15/09 - Aula 13 - Funções exponenciais e logarítmicas

Regras de Potências:

$$p^n = \underbrace{p \cdot p \cdots p}_{\text{n-vezes}}$$ se $n \in \mathbb{N} = \{0, 1, 2, \dots\}$

- $a^{\frac{1}{n}} = \sqrt[n]{a}$, $16^{\frac{1}{2}} = \sqrt{16} = 4$

- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \}$

Problema: Como calculamos $2^{\sqrt{2}}$? $\sqrt{2} \notin \mathbb{Q}$

Observamos que $\boxed{\sqrt{2} \approx 1,414213562}$ logo, $\boxed{2^{\sqrt{2}} \approx 2,6651}$

$\underbrace{1,414213562}_{\in \mathbb{Q}} = \frac{1414213562}{1000000000}$

$2^1 = 2$

$2^{1,4} = 2^{14/10} = \sqrt[10]{2^{14}} \approx 2,6390$

$2^{1,41} = 2^{141/100} = \sqrt[100]{2^{141}} \approx 2,6574$

$2^{1,414} = 2^{1414/1000} \approx 2,6647$

$2^{1,4142} = 2^{14142/10000} \boxed{\approx 2,6651}$

Seja $a \in \mathbb{R}$, $a > 0$ e β um número irracional ou seja $\boxed{\beta \neq \frac{m}{n}}$ então como podemos calcular a^β

$$\boxed{a^\beta = \lim_{n \to +\infty} a^{\beta_n}}$$

$\beta_n \in \mathbb{Q}$

$\lim_{n \to +\infty} \beta_n = \beta$

Se $a \in \mathbb{R}$, $a > 0$, $b > 0$ e $x, y \in \mathbb{R}$

$$a^x \cdot a^y = a^{x+y}$$
$$(a^x)^y = a^{xy}$$
$$a^{-x} = \frac{1}{a^x}, \quad a^{x-y} = \frac{a^x}{a^y}, \quad a^0 = 1$$
$$a^x \cdot b^x = (ab)^x$$

A função exponencial

Sendo a um número real, positivo, $a \neq 1$, define-se a função exponencial de base a por

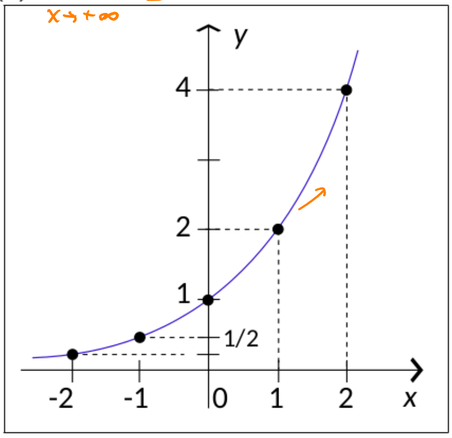
$$f(x) = a^x, \quad \text{para cada } x \in \mathbb{R}$$

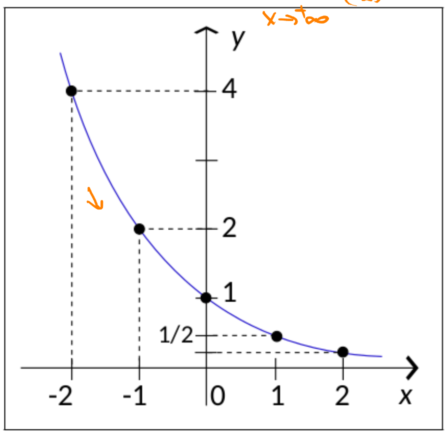
A função exponencial é contínua: $\boxed{\lim_{x \to x_0} a^x = a^{x_0}}$

Gráfico da função exponencial

$\lim_{x \to +\infty} 2^x = +\infty$

$\lim_{x \to +\infty} \left(\frac{1}{2}\right)^x = 0$





$x_1 < x_2 \Rightarrow 2^{x_1} < 2^{x_2}$

$$f(x) = 2^x$$

$2 > 1$

$x_1 < x_2 \Rightarrow \left(\frac{1}{2}\right)^{x_1} > \left(\frac{1}{2}\right)^{x_2}$

$$f(x) = \left(\frac{1}{2}\right)^x$$

$0 < \frac{1}{2} < 1$

Assumiremos também que

(i) se $a > 1$, a função $f(x) = a^x$ é crescente, com $\lim_{x \to +\infty} a^x = +\infty$;

(ii) se $0 < a < 1$, a função é decrescente, com $\lim_{x \to +\infty} a^x = 0^+ (= 0)$.

