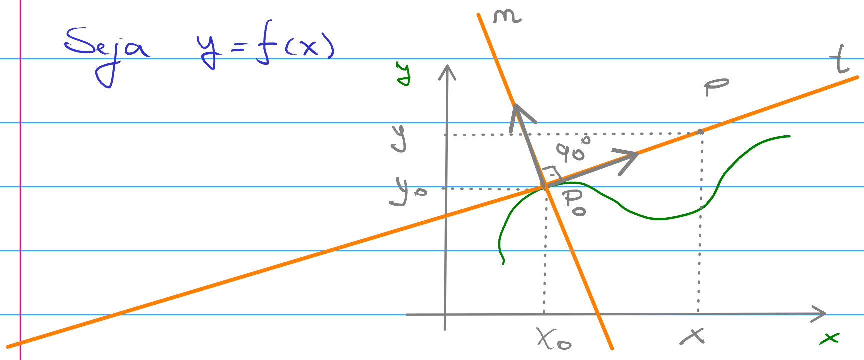
23/08 - Aula 4 - Regra da Cadeia

Seja $y = f(x)$



$m_t = \dfrac{y - y_0}{x - x_0} = f'(x_0)$

$m_t \cdot m_m = -1$

$f'(x_0) \cdot m_m = -1$

$m_m = \dfrac{-1}{f'(x_0)}$

Reta tangente: $\boxed{y - y_0 = f'(x_0) \cdot (x - x_0)}$

Reta normal: $\dfrac{y - y_0}{x - x_0} = \dfrac{-1}{f'(x_0)} \iff \boxed{y - y_0 = \dfrac{-1}{f'(x_0)} (x - x_0)}$

Regra do Produto para a derivada:

$$\boxed{(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x)}$$

Ex. Seja $f(x) = x^5 + 2x + 3$ e $g(x) = x^6 - 1$, calcule $(f(x)\,g(x))'$

Sol (I) Aplicando a propriedade distributiva

$f(x)g(x) = (x^5 + 2x + 3)(x^6 - 1) = x^{11} - x^5 + 2x^7 - 2x + 3x^6 - 3$

$\Rightarrow$

$(f(x)g(x))' = (x^{11} + 2x^7 + 3x^6 - x^5 - 2x - 3)'$

$= (x^{11})' + (2x^7)' + (3x^6)' - (x^5)' - (2x)' - \underbrace{(3)'}_{=0}$

$(x^n)' = nx^{n-1}$

$= \boxed{11x^{10} + 14x^6 + 18x^5 - 5x^4 - 2}$

(II) Regra do Produto:

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$= (x^5 + 2x + 3)' \cdot (x^6 - 1) + (x^5 + 2x + 3) \cdot (x^6 - 1)'$

$= (5x^4 + 2)(x^6 - 1) + (x^5 + 2x + 3) \cdot (6x^5)$

$$= 5x^{10} - 5x^4 + 2x^6 - 2 + 6x^{10} + 12x^6 + 18x^5$$

$$= \boxed{11x^{10} + 14x^6 + 18x^5 - 5x^4 - 2}$$

Prova da Regra do Produto.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \overset{0}{\Delta x}) - f(x)}{\underset{=0}{\Delta x}} = \frac{\overbrace{f(x) - f(x)}^{0}}{0} = \frac{0}{0}$$

$$(f(x) \cdot g(x))' = \lim_{\Delta x \to 0} \frac{\Delta(fg)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\, g(x+\Delta x) - f(x)\, g(x)}{\Delta x} \overset{*}{=}$$

Somando e subtraindo o termo $f(x)\, g(x+\Delta x)$ temos

$$\overset{*}{=} \lim_{\Delta x \to 0} \frac{f(x+\Delta x) g(x+\Delta x) - f(x) g(x+\Delta x) + f(x) g(x+\Delta x) - f(x)\, g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x))\, g(x+\Delta x) + f(x)\,(g(x+\Delta x) - g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \cdot g(x+\Delta x) + f(x) \cdot \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \right\}$$

$$= \left(\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right) \cdot \lim_{\Delta x \to 0} g(x+\Delta x) + f(x) \left(\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \right)$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x) \qquad \square$$

