

$\vdots$

$R_n = (x_m - x_{n-1}) f(c_n) = f(c_n) \Delta x_m$

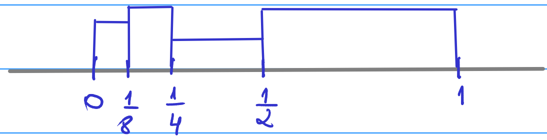
$$A \approx f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \cdots + f(c_m)\Delta x_m =$$

$$= \sum_{i=1}^{m} f(c_i)\Delta x_i$$ (Soma de Riemann)

$$P = \{(x_0, x_1, x_2, \ldots, x_m) \mid a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_m\}$$

Ex: $P = \{(0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1)\}$ é uma partição do intervalo $[0,1]$

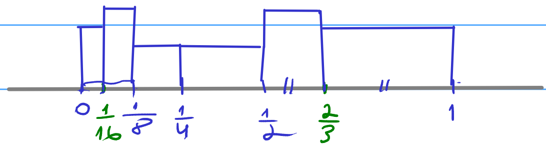
x_0, x_1, x_2, x_3, x_4



$P_1 = \{(0, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{2}{3}, 1)\}$, $P \subset P_1$

$1 - \frac{2}{3} = \frac{1}{3}$

$\frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$



$\Delta = \max\{\frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}\} = \frac{1}{3}$

$P = \{(x_0, x_1, \ldots, x_m) \mid x_0 = a < x_1 < \cdots < x_m = b\}$ partição de $[a,b]$

$\{\Delta x_1, \Delta x_2, \Delta x_3, \Delta_4, \ldots, \Delta x_m\}$

$P = \{(0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1)\}$

$\{\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4\} = \{\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\} \Rightarrow \Delta = \max\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\} = \frac{1}{2}$

$\Delta x_1 = \frac{1}{8} - 0$ $\qquad$ $\Delta x_3 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\Delta x_2 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$ $\qquad$ $\Delta x_4 = 1 - \frac{1}{2} = \frac{1}{2}$

Δ = norma da partição

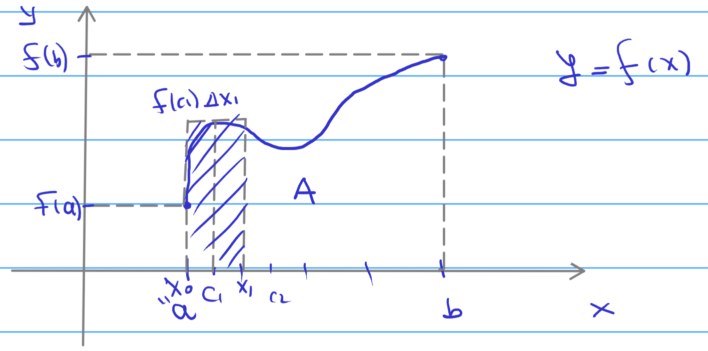
$\Delta = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta n\}$

$P \Rightarrow \Delta = 1/2$

$P \subset P_1$, $P_1 \Rightarrow \Delta = 1/3$

Escrevendo de modo mais simplificado, a *integral definida de* f*, de* a *até* b (ou *no intervalo* [a, b]) é o número real

$$\gamma = \int_a^b f(x)\,dx = \lim_{\Delta \to 0} S = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$



$$A = \int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{\Delta \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

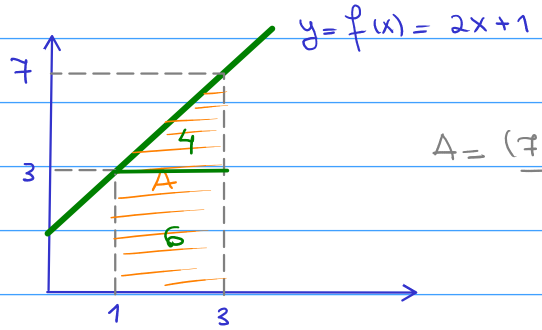
Integral definida de f(x) em [a,b]

