27/09 - Aula 18 - Regra de L'Hopital

3. Calcule y', calculando $\ln y$, expandindo o segundo membro, utilizando propriedades de logaritmos, e então derivando implicitamente.

(a) $y = \sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}$ }

Respostas. (a) $\frac{1}{3}\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}} \cdot \left(\frac{1}{x} + \frac{2x}{x^2+1} - \frac{2}{x-1}\right)$

$$y = \left(\frac{x(x^2+1)}{(x-1)^2}\right)^{\frac{1}{3}} \Rightarrow \ln y = \ln\left(\frac{x(x^2+1)}{(x-1)^2}\right)^{\frac{1}{3}} \Rightarrow \ln y = \frac{1}{3}\ln\left(\frac{x(x^2+1)}{(x-1)^2}\right)$$

* $\log_b x^\beta = \beta \log_b x$

Derivando ambos membros obtemos

$$\frac{d}{dx}\ln y(x) = \frac{d}{dx}\frac{1}{3}\ln\left(\frac{x^3+x}{(x-1)^2}\right) \Leftrightarrow$$

$$\frac{y'(x)}{y(x)} = \frac{1}{3}\cdot\left[\frac{\left(\frac{x^3+x}{(x-1)^2}\right)'}{\frac{x^3+x}{(x-1)^2}}\right] \quad (*)$$

** $(\ln y(x))' = \frac{y'(x)}{y(x)}$ $\quad -(2x^4-x^3+2x^2-x)$

$$\left(\frac{x^3+x}{(x-1)^2}\right)' = \frac{(3x^2+1)(x-1)^2-(x^3+x)\cdot 2(x-1)\cdot 1}{(x-1)^4}$$

$$= \frac{(3x^2+1)(x^2-2x+1)-2x^4+x^3-2x^2+x}{(x-1)^4}$$

$$= \frac{\cancel{3x^4}-\cancel{6x^3}+\cancel{3x^2}+\cancel{x^2}-2x+1-\cancel{2x^4}+\cancel{x^3}-\cancel{2x^2}+x}{(x-1)^4}$$

$$= \frac{x^4-5x^3+2x^2-x+1}{(x-1)^4}$$

$$\Rightarrow \frac{y'(x)}{y(x)} = \frac{1}{3}\cdot\frac{x^4-5x^3+2x^2-x+1}{(x-1)^{\cancel{4}\,2}}\cdot\frac{(x-1)^{\cancel{2}}}{x^3+x}$$

$$\frac{y'(x)}{y(x)} = \frac{1}{3}\cdot\frac{(x^4-5x^3+2x^2-x+1)}{(x-1)^2\cdot(x^3+x)}$$

$$y'(x) = \frac{1}{3}\, y(x) \cdot \frac{x^4 - 5x^3 + 2x^2 - x + 1}{(x-1)^2 \cdot x \cdot (x^2+1)} \Rightarrow$$

$$y'(x) = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}} \cdot \frac{x^4 - 5x^3 + 2x^2 - x + 1}{x \cdot (x^2+1) \cdot (x-1)^2} \quad (*)$$

Falta mostrar que $\boxed{\dfrac{x^4 - 5x^3 + 2x^2 - x + 1}{x(x^2+1)(x-1)^2} = \dfrac{1}{x} + \dfrac{2x}{x^2+1} - \dfrac{2}{x-1}}$

Teorema 13.1 (Regras de L'Hopital). *Se* $\lim_{x\to a} f(x)/g(x)$ *tem uma forma indeterminada 0/0 ou* ∞/∞*, então*

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

caso o limite $\lim_{x\to a} f'(x)/g'(x)$ *exista (sendo finito ou infinito). O mesmo vale se* a *é substituído por* a^+ *ou* a^-*, ou se* $a = +\infty$ *ou* $-\infty$.

Ex Calcule o limite fundamental $\lim_{x\to 0} \frac{\operatorname{sen} x}{x}$ aplicando L'Hopital

Sol $f(x) = \operatorname{sen} x$, $g(x) = x \Rightarrow \frac{f(x)}{g(x)} = \frac{\operatorname{sen} x}{x} \Rightarrow \frac{f(0)}{g(0)} = \frac{\operatorname{sen} 0}{0} = \frac{0}{0}$

L'Hopital $\Rightarrow \lim_{x\to 0} \frac{\operatorname{sen} x}{x} = \lim_{x\to 0} \frac{(\operatorname{sen} x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$

Logo, $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$

Ex: Calcule $\lim_{x\to 0} \frac{x - \operatorname{sen} x}{x^3} = ?$

Note que a indeterminação é do tipo $\frac{0 - \operatorname{sen} 0}{0^3} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{x - \operatorname{sen} x}{x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{(x - \operatorname{sen} x)'}{(x^3)'} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{(1-\cos x)'}{(3x^2)'} = \lim_{x\to 0} \frac{\operatorname{sen} x}{6x} = \frac{1}{6} \lim_{x\to 0} \frac{\operatorname{sen} x}{x} = \frac{1}{6} \cdot 1 = \frac{1}{6}$$

