10/11 - Aula 30 - Aplicando o repertório de técnicas de integração

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \qquad \boxed{\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C.} *$$

$$* \int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{arcsen}\frac{x}{a} + C \qquad * \int \frac{dx}{\sqrt{x^2+\lambda}} = \ln\left|x+\sqrt{x^2+\lambda}\right| + C$$

Prove que $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

$$\frac{1}{a^2-x^2} = \boxed{\frac{1}{(a+x)(a-x)} = \frac{A}{a+x} + \frac{B}{a-x}}$$ (somas parciais)

$$\frac{A}{a+x} + \frac{B}{a-x} = \frac{A(a-x)+B(a+x)}{\underbrace{(a+x).(a-x)}_{m.m.c}} = \frac{Aa - Ax + Ba + Bx}{(a+x).(a-x)} = \boxed{\frac{(B-A)x + (A+B)a}{(a+x).(a-x)}} = $$

a^2-x^2

$$= \boxed{\frac{0.x + 1}{a^2-x^2}} \Rightarrow (B-A)x + (A+B)a = 0.x + 1, \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \begin{cases} B - A = 0 \\ (A+B)a = 1 \end{cases} \iff \begin{cases} B - \cancel{A} = 0 \\ B + \cancel{A} = \frac{1}{a} \end{cases} \quad +$$

$$2B = \frac{1}{a} \Rightarrow \boxed{B = \frac{1}{2a} = A}$$

$$\frac{1}{a^2-x^2} = \frac{1}{2a}\cdot\frac{1}{a+x} + \frac{1}{2a}\,\frac{1}{a-x} \Rightarrow$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\int\frac{1}{a+x}\,dx + \frac{1}{2a}\int\frac{1}{a-x}\,dx$$

$$= \frac{1}{2a}\ln|a+x| - \frac{1}{2a}\ln|a-x| + C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{a-x}\,dx = \int \frac{1}{u}\cdot(-du) = -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln|a-x| + c$$

$\underbrace{u = a-x}$ $\Rightarrow$ $du = -dx$

Ex. Calcular $\displaystyle\int \frac{dx}{2x^2+3x+1} \overset{*}{=}$

Sol

$\boxed{(x+a)^2 = x^2 + 2.x.a + a^2}$ (obs)

$$2x^2+3x+1 = 2\left(x^2 + \frac{2.3}{2.2}.x + \frac{1}{2}\right) = 2\left(\underbrace{x^2+2.x.\frac{3}{4}+\left(\frac{3}{4}\right)^2} - \left(\frac{3}{4}\right)^2 + \frac{1}{2}\right)$$

$$= 2\left[\left(\underbrace{x^2 + 2.x.\frac{3}{4}}_{(x+\frac{3}{4})^2} + \frac{9}{16}\right) - \frac{9}{16} + \frac{1}{2}\right] \quad ; \quad \frac{1}{2} - \frac{9}{16} = \frac{8-9}{16} = -\frac{1}{16}$$

$$= 2\left[\left(x+\frac{3}{4}\right)^2 - \frac{1}{16}\right]$$

Logo,

$$\int \frac{1}{2x^2+3x+1}\,dx = \int \frac{1}{2\left[\left(x+\frac{3}{4}\right)^2 - \frac{1}{16}\right]}\,dx = \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}\,dx =$$

Seja $\boxed{u = x + \frac{3}{4}}$ $\Rightarrow$ $du = dx$

$$\int \frac{1}{u^2-a^2}\,du = -\int \frac{1}{a^2-u^2}\,du = -\frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + c$$

$$= \frac{1}{2}\int \frac{1}{u^2 - (1/4)^2}\,du \quad , a = 1/4$$

$$= -\frac{1}{\cancel{2}} \cdot \frac{1}{\cancel{2.\frac{1}{4}}} \ln\left|\frac{1/4+u}{1/4-u}\right| + c = -\ln\left|\frac{1+4u}{1-4u}\right| + c$$

$$= -\ln\left|\frac{1+4.(x+3/4)}{1-4(x+3/4)}\right| + c = -\ln\left|\frac{1+4x+3}{1-4x-3}\right| + c$$

$$= -\ln\left|\frac{4+4x}{-2-4x}\right| + c = \ldots = \ln\left|\frac{2x+1}{2x+2}\right| + c$$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(x^2+2\cdot x\cdot\frac{b}{2a}+\frac{c}{a}\right)$$

$$= a\left(\underbrace{x^2+2x\frac{b}{2a}+\left(\frac{b}{2a}\right)^2}_{=\left(x+\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right) =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2+4ac}{4a^2}\right]$$

$$= a\left[\left(x+\underbrace{\frac{b}{2a}}\right)^2-\left(\frac{\overbrace{b^2-4ac}^{\Delta}}{4a^2}\right)\right]$$ Seja $m = \frac{b}{2a}$ e $n = \frac{-\Delta}{4a^2}$

$$\Rightarrow \quad ax^2+bx+c = a(x+m)^2+n$$

Ex. $x^2+x+1 = x^2+2\cdot x\cdot\frac{1}{\underbrace{2}}+1 = \left(x^2+2\cdot x\cdot\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{4}+1$

$$= \left(x+\frac{1}{2}\right)^2+\frac{3}{4} \quad\Rightarrow\quad x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(\underbrace{x+1/2}_{u})^2+3/4}\,dx = \int \frac{1}{u^2+3/4}\,du$$

$$= \int \frac{1}{u^2+\left(\sqrt{\frac{3}{4}}\right)^2}\,du = \int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\operatorname{arctg}\left(\frac{u}{a}\right)+C$$

onde $a = \frac{\sqrt{3}}{2}$

$$= \frac{1}{\sqrt{3}/2}\cdot \operatorname{arctg}\left(\frac{x+1/2}{\sqrt{3}/2}\right)+C$$

///

Ex. Calcular $\displaystyle\int \frac{x-1}{\sqrt{1-x-x^2}}\,dx$