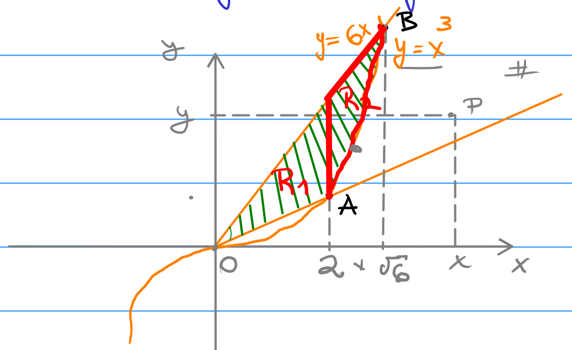
19/11 - Aula 33 - Integração de Funções Racionais

Exemplo: Calcule a área no primeiro quadrante $x \geq 0$ e $y \geq 0$ delimitada pela curva $y = x^3$, pelas retas $y = 4x$ e $y = 6x$.



$R_1 = \{(x,y) \mid 0 \leq x \leq 2$ e $4x \leq y \leq 6x\}$

$R_2 = \{(x,y) \mid 2 \leq x \leq \sqrt{2}$ e $x^3 \leq y \leq 6x\}$

Seja A a intersecção da reta $y = 4x$ com o gráfico $y = x^3$

$$\begin{cases} y = 4x \\ y = x^3 \end{cases} \Rightarrow 4x = x^3 \Rightarrow x^3 - 4x = 0 \Leftrightarrow x(x^2-4) = 0 \Rightarrow x = 0 \text{ ou } x = \pm 2$$

Seja B a intersecção da reta $y = 6x$ com o gráfico de $y = x^3$

$$\begin{cases} y = 6x \\ y = x^3 \end{cases} \Rightarrow x^3 - 6x = 0 \Rightarrow x(x^2-6) = 0 \Leftrightarrow x = 0 \text{ ou } x = \pm\sqrt{6}$$

$R_1 = \{(x,y) \mid 0 \leq x \leq 2$ e $4x \leq y \leq 6x\}$

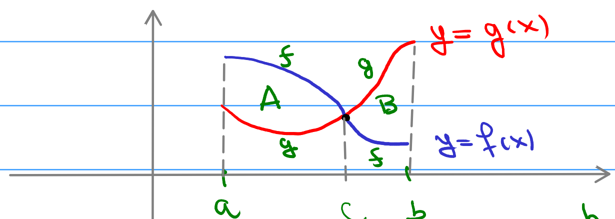
* $R_2 = \{(x,y) \mid 2 \leq x \leq \sqrt{6}$ e $x^3 \leq y \leq 6x\}$

$$\text{área de } R_1 = \int_0^2 [6x - 4x]\,dx = \int_0^2 2x\,dx = x^2\Big|_0^2 = 2^2 - 0^2 = 4$$

$$\text{área de } R_2 = \int_2^{\sqrt{6}} [6x - x^3]\,dx \overset{TFC}{=} \left[3x^2 - \frac{x^4}{4}\right]_2^{\sqrt{6}}$$

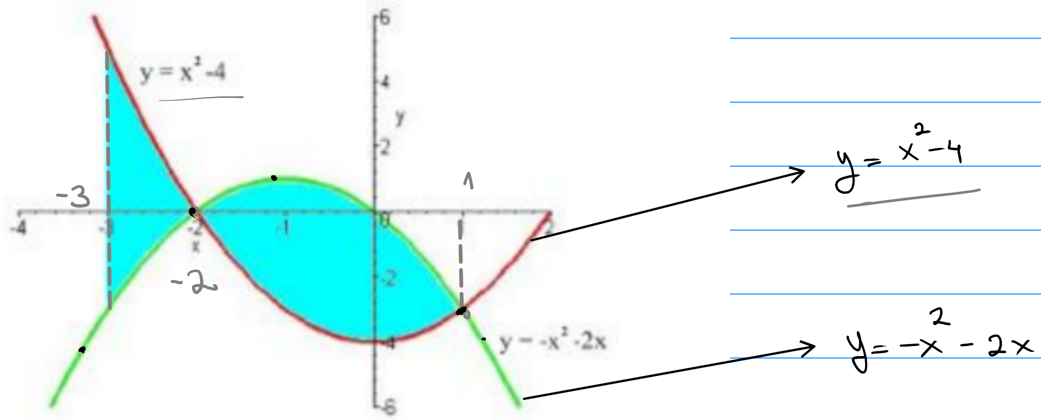
$$= 3(\sqrt{6})^2 - \frac{(\sqrt{6})^4}{4} - \left[3 \cdot 2^2 - \frac{2^4}{4}\right] = 18 - 9 - 12 + 4 = 22 - 21 = 1$$

