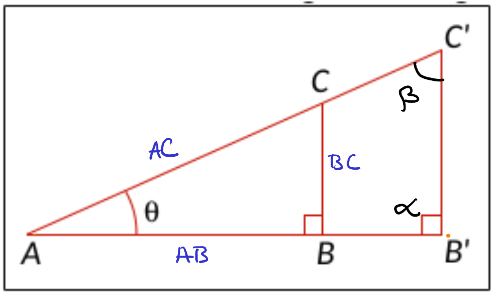
20/09 - Aula 15 - Funções Trigonométricas e o "primeiro limite fundamental"



$\alpha + \beta + \theta = 180°$

$\alpha = 90°$

ABC e AB'C'

$\cos\beta = \dfrac{B'C'}{AC'}$

$\operatorname{sen}\beta = \dfrac{AB'}{AC'}$

$\operatorname{tg}\beta = \dfrac{AB'}{B'C'}$

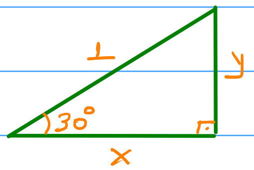
Considerando o triângulo ABC temos:

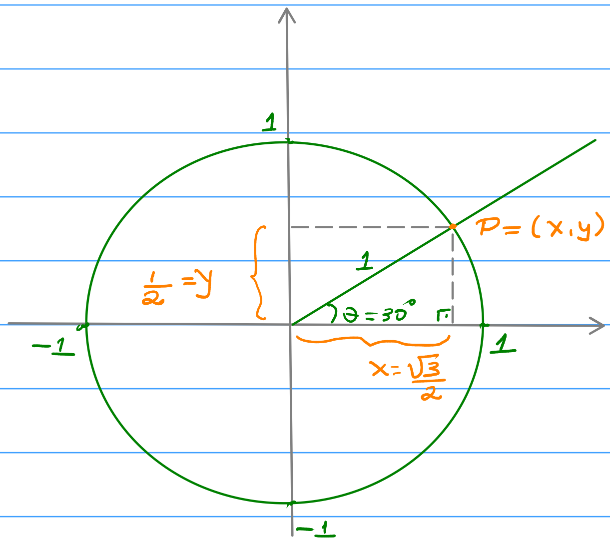
$\cos\theta = \dfrac{AB}{AC} = \dfrac{\text{cateto adj}}{\text{hip.}} = \dfrac{AB'}{AC'}$

$\operatorname{sen}\theta = \dfrac{BC}{AC} = \dfrac{\text{cateto oposto}}{\text{hip.}} = \dfrac{B'C'}{AC'}$

$\operatorname{tg}\theta = \dfrac{BC}{AB} = \dfrac{\text{cateto oposto}}{\text{cateto adjacente}} = \dfrac{B'C'}{AB'}$

θ	cos θ	sen θ	tg θ
30°	$\sqrt{3}/2$	$1/2$	$1/\sqrt{3}$
45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$



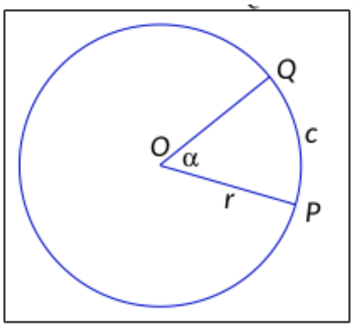


$\cos 30° = \dfrac{ca}{hip} = \dfrac{x}{1} \Rightarrow \boxed{x = \cos 30°} \Rightarrow \cos 30° = \dfrac{\sqrt{3}}{2}$

$\operatorname{sen} 30° = \dfrac{co}{hip} = \dfrac{y}{1} \Rightarrow \boxed{y = \operatorname{sen} 30°} \Rightarrow \operatorname{sen} 30° = \dfrac{1}{2}$

