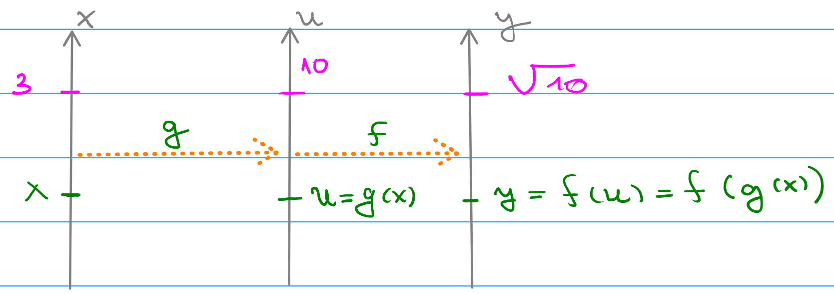
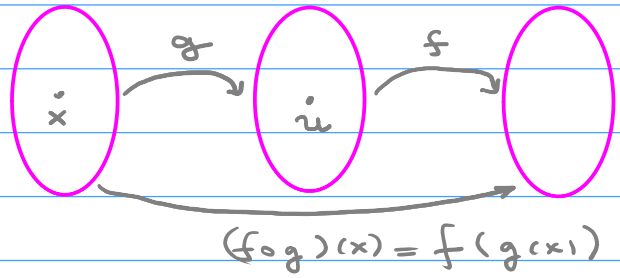
25/08 - Aula 5 - Derivada Implícita



Sejam $g(x) = x^2+1$ e $f(u) = \sqrt{u}$

$g(3) = 3^2+1 = 10$ $\qquad f(10) = \sqrt{10}$



Regra da Cadeia $\Rightarrow$

$$\boxed{(f\circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Ex. Calcule $\dfrac{d}{dx}\sqrt{x^2+1}$. $\quad$ Seja $F(x) = \sqrt{x^2+1}$

Pela definição de derivada temos

$$\frac{d}{dx}\underbrace{\sqrt{x^2+1}}_{F(x)} = \lim_{\Delta x\to 0}\frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt{(x+\Delta x)^2+1} - \sqrt{x^2+1}}{\Delta x} =$$

obs: $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) \overset{(*)}{=} a-b$ $\quad$ (Distributiva)

$$\Downarrow$$

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a} + \sqrt{b}}$$

$$= \frac{\sqrt{(x+\Delta x)^2+1} - \sqrt{x^2+1}}{\Delta x} \cdot \frac{\sqrt{(x+\Delta x)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+\Delta x)^2+1} + \sqrt{x^2+1}} \overset{(*)}{=} \frac{(x+\Delta x)^2+1 - [x^2+1]}{\Delta x\left[\sqrt{(x+\Delta x)^2+1} + \sqrt{x^2+1}\right]}$$

$$\frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 + \cancel{1} - \cancel{x^2} - \cancel{1}}{\Delta x\left[\sqrt{(x+\Delta x)^2+1} + \sqrt{x^2+1}\right]} = \frac{(2x + \Delta x)\cdot\cancel{\Delta x}}{\cancel{\Delta x}\cdot\left[\sqrt{(x+\Delta x)^2+1} + \sqrt{x^2+1}\right]}$$

Logo, $F'(x) = \lim\limits_{\Delta x \to 0} \dfrac{2x + \Delta x}{\sqrt{(x+\Delta x)^2+1} + \sqrt{x^2+1}} = \dfrac{2x}{\sqrt{x^2+1} + \sqrt{x^2+1}}$

Assim, $\dfrac{d}{dx}\sqrt{x^2+1} = \dfrac{x}{\sqrt{x^2+1}}$

Aplicando a Regra de Cadeia.

$$F(x) = f(g(x))$$

$$F'(x) = f'(\underbrace{g(x)}_{u}) \cdot g'(x) = f'(u) \cdot g'(x) \overset{*}{=} \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$g(x) = x^2+1 \Rightarrow g'(x) = 2x$

$f(u) = \sqrt{u} = u^{\frac{1}{2}} \Rightarrow f'(u) = \frac{1}{2} u^{\frac{1}{2}-1} = \frac{1}{2} \cdot u^{-1/2} = \dfrac{1}{2\sqrt{u}}$

$$f(g(x)) = \frac{1}{2\sqrt{g(x)}} = \frac{1}{2\sqrt{x^2+1}}$$

Regra: $F(x) = u(x)^m$, $m \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

$$F'(x) = \frac{d}{du}(u^m) \cdot \frac{d}{dx}(u(x)) = m \cdot u^{m-1} \cdot u'(x)$$

Logo, $\boxed{\dfrac{d}{dx}[u(x)^m] = m\, u(x)^{m-1} \cdot u'(x)}$ $\qquad \dfrac{d}{dx}[y(x)^2] = 2 \cdot y^{2-1} \cdot y' = 2y \cdot y'$

DEF: Dizemos que a variável y é definida implicitamente pela variável x se existir uma função $F(x,y)$ e uma constante $c \in \mathbb{R}$ tais que

$$\boxed{F(x,y) = c}$$

Ex (1) $x^2 + y^2 = 2$, $F(x,y) = x^2 + y^2$, F

$x^2+y^2 = 2 \Leftrightarrow F(x,y) = c$

(2) $x^3 + y^3 = x + y + xy$,

Seja $F(x,y) = x^3 + y^3 - x - y - xy$

$x^3+y^3 = x+y+xy \iff F(x,y) = 0$

$F(x,y) = 2$

$x^2 + y^2 = 2 \iff y^2 = 2 - x^2$

$\iff y = \pm\sqrt{2-x^2}$

$\iff y = y(x)$

$\frac{dy}{dx} = y'(x)$

Logo, $\boxed{x^2 + y(x)^2 = 2}$ (1)