área de R = área de R_1 + área de R_2 = 4 + 1 = 5



$$A = \int_a^b [\overbrace{g(x)}^{\text{cima}} - \underbrace{f(x)}_{\text{baixo}}]\,dx$$

(+) $A = \int_a^c [f(x) - g(x)]\,dx$ e $B = \int_c^b [g(x) - f(x)]\,dx$ (+)



$$\begin{cases} y = x^2 - 4 \\ y = -x^2 - 2x \end{cases} \Rightarrow x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \quad \div 2$$

$$\Rightarrow x^2 + x - 2 = 0$$

$$\Delta = 1^2 - 4.1.(-2) = 9 \Rightarrow x = \frac{-1 \pm \sqrt{9}}{2} \Rightarrow x = 1 \text{ ou } x = -2$$

$$A = \int_{-3}^{-2} [x^2 - 4 - (-x^2 - 2x)]\,dx + \int_{-2}^{1} [-x^2 - 2x - (x^2 - 4)]\,dx$$

$$= \ldots = ?$$

Decomposição de funções racionais em frações parciais

Função racional $\frac{p(x)}{q(x)}$ tal que grau de $q(x) \geq$ grau de $p(x)$

Se grau de $q(x)$ seja m e que

$$q(x) = (a_1x + b_1).(a_2x + b_2) + \ldots + (a_mx + b_m) = 0$$

$$x = \frac{-b_1}{a_1} \text{ ou } x = \frac{-b_2}{a_2}, \ldots, x = \frac{-b_m}{a_m} \quad (m\text{-raízes}).$$

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(a_1x + b_1)\cdot(a_2x + b_2)\ldots(a_mx + b_m)} \overset{*}{=}$$

$$\overset{*}{=} \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_m}{a_mx + b_m} \quad ; \quad A_1, \ldots, A_m \in \mathbb{R}$$

frações parciais

$$\int \frac{p(x)}{q(x)}\,dx = \int \left[\frac{A_1}{a_1x+b_1} + \cdots + \frac{A_m}{a_mx+b_m} \right] dx$$

$$= \int \frac{A_1}{a_1x+b_1}\,dx + \cdots + \int \frac{A_m}{a_mx+b_m}\,dx \overset{*}{=}$$

Note que

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\int \frac{1}{u}\,du = \frac{A}{a}\ln|u| + C$$

Seja $u = ax+b \Rightarrow du = a\,dx \Rightarrow dx = \frac{1}{a}\,du$

$$\overset{*}{=} \frac{A_1}{a_1}\ln|a_1x+b_1| + \frac{A_2}{a_2}\ln|a_2x+b_2| + \cdots + \frac{A_m}{b_m}\ln|a_mx+b_m| + C$$

Exemplo: Calcule $\int \frac{x^2-3}{(x^2-4)(2x+1)}\,dx$

$$\frac{x^2-3}{(x^2-4)(2x+1)} = \boxed{\frac{x^2-3}{(x-2).(x+2).(2x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{2x+1}}$$

$$x^2-4 = x^2-2^2 = (x-2).(x+2)$$

$$\frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{2x+1} = \frac{A(x+2)(2x+1) + B(x-2)(2x+1) + C\overbrace{(x-2)(x+2)}^{x^2-4}}{(x-2).(x+2).(2x+1)}$$

$$= \frac{A(2x^2+5x+2) + B(2x^2-3x-2) + Cx^2-4C}{(x^2-4)(2x+1)}$$

$$\boxed{= \frac{(2A+2B+C)x^2 + (5A-3B)x + (2A-2B-4C)}{\cancel{(x^2-4)(2x+1)}} = \frac{x^2-3}{\cancel{(x^2-4)(2x+1)}}}$$

$$\Leftrightarrow (2A+2B+C)x^2 + (5A-3B)x + (2A-2B-4C) = x^2 + 0.x - 3$$

$$\begin{cases} 2A + 2B + C = 1 \\ 5A - 3B = 0 \\ 2A - 2B - 4C = -3 \end{cases} \iff A = \frac{1}{20} \; ; \; B = \frac{1}{12} \; e \; C = \frac{11}{15}$$

