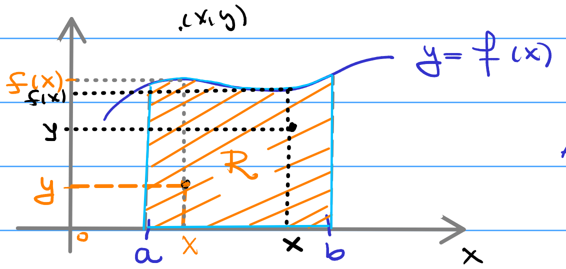
27/10 - Aula 28 - Teorema Fundamental do Cálculo

Se $f(x)$ é contínua em $[a,b]$ então $\int_a^b f(x)\,dx$ existe, $f \geq 0$ e representa a medida da área da seguinte região

$$R = \{ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b \ \text{ e } \ 0 \leq y \overset{*}{\leq} f(x) \}$$



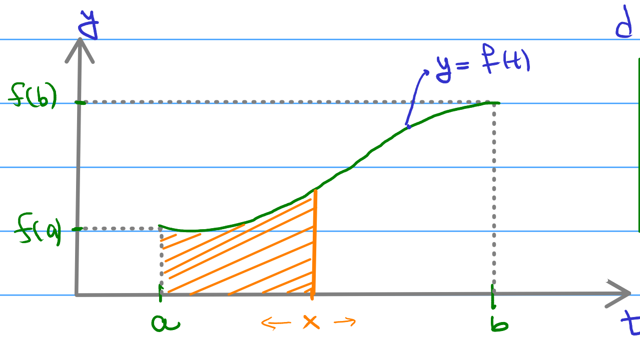
$Gr(f) = \{ (x,y) \in \mathbb{R}^2 \mid y = f(x) \}$

A = área de R

$$A = \int_a^b f(x)\,dx = \lim_{n \to \infty} S_n \ ;$$

$$S_n = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

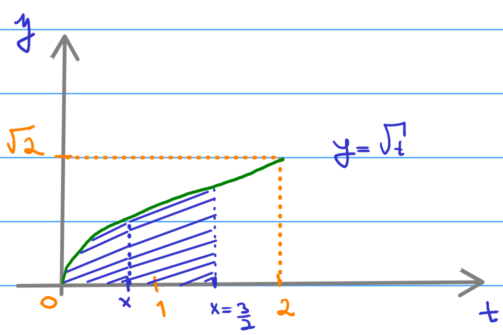
Seja $f(t)$ contínua em $[a,b]$, para cada $x \in [a,b]$ definimos



$$\boxed{\varphi(x) = \int_a^x f(t)\,dt}$$

Função Área

Ex. Seja $f(t) = \sqrt{t}$, $0 \leq t \leq 2$



$$\varphi(x) = \int_0^x f(t)\,dt = \int_0^x \sqrt{t}\,dt$$

$$\varphi(1/2) = \int_0^{1/2} \sqrt{t}\,dt =$$

$$\varphi(1) = \int_0^1 \sqrt{t}\,dt$$

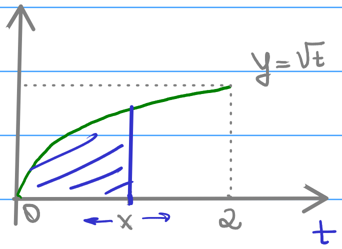
$\varphi(1/2) < \varphi(1)$

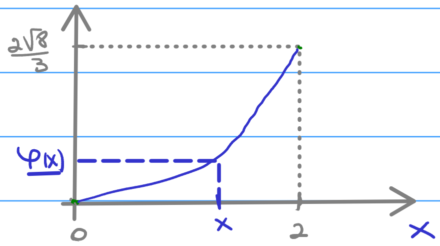
$$\varphi(x) = \int_0^x \underbrace{\sqrt{t}}_{f(t)}\,dt = F(x) - F(0) = \frac{2}{3}\sqrt{x^3} - 0 = \frac{2}{3}x^{3/2}$$

Se $f(t) = \sqrt{t} = t^{\frac{1}{2}} \Rightarrow F(t) = \dfrac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \dfrac{\sqrt{t^3}}{\frac{3}{2}} + c = \dfrac{2}{3}\sqrt{t^3} + c$

