18/10 - Aula 24 - Integração por Mudança de Variável

Seja $f(x) = e^{u(x)}$ então $f'(x) = e^{u(x)} \cdot u'(x)$. De fato,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{e^{u(x+h)} - e^{u(x)}}{h} = \frac{0}{0}$$

$$\frac{e^{u(x+h)} - e^{u(x)}}{h} = \frac{e^{\overbrace{u(x+h)}} - e^{u(x)}}{\underbrace{u(x+h) - u(x)}_{\Delta u}} \cdot \frac{u(x+h)-u(x)}{h} \overset{*}{=}$$

$$\Delta u(x) = u(x+h) - u(x) \Rightarrow \boxed{u(x+h) = \underbrace{u(x) + \Delta u}}$$

$$\overset{*}{=} \frac{e^{u(x)+\Delta u} - e^{u(x)}}{\Delta u} \cdot \frac{u(x+h)-u(x)}{h}.$$ Logo,

$$\lim_{h\to 0} \frac{e^{u(x+h)} - e^{u(x)}}{h} = \boxed{\lim_{\Delta u\to 0} \frac{e^{u+\Delta u} - e^{u}}{\Delta u}} \cdot \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} =$$

$$\boxed{\lim_{h\to 0} \Delta u(x) = \lim_{h\to 0}(u(x+h) - u(x)) = 0 \text{ pois } u(x) \text{ é contínua}}$$

$$= e^{u} \cdot u'(x) \Rightarrow \left(e^{u(x)}\right)' = e^{u(x)} \cdot u'(x),$$ regra da Cadeia. □

$F(x)$ é uma antiderivada ou uma primitiva de $f(x)$ no intervalo aberto $I =]a,b[$ se

$$F'(x) = f(x) \text{ para cada } x \in I.$$

Neste caso, denotamos

$$\int f(x)\,dx = F(x) + C$$

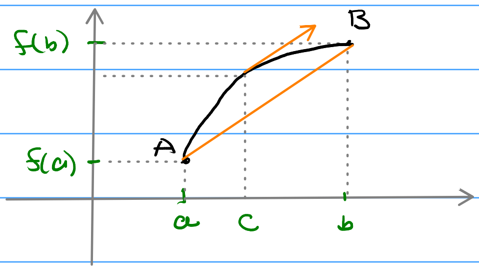
Ex: Calcule $\displaystyle\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + C \Leftrightarrow (\operatorname{arctg} x + C)' = \frac{1}{1+x^2}$

Prop. Se $F_1(x)$ e $F_2(x)$ são primitivas de $f(x)$ em $I =]a,b[$ então, existe $c \in \mathbb{R}$ tal que $F_1(x) - F_2(x) = c$, $\forall x \in I$

DEM. $(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$

$\Rightarrow F_1(x) - F_2(x) = c$, para todo $x \in I$ $\square$

TVM



f é contínua em $[a,b]$

f é derivável em $]a,b[$

$$m = \frac{f(b) - f(a)}{b-a} = f'(c), \quad a < c < b$$

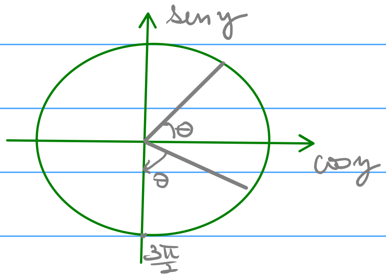
Ex.: Calcule $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx = \operatorname{arc\,sen} x + c$

basta mostrar que $(\operatorname{arc\,sen} x + c)' = \dfrac{1}{\sqrt{1-x^2}}$.

