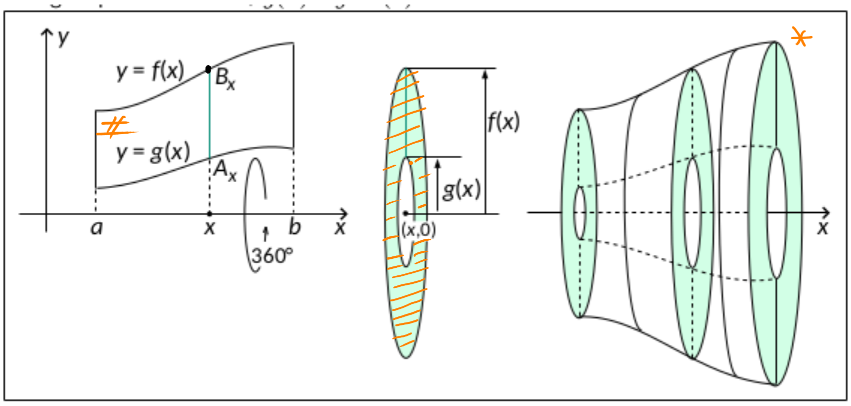
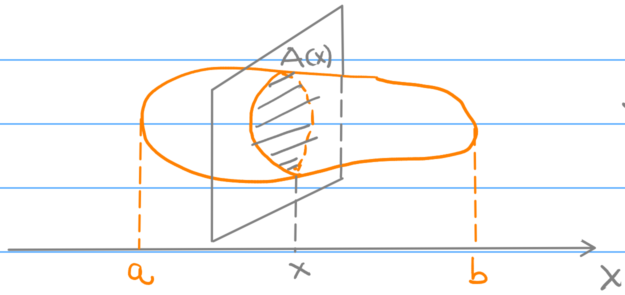
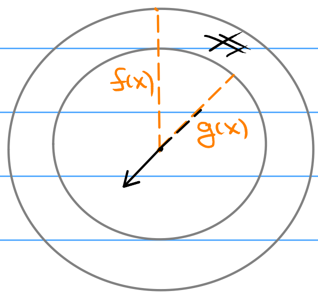
# 22/11 - Aula 34 - Aplicações da integral definida

## Volume de um sólido de revolução por fatiamento



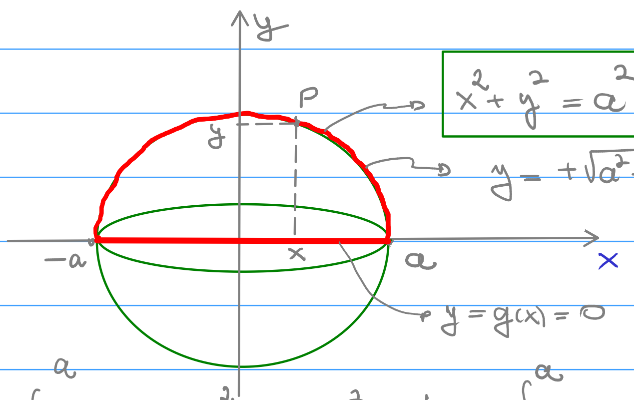

$$V = \int_a^b A(x)\,dx$$




$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2$$
$$= \pi[f(x)^2 - g(x)^2]$$
$$\Downarrow$$
$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)^2 - g(x)^2]\,dx$$

## Ex. Cálcule o volume da esfera de raio a

Note que a esfera de raio $\underline{\underline{a}}$ pode ser vista como o sólido de revolução da curva $x^2 + y^2 = a^2$, $\underline{\underline{y \geq 0}}$




$\boxed{x^2 + y^2 = a^2}$, $a > 0$

$y = +\sqrt{a^2 - x^2} = f(x)$

$y = g(x) = 0$

$$V = \int_{-a}^{a} \pi . [f(x)^2 - \underbrace{g(x)^2}_{=0}]\,dx = \int_{-a}^{a} \pi\,[\cancel{\sqrt{a^2 - x^2}}]^{\cancel{2}}\,dx$$