$(\operatorname{sen} 30°)^2 + (\cos 30°)^2 = \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{3}{4} = \dfrac{4}{4} = 1$

$$\boxed{\operatorname{sen}^2\theta + \cos^2\theta = 1}$$



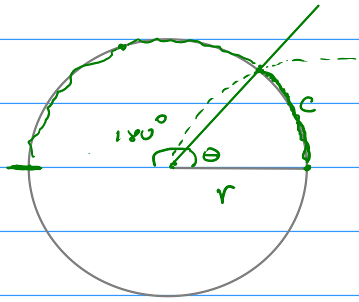
$c = \widehat{PQ}$ = arco de um círculo de raio $\underline{r}$

$c = r \cdot$ (medida de α em radianos)

Se $\alpha = 360^\circ$ = 360 graus = 2π radianos

$\Downarrow$

$360^\circ = 2\pi$ radianos



$\theta = 1$ radiano

π radianos $\cong$ 3,14 radianos

$180^\circ \longrightarrow \pi$ radianos

$360^\circ \longrightarrow 2\pi$ radianos

$180 = \pi$ radianos

$360 = 2\pi$ radianos

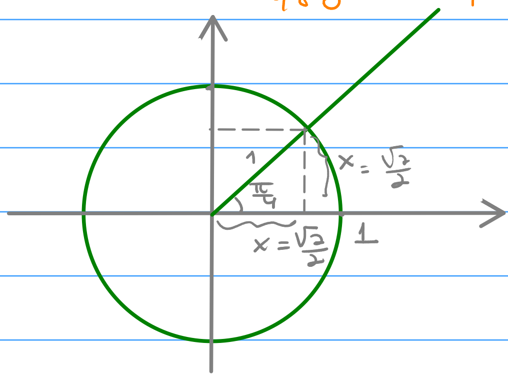
Ex. Seja $\theta = 45^\circ$ então

π —— 180°

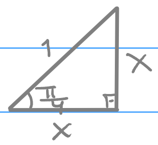
x —— 45°

$$45\pi = 180x \Rightarrow x = \frac{45\pi}{180} = \frac{\pi}{4} \Rightarrow \boxed{x = \frac{\pi}{4}}$$

$$\begin{cases} \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \text{sen}\,\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \text{tg}\,\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1 \end{cases}$$



$$\frac{\pi}{4} = 45^\circ$$



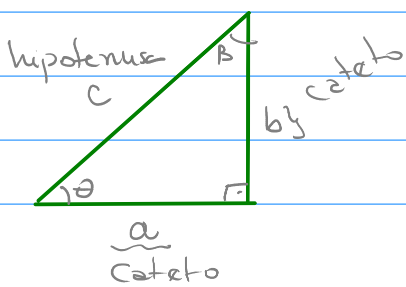
$$x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$\Rightarrow x = \frac{\sqrt{2}}{2}$$

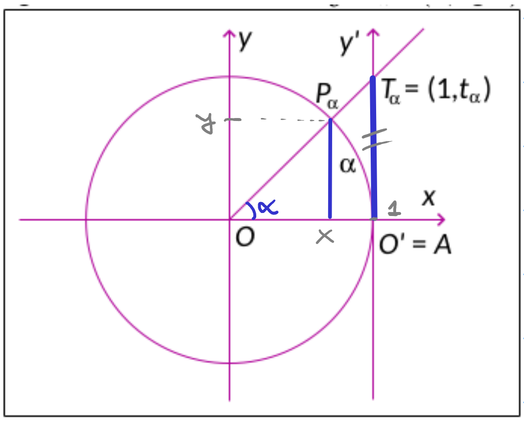
Teorema de Pitágoras:

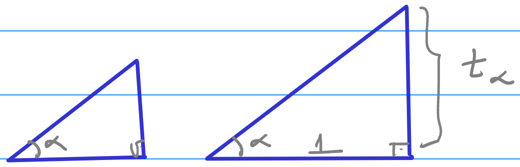
$\text{sen}^2\theta + \cos^2\theta = 1$

$\text{sen}^2\beta + \cos^2\beta = 1$



$$c^2 = a^2 + b^2$$

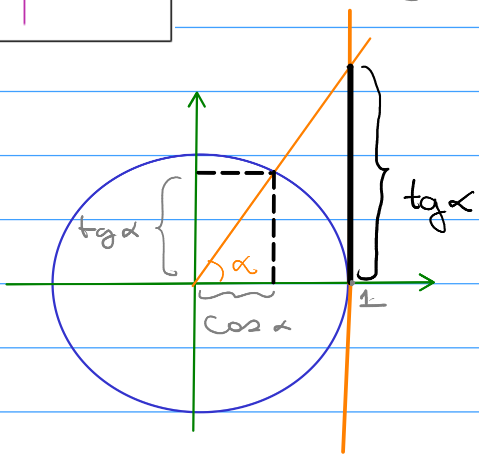




$$tg\,\alpha = \frac{co}{ca} = \frac{\operatorname{sen}\alpha}{\cos\alpha}$$

$$tg\,\alpha = \frac{co}{ca} = \frac{t_\alpha}{1} = t_\alpha = tg\,\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha}$$

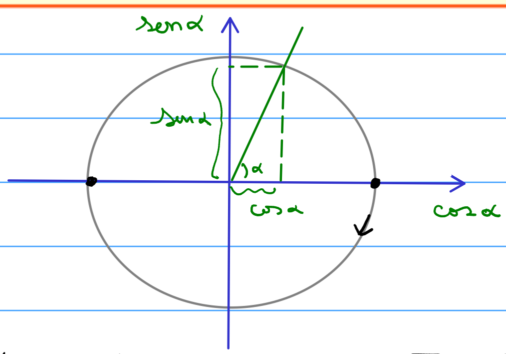
$tg\,\alpha = ?$

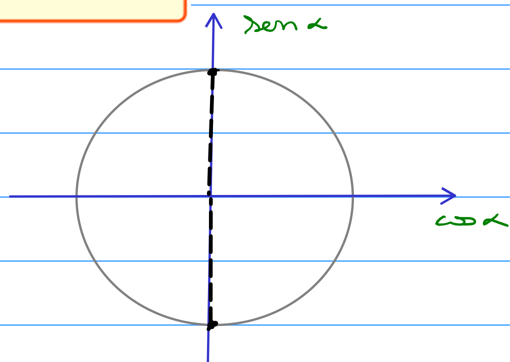


$$\text{cotangente de } \alpha = \operatorname{cotg}\alpha = \frac{\cos\alpha}{\operatorname{sen}\alpha} \qquad (\alpha \neq n\pi, \forall n \in \mathbb{Z})$$

$$\text{secante de } \alpha = \sec\alpha = \frac{1}{\cos\alpha} \qquad (\alpha \neq \frac{\pi}{2} + n\pi, \forall n \in \mathbb{Z})$$

$$\text{cossecante de } \alpha = \operatorname{cosec}\alpha = \frac{1}{\operatorname{sen}\alpha} \qquad (\alpha \neq n\pi, \forall n \in \mathbb{Z})$$





$\operatorname{sen}\alpha = 0 \iff \alpha = 0, \pi, 2\pi, 3\pi, \ldots, -\pi, -2\pi$

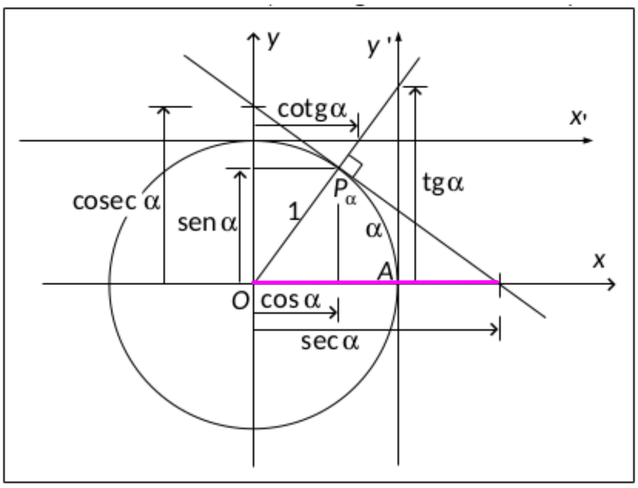
$$\boxed{\operatorname{sen}\alpha = 0 \iff \alpha = m\pi, \quad \forall m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}}$$