Logo $f(t) = \sqrt{t} \Rightarrow \varphi(x) = \frac{2}{3}x^{3/2}$ $\qquad x^a$

$\varphi(2) = \frac{2}{3}\sqrt{2^3} = \frac{2\sqrt{8}}{3}$





$$\varphi(x) = \frac{2}{3}x^{3/2} \Rightarrow \varphi'(x) = \frac{2}{3}\cdot\frac{3}{2}x^{3/2-1} = \sqrt{x} > 0$$

$$\Rightarrow \varphi''(x) = \frac{1}{2\sqrt{x}} > 0 \Rightarrow \cup$$

TFC, primeira versão: Seja f uma função contínua em $[a,b]$. Para cada $x \in [a,b]$, seja

$$\varphi(x) = \int_a^x f(t)\,dt$$

$f(t) = \ln t$ ✓
$f(x) = \ln x$ ✓
$f(u) = \ln u$ ✓

Então, $\varphi'(x) = f(x), \quad \forall x \in [a,b]$

obs: $\dfrac{d}{dx}\varphi(x) = f(x) \Leftrightarrow \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$

Em particular $\varphi(x)$ é uma primitiva de f.

Cuidado: $\displaystyle\int_a^x f(x)\,dx$, não faz sentido

obs.: $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

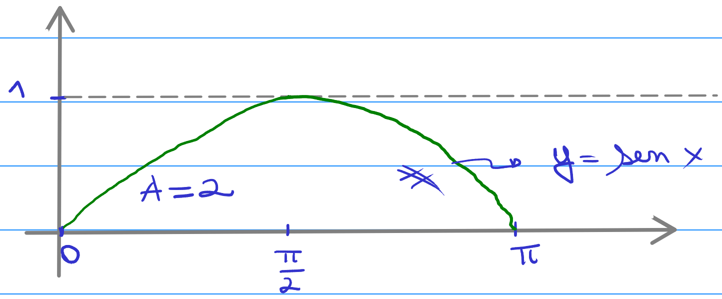
Teorema Fundamental do Cálculo-Segunda Versão

Seja f contínua em $[a,b]$ então,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

para cada $F'(x) = f(x)$

Ex. Calcule a área entre a curva $y = \operatorname{sen} x$ e o eixo x, para $0 \leq x \leq \pi$



$$A = \int_0^\pi \operatorname{sen} x\,dx = F(\pi) - F(0), \quad \text{onde } F(x) = -\cos x$$

$$= 1 - (-1) = 2$$

$$\Downarrow$$

$$F(\pi) = -\cos\pi = -(-1) = 1$$

$$F(0) = -\cos 0 = -1$$

$$A = \int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

Notação: $[F(x)]_a^b = F(b) - F(a)$

$$[-\cos x]_0^\pi = -\cos\pi - (-\cos 0) = -\cos\pi + \cos 0 = 2$$

Método da substituição para integrais definidas:

Ex: Calcule $\displaystyle\int_{-1}^1 x\sqrt{\underbrace{1+x^2}_{u}}\,dx = F(1) - F(-1)$ onde $\displaystyle F(x) = \int x\sqrt{1+x^2}\,dx = \frac{1}{3}\sqrt{(1+x^2)^3} + C$

$$\Downarrow$$

$$F(1) - F(-1) = \frac{1}{3}\sqrt{(1+1^2)^3} + C - \left[\frac{1}{3}\sqrt{(1+(-1)^2)^3} + C\right] = 0$$

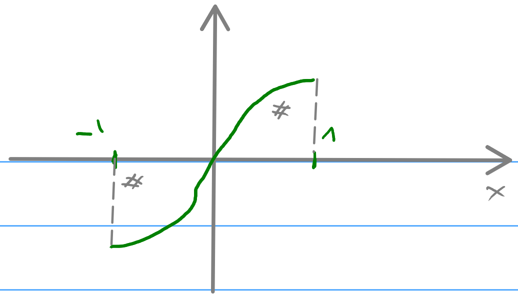
Seja $u = 1 + x^2 \Rightarrow du = 2x\,dx \Rightarrow \boxed{x\,dx = \frac{1}{2}du}$

$x = -1 \Rightarrow u = 1 + (-1)^2 = 2$

$x = 1 \Rightarrow u = 1 + 1^2 = 2$

Logo,

$$\int_{-1}^1 x\sqrt{1+x^2}\,dx = \int_2^2 \sqrt{u}\,du = 0$$



$$\int_{-1}^{1} x\sqrt{1+x^2}\,dx = 0$$