Logo,

$$\int \frac{x^2 - 3}{(x^2-4)(2x+1)}\, dx = \int \frac{1/20}{x-2}\, dx + \int \frac{1/12}{x+2}\, dx + \int \frac{11/15}{2x+1}\, dx$$

$$= \frac{1}{20}\int \frac{1}{x-2}\, dx + \frac{1}{12}\int \frac{1}{x+2}\, dx + \frac{11}{15}\int \frac{1}{2x+1}\, dx$$

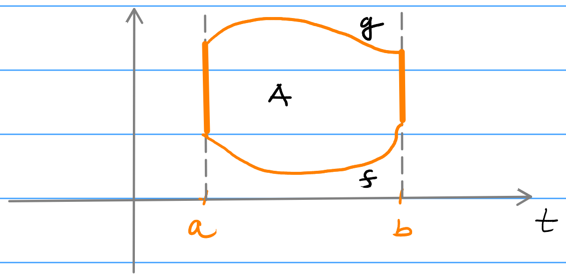
$$= \frac{1}{20} \ln|x-2| + \frac{1}{12} \ln|x+2| + \frac{11}{30} \ln|2x+1| + C$$

Aplicações da integral definida

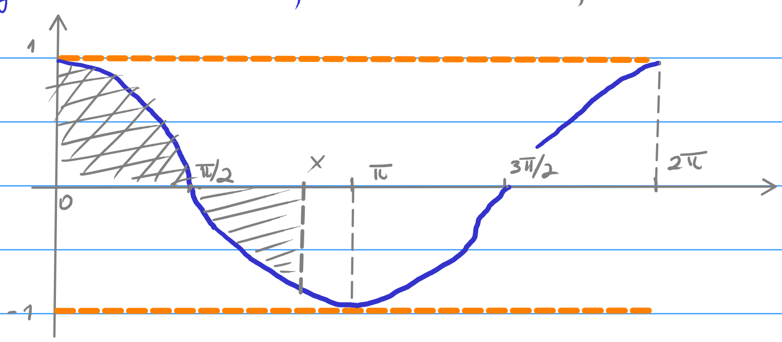
1 - Cálculo de áreas :

$$R = \{ (x, y) \mid a \leq x \leq b \;\; e \;\; f(x) \leq y \leq g(x) \}$$

$$A = \int_a^b [g(x) - f(x)]\, dx$$

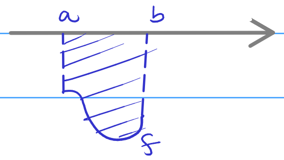


Ex. Seja $f(t) = \cos t$, $0 \leq t \leq 2\pi$, $0 \leq x \leq 2\pi$



$$F(x) = \int_0^x \cos t \, dt$$

Para cada $x \in [0, 2\pi]$ temos que $F(x)$ é a área rachurada



$$\int_a^b f(t)\,dt < 0$$

$F(x) = \int_0^x \cos t \, dt$.

(1) Quais são os zeros de $F(x)$?

(2) Quais são os pontos críticos de $F(x)$?

Respostas:

(1) $F(x) = \int_0^x \cos t \, dt \underset{TFC}{=} \operatorname{sen} t \Big|_0^x = \operatorname{sen} x - \underbrace{\operatorname{sen} 0}_{=0} = \operatorname{sen} x$

$\Rightarrow F(x) = \operatorname{sen} x = 0$, $0 \leq x \leq 2\pi \iff \boxed{x=0}$ e $\boxed{x=\pi}$ $\boxed{x = 2\pi}$

Logo $F(0) = 0$ e $F(\pi) = 0$

$F(x) = \int_0^x \cos t \, dt \Rightarrow F(0) = \int_0^0 \cos t \, dt = 0$

Se $x = \pi \Rightarrow F(\pi) = \boxed{\int_0^\pi \cos t \, dt = 0}$

(2) $F'(x) = \dfrac{d}{dx}\int_0^x \cos t \, dt \underset{TFC}{=} \cos x \Rightarrow F'(x) = \cos x$

