# 12/11 - Aula 31 - Integrais envolvendo funções trigonométricas

**Ex. 18.2.** Calcular $\displaystyle\int \frac{x-1}{\sqrt{1-x-x^2}}\,dx$

**Sol.** Aplicando o completamento de quadrados à expressão quadrática obtemos:

$$1-x-x^2 = -(x^2+x-1) = -\left(\underbrace{x^2+2\cdot x\cdot\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2}-\left(\tfrac{1}{2}\right)^2-1\right) \quad ; \quad -1-\tfrac{1}{4}=-\tfrac{5}{4}$$

$$= -\left[\left(x+\tfrac{1}{2}\right)^2-\tfrac{5}{4}\right] = -\left[\left(x+\tfrac{1}{2}\right)^2-\left(\tfrac{\sqrt5}{2}\right)^2\right]. \text{ Logo,}$$

$$\int \frac{x-1}{\sqrt{1-x-x^2}}\,dx = \int \frac{x-1}{\sqrt{\left(\frac{\sqrt5}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}}\,dx \overset{*}{=}$$

Seja $u = x+\frac{1}{2} \Rightarrow du = dx \Rightarrow x = u-\frac{1}{2} \Rightarrow x-1 = u-\frac{1}{2}-1 = u-3/2$

$$\overset{*}{=} \int \frac{u-3/2}{\sqrt{\left(\frac{\sqrt5}{2}\right)^2-u^2}}\,du = \underbrace{\int \frac{u}{\sqrt{\left(\frac{\sqrt5}{2}\right)^2-u^2}}\,du}_{I} - \frac{3}{2}\underbrace{\int \frac{1}{\sqrt{\left(\frac{\sqrt5}{2}\right)^2-u^2}}\,du}_{J} = \underbrace{I-\tfrac{3}{2}J}$$

$$I = \int \frac{u}{\sqrt{\left(\frac{\sqrt5}{2}\right)^2-u^2}}\,du \quad ; \quad w = \left(\tfrac{\sqrt5}{2}\right)^2-u^2 \Rightarrow dw = -2u\,du$$

$$\Rightarrow u\,du = -\tfrac{1}{2}\,dw$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = -\frac{1}{2}\int w^{-\frac{1}{2}}\,dw = -\frac{\cancel{1}}{\cancel{2}}\,\frac{w^{\overbrace{-\frac{1}{2}+1}^{1/2}}}{\underbrace{-\frac{1}{2}+1}_{\cancel{\frac{1}{2}}}} + C$$

$$= -\sqrt{w} + C =$$

$$= -\sqrt{\left(\tfrac{\sqrt5}{2}\right)^2-u^2} + C = -\sqrt{\tfrac{5}{4}-\left(x+\tfrac{1}{2}\right)^2} + C = -\sqrt{1-x-x^2} + C$$

$$\tfrac{5}{4}-\left(x+\tfrac{1}{2}\right)^2 = \tfrac{5}{4}-\left(x^2+2x\cdot\tfrac{1}{2}+\tfrac{1}{4}\right) = -x^2-x+1$$

$$\Rightarrow I = -\sqrt{1-x-x^2} + C$$

$$J = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - u^2}}\, du \overset{\text{tabela 18.1}}{=} \operatorname{arc\,sen}\left(\frac{u}{\left(\frac{\sqrt{5}}{2}\right)}\right) + c$$

$$= \operatorname{arc\,sen}\left(\frac{2u}{\sqrt{5}}\right) + c = \operatorname{arc\,sen}\left(\frac{2(x+1/2)}{\sqrt{5}}\right) + c = \operatorname{arc\,sen}\left(\frac{2x+1}{\sqrt{5}}\right) + c$$

Logo,
$$\int \frac{x-1}{\sqrt{1-x-x^2}}\, dx = -\sqrt{1-x-x^2} - \frac{3}{2}\operatorname{arc\,sen}\left(\frac{2x+1}{\sqrt{5}}\right) + c$$

□

Vamos agora estudar integrais da seguinte forma:
$$\boxed{\int \sin^m x \cos^n x \, dx}$$

sendo $m, n \in \mathbb{N}$

**Ex** Calcule $J = \int \operatorname{sen}^6 x \cdot \cos^5 x \, dx$ ímpar

$$J = \int \operatorname{sen}^6 x \cdot \cos^4 x \cdot \underbrace{\cos x \, dx}_{dt} \quad , \quad t = \operatorname{sen} x \quad , \quad \operatorname{sen}^2 x + \cos^2 x = 1$$

$$\cos^2 x = 1 - \operatorname{sen}^2 x \;\Downarrow\; \cos^4 x = (1-\operatorname{sen}^2 x)^2$$

$$= \int \operatorname{sen}^6 x \cdot (1 - \operatorname{sen}^2 x)^2 \cdot \cos x \, dx =$$

