20/10 - Aula 24 - Integração por Partes

29. $\int \text{tg}^4 x\, dx$. *Resposta*. $\frac{\text{tg}^3 x}{3} - \text{tg}\, x + x + C$.

Sugestão. Mostre que $\boxed{\text{tg}^4 x = \text{tg}^2 x \cdot \text{tg}^2 x = \sec^2 x \cdot \text{tg}^2 x - \sec^2 x + 1.}$ ✓

Fato: $\boxed{\sec^2 x - 1 = \text{tg}^2 x} \Leftrightarrow \sec^2 x - \text{tg}^2 x = 1$

De fato, $\sec^2 x - 1 = \dfrac{1}{\cos^2 x} - 1 = \dfrac{1 - \cos^2 x}{\cos^2 x} = \dfrac{\text{sen}^2 x}{\cos^2 x} = \text{tg}^2 x$

Note que: $\sec^2 x \cdot \text{tg}^2 x - \sec^2 x + 1 =$

$\sec^2 x\, \text{tg}^2 x - \underbrace{(\sec^2 x - 1)}_{\text{tg}^2 x} = \sec^2 x \cdot \text{tg}^2 x - \text{tg}^2 x$

$= \text{tg}^2 x \cdot (\sec^2 x - 1) = \text{tg}^2 x \cdot \text{tg}^2 x = \text{tg}^4 x.$

$$\int \text{tg}^4 x\, dx = \int (\sec^2 x \cdot \text{tg}^2 x - \sec^2 x + 1)\, dx$$

$$= \underbrace{\int \text{tg}^2 x \cdot \sec^2 x\, dx}_{I_1} - \underbrace{\int \sec^2 x\, dx}_{I_2} + \underbrace{\int 1 \cdot dx}_{I_3}$$

(I_1) Para resolver a integral I_1 aplicaremos a mudança de variável $\boxed{u = \text{tg}\, x}$. Logo, $du = (\text{tg}\, x)'\, dx = \sec^2 x\, dx$

$$\int \text{tg}^2 x \cdot \underbrace{\sec^2 x\, dx}_{du} = \int u^2\, du = \frac{u^{2+1}}{2+1} + C = \frac{1}{3} u^3 + C = \frac{1}{3} \text{tg}^3 x + C_1$$

(I_2) $\int \sec^2 x\, dx = \text{tg}\, x + C_2$

(I_3) $\int 1 \cdot dx = x + C_3$

Logo, $\int \text{tg}^4 x\, dx = \frac{1}{3} \text{tg}^3 x + C_1 - (\text{tg}\, x + C_2) + x + C_3$

$$= \frac{1}{3} \text{tg}^3 x - \text{tg}\, x + x + \underbrace{(C_1 - C_2 + C_3)}_{= C}$$

Ex 30: $\displaystyle\int \frac{dx}{\cos^2 x(3\,tg\,x+1)} = \int \frac{\sec^2 x}{3\,tg\,x+1}\cdot dx = \int \underbrace{\frac{1}{3u(x)+1}}_{f(u)}\cdot u'(x)\,dx = \int f(u)\,u'(x)\,dx$

$\int f(x)\,\underbrace{dx}_{\text{diferencial de x}} = F(x)+c \iff (F(x)+c)' = f(x)$

$$\int \underbrace{\frac{1}{\cos^2 x.(3\,tg\,x+1)}}_{f(x)}\,dx = \int \frac{1}{3\boxed{tg\,x}+1}\cdot\boxed{\sec^2 x}\,dx = \int \frac{1}{3u+1}\cdot du \overset{(*)}{=}$$

Seja $u = tg\,x$

usar que. $(tg\,x)' = \sec^2 x$ $\qquad du = \sec^2 x\,dx$

$$= \int \frac{1}{3(u+\frac{1}{3})}\,du = \frac{1}{3}\int \frac{1}{u+1/3}\,du \underset{(*)}{=} \frac{1}{3}\ln|u+\tfrac{1}{3}| + c$$

$$\overset{(*)}{=} \frac{1}{3}\ln\left|tg\,x+\frac{1}{3}\right| + c = \frac{1}{3}\ln\left|\frac{3\,tg\,x+1}{3}\right| + c$$

$$= \frac{1}{3}\cdot\ln\left(\frac{|3\,tg\,x+1|}{3}\right)+c = \frac{1}{3}\left(\ln|3\,tg\,x+1| - \ln 3\right) + c$$

$$= \frac{1}{3}\ln|3\,tg\,x+1| \underbrace{-\frac{1}{3}\ln 3 + c}, \quad \text{Seja } C_0 = C - \frac{1}{3}\ln 3 \Rightarrow$$

$$= \frac{1}{3}\ln|3\,tg\,x+1| + C_0$$

Obs.: 1) $\displaystyle\int \underbrace{tg\,x}_{u}\cdot\underbrace{\sec^2 x\,dx}_{du} = \int u.\,du = \frac{u^2}{2}+c = \frac{tg^2 x}{2}+c$

$u = tg\,x \Rightarrow$

$\frac{du}{dx} = (tg\,x)' = \sec^2 x \iff du = \sec^2 x\,dx$

2) $\displaystyle\int \frac{\overset{u}{\ln x}}{x}\,dx = \int u\,\underline{du} = \frac{u^2}{2}+c = \frac{(\ln x)^2}{2}+c$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{\ln x}{x}\,dx = \int u(x)\cdot u'(x)\,dx$$

$$\int \frac{1}{u^2+1}\cdot u'(x)\,dx \;;\quad \int \ln(u)\cdot u'(x)\,dx \;;\quad \int f(u)\cdot u'(x)\,dx$$

$$\int \frac{1}{3u(x)+1} \cdot \underbrace{u'(x)\,dx}_{du} = \int \frac{1}{3u+1} \cdot du =$$

Integração por Partes: $\boxed{\int u(x).v'(x)\,dx = u(x)v(x) - \int v(x).u'(x)\,dx}$

Aplicando a regra do produto temos:

$$(u(x).v(x))' = u'(x)v(x) + u(x).v'(x) \Rightarrow$$

$$\int (u(x)v(x))'\,dx = \int (u'(x)v(x) + u(x)v'(x))\,dx$$

$$u(x).v(x) = \underbrace{\int u'(x).v(x)\,dx}_{\text{passando para o 1º membro}} + \int u(x).v'(x)\,dx$$

$$\Rightarrow \int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

Obs. $\int f'(x)\,dx = f(x) + c$ pois $(f(x)+c)' = f'(x) \Rightarrow \int f'(x)\,dx = f(x) + c$

Ex Calcule $\int x \operatorname{sen} x\,dx$ por IPP

Sol.

$$\int x \operatorname{sen} x\,dx = \int \underbrace{x}_{u(x)} . \underbrace{(-\cos x)'}_{v'(x)}\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

$u(x) = x \Rightarrow u'(x) = 1$

$v(x) = -\cos x$

$$= x.(-\cos x) - \int -\cos x . 1\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \operatorname{sen} x + c$$

Logo, $\boxed{\int \underbrace{x \operatorname{sen} x}_{\text{integrando}}\,dx = \underbrace{-x\cos x + \operatorname{sen} x}_{\text{primitiva}} + c}$

Verificação: $(-x\cos x + \operatorname{sen} x + c)' = (-x\cos x)' + \cos x + 0$

$$= \cancel{-\cos x} - x.(-\operatorname{sen} x) + \cancel{\cos x} = x \operatorname{sen} x$$