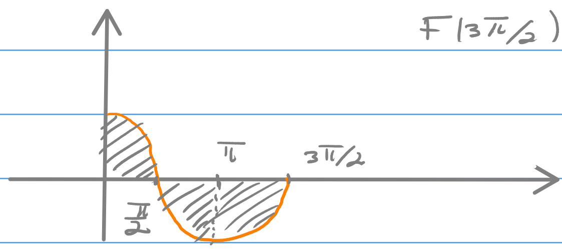
$\Rightarrow F'(x) = 0 \iff \cos x = 0 \iff x = \dfrac{\pi}{2}$ ou $x = \dfrac{3\pi}{2}$

$F''(x) = -\operatorname{sen} x \Rightarrow F''(\pi/2) = -\operatorname{sen}(\pi/2) = -1 < 0 \Rightarrow x = \dfrac{\pi}{2}$ é pto max. local

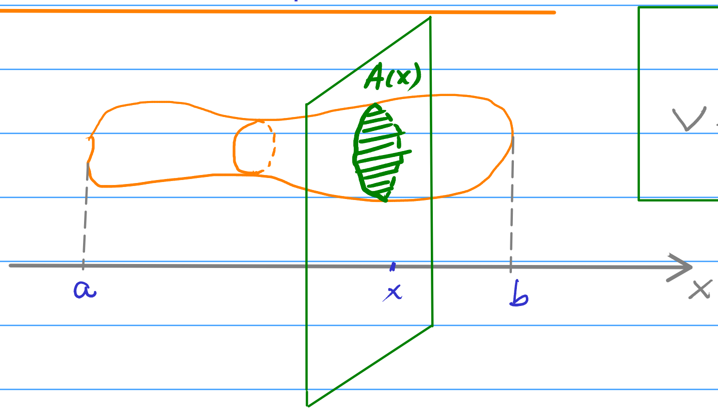
$F''(3\pi/2) = -\operatorname{sen}(3\pi/2) = -(-1) = 1 > 0 \Rightarrow \dfrac{3\pi}{2}$ é mínimo local.

$\underbrace{F(\pi/2)}_{max}$ e $\underbrace{F(3\pi/2)}_{min}$

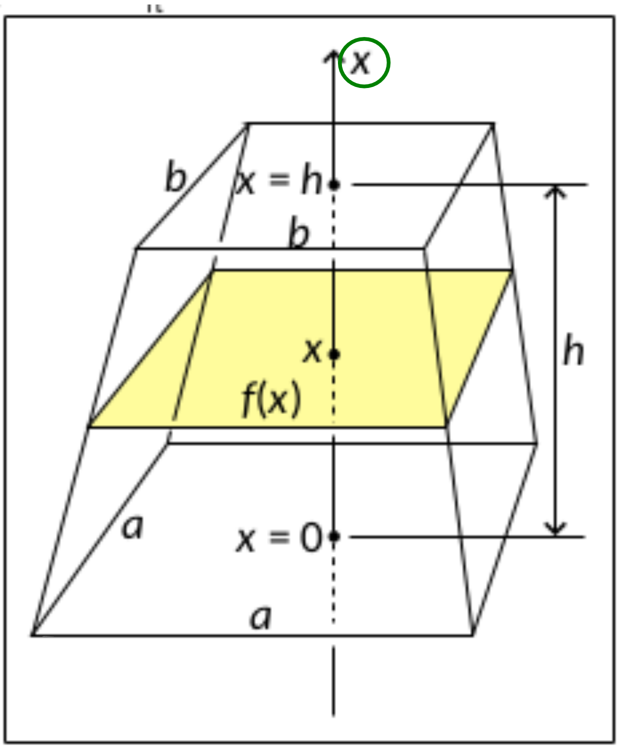
$F(3\pi/2)$

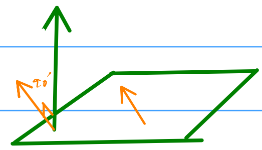


2 - Volume de um sólido por fatiamento



$$\boxed{V = \int_a^b A(x)\,dx}$$





$A(x) = a + \frac{(b-a)}{h} \cdot x$ onde

$0 \leq x \leq h$

$\Downarrow$

$$V = \int_0^h \underbrace{A(x)}_{?}\,dx$$

$$V = \int_a^h \left(a + \frac{(b-a)}{h}\,x\right) dx = \frac{h}{3}\left(a^2 + ab + b^2\right)$$