Ex Calcule $\lim\limits_{x\to+\infty} \dfrac{e^{2x}}{x^3} = \dfrac{e^{2(+\infty)}}{(+\infty)^3} = \dfrac{+\infty}{+\infty}$

Sol. $\lim\limits_{x\to+\infty} \dfrac{e^{2x}}{x^3} \overset{L'H}{=} \lim\limits_{x\to+\infty} \dfrac{2e^{2x}}{3x^2} \overset{L'H}{=} \lim\limits_{x\to+\infty} \dfrac{4e^{2x}}{6x} \overset{L'H}{=} \lim\limits_{x\to+\infty} \dfrac{8e^{2x}}{6} = +\infty$

obs. $\lim\limits_{x\to+\infty} \dfrac{e^{2x}}{x^3} = +\infty$ nos diz que $y = e^{2x}$ cresce ao infinito mais rapidamente que a função $y = x^3$

Obs. 1) $y = x^a$, $x>0 \Rightarrow y' = a\,x^{a-1}$ (regra do tombo)

2) $y = a^x$, $x\in\mathbb{R} \Rightarrow y' = a^x . \ln a$

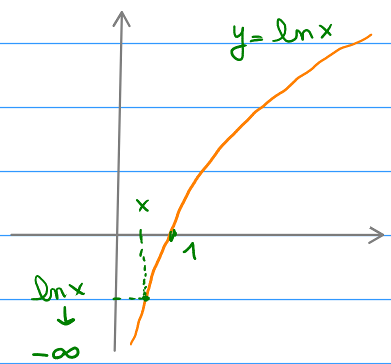
3) Seja $a = e \approx 2,71 \Rightarrow y = e^x \Rightarrow y' = e^x . \underbrace{\ln e}_{=1} = e^x$

4) Seja $y = e^{u(x)} \Rightarrow y' = e^{u(x)} . u'(x)$

se $y = e^{2x} \Rightarrow y' = e^{2x} . \underbrace{(2x)'}_{=2} = 2e^{2x}$.

Ex. Calcule $\lim\limits_{x\to0^+} x . \ln x$

Sol: $\lim\limits_{x\to0^+} x . \ln x = 0 . \underbrace{\ln 0}_{-\infty} = 0.(-\infty)$



$D = \{x\in\mathbb{R} \mid x>0\}$

$\lim\limits_{x\to0^+} x \ln x = \lim\limits_{x\to0^+} \dfrac{\ln x}{\left(\frac{1}{x}\right)} = \dfrac{-\infty}{+\infty}$, aplicamos L'H

$= \lim\limits_{x\to0^+} \dfrac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to0^+} -\dfrac{x^2}{x} = -\lim\limits_{x\to0^+} x = -0 = 0$

Logo $\boxed{\lim\limits_{x\to0^+} x \ln x = 0}$

$\underset{\substack{\downarrow\\0}}{x} . \underset{\substack{\downarrow\\-\infty}}{\ln x} \longrightarrow 0$

Seja $\lim_{x\to a} f(x) = 0$ e $\lim_{x\to a} g(x) = +\infty$ então

$\lim_{x\to a} f(x)\cdot g(x) = 0\cdot(+\infty)$ (indeterminação)

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0} \overset{(*)}{=}$$

$$g(x) \to +\infty \Rightarrow \frac{1}{g(x)} \longrightarrow \frac{1}{+\infty} = 0$$

$$\overset{*}{=} \lim_{x\to a} \frac{f'(x)}{\left(\frac{1}{g(x)}\right)'} \quad , \quad a\in\mathbb{R},\ a=-\infty,\ a=+\infty$$

Suponha que o limite $\lim_{x\to a} f(x)^{g(x)}$ tenha a forma indeterminada $0^0, \infty^0, 1^\infty$. Se $f(x)>0$

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\cdot \ln f(x)}$$

$f(x)>0 \Rightarrow \ln f(x)$ esteja definido

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\cdot \ln f(x)}$$

$$= e^{\lim_{x\to a}[g(x)\cdot \ln f(x)]} = e^{L}$$

Se $\lim_{x\to a}[g(x)\ln f(x)] = L$

Ex Calcule $\lim_{x\to 0^+} x^x = 0^0$

Sol $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\overbrace{\lim_{x\to 0^+}[x\cdot\ln x]}^{=0}} = e^0 = 1$