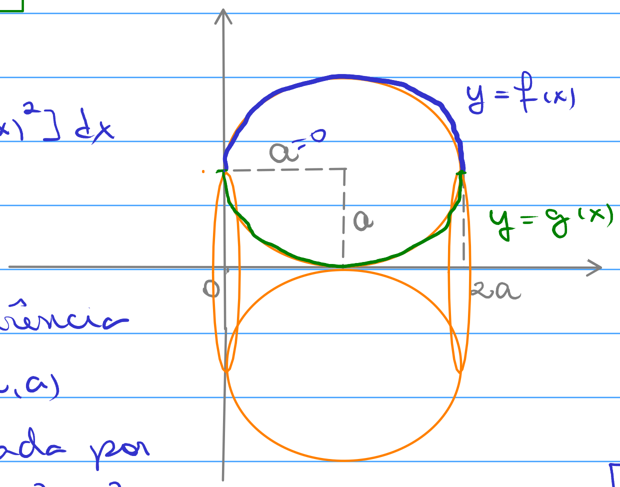
$$= \int_{-a}^{a} \pi (a^2 - x^2)\, dx = \pi \cdot \int_{-a}^{a} (a^2-x^2)\, dx \overset{TFC}{=} \pi \left[a^2 x - \frac{x^3}{3}\right]\Bigg|_{-a}^{a}$$

$$= \pi \left[ a^2 \cdot a - \frac{a^3}{3} - \left(a^2 \cdot (-a) - \frac{(-a)^3}{3}\right) \right]$$

$$= \pi \left[ a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} \right] = \pi \left[ \frac{3a^3 - a^3 + 3a^3 - a^3}{3} \right] = \frac{4}{3}\pi a^3$$

$$\Rightarrow \boxed{V = \frac{4}{3}\pi a^3}$$

$$V = \int_0^{2a} \pi \left[ f(x)^2 - g(x)^2 \right] dx$$




A eq. da circunferência
de centro $C = (a, a)$
e raio $\underline{a}$ é dada por

$$(x-a)^2 + (y-a)^2 = a^2$$

$$(y-a)^2 = a^2 - (x-a)^2$$

$$y - a = \pm\sqrt{a^2 - (x-a)^2}$$

$$y = a \pm \sqrt{a^2 - (x-a)^2}$$

$$f(x) = a + \sqrt{a^2 - (x-a)^2}$$

$$g(x) = a - \sqrt{a^2 - (x-a)^2}$$

$$V = \pi \int_0^{2a} \left[ \left(a + \sqrt{a^2 - (x-a)^2}\right)^2 - \left(a - \sqrt{a^2 - (x-a)^2}\right)^2 \right] dx$$

$$C = (0, a) \iff x^2 + (y-a)^2 = a^2$$

<u>obs</u>: $\int_{-a}^{a} \underbrace{(a^2 - x^2)}_{f(x)}\, dx = \int_{-a}^{a} f(x)\, dx = F(a) - F(-a) = \left[F(x)\right]_{-a}^{a}$