Álgebra dos limites no infinito:

$a > 1 \Rightarrow a^{+\infty} = \lim_{x \to +\infty} a^x = +\infty$

$\Downarrow$

$a^{+\infty} = +\infty$

Se $a > 1$, $a^{+\infty} = +\infty$, $a^{-\infty} = \frac{1}{a^{+\infty}} = \frac{1}{+\infty} = 0^+ \ (= 0)$

Se $0 < a < 1$, $a^{+\infty} = 0^+ \ (= 0)$, $a^{-\infty} = \frac{1}{a^{+\infty}} = \frac{1}{0^+} = +\infty$

$\pi \approx 3,14159 \Rightarrow \beta = \pi$ $\quad \{\beta_n\} \to \pi$

$\beta_1 = 3$ $\qquad \beta_4 = 3,141$

$\beta_2 = 3,1$ $\qquad \beta_5 = 3,1415$

$\beta_3 = 3,14$ $\qquad \beta_6 = 3,14159$

$\vdots$

$3^\pi = 3^\beta = \lim_{n \to \infty} 3^{\beta_n} \approx 31,544$

Função Logarítmica

$$\log_a^x = y \iff a^y = x$$

$a > 0,\ a \neq 1,\ x > 0$

$\log_a x = \alpha \iff a^\alpha = x$
$\log_a y = \beta \iff a^\beta = y$
$\Big\}\ \log_a x + \log_a y = \alpha + \beta \overset{af}{=} \log_a(xy)$ pois $a^{\alpha+\beta} = a^\alpha . a^\beta$

$= x . y$

$\Downarrow$

$a^{\alpha+\beta} = xy \iff$

$\log_a(xy) = \alpha + \beta$

$= \log_a x + \log_a y$

Sendo x e y reais positivos, z real, e $a > 0, a \neq 1$,

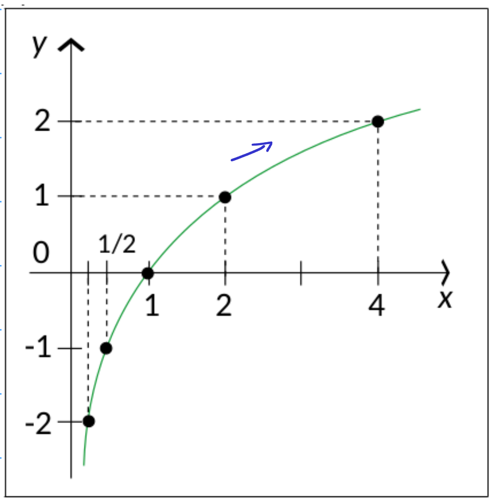
$$\log_a(xy) = \log_a x + \log_a y \quad (*)$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^z = z \cdot \log_a x$$

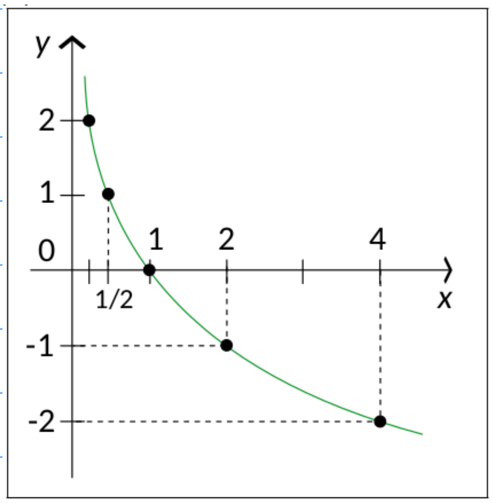
$$\log_a x^{1/z} = \frac{\log_a x}{z} \quad (\text{se } z \neq 0)$$

$$\log_a x = \frac{\log_b x}{\log_b a}, \quad (\text{se } b > 0, b \neq 1) \qquad (\text{mudança de base})$$