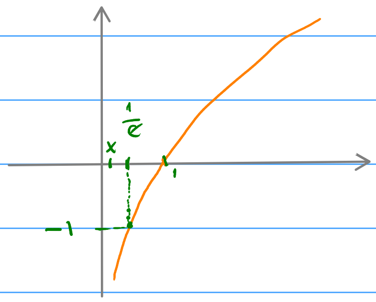
Logo, $\boxed{\lim_{x\to 0^+} x^x = 1}$

Seja $f(x) = x^x, \quad x > 0$

$$f'(x) = (x^x)' = x \cdot x^{x-1} + x^x \cdot \ln x = x^x + x^x \ln x = \underbrace{x^x}_{\neq 0} \underbrace{(1+\ln x)}_{\text{Sinal}} = 0$$

$$\Rightarrow 1 + \ln x = 0 \Rightarrow \ln x = -1 \Rightarrow \log_e x = -1 \iff x = e^{-1}$$

$\Rightarrow x = \frac{1}{e}$ é ponto crítico.



$f'(x)$: ? $\searrow$ $\nearrow$ — 0 — – – – – $\frac{1}{e}$ + + + + + + + + +

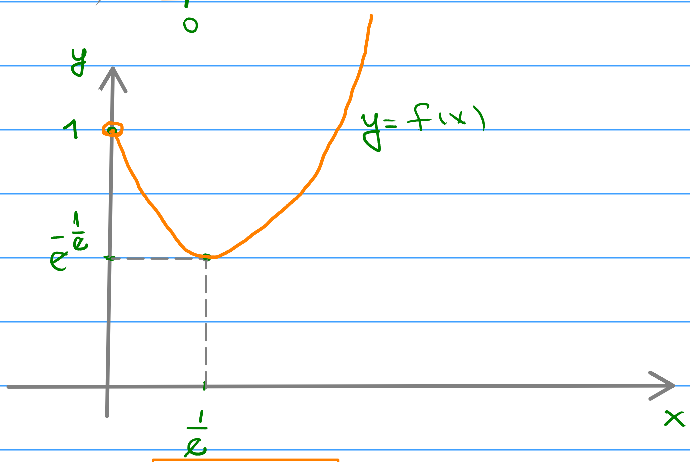
Se $\boxed{0 < x < \frac{1}{e}} \Rightarrow \ln x < -1 \Rightarrow 1 + \ln x < 0 \Rightarrow f'(x) = x^x \underbrace{(1+\ln x)}_{(-)} < 0$

Se $x > \frac{1}{e} \Rightarrow f'(x) > 0$

$$f''(x) = \left[x^x \cdot (1+\ln x)\right]' = \underbrace{(x^x)'}_{x^x(1+\ln x)} (1+\ln x) + x^x \cdot \frac{1}{x} = x^x (1+\ln x)^2 + \frac{x^x}{x}$$

$$\Rightarrow f''(x) = \underbrace{x^x}_{(+)} \cdot \left[\underbrace{(1+\ln x)^2}_{(+)} + \frac{1}{\underbrace{x}_{(+)}}\right] > 0, \quad \forall x > 0$$

$f''(x)$: 0 + + + + + + + + + + + + ∪



$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} = \left(e^{-1}\right)^{\frac{1}{e}} = e^{-1/e}$$

$$\boxed{\lim_{x \to 0^+} x^x = 1}$$

$$f(x) = \begin{cases} x^x & \text{se } x > 0 \\ 1 & \text{se } x = 0 \end{cases}$$

Fato da Vida: $\boxed{e^{-1/e} < 1}$

Exercício: Calcule as equações das retas assíntotas do gráfico da função abaixo

(a) $f(x) = \dfrac{\ln x}{\sqrt[3]{x}}$, $x > 0$ (c) $y = 2x \cdot e^{-1/x}$

Sol:

(a) Uma assíntota para a função $f(x) = \dfrac{\ln x}{\sqrt[3]{x}}$ é uma função $y = ax + b$ tal que

$$\lim_{x \to +\infty} \left[\frac{\ln x}{\sqrt[3]{x}} - (ax+b) \right] = 0$$

(I) $a = \lim_{x \to +\infty} \dfrac{f(x)}{x} = \lim_{x \to +\infty} \dfrac{1}{x} \cdot \dfrac{\ln x}{\sqrt[3]{x}} = \lim_{x \to +\infty} \dfrac{\ln x}{x^{\frac{4}{3}}} \overset{L'H}{=}$

$= \lim_{x \to +\infty} \dfrac{\frac{1}{x}}{\frac{4}{3} \cdot x^{\frac{4}{3}-1}} = \dfrac{3}{4} \cdot \lim_{x \to +\infty} \dfrac{1}{x \cdot x^{\frac{4}{3}-1}} = \dfrac{3}{4} \lim_{x \to +\infty} \dfrac{1}{\sqrt[3]{x^4}} = 0$