Seja $\boxed{t = \operatorname{sen} x}$ $\Rightarrow dt = \cos x \, dx$

$$\int t^6 \cdot (1-t^2)^2 \cdot dt = \int t^6 (1 - 2.1.t^2 + t^4)\, dt$$

$$= \int (t^6 - 2t^8 + t^{10})\, dt = \frac{t^7}{7} - 2\frac{t^9}{9} + \frac{t^{11}}{11} + c$$

$$= \frac{\operatorname{sen}^7 x}{7} - 2\frac{\operatorname{sen}^9 x}{9} + \frac{\operatorname{sen}^{11} x}{11} + c$$

□

**Método** $J = \int \operatorname{sen}^m x \cdot \cos^n x \, dx$

**caso 1:** Suponha que $m$ ou $n$ é ímpar

Seja $m = 2k+1$, $k \in \mathbb{N}$

$$\int \operatorname{sen}^{2k+1} x \cdot \cos^n x \, dx = \int \operatorname{sen}^{2k} x \cdot \cos^n x \cdot \underbrace{\operatorname{sen} x \, dx}_{dt} \quad ; \quad t = -\cos x$$

$$\int (\operatorname{sen}^2 x)^k \cdot \cos^m x \cdot \operatorname{sen} x \, dx = \int (1-\cos^2 x)^k \cos^m x \cdot \operatorname{sen} x \, dx$$

Seja $t = \cos x \Rightarrow J = \int (1-t^2)^k \cdot t^m \cdot -dt$

$$\Rightarrow J = -\int (1-t^2)^k \cdot t^m \cdot dt = \ldots$$

$\boxed{\text{Bimômio de Newton}}$: $(1-t^2)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} 1^{k-\ell} \cdot (-t^2)^\ell$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell \cdot t^{2\ell}.$$

Caso 2: $m$ e $n$ são pares $\Leftrightarrow m = 2k$ e $n = 2\ell$, $k, \ell \in \mathbb{N}$

$$I = \int \operatorname{sen}^m x \cdot \cos^n x \, dx = \int (\operatorname{sen}^2 x)^k \cdot (\cos^2 x)^\ell \, dx \stackrel{(*)}{=}$$

Relações: $\cos^2 x = \dfrac{1+\cos 2x}{2}$ e $\operatorname{sen}^2 x = \dfrac{1-\cos 2x}{2}$

$$\stackrel{(*)}{=} \int \left(\frac{1-\cos 2x}{2}\right)^k \cdot \left(\frac{1+\cos 2x}{2}\right)^\ell dx = \ldots$$

Seja $k = 2$ e $\ell = 1 \Rightarrow$

$$\int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot \left(\frac{1+\cos 2x}{2}\right) dx = \int \left(\frac{1-2\cos 2x + \cos^2 x}{4}\right) \cdot \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x + \cos^3 2x) \, dx =$$

$$= \frac{1}{8} \int dx + \frac{1}{8} \int \cos 2x - \frac{1}{8} \int \cos^2 2x + \frac{1}{8} \int \cos^3 2x \, dx$$

$$= \frac{x}{8} + \frac{\operatorname{sen} 2x}{16} - \frac{1}{8} \int \frac{1+\cos 4x}{2} \, dx + \frac{1}{8} \int \underset{\shortparallel}{\cos^2 2x} \cdot \cos 2x \, dx$$

$$+ \frac{1}{8} \int (1-\operatorname{sen}^2 2x) \underbrace{\cos 2x \, dx}_{\frac{dt}{2}}$$

$t = \operatorname{sen} 2x \Rightarrow dt = 2\cos 2x$ ...