$y(x) = \operatorname{arc\,sen} \underbrace{x}_{\text{argumento}} \Leftrightarrow \boxed{\operatorname{sen} y(x) = x} \Rightarrow (\operatorname{sen} y(x))' = (x)' \Leftrightarrow$

$\cos y(x) \cdot y'(x) = 1 \Rightarrow y'(x) = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-\operatorname{sen}^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$

$\cos^2 y + \operatorname{sen}^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1-\operatorname{sen}^2 y}$



$\cos y \geq 0 \Leftrightarrow y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ (*)

$\cos y < 0 \Leftrightarrow y \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$

Logo, $(\operatorname{arc\,sen} x)' = \begin{cases} \dfrac{1}{\sqrt{1-x^2}} & \text{se } \operatorname{arc\,sen} x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ -\dfrac{1}{\sqrt{1-x^2}} & \text{se } \operatorname{arc\,sen} x \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[\end{cases}$

Ex. Calcule $\int \frac{1}{\sqrt{3-2x}}\,dx =$

$(\sqrt{u})' = \frac{1}{2\sqrt{u}} \Rightarrow \int \frac{1}{2\sqrt{u}}\,du = \sqrt{u} + c \Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + c$

$\Rightarrow \boxed{\int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + \underbrace{2c}_{c}}$

$\int \frac{1}{\sqrt{3-2x}}\,\boxed{dx} = \int \frac{1}{\sqrt{u}}\,dx = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\right)du \overset{*}{=}$

Seja $u = 3-2x \Rightarrow du = u'(x)\,dx = -2\,.dx \Rightarrow \boxed{dx = -\frac{1}{2}\,du}$

$\overset{*}{=} -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\left[2\sqrt{u} + 2c\right] = -\sqrt{u} - c = -\sqrt{3-2x}\ \underbrace{-c}$

Logo, $\boxed{\int \frac{1}{\sqrt{3-2x}}\,dx = -\sqrt{3-2x} + c}$

$u(x) = 3-2x \Rightarrow du = \underbrace{u'(x)}_{=-2}\,dx = -2\,dx \Rightarrow \boxed{-\frac{1}{2}\,du = dx}$

$u'(x) = -2$

Ex: Calcule $\int tg\,x\,dx$

Sol

$\int tg\,x\,dx = \int \frac{\overbrace{\operatorname{sen} x\,dx}^{-du}}{\boxed{\cos x}_{=u}} = \int \frac{-du}{u} = -\int \frac{1}{u}\,du = -\ln|u| + c$

Seja $\underbrace{u = \cos x} \Rightarrow \boxed{du = -\operatorname{sen} x\,dx}$

$\Rightarrow \boxed{\int tg\,x\,dx = -\ln|\cos x| + c}$

$(\ln|\cos x|)'$

Seja $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\Rightarrow \cos x \geq 0 \Rightarrow |\cos x| = \cos x \Rightarrow (\ln \cos x)'$

$= \frac{(\cos x)'}{\cos x} = \frac{-\operatorname{sen} x}{\cos x} \Rightarrow (-\ln|\cos x| + c)' = tg\,x$ se $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$\square$

Ex. Calcular $\displaystyle\int \frac{x}{\sqrt{x^2+5}}\,dx = \sqrt{x^2+5} + c$

Seja $u = x^2+5 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$$\stackrel{*}{=} \int \frac{\frac{1}{2}du}{\sqrt{u}} = \int \frac{1}{2\sqrt{u}}\,du = \sqrt{u} + c = \sqrt{x^2+5} + c$$

Proposição 15.4. *Sendo* $\boxed{a > 0,\ e\ \lambda \neq 0,}$

1. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C.$
2. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C.$
3. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{arcsen}\frac{x}{a} + C.$
4. $\displaystyle\int \frac{dx}{\sqrt{x^2+\lambda}} = \ln\left|x + \sqrt{x^2+\lambda}\right| + C$