Regra 2.2. Se g é derivável e $g \neq 0$ então

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

$$\left(\frac{1}{g(x)}\right)' = \lim_{\Delta x \to 0} \frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x} =$$

$$\frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x} = \frac{\frac{g(x) - g(x+\Delta x)}{g(x+\Delta x)\cdot g(x)}}{\frac{\Delta x}{1}} = \frac{g(x) - g(x+\Delta x)}{\Delta x\,(g(x+\Delta x)g(x))} =$$

$$= -\frac{(g(x+\Delta x) - g(x))}{\Delta x} \cdot \frac{1}{g(x+\Delta x)\,g(x)}$$

$$\lim_{\Delta x \to 0} \frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x} = \lim_{\Delta x \to 0} \left[-\frac{(g(x+\Delta x) - g(x))}{\Delta x} \cdot \frac{1}{g(x+\Delta x)\,g(x)} \right]$$

$$= -g'(x) \cdot \frac{1}{g(x)g(x)} = -\frac{g'(x)}{g(x)^2}$$

Portanto, $\boxed{\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g(x)^2}}$

Exemplo: Mostre que $(x^{-m})' = -m.x^{-m-1}$.

Sol

$$(x^{-m})' = \left(\frac{1}{x^m}\right)' = -\frac{(x^m)'}{(x^m)^2} = \frac{-m x^{m-1}}{x^{2m}} = -m . x^{m-1} . x^{-2m} =$$

$g(x) = x^m \Leftarrow \left(\frac{1}{g(x)}\right)'$

$$= -m . x^{m-1-2m} = -m\, x^{-m-1} \qquad \square$$

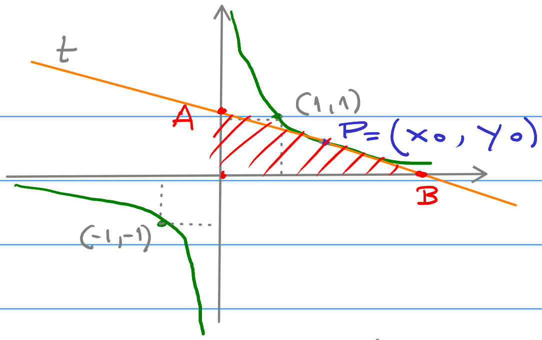
Obs: $(fgh)' = f'gh + fg'h + fgh'$

$$(\underbrace{fg}_{j}.h)' = (fg)'.h + (fg).h'$$

$$= (f'g + fg')h + fgh'$$

$$= f'gh + fg'h + fgh'$$

Ex: Seja $f(x) = \frac{1}{x}$, $x \neq 0$



$t: \frac{y - y_0}{x - x_0} = f'(x_0)$ ("$\frac{1}{x_0}$")

onde $y_0 = f(x_0) = \frac{1}{x_0}$

$$f'(x) = \left(\frac{1}{x}\right)' = (x^{-1})' = -1 \cdot x^{-1-1} = -1 \cdot x^{-2} = -\frac{1}{x^2}$$

Logo, $f'(x_0) = -\frac{1}{x_0^2}$. Assim,

$$t: \frac{y - 1/x_0}{x - x_0} = -\frac{1}{x_0^2} \iff y - \frac{1}{x_0} = \frac{-(x - x_0)}{x_0^2}$$

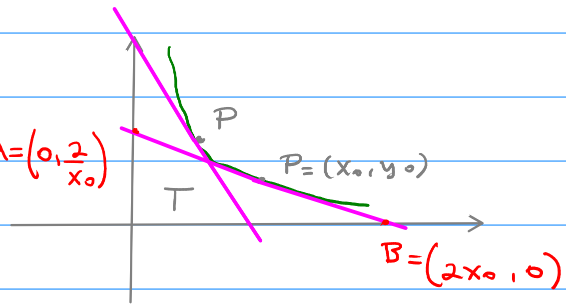
$$\Rightarrow y = \frac{1}{x_0} - \frac{1}{x_0^2}(x - x_0) \iff \boxed{y = -\frac{x}{x_0^2} + \frac{2}{x_0}} \quad (t)$$

$$y = \frac{1}{x_0} - \frac{x}{x_0^2} + \frac{x_0}{x_0^2}$$

Substituindo $x = 0$ em (t) obtemos $y = \frac{2}{x_0} \Rightarrow A = \left(0, \frac{2}{x_0}\right)$

Substituindo $y = 0$ em (t) obtemos $0 = -\frac{x}{x_0^2} + \frac{2}{x_0}$

$$\Rightarrow \frac{x}{x_0^2} = \frac{2}{x_0} \Rightarrow x = 2x_0 \Rightarrow B = (2x_0, 0)$$



$$\text{área}(T) = \frac{b \times h}{2} = \frac{2x_0 \cdot 2/x_0}{2} = \frac{4}{2} = \boxed{2}$$

Ex: Seja $y = (x^8+1)^{10}$ então calcule $\frac{dy}{dx}$.