Logo, $\boxed{a = 0}$

(II) $b = \lim_{x \to +\infty} [f(x) - \overset{0}{a}x] = \lim_{x \to +\infty} \left[\dfrac{\ln x}{\sqrt[3]{x}} - 0x \right]$

$\overset{L'H}{=} \lim_{x \to \infty} \dfrac{\frac{1}{x}}{\frac{1}{3} x^{\frac{1}{3}-1}} = 3 \cdot \lim_{x \to \infty} \dfrac{1}{\sqrt[3]{x}} = 3 \cdot 0 = 0 \Rightarrow \boxed{b = 0}$

Assíntota horizontal = $y = 0 \cdot x + 0 = 0$

Assíntota vertical = $x = 0$

(c) $y = 2x \cdot e^{-\frac{1}{x}}$, $x \in \mathbb{R} - \{0\}$

Assíntota à direita

$a = \lim_{x \to +\infty} \dfrac{\cancel{2x} e^{-\frac{1}{x}}}{\cancel{x}}$

$b = \lim_{x \to +\infty} [2x e^{-\frac{1}{x}} - ax]$

Assíntota à esquerda

$a = \lim_{x \to -\infty} \dfrac{2x e^{-1/x}}{x}$

$b = \lim_{x \to -\infty} [2x e^{-\frac{1}{x}} - ax]$

$\Downarrow$

$$a = \lim_{x\to+\infty} 2\cdot e^{-\frac{1}{x}} = \lim_{x\to+\infty} \frac{2}{e^{1/x}} = 2 \Rightarrow \boxed{a=2}$$

$$b = \lim_{x\to+\infty} [2xe^{-\frac{1}{x}} - \underbrace{2x}_{ax}] = \lim_{x\to+\infty} 2x.[e^{-1/x}-1] = +\infty.(0)$$

$$= 2.\lim_{x\to+\infty} \frac{e^{-\frac{1}{x}}-1}{\left(\frac{1}{x}\right)} \overset{L'H}{=} 2.\lim_{x\to+\infty} \frac{e^{-\frac{1}{x}}.\left(-\frac{1}{x}\right)'}{\left(-\frac{1}{x^2}\right)} = 2.\lim_{x\to+\infty} \frac{e^{-\frac{1}{x}}.\cancel{\frac{1}{x^2}}}{-\cancel{\frac{1}{x^2}}}$$

$$= -2.\lim_{x\to+\infty} e^{-1/x} = -2.1 = -2 \Rightarrow \boxed{b=-2}$$

Assíntota horizontal = $y = ax+b = 2x-2$

Assíntota vertical = $x = 0$

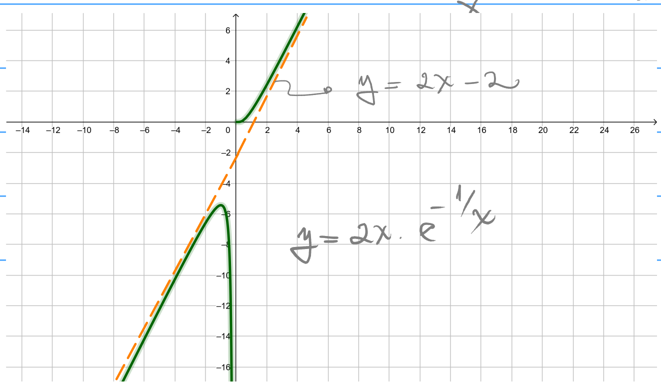
$$y = \frac{2x}{e^{\frac{1}{x}}}$$

$$\lim_{x\to0^+} \frac{2x}{e^{\frac{1}{x}}} = \frac{0}{e^{+\infty}} = \frac{0}{+\infty}$$

$$\lim_{x\to0^-} \frac{2x}{e^{\frac{1}{x}}} = \frac{0}{e^{-\infty}} = 0.e^{\infty} = 0.\infty$$

$$\lim_{x\to0^+} \frac{2x}{e^{\frac{1}{x}}} = 2\lim_{x\to0^+} \frac{1}{\frac{e^{\frac{1}{x}}}{x}} = 2.\lim_{u\to+\infty} \frac{1}{ue^{u}} = 2.\frac{1}{\infty} = 2.0 = 0$$

$$\lim_{x\to0^-} 2x.e^{-\frac{1}{x}} = 2\lim_{x\to0^-} \frac{e^{-\frac{1}{x}}}{\frac{1}{x}} = 2\lim_{u\to-\infty} \frac{e^{-u}}{u} = 2\lim_{u\to-\infty} \frac{-e^{-u}}{1} = 2.+\infty = -\infty$$



Ou seja

$\lim_{x\to0^+} 2xe^{-\frac{1}{x}} = 0$

$\lim_{x\to0^-} 2xe^{-\frac{1}{x}} = -\infty \Rightarrow x=0$ é assíntota vertical à esquerda.