$\int f(x)\,dx =$

1. $\displaystyle\int \frac{dx}{a^2+x^2} = \int \frac{1}{a^2+x^2}\,dx$, note que esta integral é parecida com $\boxed{\displaystyle\int \frac{1}{1+u^2}\,du = \operatorname{arctg} u + c}$

a é constante

x é a variável

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx \stackrel{*}{=}$$

Considere a mudança de variável $u = \frac{x}{a} \Rightarrow du = \frac{dx}{a}$

$\Rightarrow \boxed{dx = a\,du}$

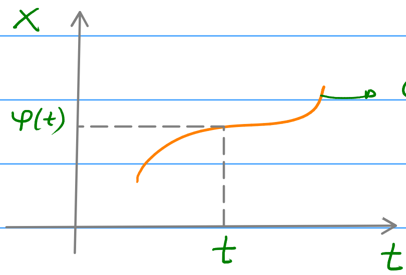
$$\stackrel{*}{=} \frac{1}{a^2}\int \frac{1}{1+u^2}\cdot a\,du = \frac{\cancel{a}}{a^{\cancel{2}}}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{a}\cdot\left\{\operatorname{arctg} u + c\right\} = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + \underbrace{\frac{c}{a}}_{c} \Rightarrow$$

$$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + c}$$

Proposição: Seja $\int f(x)\,dx = F(x) + c$ então

$$\boxed{\int f(\varphi(t)).\varphi'(t)\,dt = F(\varphi(t)) + C}$$



Gr(φ), $\varphi'(t) = \lim_{h\to 0} \frac{\varphi(t+h)-\varphi(t)}{h} \in \mathbb{R}$

H: $\left\{ \int f(x)\,dx = F(x) + c \right.$ T: $\left\{ \int f(\varphi(t)).\varphi'(t)\,dt = F(\varphi(t)) + c \right.$

H $\Rightarrow$ $\boxed{F'(x) = f(x)}$

(Queremos mostrar que $(F(\varphi(t)) + c)' \underset{RC}{=}$

$= F'(\varphi(t)).\varphi'(t) = f(\varphi(t)).\varphi'(t)$

Logo,

$$(F(\varphi(t)) + c)' = f(\varphi(t)).\varphi'(t)$$

$$\Updownarrow$$

$$\boxed{\int f(\varphi(t)).\varphi'(t)\,dt = F(\varphi(t)) + C}$$

Ex. $\int \sqrt{2t-5}\,dt = \frac{1}{2}\int \sqrt{2t-5}\,.\underbrace{2}_{\varphi'(t)}dt = \frac{1}{2}\int \sqrt{\varphi(t)}\,.\varphi'(t)\,dt =$

$x = 2t-5 \Rightarrow dx = 2dt$ onde $\varphi(t) = 2t-5 \Rightarrow \varphi'(t) = 2$

$\frac{1}{2}\int \underbrace{\sqrt{\varphi(t)}}_{f(\varphi(t))}.\varphi'(t)\,dt = \frac{1}{2}.F(\varphi(t)) + c = \frac{1}{2}F(2t-5) + C \overset{*}{=}$

onde $F(u)$ é a primitiva de $f(u) = \sqrt{u} \Rightarrow F(u) = \int u^{\frac{1}{2}}du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}u^{3/2}$

$\overset{*}{=} \frac{1}{2}.\frac{2}{3}.(2t-5)^{3/2} + c = \frac{1}{3}\sqrt[2]{(2t-5)^3} + C$

Ex. Mostre que $\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$, α cte

Sol: $\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{\cancel{(\alpha+1)}\cdot x^{\alpha+\cancel{1}-\cancel{1}}}{\cancel{\alpha+1}} = x^{\alpha}$. Logo, $\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$

Ex $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\cdot\sqrt{x^3} + C$

Logo, $\boxed{\int \sqrt{x}\,dx = \frac{2}{3}\cdot\sqrt{x^3} + C}$

Ex $\int \sqrt{2x-5}\,\underbrace{dx}_{\frac{1}{2}du} = \int \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{2}\int\sqrt{u}\,du = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\sqrt{u^3} + C \overset{*}{=}$

MV $\Rightarrow$ $u = 2x-5$ $\Rightarrow$ $du = (2x-5)'dx = 2dx \Rightarrow dx = \frac{1}{2}du$

$= \frac{1}{3}\cdot\sqrt{(2x-5)^3} + C$ $\square$