$$= \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

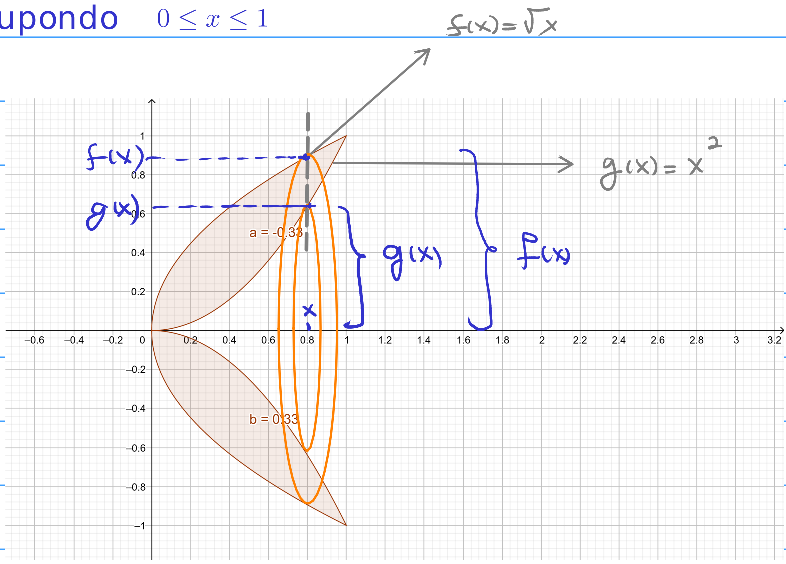
onde $F'(x) = a^2 - x^2$

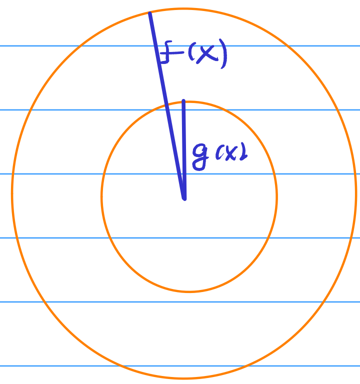
Logo, $F(x) = a^2 x - \frac{x^3}{3}$

$$F'(x) = a^2 - \frac{\cancel{3}x^2}{\cancel{3}} = a^2 - x^2 = f(x)$$

$$= a^2 \cdot a - \frac{a^3}{3} - \left(a^2(-a) - \frac{(-a)^3}{3}\right) = \ldots$$

Ex. Encontre o volume do sólido de revolução formado pela rotação em torno do eixo x da região entre os gráficos das funções $f(x) = \sqrt{x}$ e $g(x) = x^2$ supondo $0 \leq x \leq 1$




$$V = \pi\int_0^1 [f(x)]^2 - [g(x)]^2 \, dx = \pi\int_0^1 (\sqrt{x})^2 - (x^2)^2 \, dx$$

$$= \pi\int_0^1 (x - x^4)\, dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left[\frac{1}{2} - \frac{1}{5}\right] = \left(\frac{5-2}{10}\right)\pi = \frac{3\pi}{10}$$

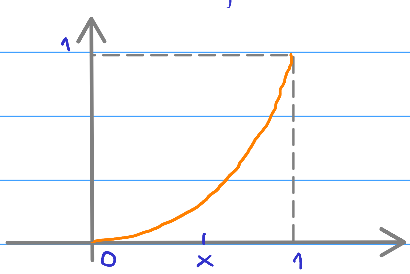
$V \approx 0{,}9424$ um$^3$

## Comprimento de uma curva

Supondo que $f'(x)$ seja contínua no intervalo [a,b], temos que o comprimento da curva $y = f(x), \quad a \leq x \leq y$ é dada por

$$\boxed{s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

Ex. Calcule o comprimento da curva $y = x^2$, $0 \leq x \leq 1$




$f(x) = x^2 \Rightarrow f'(x) = 2x$

$$S = \int_0^1 \sqrt{1 + (2x)^2}\, dx$$

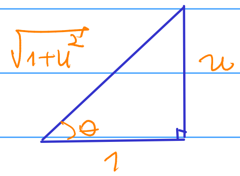
$$= \int_0^1 \sqrt{1 + (2x)^2}\, dx = \frac{1}{2}\int_0^2 \sqrt{1 + u^2}\, du$$

$u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2} du$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = 2$

$$\Rightarrow S = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du$$




$$tg\theta = \frac{co}{ca} = \frac{u}{1} \Rightarrow \boxed{u = tg\theta}$$

$$du = sec^2\theta\,d\theta \Rightarrow S = \frac{1}{2}\int_0^{arctg\,2} \sqrt{1+tg^2\theta}\cdot sec^2\theta\,d\theta \Rightarrow$$

"$\sqrt{sec^2\theta} = sec\theta$"

$u=0 \Rightarrow tg\theta = 0 \Rightarrow \theta = 0$

$u=2 \Rightarrow tg\theta = 2 \Rightarrow \theta = arctg\,2$

$$S = \frac{1}{2}\int_0^{arctg\,2} sec^3\theta\,d\theta \overset{TFC}{=} \frac{1}{2}\left[\frac{1}{2}sec\theta\cdot tg\theta + \frac{1}{2}ln|sec\theta + tg\theta|\right]_0^{arctg2}$$

$$S = \frac{1}{4}\Big[sec(arctg\,2)\cdot tg(arctg\,2) + ln|sec(arctg\,2) + tg(arctg\,2)|$$

$$- \cancel{\underbrace{sec\,0}_{=1}\cdot\underbrace{tg\,0}_{=0}} - \cancel{ln|\underbrace{sec\,0}_{1} + \underbrace{tg\,0}_{0}|}$$