Notação: $\frac{dy}{dx} = y'(x)$

Sol. $y = (x^8+1)^{10} = \sum_{k=0}^{10} \binom{10}{k} (x^8)^k \cdot 1^{10-k} = \sum_{k=0}^{10} \binom{10}{k} x^{8k}$

$$y = \binom{10}{0} + \underbrace{\binom{10}{1} x^8}_{k=1} + \underbrace{\binom{10}{2} x^{16}}_{k=2} + \cdots + \binom{10}{10} x^{80}$$

$$\frac{dy}{dx} = \binom{10}{1} \cdot 8x^7 + \binom{10}{2} \cdot 16x^{15} + \cdots + \binom{10}{10} \cdot 80x^{79}.$$ (Fato da Vida)

$y = (x^8+1)^{10}$.

Aplicando a Regra da Cadeia obtemos:

$$\frac{dy}{dx} = 10 \cdot (x^8+1)^{10-1} \cdot (x^8+1)' = 10(x^8+1)^9 \cdot 8x^7$$

$$\boxed{\frac{dy}{dx} = 80x^7(x^8+1)^9}$$

$y = (x^8+1)^{10}$; seja $u = x^8+1 \Rightarrow \frac{du}{dx} = 8x^7 \Rightarrow y = u^{10} \Rightarrow \frac{dy}{du} = 10u^9$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}} = 10u^9 \cdot 8x^7 = 80 \cdot (x^8+1)^9 \cdot x^7$$

Composição de Funções

Sejam, $y = f(x)$ e $y = g(x)$ duas funções

Podemos definir uma operação da por

$$F(x) = (f \circ g)(x) = f(g(x))$$

Ex: Sejam $f(x) = \sqrt{x}$ e $g(x) = x^2 + x + 1$

$$(f \circ g)(x) = f(g(x)) = f(x^2+x+1) = \sqrt{x^2+x+1}$$

$$f(g(x)) = \sqrt{g(x)} = \sqrt{x^2+x+1}$$

$x^2+x+1 \geq 0$

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + \sqrt{x} + 1$$
$$= x + \sqrt{x} + 1 \quad \text{(Função Irracional)}$$

$|x| = \sqrt{x^2}$; $(\sqrt{x})^{\cancel{2}} = x$ $\Rightarrow$ $x \geq 0$

$$(f \circ g)(x) = \sqrt{x^2+x+1} \Rightarrow D_{f \circ g} = \{x \in \mathbb{R} \mid x^2+x+1 \geq 0\}$$

$$(g \circ f)(x) = x + \sqrt{x} + 1 \Rightarrow D_{g \circ f} = \{x \in \mathbb{R} \mid x \geq 0\}$$

$$\frac{d}{dx}\sqrt{x^2+x+1} = \frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x)$$

$$f(x) = \sqrt{x} \Rightarrow \boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

$$g(x) = x^2+x+1 \Rightarrow g'(x) = 2x+1$$

Temos $\displaystyle \frac{d}{dx}\sqrt{x^2+x+1} = \frac{1}{2\sqrt{g(x)}} \cdot (2x+1) = \frac{2x+1}{2\sqrt{x^2+x+1}}$ ///

Regra 3.1 (Regra da Cadeia). Se $y = f(u)$ e $u = g(x)$ então

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Em outras palavras, $y = f(g(x))$, temos

$$\boxed{\frac{dy}{dx} = [f(g(x))]' = f'(g(x)) \cdot g'(x)}$$

Ex Seja $y = \left[(\underbrace{x^{50} + 2x}) ^2 + 1\right]^{10}$ calcule $\frac{dy}{dx}$

$$\frac{dy}{dx} = 10\left[(x^{50}+2x)^2+1\right]^9 \cdot 2(x^{50}+2x)^1 \cdot (50x^{49}+2)$$

$z = u^{10}$

Seja, $u = (x^{50}+2x)^2+1$ $\left.\right\}$ $\Rightarrow$ $y = u^{10} = (v^2+1)^{10}$

$v = (x^{50}+2x)$

$\Downarrow$

$u = v^2+1$

$y = z(u(v(x)))$

$\Downarrow$

$$\frac{dy}{dx} = \frac{dz}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \quad \underline{\underline{*}}$$

* $z = u^{10} \Rightarrow \frac{dz}{du} = 10 \cdot u^9$

* $u = v^2+1 \Rightarrow \frac{du}{dv} = 2v$

* $v = x^{50}+2x \Rightarrow \frac{dv}{dx} = 50x^{49}+2$

$$\frac{dy}{dx} = 10u^9 \cdot 2v \cdot (50x^{49}+2)$$

$$= 10\left[(x^{50}+2x)^2+1\right]^9 \cdot 2(x^{50}+2x) \cdot (50x^{49}+2)$$ ''