$y = \log_2 x$

$a = 2 > 1$



$y = \log_{\frac{1}{2}} x$

$a = \frac{1}{2} \Rightarrow 0 < a < 1$

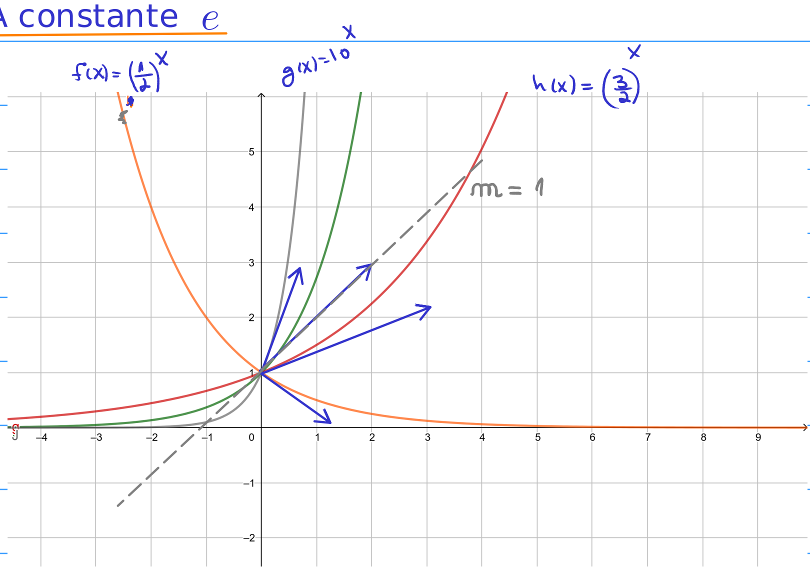
Além disso,

(i) se $a > 1$, $f(x) = \log_a x$ é crescente;

(ii) e se $0 < a < 1$, $f(x) = \log_a x$ é decrescente.

Propriedade Importante: $\boxed{a^{\log_a x} = x}$ $\quad x > 0$

A constante e



$f(x) = \frac{1}{2^x}$ ↘

$g(x) = 10^x$ ↗

$h(x) = \left(\frac{3}{2}\right)^x = \frac{3^x}{2^x}$ ↗

$f(0) = a^0 = 1$

$f(x) = a^x \ , \quad a > 0, \ a \neq 1, \ x \in \mathbb{R} \Rightarrow D(f) = \mathbb{R}.$

$0 \quad a \quad 1 \quad a$

$\Downarrow$

a^x é decrescente $\qquad a^x$ é crescente

Euler: Qual deve ser o valor de a para que a função $f(x) = a^x$ tenha reta tangente no ponto $P = (0, 1)$ coeficiente angular m=1

$a = ?$

Vamos admitir, sem demonstração, que para cada x real

$$\boxed{\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e}$$

$$\lim_{\substack{m \to +\infty \\ m \in \mathbb{N}}} \left(1 + \frac{1}{m}\right)^m = \lim_{m \to \infty} \left(\frac{m+1}{m}\right)^m \approx 2,718$$

Proposição 9.1: $\displaystyle \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$ (Pense nisso)

Proposição 9.2: $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

Prova:

$$\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x \to 0^-} (1+x)^{\frac{1}{x}} = e$$

caso 1: $x \to 0^+$

Seja $\alpha = \frac{1}{x} \Rightarrow \lim_{x \to 0^+} \overset{\alpha}{\widetilde{\frac{1}{x}}} = +\infty \Rightarrow \alpha \longrightarrow +\infty$

$$(1+x)^{\frac{1}{x}} = \left(1+\frac{1}{\alpha}\right)^{\alpha} \Rightarrow \lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = \lim_{\alpha \to +\infty} \left(1+\frac{1}{\alpha}\right)^{\alpha} = e$$

Analogamente, $\lim_{x \to 0^-} (1+x)^{\frac{1}{x}} = \lim_{\alpha \to -\infty} \left(1+\frac{1}{\alpha}\right)^{\alpha} = e$

$a^{-\infty}$

$a > 1 \;\; = 0$

$0 < a < 1 = \infty$

$a \neq 1$

Ex. Calcule $\lim_{x \to +\infty} \left(\frac{1-2x}{5-2x}\right)^{2-x}$. $\left(\frac{\frac{1}{x}-2}{\frac{5}{x}-2}\right)^{2-x} \underset{x=+\infty}{=} \left(\frac{-2}{-2}\right)^{2-(+\infty)} = 1^{-\infty} \overset{?}{=}$

Seja y tal que $\frac{1-2x}{5-2x} = 1+\frac{1}{y}$

$\Rightarrow \boxed{y = \frac{2x-5}{4}}$ (verifiquem) $\Rightarrow x = 2y + \frac{5}{2}$

$$2-\left(2y+\frac{5}{2}\right) = 2-\frac{5}{2}-2y = -2y-\frac{1}{2}$$

$$\lim_{x \to +\infty} \left(\frac{1-2x}{5-2x}\right)^{2-x} = \lim_{y \to +\infty} \left(1+\frac{1}{y}\right) \overset{*}{=}$$

$$\lim_{x \to +\infty} y = \lim_{x \to +\infty} \frac{2x-5}{4} = +\infty \qquad 1^{-1/2} = \frac{1}{\sqrt{1}} = 1$$

$$\overset{*}{=} \lim_{y \to +\infty} \left(1+\frac{1}{y}\right)^{-2y-\frac{1}{2}} = \lim_{y \to +\infty} \left[\left(1+\frac{1}{y}\right)^{y}\right]^{-2} \cdot \left(1+\frac{1}{y}\right)^{-1/2}$$

$$= e^{-2} \cdot 1 = \frac{1}{e^2}$$

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