$tg(arctg\,2) = 2$

$sec(arctg(2)) = \sqrt{1+(tg(arctg(2)))^2} = \sqrt{1+2^2} = \sqrt{5}$

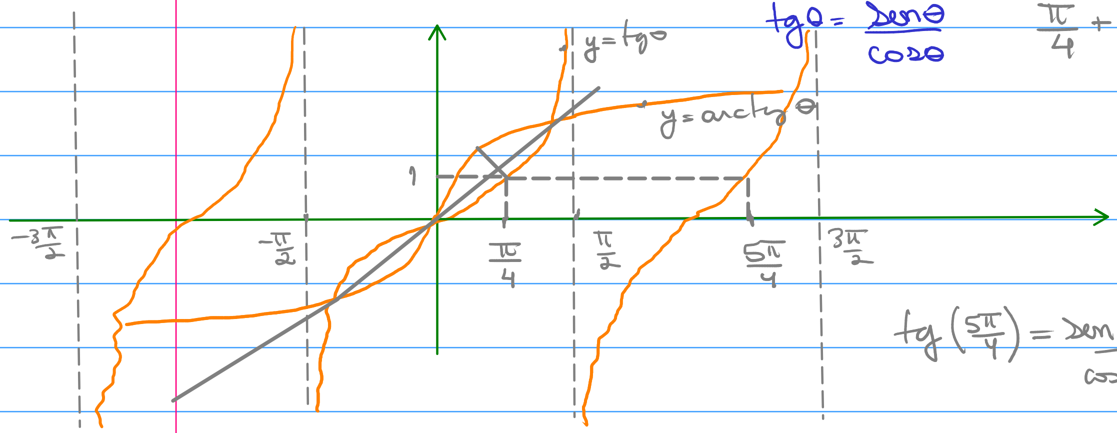
$$\Rightarrow S = \frac{1}{4}\left[\sqrt{5}\cdot 2 + ln|\sqrt{5}+2|\right] \Rightarrow$$

$$\Rightarrow \boxed{S = \frac{\sqrt{5}}{2} + \frac{1}{4}ln(\sqrt{5}+2)}$$

$tg\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

$tg\theta = \frac{sen\theta}{cos\theta}$

$\frac{\pi}{4} + \pi = \frac{5\pi}{4}$




$$tg\left(\frac{5\pi}{4}\right) = \frac{sen\left(\frac{5\pi}{4}\right)}{cos\left(\frac{5\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Ex. Calcule o comprimento da curva $y = \ln x$ , de $x = \sqrt{3}$ a $x = \sqrt{8}$

Sol

$$S = \int_a^b \sqrt{1+(f'(x))^2}\, dx \qquad f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$S = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1+\left(\frac{1}{x}\right)^2}\, dx = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{\frac{x^2+1}{x^2}}\, dx = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x}\, dx$$

Seja $\boxed{u = \sqrt{1+x^2}} \Rightarrow du = \dfrac{x}{\sqrt{1+x^2}}\, dx \Rightarrow dx = \dfrac{\sqrt{1+x^2}}{x}\, du$

$$\Rightarrow \boxed{dx = \frac{u}{x}\, du}$$

Se $x = \sqrt{3} \Rightarrow u = \sqrt{1+(\sqrt{3})^2} = 2$

Se $x = \sqrt{8} \Rightarrow u = \sqrt{1+(\sqrt{8})^2} = 3$

$$\Rightarrow S = \int_2^3 \frac{u}{x} \cdot dx = \int_2^3 \frac{u}{x} \cdot \frac{u}{x}\, du = \int_2^3 \frac{u^2}{x^2}\, du = \int_2^3 \frac{u^2}{u^2-1}\, du$$

$u = \sqrt{1+x^2} \Rightarrow u^2 = 1+x^2 \Rightarrow x^2 = u^2-1$

$\cancel{x}+1-\cancel{x}+1$

$$\frac{u^2}{u^2-1} - 1 = \frac{\cancel{u^2} - \cancel{u^2} + 1}{u^2-1} = \frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]$$

$$\frac{u^2}{u^2-1} = 1 + \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right] \Rightarrow$$

$$S = \int_2^3 1 + \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right] du$$

Resp. $1 + \frac{1}{2}\ln\left(\frac{3}{2}\right)$

$$= \left\{ u + \frac{1}{2}\left[\ln|u-1| - \ln|u+1|\right] \right\}_2^3 = \ldots$$

$$= 3 + \frac{1}{2}\left[\ln 2 - \ln 4\right]$$