## Fórmulas de redução (ou de recorrência)

**Exemplo 18.5.** *Sendo* $n \geq 2$, *deduzir a fórmula de redução*

$$\boxed{\int \sec^n x\, dx = \frac{\operatorname{tg} x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \cdot \int \sec^{n-2} x\, dx} \qquad (18.2)$$

Se $n=2 \Rightarrow \int \sec^2 x\, dx = \dfrac{\operatorname{tg} x \cdot \sec^{2-2} x}{2-1} + \cancel{\dfrac{2-2}{n-1} \int \sec^{2-2} x\, dx}$

$$\Rightarrow \boxed{\int \sec^2 x\, dx = \operatorname{tg} x}$$

Se $n=4 \Rightarrow \int \sec^4 x\, dx = \dfrac{\operatorname{tg} x\, \sec^2 x}{3} + \dfrac{2}{3} \underbrace{\int \sec^2 x\, dx}_{\operatorname{tg} x} = \dfrac{\operatorname{tg} x \cdot \sec^2 x}{3} + \dfrac{2}{3}\operatorname{tg} x$

Seja $I_n = \int \sec^n x\, dx$, aplicando integração por partes temos:

$$I_n = \int \sec^n x\, dx = \int \underbrace{\sec^{n-2} x}_{u} \cdot \underbrace{\sec^2 x\, dx}_{dv}, \quad \text{pois } n \geq 2 \Leftrightarrow n-2 \geq 0$$

$$= uv - \int v\, du = \sec^{n-2} x \cdot \operatorname{tg} x - \int \operatorname{tg} x \cdot \boxed{(n-2) \sec^{n-2} x \cdot \operatorname{tg} x}\, dx$$

$$u = \sec^{n-2} x \Rightarrow du = (n-2) \sec^{n-3} x \cdot (\sec x)' = (n-2) \sec^{n-3} x\, \sec x \cdot \operatorname{tg} x$$

$$\Rightarrow \boxed{du = (n-2) \sec^{n-2} x \cdot \operatorname{tg} x}$$

$$\Rightarrow I_n = \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \int \sec^{n-2} x\, \underbrace{\operatorname{tg}^2 x}_{\sec^2 x - 1}\, dx \Rightarrow$$

$$I_n = \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \int \sec^{n-2} x\, (\sec^2 x - 1)\, dx \Rightarrow$$

$$I_n = \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \underbrace{\int \sec^n x\, dx}_{I_n} + (n-2) \underbrace{\int \sec^{n-2} x\, dx}_{I_{n-2}} \Rightarrow$$

$$I_n = \sec^{n-2} x \cdot \operatorname{tg} x \underbrace{- (n-2) I_n} + (n-2) I_{n-2}$$

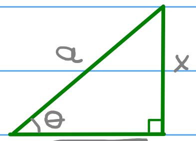
$$\underbrace{(1+n-2)}_{(n-1)} I_n = \sec^{n-2} x \cdot \operatorname{tg} x = (n-2) I_{n-2} \Rightarrow$$

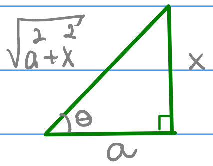
$$\boxed{I_n = \frac{\operatorname{tg} x \cdot \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}}$$

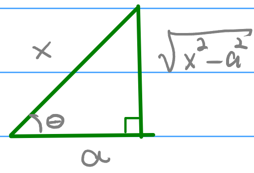
□

Vamos estudar agora integrais que envolvam uma das expressões abaixo:

$$* \quad \sqrt{a^2 - x^2} \qquad \overset{*}{\sqrt{a^2 + x^2}} \qquad \sqrt{x^2 - a^2}$$




$\beta = \sqrt{a^2 - x^2}$

$a^2 = x^2 + \beta^2$

$\Downarrow$

$\beta = \sqrt{a^2 - x^2}$

$\Rightarrow$

$$\boxed{\operatorname{sen}\theta = \frac{x}{a} \qquad \cos\theta = \frac{\sqrt{a^2-x^2}}{a} \qquad \operatorname{tg}\theta = \frac{x}{\sqrt{a^2-x^2}}}$$

$$\boxed{\operatorname{sen}\theta = \frac{x}{\sqrt{a^2+x^2}} \qquad \cos\theta = \frac{a}{\sqrt{a^2+x^2}} \qquad \operatorname{tg}\theta = \frac{x}{a}}$$

$$\boxed{\operatorname{sen}\theta = \frac{\sqrt{x^2-a^2}}{x} \qquad \cos\theta = \frac{a}{x} \qquad \operatorname{tg}\theta = \frac{\sqrt{x^2-a^2}}{a}}$$

Ex. Calcule $\displaystyle\int \sqrt{9 - x^2}\, dx = \int \underbrace{\sqrt{3^2 - x^2}}_{3\cos\theta}\, \underbrace{dx}_{3\cos\theta\, d\theta} \overset{*}{=}$