Vamos aplicar derivada implícita para calcular $\frac{dy}{dx}$

Derivando ambos membros de (1) com respeito à x obtemos

$$\frac{d}{dx}[x^2 + y(x)^2] = \frac{d}{dx}[2] \iff$$

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y(x)^2] = 0 \iff$$

$$2x + 2y(x)^{2-1} \cdot y'(x) = 0 \iff$$

$$2x + \underbrace{2y \cdot y'}_{(*)} = 0 \iff y'(x) = \frac{-2x}{2y(x)} = -\frac{x}{y(x)}$$

Como, $y(x) = \sqrt{2-x^2} \Rightarrow \boxed{y'(x) = \frac{-x}{\sqrt{2-x^2}}}$

Ex $x^3 + y^3 = x + y + xy$ (Equação Álgebrica)

$\boxed{y = y(x)} = y'(x)$

$$\frac{d}{dx}[x^3+y^3] = \frac{d}{dx}[x+y+xy]$$

$$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] = \frac{d}{dx}[x] + \frac{d}{dx}[y] + \frac{d}{dx}[xy]$$

// R.C // Regra do Produto

$$3x^2 + 3y^2 \cdot y' = 1 + y' + 1 \cdot y + x \cdot y' \Rightarrow$$

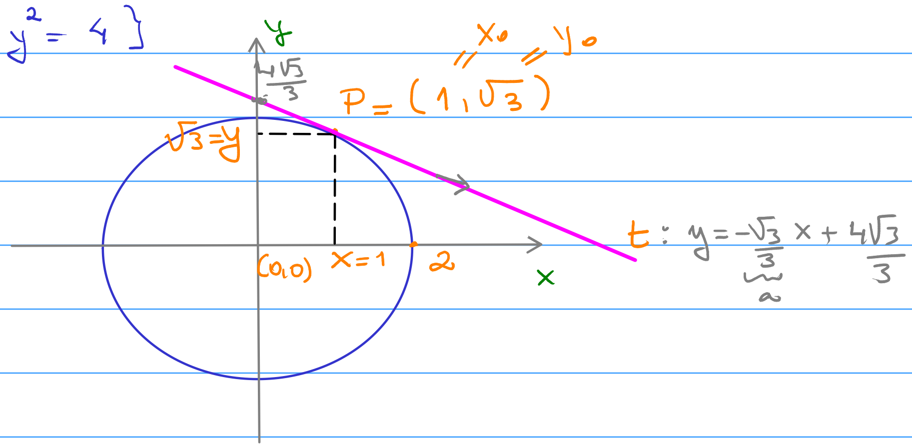
$$3x^2 + 3y^2 y' = 1 + y' + y + xy' \Rightarrow$$

$$3y^2y' - y' - xy' = 1 + y - 3x^2 \Rightarrow$$

$$(3y^2 - 1 - x)\cdot y' = 1 + y - 3x^2 \Rightarrow$$

$$\boxed{y' = \frac{1 + y - 3x^2}{3y^2 - 1 - x}}$$

Ex $C = \{(x,y) \mid x^2 + y^2 = 4\}$



$$x^2 + y^2 = 2^2$$

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

$$1^2 + y^2 = 4$$

$$y^2 = 3 \Rightarrow y = \pm\sqrt{3}$$

$$t: \quad y - y_0 = y'(x_0)(x - x_0)$$

$$y - \sqrt{3} = y'(1)(x - 1)$$

$$y'(x) = ?$$

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[4] = 0$$

$$2x + 2yy' = 0 \Rightarrow y' = \frac{-x}{y} \Leftrightarrow y'(x) = \frac{-x}{y(x)}$$

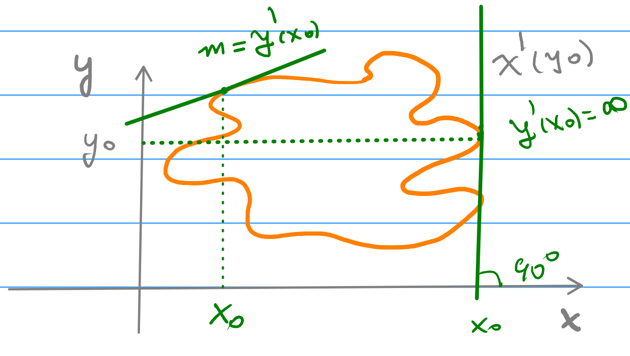
$$\Rightarrow y'(1) = -\frac{1}{y(1)} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Então,

$$\boxed{t: \quad y - \sqrt{3} = -\frac{\sqrt{3}}{3}(x - 1)}$$

$$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} + \sqrt{3}$$

$$y = -\frac{\sqrt{3}}{3}x + \frac{4\sqrt{3}}{3}$$



$$F(x,y) = C$$

$$\frac{d}{dx}F(x,y) = 0$$

$$tg\, 90° = +\infty$$

$$\frac{x - x_0}{y - y_0} = x'(y_0)$$