$\cos\alpha = 0 \iff \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots, -\frac{\pi}{2}, -\frac{3\pi}{2}, \ldots$

$$\boxed{\cos\alpha = 0 \iff \alpha = \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{Z}}$$

Seja $f(x) = tg\,x = \frac{\operatorname{sen} x}{\cos x}, \quad x \in \mathbb{R}$

$$\Rightarrow \boxed{D(f) = \left\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + m\pi, \ m \in \mathbb{Z}\right\}}$$



Relações trigonométricas:

1. $\cos^2\alpha + \text{sen}^2\alpha = 1$

2. $1 + \text{tg}^2\alpha = \sec^2\alpha$

$$1 + \left(\frac{\text{sen}\,\alpha}{\cos\alpha}\right)^2 = 1 + \frac{\text{sen}^2\alpha}{\cos^2\alpha} = \frac{\overbrace{\cos^2\alpha + \text{sen}^2\alpha}^{1}}{\cos^2\alpha} =$$

$$= \frac{1}{\cos^2\alpha} = \left(\frac{1}{\cos\alpha}\right)^2 = \sec^2\alpha$$

3. $\text{sen}\,(a+b) = \text{sen}\,a \cdot \cos b + \text{sen}\,b \cdot \cos a$

$\text{sen}\,(a-b) = \text{sen}\,a \cdot \cos b - \text{sen}\,b \cdot \cos a$

$\cos(a+b) = \cos a \cdot \cos b - \text{sen}\,a \cdot \text{sen}\,b$

$\Downarrow$

$\cos(a+a) = \cos a \cdot \cos a - \text{sen}\,a \cdot \text{sen}\,a$

$$\boxed{\cos(2a) = \cos^2 a - \text{sen}^2 a}$$

$\cos^2 a = 1 - \text{sen}^2 a$

$\text{sen}^2 a = 1 - \cos^2 a$

$\cos 2a = 1 - \text{sen}^2 a - \text{sen}^2 a$

$\Downarrow$

$$\boxed{\begin{gathered}\cos 2a = 1 - 2\,\text{sen}^2 a \\ \text{sen}^2 a = \frac{1 - \cos 2a}{2}\end{gathered}}$$

$\cos 2a = \cos^2 a - (1 - \cos^2 a)$

$\Downarrow$

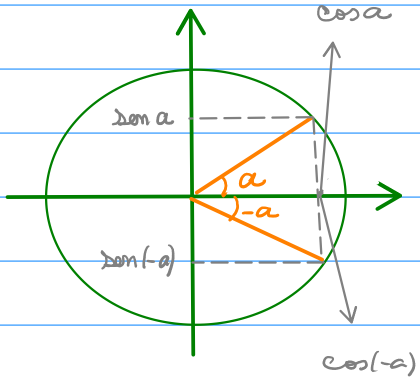
$$\boxed{\begin{gathered}\cos 2a = 2\cos^2 a - 1 \\ \cos^2 a = \frac{1 + \cos 2a}{2}\end{gathered}}$$