$\operatorname{sen}\theta = \dfrac{x}{3} \Rightarrow \boxed{x = 3\operatorname{sen}\theta} \Rightarrow dx = 3\cos\theta \cdot d\theta$

$\boxed{\cos\theta = \dfrac{\sqrt{3^2 - x^2}}{3}} \Rightarrow \sqrt{3^2 - x^2} = 3\cos\theta$

$$\overset{*}{=} \int 3\cos\theta \; 3\cos\theta \, d\theta = 9\int \cos^2\theta \, d\theta = 9\int \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\int d\theta + \frac{9}{2}\int \cos 2\theta \, d\theta = \frac{9\theta}{2} + \frac{9}{4}\operatorname{sen} 2\theta + C$$

Como $x = 3\operatorname{sen}\theta \Rightarrow \boxed{\operatorname{sen}\theta = \dfrac{x}{3}} \Rightarrow \theta = \operatorname{arcsen}\left(\dfrac{x}{3}\right)$

$$\Rightarrow \boxed{\frac{9\theta}{2} = \frac{9}{2}\operatorname{arcsen}\left(\frac{x}{3}\right)}$$

$$\operatorname{sen} 2\theta = \operatorname{sen}(\theta+\theta) = 2\operatorname{sen}\theta \cdot \cos\theta = 2\operatorname{sen}\theta \cdot \sqrt{1-\operatorname{sen}^2\theta}$$

$$= 2 \cdot \frac{x}{3} \cdot \sqrt{1-\left(\frac{x}{3}\right)^2} = \frac{2x}{3}\,\frac{\sqrt{9-x^2}}{3} = \frac{2x\sqrt{9-x^2}}{9}$$

Portanto, $\displaystyle\int \sqrt{9-x^2}\,dx = \frac{9\theta}{2} + \frac{9}{4}\operatorname{sen} 2\theta + C$

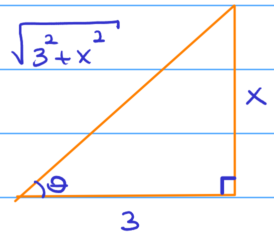
$$= \frac{9}{2}\operatorname{arcsen}\left(\frac{x}{3}\right) + \frac{\cancel{9}}{\cancel{4}_{\,2}} \cdot \frac{\cancel{2}x\sqrt{9-x^2}}{\cancel{9}} + C$$

$$= \frac{9}{2}\operatorname{arcsen}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2} + C$$

□

<u>Ex</u> Calcular $\displaystyle\int_0^3 \sqrt{9+x^2}\,dx = \int_0^3 \sqrt{3^2+x^2}\,dx$

$$\operatorname{tg}\theta = \frac{x}{3} \Rightarrow \boxed{x = 3\operatorname{tg}\theta}$$




Seja $x = 3\operatorname{tg}\theta \Rightarrow 9+x^2 = 9+(3\operatorname{tg}\theta)^2 = 9+9\operatorname{tg}^2\theta = 9(1+\operatorname{tg}^2\theta)$

$\Downarrow$

$dx = 3\sec^2\theta\,d\theta \Rightarrow 9+x^2 = 9\sec^2\theta \Rightarrow \boxed{\sqrt{9+x^2} = \sqrt{9\sec^2\theta} = 3\sec\theta}$

$$\int_0^3 \sqrt{9+x^2}\,dx = \int_0^{\pi/4} 3\sec\theta \cdot 3\sec^2\theta\,d\theta = 9\int_0^{\pi/4}\sec^3\theta\,d\theta \overset{*}{=}$$

$x=0 \Rightarrow 3\operatorname{tg}\theta = 0 \Leftrightarrow \theta = 0$

$x=3 \Rightarrow 3\operatorname{tg}\theta = 3 \Leftrightarrow \operatorname{tg}\theta = 1 \Leftrightarrow \theta = \pi/4$

$$= 9 \cdot \left[\frac{\operatorname{tg}\theta\,\sec\theta}{2} + \frac{1}{2}\underbrace{\int \sec\theta\,d\theta}_{=\ln|\sec\theta+\operatorname{tg}\theta|}\right]_0^{\pi/4} =$$

$$= \frac{9}{2}\left[\operatorname{tg}\frac{\pi}{4}\cdot\sec\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4}+\operatorname{tg}\frac{\pi}{4}\right| - \operatorname{tg}0\cdot\sec 0 - \ln|\sec 0+\operatorname{tg}0|\right] = \cdots$$

$$= \frac{9\sqrt{2}}{2} + \frac{9}{2}\ln(\sqrt{2}+1)$$