$\Delta = \max \Delta x_i$

Teorema Fundamental do Cálculo: Seja $F(x)$ uma primitiva de $f(x)$ em $[a,b]$ então

$$A \doteq \int_a^b f(x)\,dx = F(b) - F(a)$$

Ex $f(x) = 2x+1$, $[a,b] = [1,3]$



$$A = \frac{(7+3)\cdot 2}{2} = 10$$

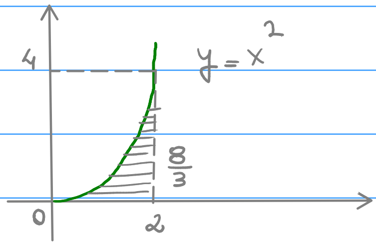
$$\text{TFC} \Rightarrow A = \int_1^3 2x+1\,dx = F(3) - F(1) = (12+\cancel{c}) - (2+\cancel{c}) = 10$$

$$F'(x) = 2x+1 \Rightarrow F(x) = x^2 + x + c$$

$$F(3) = 3^2 + 3 + c = 12 + c$$

$$F(1) = 1^2 + 1 + c = 2 + c$$

Ex Calcule $\int_0^2 x^2\,dx$



$$\int_0^2 x^2\,dx = F(2) - F(0) = \frac{8}{3} + c - c = \frac{8}{3}$$

onde $F(x) = \frac{x^3}{3} + c$

$\Downarrow$

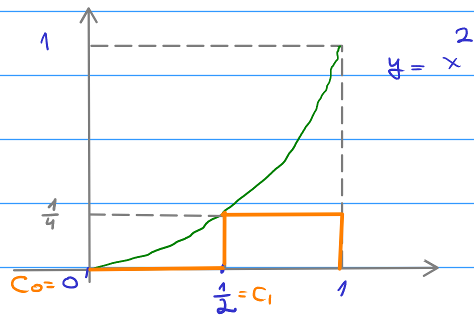
$F(2) = \frac{8}{3} + c$

$F(0) = c$

$$\int_0^2 x^2\,dx = \lim_{\Delta \to 0} \sum_{i=1}^{m} f(c_i)\,\Delta x_i$$

Ex Sendo $f(x) = x^2$, calcular $\int_0^1 f(x)\,dx$.

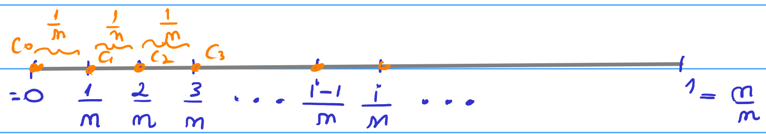
Sol



$m = 2 \Rightarrow \mathcal{P} = \{(0, \frac{1}{2}, 1)\}$

$f(c_0) = f(0) = 0$

$f(c_1) = f(\frac{1}{2}) = (\frac{1}{2})^2 = \frac{1}{4}$



$\mathcal{P} = \{(0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{i-1}{m}, \frac{i}{m}, \dots, \frac{m}{m} = 1)\}$ $m \in \mathbb{N}$ fixo

$\Rightarrow \boxed{\Delta = \frac{1}{m}}$ pois $\Delta x_1 = \Delta x_2 = \dots = \Delta x_m = \frac{1}{m}$

Tome $\boxed{c_i = \frac{i}{m}}$; $c_0 = 0$; $c_1 = \frac{1}{m}$, $c_2 = \frac{2}{n}$

$$\sum_{i=1}^{m} f(c_i)\underbrace{\Delta x_i}_{\frac{1}{m}} = \sum_{i=1}^{m} f\left(\frac{i}{m}\right)\cdot\frac{1}{m} = \sum_{i=1}^{m}\left(\frac{i}{m}\right)^2\cdot\frac{1}{m} = \sum_{i=1}^{m}\frac{i^2}{m^3}$$

$$= \frac{1^2 + 2^2 + 3^2 + \dots + m^2}{m^3} = \frac{\frac{1}{6}m(n+1)(2n+1)}{m^3}$$

Fato: $\boxed{1^2 + 2^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1)}$

$$\int_0^1 x^2\,dx = \lim_{\Delta \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n \to \infty} \frac{\cancel{n^3}\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)}{6\cancel{n^3}} = \frac{2}{6} = \frac{1}{3} \Rightarrow \int_0^1 x^2\,dx = \frac{1}{3}$$