$\cos(a-b) = \cos a \cdot \cos b + \text{sen}\,a \cdot \text{sen}\,b$

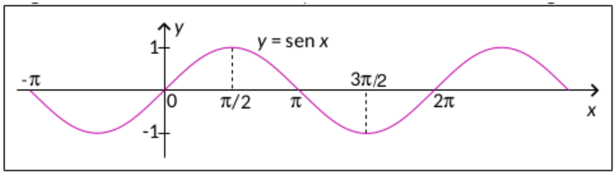
4. $\cos(-a) = \cos a$

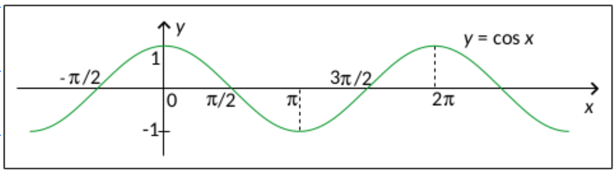
$\text{sen}\,(-a) = -\text{sen}\,a$

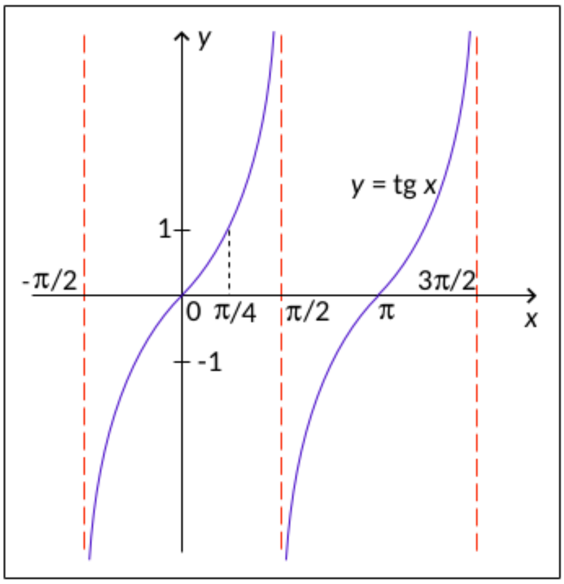
$$\text{tg}\,(-a) = \frac{\text{sen}\,(-a)}{\cos(-a)} = \frac{-\text{sen}\,a}{\cos a} = -\text{tg}\,a$$



$$\begin{cases} \text{sen}\,(-a) = -\text{sen}\,a \Rightarrow \text{sen}\,a \text{ é ímpar} \\ \cos(-a) = \cos a \Rightarrow \cos a \text{ é par} \end{cases}$$



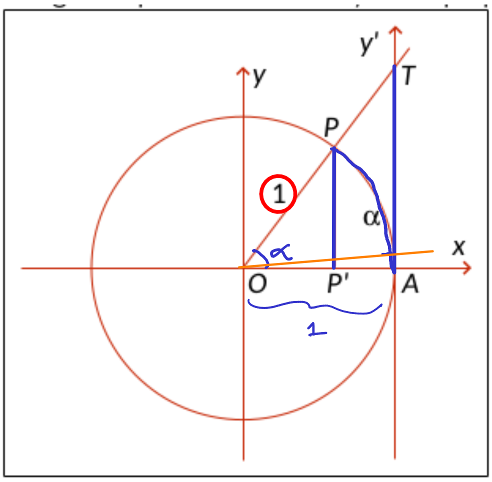




$tg(-\alpha) = -tg(\alpha)$

Proposição 11.1 (Primeiro limite fundamental).

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$



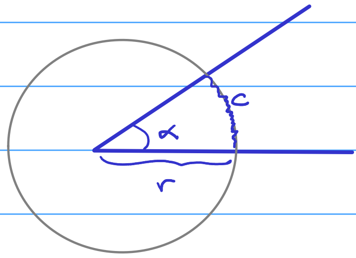
$\frac{\operatorname{sen}\alpha}{\alpha} \approx 1$ qdo $\alpha \approx 0$

$PP' < \widehat{PA} < AT$

$PP' = \operatorname{sen}\alpha$

$AT = tg\,\alpha$

$\widehat{PA} = \alpha \cdot 1 = \alpha$



$c = \alpha \cdot r$

$PP' < \widehat{PA} < AT \Rightarrow \operatorname{sen}\alpha < \alpha < tg\,\alpha \Rightarrow$

