DERIVADAS PARCIAIS DE ORDEM SUPERIOR

Definição: Sejam $z = f(x,y)$ e $(x_0,y_0) \in D_f$.

A derivada parcial de f em relação a x no ponto (x_0,y_0) é definida por:

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

Se este limite existir.

Exemplo: $f(x,y) = x^2 + y.$

$(x_0, y_0) = (2, -4)$

$$\frac{\partial f}{\partial x}(2,-4) = \lim_{\Delta x \to 0} \frac{f(2+\Delta x, -4) - f(2,-4)}{\Delta x}$$

- $f(2+\Delta x, -4) = (2+\Delta x)^2 + (-4)$

$$= \cancel{4} + 4\Delta x + \Delta x^2 \cancel{-4}$$

$$= 4\Delta x + \Delta x^2$$

- $f(2,-4) = (2)^2 - 4 = \cancel{4} - \cancel{4} = 0$

$$\frac{\partial f}{\partial x}(2,-4) = \lim_{\Delta x \to 0} \frac{4\Delta x + \Delta x^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \left[4 + \cancel{\Delta x}^{\,0} \right] = 4$$

Observe que $f(x,y) = x^2 + y$

$$\Rightarrow \frac{\partial f}{\partial x}(x,y) = 2x$$

$$\frac{\partial f}{\partial x} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial f}{\partial x}(x,y) = 2x \quad \Bigg| \quad \frac{\partial f}{\partial x}(2,-4) = 4$$

$$\boxed{f(x,y) = \begin{cases} \dfrac{x^3 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}}$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{x^4 + 3x^2y^2 + 2xy^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$\hookrightarrow$ Domínio $\left(\dfrac{\partial f}{\partial x}\right) = \mathbb{R}^2$

$$\boxed{f(x,y) = \begin{cases} \dfrac{x^3 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\[2ex] 0, & (x,y) = (0,0) \end{cases}}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} -\dfrac{2x^2 y(1+x)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\[2ex] \nexists, & (x,y) = (0,0) \end{cases}$$

Domínio $\mathbb{R}^2 \setminus \{(0,0)\}$

$$f : D_f \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\boxed{\frac{\partial f}{\partial x} := f_x} : D_{f_x} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial (f_x)}{\partial x}(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0)$$

$$= \lim_{\Delta x \to 0} \frac{f_x(x_0 + \Delta x, y_0) - f_x(x_0, y_0)}{\Delta x}$$

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$$\frac{\partial f}{\partial x} := f_x : D_{f_x} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial}{\partial x}(f_x)(x_0,y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x_0,y_0)$$

$$\frac{\partial}{\partial y}(f_x)(x_0,y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0,y_0)$$

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$$\frac{\partial f}{\partial x} := f_x : D_{f_x} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$$

$$⚠ \quad \frac{\partial}{\partial x}(f_x)(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0)$$

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$$\frac{\partial}{\partial x}(f_y)(x_0,y_0) = \overbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x_0,y_0)}^{\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)}$$

$$\frac{\partial}{\partial y}(f_y)(x_0,y_0) = \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x_0,y_0)}_{\frac{\partial^2 f}{\partial y^2}(x_0,y_0)}$$

Exemplo: $f(x,y) = xy - e^x \cos y$

- $\dfrac{\partial f}{\partial x}(x,y) = y - e^x \cos y$

- $\dfrac{\partial f}{\partial y}(x,y) = x + e^x \operatorname{sen} y$

$\dfrac{\partial^2 f}{\partial x^2}(x,y) = -e^x \cos y$

$\dfrac{\partial^2 f}{\partial y \partial x}(x,y) = 1 + e^x \operatorname{sen} y$

$\dfrac{\partial^2 f}{\partial x \partial y}(x,y) =$

$= 1 + e^x \operatorname{sen} y$

$\dfrac{\partial^2 f}{\partial y^2}(x,y) =$

$= e^x \cos y$

Exemplo: $f(x,y,z) = \ln(x^2+y^2+z^2)$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{2z}{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \frac{\partial^2 f}{\partial z \partial x}(x,y,z) =$$

$$= \frac{0.(x^2+y^2+z^2) - 2z.(2x)}{(x^2+y^2+z^2)^2}$$

$$= (-4xz)/(x^2+y^2+z^2)^2$$

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$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)}(x,y,z) = \frac{\partial^2 f}{\partial x \partial z}(x,y,z) =$$

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$$= \boxed{(-4xz)/(x^2+y^2+z^2)^2}$$

Observemos que nos dois últimos exemplos tivemos as derivadas parciais mistas iguais, isto é,

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial y}(x,y)$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial z}(x,y)$$

Definição: Uma função f é de classe C^1 quando suas derivadas parciais são contínuas. Se as derivadas parciais de segunda ordem de f são contínuas, dizemos que f é de classe C^2.

Teorema: Se $z = f(x,y)$ é de classe C^2, então suas derivadas parciais mistas são iguais, isto é,

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

Obs: Este teorema vale para funções de várias variáveis.