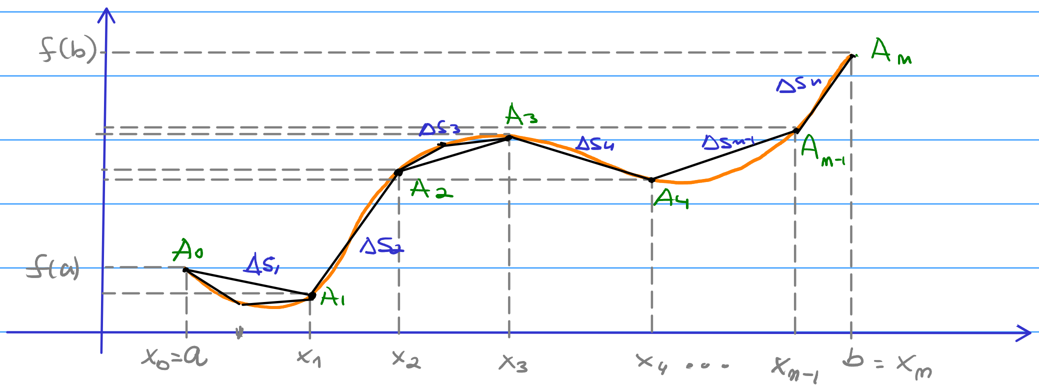
24/11 - Aula 35 - Comprimento de gráfico e Integral de Superfície de Revolução

Comprimento de curva : Seja $y=f(x)$, $a \leq x \leq b$

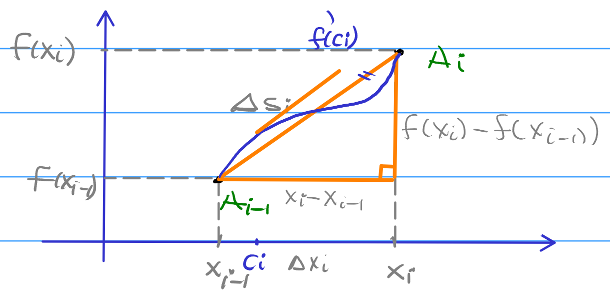
$G_r(f)=\{(x,y) \mid y=f(x) \text{ e } a \leq x \leq b\}$



$A_0=(a,f(a))$ $\qquad$ $A_i=(x_i,f(x_i))$, $0 \leq i \leq m$

$A_1=(x_1,f(x_1))$

$A_2=(x_2,f(x_2))$, ... , $A_{m-1}=(x_{m-1},f(x_{m-1}))$ e $A_m=(x_m,f(x_m))$



$$\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}} = f'(c_i)$$

$$\Delta s_i = \sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2} = \sqrt{\underbrace{(x_i-x_{i-1})^2}_{=\Delta x_i^2} \cdot \left[1+\left(\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right)^2\right]}$$

$$\sum_{i=1}^{m} \Delta s_i = \text{comprimento de linha poligonal}$$

$$\Rightarrow \Delta s_i = \sqrt{1+\left[\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right]^2} \cdot \Delta x_i \quad \text{, onde } \Delta x_i = x_i - x_{i-1}$$

$$\sum_{i=1}^{m} \sqrt{1+\left[\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right]^2} \cdot \Delta x_i \quad \text{comprimento de linha poligonal.}$$

Aplicando o Teorema do Valor médio temos que existe $c_i \in [x_{i-1}, x_i]$ tal que

$$\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i)$$

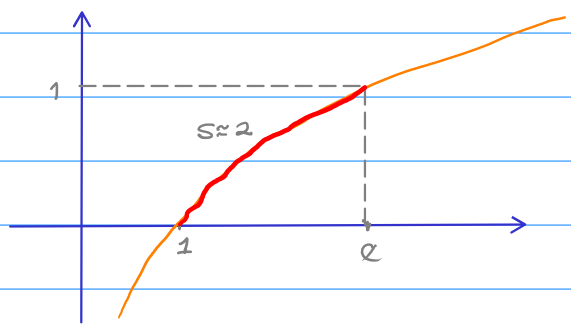
$$\sum_{i=1}^{n} \sqrt{1+\left[\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}\right]^2} \cdot \Delta x_i =$$

$$= \sum_{i=1}^{n} \sqrt{1+(f'(c_i))^2} \cdot \Delta x_i = \text{comp. linha poligonal.}$$

$$S = \lim_{|\Delta| \to 0} \underbrace{\sum_{i=1}^{n} \sqrt{1+(f'(c_i))^2} \cdot \Delta x_i}_{\text{Soma de Riemann}} = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Ex. Seja $f(x) = \ln x$, $1 \leq x \leq e$ então calcule o comprimento do gráfico $Gr(f) = \{(x,y) \mid y = \ln x \text{ e } 1 \leq x \leq e\}$

$$S = \int_1^e \sqrt{1+(f'(x))^2}\, dx = \int_1^e \sqrt{1+\left(\frac{1}{x}\right)^2}\, dx = \int_1^e \frac{\sqrt{1+x^2}}{x}\, dx \approx 2{,}0034$$

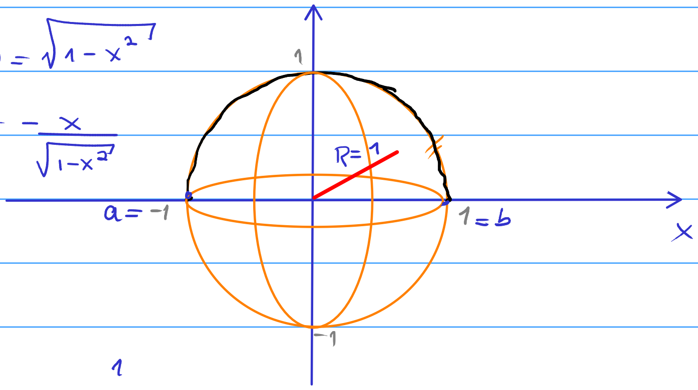


Área de Superfície de Revolução

Ex: Seja $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$

$f(x) = \sqrt{1-x^2}$

$f'(x) = -\dfrac{x}{\sqrt{1-x^2}}$

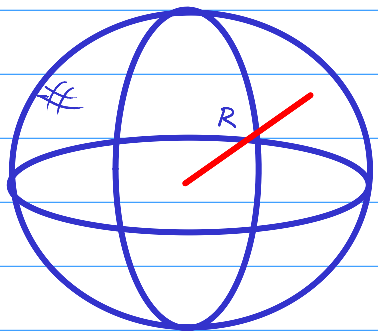


$$S = \int_a^b 2\pi f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx$$

$$S = \int_{-1}^{1} 2\pi \sqrt{1-x^2} \cdot \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx$$

$$= \int_{-1}^{1} \underbrace{2\pi}\, \sqrt{1-x^2} \cdot \sqrt{\frac{1 \cancel{-x^2} + \cancel{x^2}}{1-x^2}}\, dx = 2\pi \int_{-1}^{1} \cancel{\sqrt{1-x^2}} \cdot \frac{1}{\cancel{\sqrt{1-x^2}}}\, dx = 4\pi$$

Logo, $\boxed{S = 4\pi}$



$S = 4\pi R^2$

Ex. Calcule a área da superfície obtida pela revolução da curva $y = \operatorname{sen} x$, $0 \leq x \leq \pi$, rotacionada em torno do eixo x.