$\Rightarrow \operatorname{sen}\alpha < \alpha < \frac{\operatorname{sen}\alpha}{\cos\alpha}$ $\div \operatorname{sen}\alpha \neq 0$

$0 < \alpha < \frac{\pi}{2} \Rightarrow$ $\boxed{\operatorname{sen}\alpha > 0}$

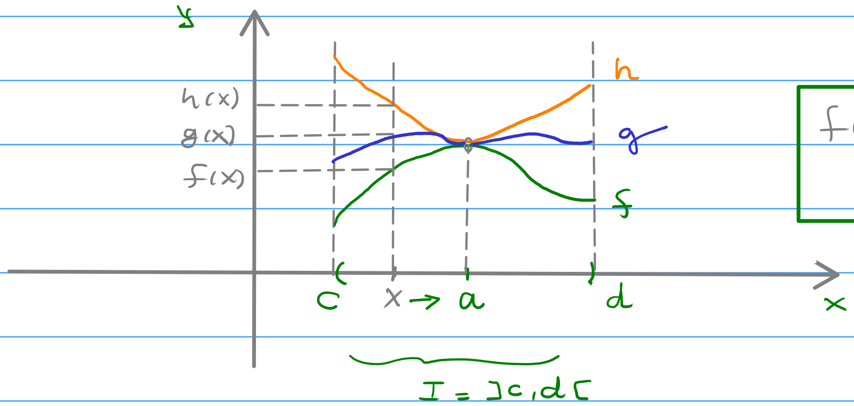
$1 = \frac{\operatorname{sen}\alpha}{\operatorname{sen}\alpha} < \frac{\alpha}{\operatorname{sen}\alpha} < \frac{\cancel{\operatorname{sen}\alpha}}{\cos\alpha} \cdot \frac{1}{\cancel{\operatorname{sen}\alpha}} = \frac{1}{\cos\alpha}$

$\Rightarrow$

$1 < \frac{\alpha}{\operatorname{sen}\alpha} < \frac{1}{\cos\alpha} \iff \boxed{\underset{\approx 1}{\cos\alpha} < \frac{\operatorname{sen}\alpha}{\alpha} < 1}$ (Pense!)

$\alpha \approx 0 \Rightarrow \cos\alpha \approx 1 \Rightarrow \frac{\operatorname{sen}\alpha}{\alpha} \approx 1 \Rightarrow$ $\boxed{\lim_{\alpha \to 0} \frac{\operatorname{sen}\alpha}{\alpha} = 1}$

Teorema 11.1 (Teorema do confronto, ou teorema do sanduíche). *Sendo* $I \subset \mathbb{R}$ *um intervalo, sendo* $a \in I$, *e* f, g *e* h *funções definidas para* $x \in I$, $x \neq a$, *se* $f(x) \leq g(x) \leq h(x)$ *para todo* $x \in I, x \neq a$, *e se* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, *então* $\lim_{x\to a} g(x) = L$. *Vale o mesmo resultado para limites laterais (neste caso,* a *pode ser o extremo inferior ou superior do intervalo* I*). Vale o mesmo resultado se* $a = +\infty$ *ou* $-\infty$.



$$\boxed{\begin{array}{c} f(x) \leq g(x) \leq h(x) \\ \forall x \in I - \{a\} \end{array}}$$

Supondo que $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

então $\lim_{x\to a} g(x) = L$

Ex Calcular $\lim_{x\to 0} \frac{tg\, 4x}{x} = \frac{0}{0}$ (indeterminação)

Sol $\frac{tg\, 4x}{x} = \boxed{\frac{4 \cdot sen\, 4x}{4x}} \cdot \frac{1}{\cos 4x} = 4 \cdot \frac{sen\, 4x}{4x} \cdot \frac{1}{\cos 4x}$

$\Rightarrow \lim_{x\to 0} \frac{tg\, 4x}{x} = \lim_{x\to 0} 4 \cdot \frac{sen\, 4x}{4x} \cdot \frac{1}{\cos 4x} \overset{*}{=}$

(*) $\lim_{x\to 0} \frac{sen\, 4x}{4x} = \lim_{u\to 0} \frac{sen\, u}{u} = 1$

$u = 4x \to 0$

(**) $\lim_{x\to 0} \frac{1}{\cos 4x} = \frac{1}{\cos 4.0} = \frac{1}{\underbrace{\cos 0}_{1}} = 1$

$\overset{*}{=} 4 . \lim_{x\to 0} \frac{sen\, 4x}{4x} . \lim_{x\to 0} \frac{1}{\cos 4x} = 4 . 1 . 1 = 4$

□

$$\lim_{x\to 0} \frac{\text{sen}\, mx}{mx} = \lim_{u\to 0} \frac{\text{sen}\, u}{u} = 1$$

$u = mx$

Ex. $\lim\limits_{x\to 0} \dfrac{1-\cos ax}{bx} = \dfrac{0}{0}$ indeterminação.

$$\frac{1-\cos a.0}{b.0} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$$

$$\frac{1-\cos ax}{bx} = \frac{1-\cos ax}{bx} \cdot \frac{1+\cos ax}{1+\cos ax}$$

$$= \frac{1-\cos^2 ax}{bx(1+\cos ax)} = \frac{\text{sen}^2 ax}{bx(1+\cos ax)}$$

$$= \frac{a}{b} \cdot \frac{1}{\ } \cdot \frac{\text{sen}\, ax}{ax} \cdot \frac{\text{sen}\, ax}{1+\cos ax}$$

$$= \frac{a}{b} \cdot \frac{\text{sen}\, ax}{ax} \cdot \frac{\text{sen}\, ax}{1+\cos ax}$$

$$\lim_{x\to 0} \frac{1-\cos ax}{bx} = \frac{a}{b} \lim_{x\to 0} \frac{\text{sen}\, ax}{ax} \cdot \lim_{x\to 0} \frac{\text{sen}\, ax}{1+\cos ax}$$

$$= \frac{a}{b} \cdot 1 \cdot \frac{0}{1+1} = \frac{a.0}{2b} = 0 \qquad \square$$

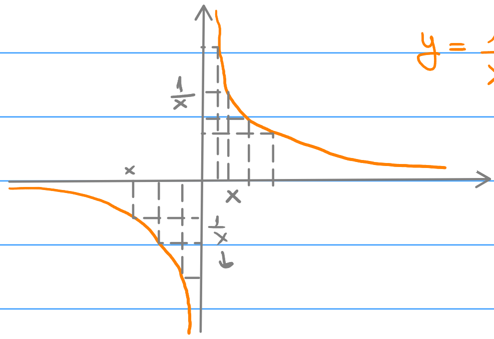
$$\lim_{x\to 0} \frac{\text{sen}\, ax}{x} = \lim_{x\to 0} a \cdot \frac{\text{sen}\, ax}{ax}$$

$$= a \lim_{x\to 0} \frac{\text{sen}\, \overset{u}{\overline{ax}}}{\underset{}{\overset{u}{\overline{ax}}}}$$

$$= a \cdot \lim_{u\to 0} \frac{\text{sen}\, u}{u} = a \cdot 1 = a$$

4. Sendo $f(x) = 2^{\frac{1}{x}}$, calcule os limites laterais $\lim_{x\to 0^+} f(x)$ e $\lim_{x\to 0^-} f(x)$.

Resposta. $+\infty$ e 0, respectivamente.



$y = \frac{1}{x} \Rightarrow \lim_{x\to 0^+} \frac{1}{x} = +\infty$ e $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$$\lim_{x\to 0^+} 2^{\frac{1}{x}} = 2^{+\infty} = +\infty$$

$$\lim_{x\to 0^-} 2^{\frac{1}{x}} = 2^{-\infty} = \frac{1}{2^{+\infty}} = \frac{1}{+\infty} = 0$$